A RESTRICTED CONLEY INDEX AND ROBUST DYNAMICS OF COUPLED OSCILLATOR SYSTEMS

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I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

Signature: ——————.

Abstract

In this thesis, we explore the robustness of heteroclinic cycles which can appear as solutions of dynamical systems subject to certain constraints. We develop a method, using topological notions, to inspect the dynamics of simple heteroclinic cycles; in particular, in the first part of the thesis we develop a "restricted Conley index". This tool is defined by restricting the general Conley index to specific invariant subspaces associated with constraints on the vector field. The resulting restricted Conley index allows us to find connections that are robust to perturbations that respect these constraints. In the second part of this thesis we study the dynamical problem of designing a system of globally coupled oscillator systems with a specific structure. We extend some known results on conditions for stability of cluster states in these systems and, as an example, we give sufficient stability conditions for cluster states with three non-trivial clusters, $(2, 2, 2)-$cluster states, in a system of six globally coupled oscillators. We show that robust heteroclinic cycles connecting these states can appear as a result of our investigation on dynamics of a systems of six globally coupled oscillators, and we use the restricted Conley index to investigate the robustness of heteroclinic cycles between three nontrivial clusters.

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Chapter 1

Introduction

A heteroclinic cycle is a type of solution for nonlinear dynamical systems known to occur particularly in systems with symmetry. These cycles appear as a result of a series of connections between equilibria via their stable and unstable manifolds and in certain contexts heteroclinic cycles can be asymptotic stable and robust. We begin this introductory chapter with the basic concepts of nonlinear dynamics and, in particular, heteroclinic cycles. In the second section, we discuss globally coupled systems which have been used to describe dynamics resulting from connections between two or more cluster states to form heteroclinic cycles. Coupled systems showing oscillatory behaviour are often used as models in biological, chemical, and physical problems when the systems considered show periodic behaviour (Lotka-Volterra equations, neural oscillations, coupled pendulums). If two or more of these systems are coupled together (where the coupling might be unidirectional or bidirectional), much more complicated behaviour can appear. Each equation in these systems has a limit cycle as a solution and, depending on the type and details of the coupling, more complex spatio-temporal behaviour can be seen in these systems. Spatio-temporal dynamics includes, for example, when the time evolution of two or more oscillators are periodic with the same period (synchronization). Also, the converse issue of designing a suitable coupling function to ensure the existence (and even stability) of the cluster state described is briefly discussed in the same section. Generally, the invariant subspaces characterized by solutions of dynamical systems are studied either analytically or topologically. In this thesis, we use a topological method to understand the dynamics of the invariant subspaces. This is done by analyzing the topology of such subspaces by dividing them to invariant sets. A particular solution can be the heteroclinic cycles solution; a heteroclinic cycle and its dynamics can be understood by investigating the topology of the invariant sets that form the cycle. The tool for this investigation is the Conley index which, we introduce in the third section. The use of the Conley index is subject to some constraints which will also be discussed.

1.1 Robust heteroclinic cycles for dynamical systems

Systems of ordinary differential equations are a vital tool in the field of applied mathematics. Solutions for these systems can be used to reflect the dynamical behaviour of these systems and to give predictions of long-term behaviour. Consider a system of ODEs given by:

$$\frac{dx}{dt} = f(x); \quad x \in M \ ; \ \ t \in \mathbb{R} \tag{1.1}$$

where f is a smooth C^1 vector field on a smooth manifold M. Consider its associated flow $\phi_f : \mathbb{R} \times M \longrightarrow M$ generated by solutions of such system. For any point $x \in M$, its omega limit set $\omega(x)$ and alpha limit set $\alpha(x)$ are defined by:

$$\omega(x) = \{y \in M : y = \lim_{n\to\infty} \phi_f(t_n, x) \ \ as \ \ t_n \to \infty\}.$$

$$\alpha(x) = \{y \in M : y = \lim_{n\to\infty} \phi_f(t_n, x) \ \ as \ \ t_n \to -\infty\}.$$

Different types of solutions can be found for these systems, and so we introduce the following definitions:

Definition 1.1 *A constant solution q of the system (1.1) where $f(q) = 0$ is called an equilibrium (fixed point).*

Definition 1.2 *A solution $P(t)$ of (1.1) is called a periodic orbit (cycle) if it is a closed curve that is not equilibrium.*

Note that for both equilibria q and periodic orbits $P(t)$ we have

$$\alpha(q) = \omega(q) = q,$$

and

$$\alpha(P(0)) = \omega(P(0)) = P.$$

Definition 1.3 *A solution curve $h(t)$ of the system (1.1) is called a* homoclinic orbit *if it connects an equilibrium point q to itself ($h(0) \neq q$), i.e, if*

$$\alpha(h(0)) = \omega(h(0)) = q$$

Definition 1.4 *A solution curve $h(t)$ of the system (1.1) is called a* heteroclinic orbit *between equilibria if it connects two equilibria $q_1 \neq q_2$, In this case*

$$\omega(h(0)) = q_1 \quad \textit{and} \quad \alpha(h(0)) = q_2.$$

Similarly, one can define a heteroclinic orbit between periodic orbits.

Definition 1.5 *A finite collection Σ of heteroclinic orbits which connect a finite number of equilibria $\{q_1, ..., q_n\}$ in a cycle for the system (1.1) is called a* heteroclinic cycle. *That is*

$$q_1 \to q_2 \to ... \ \to q_n \to q_1$$

where $q_i \to q_j$ means there exists (at least one) heteroclinic orbit from q_i to q_j for all $i, j \in \mathbb{N}$.

Definition 1.6 (invariant set) *A subset $X \subset M$ is called invariant under the dynamics of (1.1) if $x_0 \in X$ implies $\phi(t, x_0) \in X$ for all $t \in \mathbb{R}$.*

Examples of flow−invariant subsets are the sets $\omega(x), \alpha(x)$. A linear invariant set is called an *invariant subspace.*

The simplest form of heteroclinic cycles are those that connect between a set of equilibria, but they can also connect periodic or aperiodic dynamics. For example, heteroclinic cycles have been observed in models of physical systems where the connections are between chaotic sets. Dellnitz et al. in [40] showed how can these cycles occur and in [46] introduced stability results for these type of cycles. A dynamical interpretation of homoclinic and heteroclinic connections can be expressed by saying that, two important connection behaviours can occur near steady solutions for dynamical systems. The simplest one is a homoclinic connection where a branch of the unstable manifold coincides with a branch of the stable manifold for the same equilibrium. The more complex one is a heteroclinic connection in which a branch of the unstable manifold for one equilibrium intersects with a branch of the stable manifold for another equilibrium [69]. While the homoclinic cycle consists of a saddle equilibrium that connects to itself via homoclinic connection, the heteroclinic cycle consists of at least two saddles and a chain of heteroclinic connections connecting them (see [131]). In fact, systems with symmetry or with other constraints that lead to the appearance of invariant solution subspaces and such subspaces can be robustly attracting under perturbations that respect those constraints. Hence the presence of symmetry or any other constraints (systems with prescribed invariant subspaces, e.g., population models) on the system considered preserve the heteroclinic cycles to the perturbations that respect these constraints, and the heteroclinic cycles are called robust. Many articles have investigated robust heteroclinic cycles; an important study was that by Guckenheimer and Holmes in 1988 [62], who analyzed an example of a robust heteroclinic cycle connecting three hyperbolic equilibria in three-dimensional space. These heteroclinic cycles were observed in a nonlinear system by May and Leonard in 1975 [114]. An early article (1979) by Busse and Clever [24] also introduced an ODE model where a robust heteroclinic solution can be found. Moreover, these cycles can be robustly attracting [77] to perturbations that respect these symmetries or constraints. For example, the dynamics of some biological systems (Lotka-Volterra population models in ecology and game dynamics) provide a robust heteroclinic cycles [114, 66]. The dynamics of winnerless competition also gives attractors that are robust heteroclinic cycles.

The stability of heteroclinic cycles between two or more equilibria depends on the eigenvalues of linearization of the vector field in the neighbourhood of each equilibria within the cycle. Many stability types with different strength have been explored by several authors. The strongest stability for heteroclinic cycles is the asymptotic stability [78, 68, 48], where all trajectories started in the neighbourhood of the cycle are attracted to it. Sufficient and necessary conditions for asymptotic stable heteroclinic cycles are given for different examples in [4, 5, 6]. A heteroclinic cycle with weaker stability properties can be seen in other systems, for example, essential asymptotic stability in [95, 86, 79]. Even weaker stable heteroclinic cycles have been studied in [75, 21]. The absence (loss) of stability or the breaking of symmetry leads to bifurcation of the cycle where another heteroclinic cycle may appearance, as well as periodic orbits. Depending on the stability strength, robust heteroclinic cycles may have an attraction feature. Asymptotic stable robust heteroclinic cycles attract nearby trajectories while in a weaker stable heteroclinic cycles attract trajectories in a weaker sense, (Krupa in [77] discusses stability of robust heteroclinic cycles and reviews results up to that time). In this thesis we are going to investigate the robustness of heteroclinic cycles, Chapters 5 and 7, but we will not research on the attraction of these cycles.

An equilibrium with two or more unstable directions that connect to two or more different equilibria and form two (or more) heteroclinic cycles leads to the appearance of a heteroclinic network, defined formally in the following:

Definition 1.7 *A* heteroclinic network *is a connected union of heteroclinic cycles.*

Many systems have used heteroclinic networks to model their dynamics [10, 18, 108, 13, 120]; also, several systems have robust heteroclinoc networks dynamics [67, 77, 114].

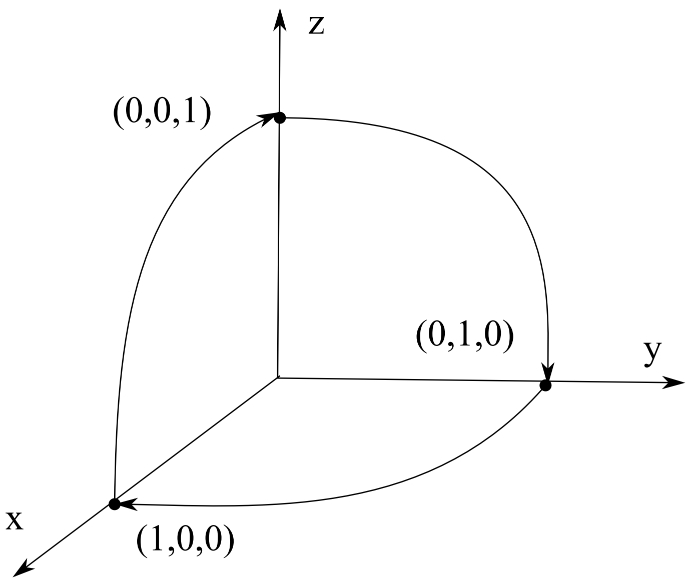

Figure 1.1: Guckenheimer-Holmes robust heteroclinic cycle connecting the equilibria $(1,0,0),(0,1,0),(0,0,1)$ via connections in XY, YZ and $XZ-$ plan respectively.

One of the early heteroclinic networks considered by Kirk and Silber in [75] was formed by two heteroclinic cycles each consisting of three equilibria in four-dimension space where they have shown that, even if neither of the two cycles within the network are stable there may be weaker stability for the network. Other attracting robust heteroclinic networks have been found in the dynamics of Lotka-Volterra systems [77], in symmetric systems [59], and in systems of globally coupled phase oscillators with four or more oscillators (see [14] and other references there in).

Heteroclinic cycles solutions have been observed in a class of systems, namely, globally coupled oscillators system, where the heteroclinic cycles appear as a result of interactions between clusters of oscillators. However, the existence of at least one common equilibrium for two of such heteroclinic cycles results in a heteroclinic network. Therefore, such systems are used to model this sort of dynamical behaviors. In the next section we introduce an overview of the dynamics of globally coupled oscillator systems.

1.2 Globally coupled oscillator systems

The dynamics of coupled nonlinear systems has been the subject of a vast number of articles, in particular for coupled identical oscillators where the individual systems show periodic behaviour. A very early study was conducted by Huygens in 1673 when he noticed the synchronization between pendulums of two clocks. Coupled systems were also used by Van der Pol in 1928 to model biological and physical systems. They have been used for modeling various biological systems, as in [23, 70, 109, 61, 76, 9], and [80, 81, 125, 45, 83, 82, 116].

In contrast to the case of uncoupled oscillators where each oscillator has its own dynamics, in globally (all-to-all) coupled oscillators, even with weak coupling, much more interesting dynamics can appear due to connectivity.

To study the dynamics of this class of coupled systems, the phase oscillators models define the coupled system in terms of the phase of each oscillator (coupled phase model) where the phase of one oscillator is fixed and the phases of all other oscillators are computed with respect to the fixed one [23, 16, 64].

A model introduced by Kuramoto in [80] was revolutionary in this area for understanding the dynamics of globally coupled oscillators systems. Dynamics of chemical and biological systems have been modeled using globally coupled oscillators systems [36, 80, 81]:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)), \tag{1.2}$$

where θ_i is the phase state of the i^{th} oscillator, ω_i is the natural frequency for each oscillator, K is the coupling strength, and N is the number of oscillators.

Besides the properties of individual oscillators (including natural frequency, phase, and frequency), the number of oscillators, the coupling strength, and the type of coupling between oscillators all have an effect on the dynamics of globally coupled nonlinear systems. Indeed, the coupling between the oscillators (which is expressed by a coupling function) can give complicated dynamics: different types of synchronization, stable cluster states, switching between different cluster states. More details on the dynamics of globally coupled oscillators can be found in [126, 105, 3]. A cluster state is the state where the oscillators partition into smaller collections such that oscillators within each class of partition behave the same; however, three or more saddle cluster states can be part of a robust heteroclinic cycle [10, 64]. Geometrically, as the topology behaviour of each oscillator is a periodic orbit, so in the coupled oscillator systems the stable cluster states can be viewed as a limit cycle. A general approach to investigate the stability of cluster states is to determine the eigenvalues of the linearization of the vector field in the neighbourhood of each cluster [105]. However, dynamical systems with symmetric properties (also called equivariant) have special behavior, as discussed in [60]. In [16] one can find a description for the geometry of the phase space of an equivariant coupled system. In this thesis we consider globally coupled oscillator systems that are symmetric systems, in Chapter 6 we will assume that the system is symmetric under the continuous rotation S^1 and the full permutation group S_N (the symmetric group of all permutations on N symbols).

Coupled nonlinear systems of chaotic oscillators have even more difficult dynamics, which has been studied in [136]; such systems are called chaotic systems. The synchronization dynamics of these chaotic systems is called a chaotic attractor; the chaotic attractors with different types of synchronizations have been considered in [57].

The aim of the articles we have mentioned so far in this section is to study the cluster dynamics of globally coupled oscillator systems with a specific coupling function [15, 23, 109], [2, 18, 108]. At the same time, there are other, different models where the robust heteroclinic cycles have been found [135, 15]. The converse of this approach was considered by the authors of [105], where they introduced a set of constraints on the coupling function and its first derivative in order to ensure the existence and stability of a specific stable cluster state. They generalized the stability results of a special class of cluster state introduced in [23, 104] to general cluster states. In order to ensure both the transversal and tangential stability conditions, they derived inequalities that ensured the stability of the assumed cluster state. They used the idea of using Fourier series expansion of the coupling function [38, 104] (because of the periodicity of coupling function). In particular, the same article mentioned the problem of designing the coupling function associated with a non-trivial stable 3-cluster state in a general context. In this thesis we will extend this work in Chapter 6

As indicated in the previous section, we will use topological notions to analyze invariant subspaces of solutions for constrained dynamical systems. The next section will be devoted to introducing the topological concepts which we will need in this thesis.

1.3 Topological methods for invariant sets

The topology or geometry of solutions can be used to study the dynamics of systems [106]. In fact, beside other methods, understanding the topology of solution subspaces can be used to predict dynamical behaviour. A widely used topological tool is the Conley index [29], which has been used very successfully in dynamical systems to predict the dynamics inside specific invariant subspaces by dividing them into their invariant sets. The Conley index is defined for an isolated invariant set using homology and/or homotopy theory to characterize the topology of such invariant sets from which one can deduce information about the solutions. This topological tool was first discussed in a paper by Conley and Easton [30]. The use of the Conley index to understand the dynamics of invariant subspaces has been the aim of many papers [49, 90, 93, 94, 97, 98, 99, 129]. Conley in his book: *"Isolated invariant sets and the Morse Index"* [29] introduced many general aspects and properties of the index. Afterwards, Salamon [117] simplified and generalized some proofs from [29]. A series of papers by Conley and Smoller [34, 33, 27, 32] used this index in an application to show the existence of shock waves. Furthermore, other authors have made modifications to the Conley index on non-compact spaces for semiflow (Rybakowski [115]) or the existence of Conley's connection matrices (Franzosa [52, 54, 53, 55, 113, 111]), the relation between the transition matrices and the existence of co-dimension one connecting orbits [110, 92, 56].

The theory of applying the Conley index to discrete dynamics has been studied by Robbin, Salamon and Morzek [102, 112, 103, 128, 39, 39, 51], while Floer Homology was introduced by Floer in [49]. Moreover, in numerical applications the use of the Conley index produces an economic tool one can use to study the dynamics [100, 99, 130]. We are interested in developing a Conley index for heteroclinic cycles (which are a solution for different forms of dynamical systems) to determining whether or not these cycles are robust. We were inspired by the papers of Kappos [72, 73], who provides a very readable review, although it was not an early paper on this subject, it does discuss conditions for the existence of heteroclinic orbits that we have generalized here to heteroclinic cycles.

1.4 Thesis overview

This thesis studies two basic dynamical issues. Namely, how to use topology to analyze the dynamics of a system and how to design a dynamical system with a prescribed topology.

The first part of this thesis deals with the investigation of the topological properties of invariant sets. Our tool for this analysis is a type of Conley index. A general introduction to the Conley index with other topological notions is given in Chapter 2. In Chapter 3 we present a new concept, the "Restricted Conley Index". We show theoretical results for this tool, including that it is well-defined and other properties that allow us to use it to study robustness property for some dynamics, specifically robustness of heteroclinic connections. These results are considered on systems under the constraint that they are defined on a compact manifold and the existence of the Morse function in the neighbourhood of subspace under consideration. Our main Theorem 3.15 in this chapter assumes an isolation property on the intersection set of heteroclinic orbits and the chosen invariant set.

Chapter 4 presents a general introduction to equivariant (symmetric) dynamical systems. We start this chapter by reviewing the action of Lie group on a general manifold, followed by results on an orthogonal compact Lie group action on $\mathbb{R}^n$. However, we introduce an example 4.1 of an action of the permutation group S_6 on $\mathbb{R}^n$ with its lattice diagram 4.1 which we will use in Chapter 6. In Section 4.3 we review the action of a compact orthogonal Lie group on nonlinear systems of ordinary differential equations, and its consequences when analyzing the dynamics of these systems. Next, in Chapter 5 we use our results from Chapter 3 both in a general context and in some special cases. The general context is to apply the restricted Conley index to an invariant set which contains a heteroclinic cycle of hyperbolic equilibria. We also introduce our tool to show robustness of a special known system of heteroclinic cycle (the Guckenheimer-Holmes heteroclinic cycle) and a heteroclinic network (the Kirk-Silber heteroclinic network). Also, we discuss a special case of implications between robustness of heteroclinic connections via a restricted Conley index and previous results of other authors using transversality.

The second part of this thesis starts with Chapter 6, in which we introduce a strategy to design a system of six coupled oscillators with prescribed dynamics. We start by assuming the existence of the dynamically complex stable cluster state and state a set of conditions should such system satisfy to have the assumed dynamics. We derive novel conditions on stability including both transverse and tangent stability. Also, we report some numerical results where we find values of Fourier coefficients that result in the desired dynamics.

Chapter 7 joins the two parts of the thesis by computing the restricted Conley index defined in Chapter 3 for heteroclinic cycles of cluster states shown in Chapter 6. This chapter starts by studying dynamics of saddle cluster states for different partitions of the six oscillators and then we give the scheme for the construction of heteroclinic cycles formed by the symmetry of the system. In this chapter we show the robustness of these cycles via the restricted Conley index. We end the chapter with general discussion of the design of a coupling function gives any non-trivial stable $3-$cluster state.

Finally, in Chapter 8 we give a discussion of other possible studies related to our investigation on the restricted Conley index and design the coupling function. In fact, if we can define the restricted Conley index for dynamical systems with less constraints, the restricted Conley index will apply more generally and make a general equivalence between transversality condition and the restricted Conley index for robustness of dynamics. We also discuss the possibility of using the restricted Conley index to show robustness of certain non transversal intersections for higher dimensions (≥ 2) subspaces. We discuss, too possible generalization of our results for design of a non-trivial stable $3-$cluster state in coupled oscillator systems. Moreover, in terms of tangential stability for non- trivial $m-$cluster states ($m \geq 3$) more investigations are needed to design the coupling function. The problem of designing a coupling function for a certain cluster state is also discussed.

Chapter 2

Topology of dynamical systems

This introductory chapter reviews some basic geometric concepts and constructions that are used to understand the behaviour of dynamical systems. Such approaches are known as the algebraic topology or geometry of dynamical systems. The two subjects homology and homotopy are about studying the nature of a topological space via an algebraic structure (group or sequence of groups). These algebraic structures give information about the type and number of shapes that exist within the topological space. One of the topological tools that apply to isolated invariant sets is the Conley index, which can be characterized using homotopy, homology or cohomology. The definition we adopt here is based on homology. As the following sections will show, the Conley index is a powerful tool that can be used to understand the global behaviour of flows. In his book [29], Conley introduces a detailed theory of this index, and many articles have explained features of the index as a generalization of the Morse index and how it can be used to understand dynamics, see Chapter 1 and [117, 33, 30].

2.1 Homology theory

We give a brief discussion of the relevant homology theory for convenience, as this is not typically studied in dynamical systems. For more details and proofs, many sources can be found: for example [35, 65, 107]. Generally, a homology group can be defined on any topological space, but here we will only consider homology groups for subsets of the Euclidean space $\mathbb{R}^n$.

2.1.1 The simplex, boundary operator, chain complex

We begin by clarifying the meaning of geometric independence, which is needed to define the key concept of this section, the *s*implex. A set of $p+1$ points $\{a_0, \cdots, a_p\} \subset \mathbb{R}^n$ is geometrically independent if the set of vectors $\{a_1 - a_0, \cdots, a_p - a_0\}$ are linearly independent [119].

Definition 2.1 *[119] For a set $A = \{a_0, ..., a_p\}$ of $p+1$ points in $\mathbb{R}^n$, the* p-simplex *on A is the set of all subsets of A, and is denoted by $\sigma = \langle a_0, \ldots, a_p \rangle$.*

Geometrically, a $p-$*simplex* for a set of geometrically independent points $\{a_0, ..., a_p\}$ in $\mathbb{R}^n$ is the smallest convex set containing them.

Each of the $a_0, ..., a_p$ is called a *vertex* and the number p is the *dimension* of σ. A $p-$simplex σ is called *standard* if it can be expressed as:

$$\sigma = \{(a_0, a_1, ..., a_p) : \quad \sum_{r=0}^{p} a_r = 1 \quad a_r \geq 0\}.$$

For an $p-$simplex, any non-empty subset of m points $(m < p)$ defines an $m-$simplex and is called a *face* of the simplex. Generally, the number of m faces in an $p-$simplex is given by the binomial $\binom{p+1}{m+1}$ if we take orientation appropriately. The union of all the faces of σ is known as the *boundary of* σ, denoted by $\partial\sigma$, and can be defined as:

$$\partial(\{a_{i_0}, \cdots, a_{i_p}\}) = \sum_{j=0}^{p} (-1)^j \{a_{i_0}, \cdots, \hat{a_{i_j}}, \cdots, a_{i_p}\}.$$

From the definition of the faces of a simplex, the following theorem can be proved.

Theorem 2.1 *[35] For any $p-$simplex σ, $\partial\partial(\sigma) = 0$.*

Definition 2.2 (Simplicial complex) *A* simplicial complex Δ *is a finite collection of simplices which satisfy the following conditions:*

1. *Any two simplices within Δ are either disjoint or their intersection is a face of both of them.*

2. *Each face of a simplex within Δ is in Δ.*

The *dimension of a simplicial complex* Δ is the largest positive integer d such that Δ has a $d-$simplex. The simplicial complex of all simplices in Δ of dimension $\leq r$ is called the $r-$skeleton of Δ $(r > 0)$. From the definition of simplicial complex and Theorem 2.1, the boundary of simplicial complex is zero if it is so for all of its simplices.

A simplicial complex is called *oriented* if it has a total order on its vertices. From now on any assumed simplicial complex is supposed to be oriented.

Definition 2.3 *For a simplicial complex Δ, define the* union of its elements *as*

$$|\Delta| = \cup_{\sigma\in\Delta}\sigma.$$

There are natural ways to define a topology on $|\Delta|$ (e.g., the Euclidean subspace topology)

Definition 2.4 *Let X be a topological space, a* triangulation *of X is defined as a homeomorphism $h : |\Delta| \to X$ where Δ is a simplicial complex.*

2.1.2 Fundamental aspects of group theory

Now, we turn to the problem of associating a group to a topological space where the associated group is called homology group. Intuitively, a collection of special class of groups that are related together in a way which leads to the definition of homology group. We now introduce basic ideas from the theory of groups.

Recall that if G_1, G_2 are two groups, the *kernel* of a group homomorphism $f : G_1 \to G_2$ is defined as $\ker f = \{g \in G_1 : \ f(g) = e; \ \textit{where } e \textit{ is the identity of } G_2\}$ and the *image* of a homomorphism f is defined as $\mathrm{im} f = \{f(g) : g \in G_1\}$. A bijective homomorphism is called an *isomorphism* and is denoted as $G_1 \cong G_2$. For any group homomorphism f there is an isomorphism $G_1/\ker f \cong \mathrm{im} f$.

Definition 2.5 *Let $(G, +)$ be an abelian group. If there is a finite set of elements $g_1, g_2, \cdots, g_n \in G$ which freely generate G; (i.e., any $g \in G$ can be uniquely expressed as $g = k_1 g_1 + \cdots + k_n g_n$ where $k_i \in \mathbb{Z}$ for all i), then we say that G is* free abelian group.

The elements $\{g_1, g_2, \cdots, g_n\}$ are called a basis of G and their number is known as the *rank* of G.

Definition 2.6 *Let $G_i; \;\; i = 1, 2, 3$ be groups, we say a sequence of the following homomorphisms*

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3$$

is exact *if $imf_1 = \ker f_2$.*

This implies that $f_2 \circ f_1 = 0$. An exact sequence that connects three groups and such that f_1 is monomorphism and f_2 is epimorphism is called a *short exact sequence.* In this case the quotient group can be defined as $G_2/f_1(G_1)$. Moreover, a short exact sequence in which each group is abelian can be written as

$$0 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \longrightarrow 0.$$

Generally, for an infinite set of groups $G_i; \;\; i \in N$, a sequence of group homomorphisms

$$\cdots \longrightarrow G_{i+1} \xrightarrow{f_{i+1}} G_i \xrightarrow{f_i} G_{i-1} \longrightarrow \cdots$$

is called an *exact sequence* if it is exact at G_i for all i (more details in [35]).

2.1.3 Simplicial and singular homology

Definition 2.7 *Let Δ be an oriented simplicial complex and $\sigma \in \Delta$ is an $n-$simplex. The set of linear combinations of finitely many oriented $n-$simplex σ defined by*

$$C_n(\Delta) = \left\{ \sum_i c_i\sigma_i : \ \sigma_i \ is \ n - simplex \ in \ \Delta \ and \ c_i \in \mathbb{Z}; \quad i \in \mathbb{N} \right\}$$

is called a simplicial $n-$chain.

Elements of $C_n(\Delta)$ are called chains. The set $C_n(\Delta)$ forms a group under an operation of chain addition, it is called a *simplicial $n-$chain*.

Theorem 2.2 *$C_n(\Delta)$ is an abelian group generated by the $n-$simplices of Δ [118].*

Define a boundary operator ∂_i; for each $i \in \mathbb{N}$; between any successive groups $C_i(\Delta)$ as:

$$\cdots \longrightarrow C_{i+1}(\Delta) \xrightarrow{\partial_{i+1}} C_i(\Delta) \xrightarrow{\partial_i} C_{i-1}(\Delta) \xrightarrow{\partial_{i-1}} C_{i-2}(\Delta) \longrightarrow \ldots.$$

Such a boundary operator sends each $i-$simplex into its boundaries. The kernel of ∂_i; $(\ker(\partial_i))$, is a subgroup in $C_i(\Delta)$ and is called *the cycles*. (a cycle is a chain such that its boundary is zero) whilst the elements of $\mathrm{im}(\partial_{i+1})$ are called boundaries.

The $(C_i(\Delta), \partial_i)$ forms a simplicial chain complex and $\mathrm{im}(\partial_{i+1}) \subset \ker(\partial_i)$. A quotient group $H_i(\Delta)$ can be defined as follows:

$$H_i(\Delta) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1}).$$

Such a group is called the *simplicial i^{th} homology group of* Δ.

The rank of the simplicial homology group $H_i(\Delta)$ defines a number of shapes in Δ of dimension i. The elements of $H_i(\Delta)$ are called homology classes. Two cycles are called *homologous* if they represent the same homology class, in other words two $i-$cycles are homologous if they are the boundary of $i + 1-$chain.

These groups help us to understand the topological properties of topological spaces; furthermore, in the following we extend the concept of homology groups; namely, singular homology groups, which are based on building a topological space X from standard simplices rather than proper simplices [65]. In fact, a singular homology is more general than a simplicial one in that it can be applied to more general topological spaces.

Definition 2.8 (Singular homology group) *[65] Let σ be a standard $n-$simplex and X be a topological space. A continuous map $\tau : \sigma \longrightarrow X$ is called a* singular $n-$simplex.

In the same manner one builds the algebraic structure of simplicial homology groups one can construct the singular homology groups as follows [65].

A finite linear combination $\sum_i c_i\tau_i$ for $i \in \mathbb{N}$ is called a singular $n-$chain and the set $S_i(X) = \{\sum_i c_i\tau_i : \tau_i \textit{ is a singular } n - \textit{simplex and } c_i \in \mathbb{Z};\ i \in \mathbb{N}\}$ form an abelian group. In order to define the singular homology groups one needs to define the boundary operators on the abelian groups $S_i(X)$ (for $i \in N$) as

$$\partial_i : S_i(X) \longrightarrow S_{i-1}(X)$$

such that

$$\partial_i \circ \partial_{i+1} = 0 \quad \forall i \in \mathbb{N}.$$

Then for all $i \in \mathbb{N}$, the pair $(S_i(X), \partial_i)$ defines the $i^{th}-$singular homology groups on X as follows:

Definition 2.9 *[65] The $i^{th}-$*singular homology group *on X; H_i^s; is defined as:*

$$H_i^s = \ker(\partial_i)/im(\partial_{i+1}).$$

These singular homology groups define different classes of singular $i-$cycles of X which enclose shapes which are not boundaries of singular $i+1-$chains (i.e., different elements in H_i^s refer to different homology classes).

Simplicial and singular homology groups have many similarities (they are the same when applied to topological spaces of polyhedra type), although the singular homology group has the advantage that it can be applied to any topological space [35]. Various other types of homology groups have been defined and discussed, for example in [26].

There is a dual concept to the homology group called the *cohomology group*, which is defined in a dual way and which satisfies similar properties to homology. The main difference is that in cohomology the boundary operators are defined in ascending order where this difference leads relatively to extra structure [65].

2.2 Homotopy theory

In this section we briefly introduce how to study the structure of topological spaces by means of embeddings curves in the space. Such an investigation is called the "Homotopy theory" which is concerned with studying topological properties up to equivalence via continuous deformation. In the following we review the basics of homotopy and its relations to homology; references can be found on homotopy theory and related concepts in [20, 22, 65].

In Chapters 3 and 4 homology groups are used to understand the properties of pointed topological spaces under homotopy equivalence. In order to do this, we first show why homology groups can be used to understand equivalence under homotopy [87].

2.2.1 Homotopy equivalence

Homotopy is a weaker topological relation between different topological spaces when compared with homeomorphism, (two spaces X and Y are called *homeomorphic* if there exists a homeomorphism $f : X \to Y$, denoted $X \cong Y$. A bijective map $f : X \to Y$ such that f and f^{-1} are continuous is called a *homeomorphism*). Two homeomorphic spaces coincide in their topological properties, but this is not necessarily the case for homotopy spaces. However, homeomorphic spaces implies they are homotopy equivalent but not vic-versa

Definition 2.10 *Let $f_1 : X \to Y$ and $f_2 : X \to Y$ be two continuous functions between topological spaces X, Y. Then f_1 and f_2 are* homotopic *if there is a continuous function $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f_1$ and $F(x, 0) = f_1$ for any $x \in X$.*

Homotopic equivalence of functions is denoted by $f_1 \simeq f_2$. A function that is homotopic to a constant function is called a null-homotopic. Note that homotopy is an equivalence relation on the set of all continuous functions [35].

Definition 2.11 *Two topological spaces X, Y are said to be* homotopy equivalent *(have the same homotopy type) if there are two continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to I_Y (the identity function on Y) and $g \circ f$ is homotopic to I_X (the identity function on X).*

In other words, two topological spaces are homotopy equivalents if they can be transformed into one another by continuous deformation (expanding or shrinking).

Definition 2.12 *A topological space is called* contractible *if it is homotopically equivalent to a point.*

2.2.2 Path, loop, fundamental group

Definition 2.13 *Let X be a topological space and $I = [0, 1]$. A continuous map $P : I \to X$ is called a* path *in X.*

For any two points $x_1, x_2 \in X$, two paths P_1, P_2 that connect x_1 and x_2 are *homotopic* within a subset $S \subset X$ if there exists a continuous function $F : I \times I \longrightarrow S$ such that $F(s, 0) = P_1(s)$, $F(s, 1) = P_2(s)$ and $F(0, t) = x_1$, $F(1, t) = x_2$ for all s, t and $S \subset X$. In such a case, we say that the function F is a *continuous deformation* of P_1 to P_2. We denote by $[P]$ the class of all paths that are homotopy equivalent to the path P.

Definition 2.14 *A topological space X is called* path connected *if any two points in X have at least one path connecting them. A path $P : I \to X$ with $P(0) = P(1) = x_0$ (for some $x_0 \in X$) is called a* loop. *The point x_0 is said to be the base point for this loop.*

In the set of all homotopy equivalence classes for loops in a space X, let p_1, p_2 be any two loops through the base point x_0. The product of the two loops p_1, p_2 is defined by

$$p_1.p_2 = +\begin{cases} p_1(2t) & \text{if } \ 0 \le t \le \frac{1}{2} \\ p_2(2t-1) & \text{if } \ \frac{1}{2} \le t \le 1 \end{cases}$$

Assume that their homotopy equivalence classes are $[p_1], [p_2]$, a product operation is defined on the set of all equivalence classes by:

$$[p_1][p_2] = [p_1 p_2].$$

This can be used to define the fundamental group as follows:

Definition 2.15 *Let X be a topological space and $x_0 \in X$. The group $\pi_1(X, x_0) = \{[p_i] : \ p_i \text{ is a loop with base point } x_0; \ \ i \in N\}$ is called a* fundamental group *of X with base point x_0.*

If X is path connected then $\pi_1(X, x) \cong \pi_1(X, y)$ for any base points $x, y \in X$, [65]. In this case we drop dependence on the base point x and write $\pi_1(X)$.

Definition 2.16 *A* simply connected space *X is a path connected space such that $\pi_1(X)$ is trivial.*

In general, the fundamental group is not abelian, but if X is a path connected topological space (e.g. $\mathbb{R}^n$) then an isomorphism between its abelianization and its first singular homology group is as in the following theorem. (For any group G its abelianization G^{ab} is defined by the quotient group $G^{ab} := G/[G, G]$ where $[G, G] := \{[g_1, g_2] : [g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2 \ for \ all \ g_1, g_2 \in G\}$ is the smallest subgroup of G containing such elements).

Theorem 2.3 *[65, Thoerem 2A.1] For any topological space X there is a natural homomorphism $h : \pi_1(X) \to H_1^s(X)$. Moreover, if X is a path connected topological space then h defines an isomorphism and $\pi_1\ (X)^{ab} \cong H_1^s(X)$.*

Proposition 2.4 *[35]*

1. *Fundamental groups associated with homeomorphic topological spaces are isomorphic (that is, the fundamental group is a topological invariant of a space).*
2. *Homotopic topological spaces have isomorphic fundamental groups.*

In the following we introduce a generalization of the fundamental group that there are groups of higher dimensions called homotopy groups. In fact, homotopy groups of dimension more than one are abelian and there is an exact sequence of a particular type of homotopy groups. Although it is typically difficult to compute homotopy groups of higher dimensions, many important results have been introduced (e.g [65]) using its commutativity property. In the following we state the theorem that connects homotopy and homology groups of a topological space.

Definition 2.17 (Homotopy group, [65]) *Let X be a topological space $x_0 \in X$, $I^n = I\times \cdots \times I$ ($n-$times). Define the set of homotopy classes $\pi_n(X, x_0) = \{[f]\ | f : (I^n, \partial I^n) \to (X, x_0)$ and $f_t(\partial I^n) = x_0\ for\ all\ t\}$. Under a well-defined sum operation in $\pi_n(X, x_0)$; for $n > 1$, the abelian group conditions are satisfied. The group $\pi_n(X, x_0)$ is called the* homotopy group *of (X, x_0).*

Theorem 2.5 (Hurewicz isomorphism [65]) *If X is a simply connected space, then $\Pi_n\ (X) \cong H_n^s(X)$ for all $n \geq 2$.*

Definition 2.18 *For a topological space X and any point $x_0 \in X$, a* pointed topological space *is a pair (X, x_0).*

Definition 2.19 *For any two pointed spaces* $(X, x_0), (Y, y_0)$*, the* wedge sum of X *and* Y *is defined by*

$$X \vee Y = x \sqcup Y / \sim$$

where $\sqcup$ *represents disjoint union, and* $\sim$ *is the equivalence relation that identifies the associated base points.*

In algebraic topology many concepts are homotopy invariant: if two spaces are homotopy equivalent then if one is path connected so is the other. The following is an important theorem that relates homology and homotopy.

Theorem 2.6 *[65, Theorem 2.10] If* X *and* Y *are homotopy equivalent then the homology (cohomology) group of one is isomorphic to the homology (cohomology) group of the other. i.e.,*

$$H_i(X) \cong H_i(Y) \quad (homology \quad groups);$$

$$H_i^*(X) \cong H_i^*(Y) \quad (cohomology \quad groups)$$

for all $i \in \mathbb{N}$.

We will use this theorem in later chapters when trying to describe the topology of complicated topological spaces. This is done by defining the homotopy type of such topological spaces and finding the homology group for the simpler space which is equivalent to the original up to homotopy by Theorem 2.6.

In general, it is not easy to compute the homotopy groups (even the fundamental group), and it is even more difficult in higher dimensions.

2.3 Morse theory and gradient-like vector fields

This section is devoted to discussing a particular type of vector field associated with ordinary differential equations; namely, the gradient-like vector field, which is used to study the topology of manifolds. Further detail on the theory of gradient-like vector fields is introduced in Section 2.3.1. Applications of this theory to finite dimensional smooth compact manifolds was introduced by Smale [121]. In order to ensure the existence of gradient-like vector fields, a special class of real-valued function called the Morse function need to be defined. We start by describing the Morse function and its basic properties.

2.3.1 Morse function

The concept of the Morse function was first introduced by Morse in [101]. Other good references are [19, 96], and related articles are cited in [50]. In the following we assume that M is a compact $m-$dimensional differentiable manifold.

Definition 2.20 *Let $P : M \to \mathbb{R}$ be a real-valued smooth function on an m-dimensional differentiable manifold M. A point $q \in M$ is called a* critical point *of P if $\nabla P(q) = 0$.*

A critical point q is called *non-degenerate* if the Hessian matrix at q

$$[H(q)]_{ij} = \left[\frac{\partial^2 P}{\partial x_i \partial x_j}\right]$$

is non-singular. Otherwise (i.e., if the Hessian matrix at q is singular) q is called a *degenerate critical point.*

Definition 2.21 *A real-valued function $P : M \to \mathbb{R}$ is called a* Morse function *if it does not have any degenerate critical points.*

In general, Morse functions exist on any smooth manifolds; a generic smooth function is a Morse function [88].

Theorem 2.7 *[121, Smale theorem] Every compact smooth manifold M admits a Morse function $f : M \to \mathbb{R}$.*

Definition 2.22 *The* Morse index *of a non-degenerate critical point q is the number of negative eigenvalues of its Hessian matrix.*

Sometimes, we simply refer to the Morse index of a non$-$degenerate critical point q as its index.

Lemma 2.8 *[88, Morse Lemma] Let $P : M \to \mathbb{R}$ be a Morse function which has a critical point q of index k. Then in a neighbourhood U of q there are coordinates $(v^1, \dots, v^n)$ such that $v^i(q) = 0 \quad \forall i$ (i.e. q corresponding to $(0, \cdots, 0)$) and inside $U \quad P((v^1, \dots, v^n) = P(q) - (v^1)^2 - \cdots - (v^k)^2 + (v^{k+1})^2 + \cdots + (v^n)^2$.*

Lemma 2.8 says that, starting at the critical point q, the Morse function decreases in all k directions and increase in the other coordinate directions. We conclude that the index k of a non$-$degenerate critical point q describes the behavior of the Morse function P. This helps one compute the homotopy type of the manifold.

2.3.2 Gradient-like vector fields

Let $P : M \to \mathbb{R}$ be a Morse function defined on a closed m-dimension differentiable manifold M and q be a critical point of P.

Definition 2.23 *[137] Let $P : M \to \mathbb{R}$ be a Morse function on a compact smooth manifold M such that $crit_P$ denotes the set of all critical points of P. A C^∞ vector field f on M is called a* gradient-like vector field *of P if the following conditions are satisfied:*

1. *$f(q) = 0$ for all $q \in crit_P$;*
2. *$f(x)P > 0$ for all $x \notin crit_P$;*
3. *we can pick coordinates $(x_1, \cdots, x_m)$ so that locally (near q) P is given by:*

$$P = -x_1^2 - ... - x_k^2 + x_{k+1}^2 + ... + x_m^2 + P(q). \tag{2.1}$$

More generally, f is called gradient-like if it is gradient-like for some Morse function P. As stated in Theorem 2.7 in a compact manifold a generic function is a Morse function, then a gradient-like vector field can be defined on any compact manifold. To get closer to our goal of analyzing the dynamics of systems of ordinary differential equations via topology of their solution subspaces, we now introduce some concepts related to invariant sets and some results used to analyze them [30].

2.4 The topology of isolated invariant sets

Nonlinear systems display a wide variety of solutions, from simple fixed points to more complicated chaotic invariant sets. In the following we give a quick review of an important class of invariant sets that isolates by appropriate choice of neighbourhood (see [30] for more details).

2.4.1 Isolating neighbourhoods and isolated invariant sets

Let f be a smooth (C^1) vector field on a smooth manifold M and consider its associated flow $\phi_f : \mathbb{R} \times M \longrightarrow M$ generated by solution of the system of ordinary differential equation:

$$\frac{dx}{dt} = f(x); \quad x \in M \ ; \ \ t \in \mathbb{R} \tag{2.2}$$

The limiting behaviour of $x \in M$ with respect to the flow ϕ_f can be understood in terms of the $\omega-$limit set and the $\alpha-$limit set defined as follows:

$$\omega(x) = \{m \in M : \lim_{k\to\infty} \phi_f(t_k, x) = m \ as \ t_k \to \infty\}$$

$$\alpha(x) = \{m \in M : \lim_{k\to\infty} \phi_f(t_k, x) = m \ as \ t_k \to -\infty\}.$$

A set $S \subset M$ is called invariant for the flow ϕ_f if

$$\phi_f(t, S) = S \ \ for \ all \ \ t \in \mathbb{R}.$$

Definition 2.24 *A compact set $N \subset M$ such that any of its points which has its entire orbit in N is in the interior of N (i.e., if $n \in N$ such that $\phi_f(t,n) \in N$ for all $t \in \mathbb{R}$ then $n \in Int(N)$; where $Int(N)$ denotes the interior set of N) is called an* isolating neighbourhood *for the flow ϕ_f.*

We define the maximal invariant set S(N) within N to be:

$$S(N) = \{x \in N : \phi_f(t,x) \in N \; for \; all \; t \in \mathbb{R}\}.$$

We should note that S(N) may be empty.

Definition 2.25 *A subset N has an* internal tangent *if there exists $x \in \partial N, \epsilon \geq 0$ such that $\phi(t,x) \in \bar{N}$ for any $t \in (-\epsilon, \epsilon)$, where $\bar{N}$ denotes the closure set on N.*

Definition 2.26 *An isolating neighbourhood N is called an* isolating block *if there are no internal tangencies of the flow to the boundary of N.*

Definition 2.27 *An invariant set S is called* isolated *if it is the maximal invariant set in some isolating neighbourhood (block) N, denoted by $S(N)$.*

From now on we assume manifold M to be a compact metric space.

An isolating neighbourhood is robust to perturbation of the flow in the compact open topology. In [29] it has been proven that if N is an isolating neighbourhood for a flow ϕ defined on a compact metric space M, then there exists a neighbourhood V of N in the compact open topology such that N remains an isolating neighbourhood for all flows in V.

2.4.2 Attractor-repeller decomposition

The simplest isolated invariant sets (such as fixed points or periodic orbits) are simple trajectories, but this is not the case with more complicated invariant sets [8]. Morse decomposition is a method to decompose an invariant set into smaller invariant sets and connecting orbits between these sets, see for example in [122, 84, 91]. Such a decomposition allows to describe the global dynamics of flows on a compact metric space. This section discusses the Morse decomposition of an isolated invariant set, and as a particular case, the decomposition of an invariant set into an attractor−repeller pair [29, 89].

Definition 2.28 *The* Morse decomposition *of an isolated invariant set S is a finite collection of mutually disjoint isolated compact invariant sets $\{\mu_i(S)\}_{i\in\mathbb{N}}$ where we can define an order partial relation $\prec$ on it as:*
$\mu_i \prec \mu_j$ if there are $\mu_{k_0} = \mu_i, \mu_{k_1}, \cdots, \mu_{k_l} = \mu_j$ and $x_1, x_2, \cdots, x_l \in M$ with $\alpha(x_l) \subset \mu_{k_{j-1}}$ and $\omega(x_l) \subset \mu_{k_j}$ for $j = 1, \cdots, l$.
Moreover, if $x \in S \setminus \mu(S)$ there exist $x_1, x_2 \in \mu(S)$, $x_1 < x_2$ such that $\omega(x) = x_2$ and $\alpha(x) = x_1$.

Each element of $\mu(S)$ is called a *Morse set* and $S\backslash\mu(S)$ the set of orbits connect x_1 and x_2. The simplest non-trivial Morse decomposition is an attractor-repeller decomposition; more details can be found in [89, 29].

Definition 2.29 *Consider an invariant set $S \subset M$, a compact set $A \subset S$ is called an* attractor *in S if there exists a neighbourhood U of A in S such that $A = \omega(U \cap S)$. The dual of A in S is called a* repeller *and defined as $A^* = \{x \in S : \omega(x) \cap A = \emptyset\}$.*

The set $C(A^*, A; S)$ denotes the set of *connecting orbits* from A^* into A within S and defined as:

$$C(A^*, A; S) = \{x \in S : \omega(x) \subseteq A, \omega^* \subseteq A^*\}$$

Definition 2.30 *Let A be an attractor and A^* be the corresponding repeller for an invariant set S; the pair (A, A^*) is called an* attractor-repeller pair decomposition *of S. In this case we can write S as a decomposition:*

$$S = A^* \cup C(A^*, A; S) \cup A.$$

If an invariant set S is isolated, then so are A and A^*.

Chapter 3

The Conley index and a restricted Conley index

The Conley index is a powerful tool for the characterization of invariant sets, for global bifurcations and, in particular, for proving the existence of connecting orbits in flows. We describe a setting where robust heteroclinic connections can appear and use a restricted Conley index to prove some necessary conditions for robustness. Although many results can be obtained using standard transversality arguments, we suggest that these techniques will extend to more general situations. However, to our knowledge it has not been used previously to show the persistence of heteroclinic networks for constrained systems.

The aim of this chapter is show that there is a simple generalization of the Conley index, known as a "restricted Conley index", for constrained systems that allows one to give alternative proofs to those based on the transversality of robustness of heteroclinic cycles in such contexts. This chapter is organized as follows. In Section 3.2 we introduce the restricted Conley index (a brief discussion of the structural stability of Morse-Smale vector fields in more detail is provided in an appendix). In Section 3.3 we show that the satisfaction of conditions that ensure existence of heteroclinic orbit (at least one) between two hyperbolic equilibria using the Conley index. However, under even more constraints we prove the robustness of such a heteroclinic orbit using our tool, a restricted Conley index. Using the restricted Conley index to show the robustness of heteroclinic orbits that do not intersect transversely is discussed in the last Section 3.4.

3.1 The Conley index

The decomposition of isolated invariant sets as shown in the previous chapter gives a simplification of the isolated invariant set and will make it easier to understand their dynamics intuitively. The topological tool we develop in this thesis is an extension of the Conley index. This section is an introduction to the Conley index for flows on locally compact spaces. For more details see [29, 117, 123]. The Conley index can be defined for either an isolated invariant set, or for an isolating block as it is shown to be equivalent for all isolating blocks for the same invariant set.

The main results of Conley index theory are that this index is well-defined and robust (continues) under continuous changes to the flow. Various authors [73, 100] give illustrative examples of computation and applications of the Conley index, one of the most fundamental being that if the index of an isolating neighbourhood N is non-trivial, then $S(N)$ is non-trivial

As various authors working on the Conley index have used different but equivalent notation (and indeed definitions) for the Conley index [29, 117, 100], here we fix on a notation close to that used by Kappos in [72, 73].

3.1.1 Preliminaries to Conley index for flow

Let $\mathcal{X}(\mathbb{R}^n)$ be the set of C^2 vector fields on $\mathbb{R}^n$; ($\mathbb{R}^n$ is a locally compact metric space); (you may assume this is globally Lipschitz to ensure that the flow is defined for all positive times, though this will not be necessary for the discussion). Consider the flow $\phi_f(t, x)$ generated by the solutions of

$$\frac{dx}{dt} = f(x) \tag{3.1}$$

where $f \in \mathcal{X}(\mathbb{R}^n)$.

Suppose $N \subset \mathbb{R}^n$ is some isolating neighbourhood and let $S(N)$ be the isolated invariant set in N.

Definition 3.1 *A subset $A \subset N$ is called* positively invariant *in N if for any $x \in A$ and $\phi([t, 0], x) \subset N$ then $\phi([t, 0], x)) \subset A$ (i.e. orbits from A must stay in A as long as they remain in N).*

Definition 3.2 *For a flow ϕ on $\mathbb{R}^n$, let N be an isolating neighbourhood of an isolated invariant set $S(N)$. A subset $A \subset N$ is an* exit set *of N if:*

1. *A is positively invariant in N;*
2. *$N \setminus A$ isolates S;*
3. *Orbits leaving N must go through A.*

Definition 3.3 *A pair (N, A) is called an* index pair *of $S(N)$, where A be an exit set of N as defined in definition 3.2.*

The (homotopy) Conley index of an invariant set is defined as follows [72, 73]:

Definition 3.4 *Suppose $N \subset \mathbb{R}^n$ is some isolating neighbourhood, $S(N)$ be the isolated invariant set in N and A be an exit set of N. The (homotopy)* Conley index, *$C(S)$, of the isolated invariant set $S \subset \mathbb{R}^n$ is defined to be the equivalence class of the pointed space $(N/A, [A])$ under homotopy equivalence, (i.e., the homotopy type of the pointed topological space).*

By this, we mean that we consider all homotopic spaces to N/A where A is "collapsed" to a point as equivalent. Because any isolated invariant set is contained in an isolating neighbourhood, one can also define $C(N)$ more generally for an isolating neighbourhood as follows:

$$C(N) = C(N, \phi) \sim C(S(N, \phi)).$$

Due to the difficulty of working on homotopy classes, and since homotopic spaces have isomorphic homology groups (Theorem 2.6), up to homotopy equivalence, one can define the Conley index using homology concepts. As mentioned previously that the singular homology is more general in terms of topological spaces, so we will define Conley index using the concept of singular homology as follows:

The *singular homology Conley index* is defined to be [29]:

$$CH_i^s(S) := H_i^s(N/A, [A]) \approx H_i^s(N, A); \quad i \in \mathbb{N}$$

Since the homotopy Conley index is the homotopy type of a topological space, so a consequence of Theorem 2.6, the singular homology Conley index defined as above. The definition of the Conley index is well−defined [29, 8] and is computable. One early existence result that can be stated is the following:

Theorem 3.1 *[117, Wazewski Property] Let N be an isolating neighbourhood such that its Conley index is not homotopically equivalent to the trivial space, then N encloses a non−trivial isolated invariant set. (i.e., if $C(N) \not\approx 0$ then $S(N) \neq \emptyset$).*

The converse of Theorem 3.1 is not true; for example, the Conley index of an isolating neighbourhood N which contains a saddle and a sink fixed point is homotopy equivalent to the trivial space.

Using this property one can conclude the existence of an isolated invariant set; in any of references on Conley index mentioned above one can find the proof of an application of this results which says that if $C(S(N)) \approx \mathbb{Z}$, then $S(N))$ contains a fixed point. There are also more general results concerning the existence of more complicated isolated invariant sets [29].

However, there are other theorems we state them below which use Conley index to show (predict) the existence of orbits connecting different isolated invariant sets [32, 124, 34]. All of the following existence theorems are based on the following result.

Theorem 3.2 *[8] If S_1, S_2 are two disjoint isolated invariant sets and $S = S_1 \cup S_2$ then*

$$C(S) = C(S_1 \cup S_2) = C(S_1) \vee C(S_2).$$

Theorem 3.3 *[124, 72] Let N be an isolating neighbourhood for the gradient−like vector field ϕ such that exactly two equilibria are in N, at least one of which is hyperbolic. Assuming that $S(N)$ is the maximum invariant set in N such that $C(S(N)) = 0$ then there exists a connection orbit joining the equilibria.*

Theorem 3.4 *[32, 72] Let q_1, q_2 be two equilibria contained in an isolating neighbourhood N for the flow ϕ. If*

$$C(S(N)) \neq C(q_1) \vee C(q_2)$$

then there exists a connection orbit of ϕ different from q_1 and q_2 within N. Moreover, if ϕ is gradient−like in N then the orbit connects q_1 and q_2.

Theorem 3.5 *[89] Let S be a compact invariant set with an attractor-repeller pair (A, A^*). Then S is an isolated in X if and only if S is a topological embedding of a zero-sphere or a k-sphere, or $k = 1$ and S is a topological embedding of the unit interval.*

3.1.2 Singular connecting orbits in attractor-repeller pairs

As an attractor-repeller decomposition is a method by which one can divide an invariant set into simpler sub-invariant sets, applying the Conley index on these small invariant sets reveals the dynamics of the whole invariant set.

Recall that if S is an isolated invariant set with attractor-repeller pair (A, A^*) then

$$S = A \cup C(A, A^*; S) \cup A^*$$

where $C(A, A^*; S)$ is the set of connecting orbits from A^* into A in S. Therefore, the Conley index can be computed for each of S, A, A^* [85]. In particular, in the next chapter we will analyze the dynamics of an isolated invariant set S which contains two hyperbolic equilibria q_1, q_2 and the set $C(q_1, q_2; S)$ of connections orbits between q_1 and q_2. Because Conley indices can be computed for q_1, q_2, we still need to study the set of connecting orbits by analyzing each connecting orbit individually using the separations of the connecting orbit set. In this subsection we review some general conditions from [89] under which a connecting orbit in an attractor repeller pair is isolated and the consequences. Also, some results on connecting orbits between two hyperbolic equilibria are reviewed.

Definition 3.5 *[89] Let $J \subset \mathbb{N}$, X be a topological space. A collection of disjoint open subspaces of X; $\{x_i : \ i \in J\}$, such that $X = \cup_{i \in J} x_i$ is called an $J-$*separation *of X.*

However, for the flow ϕ_f on $\mathbb{R}^n$ an *invariant $J-$separation* is a $J-$separation such that each x_i is an invariant under the flow.

Remark 3.1 *[89]*

1. *If $\{s_i\}_{i \in J}$ is an invariant $J-$separation of a compact invariant set S then J is finite.*
2. *If $\{s_i\}$ is an invariant separation of a compact invariant set S then S is isolated if and only if each s_i is isolated.*

For a compact invariant set S in $\mathbb{R}^n$ with attractor-repeller pair (A, A^*), the previous two remarks can be restated as follows:

Lemma 3.6 *[89, Corollary 2.2]*

- *If $\{C_i\}_{i \in J}$ is an invariant $J-$separation of $C(A, A^*; S)$ then J is finite.*
- *If the set S is isolated and $\{C_i\}_{i \in J}$ is a separation of the set $C(A, A^*; S)$ then each $S_i = A \cup C_i \cup A^*$ is isolated for each $i \in J$.*

Using the separation defined in Lemma 3.6, the homology Conley index can be defined for S with an attractor-repeller pair as follows [89]. For an isolated invariant set S with attractor-repeller pair (A, A^*) and the set $C(A, A^*; S)$ of connecting orbit, one can compute the homology Conley index for S and A, A^* as well because they are both isolated. Using results from Kurland [85] there exist compact subsets $N_0 \subseteq N_1 \subseteq N_2$ such that (N_2, N_0) is an index pair for S in $\mathbb{R}^n$, (N_1, N_0) is an index pair for A in $\mathbb{R}^n$, and (N_2, N_1) is an index pair for A^* in $\mathbb{R}^n$ (Figure 3.1 illustrates an index triple of two saddle points x_1, x_2 for a planar flow where (N_2, N_0) is the index pair of the maximum invariant set S containing them in $\mathbb{R}^2$, (N_1, N_0) is the index pair of x_1 in $\mathbb{R}^2$ and (N_2, N_1) is the index pair of x_2 in $\mathbb{R}^2$). These compact subsets $N_0 \subseteq N_1 \subseteq N_2$ form an index triple of $(S; A, A^*)$, Kurland in [85] relates between the homotopy Conley indices of S, A, A^* by construct a long coexact sequence. However, Franzosa in [52] constructs an exact sequence relating the homology Conley indices of S, A, A^*. Assume N_0, N_1, N_2 are as describe above, there exists an exact sequence

$$\cdots \xrightarrow{\partial} H_*(N_1, N_0) \xrightarrow{i_*} H_*(N_2, N_0) \xrightarrow{j_*} H_*(N_2, N_1) \xrightarrow{\partial} \cdots$$

There exists an exact sequence between homology Conley indices of attractor-repeller sequence of $(A, A^*; S)$ as follows:

$$\cdots \to CH_i(X; A) \longrightarrow CH_i(X; S) \longrightarrow CH_i(X; A^*) \to \cdots$$

Moreover, for each S_k, there exists an exact sequence between homology Conley indices of attractor-repeller sequence for each $(A, A^*; S_k)$

$$\cdots \to CH_i(X; A) \longrightarrow CH_i(X; S_k) \longrightarrow CH_i(X; A^*) \to \cdots$$

The simplest example of an attractor-repeller pair is that which connects two hyper-

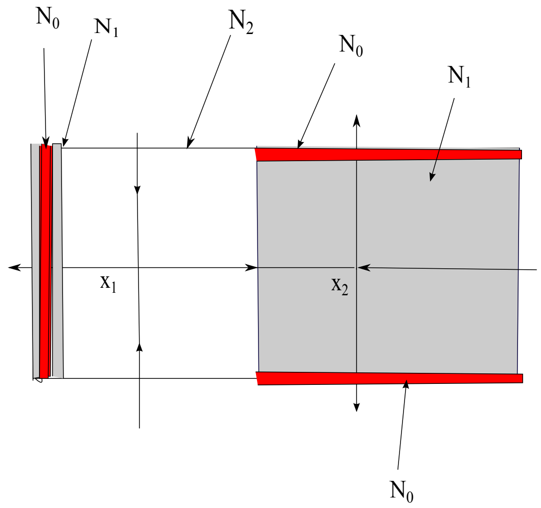

Figure 3.1: An illustration of compact sets N_0, N_1, N_2 that used to define the index triple needs to compute the homology Conley indices of two hyperbolic equilibria x_1 and x_2 in $\mathbb{R}^2$ and the whole isolated invariant set S that contains them. N_2 is represented by the whole figure; the set N_1 is represented by the gray area and the set N_0 is represented by the red area

bolic equilibria q_1, q_2 (the definitions of hyperbolic equilibrium and its unstable and stable manifolds are stated in Chapter 4; section 4.3) . The following theorem states sufficient conditions which lead to isolated subinvariant sets $S_k = q_1 \cup C_k \cup q_2$ within S for each $k \in \mathbb{N}$.

Theorem 3.7 *[89, Theorem 3.1] Let q_1, q_2 be two hyperbolic equilibria in a manifold X such that $dim(W^u(q_1)) = r + 1$ and $dim(W^u(q_2)) = r$ with transverse intersection between $W^u(q_1)$ and $W^s(q_2)$. Then every connecting orbit $C_i \in C(q_1, q_2; S)$ has an isolated invariant set; for each $k \in \mathbb{N}$.*

As a consequence of this theorem, the set $S = cl(C(q_1, q_2; X))$ is an isolated invariant set containing finitely many connecting orbits between q_1 and q_2 including the two hyperbolic equilibria. Even in the case of non-transverse intersection between $W^u(q_1)$ and $W^s(q_2)$, the set $(S; q_1, q_2)$ is isolated and the set of connections $C(q_1, q_2; S)$ is connected. Under small perturbation of the flow the transverse intersection can be achieved [89].

In the next section, we introduce the continuity property of the Conley index. As the existence results in the next subsection use the cohomology Conley index so that is what we will use.

3.1.3 Continuation of the Conley index

We summarize some basic results concerning the continuation of the Conley index from [29, 117, 100]. Continuation here means the persistence of the Conley index for the same isolated invariant set under small continuous perturbation of flow within a compactly parametrized space. Let X be a locally compact metric space, Λ be a compact, locally contractible, connected metric space ("parameter space") and the parametrized flow $\Phi : \mathbb{R} \times X \times \Lambda \longrightarrow X \times \Lambda$ defined by $\Phi(t, x, \lambda) := (\phi^\lambda(t, x), \lambda)$. Then new notations will be defined as ϕ^λ a continuous parametrized family of flow $\phi^\lambda : \mathbb{R} \times X \to X$; and $N^\lambda := N \cap (X \times \lambda)$ for any isolating neighbourhood $N \subset X \times \Lambda$.

We define projection maps $\pi_X : X \times \Lambda \to X$ and $\pi_\Lambda : X \times \Lambda \to \Lambda$.

Proposition 3.8 *[117] Let N be an isolating neighbourhood for the flow ϕ^{λ_0}. Then there is $\epsilon > 0$ such that N remains an isolating neighbourhood for ϕ^λ for any $\lambda \in \Lambda$ whenever $d(\lambda, \lambda_0) < \epsilon$.*

Theorem 3.9 *[8, Theorem 2.3.2] Let N be an isolating neighbourhood for a flow ϕ^{λ_0}. Then there is an $\epsilon > 0$*

$$CH^s(N, \phi^\lambda) \approx CH^s(N, \phi^{\lambda_0})$$

for any $\lambda \in \Lambda$ such that $d(\lambda, \lambda_0) < \epsilon$.

Definition 3.6 *Let S_{λ_i} be an isolated invariant set for the family of flows ϕ^{λ_i} where $\lambda_i \in \Lambda$; $i \in \mathbb{N}$. We say that $S_{\lambda_1}(N^{\lambda_1}, \phi^{\lambda_1})$ and $S_{\lambda_2}(N^{\lambda_2}, \phi^{\lambda_2})$ are related by continuation if there is an isolating neighbourhood $N \subset X \times \Lambda$ of the parametrized flow Φ.*

The $S_{\lambda_1}(N^{\lambda_1}, \phi^{\lambda_1}), S_{\lambda_2}(N^{\lambda_2}, \phi^{\lambda_2})$ denote the maximal invariant sets inside $(N^{\lambda_1}, \phi^{\lambda_1})$ and $(N^{\lambda_2}, \phi^{\lambda_2})$, respectively.

Theorem 3.10 *[8]*

$$CH_i^s(S_{\lambda_j}) \approx CH_i^s(S_{\lambda_k}).$$

whenever S_{λ_j} *and* S_{λ_k} *are related by continuation,* $j, k \in \mathbb{N}$.

3.2 A restricted Conley index for a flow with invariant subspaces

Suppose that I is an affine subspace in $\mathbb{R}^n$ such that I is *invariant* for the flow ϕ_f generated by the ODE on $\mathbb{R}^n$

$$\frac{dx}{dt} = f(x) \tag{3.2}$$

where $f \in \mathcal{X}(\mathbb{R}^n)$ is a smooth vector field on $\mathbb{R}^n$. Note that this is the case if and only if $x \in I$ implies that $f(x) \in T_x I$ (the tangent space of I at x). We write $f \in \mathcal{X}_I$ to be the subset of vector fields in $\mathcal{X}$ that leave I invariant and start with some basic observations.

Lemma 3.11 *Suppose* N *is an isolating neighbourhood for* $f \in \mathcal{X}_I$ *and* $N \cap I \neq \emptyset$. *Then* $I \cap N$ *is an isolating neighbourhood for* f_I.

Proof: Note that $I \cap N$ is open relative to I as I is an affine subspace. Moreover, $f \in \mathcal{X}_I$ implies that the maximal invariant set $S(N \cap I) = S(N) \cap I$; to see this, first note that $S(N \cap I) \subset S(N) \cap I$ as $S(I) = I$. Now consider $x \in S(N) \cap I$. This implies that $\phi_f(x, t) \in N \cap I$ and so $x \in S(N \cap I)$. Hence $S(N \cap I) = S(N) \cap I \subset Int(N \cap I)$ and so the neighbourhood is isolating. **QED**

The converse of Lemma 3.11 is not true. One can find a set N and f such that $I \cap N$ is an isolating neighbourhood but N is not; similarly, $S(N) \cap I$ is an isolated invariant set for f_I, but not necessarily f. Deviating slightly from the notation we have used in Chapter 2, we use $C^f(N)$ to denote the (homotopy) Conley index for the flow generated by f for the isolating block N.

Definition 3.7 *For the flow ϕ_f, we define the* Conley index for N restricted to I *by*

$$C_I^f(N) = C^{f_I}(N \cap I)$$

if N is an isolating block, and similarly

$$C_I^f(S) = C^{f_I}(S \cap I)$$

if S is an isolated invariant set for the flow.

Lemma 3.12 *If N is an isolating block for the flow generated by $f \in \mathcal{X}_I$ and $N \cap I \neq \emptyset$, then the restricted Conley index $C_I(N)$ is well defined.*

Proof: By Lemma 3.11, $N \cap I$ is an isolating neighbourhood for f_I. Moreover, if N has no internal tangencies then so for $N \cap I$, meaning it is an isolating block for f_I. Hence we can define the Conley index $C^{f_I}(N \cap I)$ of the flow restricted to I. **QED**

Theorem 3.13 *Suppose that $f \in \mathcal{X}_I$ leaves I invariant and N is an isolating neighbourhood for f with $N \cap I \neq \emptyset$. Then $C_I(N)$ is robust to perturbations of f within $\mathcal{X}_I$.*

Proof: Standard results on the continuation of the Conley index (e.g., [117])[Lemma 6.1] show that if N is an isolating neighbourhood for the compactly parametrized f_λ with $f_0 = f$ then there is a neighbourhood of $\lambda = 0$ such that N remains an isolating neighbourhood for ϕ_{f_λ}. By continuation of the Conley index (e.g., [100]) [Theorem 3.10] and Lemma 3.11 $N \cap I$ will remain an isolating neighbourhood perturbations of $\phi_f(t, x)$, $C_I^f(N)$ will remain the same for these perturbations; (see also Appendix 3.1.3). **QED**

As discussed in the Chapter 2, another useful feature of the Conley index theory is the ability to decompose isolated invariant sets with difficult internal structures into sub invariant sets and connecting orbits between sets. The simplest case of this decomposition is the attractor-repeller pair.

3.3 The restricted Conley index and attractor-repeller decompositions

Earlier work [29, 31] used criteria based on the Conley index for gradient-like vector fields to prove the existence of heteroclinic connections between different isolated invariant sets. In this section we use the idea of the attractor-repeller decomposition of an isolated invariant set to prove some theoretical results for the generated flow. We need these results in order to fit the restricted Conley index to previous results which have used Conley index to show existence of heteroclinic connections. To do this we need the following result:

Theorem 3.14 *Suppose that x_0 and x_1 are hyperbolic equilibria for the flow associated to $f \in \mathcal{X}(M)$ and suppose that there is a set of connection orbits*

$$S = W^u(x_0) \cap W^s(x_1)$$

such that $S \cup \{x_0, x_1\}$ is a compact invariant set with an attractor-repeller x_0, x_1. If $S \cup \{x_0, x_1\}$ is isolated then f is gradient-like on some neighbourhood of S.

Proof: Consider a compact neighbourhood of $S \cup \{x_0, x_1\}$ that is a union of three parts; two compact neighbourhoods $N_{0,1}$ that are neighbourhoods of $x_{0,1}$ such that the standard form (2.1) can be chosen for a Morse function, and a neighbourhood N_2 of the connecting orbit S. Because $\tilde{S} = S \setminus (N_0 \cup N_1)$ is compact and consists only of transient trajectories, there will be a maximum duration for which trajectories remain in $\tilde{S}$ and, by continuity, in any small enough neighbourhood N_2 of $\tilde{S}$. If we choose a Morse function on N_2 that is the given by the time since entry into N_2, then we can interpolate between the Morse function on the three neighbourhoods to obtain a Morse function that is valid on $N = N_0 \cup N_1 \cup N_2$. **QED**

We remark that there are plenty of robust heteroclinic cycles such that the set of connecting orbits $S \cup \{x_0, x_1\}$ are not compact isolated invariant sets; for example, see [11].

In particular, Smoller and Conley (as we restated in Theorem 3.4 [31]) showed that if there is an isolating neighbourhood N containing two hyperbolic equilibria x_i and x_{i+1} with

$$C(N) \neq C(x_i) \vee C(x_{i+1}) \tag{3.3}$$

for a gradient-like vector field, then there exists (at least one) orbit in N that connects x_i and x_{i+1}. However, for connections in robust heteroclinic cycles, one typically has

$$C(N) = C(x_i) \vee C(x_{i+1}) \tag{3.4}$$

and so one cannot conclude the connection is robust by naive application of this result. However, the next result aims to show that by considering a restricted Conley index, one may use existing results on invariance of the Conley index to show robustness, even in cases where (3.4) holds. We review some relevant results on the continuation of the Conley index of an isolated invariant set under perturbations of the flow using parametrized local flow in Appendix 3.1.3.

We motivate our results by the following example, before showing how we use the restricted Conley index to give sufficient criteria for the robustness of heteroclinic connection.

Example 3.1 *We consider the flow indicated by a planar vector field sketched in Figure 3.2.*

Consider an ODE $\dot{x} = f(x)$ where $x \in \mathbb{R}^2$ and x_1, x_2 are equilibria of the saddle type for the flow ϕ_f generated by the vector field f. The Conley indices of each of these hyperbolic equilibria (with unstable manifold of dimension one) is a homotopy sphere of dimension one; so the Conley index of each equilibrium is a pointed circle. If N is an isolating block

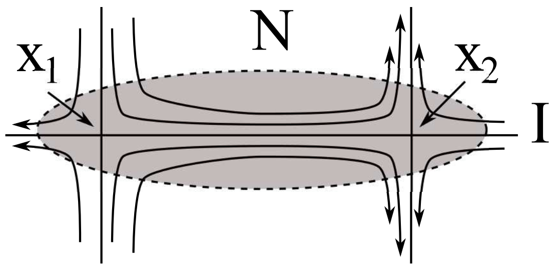

Figure 3.2: A connection between two saddle points for a planar flow, with an isolating neighbourhood N. This can be shown to exist using the restricted Conley index $C_I(N)$ for the invariant subspace I, but not using the usual index $C(N)$.

that contains x_1 and x_2 as in Figure 3.2, then $C(N)$ clearly has the homotopy type of a figure eight. On the other hand, if we restrict the flow to the invariant set I then we can compute the restricted Conley indices: $C_I(x_1)$ is one-sphere (x_1 is saddle), $C_I(x_2)$ is two points (x_2 is a sink with an empty exit set) while $C_I(N \cap I)$ is a single point and (3.3) holds, meaning that the existence of the connection can be checked by this criterion.

Theorem 3.15 *Suppose that, for some $f \in \mathcal{X}_I$, $\phi_f(t,x)$ has hyperbolic equilibria x_1 and x_2 and that S is the set of connections between these equilibria. If $S \cap I$ is a non-empty isolated invariant set of connections, then $f_{I\cap N}$ is locally a gradient-like vector field. Moreover, if*

$$C_I(S) \neq C_I(x_1) \vee C_I(x_2) \tag{3.5}$$

then the connection is robust to perturbations in $\mathcal{X}_I$.

Proof: Choose a neighbourhood N that is an isolating block for the connection $S \cap I$. By Lemma 3.11, this is an isolating block for the connection, also for the flow ϕ_f restricted to I. We now pick isolating neighbourhoods N_1 and N_2 for the hyperbolic equilibria x_1 and x_2. By Theorem 3.13 and the Hartman-Grobman theorem, there will be invariant sets $\tilde{x}_i$ that are continuations of x_i such that

$$C_I(N) \neq C_I(\tilde{x}_1) \vee C_I(\tilde{x}_2).$$

By Theorem 3.14, the vector field is gradient-like, this property is also robust, hence the result of Smoller and Conley [124] (which stated in [72, Theorem 3]) for nearby parametrized perturbations of f implies there will be connecting orbits $\tilde{S}$ between $\tilde{x}_1$ and $\tilde{x}_2$; the connection is robust. **QED**

We do not prove continuity of the set of connections, but we note that by choosing N to be arbitrarily small, if the connections are isolated, then the location of the connection can be chosen to depend continuously on variations in f.

3.4 The restricted Conley index and robustness with non-transverse intersection

Although the main results in the manuscript in Chapter 3 so far can be obtained using transversality arguments, there are a number of possible extensions where the restricted Conley index may be more useful. The following is an example of the non-transverse intersection where we cannot use the traditional robustness test where the restricted Conley index is applicable.

Example 3.2 *For the flow generated by a gradient-like vector field $f \in \mathcal{X}_I$, suppose that x_1, x_2 are hyperbolic equilibria in $\mathbb{R}^4$ such that $dim^u(x_1) = 2$ and $dim^s(x_2) = 2$ which intersect non-transversely with the $3-$invariant subspace I. Although there are different patterns for their connection scheme (details on the connection scheme will be discussed in Section 5.1 of this thesis) in the following we try to discuss a case where we can find out how many components do exist sets have for $dim^u(x_1) \cap I)$, $dim^u(x_2) \cap I)$ and $S \cap I)$. However, the robustness of the heteroclinic connections can be justified using the restricted Conley index condition in Theorem 3.15. We illustrate this in Figure 3.3.*

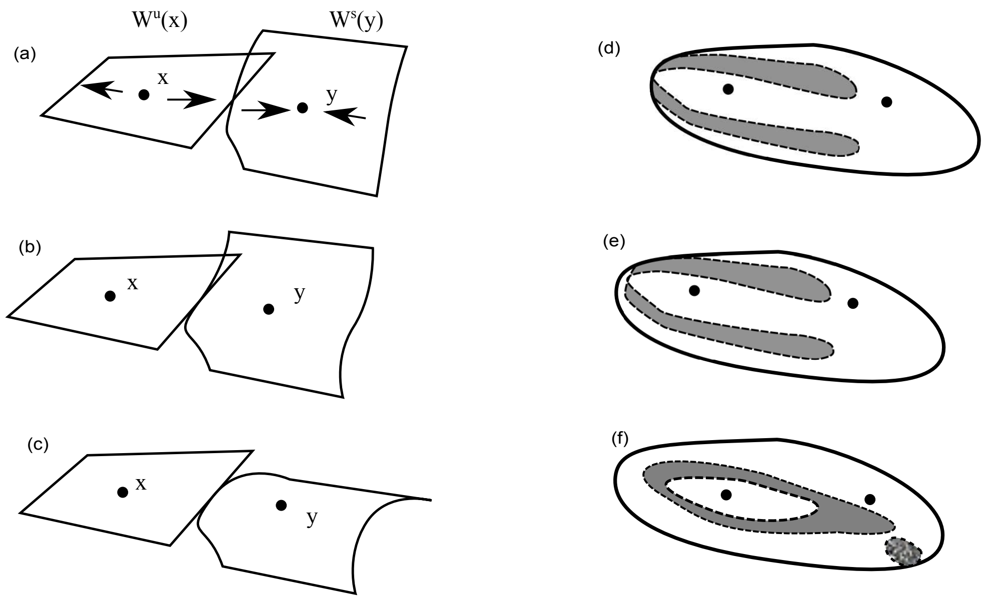

Figure 3.3: Schematic diagrams showing transverse (a,d) and nontransverse (b,c,e,f) intersections between two invariant manifolds. Diagrams (a,b,c) show an intersection of 2D unstable and a 2D stable manifold for the equilibria x and y in 3D phase space; (d,e,f) show the topology of the exit set (dashed lines/shaded areas) implied by these intersections. Note that in the case (b,e), there is a non-trivial exit set, meaning that the configuration can be shown to have a robust intersection by use of the Conley index, even though the intersection is not transverse.

Chapter 4

Symmetric dynamical systems

A mathematical characterization of the symmetry of a system is defined as a symmetry group of transformations. These symmetries can be found in both ordinary and partial differential equations, but in this chapter we review the symmetry of system of ordinary differential equations which relate to this thesis. Basically, the symmetry of a system is a transformation which sends any solution of a differential equation to another solution of the same equation. [134]

4.1 Lie groups and compact orthogonal actions on $\mathbb{R}^n$

The study of continuous groups was undertaken in the 20th-century by the mathematician Souphus Lie. The theory of Lie groups becomes important in many mathematical or physical applications because they appear as groups of symmetry of various systems. Lie used group theory to describe the symmetries of analytic systems; more details can be found in [43, 58, 132, 133]. The fact that a Lie group G is a non-empty set such that G is both a group and a (smooth) manifold, means that both algebraic and analysis properties can be used. In particular, in this chapter we review some properties of a compact orthogonal action of a Lie group on the Euclidean space $\mathbb{R}^n$ and the implications for dynamics of ODES. First we give general definitions of a Lie group acting on a manifold M.

Definition 4.1 *A topological space M with a C^∞ differential structure is called a* differentiable manifold *(smooth).*

Definition 4.2 *[63] Let Γ be a differentiable manifold and f be a multiplication map defined as $f : \Gamma \times \Gamma \to \Gamma$. We write $f(\gamma_1, \gamma_2)$ as $\gamma_1\gamma_2$. Then Γ is called a* Lie group *if it is a group and f is differentiable (C^∞).*

Γ is called a *compact Lie group* if it is a compact (bounded and closed) differentiable manifold. Any finite group is a $0-$dimensional Lie group.

Proposition 4.1 *[44, 1]*

- *A* homomorphism (isomorphism) of compact Lie groups *is a differentiable group homomorphism (isomorphism).*
- *A closed subgroup (a subgroup which is a compact smooth submanifold) of a compact Lie group is a compact Lie subgroup.*

Definition 4.3 *[1] A* smooth action of a compact Lie group Γ on a manifold M *is a differentiable map $L : \Gamma \times M \to M;\quad (\gamma, x) \longmapsto L_\gamma(x)$ such that:*

- *$L_e(x) = x$ for all $x \in M$ and the identity e of Γ;*
- *$L_{\gamma_1 * \gamma_2}(x) = L_{\gamma_1}(L_{\gamma_2}(x))$ for all $\gamma_1,\ \gamma_2 \in \Gamma$ and $x \in M$.*

The manifold M is called $\Gamma-$manifold.

In other words, an action can be determined as a representation of Γ in the automorphism group of M.

Definition 4.4 *[60, Isomorphic actions] Let the Lie group* Γ *acts on two* $m-$*dimensional manifolds* M_1, M_2*. If there exists an isomorphism* $h : M_1 \to M_2$ *such that*

$$h(\gamma m) = \gamma h(m)$$

for all $m \in M,\ \gamma \in \Gamma$*, then we say that* M_1 *and* M_2 *are* $\Gamma-$isomorphic.

After this quick look at the general definition of a Lie group and its action on manifolds, we introduce a special case of Lie group; namely, the orthogonal compact Lie group $O(n)$ and its action on the Euclidean space $\mathbb{R}^n$.

Definition 4.5 *Let* $n \in \mathbb{N}^+$*, the following group of* $n \times n$ *real matrices with given conditions:*

$$O(n, \mathbb{R}) = \{A : \ A \ is \ n \times n \ invertible \ real \ matrix \ such \ that \ AA^T = I\}$$

is called an orthogonal group.

A^T denotes the transpose matrix of A and I the identity matrix. The real orthogonal group $O(n, \mathbb{R})$ is a *compact Lie group* on an Euclidean space $\mathbb{R}^n$, we will denote such a group as $O(n)$.

Definition 4.6 *The action of* $O(n)$ *on* $\mathbb{R}^n$ *is defined as a smooth map* $L : O(n) \times \mathbb{R}^n \to \mathbb{R}^n$*;* $L(A, x) \to Ax$ *for all* $A \in O(n)$ *and* $x \in \mathbb{R}^n$*; such that* $L_I(x) = x$ *for all* $x \in \mathbb{R}^n$ *(*I *denotes the identity of* $O(n)$*) and* $L_{A*B}(x) = L_A(L_B(x))$*for all* $A, B \in O(n)$ *and* $x \in \mathbb{R}^n$*.*

Given this action of $O(n)$ on the Euclidean space $\mathbb{R}^n$, we state the following basic definitions in our context.

Definition 4.7 *[1] For each* $x \in \mathbb{R}^n$ *define its orbit under the action* L *of* $\Gamma \subset O(n)$ *as the following subset of* $\mathbb{R}^n$*:*

$$orb_x = \{L_\gamma(x) : \ \gamma \in \Gamma\}.$$

orb_x *is called the* group orbit *of* x.

The set of all orbits of $\mathbb{R}^n$ under the action of Γ is called *the orbit space* and denoted by $\mathbb{R}^n/\Gamma$.

Definition 4.8 *[59] An* isotropy subgroup Σ_x *for each* $x \in \mathbb{R}^n$ *and a group* $\Gamma \subset O(n)$ *is defined as follows:*

$$\Sigma_x = \{\gamma \in \Gamma : \gamma.x = x\}.$$

Such an isotropy subgroup defines the symmetry of a point x (or any other solution), and, moreover we will see throughout the thesis that it can be used to find other solutions for the system with given symmetries.

Lemma 4.2 *[59] For any* $x \in \mathbb{R}^n$ *and* $\gamma \in \Gamma \subset O(n)$, *the two elements* $x, \gamma.x$ *lie on the same orbit and have conjugate isotropy subgroups*

$$\Sigma_{\gamma.x} = \gamma \Sigma_x \gamma^{-1}$$

The set of all isotropy subgroups that are conjugate to Σ_x is called the *conjugacy class* of Σ_x; denoted by $[\Sigma_x] = \{\Sigma_y : \ \Sigma_y = \gamma \Sigma_x \gamma^{-1} \ for \ some \ \gamma \in \Gamma; \ y \in \mathbb{R}^n\}$. This means that points of the same group orbit have isotropy subgroups that are in the same conjugacy class and have the same existence and stability properties.

Theorem 4.3 *Assume that a (finite) group* $\Gamma \subset O(n)$ *acts on* $\mathbb{R}^n$. *For any* $x \in \mathbb{R}^n$ *and* $\gamma \in \Gamma$ *we have:*

- $|\Sigma_x| = |\Sigma_{\gamma x}|$
- $|orb_x| = \frac{|\Gamma|}{|\Sigma_x|}$

On the set of all conjugacy classes $\{[\Sigma_x] : x \in \mathbb{R}^n\}$ a partial ordering relation $\prec$ is defined as follows:

Let $G = [\Sigma_{x_i}]$ and $H = [\Sigma_{x_j}]$ be conjugacy classes of isotropy subgroups Σ_{x_i} and Σ_{x_j} of Γ; where $i, j \in \mathbb{N}$; then

$$G \prec H \Leftrightarrow \Sigma_{x_i} \subseteq \Sigma_{x_j}$$

for some $i, j \in \mathbb{N}$ [59].

Definition 4.9 *[59] The* lattice of isotropy subgroups *of $\Gamma \subset O(n)$ is the set of all conjugacy classes of isotropy subgroups of Γ , partially ordered by $\prec$.*

So by using the isotropy lattice which draws a picture of solutions for the system with a given symmetry, one can find other solutions for the system under the given symmetry. However, subspaces where these solutions exist can be found which are called fixed−point spaces and defined as follows:

Definition 4.10 *[59] Let $\Gamma \subset O(n)$ act on $\mathbb{R}^n$. A* fixed-point space *Fix(Σ) of a subgroup $\Sigma \subset \Gamma$ is defined as:*

$$Fix(\Sigma) = \{x \in \mathbb{R}^n : \sigma x = x \ for \ all \ \sigma \in \Sigma\}.$$

Fixed−point spaces are linear subspaces of $\mathbb{R}^n$ (they are closed under addition and scalar multiplication).

Note that, in applications of group actions there are different type of actions: self action, a group acting on itself by conjugation, or a group acting on its quotient group. In the following we introduce some results concerning permutation action and an example of the permutation action of the group S_6 on $\mathbb{R}^6$.

Definition 4.11 *[71] For any finite set $X = \{x_1, \cdots, x_n\}$ a group structure can be built from the set of all bijective functions on it under composition operation as follows:*

$$S_n = \{f : X \to X : f \ is \ bijective \ on \ X\}.$$

Such a group is called a symmetric group *on X with $|S_n| = n!$.*

Theorem 4.4 *[71, Cayley's Theorem] Every finite group of order n is isomorphic to a subgroup of S_n.*

As a requirement of our investigation on the system of six coupled nonlinear oscillators in Chapter 6, we introduce the following example of the action of S_6 on $\mathbb{R}^6$.

Example 4.1 *An action of the permutation group S_6 on the set $\mathbb{R}^6$ is defined by a homomorphism:*

$$S_6 \times \mathbb{R}^6 \to \mathbb{R}^6$$

written as $L_\gamma(x) = \gamma x$ for each $\gamma \in S_6$ and $x \in \mathbb{R}^6$. The orbit of $x \in \mathbb{R}^6$ under the action of S_6 is defined to be the set:

$$orb_x = \{\gamma x : \gamma \in S_6\}.$$

The isotropy subgroup of $x \in \mathbb{R}^6$ is given by:

$$\Sigma_x = \{\gamma \in S_6 : \gamma x = x\}.$$

The fixed−point subspace for Σ_x is given by:

$$Fix(\Sigma_x) = \{x \in \mathbb{R}^6 : \sigma x = x \ for \ all \ \sigma \in \Sigma\}.$$

Under the permutation action of S_6, the set $\mathbb{R}^6$ divides into a partition:

$$P = \{p_i : i \in \mathbb{N} \ and \ i \leq 6\}$$

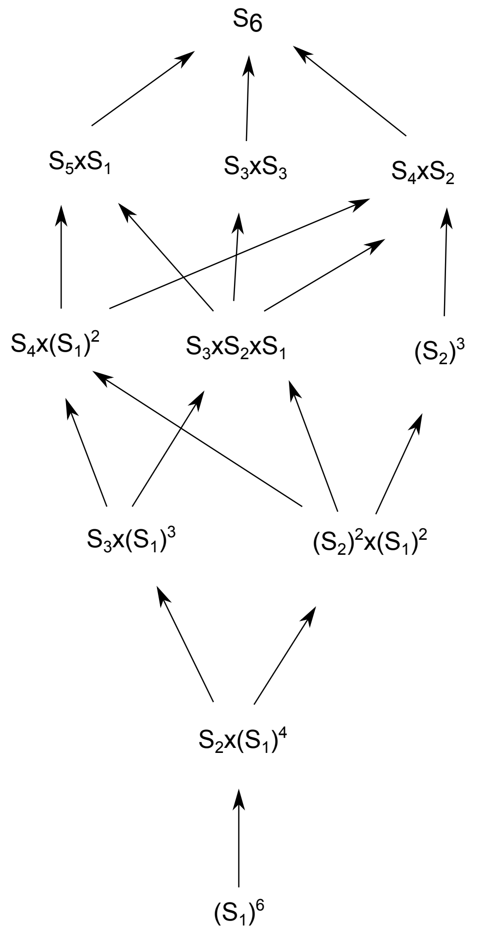

Figure 4.1: Isotropy lattice of S_6 acting by permutation on $\mathbb{R}^6$, (see Chapter 6 and Table 6.1 for more details).

such that we conclude that the isotropy lattice of S_6 contains the following conjugacy classes: $[P_1] = \{6\}, [P_2] = \{5,1\}, [P_3] = \{4,2\}, [P_4] = \{4,1,1\}, [P_5] = \{2,2,2\}, [P_6] = \{3,3\}, [P_7] = \{3,2,1\}, [P_8] = \{3,1,1,1\}, [P_9] = \{2,1,2,1\}, [P_{10}] = \{1,1,1,2,1\}, [P_{11}] = \{1,1,1,1,1,1\}$, as illustrated in Figure 4.1. Using Theorem 4.3, the size of orbit in each conjugacy class can be computed using the following:

$$\frac{|S_6|}{|\Sigma_x|}$$

for any $x \in [P_i];\ 1 \leq i \leq 6$.

4.2 Symmetries of systems of ordinary differential equations

In this section we review the application of group representations to nonlinear dynamical systems (systems which are symmetric under the action of a group are called symmetric (equivariant) dynamical systems). Dynamics of equivariant systems have some implications due to being symmetric and we will discuss these results in this section; this approach has been used, for example, in investigations of biological systems [28]

Definition 4.12 *[60] Consider the system of ordinary differential equations:*

$$\frac{dx}{dt} = f(x). \tag{4.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞-smooth vector field on a manifold $\mathbb{R}^n$ and assume that $\Gamma \subset O(n)$ acts on f. A vector field f is called equivariant *under the action of Γ in $\mathbb{R}^n$ if the following condition is satisfied:*

$$f(\gamma x) = \gamma f(x)$$

for all $\gamma \in \Gamma$; $x \in \mathbb{R}^n$. f is called $\Gamma-$equivariant.

As a definition of fixed$-$point space was introduced in the previous section, in the following an important property of such spaces is given:

Theorem 4.5 *[59] Consider the system (4.1) which is equivariant under the action of a group $\Gamma \subset O(n)$ and $Fix(\Sigma)$ be a fixed$-$point space for a subgroup $\Sigma \subset \Gamma$. Then $Fix(\Sigma)$ is* flow$-$invariant. *In notation:*

$$\phi_f(t, Fix(\Sigma)) \subseteq Fix(\Sigma).$$

Corollary 4.6 *The set of all omega limit sets and alpha limit sets give the possible long-term dynamics of f. Although the trajectory of a solution x remains in the fixed−point space of its isotropy subgroup, its omega limit set and alpha limit set typically do not.*

In the introduction of their article, Field et al. in [47] discussed whether, for example, the set $X = \{a\}$ represents an omega limit set of f that contains one point then $\Sigma_X = \Sigma_a$. However, the isotropy subgroup of a solution x becomes larger as its omega limit set has more intensive dynamic [42].

4.3 Nonlinear systems of ordinary differential equations

Typically, only linear systems have solutions that can be determined explicitly, in contrast to nonlinear systems, where they cannot. In the real world, almost all problems are generally modelled using nonlinear systems. In general, investigation of the behavior of nonlinear systems at a specific solution is done by approximating the nonlinear system into a linear system around this solution. This procedure is known as *linearization* and is defined in the following.

Definition 4.13 (Linearization) *Let the function $f : \mathbb{R}^n \to \mathbb{R}^n$ be an $n-$dimensional differentiable function at some point $x^* \in \mathbb{R}^n$, then the* linearization *of f at x^* is the approximation function*

$$L(x) = f(x^*) + Df(x^*)(x - x^*).$$

Where $Df = \partial f_i/\partial x_j$; $i, j = 1, 2, ..., n$ and Df is called the Jacobian matrix *of f. Note that $|f(x) - L(x)| = o(|x - x^*|)$.*

Suppose that the point $x^* \in \mathbb{R}^n$ is an equilibrium of f. Then the behavior of the system (4.1) around x^* is characterized by the linearization of the system about x^*.

Indeed for x near x^* we have

$$\frac{dx}{dt} = f(x(t)) \approx f(x^*) + Df(x^*)(x(t) - x^*).$$

From now on, in this thesis we consider the solution $q = x^*$ to be an equilibrium point.

Definition 4.14 *A solution of the system (4.1) is called* hyperbolic *if all eigenvalues of its Jacobian have non-zero real parts.*

The unstable and stable manifolds of a hyperbolic equilibrium point q are defined by:
$W^u(q) = \{x \ : \ \phi_f(t, x) \to q \text{ as } t \to -\infty \ \}$;
$W^s(q) = \{x \ : \ \phi_f(t, x) \to q \text{ as } t \to \infty \ \}$. These manifolds are flow-invariant. Note that if there is a connection from q_1, q_2, where q_1 and q_2 are hyperbolic equilibria, then the connection lies within $W^u(q_1) \cap W^s(q_2)$.

4.4 Symmetry and linearized systems

As discussed in the previous section, the behavior of (4.1) around an equilibrium solution q is determined by its linearization $(Df)_q$ at such a solution; q is stable if all eigenvalues of $(Df)_q$ have negative real part, some positive and negative eigenvalues of $(Df)_q$ means that q is unstable. We end this chapter as we started it by discussing some properties of symmetric systems [60]. Now we assume that the system (4.1) is symmetric under the action of $\Gamma \subset O(n)$ and has an equilibrium point q such that $\Sigma_q \subset \Gamma$ is the isotropy subgroup of q. Before we discuss the stability properties of q, we need to review some related results in the following:

Definition 4.15 *[60] Let $\Gamma \subset O(n)$ act on $\mathbb{R}^n$ such that the only $\Gamma-$invariant subspaces of $\mathbb{R}^n$ are $\{0\}$ and $\mathbb{R}^n$, then such action is called* irreducible.

More generally, any $\Gamma-$invariant subspace $V \subset \mathbb{R}^n$ is $\Gamma-$*irreducible* if Γ acts irreducibly on V.

Theorem 4.7 *[60, Theorem of Complete Reducibility] Let $\Gamma \subset O(n)$ act on $\mathbb{R}^n$, then*

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

where each $V_1, \cdots, V_k$ is a $\Gamma-$irreducible subspace of $\mathbb{R}^n$.

Generally, the decomposition of $\mathbb{R}^n$ in Theorem 4.7 is not unique.

Theorem 4.8 *[60] For a compact Lie group Γ acting on $\mathbb{R}^n$ we have*

1. *There are a finite number of distinct $\Gamma-$irreducible subspaces $A_1, \cdots, A_k$ of $\mathbb{R}^n$, up to $\Gamma-$isomorophism.*
2. *$\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$ where V_k be the sum of all $\Gamma-$irreducible subspaces V of $\mathbb{R}^n$ such that V is $\Gamma-$isomorphic to A_K.*

The subspaces V_k are called *isotypic components* of $\mathbb{R}^n$, of type A_k; the isotypic components are uniquely determined.

Lemma 4.9 *[60] Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map that is $\Gamma-$equivariant, where Γ be a compact Lie group acting on $\mathbb{R}^n$. If $V \subset \mathbb{R}^n$ is a $\Gamma-$irreducible subspace, then $f(V)$ is $\Gamma-$invariant. However, either $f(V) = \{0\}$ or the action of Γ on V and $f(V)$ are isomorphic.*

Theorem 4.10 *[60] Assume that a compact Lie group Γ acting on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $\Gamma-$equivariant map. Decompose $\mathbb{R}^n$ into isotypic components $R^n = V_1 \oplus \cdots \oplus V_k$ then*

$$f(V_r) \subset V_r; \quad r = 1, \cdots, k.$$

In the following we introduce the properties of linearization $(Df)_q$ of $\Gamma-$equivariant nonlinear map $f : \mathbb{R}^n \to \mathbb{R}^n$ at a solution $q \in R^n$ in terms of matrix representation [60]. Since

$$f(\gamma q) = \gamma f(q); \qquad \gamma \in \Gamma.$$

Then by differentiating both sides

$$(Df)_{\gamma q}\gamma = \gamma(Df)_q. \tag{4.2}$$

So for all $\sigma \in \Sigma_q$ the equation (4.2) is written as:

$$(Df)_q\sigma = \sigma(Df)_q. \tag{4.3}$$

Hence the linearization $(Df)_q$ commutes with Σ_q. Using Theorems **??** and 4.10, $\mathbb{R}^n$ can be decomposed into isotypic components as

$$\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$$

such that

$$(Df)_q(V_r) \subset V_r$$

for all $r = 1, \cdots, k$. In other words, the action of symmetry on $(Df)_q$ implies the existence of invariant subspaces for the linearized system [60]. Throughout this thesis we consider the action of compact orthogonal Lie group on $\mathbb{R}^n$ for the system (4.1), then the linearization $(Df)_q$ is block diagonal.

Chapter 5

Restricted Conley index and heteroclinic cycles

It has been shown in other work on the Conley index [31, 124] that under certain conditions it showed the existence of (at least one) heteroclinic orbit between two hyperbolic equilibria. In [31] it was shown that this result can be achieved by comparing the wedge sum of the Conley indices for the two equilibria and that of the isolating neighbourhood within which both are found. In this chapter we apply our results from Chapter 3 on the "restricted Conley index" to show the robustness of heteroclinic orbits and hence the robustness of heteroclinic cycles. We find that we can use these results in two different contexts [75, 62], for heteroclinic cycles and heteroclinic network.

5.1 Robust heteroclinic cycles

We briefly introduce the idea of a connection scheme for a heteroclinic cycle in a constrained system [12] before giving the main result, Theorem 5.1, that gives a criterion for the robustness of connections. Suppose that (as in symmetric or Lotka-Volterra dynamics) the ODE:

$$\frac{dx}{dt} = f(x) \tag{5.1}$$

where $f \in \mathcal{X}(\mathbb{R}^n)$ leaves some set of affine subspaces invariant. Suppose that $\mathcal{I}$ is some set of affine subspaces of $\mathbb{R}^n$

$$\mathcal{I} = \{I_i \ : \ i = 1, ..., r\}$$

where, for example, we can write $I_i = \{x \in \mathbb{R}^n : A_i x = b_i\}$, A_i is a real-valued $n \times n$ matrix and $b_n \in \mathbb{R}^n$. We assume that there are only finitely many invariant subspaces and that this set is closed under intersection, i.e., if I_i and I_i are in $\mathcal{I}$, then $I_i \cap I_j \in \mathcal{I}$; by convention we assume that $\mathbb{R}^n \in \mathcal{I}$. We define the set of vector fields in $\mathcal{X}$ respecting $\mathcal{I}$ as

$$\mathcal{X}_{\mathcal{I}} = \bigcap_i \mathcal{X}_{I_i}$$

so that if $f \in \mathcal{X}_{\mathcal{I}}$ then all of the $I \in \mathcal{I}$ are f-invariant subspaces.

Suppose that $\{x_i : i = 1, ..., p\}$ are hyperbolic equilibria for some $f \in \mathcal{X}_{\mathcal{I}}$. Assume that there is a *heteroclinic cycle* connecting the x_i, then for each i there is a *heteroclinic connection* from x_i to x_{i+1}, i.e., a non-empty invariant set

$$S_i \subseteq W^u(x_i) \cap W^s(x_{i+1})$$

(taking the subscripts modulo p). We write the cycle as

$$\Sigma = \bigcup_i \{x_i\} \cup \{S_i\}.$$

As in [12], we define a *connection scheme* for the cycle Σ to be the sequence of affine subspaces $I_{c(i)}$ such that for each $S_i \subset I_{c(i)}$:

$$\cdots \to x_i \stackrel{I_{c(i)}}{\to} x_{i+1} \to \cdots . \tag{5.2}$$

Theorem 5.1 *Let Σ be a heteroclinic cycle for a constrained system $f \in \mathcal{X}_{\mathcal{I}}$ where the $S_i \cup \{x_i, x_{i+1}\}$ are isolated invariant sets. Suppose that there is a connection scheme (5.2)* *for this cycle such that for all i we have*

$$C_{I_{c(i)}}(\overline{S}_i) \neq C_{I_{c(i)}}(x_i) \vee C_{I_{c(i)}}(x_{i+1}), \tag{5.3}$$

then the cycle Σ is robust *to perturbations in $\mathcal{X}_\mathcal{I}$.*

Proof: For each i there are isolating blocks N_i and open neighbourhoods G_i of f in $\mathcal{X}_\mathcal{I}$ such that the equilibria x_i and x_{i+1} continue and

$$C^f_{I_{c(i)}}(N_1) = C^g_{I_{c(i)}}(N_2)$$

for any $g \in G_i$. This means that we can apply Theorem 3.15 to give the existence of a perturbed connection $\tilde{S}_i$ for neighbourhoods of 0 in compactly parametrized perturbations. Taking the intersection of these neighbourhoods gives an open neighbourhood where the cycle exists - i.e., the heteroclinic cycle is robust. **QED**

This result can be used to show a special case of [12, Theorem 1] as follows:

Corollary 5.2 *Suppose that $f \in \mathcal{X}_\mathcal{I}$ has a heteroclinic cycle between hyperbolic equilibria x_i with connection scheme (5.2), is gradient-like in a neighbourhood of each connection, and is such that the connections all satisfy*

$$\dim(W^u(x_i) \cap I_{c(i)}) > 0, \quad \dim(W^s x_{i+1} \cap I_{c(i)}) = \dim(I_{c(i)}) \tag{5.4}$$

(i.e., there is a saddle-sink connection). Then the cycle is robust to perturbations in $\mathcal{X}_\mathcal{I}$.

Proof: Note that for f restricted to $I_{c(i)}$ the isolated equilibria x_i and x_{i+1} have Morse index $m_j = \dim(W^u(x_j) \cap I_{c(i)})$ for $j = i, i+1$. This means that the $C_{I_{c(i)}}$ restricted Conley indices for $x_{i,i+1}$ are homotopy $m_{i,i+1}$-spheres. The second assumption in (5.5) implies that $m_{i+1} = 0$ and so $C_{I_{c(i)}}(x_{i+1})$ is a zero-sphere (two points). Hence $C_{I_{c(i)}}(x_i) \vee$

$C_{I_{c(i)}}(x_{i+1})$ is disconnected but not simply two points. On the other hand, either $C_{I_{c(i)}}(\overline{q}_i)$ has an exit set and is connected, or it is two points. Either way, (5.3) must hold and so we get robustness of the cycle by application of Theorem 5.1. **QED**

It would be nice to be able to extend this argument to the more general case as in [12, Theorem 1] when

$$\dim(W^u(x_i) \cap I_{c(i)}) + \dim(W^s(x_{i+1} \cap I_{c(i)}) \geq \dim(I_{c(i)}) + 1. \tag{5.5}$$

In $\mathbb{R}^n$, transverse intersection means that $dim(W^u(x_i) \cap W^s(x_{i+1})) \geq 1$. However, a restriction to an invariant subspace I means that $dim(W^u(x_i) \cap W^s(x_{i+1}) \cap I)) \geq 1$ which leads to limit our investigation due to many possibility of connection schemes.

5.2 Examples of robustness for heteroclinic cycles

In this section we illustrate an application of the restricted Conley index as an alternative to the standard methods for discussing the robustness of two well-studied heteroclinic cycles - the Guckenheimer-Holmes [62] and Kirk-Silber [75] cycles.

5.2.1 The Guckenheimer-Holmes heteroclinic cycle

For a manifold $M = \mathbb{R}^3$ and a Lie group $G \subset O(3)$ (finite subgroup) acting on $\mathbb{R}^3$; define $X_G(\mathbb{R}^3)$ to be the space of C^1-vector fields on $\mathbb{R}^3$ equivariant with respect to a symmetry group G. In [62], Guckenheimer and Holmes show an existence of an open set $U \subset X_G(\mathbb{R}^3)$ of topologically equivariant vector fields such that each vector field $V \in U$ has a robust heteroclinic cycle. This system has been studied by other authors (see for example [7, 17]). Consider the system of equations

$$\begin{aligned}\dot{x} &= x(\lambda + ax^2 + by^2 + cz^2)\\ \dot{y} &= y(\lambda + ay^2 + bz^2 + cx^2)\\ \dot{z} &= z(\lambda + az^2 + bx^2 + cy^2)\end{aligned} \tag{5.6}$$

where parameters a, b, c, λ satisfy

$$\lambda = 1; \quad a + b + c = -1; \quad -1/3 < a < 0; \quad c < a < b < 0. \tag{5.7}$$

As shown in [62], the system (5.6) for these parameter values has a heteroclinic cycle consisting of three equilibria q_i on the coordinate axes and connections c_i; for $i = 1, 2, 3$ in the coordinate planes connect these equilibria. Moreover, all trajectories near the coordinate axes or near the coordinate planes are attracted to the heteroclinic cycle, and the cycle is robust to any perturbation that preserves the symmetries G generated by reflections in the coordinate planes and by cyclic permutation of the axes (in fact it is robust to the subgroup just generated by reflection in coordinate planes).

More precisely, the following three points are equilibria of the system (5.6):

$$q_1 = (0, \sqrt{-\lambda/a}, 0), \quad q_2 = (\sqrt{-\lambda/a}, 0, 0), \quad q_3 = (0, 0, \sqrt{-\lambda/a}),$$

and we will use the following notation to refer to the possible invariant sets under the flow generated by the system (5.6) that preserve the symmetries G

- $\mathbb{R}^3$: the whole space
- $I_{x,y,z}$: the x, y and z coordinate axes
- $I_{xy,xz,yz}$: the xy, xz and yz coordinate planes

Although the vector field is not gradient-like in $\mathbb{R}^n$ (no gradient-like field can have a heteroclinic cycle and so (5.6) cannot be globally gradient-like), this vector field is gradient-like when restricted to the coordinate planes:

Lemma 5.3 *The flow generated by (5.6) is a globally gradient-like field on the coordinate planes $I_{xy,xz,yz}$.*

Proof: We exhibit an explicit Morse function for the flow restricted to I_{xy} as follows

$$P_{I_{xy}}(x,y) = -\left[x^2y^2/2 + \lambda x^2/2b + ax^4/4b + \lambda y^2/2c + ay^4/4c\right]$$

To see that this is a Morse function, we calculate

$$\begin{aligned}
\frac{dP}{dt} &= \frac{\partial P}{\partial x}(\dot{x}) + \frac{\partial P}{\partial y}(\dot{y}) \\
&= -(xy^2 + \frac{\lambda x}{y} + \frac{ax^3}{b})(x(\lambda + ax^2 + by^2)) - (yx^2 + \frac{\lambda y}{c} + \frac{ay^3}{c})(y(\lambda + ay^2 + cx^2)) \\
&= -(x^2y^2 + \frac{\lambda x^2}{b} + \frac{ax^4}{b})(\lambda + ax^2 + by^2) - (x^2y^2 + \frac{\lambda y^2}{c} + \frac{ay^4}{c})(\lambda + ay^2 + cx^2)) \\
&= \frac{-x^2}{b}(\lambda + ax^2 + by^2)^2 - \frac{y^2}{c}(\lambda + ay^2 + cx^2)^2 \qquad (5.8)
\end{aligned}$$

Under the conditions (5.7), and in particular $b, c < 0$, the function (5.8) clearly satisfies $\frac{dP}{dt} > 0$ away from equilibria and $\frac{dP}{dt} = 0$ at equilibria. Similarly, we find Morse functions for the flow restricted to the invariant subspace $I_{yz,xz}$. **QED**

Generally, in a heteroclinic cycle for any two equilibria q_1, q_2 and the maximum invariant set S containing q_1 and q_2 we have:

$$C(q_1) \vee C(q_2) = C(S)$$

where $C(q_i)$ is the Conley index for an equilibrium point q_i ($i = 1, 2$). This Conley index, however, cannot be used to prove the robustness of the heteroclinic cycle. On the other hand, using our result we aim to show that for any q_1, q_2, S we have:

$$C_I(q_1) \vee C_I(q_2) \neq C_I(S)$$

where C_I is the Conley index restricted to any invariant set I of the system (5.6). So from this result the robustness can be achieved.

Starting with the Jacobian of the linearization of (5.6) as follows:

$$Jac = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1-c/a & 0 \\ 0 & 0 & 1-b/a \end{bmatrix},$$

under the conditions (5.7), the Jacobian matrix Jac_{q_i}; for $i = 1, 2, 3$; has one positive eigenvalue. Thus the Conley index for each equilibrium is a pointed circle, however if we choose S to be the maximum isolating set containing q_i, q_j, $i, j = 1, 2, 3; i \neq j$ and use a solid cylinder to isolate it, then the exit set has three components and its Conley index is $\Sigma^1 \vee \Sigma^1$ (Σ is the one-sphere).

Take an invariant set I_x, then $C_{I_x}(q_1)$ is a pointed circle; $C_{I_x}(q_2)$ is two points; $C_{I_x}(N)$ is a single point. Then the result

$$C_{I_x}(q_1) \vee C_{I_x}(q_2) \neq C_{I_x}(N)$$

ensures the robustness of the heteroclinic cycle.

5.2.2 The Kirk-Silber heteroclinic network

As a less trivial example, we apply our method to determine the robustness of a known heteroclinic network comprising two heteroclinic cycles [75]. This model provides an example of a heteroclinic network which has strong stability properties although neither of the cycles lying within it does. The existence of a common heteroclinic connection between the two cycles makes this network interesting because it is impossible for either of the cycles to attract all nearby trajectories. More precisely, consider the equivariant

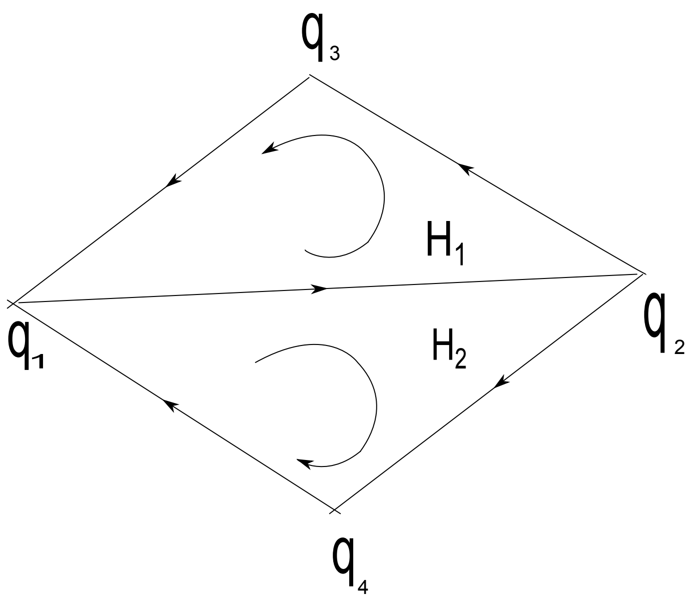

Figure 5.1: Kirk-Silber robust heteroclinic network.

system

$$\dot{x} = f(x) \quad for \quad x \in \mathbb{R}^4 \tag{5.9}$$

where $f : \mathbb{R}^4 \to \mathbb{R}^4$ is a smooth vector field symmetric under the action of $(Z_2)^4$ generated by reflections in the coordinate hyperplanes. This symmetry will leave the following subspaces invariant under the flow generated by (5.9):

- X_i where $1 \leq i \leq 4$; the four coordinate axis.
- X_{ij} where $1 \leq i; j \leq 4; i \neq j$; the six planes.
- X_{ijk} where $1 \leq i; j; k \leq 4;\ i \neq j \neq k$; the four hyperplanes.

As shown in [75], the system (5.9) has two robust heteroclinic cycles (5.1) formed by four non-trivial hyperbolic equilibria $q_i, 1 \leq i \leq 4$ of saddle type (an equilibrium for each axis). One cycle (call it H_1) consists of $q_1 \to q_2 \to q_3 \to q_1$, and lies on X_{123} space. The other cycle (call it H_2) consists of $q_1 \to q_2 \to q_4 \to q_1$ and lies in the X_{124} hyperplane. Moreover, each connection from q_i into q_j lies in X_{ij}.

We noticed that the example introduced in [74] produced a heteroclinic network by assuming ($\omega = 0$) which consisting of two heteroclinic cycles with the same dynamics as the Kirk-Silber heteroclinic network in [75]. In the following we study such an example and use the restricted Conley index to show the robustness of the heteroclinic network. Consider the system on $\mathbb{R}^4$

$$\begin{aligned}
\dot{x_1} &= x_1(1 - x_1^2 - Ex_2^2) \\
\dot{x_2} &= x_2(1 - x_2^2 - Fx_3^2 - Gx_4^2) \\
\dot{x_3} &= x_3(1 - x_3^2 - Ex_1^2 + Hx_2^2 - Dx_4^2) \\
\dot{x_4} &= x_4(1 - x_4^2 - Ex_1^2 - Dx_3^2)
\end{aligned} \tag{5.10}$$

where parameters satisfy that

$$E, F, G, D, \frac{F+G}{D+1} \in (1,3); \qquad H > 0 \tag{5.11}$$

Although no global gradient-like vector field can be defined for the heteroclinic network under consideration, using Theorem 3.14, we ensure the existence of a Morse function in the neighbourhood of each heteroclinic connection within the network and therefore the restricted Conley index can be applied to the network.

The Jacobian matrix for this system is given by:

$$Jac = \begin{bmatrix} 1-3x_1^2-Ex_2^2 & -2Ex_1x_2 & 0 & 0 \\ 0 & 1-3x_2^2-Fx_3^2-Gx_4^2 & -2Fx_2x_3 & -2Gx_2x_4 \\ -2Ex_1x_3 & 2Hx_2x_3 & 1-3x_3^2-Ex_1^2-Dx_4^2 & -2Dx_3x_4 \\ -2Ex_1x_4 & 0 & -2Dx_3x_4 & 1-3x_4^2-Ex_1^2-Dx_3^2 \end{bmatrix},$$

Under the parameters conditions (5.11) we found that for the equilibria $q_i; \quad i = 1,3,4$ the associated Jacobian matrix has one positive eigenvalue, while at q_2 it has two positive eigenvalues. Implies that the dimension of unstable manifold for each equilibrium must be as follows:

$$\dim(W^u(q_1)) = 1, \quad \dim(W^u(q_2)) = 2, \quad \dim(W^u(q_3)) = \dim(W^u(q_4)) = 1.$$

So by [29], the Conley index for each equilibrium can be computed to be $C(q_i)$ is a $1-$sphere, S^1, for $i = 1, 3, 4$, while $C(q_2)$ is the $2-$sphere S^2.

In the following we will use $N_{q_i q_j}$ to denote an isolating neighbourhood containing the maximal invariant set containing q_i, q_j. In the cycle H_1 if $N_{q_1 q_2}$ isolate q_1 and q_2 then its Conley index $C(N_{q_1 q_2})$ is $S^2 \vee S^1$.

Similarly, for q_2 and q_3 the Conley index of $N_{q_2 q_3}$ is $C(N_{q_1 q_2}) = S^2 \vee S^1$. Also, for $N_{q_3 q_4}$ isolate q_3 and q_1 the Conley index $C(N_{q_3 q_1}) = S^1 \vee S^1$. In all the previous cases we notice that $C(N_{q_i q_j}) = C(q_i) \vee C(q_j)$ where $i, j \in 1, 2, 3, 4 \quad i \neq j$ so these results cannot tell us anything about the robustness of heteroclinic connections.

On the other hand, the restricted Conley index can be used to do this. For the cycle H_1, if we compute the restricted Conley index to the smallest invariant set containing the connection we end up with the following result:

Lemma 5.4 *The system (5.10) with the set of conditions of its parameters (5.11) has a robust heteroclinic network consisting of two cycles H_1, H_2.*

Proof:

In the heteroclinic cycle H_1 we have, for q_1 and q_2, the Conley index restricted to an invariant plane X_{12} are $C_{X_{12}}(q_1)$ is a one-sphere S^1 and $C_{X_{12}}(q_2)$ is a two points while $C_{X_{12}}(N_{q_1 q_2})$ is zero-sphere.

For q_2 and q_3, the Conley index restricted to the invariant set X_{23} are $C_{X_{23}}(q_2)$ is one-sphere S^1 and $C_{X_{23}}(q_3)$ is two points while $C_{X_{23}}(N_{q_2 q_3})$ is zero-sphere.

For q_3 and q_1, the Conley index restricted to the invariant set X_{31} are $C_{X_{31}}(q_3)$ is one-sphere S^1 and $C_{X_{31}}(q_1)$ is zero-sphere while $C_{X_{31}}(N_{q_3 q_1})$ is a zero-sphere. Hence the heteroclinic cycle H_1 is robust using our result on the restricted Conley index Theorem 5.1.

Similarly, we can prove robustness of the heteroclinic cycle H_2 and hence the heteroclinic network is robust. **QED**

Chapter 6

Cluster states for coupled oscillator systems

Coupled nonlinear oscillator models exhibit a complex dynamics which has been observed in a wide range of different fields, including physical, technological [9, 25, 61] and biological [109]. In fact, studies on coupled nonlinear oscillators systems divide into two issues: the first concerns the dynamics of these systems [15, 23, 127], and the second focuses on a systematic method to design these types of systems with specified behaviour. This chapter will investigate to some extent questions related to both of these issues. In particular, we give improved conditions from [105] that ensure the existence and stability of non$-$trivial 3$-$cluster state for a system of globally coupled oscillators. In the first two sections we review the basic conditions of designing stable cluster states for these systems [105]. In the remaining sections we introduce our novel results (Section 6.2.2) on a complex cluster states for the system of six coupled oscillators and satisfy the existence and stability conditions; we also show the existence of Fourier expansion for the associated coupling function.

6.1 Previous results

We first recall from [105] some conditions on the coupling function and its first derivative that ensure the existence and stability of desired cluster state in the system (6.1). The number of coupled oscillators and the number of clusters impact on a number of conditions that the coupling function and its first derivative should satisfy. A coupling function associated with a stable 3-cluster state with an arbitrary number of oscillators and an arbitrary cluster size is discussed in [105]. Here we derive the conditions of the coupling function and its first derivative to design a stable 3-cluster state with a specific number of oscillators and cluster size.

Let us consider N nonlinear oscillators that are all-to-all coupled with the following governing system:

$$\dot{\theta}_i = \omega + \frac{1}{N}\sum_{j=1}^{N} g(\theta_i - \theta_j) \tag{6.1}$$

where $\dot{\theta} \in [0, 2\pi)$ is the rate of phase change of the i^{th} oscillator with respect to time, $i = 1, \cdots, N$ and $g : [0, 2\pi) \to \mathbb{R}$ is C^2 periodic nonlinear coupling function with period 2π. The system (6.1) is equivariant under continuous rotations in S^1, that is, solutions are invariant under the transformation

$$(\theta_1, \cdots, \theta_N) \longrightarrow (\theta_1 + \xi, \cdots, \theta_N + \xi)$$

for $\xi \in [0, 2\pi)$. Also the system (6.1) is equivariant under the full permutation symmetry S_N,

$$(\theta_1, \cdots, \theta_N) \longrightarrow (\theta_{\rho(1)}, \cdots, \theta_{\rho(N)})$$

where $\rho \in S_N$. For N oscillators consider a partition $\mathcal{P} = \{p_1, p_2, ..., p_M\}$ where $1 \leq M \leq N$. Each of p_k form a cluster with size $|p_k| = m_p$ for $k = 1, ..., M$. So the partition $\mathcal{P}$ will introduce M-clusters such that $\sum_{k=1}^{M} |p_k| = N$ and each oscillator belongs to one and only one cluster ($p_k \cap p_l = \emptyset, k \neq l$).

In the state space Π^N let us define a subspace $\Pi^N_{\mathcal{P}}$ in which clustering occurs as follows:

$$\Pi^N_{\mathcal{P}}=\{\theta \in \Pi^N\text{: If there exists } k \text{ such that } i,j \subset p_k \text{ then } \theta_i = \theta_j\}.$$

All $\Pi^N_{\mathcal{P}}$ are invariant for the dynamics of the system (6.1) under S_N symmetry as they are conjugated to fixed point spaces for $S_{p_1} \times S_{p_2} \times \cdots \times S_{p_M}$.

To study these invariant subspaces $\Pi^N_{\mathcal{P}}$ where cluster states happen, we restrict the model to $\Pi^N_{\mathcal{P}}$ as follows.

As θ_i represents an oscillator phase, Ψ_p will be used for the cluster phase, so the phase equation (the rate of change of the phase of a specific cluster depends on its frequency and the coupling function of the differences between the phase of this cluster and all other clusters phases) is given by:

$$\dot{\Psi}_k = \omega + 1/N \sum_{l=1}^{M} m_l g(\Psi_k - \Psi_l), k = 1, ..., M. \tag{6.2}$$

The behaviour of a simple cluster state described as a periodic orbit of oscillators in the state space, can be expressed as the following periodic solution:

$$\Psi_k = \phi_k + \Omega t \quad k = 1, ..., M \tag{6.3}$$

where $\phi_k \in \Pi$ represents the different phases of clustering on the periodic orbit, $\Omega \in \mathbb{R}^+$ is the frequency of the oscillator.

By substituting (6.3) for (6.2) we get the following system for ϕ_k and Ω:

$$\Omega = \omega + 1/N \sum_{l=1}^{M} m_l g(\phi_k - \phi_l) \tag{6.4}$$

where $k = 1, ..., M$. Computing the difference between the first equation ($k = 1$) and each of the following ones ($k = 2, 3, ..., M$) will give:

$$\sum_{l=1}^{M} m_l \left(g(\phi_k - \phi_l) - g(\phi_1 - \phi_l)\right) = 0. \tag{6.5}$$

In other words, the existence of a $p^{th}-$cluster depends on the differences between the coupling functions of all differences between other phase clusters and the phase of the underlying cluster and the coupling function of the differences between the first cluster and the phase of the underlying cluster.

Defining $g(\phi_k - \phi_k) =: g_0$ and $g(\phi_k - \phi_l) =: g_{kl} \quad (k \neq l)$ means that we can rewrite the system (6.5) as:

$$\sum_{l=1}^{M} m_l(g_{kl} - g_{1l}) = 0, \quad k = 2, ..., M. \tag{6.6}$$

Equations (6.6) give conditions that the coupling function should satisfy for the existence of cluster state (further details can be found in [105]).

To study cluster state stability, general stability results have been introduced in [105] for an arbitrary cluster state. In particular, a cluster state is stable in both the following senses:

- **Tangential stability:** To determine the stability of the same cluster to the change in its phases that respect its size which is induced by linearizing the system (6.2) about the periodic orbit (6.3), $\Psi_k = \omega t + \phi_k$ and assume that $\chi = \Psi - (\phi_k + \Omega t)$ gives the following:

$$\dot{\chi_k} = \frac{1}{N}\sum_{l=1}^{M} m_l g'(\phi_k - \phi_l)(\chi_k - \chi_l) = \sum_{l=1}^{M} T_{k,l}\chi_l \tag{6.7}$$

  where, as stated in [105], T is a matrix that can be written as:

$$T_{k,l} = \frac{1}{N}\left[\delta_{k,l}\left(\sum_{r=1, r\neq k}^{M} m_r g'_{k,r}\right) - (1 - \delta_{k,l} m_l g'_{k,l})\right]$$

  where δ is the Kronecker delta and $g'_{kl} =: g'(\phi_k - \phi_l),\ g'_0 =: g'(0)$.

  The tangent matrix T has M eigenvalues (including one trivial value) and for tangential stability we require that all the other tangent eigenvalues have negative real parts. This means that a p^{th} cluster is tangentially stable if:

$$Re(\lambda_k^{tang}) < 0, \qquad k = 2, ..., M. \tag{6.8}$$

- **Transverse stability:** To determine the stability of a cluster to the change of phases that change in its size and is obtained by linearizing (6.1) about $\eta_i = \theta_i - (\phi_k + \Omega t)$, we will introduce the following :

  $$\dot{\eta_u} = \sum_{v=1}^{s_k} S_{u,v}^{(k)} \ \eta_v,$$

  where (as stated in [105]) S is a matrix given by

  $$S_{u,v}^{(k)} = \frac{1}{N}\left[\delta_{u,v}\left(\sum_{r=1}^{M} m_r g'_{k,r} - g'_0\right) - (1 - \delta_{u,v})\, g'_0\right]$$

  with $(M+k)$ real eigenvalues which must be negative for a transverse stable cluster state. So a cluster state is transversely stable if:

  $$\lambda_{M+k}^{tran} < 0, \quad k = 1, ..., W. \tag{6.9}$$

  where W is the number of clusters with more than one oscillator.

A coupling function associated with a desired stable cluster state as defined in the previous section is periodic. In order to satisfy the conditions stated above (6.6), (6.8), and (6.9) the coupling function together with its first derivatives needs to satisfy the total of $2M - 2 + W$-conditions, where W is the number of clusters with more than one cluster and $0 \leq W \leq M$ [105]. Due to its periodicity with period 2π, it is useful to approximate $g(\phi)$ as a Fourier series. This was used in [37], [104] to study the dynamics of globally coupled oscillator systems and [105] has stated the lower bound of the number of Fourier modes that are required to find the coupling function. In [105] it was proven that if the coupling function g approximates using Fourier series with R modes as follows:

$$g(\zeta) = \sum_{r=1}^{R} \left(c_r cos(r\zeta) + s_r sin(r\zeta)\right). \tag{6.10}$$

then to satisfy all conditions in (6.6), (6.8), (6.9) with $(M-1+\frac{W}{2})$ unknowns, in general we need to choose the number of Fourier modes R to be:

$$R \geq M - 1 + \frac{W}{2}. \tag{6.11}$$

We define an isotropy subgroup of an equilibrium point q for the system (6.1) as

$$\Sigma_q = \{s \in S_N : sq = q\}.$$

with the associated fixed-point subspace defined as

$$Fix(\Sigma_q) = \{x \in \mathcal{A}^N : sx = x, \forall s \in \Sigma_q\}.$$

In the next section we consider a stable cluster state that has more oscillators than were used in [15]. We introduce theoretical conditions sufficient for a stable $3-$cluster state for a special case of the system of globally coupled oscillators 6.1 with six oscillators. However, we show that although solving a system of equations that leads one to identify the coupling function in terms of Fourier expansion is not an easy task, numerical observations show the existence of the desired coupling function.

6.2 Construction of stable 3-cluster states

Here we consider a cluster state where six coupled nonlinear oscillators are divided into three clusters of two oscillators in each cluster. This system exhibits more complex dynamics due to the increased size of each cluster, which makes possible a greater range of dynamics. Assuming the existence of this cluster state, we find conditions on the associated coupling function. Furthermore, we find conditions on the first derivative of such a coupling function that imply both transverse and tangential stability.

Consider the phase space consisting of six oscillators that are all-to-all coupled via the coupling function $g : [0, 2\pi) \to \mathbb{R}$ and an equivariant under continuous rotation S^1 and also full permutation symmetry S_6 given by:

$$\dot{\theta}_i = \omega + \frac{1}{6}\sum_{j=1}^{6} g(\theta_i - \theta_j). \tag{6.12}$$

with a set of invariant subspaces defined by:

$$\Pi^6_{\mathcal{P}}=\{\theta \in \Pi^6 : \text{If there exists } k \text{ such that } i,j \subset p_k \text{ then } \theta_i = \theta_j\}.$$

on which the partition $\mathcal{P} = \{p_1, p_2, p_3\}$ divides the oscillators into three non-trivial groups (clusters) each of size 2.

We rewrite the model (6.12) in the phase cluster space form Ψ_k as follows:

$$\dot{\Psi}_k = \omega + \frac{1}{6}\sum_{l=1}^{3} m_l g(\Psi_k - \Psi_l), \quad k = 1, 2, 3 \tag{6.13}$$

A symmetry group S_6 has nine possible isotropy subgroups; we find all its possible associated fixed-point subspaces for a system with six oscillators which is equivariant under S_6 as listed in the Table 6.1.

Assume the existence of θ which recognizes a stable 3-cluster partition as a periodic orbit. Then all set of conditions (6.6), (6.8), and (6.9) satisfy as follows:

$$\sum_{l=1}^{3} m_l(g(\phi_k - \phi_l) - g(\phi_1 - \phi_l) = 0 \quad k = 2, 3. \tag{6.14}$$

which produces the following two existence equations:

$$\begin{aligned} &2(g_{21} - g_0) + 2(g_0 - g_{12}) + 2(g_{23} - g_{13}) = 0 \\ &2(g_{31} - g_0) + 2(g_{32} - g_{12}) + 2(g_0 - g_{13}) = 0 \end{aligned} \tag{6.15}$$

Isotropy subgroup Σ	**dim(Fix(Σ)**	**Representive point**	**# conjugate subspaces**	**Orbit size**
S_6	1	$(\theta_1, \theta_1, \theta_1, \theta_1, \theta_1, \theta_1)$	1	1
$S_5 \times S_1$	2	$(\theta_1\theta_1\theta_1\theta_1\theta_1\theta_2)$	6	6
S_3^2	2	$(\theta_1\theta_1\theta_1\theta_2\theta_2\theta_2)$	10	20
$S_4 \times S_2$	2	$(\theta_1\theta_1\theta_1\theta_1\theta_2\theta_2)$	15	15
$S_4 \times S_1^2$	3	$(\theta_1\theta_1\theta_1\theta_1\theta_2\theta_3)$	30	60
$S_2 \times S_3 \times S_1$	3	$(\theta_1\theta_1\theta_2\theta_2\theta_2\theta_3)$	60	60
S_2^3	3	$(\theta_1\theta_1\theta_2\theta_2\theta_3\theta_3)$	15	90
$S_2^2 \times S_1^2$	4	$(\theta_1, \theta_1, \theta_2\theta_2, \theta_3, \theta_4)$	45	180
$S_3 \times S_1^3$	4	$(\theta_1\theta_1\theta_1\theta_2\theta_3\theta_4)$	20	120
$S_2 \times S_1^4$	5	$(\theta_1\theta_1\theta_2\theta_3\theta_4\theta_5)$	15	360
S_1^6	6	$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$	1	720

Table 6.1: Fixed-points subspaces of S_6 and its conjugates, see Figure 4.1 for the isotropy lattice of S_6.

Conditions (6.8) of tangent stability correspond to eigenvalues of the matrix T :

$$T = \frac{2}{6} \begin{bmatrix} g'_{12} + g'_{13} & -g'_{12} & -g'_{13} \\ -g'_{21} & g'_{21} + g'_{23} & -g'_{23} \\ -g'_{31} & -g'_{32} & g'_{31} + g'_{32} \end{bmatrix} \tag{6.16}$$

For tangent stability of the 3-cluster state, in addition to the trivial tangent eigenvalue of the matrix T there are two tangent eigenvalues given by:

$$\begin{aligned} \lambda_2^{tang} &= \tfrac{1}{2}(\mu + i\sqrt{\nu - \mu^2}); \\ \lambda_3^{tang} &= \tfrac{1}{2}(\mu - i\sqrt{\nu - \mu^2}). \end{aligned} \tag{6.17}$$

where

$$\mu = \frac{2}{6}(g'_{12} + g'_{13} + g'_{21} + g'_{23} + g'_{31} + g'_{32}). \tag{6.18}$$

$$\nu = \frac{16}{36}\left((g'_{21} + g'_{23})(g'_{31} + g'_{32}) - g'_{32}g'_{23} + (g'_{12} + g'_{13})(g'_{31} + g'_{32}) - g'_{31}g'_{13} + (g'_{12} + g'_{13})(g'_{21} + g'_{23}) - g'_{21}g'_{12}\right)$$

Or we can rewrite ν as

$$\nu = \frac{16}{36}\left[g'_{32}(g'_{12}+g'_{13}) + g'_{23}(g'_{12}+g'_{13}) + g'_{31}(g'_{21}+g'_{23}) + g'_{21}(g'_{32}+g'_{13}) + g'_{12}g'_{31}\right] \quad (6.19)$$

As noticed in [105], tangent eigenvalues depend only on g'_{ij} for $i \neq j$, Table (6.2) reviews the results from [105] on tangent stability conditions in terms of values of μ and ν.

Eigenvalue	$\nu - \mu^2 < 0$, $\mu < 0$	$\nu - \mu^2 > 0$, $\mu < 0$, $\nu < 0$	$\nu - \mu^2 < 0$, $\mu < 0$, $\nu > 0$
λ_2^{tang}	stable(real)	stable	stable
λ_3^{tang}	stable(real)	unstable	stable

Table 6.2: Tangent stability conditions [105].

Transverse stability conditions (6.9) can be satisfied by finding the eigenvalues of matrix $S_{u,v}^{(k)}$ which can be written as:

$$\begin{aligned}
\lambda_4^{tran} &= \frac{2}{6}(g'_0 + g'_{12} + g'_{13}) \\
\lambda_5^{tran} &= \frac{2}{6}(g'_{21} + g'_0 + g'_{23}) \\
\lambda_6^{tran} &= \frac{2}{6}(g'_{31} + g'_{32} + g'_0).
\end{aligned} \quad (6.20)$$

In brief, we can design a coupled oscillator system that has a stable (2,2,2)−cluster state if the following set of equations and inequalities are satisfied:

$$
\begin{aligned}
2(g_{21}-g_0)+2(g_0-g_{12})+2(g_{23}-g_{13}) &= 0;\\
2(g_{31}-g_0)+2(g_{32}-g_{12})+2(g_0-g_{13}) &= 0;\\
Re(\lambda_2^{tang}) = Re(\frac{1}{2}(\mu+i\sqrt{\nu-\mu^2})) &< 0;\\
Re(\lambda_3^{tang}) = Re(\frac{1}{2}(\mu-i\sqrt{\nu-\mu^2})) &< 0;\\
Re(\lambda_4^{tran}) = Re(\frac{2}{6}(g_0'+g_{12}'+g_{13}')) &< 0;\\
Re(\lambda_5^{tran}) = Re(\frac{2}{6}(g_{21}'+g_0'+g_{23}')) &< 0;\\
Re(\lambda_6^{tran}) = Re(\frac{2}{6}(g_{31}'+g_{32}'+g_0')) &< 0.
\end{aligned}
\tag{6.21}
$$

Using the results discussed in the previous section, which have been proved in [105], one can use a Fourier series to approximate the coupling function. As a consequence of this approximation, one can design the desired coupling function by finding (approximately) its associated Fourier coefficients. In contrast, we also suggest values for these coefficients that lead to the occurrence of stable $3-$cluster states. In the following parts of this chapter we discuss these two matters.

6.2.1 Fourier series for stable $3-$cluster states

The coupling function of stable 3-cluster states can be approximated using Fourier series with a number of modes R more than or equal to 3 [105], and here we will use four modes:

$$
g(\zeta) = \sum_{r=1}^{4} \left(c_r \cos(r\zeta) + s_r \sin(r\zeta)\right). \tag{6.22}
$$

Eight constants (i.e. four Fourier modes) will be used to represent the coupling function for our system (6.12).

Assume that $A_r = c_r \cos(r\zeta)$ and $B_r = s_r \sin(r\zeta)$ $(1 \le r \le 4)$ and substitute with the Fourier form of the coupling function (6.22) in existence equations (6.15) and tangent equations we get:

$$0 = \sum_{r=1}^{4} [A_r + B_r] . \tag{6.23}$$

Using the Fourier approximation of the coupling function in the existence equations of the 3−cluster state (6.15), the tangent eigenvalues equations (6.17), and the equations of transverse eigenvalues (6.20) will give a system of seven equations with eight unknowns. The nonlinear expressions for the tangent eigenvalues complicate the analytic solution for this system. Rather, we derive clear conditions for tangent and transverse stability of the 3−cluster state. These conditions are based on our analysis of the tangent eigenvalues equations and the equations of transverse eigenvalues for system of nonlinear six globally coupled oscillator where they depend on the coupling function and its first derivative.

6.2.2 Stability for three-cluster states

Consider the stability of a periodic 3−cluster state where all clusters are non-trivial (i.e., the clusters have sizes $m_r \geq 2$ for $r = 1, 2, 3$, where $m_1 + m_2 + m_3 = N$). This implies that we must consider $N \geq 6$ oscillators, and we will focus in the later sections especially on the case $N = 6$. Recall from [105] that (6.8) for tangential stability is determined by the eigenvalues of the matrix T

$$T = \frac{1}{N} \begin{bmatrix} a_2 g'_{12} + a_3 g'_{13} & -a_2 g'_{12} & -a_3 g'_{13} \\ -a_1 g'_{21} & a_1 g'_{21} + a_3 g'_{23} & -a_3 g'_{23} \\ -a_1 g'_{31} & -a_2 g'_{32} & a_1 g'_{31} + a_2 g'_{32} \end{bmatrix} \tag{6.24}$$

which has $\lambda_1^{tang} = 0$, and the remaining eigenvalues

$$\begin{aligned} \lambda_2^{tang} &= \tfrac{1}{2}(\mu + i\sqrt{\nu - \mu^2}) \\ \lambda_3^{tang} &= \tfrac{1}{2}(\mu - i\sqrt{\nu - \mu^2}) \end{aligned} \tag{6.25}$$

where

$$\mu = \frac{1}{N}(a_2 g'_{12} + a_3 g'_{13} + a_1 g'_{21} + a_3 g'_{23} + a_1 g'_{31} + a_2 g'_{32}). \tag{6.26}$$

and

$$\begin{aligned}\nu =& \frac{4}{N^2}\left((a_1 g'_{21} + a_3 g'_{23})(a_1 g'_{31} + a_2 g'_{32}) - a_2 a_3 g'_{32} g'_{23}\right.\\ &+ (a_2 g'_{12} + a_3 g'_{13})(a_1 g'_{31} + a_2 g'_{32}) - a_1 a_3 g'_{31} g'_{13}\\ &\left.+ (a_2 g'_{12} + a_3 g'_{13})(a_1 g'_{21} + a_3 g'_{23}) - a_1 a_2 g'_{21} g'_{12}\right).\end{aligned} \tag{6.27}$$

which we note can be written in the form

$$\begin{aligned}\nu =& \frac{4}{N^2}\left[a_1^2 g'_{21} g'_{31} + a_2^2 g'_{12} g'_{32} + a_3^2 g'_{13} g'_{23}\right.\\ &+ a_1 a_2 \left(g'_{21} g'_{32} + g'_{12} g'_{31}\right)\\ &+ a_1 a_3 \left(g'_{23} g'_{31} + g'_{13} g'_{21}\right)\\ &\left.+ a_2 a_3 \left(g'_{13} g'_{32} + g'_{12} g'_{23}\right)\right].\end{aligned} \tag{6.28}$$

This simplification allows one to verify the following new result:

Lemma 6.1 *Suppose there is a periodic three-cluster state such that $g'_{ij} < 0$ for all $i \neq j$. Then the cluster is tangentially stable with complex contracting eigenvalues.*

Proof: Note that if $g'_{ij} < 0$, then all terms in (6.26) are negative, and so $\mu < 0$ while the terms in (6.28) cancel leaving only positive terms, so $\nu > 0$. Hence the eigenvalues (6.25) are complex with negative real parts and so the cluster is tangentially stable. **QED**

We give new results for the transverse stability as follows. Firstly, note that the transverse eigenvalues are

$$\begin{aligned}\lambda_4^{tran} &= \frac{1}{N}(a_1 g'_0 + a_2 g'_{12} + a_3 g'_{13})\\ \lambda_5^{tran} &= \frac{1}{N}(a_1 g'_{21} + a_2 g'_0 + a_3 g'_{23})\\ \lambda_6^{tran} &= \frac{1}{N}(a_1 g'_{31} + a_2 g'_{32} + a_3 g'_0)\end{aligned} \tag{6.29}$$

and so define

$$\lambda_1 = \frac{1}{a_1}(a_2 g'_{12} + a_3 g'_{13}), \quad \lambda_2 = \frac{1}{a_2}(a_1 g'_{21} + a_3 g'_{23}), \quad \lambda_3 = \frac{1}{a_3}(a_1 g'_{31} + a_2 g'_{32}).$$

Without loss of generality (by renumbering the clusters), we can assume that

$$\lambda_3 < \lambda_2 < \lambda_1 \tag{6.30}$$

and the following new results hold:

Theorem 6.2 *Suppose there is a periodic 3−cluster state with non-trivial clusters and assume (6.30). Without counting the multiplicity of the transverse eigenvalues for each cluster, if the state is tangentially stable then we can classify the stability as follows:*

- *If $g'_0 < -\lambda_1$ then all transverse eigenvalues are negative and the state is transversely stable*
- *If $-\lambda_1 < g'_0 < -\lambda_2$ then the state has precisely one positive transverse eigenvalue (one unstable cluster)*
- *If $-\lambda_2 < g'_0 < -\lambda_3$ then the state has precisely two positive transverse eigenvalues (two unstable clusters)*
- *If $-\lambda_3 < g'_0$ then all transverse eigenvalues are positive (all the three clusters are transversely unstable)*

Proof: The conclusions follows from noting that (6.30) and the conditions on g'_0 ensure that (6.29) have zero, one, two or three positive transverse eigenvalues (not counting multiplicity). **QED**

We can use this to prove the following bifurcation theorem for families of coupling functions:

Theorem 6.3 *For a given set of phase differences, if the family of coupling functions $g(\phi)$ satisfies the set of existence and stability equations and inequalities in (6.21). Then the system (6.12) has a periodic 3−cluster state with non-trivial clusters and certain stability properties. However, there is a parameterized perturbation of the coupling function*

$$g_\lambda(\phi) = g(\phi) + \lambda h(\phi)$$

and $\lambda_3 \leq \lambda_2 \leq \lambda_1$, such that for all λ the cluster state is tangentially stable and:

1. *If $g_0' + \lambda < -\lambda_1$ then the cluster state is also transversely stable*
2. *If $-\lambda_1 < g_0' + \lambda < -\lambda_2$ then precisely one cluster is unstable*
3. *If $-\lambda_2 < g_0' + \lambda < -\lambda_3$ then precisely two clusters are unstable*
4. *If $-\lambda_3 < g_0' + \lambda$ then all clusters are unstable*

Proof: For a specific 3−cluster state there is an $\epsilon > 0$ such that $|\phi_j - \phi_k| > \epsilon$ for any $j \neq k$. Now consider a smooth periodic function h such that $h(\phi) = 0$ for all $\epsilon < |\phi| \leq \pi$, $h(0) = 0$ and $h'(0) = 1$. Then $g_{\lambda;ij} = g_{i,j}$, $g'_{\lambda;ij} = g'_{i,j}$ for all $i \neq j$, $g_\lambda(0) = g(0)$ and

$$g'_{\lambda;0} = g'_0 + \lambda.$$

Hence the existence condition (6.6) and the tangential stability conditions do not depend on λ, while Theorem 6.2 can now be applied to prove the possible cases depending on λ. **QED**

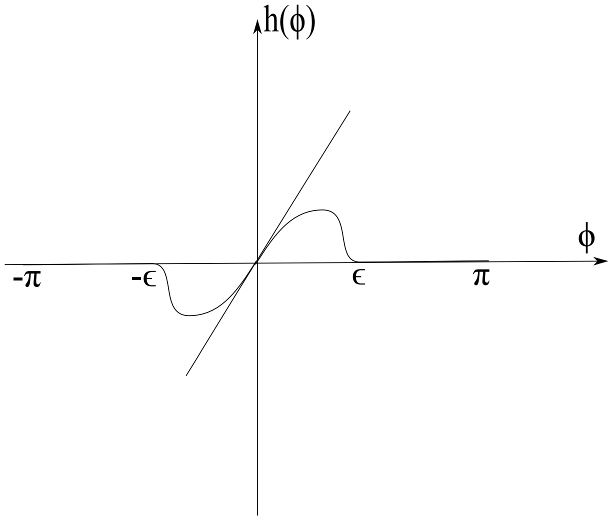

Figure 6.1: An illustration of the function $h(\phi)$ used to perturb the coupling functions in Theorem 6.3.

6.3 Three−cluster states and the existence of appropriate coupling functions

A coupling function is constructable for any stable cluster state with specific cluster phases. In [105] the authors introduced a case of designing a coupling function associated with equidistant stable clusters. This was introduced to go beyond the difficulty of finding Fourier coefficients for systems with a large number of clusters. So, by considering constraints on cluster phases in the article [105] it was shown that for stable cluster state with any number of clusters one can choose harmonic and localized coupling function. This partition of the set of oscillators depends on the initial condition. (For example, different initial conditions may lead to the same cluster state with different partitions while the same initial condition may lead to different cluster states [105]).

In this thesis, as we have discussed in Section 6.2, the design of coupling function for stable non-trivial 3−cluster state is complicated due to the nonlinearity of tangential stability equations that we need to have this relatively large number of stable clusters. However, numerically using MATLAB we have found two slightly different sets of Fourier coefficients (see Table 6.3) and by looking at time series of coupled six oscillators system for the same initial condition, we observe different 3−cluster states with short transition times of connection between these different 3−cluster states; (see results in Figure 6.2). However, the xppaut program shows that the set of parameters with a cycle between the $(2, 2, 2)-$cluster states in non-empty..

Moreover, to show that our assumed Fourier coefficients, shown in Table 6.3, can be used to design stable 3−cluster state, we consider the coupled system (6.13) and use the clusters phases values and their tangent and transversal eigenvalues associated with the Fourier coefficients in Table 6.3 and we found that they imply the same dynamical behavior in Figure 6.2.

Note that, for the same coupling function of stable 3−cluster states with the same values as in Table 6.3 but different initial conditions, we have either a chain of different cluster states or chaotic behavior as shown in Figure 6.3.

Values of Fourier parameters	c_1	c_2	c_3	c_4	s_1	s_2	s_3	s_4
Case1	0.31185	0.37096	0	0.99008	0.10793	0.58180	0	−0.14053
Case2	0.31185	0.39	0	0.99008	0.10793	0.58180	0	−0.14053

Table 6.3: Fourier coefficients of the coupling function used to construct 3-cluster states.

Properties of $(2, 2, 2)$**-cluster solution**	$\Psi_1 - \Psi_3$	$\Psi_2 - \Psi_3$	λ_i^{tang}	λ_i^{tran}
Case1	1.70136	4.75731	0, −0.447290, −1.46903	−1.30703, −0.060143, 0.163551
Case2	1.7087	4.7761	0, −0.51022, −1.39014	−1.279837, 0.036919, 0.025683

Table 6.4: Properties of 3-cluster states for the parameters in Table 6.3.

We conclude from these observations that the desired coupling function to construct a stable non-trivial cluster state with prescribed phases is exist although it is not easy to

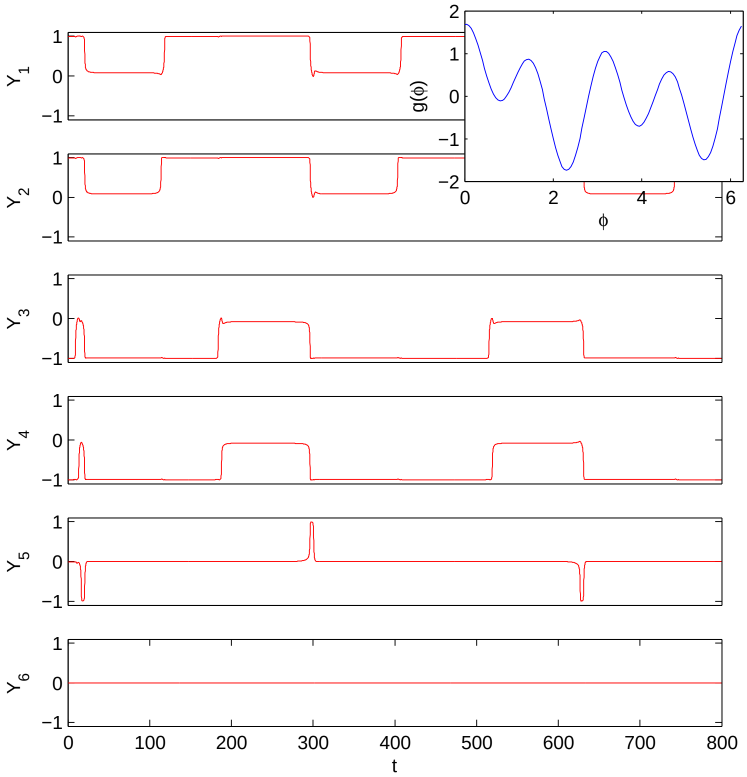

Figure 6.2: Example: case 1. The inset shows that coupling function $g(\varphi)$ while the timeseries show the evolution of the phase differences relative to the 6th oscillator; $y_k = \sin(\varphi_k - \varphi_6)$ as a function of t. Observe that the six oscillators synchronize into three clusters for most of the time, but there are short times when the clusters break along a connection.

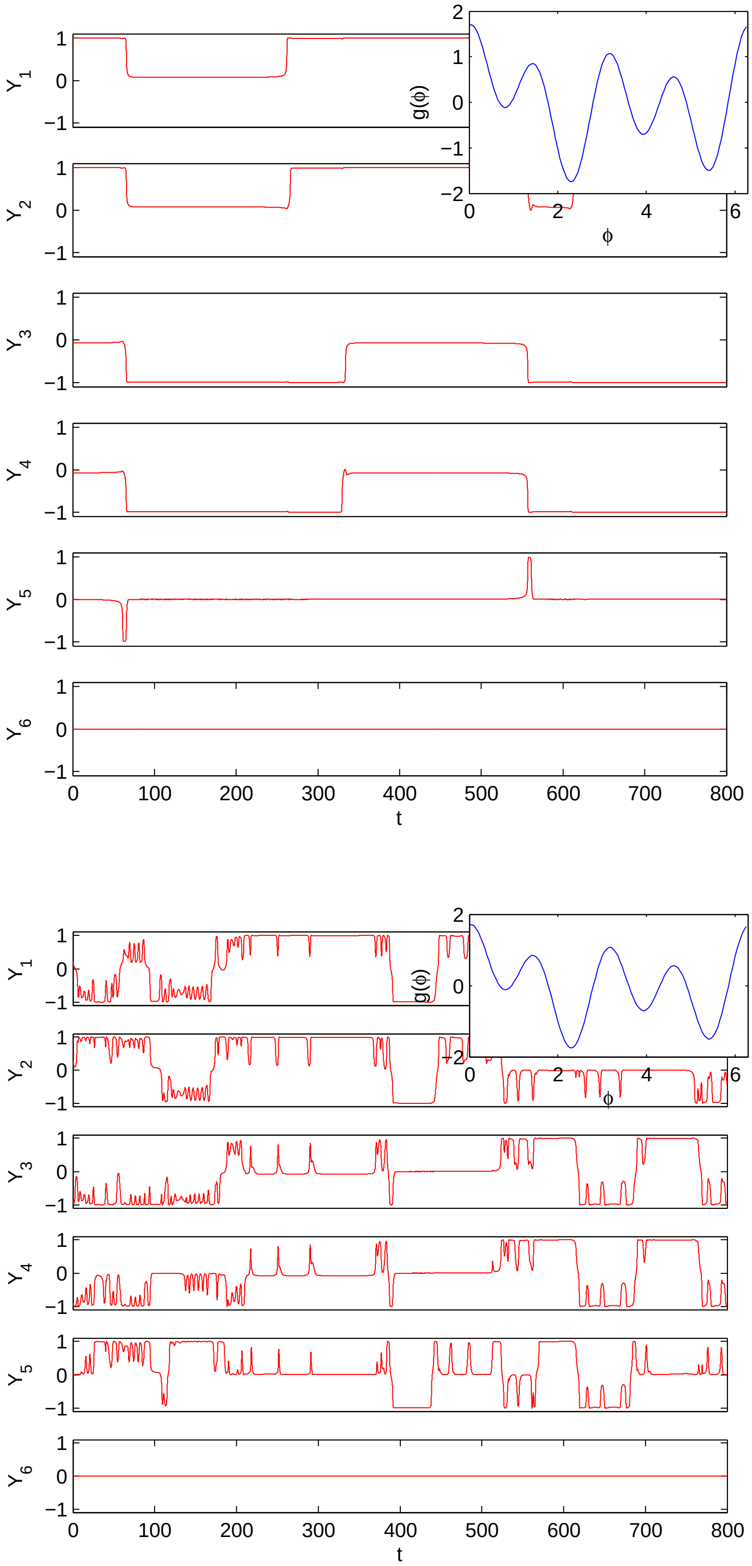

Figure 6.3: Example: case 2 timeseries: two different initial conditions.

solve the system of existence and stability conditions that we need to find the coupling function due to the non-linearity of such system. In Theorem 6.3 we show the existence of stable $3-$cluster state under the assumption of existence of coupling function with a constrained phases. However, using MATLAB we illustrate in this section the problem of solve the set of equations and inequalities (6.21) that ensure the existence of stable non-trivial $3-$cluster state. Even using MATLAB to solve the system (6.21) and find more examples of spaces of solutions for the system of six coupled oscillators systems (6.12), we found that it is difficult to do due to the large number of Fourier modes required

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Chapter 7

The restricted Conley index and heteroclinic cycles for cluster states

In this chapter we apply the restricted Conley index from Chapter 3 to study the robustness of non-trivial clusters states in systems of globally coupled oscillators. In particular, we investigate the robustness of heteroclinic cycle of $(2, 2, 2)-$cluster state in an equivariant system consisting of six all-to-all coupled oscillators by considering the heteroclinic cycle whose existence we have shown in Chapter 6 due to connections between saddle $(2, 2, 2)-$cluster state. The results introduced in the following sections relate to the computations introduced in the previous chapter. We also study the restricted Conley index for the cycle by following the connection scheme of heteroclinic cycle of the $(2, 2, 2)-$cluster state. The considered system has other $3-$cluster states with different partitions for the oscillators; indeed simpler dynamics occur in the existence case of trivial cluster (cluster with one oscillator), see for example [15]. The same conclusions can be stated regarding the robustness for dynamical systems which introduced the same dynamics.

7.1 Nonlinear globally coupled oscillators system and saddle cluster behaviour

In this section we essentially recall the globally coupled phase oscillator model for six oscillators that we studied in Chapter 6.

Consider the system of globally coupled oscillators

$$\dot{\theta}_i = \omega + \frac{1}{6}\sum_{j=1}^{6} g(\theta_i - \theta_j). \tag{7.1}$$

The solution of this model represents a periodic orbit with period 2π ($6-$torus Π^6 with symmetry S_6 and S^1) . We shall consider the phase difference model associated with this system (due to the continuous rotational symmetry S^1). Define the coupling function using the following harmonics:

$$g(\zeta) = \sum_{r=1}^{4} \left(c_r \cos(r\zeta) + s_r \sin(r\zeta)\right).$$

and fixing cluster phases. Then one can find the Fourier coefficients such that the system (7.1) has robust solutions [105]. Showing this robustness analytically is what the following sections will discuss. In particular, we are interested in the heteroclinic cycles formed by connections between $3-$cluster states. Fixed$-$point subspaces in Table 6.1 addressed the subspaces where the different cluster states occur. To analyze the dynamics of this type of heteroclinic cycles we thus restrict the system (7.1) to the subspaces where the $3-$cluster states appear. Following the same assumptions as in [105] and denoting the phase of k^{th} cluster by Ψ_k where $k = 1, 2, 3$, we rewrite the system (7.1) as:

$$\dot{\Psi}_k = \omega + \frac{1}{6}\sum_{l=1}^{3} m_l g(\Psi_k - \Psi_l). \tag{7.2}$$

A heteroclinic cycle between $3-$cluster states occur in two cases. First, the $(2, 2, 2)$-

cluster states that are symmetrically related by $(S_2)^3$. Second, the $(3,2,1)-$cluster states which are symmetrically related by $S_3 \times S_2 \times S_1$ and seems to have the same dynamics has been studied in detail in [15]. Other types of heteroclinic cycles have also been observed in the numerical results that are symmetrically related to $2-$cluster states; for example, the six oscillators divided into two clusters (under $(S_3)^2$ symmetry), and the other cycle related to two cluster states, one consisting of four oscillators and the other of two (under the $S_4 \times S_2$ symmetry).

In the following we concentrate on analyzing the dynamics of the heteroclinic cycle that are formed by connections between non-trivial $3-$cluster states.

7.2 Robust heteroclinic cycles of $3-$cluster states

7.2.1 Dynamics of saddle (2,2,2)-cluster state

Using the set of parameters in Table 6.3 which yield $3-$cluster states with at least one positive transverse eigenvalue, as stated in the Table 6.4, however, there are $\frac{6!}{2!2!2!} = 90$ different $3-$cluster states obtained by permutation symmetry S_6.

If θ_i denotes the phase of the i^{th} oscillator, the invariant subspace of the $(2,2,2)-$cluster state can be expressed as

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_1 \\ \theta_2 \\ \theta_2 \\ \theta_3 \\ \theta_3 \end{pmatrix} \tag{7.3}$$

where the first and second oscillators are in one cluster ($\star-$cluster), the third and fourth oscillators are in one cluster ($\circ-$cluster), and the fifth and sixth oscillators are in one cluster ($\bullet-$cluster). Assume that the phase of $\star-$cluster is $\phi_\star$, the phase of $\circ-$cluster is $\phi_\circ$ and the phase of $\bullet-$cluster is $\phi_\bullet$. Then the $(2,2,2)-$cluster subspace (7.3) can expressed as:

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{pmatrix} = \begin{pmatrix} \phi_\star + \Omega t \\ \phi_\star + \Omega t \\ \phi_\circ + \Omega t \\ \phi_\circ + \Omega t \\ \phi_\bullet + \Omega t \\ \phi_\bullet + \Omega t \end{pmatrix} \tag{7.4}$$

for some $\phi_\star < \phi_\circ < \phi_\bullet < \phi_\star + 2\pi$.

So we have three different cluster phases $\psi_\star, \psi_\circ, \psi_\bullet$.

As mentioned above, the continuous rotational symmetry S^1 can be used to fix one cluster phase and study the phase differences of the other two phases corresponding to the fixed one. So to analyze the $3-$cluster state of type $(2,2,2)$, we fix the first cluster ($\star-$cluster) and assume that:

$$\alpha := \psi_\circ - \psi_\star$$

$$\beta := \psi_\bullet - \psi_\star$$

such that $\psi_1 = \psi_2 = o$; $\psi_3 = \psi_4 = \alpha$; $\psi_5 = \psi_6 = \beta$. In the following we analyze heteroclinic cycles connected to three $3-$cluster states.

7.2.2 Interactions between saddle $(2,2,2)-$cluster states

Assume that P_k represents $3-$cluster states that are related by $(S_2)^3$ symmetry to form a heteroclinic cycle; $k = 1,2,3$, then

- $P_1 = (0,0,\alpha,\alpha,\beta,\beta)$;

- $P_2 = (\beta,\beta,0,0,\alpha,\alpha)$;

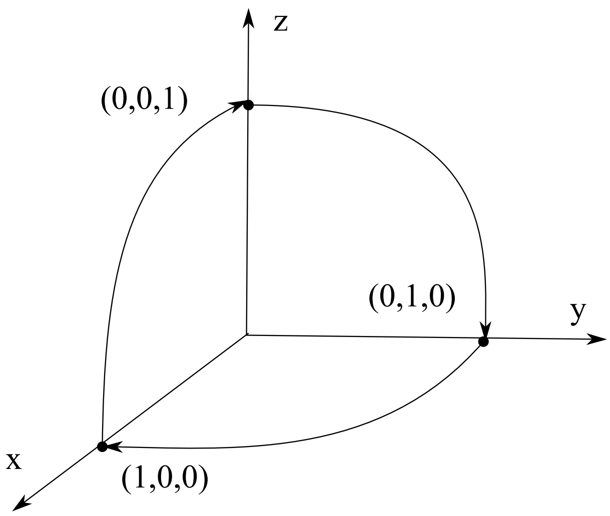

Figure 7.1: Schematic diagram showing the heteroclinic cycle between $(2, 2, 2)$-cluster states for six oscillators, as shown in Figure 6.2. The connections between the three symmetrically related $(2, 2, 2)$- cluster states P_i for $i = 1, 2, 3$ are contained within $S_2 \times S_2$ fixed point subspaces shown here by I_i for $i = 1, 2, 3$.

- $P_3 = (\alpha, \alpha, \beta, \beta, 0, 0)$.

In the following we analyze how such heteroclinic cycle as shown in Figure 6.1 can appear. Each of these 3−cluster states P_1, P_2, P_3 has at least one positive transverse eigenvalue, starting with the 3−cluster state $P_1 = (0, 0, \alpha, \alpha, \beta, \beta)$ (which has at least one positive transverse eigenvalue) and, assuming that the third cluster is unstable, there will be a connection (heteroclinic orbit) $c_{P_1P_2}$ in the subspace that has $(S_2)^2 \times (S_1)^2$ symmetry of the form $(\gamma, 0, 0, \alpha, \alpha, \beta) = I_{P_1P_2}$ which connects P_1 to P_2 (P_2 is stable in $I_{P_1P_2}$). On the other hand, the cluster state P_2 connects symmetrically with P_3 via its one−dimensional unstable manifold $c_{P_2P_3}$, which lies within the invariant subspace of the form $(\beta, \gamma, 0, 0, \alpha, \alpha) = I_{P_2P_3}$ (the cluster state P_3 is stable in $I_{P_2P_3}$). Moreover, P_3 has one unstable cluster which is split in the subspace $(\alpha, \alpha, \beta, \gamma, 0, 0) = I_{P_3P_1}$ and connects to its $(S_2)^3$ symmetrically image P_1 via a connection $c_{P_3P_1}$. We end with the heteroclinic cycle $\Sigma_{(2,2,2)}$; see (7.1), which has the following connection scheme:

$$P_1 \xrightarrow{I_1} P_2 \xrightarrow{I_2} P_3 \xrightarrow{I_1} P_1 \tag{7.5}$$

Using this existence verification we state the following theorem which relies on a numerical observation of a connection in one case to show robustness.

Theorem 7.1 (Robustness of saddle 3-cluster state) *Given a globally coupled oscillators system (7.1). If the coupling function $g(\Psi)$ for its associated coupled phase oscillator system (7.2) satisfies the existence conditions (6.15) of $(2,2,2)-$cluster states whose was given in Theorem 6.3, then the system has a robust heteroclinic cycle between $(2,2,2)-$cluster states.*

Proof: Consider the coupling function

$$g(\zeta_i) = \sum_{r=1}^{4} \left[c_r cos(r\zeta_i) + s_r sin(r\zeta_i)\right]; \; i = 1,2$$

such that $\psi_1 = \phi_\circ - \phi\star$; and $\psi_2 = \phi_\bullet - \phi_\star$.

With the set of Fourier coefficients given in Table 6.3 in the previous chapter. Then such system (7.1) has a heteroclinic cycle of $(2,2,2)-$cluster states. Using our results from Chapter 5, Theorem 5.1 predicts the robustness of these heteroclinic connections using the restricted Conley index. **QED**

As we showed in Chapter 6 (section 6.3) that using Matlab one can show there exists a coupling function for the system of six coupled oscillators that gives rise to a stable non-trivial $3-$cluster state. From Theorem 7.1, we can state the same statement about the existence of coupling function for the system (6.12) to have robust heteroclinic cycle between $(2,2,2)-$cluster states. Here we compute the restricted Conley index for each component in the heteroclinic cycle of the $(2,2,2)-$cluster state.

Consider the heteroclinic cycle $\Sigma_{(2,2,2)}$ which consists of three symmetrically related $(2,2,2)-$cluster states, and in which each of its components has one$-$dimensional unstable manifold. From the above discussion $dim(W^u(P_i) \cap I_{P_iP_j}) = 1$; $dim(W^u(P_j) \cap I_{P_iP_j}) = 0$ and if $S_{P_iP_j} = P_i \cup c_{P_iP_j} \cup P_j$ for $i,j = 1,2,3$, then the restricted Conley index for each heteroclinic connection within $\Sigma_{(2,2,2)}$ are as follows:

- $C_{I_{P_1P_2}}(P_1) = S^1$;
- $C_{I_{P_1P_2}}(P_2) = \{a, b\}$;
- $C_{I_{P_1P_2}}(S_{P_1P_2}) = S^0$;
- $C_{I_{P_2P_3}}(P_2) = S^1$;
- $C_{I_{P_2P_3}}(P_3) = \{a, b\}$;
- $C_{I_{P_2P_3}}(S_{P_2P_3}) = S^0$;
- $C_{I_{P_3P_1}}(P_3) = S^1$;
- $C_{I_{P_3P_1}}(P_1) = \{a, b\}$;
- $C_{I_{P_3P_1}}(S_{P_3P_1}) = S^0$.

In all the previous cases we can see that:

$$C_{I_{P_iP_j}}(S_{P_iP_j}) \neq C_{I_{P_iP_j}}(P_i) \vee C_{I_{P_iP_j}}(P_j).$$

From Theorem 5.1 we can conclude that the heteroclinic cycle $\Sigma_{(2,2,2)}$ is robust to perturbations that respect invariant subspace $I_{P_1P_2}$.

7.2.3 Other possible heteroclinic cycles

Another possible heteroclinic cycle of 3−cluster states that we have not investigated numerically but which may existis formed by symmetry between cluster states of type $(3, 2, 1)$ where three oscillators cluster together to form one cluster, two oscillators cluster together to form another cluster, and the other oscillator forms a trivial cluster with a single oscillator. Indeed, via $(S_3 \times S_2 \times S_1)$ symmetry all the three cluster states of type $(3, 2, 1)$ are characterized as heteroclinic cycles, this has dynamics analogue to the case of system of coupled five oscillators in [15]. The heteroclinic cycle considered by such cluster states may be form in two different ways depending on which nontrivial cluster is non stable. For example, if we assume the 3−cluster state $P_1 = (0, \beta, \beta, \alpha, \alpha, \alpha)$ such that the cluster with two oscillators represents the unstable cluster. In this case there can be one−dimensional unstable manifold in the invariant subspace $I_{P_1P_2} = (0, a, b, c, c, c)$ which connects P_1 to $P_2 = (\alpha, \alpha, \alpha, 0, \beta, \beta)$. Again, under the $(S_3 \times S_2 \times S_1)$ symmetry the 3−cluster states P_2 and $P_3 = (\beta, \beta, \alpha, \alpha, \alpha, 0)$ are connected by the connection orbit in the invariant subspace $I_{P_2P_3} = (c, c, c, 0, a, b)$. Finally, P_3 connects with P_1 via a connection lying in $I_{P_3P_1}$ to form a heteroclinic cycle of three cluster states.

Now, we consider possible heteroclinic cycles consisting of the same kind of clustering as the previous cycle, but with different dynamics. Assume that the cluster with three oscillators is the unstable one, then the dimension of the unstable manifold might be one or two. Starting with the 3−cluster state $P_1 = (0, \beta, \beta, \alpha, \alpha, \alpha)$ with two−dimensional unstable manifold then the heteroclinic orbit lies within the invariant subspace of the form $I_{P_1P_2} = (0, a, a, b, c, d)$ and connects P_1 to the 3−cluster state $P_2 = (\alpha, \alpha, \alpha, 0, \beta, \beta)$. Similarly, from the 3−cluster state $P_2 = (\alpha, \alpha, \alpha, 0, \beta, \beta)$ there is a heteroclinic orbit in the invariant subspace $I_{P_2P_3} = (a, b, c, 0, d, d)$ which connects to the 3−cluster state $P_3 = (\beta, \beta, \alpha, \alpha, \alpha, 0)$, and finally the heteroclinic cycle is formalized by connection between P_3 and P_1 via $I_{P_3P_1}$.

Moving to the other case of the same 3−cluster state $P_1 = (0, \beta, \beta, \alpha, \alpha, \alpha)$ but with one−dimensional unstable manifold so the connection scheme can be explained. There is one connection in the invariant subspace $I_{P_1P_2} = (0, a, a, b, b, c)$ which connects P_1 to $P_2 = (\alpha, \alpha, \alpha, 0, \beta, \beta)$. Although P_2 is an attractor in $I_{P_1P_2}$, on the other side it has one−unstable direction which connects P_2 to $P_3 = (\beta, \beta, \alpha, \alpha, \alpha, 0)$ via a heteroclinic orbit in the invariant subspace $I_{P_2P_3} = (a, a, b, 0, c, c)$. Another connection orbit in $I_{P_3P_1}$ will connect P_3 to P_1 and this composes a heteroclinic cycle with a different connection scheme.

We can find other heteroclinic cycles of 2−cluster states formed by either $S_3 \times S_3$

symmetry between two 2−cluster states $P_1 = (\alpha, \alpha, \alpha, \beta, \beta, \beta)$, or by $S_4 \times S_2$ symmetry between two 2−cluster states $P_1 = (\alpha, \alpha, \alpha, \alpha, \beta, \beta)$. The difficulty of dynamics for these heteroclinic cycles is the result of the growth of their unstable manifolds, which leads to a greater variety of possible connections.

Chapter 8

Conclusion

8.1 Summary of the thesis' findings

This thesis set out to investigate the concept of robustness of dynamical systems, in particular for heteroclinic cycles. Constraints and motivation for robust heteroclinic solutions of dynamical systems and some implications of the existence of robust heteroclinic cycles have been discussed in Section 1.1; Chapter 1. We have investigated this problem by introducing a constrained definition of the Conley index with respect to an invariant subspace in Chapter 3. Basically, to use the restricted Conley index to study the topology of an invariant set we need to isolate such invariant set using a constrained neighbourhood and moreover need to define a gradient-like vector field for the flow generated by the solution of the considered system in this neighbourhood. In Lemma 3.11 we showed that for any invariant subspace for the flow generated by a vector field and any isolating neighbourhood for the same flow, their intersection is also an isolating neighbourhood for the same flow whenever their intersection is not empty. However, Theorem 3.13 proves the robustness of the Conley index when restrict to an invariant subspace. Using results from [89] to divide an invariant set which contains at least one heteroclinic orbit connecting two hyperbolic equilibria into a set of isolated orbits, in Theorem 3.14 we have satisfied a gradient-like condition in the neighbourhood of an isolated invariant set. We use the restricted Conley index to show the robustness of heteroclinic orbit between two hyperbolic equilibria of dynamical systems in Theorem 3.15. Also, we set an example 3.2 in $\mathbb{R}^4$ where the intersection between the stable and unstable manifolds of two hyperbolic equilibria is not transverse, and show robustness of heteroclinic orbits connecting them using the restricted Conley index. In Chapter 5 we show robustness of a heteroclinic cycle using the restricted Conley index. Section 5.1 reviews the concept of the connection scheme for a heteroclinic cycle which has been introduced in [12]. We show the robustness of a heteroclinic cycle with a prescribed connection scheme that appear in a constrain dynamical system (discussed in Chapter 3), using the restricted Conley index, in Theorem 5.1. The condition for a robust heteroclinic cycle we have introduced in Theorem 5.1 is different from that has been introduced in [12], we try to prove an equivalence between them but it seems to be not an easy. However, we prove a special case of implication in Corollary 5.2. In Sections 5.2.1 and 5.2.2 we apply the condition for a robust heteroclinic cycle using the restricted Conley index on Guckenheimer-Holmes heteroclinic cycle and Kirk-Silber heteroclinic network. However, we use the topology to show robustness of heteroclinic connections between cluster states in systems of globally coupled oscillators. Therefore, we dedicated part of this thesis investigating the question which has been stated by other authors where one assumes the existence of a particular behavior (in fact, stable cluster state) in a globally coupled oscillators dynamical system and seek the coupling function of the system and identify which conditions should the appropriate coupling function satisfy. To our knowledge, the problem of designing the coupling function was considerd for the case of non-trivial stable $2-$cluster state for globally coupled oscillators system in [105] where the authors designed the desired coupling function using conditions introduced in the same article, and this was, in general, the largest non-trivial cluster state. We have investigated designing the coupling function to identify non-trivial stable $3-$cluster state which needs to consider system of at least six globally coupled oscillators using the general conditions stated in [105]. However, we set new stability conditions for such cluster state using first derivatives of the associated coupling function rather than the previous conditions which based on study their tangent and transverse eigenvalues. The thesis has researched on two different themes, the first has investigated on Chapters 3, 5, the second has discussed in Chapter 6. In Chapter 7 we connect them through apply the theoretical results from our first theme on the system designed with a prescribed behavior from our second theme. In fact the thesis sought to answer these questions:

1. Can we investigate the robustness of heteroclinic cycles using topological methods as Conley index, and; what constraints should the dynamical systems have to satisfy to study the robustness using these topological concepts?
2. Can we design a globally coupled six oscillators system with a prescribed stable $3-$cluster state?
3. How can we show the robustness of the heteroclinic cycle of non-trivial $3-$cluster state using these topological methods?

The main results achieved are in Chapters 3, 5, 6, and 7. In Sections 3.2, 3.3, and 3.4 of Chapter 3, a special case of the Conley index, the "restricted Conley index" is defined by restricting the computation of the index with respect to the intersection between an isolating neighbourhood of the considered solution and the appropriate invariant subspace. We introduce a condition for the robustness of heteroclinic orbits that connect two hyperbolic equilibria of the flow generated by gradient-like vector field using the restricted Conley index. Therefore, the computed index within an invariant subspace is resistant to the perturbations of vector field. We achieved our theoretical result by decompose the isolated invariant set using attractor-repeller decomposition and, since one cannot generally isolate the set of connections between two components of such attractor-repeller decomposition, we assumed the compactness of this set and showed the existence of a gradient-like vector field locally, which leads us to answer part of the first question regarding to constraint of the system to adjust it with the restricted Conley index theory. We have supported these findings by introducing a simple example in $\mathbb{R}^2$ which clarifies the calculations of the restricted Conley index.

As the theory of the Conley index uses the C^0 topology and the results of continuation of the Conley index with respect to parametrized space, stronger statements on the robustness on the Conley index cannot be made. However, we note that according to personal communication with K.Mischaikow, it is presumably true that similar statements can be made for robustness to neighbourhoods of $f \in \mathcal{X}$ in the C^2 topology.

Up to this point we have presented the restricted Conley index as an alternative method to show robustness of heteroclinic orbits. To make the restricted Conley index applicable in the case of failure of the other methods which need transverse intersection condition to show the robustness of connections, we close the Chapter 3 with an example where the restricted Conley index may be the only way to show robustness of the heteroclinic orbit. An example shows how a connection between two hyperbolic equilibria in $\mathbb{R}^3$ with two$-$dimensional stable and unstable manifolds that intersect nontransversally can be robust.

In Chapter 5, Section 5.1, we use the restricted Conley index to show robustness of heteroclinic cycles by applying it separately to each heteroclinic connection within the heteroclinic cycle. In Section 5.2, we discuss the well-known Guckenheimer-Holmes and Kirk-Silber robust heteroclinic networks. For the Guckenheimer-Holmes cycle we exhibit a gradient-like vector field, but for heteroclinic network of Kirk-Silber we cannot find such a gradient-like field on the invariant subspaces.

On the other hand, results introduced in chapter 5 clarify the contradiction between statements of two theorems about the relation between the wedge sum of the Conley indices of two hyperbolic equilibria and the Conley index of the maximal invariant set containing them. We clarified this by showing that, for every vector field with two hyperbolic equilibria, assuming the isolated non-empty invariant set of connections between two such equilibria allows us to define a gradient-like vector field by restricting the vector field to the intersection invariant subspace

Chapter 6, Section 6.2, discusses our other question about designing a system of globally coupled oscillators under an assumed behavior as heteroclinic cycles which appear to be a solution of different dynamical systems including globally coupled oscillators systems. We assumed the existence of a stable $(2, 2, 2)-$cluster state in a system with six all-to-all coupled oscillators and set the conditions for its existence and stability. In order to design this system with an assumed dynamics, seven equations (including two nonlinear equations of tangent eigenvalues) must be solve but we found that it is not easy to do so analytically, even when we expand the coupling function using Fourier series (in section 6.3) where the number of unknowns was greater than the number of equations. But numerically, we observed that, for some initial condition we can find the desired coupling function by observing time series of the coupled six oscillators until the occurrence of $(2, 2, 2)-$cluster state and pick up their associated Fourier coefficients. However, conversely by using another approach, we showed the existence of such cluster state using values of Fourier coefficients which we obtained earlier. Also, in Section 6.2 we conclude general stability results for a non-trivial $3-$cluster state. If all the first derivatives of the coupling function are negative then all three clusters are tangentially stable; we also noted that, by decomposing transverse eigenvalues and assuming some inequalities, we made implication results between tangential and transverse stability. The advantage of these conditions is that one can avoid solving system consisting of existence and stability (tangent and transverse) equations to design the desired coupling function, which is almost analytically impossible, and one can instead assume these conditions on the chosen coupling function.

In Chapter 7 we connect two themes by showing a robustness result for heteroclinic cycles of saddle $3-$cluster state using the restricted Conley index. Three saddle $(2,2,2)-$cluster states related via S_6 symmetry, in particular S_2^3 symmetry, can form a heteroclinic cycle where these cycles are formed by the intersection between the unstable and stable manifolds of any of these clusters such that the connection lies within one of the invariant subspaces shown in Table 6.1; details of such dynamics are presented in Section 7.2 of Chapter 7. In Theorem 7.1 we show robustness of heteroclinic cycles of $(2,2,2)-$cluster state using our topological definition of the restricted Conley index. We follow this theorem with a discussion of the exist set components in each of such $3-$cluster state and find their restricted Conley index. We investigate the robustness equality (5.3) in Theorem 5.1 which we proved in Chapter 5. Dynamics of other heteroclinic cycles of $3-$cluster states with at least one trivial cluster are briefly discussed in the same section.

8.2 Further work

To prove robustness of the restricted Conley index in Theorem 3.13 we use the continuation property of the Conley index for an isolated invariant set which is investigated for a perturbation of flow with respect to compactly parametrized space. Demonstrating stability to more general C^2-perturbations of the vector field rather than an open set of perturbations of a compactly parametrized system is still an open issue. However, stronger and more flexible results could be achieved when the vector field has a continuous second derivative and also develop more results for the dynamics of the system using the Conley index.

We showed the existence of a locally gradient-like vector field for systems under some constraints in Theorem 3.14, which enabled us to use the result for existence of heteroclinic orbit (at least one) that connects two hyperbolic equilibria. However, not all systems have a global gradient-like vector field (e.g., gradient-like cannot define for smooth vector fields with closed orbits). Hence, an investigation into defining a local gradient-like vector for smooth vector fields might make it possible to use the topology of invariant subspaces to show the existence of heteroclinic orbits. Or, another possible question for study is whether the existence of heteroclinic connections between two hyperbolic equilibria can be proved for a gradient vector field rather than a gradient-like field. (Other conditions on the vector field may need to be investigated).

However, structural stability for the restricted Conley index rather than parametrized robustness may make the restricted Conley index applicable in a wider context. Also, there are a number of possible extensions where the restricted Conley index may be useful, for example, for heteroclinic cycles between chaotic saddles (cycling chaos [41]), especially in the non-hyperbolic case.

Although we provide an example where we prove robustness of heteroclinic connections in the case of nontransverse intersection in Example 3.2, we also proved an implication between a particular case of transverse intersection in corollary 5.2 and our results of robust heteroclinic cycle in Theorem 5.1 using the restricted Conley index. However, it would be useful to prove an implication between the more general case of the dimension condition (transverse intersection) introduced in [12, Theorem 1]

$$\dim(W^u(x_i) \cap I_{c(i)}) + \dim(W^s(x_{i+1}) \cap I_{c(i)}) > \dim(I_{c(i)}) \tag{8.1}$$

and the condition of the restricted Conley index

$$C_{I_{c(i)}}(\overline{S}_i) \neq C_{I_{c(i)}}(x_i) \vee C_{I_{c(i)}}(x_{i+1}) \tag{8.2}$$

for robustness of connections.

The restricted Conley index makes a contribution to existence and robustness results in two ways, even in the case of transverse intersection one can use the restricted Conley index to support transverse results. Further research is needed to fully understand the relation between these two approaches. The other way appears in the non-transverse intersection which is still need more general analytic study.

Finally, there are many unsolved problems related to the design of coupled oscillators systems with prescribed behaviors (in particular cluster states). Until now the problem of designing the coupling function associated with large number (greater than 2) of stable non-trivial cluster states was solved numerically by following the time series of the coupled oscillators until the occurrence of the required stable cluster state and then defining the coupling function (using the Fourier expansion form of the coupling function). However, we try to analyze the state of designing the coupled function where the number of clusters is more than or equal to 3 using the set of existence and eigenvalues equations and we found that it is not easily solvable. If we suppose the existence of stable $k-$cluster state ($k \geq 3$), further studies are needed, especially in terms of the tangent eigenvalues equations because the complication of define the coupling function comes from the non-linearity of the tangent eigenvalues. For the clustering behavior discussed in Chapter 6, the stability conditions for non-trivial $3-$cluster state in the system of globally coupled oscillators can apply to systems of any number of coupled oscillators, regardless of the size of each cluster. Other dynamical behaviour we have observed by following the time series for the coupled oscillators is that different cluster states with different clusters size are possible using the same coupling function, so it would be useful if one can design the coupling function that restricts the dynamics of the system to have a certain $m-$cluster states. Also, one more question to answer is whether characterized stability of heteroclinic cycles is an attractor or not, many researches have investigated conditions for attraction (see [77]). So one can use the existence results to examine the attraction of heteroclinic cycles of $3-$cluster states introduced in Chapter 7.

Appendix A

Continuity of the Conley index

In Chapter 3, we use the property that, under small perturbations of the flow, the Conley index for an isolated invariant set is continuous (invariance). Here we briefly review basic related notions that we need to clarify what parametrized flows means and then we restate the continuation theorem for the Conley index of the isolated invariant sets; our reference to the following results is [117].

Define the folowing notations: Λ: the parameter space which is a locally contractible, compact, connected metric space. X: a locally compact metric space. ϕ: a flow and $X \times \Lambda \subset \phi$ is a local flow. $\pi_X : X \times \Lambda \to X$ and $\pi_\Lambda : X \times \Lambda \to \Lambda$ are the canonical projection maps.

Lemma A.1 *[117] Let $N \subset X$ be a compact set, then the set of parameters $\lambda \in \Lambda$ which makes $N \times \lambda$ an isolating neighbourhood in $X \times \lambda$ is open in Λ, denote this set as $\Lambda(N)$.*

Definition A.1 *[117] For a compact set $S \subset X$ and $\lambda \in \Lambda$ such that $S \times \lambda$ is an isolated invariant set in $X \times \lambda$, the set of* isolated invariant sets *in $X \times \lambda$ is defined by:*

$$\mathcal{N} = \{S \times \lambda : S \times \lambda \;\; is \;\; an\ isolated \;\; invariant \;\; set\ in\ X \times \Lambda\}.$$

Also, we can define the map $\Psi_N : \Lambda(N) \to \mathcal{N}$ which takes each parameter $\lambda \in \Lambda(N)$

onto its associated isolated invariant set $S(N \times \lambda)$. The topology on $\mathcal{N}$ is generated by the sets $\{\Psi_N(U) : N \subset X \textit{ is compact}, \quad U \subset \Lambda(N) \textit{ is open}\}$.

The following are results from [117, Lemma 6.2]:

Corollary A.2 *1. The projection map $\pi_\Lambda : \mathcal{N} \to \Lambda$ is a local homeomorphism.*

2. The inverse of the restriction of π_Λ to the neighbourhood of $S \times \lambda$ in $\mathcal{N}$ is the map $\sigma_N : \Lambda(N) \to \mathcal{N}$.

For any compact sets $N \subset X \times \Lambda$ and $W \subset \Lambda$, define:

$$N(W) = N \cap X \times W.$$

However, if S is an isolated invariant set in $X \times \Lambda$, then $S(W)$ denotes an isolated invariant set in $X \times W$

Remark A.1 *Let S be an isolated invariant set in $X \times \Lambda$ and $W \subset \Lambda$ be a compact set. Then for any index pair (N, N^e) for S in $X \times \Lambda$ and any such W, the sets $(N(W), N^e(W))$ form an index pair for the isolated invariant set $S(W)$ in $X \times W$.*

Proposition A.3 *Let S be an isolated invariant set in $X \times \Lambda$ with an index pair (N, N^e), then there is a morphism defined by the canonical injection map $j(\lambda) : N(\lambda)/N^e(\lambda) \to N/N^e$ between the corresponding connected simple systems, independent of the choice of the index pair.*

The following theorem is the main result concerning the continuation for the Conley index, more details in [117, Theorem6.7].

Theorem A.4 *[117] Let $\lambda_0 \in \Lambda$ and W be a compact, contractible neighbourhood of λ_0 in Λ that satisfies the set of conditions in [117, Lemma6.6]. Then for an index pair (N, N^e) of S in $X \times \Lambda$, the injection map $j_W(\lambda) : N(\lambda)/N^e(\lambda) \to N(W)/N^e(W)$ is a local isomorphism (i.e., the injection map is homotopy equivalent for every $\lambda \in W$ and has a homotopy inverse).*

Lemma A.5 *Let $N^f \subset X$ be a compact subset for the flow f and $O^f \subset \Lambda(N^f)$. If g is a flow with $|f - g| < \epsilon$ for some $\epsilon > 0$, then*

1. *$\Lambda(N^f, N^g) = \{\lambda \in \Lambda(N^f) \cap \Lambda(N^g) : S(N^f \times \lambda) = S(N^g \times \lambda\}$ is open set in Λ;*

2. *$\Psi_{N^f}(O^f) \cap \Psi_{N^g}(O^g) = \Psi_{N^f}(O^f \cap O^g \cap \Lambda(N^f, N^g))$;*

3. *$\Psi_{N^f} : \Lambda(N^f) \to \mathcal{N}$ is continuous.*

Corollary A.6 (of Theorem ??) *For any system parametrized by Λ and N such that N is an isolating block for $\lambda = 0$, there is an open neighbourhood of 0 in Λ such that the Conley index of $S(N \times \lambda)$ is independent of λ.*

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