Essays in Financial Economics

Submitted by Giuseppe Bova to the University of Exeter
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DEDICATED TO:

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Abstract

We present in this thesis three distinct models in Financial Economics. In the first chapter we present a pure exchange economy model with collateral constraints in the spirit of Kiyotaki and Moore (1997). As a first result in this chapter we prove the existence of an equilibrium for this type of economies. We show that in this type of models bubbles can exist and provide a bubble example in which the asset containing the bubble pays positive dividends. We also show for the case of high interest rates the equivalence between this type of models and the Arrow-Debreu market structure.

In the second chapter we present a model with limited commitment and one-side exclusion from financial markets in case of default. For this type of models we prove a no-trade theorem in the spirit of Bulow and Rogoff (1989). This is done for an economy with and without bounded investment in a productive activity.

The third chapter presents a 2 period economy with complete markets, and 250 states of the world and assets. For this economies we generate a sequence of observed returns, and we show that a market proxy containing only 80% of the assets in the economy provides similar results as the true market portfolio when estimating the CAPM. We also show that for the examples we present a vast amount of observations is required in order to reject the CAPM. This raises the question what the driving force behind the bad empirical performance of the CAPM is.
This thesis presents three essays in Financial Economics which build upon General Equilibrium theory. In this work we address the topics of collateral constraints, bubbles, default with one-sided exclusion and the testability of the Capital Asset Pricing Model (CAPM). All three works are essentially deviations from the workhorse model in General Equilibrium theory; the Arrow-Debreu model. To highlight the connection between the models, we will proceed by outlining the major developments that led to the Arrow-Debreu model and its most relevant extensions.

General Equilibrium theory seeks to address economic questions through rigorous application of mathematics. Its main objective is to unite decision making of the agents with the coordination of markets. As such it has become a powerful tool for analysis in Financial Economics.

The field of General Equilibrium theory was pioneered by Léon Walras, see for example Walras (1926). The main idea of a General Equilibrium is that in an economy prices influence agent choices and, given that resources are in scarce supply, the aggregate of agent choices influences prices. Depending on the fundamentals of the economy, e.g. consumer preferences, production technology, endowments, et cetera, solving for the equilibrium allocation and the equilibrium price vector can be very difficult and, if certain conditions are not fulfilled, it is not guaranteed that a solution exists. In its modern form, General Equilibrium theory developed around the works of Arrow and Debreu (1954) and McKenzie (1954). These works provided equilibrium existence theorems based on Kakutani’s fixed point theorem and distinguished themselves for leading the transition from Calculus based methods to Topology based methods. A broad exposition of what came to be know as the Arrow-Debreu, or Arrow-Debreu-McKenzie model, can be found in Debreu (1959).

The Arrow-Debreu model is a model with a finite amount of time periods, a finite amount of consumers or households and a finite amount of firms producing a finite amount of goods. Households observe prices and state their demand for goods.
They also own shares in firms such that profits are redistributed back to households. Firms observe prices and state their demand for “intermediate” goods and supply of “final” goods in order to maximise their profits. If markets clear, i.e. if for each good the supply of goods is equal to the demand for goods, the price vector that clears the markets together with the associated demand and supply for goods is called an equilibrium. In this model labour is treated as a good. At a positive wage, which is the price for the good labour, a household may have a negative demand for labour. This means the household is supplying labour to firms. Goods are not in general classified as intermediate and final goods, but depending on the technology available to a firm a certain good is an intermediate good or a final good for the firm. The model has essentially an implementation of a stock market, since agents can trade the shares of firms among them.

In brief, an allocation with an associated price vector is a General Equilibrium if at given prices the following conditions hold:

- For any household there does not exist a bundle of goods which is affordable and preferred, i.e. agents are choosing the optimal consumption bundle among the feasible choices.
- For any firm there does not exist a production plan which is technological feasible and yields higher profits, i.e. firms are profit maximising.
- There does not exist a good for which the demand exceeds the supply or the supply exceeds the demand, i.e. markets clear.

One of the most useful abstractions in Arrow and Debreu (1954) is that the same physical good at a different location or at a different time is treated in the model as a different good. This notion can be extended to probability spaces through the definition of Arrow Securities. Arrow Securities have been introduced by Arrow (1952). They consist in assets that pay one unit of the numéraire good in exactly one state of the world and zero in all other states. If for each state of the world one of these Arrow Securities exists, the market is called complete and the equilibrium existence proof in Arrow and Debreu (1954) can easily be extended to capture uncertainty in this economy, as shown in Debreu (1959). For the case where markets are incomplete, Radner (1972) proved that under certain conditions an equilibrium can still exist. Radner’s work highlights also the difference between an Arrow-Debreu market structure, where trade takes place only in the first period for all successive periods, and sequential markets, where trade takes places from period to period in a sequential fashion.

A second major extension of the standard model is due to Peleg and Yaari (1970), and Bewley (1972). The paper of Peleg and Yaari extends the standard model for an exchange economy to countable infinite many commodities or time periods. Bewley
accomplishes the same task for an economy with production. Bewley’s paper is especially relevant for chapter 1 of this thesis, where we use the same principle of proving the existence of an equilibrium by truncating the economy and considering the limit of equilibria for the truncated economies.

The first chapter presents an exchange economy with collateral requirements in the spirit of Kiyotaki and Moore (1997). When considering infinite lived agents, who have the opportunity to save or borrow, a Transversality Condition has to be imposed that forbids agents to choose trading sequence in which they continue rolling over their debts and effectively never repay it. Given that it is not always possible to give an economic meaningful interpretation of those Transversality Conditions, borrowing constraints can be used to implement a Transversality Condition. For this task, collateral constraints offer very intuitive borrowing limits that a very common in contracts, as for example mortgages. We present in this chapter an equilibrium existence result that extends the current literature on collateral constraints and provides the foundations for models like Kiyotaki and Moore (1997). Given that borrowing constraints can give rise to bubbles, we show that bubbles in this model can exist and provide a bubble example in the spirit of Kocherlakota (1992), and Huang and Werner (2000). A particularity of this bubble example is that, the asset containing the bubble is not fiat money but an asset paying positive dividends. For this model we show also an equivalence between our framework and an Arrow-Debreu market structure. This equivalence holds for the case of high interest as defined by Alvarez and Jermann (2000). We provide a no bubble theorem that rules out bubbles under high interest. This implies that our bubble example exist in sequential markets but not in an Arrow-Debreu market structure.

The second chapter presents a model with limited commitment and one-sided exclusion in case of default. In this chapter agents can invest in a productive activity that yields a return in units of the consumption good. Agents have the same productivity and we consider two version of the model. In the first version we assume that agents’ investment is only limited by their budget constraint, in the second version we assume that there exists an exogenous bound on investments, which we normalise to unity. We allow agents to borrow from each other but we assume that they cannot commit to their repayment promises and have the option to default on their debts, i.e. we have limited commitment in the model. The punishment for defaulting consists in a one-sided exclusion from financial markets, i.e. after defaulting an agent can save but not borrow. We assume that agents can observe the net debt position of the other agents in the economy and that they will not lend to an other agent if her/his next period endowment is not sufficient to cover all outstanding debt. In this framework we show that agents have no incentive to repay their debts; and thus the only equilibrium without default is a no-trade equilibrium in the spirit of Bulow and Rogoff (1989). By constructing an appropriate example, we show that
this result can have welfare implication in the economy with bounded investment and the resulting outcome is not Pareto efficient. This result is consequential in the study of financial markets and its welfare implications. We believe that this result could also be an important reference point in development economics, especially where market structures that regulate default are missing.

In the third chapter we use the notion of Arrow Securities to solve a two period economy with 250 assets and 250 states of the world. We do this for two versions of the economy, one in which agents have quadratic utility functions and the other in which they have logarithmic utility functions. It is known that under quadratic utility the CAPM holds. In contrast to, it does not hold in general under logarithmic utility. For each economy we construct a random sequence of observed returns and compare the performance of an OLS estimation of the CAPM based on the true market portfolio against an estimation based on a market proxy containing only 80% of the assets in the economy. We find that for both economies the estimations based on the market proxy are close to the estimations based on the market portfolio. As a consequence of this, it is very likely that using a market proxy does not distort the CAPM tests. On the other hand, we find also that for the examples we present, the deviations from the predictions of the CAPM are very small when we use logarithmic utility. In this case we get only a significant rejection of the CAPM after increasing the sequence of observed returns to an amount that is hardly feasible in tests based on field data. This simulation is designed to test the implication of Roll’s critique (see Roll, 1977) on the estimation of the CAPM. Our model is a completely artificial economy in the sense that it contains no noise except for rounding errors in the software package we are using. As such, our results raise the question whether a test of the CAPM is truly feasible with real world data, given that it requires so many observations in a well behaved data set as ours. Both of our findings are connected to each other. On the one hand the errors produced by the market proxy are negligible, on the other hand the true deviations from the CAPM are also small, such that an extensive amount of observations is required to distinguish between an economy where the CAPM holds and one where the CAPM does not hold.

We can show the connection of the three models presented by departing from the Arrow-Debreu model. The model of the first chapter, is a pure exchange economy. That is, compared to the Arrow-Debreu model we assume that the consumption good is not produced by firms, but each household has its individual sequence of endowments, which specifies how many units of the consumption good the agent is endowed with at the beginning of each period. In the second chapter the future endowment depends on the past investment. One can think of the endowments being produced by trees. Depending on the economy we are considering, agents can only plant a finite amount of these trees. Trees cannot be traded among the agents, but as in the first chapter agents exchange promises on future delivery of
the consumption good.

The models presented in the first and second chapter are closely related to each other. The key differences are the exogenous endowment stream, which the agents face at the beginning of the first model, and the punishment in case of default. The first model is an exchange economy in which agents smooth their consumption depending on the endowment stream they face. It is possible to observe a high amount of trade if the agents face very variable endowment streams, especially if there exists a group of agents that faces a relative high endowment in periods where another group faces a relative low endowment. The main role of the collateral requirement in the first model is to discourage agents to default. So, in the first model there will never exist an equilibrium in which an agent has an incentive to default. In the Arrow-Debreu market structure this implies that all equilibria are self-fulfilling.

In the second model, the only difference between agents is their initial wealth. This implies that at the beginning of the period, there exist a group of agents which is relative wealthy and one group which is relative poor. In principle, one would expect that, given the decreasing marginal utility, the relative rich agent will lend to the relative poor agent. This does not occur because the punishment considered is too weak and the borrower is always better-off by defaulting at some period. In an Arrow-Debreu market structure this implies that the only equilibria that are not self-fulfilling are no-trade equilibria.

The main questions that arises from a comparison of the two models are; whether we would observe trade by introducing collateral constraints with a punishment consisting in a seizure of the collateral in the second model, and whether we could categorically exclude default in the first model, when considering a one-sided exclusion as punishment instead of a seizure of the collateral.

Intuitively speaking, we can see from the proof of the no-trade equilibrium in the second chapter, that trade can arise when considering the different punishment, i.e. when agents lose their collateral in case of default. The intuition for the no-trade result is that at some point in time the present value of the debt will start to decrease, since the agent can not extend its debt further. Thus, the agent is effectively starting to repay its debt. The collateral constraint would imply that the present value of the collateral is equivalent to the present value of the outstanding debt. From this would follow that in terms of present value the benefit of defaulting is entirely cancelled out by the punishment which would go to the creditor.

In the case of the first model, one has to consider that when the collateral constraint is imposed, there exists a long-lived asset that through a non-arbitrage argument can be shown to be equivalent to the short-lived asset. By changing the punishment, this equivalence would not hold any longer and the question would arise, whether the short-lived asset will be traded at a premium because of the risk
of default. Intuitively speaking, one can exclude that the two assets are equivalent because if the return would be the same, then an agent might consider defaulting on the promises involved in the short-lived asset and invest the amount on which she/he defaulted in the long-lived asset. So, to answer this question, one would have to look at the new choice sets of the agents, which in this case, might be non-convex. This non-convexity can add further difficulties to the equilibrium existence proof. In this sense the model sacrifices a realistic feature of economies for to achieve greater simplicity.

In the third chapter we consider a two period economy in which agents are endowed with an initial amount of the consumption good that can be used to purchase shares in trees. The endowment of the consumption good in the second period is again produced by trees. We also assume that there a different type of trees and states of the world. The payoff of each tree depends on the state of the world that occurs, such that each tree has its individual payoff vector. The payoff vectors of the trees are all linearly independent and the market is complete. This model is essentially the classical Arrow-Debreu model, with two differences. The first difference is that goods are not produced by firms, but by trees, whose output is random. This implies that the firms profit maximisation problem is replaced by a distribution of aggregate output. The second is that there is no labour in the economy, so households have simply to choose their investment strategies.
1.1 Introduction

When considering infinite horizon models, a Transversality Condition needs to be imposed to ensure the optimality of the consumers’ optimisation problem. This condition is analogous to the terminal condition in finite horizon models. When considering a finite horizon model, agents are usually required to save or borrow nothing in the last period. This is because it cannot be optimal for an agent to keep savings at the end of the last period, since she/he cannot consume them in the successive periods. Similarly, an agent cannot accumulate debts at the end of the last period, since there is no possibility to repay them. In such a case the problem would be misspecified, given that the agent would essentially receive a gift. In infinite horizon models there exists no terminal period, and one needs to impose a condition that regulates the limit of the trading sequence. As in the finite horizon model, it cannot be an optimal solution if the limit of the present value of savings is positive, and it cannot be a valid solution if the limit of the present value is negative, i.e. the agent effectively does not pay its debts. Especially the later case is problematic in infinite horizon models, since the agent is running a Ponzi scheme, which implies that the agent is repaying old debt obligations by issuing new debt, such that effectively she/he is never repaying the debt. Thus, to rule out Ponzi schemes and guarantee the optimality of the solution of the consumers’ optimisation problem, the Transversality Condition must hold; it is a necessary condition. Strictly speaking, it would not be enough to impose a No Ponzi Game Condition since this would not rule out that an agent saves too much, in the sense that she/he could increase the utility by saving less. The problem of over-saving was addressed, by Malinvaud (1953, 1962). In his paper the author extends the notion
of Pareto efficiency, by demonstrating, that in an efficient allocation for a infinite horizon problem with capital accumulation, the present value of future consumption and future capital tend to zero as time tends to infinity (Malinvaud, 1953, p. 251). As pointed out by Benveniste (1976), the proof presented by Malinvaud actually shows that only the limit inferior is required to converge to zero for the program to be efficient.

Usually, efficiency questions in General Equilibrium models are addressed by considering an optimisation problem of a social planner. In our model we have difficulties in addressing efficiency questions because the borrowing constraint is endogenous and depends on prices, while the social planner does not consider prices. This implies that we have difficulties in formulating the correct optimisation problem for a social planner and cannot address efficiency questions in the classical way.

To rule out Ponzi schemes, borrowing constraints are frequently used in the literature. Broadly speaking, the literature distinguishes between *implicit* borrowing constraints and *explicit* borrowing constraints. Implicit borrowing constraints limit the rate at which debt can grow. Essentially, they are a direct way of imposing a no-Ponzi game condition by ruling out all borrowing sequences that are not conform with this condition. Explicit borrowing constraints impose a direct bound on the debt values for each period. This bounds will rule out Ponzi schemes, since they impose an upper bound on debts, whose present value converges to zero. The first approach has the advantage of being more general and allowing for a direct comparison between the Arrow-Debreu market structure and sequential markets. The second is more intuitive and easier to interpret economically.

There exists a vast literature which focuses on Transversality Conditions in infinite horizon models. A recent exposition can be found in Martins-da-Rocha and Vailakis (2012). A main issue highlighted in this literature is whether a common evaluation for debt exists, as it is the case in complete markets. When a common evaluation exists, one can impose a solvency requirement that implies that the present value of the debt can never exceed the present value of the future endowment or income.

In this model there exists a common evaluation of debt, but because we allow for the possibility of default we impose a more stringent borrowing constraint that rules out default at equilibrium. We do this by imposing a collateral constraint in the spirit of Kiyotaki and Moore (1997). This offers an intuitive way of eliminating strategic default and allows us to study the use of collateral as an instrument for risk sharing. The provision of collateral is one of the most important means of securing loans and, if one considers the mortgage market it is one of the most widespread. In the wake of the financial crisis of 2008 collateral backed loans have been in the focus of the public attention and are perceived as one of the major causes of the crises. In spite of this negative attention, it has to be mentioned that collateral as a
mean of securing loan has two major benefits. First, the use of collateral decreases incentives of strategic default. Second, it allows for more risk sharing among agents. These benefits become obvious if one considers that especially for young individuals it becomes easier to acquire a property by means of secured loans. The gap between the public’s perception and the obvious benefits of collateral backed assets highlights the need of research in this subject.

This chapter contributes to this field by studying a pure exchange, deterministic economy with infinitely lived agents and a single consumption good. Loans are protected by collateral constraints in the spirit of Kiyotaki and Moore (1997). For each loan, agents have to provide the equivalent value in units of a long-lived asset as collateral. In case of default, the lender has the right to seize the collateral as reparation. The implication of this is that there will be no default at equilibrium.

Our first contribution consist in proving the existence of an equilibrium for these type of economies. To do this, we use a similar argument as Bewley (1972). We truncate the economy and consider first a finite period economy. For this truncated economy we use the fixed point theorem provided by Gale and Mas-Colell (1975, 1979) to prove the existence of an equilibrium. By moving the truncation period towards infinity, we are able to show that the equilibrium in the truncated economy converges to an equilibrium for the infinite period economy. This equilibrium existence proof serves as a reference point for works that study models with collateral constraints and can be a point of departure to prove the existence of an equilibrium in Kiyotaki and Moore (1997).

As shown by Kocherlakota (1992), it is possible that in sequential markets with borrowing constraints bubbles can arise at equilibrium. Our second contribution consists in showing that in the model we propose, the collateral constraint can give rise to a bubble. We also present a bubble example in the spirit of Kocherlakota (1992) and Huang and Werner (2000).

Bubbles are not a novelty in General Equilibrium models. Tirole (1985) shows that in Overlapping Generations models bubbles occur when the resources in the economy grow fast enough to enable agents to trade the bubble. This is also a crucial condition in our model and, similarly to Huang and Werner (2000), we are able to exclude bubbles when the limit of the present value of the aggregate endowment is finite. A key assumption in Tirol’s model is that savings are nonproductive, in the sense that savings do not increase a production factor like capital and as a consequence, do not influence future aggregate output directly. In the model we propose, future endowment is exogenously given and thus the saving decisions have no effect on the amount of the consumption good available in each period. The importance of this assumption is highlighted in Becker et al. (2012), where the authors show that bubbles cannot occur in a Ramsey economy with borrowing constraints similar to ours. Given that the model proposed by Becker et al. (2012)
is a growth model, the amount that the economy saves on aggregate determines the
return on savings and the amount of available consumption in each period. The
proof of Theorem 7 in Becker et al. (2012) shows that this excludes bubbles.

The key difference between our findings and Tirol’s findings lies in the interpre-
tation of the bubble. Tirol shows that bubbles can help restore the Golden Rule
level of savings and avoid over accumulation of capital. The intuition for this result
is that in an Overlapping Generations model the young generations acquire the asset
containing the bubble, foreseeing that they can resell it to the next generation when
they are old. This provides a further opportunity to smooth consumption. In the
case where the asset containing the bubble pays no dividends, it can be called fiat
money. The money in this economy is then not needed for conducting transactions
but for storing value. This intuition does not hold in our model. In the example
that we provide, the bubble occurs because the borrowing constraints does not per-
mit the constrained agents to short-sell the asset containing the bubble. From this,
it follows that the driving force behind our bubble example is the borrowing con-
straint. It is difficult to establish whether this has particular welfare implications
as in Tirol’s paper, since in our model we would have to consider that a possible
relaxation of the borrowing constraint could lead to default.

As third contribution, we establish an equivalence results between a sequential
market equilibrium and an Arrow-Debreu market equilibrium for the case of high
interest rates.\(^1\) We can show that under high interest rates no bubble can exist.
Thus the bubble example we construct applies only to sequential market structures.

The structure of the paper is the following. In section 1.2 we will present the
model an relate our notation to similar models in the literature. Section 1.3 dis-
cusses properties of the equilibrium and the bubble example. Section 1.4 derives
the equivalence between the sequential market equilibrium and the Arrow-Debreu
market equilibrium. Section 1.5 presents the equilibrium existence proof. The final
section contains concluding remarks and suggestions for further research.

1.2 The model

1.2.1 Agents and commodities

There is a single non-durable consumption good available for trade at every period
t \( t \in T = \{0, 1, \ldots\} \). We let \( p = (p_t)_{t \in T} \) denote the sequence of spot prices where
\( p_t \in \mathbb{R}_+ \) denotes the price of the consumption good at period \( t \).

There is a finite set \( I \) of infinitely lived agents. Each agent \( i \in I \) is characterized
by an endowment sequence \( \omega^i = (\omega^i_t)_{t \in T} \) where \( \omega^i_t \in \mathbb{R}_+ \) denotes agent’s \( i \) endowment
available at period \( t \). Each agent chooses a consumption sequence \( c^i = (c^i_t)_{t \in T} \) where

\(^1\)The terminology follows Alvarez and Jermann (2000).
\( c_i \in \mathbb{R}_+ \). The utility function \( U^i : \mathbb{R}_+^\infty \rightarrow [0, +\infty] \) is assumed to be time-additively separable, i.e.,

\[
U^i(c^i) = \sum_{t \in T} \beta_i^t u_i(c^i_t),
\]

(1.2.1)

where \( u_i : \mathbb{R}_+ \rightarrow [0, +\infty) \) represents the instantaneous utility function at period \( t \) and \( \beta_i \in (0, 1) \) is the discount factor. Given a consumption sequence \( c^i = (c^i_t)_{t \geq 0} \) the continuation utility \( U^i_t(c) \) beginning at period \( t \) is defined as follows

\[
U^i_t(c^i) \equiv \sum_{s \geq t} \beta_i^{s-t} u_i(c^i_s).
\]

(1.2.2)

### 1.2.2 Financial markets

There is a short-lived asset available for trade whose return amounts to \( p_t \) units of the consumption good at each period \( t \). We let \( q = (q_t)_{t \in T} \) denote the asset price sequence where \( q_t \in \mathbb{R}_+ \) represents the asset price at period \( t \). For every agent \( i \), we denote by \( z^i_t \in \mathbb{R} \) her/his net financial position on the short-lived asset at period \( t \). Each agent \( i \) starts with a zero position on the asset, i.e., \( z^i_{t-1} = 0 \).

There is in addition a long-lived asset that pays its owner a dividend \( \xi_t \in \mathbb{R}_+ \). The physical asset is in positive net supply and its stock is normalized to unity. For each agent \( i \), we denote by \( x^i_t \in \mathbb{R}_+ \) the amount of claims held by the agent at period \( t \). Initial endowments are positive and sum up to 1, i.e., \( x^i_{t-1} > 0 \) and \( \Sigma_{i \in I} x^i_{t-1} = 1 \). We let \( r = (r_t)_{t \in T} \) denote the long-lived asset price sequence where \( r_t \in \mathbb{R}_+ \) represents the price at period \( t \).

The amount of the short-lived asset an agent can short-sell is observable and subject to an upper bound. The bound is endogenous and follows from the institutional environment in this economy. We assume that commitment is limited. Agents cannot be forced to repay their debts. If agent \( i \) has being short on the short-lived asset at period \( t \), that is, if her/his net financial position is \( z^i_t < 0 \), she/he should deliver the promise \(-p_{t+1}z^i_t \) at period \( t + 1 \). However, agent \( i \) may decide to default and choose to deliver a quantity \( d^i_{t+1} \) in units of the consumption good which is strictly less that her/his obligation \(-p_{t+1}z^i_t \). In this case, an institution can seize agent’s \( i \) long-lived asset holdings plus the dividend and transfer their ownership to creditors. Lenders keep track of borrowers’ asset positions and the provision of credit never exceeds the value \([r_{t+1} + p_{t+1}\xi_{t+1}]x^i_t \). In other words, agents collateralize their short-lived asset position and go short up to the value of their long-lived asset holdings. In this setting, each agent \( i \) faces the following constraints

\[
\forall t \in T, \quad [r_{t+1} + p_{t+1}\xi_{t+1}]x^i_t \geq -p_{t+1}z^i_t.
\]

Given this constraint, no agent has an incentive to default at equilibrium. This is in line with Kiyotaki and Moore (1997), where default occurs only after the shock.
From this point of view our model misses on a major stylized fact, which is the occurrence of default in real economies. The presence of uncertainty is fundamental to generate default at equilibrium. Kubler and Schmedders (2003) use a similar collateral constraint to test welfare implications of default. In their model there exists an event tree such that default may occur at some nodes of the tree, i.e. after a particular history. The existence of default in their model relies on the fact that at any successive node there exists at least one event where the agent repays its debt. Otherwise agents would anticipate that a particular agent would in any case default and would not issue the loan. Our model can be understood as an event tree with only one ramification, i.e. only one history can occur. This implies that if there would exits a period in which an agent would default, then by individual rationality no other agent would issue a loan to her/him. Thus, our decision to eliminate default at equilibrium is dictated by the fact that we are using a non stochastic model.

1.2.3 Budget constraints

Let \( A \) be the space of sequences \( a = (a_t)_{t \in T} \) with

\[
a_t = (c_t, z_t, x_t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+.
\]

Agent \( i \)'s choice \( a^i = (c^i, z^i, x^i) \in A \) must satisfy the following constraints:

- budget constraints:
  \[
  \forall t \in T, \quad p_t c^i_t + q_t z^i_t + r_t x^i_t \leq p_t \omega^i_t + [r_t + p_t \xi_t] x^i_{t-1} + p_t z^i_{t-1}; \quad (1.2.3)
  \]

- borrowing constraints:
  \[
  \forall t \in T, \quad [r_{t+1} + p_{t+1} \xi_{t+1}] x^i_t \geq -p_{t+1} z^i_t. \quad (1.2.4)
  \]

- no short-selling of the long-lived asset:
  \[
  \forall t \in T, \quad x^i_t \geq 0. \quad (1.2.5)
  \]

1.2.4 The equilibrium concept

We denote by \( \Pi \) the set of sequences of prices \((p, q, r)\) satisfying

\[
\forall t \in T, \quad (p_t, q_t, r_t) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+. \quad (1.2.6)
\]

Given a sequence of prices \((p, q, r) \in \Pi\) we denote by \( B^i(p, q, r) \) the set of plans \( a = (c, z, x) \in A \) satisfying constraints (1.2.3), (1.2.4) and (1.2.5).

\[\text{By convention we let } a_{-1} = (c_{-1}, z_{-1}, x_{-1}) \text{ with } c_{-1} = 0 \text{ and } z_{-1} = 0.\]
**Definition 1.2.1.** A *competitive equilibrium* for the economy

\[ \mathcal{E} = (u_i, \beta_i, \omega_i, x^i_{-1}, z^i_{-1})_{i \in I} \]

is a family of prices \((p, q, r)\) \(\in \Pi\) and an allocation \(a = (a^i)_{i \in I}\) with \(a^i \in A\) such that

(a) for every agent \(i\), the plan \(a^i\) is optimal among the budget feasible plans, i.e., \(a^i\) maximizes (1.2.1) and \(a^i \in B^i(p, q, r)\).

(b) commodity markets clear at every period, i.e.,

\[ \forall t \in T, \quad \sum_{i \in I} c^i_t = \sum_{i \in I} \omega^i_t + \xi_t; \quad (1.2.7) \]

(c) the asset markets clear at every period, i.e.,

\[ \forall t \in T, \quad \sum_{i \in I} z^i_t = 0; \quad (1.2.8) \]

and

\[ \forall t \in T, \quad \sum_{i \in I} x^i_t = 1. \quad (1.2.9) \]

The set of allocations \(a = (a^i)_{i \in I}\) in \(A\) satisfying the market clearing conditions (1.2.7), (1.2.8) and (1.2.9) is denoted by \(F\). Each allocation in \(F\) is called physically feasible. A plan \(a^i \in A\) is called physically feasible if there exists a physically feasible allocation \((b^i)_{i \in I} \in F\) such that \(a^i = b^i\).

**1.2.5 Assumptions**

It should be clear that these assumptions always hold throughout the paper.

(A.1) For each agent \(i \in I\), the sequence of endowments \(\omega^i\) belongs to the interior of \(\ell^+_{\infty}\).

(A.2) For every period \(t\), the utility function \(u_i\) is concave, continuous, \(C^1\), strictly increasing on \(\mathbb{R}_+\) with \(u_i(0) = 0\) and satisfies

\[ \lim_{c \to 0} u'_i(c) = +\infty. \]

(A.3) The dividend sequence \((\xi_t)_{t \in T}\) belongs to \(\ell^+_{\infty}\) and satisfies the following property

\[ \forall t \geq 0, \exists \tau > t \text{ such that } \xi_{\tau} > 0. \]
1.2.6 Relation to the literature

The model is closely related to the one analysed by Kiyotaki and Moore (1997). Kiyotaki and Moore abstract from the aforementioned general equilibrium structure. Their baseline set up can be summarized as follows:

a. There are two consumers: the farmer and the gatherer.

b. There are two commodities: one durable (land) and one perishable (fruits).

The numéraire good is the fruit and the price of land is denominated in units of the fruit. Land does not depreciate and has a fixed total supply. To facilitate the comparison between the two settings let $x_i^t$ denote the amount of land hold by consumer $i$ at period $t$ and let $c_i^t$ denote the quantity of fruits consumed by agent $i$ at period $t$. Let $p_t$ denote the price of fruits and $b_t$ denote the amount of fruits the farmer borrows at period $t$. The amount (in units of the fruit) the farmer should pay at period $t+1$ per fruit borrowed is endogenous and is denoted by $R_{t+1}$. Commitment is limited in this economy. By investing in land, the farmer can work and produce fruits to repay back her/his debt. However, the farmer may decide not to work. In that case, the lender (the gatherer) is not sure whether the farmer will pay back her/his debt. The gatherer accepts to lend fruits only if the farmer gives land as collateral.

To relate this setting with our set up one could consider that there is one short-lived asset, paying in units of the fruit, that the farmer can short-sell. Assume that there is a number $C_t > 0$ representing the units of land collateralized for one unit of the asset sold at period $t$. Let us further consider that the asset pays one unit of the fruit next period and denote by $q_t$ the unit price of the asset at period $t$ in units of the fruit. The price of the land in units of fruit is denoted by $r_t$. If the farmer sells short $z_t < 0$ units of the asset she/he has to put as collateral $-C_t z_t$ units of land. At time $t$, the farmer receives $-q_t z_t$ units of the fruit and she/he should deliver $-p_{t+1} z_t$ units of the fruit at period $t+1$.

Kiyotaki and Moore (1997) model the financial markets choosing the parametrization $(b_t, R_{t+1})$ where $R_{t+1}$ is the gross interest rate. The correspondence between the two settings is given by the equations

\[ b_t = -q_t z_t \quad \text{and} \quad R_{t+1} = p_{t+1}/q_t. \]  

(1.2.10)

In the parametrization of Kiyotaki and Moore (1997), the farmer should put as a collateral $C_t R_{t+1} b_t$ units of land when she/he asks for a loan $b_t$. Therefore, the farmer faces the following constraint

\[ C_t R_{t+1} b_t \leq x_t. \]  

(1.2.11)
If there is no other enforcement mechanism, the farmer will always deliver

$$\min\{R_{t+1}b_t, r_{t+1}C_tr_{t+1}b_t\} = R_{t+1}b_t \min\{1, r_{t+1}C_t\}. \quad (1.2.12)$$

A borrower defaults at period $t + 1$ if $r_{t+1}C_t < 1$ but repays fully her/his debt if $r_{t+1}C_t \geq 1$.

In Kiyotaki and Moore (1997), the gatherer allows the farmer to borrow $b_t$ units of fruit only if

$$R_{t+1}b_t \leq r_{t+1}x_t \quad (1.2.13)$$
i.e., the amount the farmer should repay cannot be strictly larger than the market value of the land she/he owns. That is, the collateral level is endogenously determined as follows:

$$C_t = \frac{1}{r_{t+1}}. \quad (1.2.14)$$

In other words, the collateral is chosen such that no borrower has incentives to default. In a sense the collateral level is chosen to be the less stringent (in terms of borrowing restrictions it imposes) among those levels that ensure no default.

Using equation (1.2.10), (1.2.13) implies that

$$r_{t+1}x_t \geq -p_{t+1}z_t. \quad (1.2.15)$$
Equation (1.2.15) is similar to equation (1.2.4). The difference rests on the fact that in Kiyotaki and Moore (1997) only the asset holdings (not the dividend) can be seized.\(^3\)

We can also relate our model to the one proposed by Kubler and Schmedders (2003).\(^4\) The crucial difference with respect to our setup rests on the specification of collateral levels. Using our notation, agents in Kubler and Schmedders (2003) face the following constraints

$$\forall t \in T, \quad x_t \geq -C_t z_t. \quad (1.2.16)$$

The collateral requirement $C_t$ may depend on the current period endogenous variables i.e., $C_t = f_t(p_t, r_t, (c'_i, x'_i)_{i \in I})$ for some continuous function $f_t$. Contrary to our setup, the constraints in (1.2.16) do not exclude default at equilibrium. Given $t$-period equilibrium prices and allocations, if the collateral level $C_t$ is strictly less than $1/r_{t+1}$ then there will be default at period $t + 1$.

\(^3\)To be more specific in Kiyotaki and Moore (1997) agents use the land to produce fruits using a production technology. It is assumed that the output of their production (the equivalent to our dividend) cannot be seized.

\(^4\)Kubler and Schmedders (2003) consider a variant of the model proposed by Araujo et al. (2002) in which collateral takes the form of long-lived assets instead of durable and storable goods.
1.3 Equilibrium asset pricing

Let \((p, q, r) \in \Pi\) and an allocation \(a = (a^i)_{i \in I}\) with \(a^i \in A\) be an equilibrium for the economy \(E\). For any agent \(i \in I\) there exists a sequence of multipliers \((\lambda^i_t)_{t \geq 0}\) and \((\mu^i_t)_{t \geq 0}\) associated with the constraints (1.2.3) and (1.2.4) such that the following first order conditions hold:

(a) \(\lambda^i_t p_t = \beta^i_t u_i'(c^i_t)\);
(b) \(-\lambda^i_t r_t + (\lambda^i_{t+1} + \mu^i_t)[r_{t+1} + p_{t+1} \xi_{t+1}] + \zeta^i_t = 0;\)
(c) \(-\lambda^i_t q_t + p_{t+1}(\lambda^i_{t+1} + \mu^i_t) = 0;\)
(d) \(\lambda^i_t[p_t c^i_j + q_t z^i_j + r_t x^i_t - p_t \omega^i_t - (r_t + p_t \xi_t) x^i_{t-1} - p_t z^i_{t-1}] = 0;\)
(e) \(\mu^i_t[(r_{t+1} + p_{t+1} \xi_{t+1}) x^i_t + p_{t+1} z^i_t] = 0;\)
(f) \(\zeta^i_t x^i_t = 0.\)

**Proposition 1.3.1.** For any period \(t \in T\) we have \(p_t > 0\), \(q_t > 0\) and \(r_t > 0\). Moreover, the equilibrium price sequences \((p, q, r)\) satisfy the following non-arbitrage relation

\[
\frac{r_t}{q_t} = \left[\frac{r_{t+1}}{p_{t+1}} + \xi_{t+1}\right] \quad (1.3.1)
\]

and \(\forall i \in I\) we have \(\zeta^i_t = 0.\)

**Proof.** The fact that \(p_t > 0\) and \(q_t > 0\) are strictly positive follows from the strict increasingness of \(u_i\) and the first order conditions (a) and (c). Combining (b) and (c) gives

\[
\frac{r_t}{q_t} = \left[\frac{r_{t+1}}{p_{t+1}} + \xi_{t+1}\right] + \frac{\zeta^i_t}{\lambda^i_t}.
\]

Market clearing implies that \(x^i_t > 0\) for some agent \(j \in I\). Therefore, we have \(\zeta^j_t = 0\) and condition (1.3.1) holds for this agent \(j\). Given that each agent faces the same prices and dividend stream, (1.3.1) must hold for every agent. From this follows that \(\zeta^i_t = 0, \forall i \in I, \forall t \in T.\)

Assumption (A.3) implies that there exists a subsequence \((\xi_{t_k})_{k \geq 1}\) such that \(\xi_{t_k} > 0\) for all \(k \geq 1.\) Fix a date \(\tau \in T\) such that \(\xi_{\tau+1} > 0.\) Summing up over \(i\) the constraints (1.2.4) and use market clearing in security and asset markets (i.e., equations (1.2.8) and (1.2.9)) we get that

\[
r_{\tau+1} + p_{\tau+1} \xi_{\tau+1} > 0.
\]

If \(r_\tau \leq 0,\) then from the first order condition (b) we must have \(\lambda^i_{\tau+1} + \mu^i_\tau \leq 0.\) This is a contradiction since \(\lambda^i_{\tau+1} > 0\) (due to the Inada condition on \(u_i\)) and \(\mu^i_\tau \geq 0.\)
Therefore, \( r_\tau > 0 \) and using (1.3.1) we conclude that \( r_t > 0 \) for any \( t < \tau \). Following a recursive procedure we can show that \( r_t > 0 \) for all \( t \in \mathcal{T} \).

We can normalise the price sequence of the consumption good by setting \( p_t = 1, \forall t \in \mathcal{T} \). Since the equilibrium prices are strictly positive, market clearing in asset markets implies that at every period \( t \) there is at least some agent \( i \) for which the constraint (1.2.4) is not binding. The following proposition shows that non-binding agents have the highest marginal rate of substitution. That is, at any period \( t \), those agents equalise their marginal rates of substitution.

**Proposition 1.3.2.** If for some \( j \in I \)

\[ [r_{t+1} + \xi_{t+1}]x_t^j > -z_t^j \]

then

\[ \beta_j \frac{u'_j(c_{t+1}^j)}{u'_j(c_t^j)} = \max_{i \in I} \beta_i \frac{u'_i(c_{t+1}^i)}{u'_i(c_t^i)}. \]

**Proof.** It follows from a variational argument that is based on the first order conditions. \( \square \)

Propositions (1.3.1) and (1.3.2) have implications for asset pricing. In particular, we get that for any period \( t \in \mathcal{T} \)

\[ q_t = \max_{i \in I} \frac{\lambda_{t+1}^i}{\lambda_t^i} = \max_{i \in I} \beta_i \frac{u'_i(c_{t+1}^i)}{u'_i(c_t^i)} \quad (1.3.2) \]

and

\[ r_t = \max_{i \in I} \beta_i \frac{u'_i(c_{t+1}^i)}{u'_i(c_t^i)} [r_{t+1} + \xi_{t+1}]. \quad (1.3.3) \]

Equations (1.3.2) and (1.3.3) imply that it is possible to find a common evaluation for calculating the present value of debt. Nevertheless, the market is not complete since the constrained agents cannot trade in such a manner to equate their marginal rate of substitution of current and future consumption to the price vector. Thus, in this case the market is sequentially incomplete.

A recursive argument shows that

\[ r_0 = \sum_{t=1}^{T} Q_0^t \xi_t + Q_0^T r_T \quad (1.3.4) \]

where

\[ Q_0^0 = 1 \quad \text{and} \quad \forall t \geq 1, \ Q_0^t = \prod_{\sigma=0}^{t-1} q_\sigma \quad (1.3.5) \]

denotes the Arrow-Debreu price that is equal to the present value (as of period 0) of one unit of consumption at date \( t \). By construction, we have that \( q_t = Q_0^{t+1}/Q_0^t \).
Since (1.3.1) holds, we get the following recursive equation

$$\forall t \in \mathcal{T}, \quad Q_{0}^{t+1} = \frac{r_t}{r_{t+1} + \xi_{t+1}} Q_{0}^{t}. \quad (1.3.6)$$

Moreover, observe that for any \( t \) we have

$$\frac{\lambda_t r_t}{\lambda_0} \leq Q_0^t r_t \leq r_0. \quad (1.3.7)$$

The fundamental value of the long-lived asset at period 0, denoted \( FV_0 \), is defined as the present value of the dividend using as a discount factor the sequence of Arrow-Debreu prices \( (Q_0^t)_{t \geq 1} \) i.e.,

$$FV_0 = \sum_{t=1}^{\infty} Q_0^t \xi_t. \quad (1.3.8)$$

If at equilibrium the price of the asset at period 0, \( r_0 \), equals its fundamental value \( FV_0 \), then we say that there is no sequential price bubble. Otherwise, there is a price bubble equal to the difference between \( r_0 \) and \( FV_0 \). A necessary and sufficient condition for no sequential price bubbles is that

$$\lim_{T \to \infty} Q_0^T r_T = 0. \quad (1.3.9)$$

**Proposition 1.3.3.** For any agent \( i \in I \), the sequence of Lagrange multipliers \( (\lambda_t^i)_{t \in \mathcal{T}} \) is summable (i.e., it belongs to \( \ell_1^+ \)).

**Proof.** Fix a period \( T \). Multiplying both sides of (1.2.3) by the multiplier \( \lambda_t^i \), summing over all \( t \) from 0 to \( T \) and using the first order conditions (b), (c) and (e) produces

$$\sum_{t=0}^{T} \lambda_t^i c_t^i + \lambda_T^i [q_T z_T^i + r_T x_T^i] = \sum_{t=0}^{T} \lambda_t^i \omega_t^i + \lambda_0^i [r_0 + \xi_0] x_{-1}^i. \quad (1.3.10)$$

From (1.3.7) we have

$$\frac{\lambda_T^i r_T}{\lambda_0^i} \leq Q_0^T r_T \leq r_0.$$

Let \( a_t^i \) defined by

$$a_t^i \equiv x_t^i + \frac{q_t}{r_t} z_t^i.$$

Observe that \( a_t^i \geq 0 \) (this is because of the collateral constraint (1.2.4)) and \( \sum_{i \in I} a_t^i = 1 \) (because of market clearing). One has

$$\lambda_T^i q_T z_T^i + \lambda_T^i r_T x_T^i = \lambda_T^i r_T a_T^i \leq \lambda_0^i r_0. \quad (1.3.11)$$
Equations (1.3.10) and (1.3.11) together with the concavity of $u_i$ imply that

$$
\sum_{t=0}^{T} \lambda_i^t \omega_i^t \leq \lambda_0^t r_0 + \sum_{t=0}^{T} \lambda_i^t c_i^t = \lambda_0^t r_0 + \sum_{t=0}^{T} \beta_i^t (u_i(c_i^t) - u_i(0)) \\
\leq \lambda_0^t r_0 + \sum_{t=0}^{T} \beta_i^t u_i(W)
$$

where $W = \max_{i \in I} ||\omega^i|| + ||\xi||$. Since $\omega^i$ belongs to the interior of $\ell^+_{\infty}$ (Assumption A1), it follows that $(\lambda_i^t)_{t \in T}$ belongs to $\ell^+_T$.

**Proposition 1.3.4.** For any agent $i \in I$, the sequence $(\lambda_i^t r_t a_t^i)_{t \in T}$ converges to 0, i.e., $\lim_{t \to \infty} \lambda_i^t r_t a_t^i = 0$.

**Proof.** Observe that $(\lambda_i^t r_t)_{t \in T}$ is a non-increasing sequence (this follows from the first order condition (b)), so it has a limit denoted $\eta \geq 0$. Observe also that the last part of the proof of Proposition (1.3.3) (see the last inequalities) shows that $\sum_{t=0}^{\infty} \lambda_i^t \omega_i^t$ and $\sum_{t=0}^{\infty} \lambda_i^t c_i^t$ are both finite.

Assume by contradiction that $\lim_{t \to \infty} \lambda_i^t r_t = \eta > 0$. Since $\lambda_i^t \to 0$, then $r_t \to +\infty$ and $\frac{q_t}{r_t} \to 0$ (because of the non-arbitrage condition (1.3.1)). Recall that

$$
a_t^i = x_t^i + \frac{q_t}{r_t} z_t^i.
$$

It follows from (1.3.10) that $\lim_{t \to \infty} a_t^i$ exists and it is finite. We claim that in this case $\lim_{t \to \infty} a_t^i = 0$.

Indeed, if $\lim_{t \to \infty} a_t^i > 0$, there exists $T$ and $\varepsilon > 0$ strictly smaller than $\eta$ such that

$$
\forall t \geq T, \quad x_t^i + \frac{q_t}{r_t} z_t^i > \frac{\eta \varepsilon}{\eta - \varepsilon} \lambda_0^i.
$$

Consider the following asset plan $(\tilde{z}_t^i)_{t \in T}$ defined as follows

$$
\tilde{z}_t^i = \begin{cases} 
    z_t^i & \text{if } t < T \\
    z_t^i - \frac{\eta \varepsilon}{Q_0^t} & \text{if } t \geq T.
\end{cases}
$$

The non-arbitrage condition (1.3.1) and the definition of $q_t$ imply that

$$
\forall t \geq T, \quad q_{t+1} \frac{\eta \varepsilon}{Q_0^{t+1} q_{t+1}} = \frac{\eta \varepsilon}{Q_0^t q_t}.
$$

Moreover,

$$
\forall t \geq T, \quad [r_{t+1} + \xi_{t+1}] x_t^i \geq -\tilde{z}_t^i \quad \text{(5)}
$$

This is because $q_t = Q_0^{t+1}/Q_0^t$ and (1.3.6) imply that the constraint is equivalent to $x_t^i + \frac{q_t}{r_t} \tilde{z}_t^i \geq$
It follows that the plan \((\tilde{z}_t^t)_{t \in T}\) allows us to finance a consumption sequence \(\tilde{c}^t\) that differs from \(c^t\) after the period \(T - 1\) and enjoy higher utility, i.e., \(U^t(\tilde{c}^t) > U^t(c^t)\). Obviously this contradicts the optimality of \((c^t, z^t, x^t)\). Therefore, \(\lim_{t \to \infty} a_t^i = 0\).

Assume that \(\lim_{t \to \infty} \lambda_t^i r_t = 0\) and observe that from equation (1.2.4) and the market clearing conditions follows \(a_t^i \in [0, 1], \forall i \in I\). In this case it follows immediately that

\[
\lim_{t \to \infty} \lambda_t^i r_t a_t^i = 0, \forall i \in I.
\]

\(\square\)

**Theorem 1.3.1.** The following statements are equivalent:

(i) There is no sequential price bubble

(ii) \(\lim_{T \to \infty} Q_T^0 r_T = 0\)

(iii) \(\forall i \in I, \lim_{T \to \infty} \sum_{t=0}^{T} Q_t^0(c_t^i - \omega_t^i)\) exists.

**Proof.** (i)\(\Leftrightarrow\) (ii): This is because (ii) is a necessary and sufficient condition for no sequential price bubbles.

(ii)\(\Rightarrow\) (iii): Recall that

\[
a_t^i = x_t^i + \frac{q_t}{r_t} z_t^i
\]

and \(a_t^i \geq 0, \sum_{i \in I} a_t^i = 1\). Multiplying both sides of (1.2.3) by \(Q_0^i\), summing over all \(t\) from 0 to \(T\) and using the recursive relation (1.3.6) we get

\[
\sum_{t=0}^{T} Q_t^0(\omega_t^i - c_t^i) + (r_0 + \xi_0) x_{-1}^i = Q_T^0 r_T a_T^i.
\]  

(1.3.12)

Hence

\[
\lim \inf \sum_{t=0}^{T} Q_t^0(\omega_t^i - c_t^i) + (r_0 + \xi_0) x_{-1}^i \geq 0
\]

and

\[
\lim \sup \sum_{t=0}^{T} Q_t^0(\omega_t^i - c_t^i) + (r_0 + \xi_0) x_{-1}^i \leq \lim_{T \to \infty} Q_T^0 r_T.
\]

If (ii) holds, then

\[
\lim \sup \sum_{t=0}^{T} Q_t^0(\omega_t^i - c_t^i) = \lim \inf \sum_{t=0}^{T} Q_t^0(\omega_t^i - c_t^i).
\]

Moreover, using (1.3.7) and the fact that \(\lim_{t \to \infty} \lambda_t^i r_t = \eta\) gives

\[
x_t^i + \frac{q_t}{r_t} z_t^i = x_t^i + \frac{q_t}{r_t} z_t^i - \frac{\eta \epsilon}{r_t Q_0^i} \geq x_t^i + \frac{q_t}{r_t} z_t^i - \frac{\eta \epsilon \lambda_0}{\lambda_t^i r_t} \geq x_t^i + \frac{q_t}{r_t} z_t^i - \frac{\eta \epsilon \lambda_0}{\eta - \epsilon} \lambda_t^i > 0.
\]
(iii)⇒ (i): Assume that \( \lim_{T \to \infty} \sum_{t=0}^{T} Q_0^i (c_i^t - \omega_i^t) \) exists and that (i) is not true. It follows from (ii) that \( Q_0^i r_T \to \eta > 0 \). In this case, using (1.3.12), we get that \( \lim_{t \to \infty} a_i^t \) exists. From Proposition (1.3.4), follows then that \( \lim_{t \to \infty} a_i^t = 0 \) and we have the following contradiction

\[
1 = \lim_{t \to \infty} \sum_{i \in I} a_i^t = \sum_{i \in I} \lim_{t \to \infty} a_i^t = 0.
\]

The following corollary is in the spirit of Theorem 6.1 in Huang and Werner (2000). Roughly speaking it states that bubbles are excluded if interest rates (the inverse of Arrow-Debreu prices) are greater than the growth rate of economy’s aggregate endowment.

**Corollary 1.3.1** (Huang-Werner, 2000). Assume that \( \sum_{t=0}^{\infty} Q_0^i \omega_t < +\infty \), where \( \omega_t = \sum_{i \in I} \omega_i^t \). Then, there is no sequential price bubble.

**Proof.** Fix some agent \( i \in I \). From equation (1.3.12) we have that \( \sum_{t=0}^{\infty} Q_0^i c_i^t < +\infty \). Therefore, \( \lim_{T \to \infty} \sum_{t=0}^{T} Q_0^i (c_i^t - \omega_i^t) \) exists in which case we can use Theorem 1.3.1 to prove the result.

The sufficient condition in Corollary 1.3.1 involves endogenous variables (Arrow-Debreu prices) and as result is difficult to verify in workable examples. We next provide a sufficient condition for ruling out bubbles that expressed only in fundamentals.

**Corollary 1.3.2.** Let \( k > 0 \). If \( \xi_t \geq k \max_{i \in I} \omega_i^t \) for any period \( t \in T \), then there is no sequential price bubble.

**Proof.** From equation (1.3.4) we have \( \sum_{t=1}^{\infty} Q_0^i \xi_t \leq r_0 \). Under our assumption this implies that \( k \sum_{t=1}^{\infty} Q_0^i \omega_i^t \leq \sum_{t=1}^{\infty} Q_0^i \xi_t \leq r_0 \) for any \( i \in I \). We apply next Corollary 1.3.1.

The next result is in the spirit of Proposition 4 in Kocherlakota (1992). It states that a sequential price bubble can occur only if there exists an agent who is infinitely rich.

**Corollary 1.3.3** (Kocherlakota, 1992). Assume that there exists a sequential price bubble. Then, there exists an agent \( j \) such that \( (a_i^j)_{i \in T} \) has no limit. For any such agent, we have that \( \sum_{t=0}^{\infty} Q_0^j \omega_i^j = +\infty \).

**Proof.** Assume that for any \( i \in I \), \( (a_i^j)_{i \in T} \) has a limit. It follows from (1.3.12) that \( \lim_{T} \sum_{t=0}^{T} Q_0^j (\omega_i^j - c_i^j) \) exists, in which case we have no price bubble: a contradiction. Let \( j \) be such that \( (a_i^j)_{i \in T} \) has no limit and assume that \( \lim_{T} \sum_{t=0}^{T} Q_0^j \omega_i^j < +\infty \). In this case, \( \sum_{t=0}^{\infty} Q_0^j c_i^j < +\infty \) and \( \lim_{T} \sum_{t=0}^{T} Q_0^j (c_i^j - \omega_i^j) \) exists. From (1.3.12),
\( \lim_T Q_0^T r_T a_T^j \) exists, and since \( Q_0^T r_T \to \eta > 0 \) (by the assumption of a sequential price bubble), \( \lim_T a_T^j \) exists too: a contradiction.

\[ \boxed{\text{Remark 1.3.1.}} \] Kocherlakota (1992) (see Proposition 4) provides an alternative proof of Corollary 1.3.3. However, to prove the result, he claims that if a sequence \((a_t)_{t \in T}\) is bounded and has no limit, then there must exists a subsequence \((t_n)_{n=1}^{\infty}\) and a positive constant \(b\) such that \((a_{t_n} - a_{t_{n-1}}) > b\) for all \(n\). As the following counterexample illustrates, this claim is not correct.

We construct a sequence as follows: \(a_1 = 1\) and \(a_t = a_{t-1} + 1/t\) for \(2 \leq t \leq \tau_1\), where \(\tau_1\) is the first period such that \(a_{\tau_1} \geq 2\) (such a period always exists). Define next \(a_t = a_{t-1} - 1/t\) for \(\tau_1 + 1 \leq t \leq \tau_2\), where \(\tau_2\) is the first date such that \(a_{\tau_2} < -2\) (such a period always exists). Then, you define \(a_t = a_{t-1} + 1/t\) until the first date you reach the value 2 and so on so forth.

The sequence \((a_t)_{t \in T}\) constructed in this way is bounded, does not converge, and for any subsequence \((t_n)_{n=1}^{\infty}\) we have \(\lim_n (a_{t_n} - a_{t_{n-1}}) = 0\).

We present next an example of a sequential equilibrium with price bubble. An interesting feature of our example, as opposed to the example of Kocherlakota (1992) and Huang and Werner (2000), is that the asset is not fiat money and there is no growth.

\[ \text{Example 1.3.1.} \] We consider an economy with two infinitely-lived agents \((i = 1, 2)\) characterised by the same CRRA instantaneous utility function

\[ u(c) = \frac{c^{1-\theta}}{1-\theta}, \quad \theta > 0, \]

having common discount factor \(\beta \in (0, 1)\) and initial asset holdings \(x_{i-1}^i = 1/2\) for \(i = 1, 2\). The asset’s dividend process \((\xi_t)_{t \in T}\) satisfies: \(\xi_0 > 1\) and \(\sum_{t=1}^{\infty} \xi_t \leq 1\). Denote by \(\gamma\) the following constant

\[ \gamma = \frac{1}{\beta^{1/\theta}} \]

and consider the sequence \((b_t)_{t \in T}\) defined as follows: \(b_0 = 1, b_1 = 1 - \xi_1, \ldots, b_t = 1 - \sum_{s=1}^{t} \xi_t, \ldots\).

The endowments are specified as follows:

Observe that for both agents we have \((\omega_t^i)_{t \in T} \in \text{int} \ell_\infty^+\). We chose the price sequence \(r\) to be equal to the process \(b\), i.e., \(r_t = b_t\) for all \(t\). There is trade only on the long lived asset, i.e, for \(i = 1, 2\), \(z_t^i = 0\) for all \(t \in T\). The price sequence \(q\) is given by \(q_t = 1\) for all \(t\). The equilibrium values of the Lagrange multipliers \((\mu_t^i)_{t \in T}\) are specified in table 1.2. The consumption and long-lived asset allocations are specified in table 1.3.

\[ \text{---We are grateful to Paulo Klinger Monteiro for the idea underlying the construction of this counterexample.---} \]
\[ \begin{array}{c|c|c}
\text{Endowments for agent 1 } (\tau \geq 1) & \text{Endowments for agent 2 } (\tau \geq 2) \\
\hline
\omega^0_1 = 1 & \omega^0_1 > 0 \\
\omega^1_1 + b_0 = \beta^{1/\theta}(\omega^1_1 + \xi_0) \times 1/2 & \omega^2_1 - b_1 = \beta^{1/\theta}(\omega^2_1 + (b_0 + \xi_0) \times 1/2) + \gamma \\
\omega^1_2 - b_2 = \beta^{2/\theta}(\omega^1_2 + \xi_0) \times 1/2 + \gamma & \omega^2_2 + b_1 = \beta^{2/\theta}(\omega^2_2 + (b_0 + \xi_0) \times 1/2) + 1 \\
\omega^1_{2t+1} + b_{2t} = \beta^{1/\theta}(\omega^1_{2t} - b_{2t}) & \omega^2_{2t-1} - b_{2t-1} = \beta^{2/\theta}(\omega^2_{2t-1} - b_{2t-3}) + \gamma \\
\omega^1_{2t+2} - b_{2t+2} = \beta^{1/\theta}(\omega^1_{2t+1} + b_{2t+2}) + \gamma & \omega^2_{2t} + b_{2t-1} = \beta^{2/\theta}(\omega^2_{2t-2} + b_{2t-3}) + 1 \\
\end{array} \]

Table 1.1: Endowments

<table>
<thead>
<tr>
<th>Multipliers for agent 1</th>
<th>Multipliers for agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu^1_0 = 0 )</td>
<td>( \mu^2_0 = \lambda^2_0 - \lambda^2_1 )</td>
</tr>
<tr>
<td>( \mu^1_1 = \lambda^1_1 - \lambda^1_2 )</td>
<td>( \mu^2_1 = 0 )</td>
</tr>
<tr>
<td>( \mu^2_{2t} = 0 )</td>
<td>( \mu^2_{2t} = \lambda^2_{2t} - \lambda^2_{2t+1} )</td>
</tr>
<tr>
<td>( \mu^1_{2t+1} = \lambda^{1}<em>{2t+1} - \lambda^{1}</em>{2t+2} )</td>
<td>( \mu^2_{2t+1} = 0 )</td>
</tr>
</tbody>
</table>

Table 1.2: Lagrange multipliers \((\mu^t_i)_{t \in \mathcal{T}}\)

<table>
<thead>
<tr>
<th>Allocations for agent 1</th>
<th>Allocations for agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^1_{2t} = 1, x^1_{2t+1} = 0 )</td>
<td>( x^2_{2t} = 0, x^2_{2t+1} = 1 )</td>
</tr>
<tr>
<td>( c^1_0 = (\omega^0_1 + \xi_0) \times 1/2 )</td>
<td>( c^2_0 = \omega^2_0 + (\xi_0 + r_0) \times 1/2 )</td>
</tr>
<tr>
<td>( c^1_1 = \omega^1_1 + r_0 )</td>
<td>( c^2_1 = \omega^2_1 - r_1 )</td>
</tr>
<tr>
<td>( c^1_{2t} = \omega^1_{2t} - r_{2t} )</td>
<td>( c^2_{2t} = \omega^2_{2t} + r_{2t-1} )</td>
</tr>
<tr>
<td>( c^1_{2t+1} = \omega^1_{2t+1} + r_{2t} )</td>
<td>( c^2_{2t+1} = \omega^2_{2t+1} - r_{2t+1} )</td>
</tr>
</tbody>
</table>

Table 1.3: Equilibrium allocations

To see that these prices and allocations are an equilibrium, we first note that markets clear at every date, the budget and short-sales constraints are all satisfied. Moreover, the first order conditions as well as the Transversality Condition \( \lim_{\tau \rightarrow \infty} \beta^\tau u'(c^t_i)x^t_i = 0 \) are also satisfied.

Observe that
\[
\forall t \in \mathcal{T}, \quad Q^{t+1}_0 = \frac{r_t}{r_{t+1} + \xi_{t+1}} Q^t_0 = Q^t_0.
\]
and \(Q^0_0 r_t \rightarrow \overline{\tau} = 1 - \sum_{t=1}^{\infty} \xi_t \). If \( \sum_{t=1}^{\infty} \xi_t < 1 \) then \( \overline{\tau} > 0 \) and then there is a sequential price bubble. If \( \sum_{t=1}^{\infty} \xi_t = 1 \) then \( \overline{\tau} = 0 \) and bubbles are excluded.

The intuition of the above example is similar as in Example 1 by Kocherlakota (1992). The borrowing constraint does not permit agents to permanently reduce their asset holdings, because it will bind in every other period. At the same time the endowments grow fast enough to permit agents to purchase the asset when they are unconstrained.

We close this section by introducing an equivalent competitive equilibrium concept that involves wealth constraints. This concept will appear useful when we prove existence of a competitive equilibrium.

Assume that there exists an equilibrium for the economy \( \mathcal{E} \). The absence of arbitrage opportunities requires that at any period \( t \in \mathcal{T} \) the equilibrium price
sequences \((p, q, r)\) satisfy (1.3.1), i.e.,

\[
\forall t \in T, \quad \frac{r_t}{q_t} = \left[ \frac{r_{t+1}}{p_{t+1}} + \xi_{t+1} \right].
\]

Denote

\[
\forall t \in T, \quad \theta^i_{t-1} = \left[ \frac{r_t}{p_t} + \xi_t \right] x^i_{t-1} + z^i_{t-1}.
\]

Using this new variable, constraints (1.2.3) and (1.2.4) can be written as follows:

(a) budget constraints:

\[
\forall t \in T, \quad p_t c^i_t + q_t \theta^i_t \leq p_t \omega^i_t + p_t \theta^i_{t-1}.
\] (1.3.13)

(b) wealth constraints:

\[
\forall t \in T, \quad \theta^i_t \geq 0.
\] (1.3.14)

We define next a competitive equilibrium with wealth constraints.

**Definition 1.3.1.** An equilibrium with wealth constraints consists of prices \((p, q, r)\) \(\in \Pi\) together with consumption and wealth profiles \((c^i, \theta^i)_{i \in I}\) such that:

(a) for each agent \(i\), \((c^i, \theta^i)\) is optimal among all plans satisfying constraints (1.3.13), (1.3.14);

(b) commodity markets clear, i.e.,

\[
\forall t \in T, \quad \sum_{i \in I} c^i_t = \sum_{i \in I} \omega^i_t + \xi_t;
\] (1.3.15)

(c) asset market clears, i.e.,

\[
\forall t \in T, \quad \sum_{i \in I} \theta^i_t = \frac{r_{t+1}}{p_{t+1}} + \xi_{t+1};
\] (1.3.16)

(d) non-arbitrage relation, i.e.,

\[
\forall t \in T, \quad \frac{r_t}{q_t} = \left[ \frac{r_{t+1}}{p_{t+1}} + \xi_{t+1} \right].
\] (1.3.17)

**Proposition 1.3.5.** A sequence of prices \((p, q, r)\) \(\in \Pi\) together with an allocation \((c^i, \theta^i)_{i \in I}\) is a competitive equilibrium with wealth constraints if and only if \((p, q, r)\) \(\in \Pi\) and \((c^i, z^i, x^i)_{i \in I}\) is a competitive equilibrium for the economy \(E\).
1.4 Arrow-Debreu markets

In this section we state and prove two equivalence results. We show that consumption allocations in a sequential equilibrium with limited commitment can be implemented in a Arrow-Debreu trading environment with limited commitment in which agents trade only once facing a single budget constraint and a sequence of self-enforcing constraints. Equivalently, Arrow-Debreu equilibria with limited commitment can be implemented in a sequential trading environment having the described collateralised institutional structure.

1.4.1 Limited commitment

Markets open only at period 0 and agents make arrangements on future consumption claims. A possible outcome \( (c^i)_{i \in I} \) is resource feasible provided that

\[
\forall t \in T, \quad \sum_{i \in I} c^i_t = \sum_{i \in I} \omega^i_t + \xi_t.
\]

The allocation \( (c^i)_{i \in I} \) may involve promises (this is the case if \( (c^i)_{i \in I} \neq (\omega^i*)_{i \in I} \) where \( \omega^i* = \omega^i_t + x^i_{t-1}\xi_t \)). We assume that there is no commitment: agent \( i \) fulfils her/his promises only if it is optimal for her/him. In other words, agent \( i \) may default. Her/His decision (defaulting or not) depends on the consequences of default (i.e., the punishment). We assume that the punishment for defaulting at some period \( t \) is the confiscation of agent’s asset holdings as well as the agent looses the ability to finance current consumption based on her/his future wealth. More precisely, if agent \( i \) defaults at period \( t \), she/he faces the following maximization problem:

\[
V^i_t(Q_0) = \max_{c^i} U^i_t(c^i) \quad \text{s.t.} \quad c^i \in B^i_{t, \text{out}}(Q_0)
\]

where \( U^i_t(c^i) \) is the continuation utility as defined in (1.2.2) and

\[
B^i_{t, \text{out}}(Q_0) \equiv \left\{ c^i : \sum_{s=t}^{\infty} Q^s_0(c^i_s - \omega^i_s) \leq 0 \text{ and } \forall \tau \geq t, \sum_{s=t}^{\tau} Q^s_0(c^i_s - \omega^i_s) \leq 0 \right\}.
\]

Remark 1.4.1. We recall that \( Q^0_t \) is the price at period 0 of the contract delivering one unit of the good at period \( t > 0 \) (with \( Q^0_0 = 1 \)). We restrict our attention to Arrow–Debreu price processes \( Q_0 = (Q^i_0)_{i \in T} \) such that initial endowment processes have finite values, i.e.,

\[
\forall i \in I, \quad \sum_{t=0}^{\infty} Q^i_0 \omega^i_t < \infty.
\]

An Arrow–Debreu price sequence \( Q_0 \) satisfying the above property is said to have
high interest rates. We also say that a price sequence $q$ defined by $q_t = Q_{0}^{t+1}/Q_{0}^{t}$ has high interest rates if the associated Arrow-Debreu price sequence $Q_0$ has high interest rates.

**Remark 1.4.2.** Let $d_{i,t}^{\text{out}}(Q_0)$ denote the demand set associated with the budget set $B_{i,t}^{\text{out}}(Q_0)$, i.e.,

$$d_{i,t}^{\text{out}}(Q_0) \equiv \arg\max \{ U_i^t(c) : c \in B_{i,t}^{\text{out}}(Q_0) \}.$$ 

We get that

$$\sum_{s=t}^{\infty} Q_0^s(c_i^s - \omega_i^s) = 0 \text{ and } \forall \tau > t, \sum_{s=\tau}^{\infty} Q_0^s(c_i^s - \omega_i^s) \geq 0.$$

Indeed, assume the contrary, i.e., $\sum_{s=t}^{\infty} Q_0^s(c_i^s - \omega_i^s) = k < 0$, and let $\varepsilon > 0$ satisfy $k < -3\varepsilon$. There exists $T \geq t$ such that for all $\tau \geq T, \sum_{s=t}^{\tau} Q_0^s(c_i^s - \omega_i^s) - k \leq \varepsilon$.

Consider next the sequence $d^i$ defined as follows:

$$d_{s}^i = c_{s}^i, \quad s \neq T,$$

$$d_{T}^i = c_{T}^i + \frac{\varepsilon}{Q_{0}^{t}}.$$

We get that

$$\sum_{s=t}^{\infty} Q_{0}^{s}(d_{s}^i - \omega_{s}^i) = k + \varepsilon < -2\varepsilon < 0.$$

Observe that:

$$\text{for } t \leq \tau < T, \sum_{s=t}^{\tau} Q_{0}^{s}(d_{s}^i - \omega_{s}^i) = \sum_{s=t}^{\tau} Q_{0}^{s}(c_{s}^i - \omega_{s}^i) \leq 0,$$

$$\text{for } \tau \geq T, \sum_{s=t}^{\tau} Q_{0}^{s}(d_{s}^i - \omega_{s}^i) = \sum_{s=t}^{\tau} Q_{0}^{s}(c_{s}^i - \omega_{s}^i) + \varepsilon \leq k + 2\varepsilon < -\varepsilon < 0.$$

Thus, $d^i \in B_{i,t}^{\text{out}}(Q_0)$. However, $U_i^t(d^i) > U_i^t(c^i)$: a contradiction. Hence, $\sum_{s=t}^{\infty} Q_0^s(c_i^s - \omega_i^s) = 0$. Since $\sum_{s=t}^{\tau} Q_0^s(c_i^s - \omega_i^s) \leq 0$ for all $\tau \geq t$, we obtain that $\sum_{s=t}^{\infty} Q_0^s(c_i^s - \omega_i^s) \geq 0$ for all $\tau > t$.

A self-enforcing consumption process is defined as follows.

**Definition 1.4.1.** A consumption sequence $c^i$ for agent $i$ is **self-enforcing** if at every period $t > 0$ agent $i$ has no incentive to default, i.e., $U_i^t(c^i) \geq V_i^t(Q_0)$.

Given a period $\tau > 0$, the condition $U_i^\tau(c^i) \geq V_i^\tau(Q_0)$ is called the self-enforcing constraint at period $\tau$. We denote by $\text{SC}_i^\tau$ the set of all consumption sequences $c^i$ satisfying the self-enforcing constraints at every period $\tau > t$. Observe that a consumption process is self-enforcing if it belongs to $\text{SC}_0^\tau$. 

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The self-enforcing constraints correspond to a requirement of interim individual rationality, which implies that no borrower has an incentive to default. At the initial period there is no promise involved in the allocation of resources. However, an agent $j$ may decide to reject or block the allocation $(c^i)_i \in I$ if $U^j_0(c^i) < U^j_0(\omega^*)$. Naturally, we should avoid such a situation, therefore we will say that an allocation $(c^i)_i \in I$ is individually rational if at the initial period 0, $U^i_0(c^i) \geq U^i_0(\omega^*)$ for each $i \in I$.

1.4.2 Arrow–Debreu equilibrium with limited commitment

We next adapt the Arrow–Debreu competitive equilibrium concept to the framework of limited commitment.

**Definition 1.4.2.** An Arrow–Debreu competitive equilibrium with initial asset holdings $(x^i_{t-1})_i \in I$ is a family $(Q^i_0, (c^i)_i \in I)$ of an Arrow–Debreu price process $Q_0$ with high interest rates and a resource feasible and self-enforcing allocation $(c^i)_i \in I$ such that $c^i$ is optimal among all self-enforcing consumption processes $d^i$ satisfying the budget restriction

$$\sum_{t=0}^{\infty} Q^i_0(d^i_t - \omega^i_t) \leq x^i_{t-1} PV_0(Q_0, \xi)$$

where $PV_0(Q_0, \xi) = \sum_{t=0}^{\infty} Q^i_0 \xi_t$ is the present value of the dividend stream $(\xi_t)_{t \in T}$.

Given a period $t \in T$, a price process $Q_0$ with high interest rates and a real number $b_t \geq 0$, we denote by $B^i_{t, \text{ad}}(Q_0, b_t)$ the set of consumption processes $c^i$ satisfying

- the truncated budget restriction

$$\sum_{s=t}^{\infty} Q^i_0(c^i_s - \omega^i_s) \leq Q^i_0 b_t$$

- the self-enforcing constraints, i.e., $c^i \in SC^i_t$ or equivalently

$$\forall \tau > t, \quad U^j_\tau(c^i) \geq V^j_\tau(Q_0).$$

It is important to notice that we do not restrict the consumption process $c^i$ to satisfy the participation constraint at the initial period $t$.

We denote by $d^i_{t, \text{AD}}(Q_0, b_t)$ the demand set associated with the budget set $B^i_{t, \text{AD}}(Q_0, b_t)$ defined by

$$d^i_{t, \text{AD}}(Q_0, b_t) \equiv \text{argmax}\{U_t(c) : c \in B^i_{t, \text{AD}}(Q_0, b_t)\}.$$
period $t > 0$. Since the endowment process $\omega^{i*}$ belongs to $B_{0}^{i, AD}(Q_{0}, x_{-1}^{i} PV_{0}(Q_{0}, \xi))$, we must have $U_{0}^{i}(c^{i}) \geq U_{0}^{i}(\omega^{i*})$, implying that $c^{i}$ is individually rational. This property implies that if $Q_{0}$ is an Arrow–Debreu price process with high interest rates and $(c^{i})_{i \in I}$ is resource feasible, then $(Q_{0}, (c^{i})_{i \in I})$ is an Arrow–Debreu equilibrium if and only if we have

$$\forall i \in I, \quad c^{i} \in d_{0}^{i, AD}(Q_{0}, x_{-1}^{i} PV_{0}(Q_{0}, \xi)).$$

### 1.4.3 Equivalence results

We are now ready to state our equivalence results.

**Theorem 1.4.1.** Let $(q, r, (c^{i}, x^{i}, z^{i})_{i \in I})$ be a competitive equilibrium of the economy $E$ such that $q$ has high interest rates. Then, $(Q_{0}, (c^{i})_{i \in I})$ is an Arrow–Debreu equilibrium with initial asset holdings $(x_{-1}^{i})_{i \in I}$ and $Q_{0}$ is the Arrow–Debreu price sequence associated to $q$.

We also have the converse implication.

**Theorem 1.4.2.** If $(Q_{0}, (c^{i})_{i \in I})$ is an Arrow–Debreu equilibrium with initial asset holdings $(x_{-1}^{i})_{i \in I}$ and high interest rates, then $(q, r, (c^{i}, x^{i}, z^{i})_{i \in I})$ is a competitive equilibrium of the economy $E$ where:

- $q$ is the price sequence associated to $Q_{0}$, i.e.,
  $$\forall t \in \mathcal{T}, \quad q_{t} = \frac{Q_{0}^{t+1}}{Q_{0}^{t}};$$

- $r$ is an asset price sequence with no bubbles, i.e.,
  $$\forall t \in \mathcal{T}, \quad r_{t} = \frac{1}{Q_{0}^{t}} \sum_{s=t+1}^{\infty} Q_{0}^{s} \xi_{s};$$

- $\forall i \in I, \forall t \in \mathcal{T}, (x_{t}^{i}, z_{t}^{i})$ are chosen such that $a_{t}^{i} = x_{t}^{i} + \frac{r_{t}}{r_{t}^{i}} z_{t}^{i}$ where
  $$r_{t} a_{t}^{i} = \frac{1}{Q_{0}^{t}} \sum_{s=t+1}^{\infty} Q_{0}^{s} (c_{s}^{i} - \omega_{s}^{i}).$$

In order to prove Theorems 1.4.1 and 1.4.2 we first prove some intermediate properties.

**Claim 1.4.1.** Let $Q_{0}$ be an Arrow–Debreu price sequence with high interest rates. Fix a period $t$. We have:

$$c^{i} \in SC_{t}^{i} \implies \forall \tau > t, \quad \sum_{s=\tau}^{\infty} Q_{0}^{s} (c_{s}^{i} - \omega_{s}^{i}) \geq 0.$$
Proof. Let \( c^i \in \text{SC}_i^t \) and assume that there exists some \( \tau > t \) such that
\[
\sum_{s=\tau}^{\infty} Q_0^s(c^i_s - \omega^i_s) < 0.
\]
Since \( U_\tau^i(c^i) \geq V_\tau^i(Q_0) \), there exists \( \tau_1 \geq \tau \) such that
\[
\sum_{s=\tau}^{\tau_1} Q_0^s(c^i_s - \omega^i_s) > 0.
\]
Indeed, if this is not the case, then \( c^i \in d^i_{\tau}(Q_0) \). By Remark 1.4.2 it should be the case that
\[
\sum_{s=\tau}^{\infty} Q_0^s(c^i_s - \omega^i_s) = 0,
\]
which is a contradiction.

Similarly, since \( U_{\tau_1+1}^i(c^i) \geq V_{\tau_1+1}^i(Q_0) \), there exists \( \tau_2 \geq \tau_1 + 1 \) such that
\[
\sum_{s=\tau_1+1}^{\tau_2} Q_0^s(c^i_s - \omega^i_s) > 0.
\]
We can therefore find a sequence \( (\tau_n)_{n \geq 1} \) such that
\[
\forall n \geq 1, \sum_{s=\tau}^{\tau_n} Q_0^s(c^i_s - \omega^i_s) > 0.
\]
Taking the limit an \( n \) goes to infinite proves the claim. \( \square \)

Given a price sequence \( r \) and a real number \( b_i \geq 0 \), we denote by \( B^i_t(r, b_i) \) the set of all pairs \((c^i, a^i) \in \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \) satisfying

- the budget restrictions
\[
c^i_t + r_i a^i_t \leq \omega^i_t + b_i,
\]
\[
\forall \tau > t, \quad c^i_\tau + r_{\tau} a^i_\tau \leq \omega^i_\tau + [r_{\tau} + \xi_{\tau}] a^i_{\tau-1}; \quad (1.4.1)
\]
- the borrowing constraints \( a^i_{\tau} \geq 0 \) for every period \( \tau \geq t \).

We also denote by \( d^i_t(r, b_i) \) the demand set associated with the budget set \( B^i_t(r, b_i) \) and defined by
\[
d^i_t(r, b_i) \equiv \text{argmax}\{U^i_t(c): \ c \in B^i_t(r, b_i)\}.
\]

Claim 1.4.2. Let \( Q_0 \) be an Arrow–Debreu price sequence with high interest rates and \( r \) be a price sequence that satisfies the recursive equation (1.3.6). Fix a period \( t \) and some \( b_i \geq 0 \). We have:
\[
c^i \in B^{i, \text{AD}}_t(Q_0, b^i_t) \implies (c^i, a^i) \in B^i_t(r, b^i_t)
\]
where the portfolio sequence \( a^i_t \) is given by

\[
\forall \tau \geq t, \quad r_\tau Q_0^i a^i_\tau = \sum_{s=\tau+1}^{\infty} Q_0^i (c_s^i - \omega_s^i).
\]

**Proof.** Let \( c^i \in B_t^{i,\text{AD}}(Q_0, b_t^i) \) and define the sequence \( a^i_t \) as specified in the claim. Because of Claim 1.4.1 we have that \( a^i_\tau \geq 0 \). Recall that \( c^i \) satisfies

\[
Q_0^i (c^i_t - \omega_t^i) + \sum_{s=t+1}^{\infty} Q_0^i (c_s^i - \omega_s^i) \leq Q_0^i b_t^i
\]

implying that

\[
c_t^i - \omega_t^i + r_t a_t^i \leq b_t^i.
\]

Now fix a period \( \tau > t \). Observe that by the definition of the process \( a^i_t \) we have

\[
r_\tau Q_0^i a^i_\tau = \sum_{s=\tau+1}^{\infty} Q_0^i (c_s^i - \omega_s^i)
\]

\[
= \sum_{s=\tau}^{\infty} Q_0^i (c_s^i - \omega_s^i) - Q_0^i (c_\tau^i - \omega_\tau^i)
\]

\[
= r_{\tau-1} Q_0^{i-1} a^{i-1}_{\tau-1} - Q_0^i (c_\tau^i - \omega_\tau^i).
\]

Since \( r \) satisfies the recursive equation (1.3.6) we get that

\[
c_\tau^i - \omega_\tau^i + r_\tau a_\tau^i = [r_\tau + \xi_\tau] a^{i-1}_{\tau-1}.
\]

We have thus proved that the sequential budget restrictions are satisfied for every period \( \tau \geq t \).

**Claim 1.4.3.** Let \( Q_0 \) be an Arrow-Debreu price sequence with high interest rates. Fix a period \( t \) and some \( b_t^i \geq 0 \). We have:

\[
c^i \in d_t^{i,\text{out}}(Q_0) \implies c^i \in B_t^{i,\text{AD}}(Q_0, b_t^i)
\]

**Proof.** Let \( c^i \in d_t^{i,\text{out}}(Q_0) \). Since \( b_t^i \geq 0 \) we get (this is because of Remark 1.4.2) that

\[
\sum_{s=t}^{\infty} Q_0^i (c_s^i - \omega_s^i) \leq Q_0^i b_t^i.
\]

We next show that \( c^i \) is self-enforcing, i.e., \( c^i \in \text{SC}_t^i \). Assume the contrary. That is, assume there exists a period \( \tau > t \) such that

\[
U_\tau^i(c^i) < V_\tau^i(Q_0) = U_\tau^i(d^i)
\]

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where \( d^i \) satisfies
\[
\sum_{s=\tau}^{\infty} Q_0^s(d^i_s - \omega_s^i) \leq 0, \quad \text{and} \quad \forall \sigma \geq \tau, \quad \sum_{s=\sigma}^\sigma Q_0^s(d^i_s - \omega_s^i) \leq 0
\]

Let
\[
\hat{c}^i_s = c^i_s, \quad \text{for } s = 0, \ldots, \tau - 1,
\]
\[
\hat{c}^i_s = d^i_s, \quad \text{for } s \geq \tau.
\]

Observe that:
\[
\text{for } t \leq \sigma < \tau, \quad \sum_{s=t}^\sigma Q_0^s(\hat{c}^i_s - \omega_s^i) = \sum_{s=t}^\tau Q_0^s(c^i_s - \omega_s^i) \leq 0,
\]
\[
\text{for } \sigma \geq \tau, \quad \sum_{s=t}^\sigma Q_0^s(\hat{c}^i_s - \omega_s^i) = \sum_{s=t}^\tau Q_0^s(c^i_s - \omega_s^i) + \sum_{s=\tau}^\sigma Q_0^s(d^i_s - \omega_s^i) \leq 0.
\]

This implies that \( \hat{c}^i \in B^{i,\text{AD}}(Q_0, b^i_t) \). Since \( U^i_t(\hat{c}^i) > U^i_t(c^i) \), this contradicts the fact that \( c^i \in d^{i,\text{out}}_t(Q_0) \). \( \Box \)

**Claim 1.4.4.** Let \( Q_0 \) be an Arrow–Debreu price sequence with high interest rates and \( r \) be a price sequence that satisfies the recursive equation (1.3.6). Fix a period \( t \) and some \( b^i_t \geq 0 \). We have:

\[
(c^i, a^i) \in d^i_t(r, b^i_t) \implies c^i \in d^{i,\text{AD}}_t(Q_0, b^i_t).
\]

**Proof.** Let \((c^i, a^i) \in d^i_t(r, b^i_t)\). We first show that \( c^i \) belongs to \( B^{i,\text{AD}}(Q_0, b^i_t) \). Multiplying the budget constraints (1.4.1) by \( Q_0^i \) and then sum up to a period \( T > t \) we get

\[
\sum_{s=t}^T Q_0^i(a^i_s - \omega_s^i) + r_T Q_0^T a_T^i \leq Q_0^T b_T^i.
\]

Since \( a^i_T \geq 0 \), passing to the limit when \( T \) goes to infinite we get

\[
\sum_{s=t}^{\infty} Q_0^i(c^i_s - \omega_s^i) \leq Q_0^T b_T^i.
\]

In order to prove that \( c^i \) belongs to \( B^{i,\text{AD}}(Q_0, b^i_t) \) we still have to show that \( U^i_t(c^i) \geq V^i_\tau(Q_0) \) for every \( \tau > t \).

Fix \( \tau > t \) and let \( \tilde{c}^i \) be such that

\[
\tilde{c}^i \in d^{i,\text{out}}_\tau(Q_0).
\]
Using Claim 1.4.2 and Claim 1.4.3 we can conclude that

\[(\tilde{c}^i, \tilde{a}^i) \in B^i_t(r, b^i_t)\]

where \(\tilde{a}^i\) is a portfolio sequence given by

\[\forall \sigma \geq \tau, \quad r^i_{0}Q^i_0\tilde{a}^i_\sigma = \sum_{s=\sigma+1}^{\infty} Q^i_s(\tilde{c}^i_s - \omega^i_s).\]

Since \((c^i, a^i)\) is assumed to belong to the sequential demand set \(d^i_t(r, b_t)\), it follows from Bellman’s Principle of Optimality\(^7\) that \((c^i, a^i)\) also belongs to \(d^i_t(r, b^i_t)\). In particular, we must have

\[U^i_t(c^i) \geq U^i_t(\tilde{c}^i) = V^i_t(Q_0).\]

We have thus proved that \(c^i\) belongs to \(B^{iAD}_t(Q_0, b^i_0)\).

Now let \(\tilde{c}\) be a consumption process in \(d^{AD}_0(Q_0, b^i_0)\). Applying Claim 1.4.2 we have \((\tilde{c}, \tilde{a}^i) \in B^i_t(r, b_t)\) where the process \(\tilde{a}^i\) is defined according to Claim 1.4.2. Since \((c^i, a^i)\) belongs to the demand set \(d^i_t(r, b_t)\) we must have \(U^i_t(c^i) = U^i_t(\tilde{c}^i)\). This proves the claim. \(\square\)

**Proof of Theorem 1.4.1.** Let \((q, r, (c^i, x^i, z^i))_{i \in I}\) be a competitive equilibrium of the economy \(E\). Let \(b^i_0 = [r_0 + \xi_0]x^i_{-1}\) and \(a^i_t = x^i_t + \frac{\xi}{r}z^i_t\) for all \(t \in T\). Observe that the budget restriction (1.2.3) reduce to the budget restrictions (1.4.1) and the collateral constraints (1.2.4) imply that \(a^i_t \geq 0\) for all \(t \in T\). Therefore,

\[\forall i \in I, \quad (c^i, a^i) \in d^{AD}_0(r, b^i_0)\]

Claim 1.4.4 implies that \(c^i \in d^{AD}_0(Q_0, b^i_0)\). Since \(q\) has high interest rates, there are no bubbles, i.e.,

\[r_0 + \xi_0 = PV_0(Q_0, \xi).\]

Therefore, \(c^i \in d^{AD}_0(Q_0, x^i_{-1}PV_0(Q_0, \xi))\) establishing optimality of the consumption allocation. Market clearing is obvious. \(\square\)

**Proof of Theorem 1.4.2.** Let \((Q_0, (c^i))_{i \in I}\) be an Arrow–Debreu equilibrium with high interest rates and let \(q, r\) and \(a^i\) be as specified in the statement of the theorem. We show that

\[\forall i \in I, \quad (c^i, a^i) \in d^{AD}_0(r, b^i_0)\]

with \(b^i_0 = [r_0 + \xi_0]x^i_{-1}\). Since \(Q_0\) has high interest rates we have \(PV_0(Q_0, \xi) = r_0 + \xi_0\). It follows that \(c^i \in d^{AD}_0(Q_0, x^i_{-1}PV_0(Q_0, \xi))\), so we can apply Claim 1.4.2 to conclude

---

\(^7\)Principle of Optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision (see Bellman, 1972, p.83).
that \((c^i, a^i)\) belongs to \(B^0_i(r, b^0_i)\). Now let \((\tilde{c}^i, \tilde{a}^i)\) be in the demand set \(d^0_i(r, b^0_i)\). Applying Claim 1.4.4, we get that \(\tilde{c}^i\) belongs to \(d^0_0(Q_0, b^0_i)\). Since \(c^i\) also belongs to the same set, we must have \(U^0_0(c^i) = U^0_i(\tilde{c}^i)\), proving the claim.

Let the allocations \((x^i_t, z^i_t)_{t \in T}\) be such that \(a^i_t = x^i_t + \frac{r^0 t}{r^0} z^i_t\). We next show that markets clear. Using the budget restrictions of period \(t = 0\) and the fact that \(c^i\) is resource feasible gives

\[
\sum_{i \in I} a^i_0 = \frac{r_0 + \xi_0 - \sum_{i \in I} (c^i_0 - \omega^i_0)}{r_0} = 1.
\]

It then follows that \(\sum_{i \in I} x^i_0 = 1\) and \(\sum_{i \in I} z^i_0 = 0\). Following a similar argument shows that for all \(t \geq 1\) we have

\[
\sum_{i \in I} a^i_t = \frac{r_t + \xi_t - \sum_{i \in I} (c^i_t - \omega^i_t)}{r_t} = 1,
\]

implying that \(\sum_{i \in I} x^i_1 = 1\) and \(\sum_{i \in I} z^i_1 = 0\).

\[
1.5 \text{ Existence of a competitive equilibrium}
\]

\subsection*{1.5.1 Existence in the finite horizon case}

Given the family of fundamentals \((U^i, \beta^i, \omega^i, x^i_{t-1}, z^i_{t-1})_{i \in I}\), our first objective is to prove that an equilibrium exists in the finite horizon case. The proof is rather involved, so that we decompose it in several steps.

**Step 1:** *Introducing an auxiliary economy \(\hat{E}^T\).*

Fix a period \(T\) and denote by \(\Pi^T\) the set of price sequences \((p, \kappa)\) such that

\[
\Pi^T = \{(p, \kappa) \in \mathbb{R}^\infty \times \mathbb{R}^\infty_+ : \forall t > T, \ p_t = 0 \text{ and } \kappa_t = 0\}.
\]

We subsequently define the sets

\[
C^{i,T} = \{c^i \in \mathbb{R}^\infty_+ : \forall t > T, \ c^i_t = 0\}
\]

and

\[
\Phi^{i,T} = \{\varphi^i \in \mathbb{R}^\infty_+ : \forall t > T, \ \varphi^i_t = 0\}.
\]

Given prices \((p, \kappa) \in \Pi^T\) we denote by \(B^{i,T}(p, \kappa)\) the set of plans \((c^i, \varphi^i)\) \(\in C^{i,T} \times \Phi^{i,T}\) satisfying the following solvency constraints

\[
\forall t \in \{0, \ldots, T\}, \ p_t c^i_t + p_t \varphi^i_t \leq p_t \omega^i_t + \kappa_t \varphi^i_{t-1}.
\]

where \(\varphi^i_{t-1} > 0\) is given.
Given the profile of utilities, discount factors, endowments and the distribution \((\varphi^i_{-1})_{i \in I}\), we can consider the \(T\)-period economy 

\[
\hat{E}^T = (C^i, T, \Phi^i, T, u_i, \beta_i, \omega^i, \varphi^i_{-1})_{i \in I}
\]

where \(\forall i \in I, \omega^i \in C^i, T\) and \(U^i : C^i, T \to [0, \infty]\).

**Definition 1.5.1.** An equilibrium for the economy \(\hat{E}^T\) consists of prices \((p, \kappa) \in \Pi^T\) together with allocations \((c^i, \varphi^i)_{i \in I} \in \prod_{i \in I} (C^i, T \times \Phi^i, T)\) such that

(a) for each agent \(i\), \((c^i, \varphi^i)\) is optimal among all plans satisfying the constraints (1.5.1);

(b) commodity markets clear, i.e.,

\[
\forall t \in \{0, \ldots, T\}, \sum_{i \in I} c^i_t = \sum_{i \in I} \omega^i_t + \xi_t;
\]

(c) For every period \(t \in \{0, \ldots, T\},\)

\[
\left[ p_t \left( \sum_{i \in I} \varphi^i_t + \xi_t \right) - \kappa_t \sum_{i \in I} \varphi^i_{t-1} \right] = 0.
\]

Given a real number \(\rho > M\) where \(M \equiv (T + 1) \times \max_{i \in I} ||\omega^i|| + ||\xi||\) let

\[
B_T(0, \rho) = \{x \in \mathbb{R}^\infty_+ : x_t \leq \rho \text{ for } t \in \{0, \ldots, T\} \text{ and } x_T = 0 \text{ for } t > T\}.
\]

Consider subsequently the following sets

\[
\tilde{C}^i, T = C^i, T \cap B_T(0, \rho), \quad \tilde{\Phi}^i, T = \Phi^i, T \cap B_T(0, \rho).
\]

Given \(c^i \in \tilde{C}^i, T\) we denote by \(P^i(c^i)\) the set of bundles strictly preferred to \(c^i\) by agent \(i\), i.e.,

\[
P^i(c^i) = \{c' \in \tilde{C}^i, T : c' \succ c^i\},
\]

where \(\succ\) is the preference relation on \(\tilde{C}^i, T\) represented by the utility function \(U^i : \tilde{C}^i, T \to [0, \infty]\).

We next prove existence of an equilibrium for the bounded economy

\[
\hat{E}^T(\rho) = (\tilde{C}^i, T, \tilde{\Phi}^i, T, U^i, \beta_i, \omega^i, \varphi^i_{-1})_{i \in I}.
\]

**Step 2: Existence of equilibrium in the bounded economy \(\hat{E}^T(\rho)\).**

**Proposition 1.5.1.** For any \(\rho > M\) the economy \(\hat{E}^T(\rho)\) has an equilibrium \((p, \kappa, (c^i, \varphi^i)_{i \in I})\)
such that \( p_t > 0, \kappa_t > 0, c_i^t > 0, \sum_{i \in I} \varphi_{i-1}^t > 0 \) for \( t = 0, \ldots, T \) and \( \varphi_T^t = 0 \).

**Proof.** Let

\[
\tilde{\Pi}^T = \{(p, \kappa) \in \Pi^T : \forall t \in [0, \ldots, T], \ |p_t| \leq 1, \ \kappa_t \in [0, 1]\}.
\]

For every \( i \), we introduce the “modified” budget correspondences defined on \( \tilde{\Pi}^T \) as follows

\[
B_{\gamma}^{i,T}(p, \kappa) = \{(c^i, \varphi^i) \in \tilde{C}^{i,T} \times \tilde{\Phi}^{i,T} : \forall t \in [0, \ldots, T], \ p_t c_i^t + p_t \varphi_i^t \leq p_t \omega_i^t + \kappa_t \varphi_{i-1}^t + \gamma_t(p, \kappa)\}
\]

\[
B_{\gamma}^{i,T}(p, \kappa) = \{(c^i, \varphi^i) \in \tilde{C}^{i,T} \times \tilde{\Phi}^{i,T} : \forall t \in [0, \ldots, T], \ p_t c_i^t + p_t \varphi_i^t < p_t \omega_i^t + \kappa_t \varphi_{i-1}^t + \gamma_t(p, \kappa)\},
\]

where \( \gamma_t(p, \kappa) = 1 - \min\{1, |p_t| + \kappa_t\} \).

The proof follows from a series of claims.

**Claim 1.5.1.** For every \( i \in I \), for every \( (p, \kappa) \in \tilde{\Pi}^T \), \( B_{\gamma}^{i,T}(p, \kappa) \) is nonempty and convex. Its closure is \( B_{\gamma}^{i,T}(p, \kappa) \).

**Proof.** It is straightforward. \( \square \)

We subsequently introduce an additional agent by setting the following reaction correspondences defined on \( \tilde{\Pi}^T \times \prod_{i \in I}(\tilde{C}^{i,T} \times \tilde{\Phi}^{i,T}) \)

\[
\psi^i(p, \kappa, c, \varphi) = \begin{cases} 
B_{\gamma}^{i,T}(p, \kappa) & \text{if } (c^i, \varphi^i) \notin B_{\gamma}^{i,T}(p, \kappa) \\
B_{\gamma}^{i,T}(p, \kappa) \cap (P^i(c^i) \times \tilde{\Phi}^{i,T}) & \text{if } (c^i, \varphi^i) \in B_{\gamma}^{i,T}(p, \kappa)
\end{cases}
\]

\[
\psi^0(p, \kappa, c, \varphi) = \left\{ (p', \kappa') \in \Pi^T : \sum_t (p_t' - p_t) \left[ \sum_{i \in I} (c_i^t - \omega_i^t) - \xi_t \right] + \sum_t (\kappa_t' - \kappa_t) \left[ p_t \left( \sum_{i \in I} \varphi_i^t + \xi_t \right) - \kappa_t \sum_{i \in I} \varphi_{i-1}^t \right] > 0 \right\}.
\]

**Claim 1.5.2.** For every \( i \in I \cup \{0\} \), \( \psi^i \) is a convex mapping that has an open graph in \( \tilde{\Pi}^T \times \prod_{i \in I}(\tilde{C}^{i,T} \times \tilde{\Phi}^{i,T}) \).

**Proof.** It is straightforward. \( \square \)

From Gale and Mas-Colell (1975, 1979) fixed point theorem there exists \((\tilde{p}, \tilde{\kappa}, (\tilde{c}^i, \tilde{\varphi}^i)_{i \in I})\) such that

\[
\forall i \in I, \ (\tilde{c}^i, \tilde{\varphi}^i) \in B_{\gamma}^{i,T}(\tilde{p}, \tilde{\kappa}); \quad (1.5.4)
\]

\[
\forall i \in I, \ (c^i, \varphi^i) \in B_{\gamma}^{i,T}(\tilde{p}, \tilde{\varphi}^i) \Rightarrow \sum_t \beta_t^i u_i(c_i^t) \leq \sum_t \beta_t^i u_i(\tilde{c}_i^t); \quad (1.5.5)
\]

\(^8\)Equilibrium prices \((p, \kappa)\) as well as equilibrium allocations \((c, \varphi)\) depend on the choice of \( \rho \).

For notational convenience we do not make at this point this dependence explicit.
\[
\forall (p, \kappa) \in \tilde{\Pi}^T, \quad \sum_t (p_t - \tilde{p}_t) \left[ \sum_{i \in I} (\tilde{c}_t^i - \omega_t^i) - \xi_t \right]
\]
\[+ \sum_t (\kappa_t - \tilde{\kappa}_t) \left[ \tilde{p}_t \left( \sum_{i \in I} \varphi_t^i + \xi_t \right) - \tilde{\kappa}_t \sum_{i \in I} \varphi_{t-1}^i \right] \leq 0. \quad (1.5.6)
\]

(1.5.4) and (1.5.5) imply that at equilibrium every agent \(i\) chooses an allocation inside her/his budget set and that there does not exist any other allocation inside the budget set that yields a higher utility. (1.5.6) implies that at equilibrium the reaction correspondence of the additional agent is empty, i.e. \(\psi^+(\tilde{p}, \tilde{\kappa}, \bar{c}, \varphi) = \emptyset\).

**Claim 1.5.3.** Each \((\bar{c}^i, \varphi^i)\) is optimal in \(B^i_T(\tilde{p}, \tilde{q})\), i.e.,
\[
\forall i \in I, \quad (\bar{c}^i, \varphi^i) \in B^i_T(\tilde{p}, \tilde{\kappa}) \Rightarrow \sum_t \beta^i_t u_t(\bar{c}_t^i) \leq \sum_t \beta^i_t u_t(\bar{c}_t^i).
\]

**Proof.** It is straightforward \(\square\)

**Claim 1.5.4.** For all \(t \in \{0, \ldots T\}\)
\[\begin{align*}
\bullet & \sum_{i \in I} (\bar{c}_t^i - \omega_t^i) - \xi_t = 0; \\
\bullet & [\tilde{p}_t (\sum_{i \in I} \bar{\varphi}_t^i + \xi_t) - \tilde{\kappa}_t \sum_{i \in I} \bar{\varphi}_{t-1}^i] = 0; \\
\bullet & \tilde{p}_t > 0, \tilde{\kappa}_t > 0, \gamma(\tilde{p}_t, \tilde{\kappa}_t) = 0 \text{ and } \sum_{i \in I} \bar{\varphi}_{t-1}^i > 0.
\end{align*}\]

**Proof.** It follows from (1.5.6) that \(\forall (p, \kappa) \in \tilde{\Pi}^T, \forall t \in \{0, \ldots T\}\)
\[
\tilde{p}_t \left[ \sum_{i \in I} (\bar{c}_t^i - \omega_t^i) - \xi_t \right] \leq \tilde{p}_t \left[ \sum_{i \in I} (\bar{c}_t^i - \omega_t^i) - \xi_t \right] \quad (1.5.7)
\]
and
\[
\tilde{\kappa}_t \left[ \tilde{p}_t \left( \sum_{i \in I} \bar{\varphi}_t^i + \xi_t \right) - \tilde{\kappa}_t \sum_{i \in I} \bar{\varphi}_{t-1}^i \right] \leq \tilde{\kappa}_t \left[ \tilde{p}_t \left( \sum_{i \in I} \bar{\varphi}_t^i + \xi_t \right) - \tilde{\kappa}_t \sum_{i \in I} \bar{\varphi}_{t-1}^i \right]. \quad (1.5.8)
\]

If \(\sum_{i \in I} (\bar{c}_t^i - \omega_t^i) - \xi_t \neq 0\), from (1.5.7), \(|\tilde{p}_t| = 1, \gamma(\tilde{p}_t, \tilde{\kappa}_t) = 0\) and
\[
\tilde{p}_t \left[ \sum_{i \in I} (\bar{c}_t^i - \omega_t^i) - \xi_t \right] > 0. \quad (1.5.9)
\]

If \(\tilde{p}_t = -1\), from Claim 3.3, \(\forall i \in I, \bar{c}_t^i = \rho\) and
\[
\sum_{i \in I} (\bar{c}_t^i - \omega_t^i) - \xi_t = T \rho - \sum_{i \in I} \omega_t^i - \xi_t > 0,
\]
which gives a contradiction to (1.5.9). Hence, it must be that \(\tilde{p}_t = 1\).
If $\bar{p}_t(\sum_{i \in I} \bar{\varphi}^i_t + \xi_t) - \bar{\kappa}_t \sum_{i \in I} \bar{\varphi}^i_{t-1} < 0$, from (1.5.8), $\bar{\kappa}_t = 0$ and
\[
\bar{p}_t \left( \sum_{i \in I} \bar{\varphi}^i_t + \xi_t \right) < 0
\]
which is a contradiction. If $\bar{p}_t(\sum_{i \in I} \bar{\varphi}^i_t + \xi_t) - \bar{\kappa}_t \sum_{i \in I} \bar{\varphi}^i_{t-1} \geq 0$, from the “modified” budget constraints and (1.5.9) we have
\[
0 < \bar{p}_t \left[ \sum_{i \in I} (\bar{c}^i_t - \omega^i_t) - \xi_t \right] + \bar{p}_t(\sum_{i \in I} \bar{\varphi}^i_t + \xi_t) - \bar{\kappa}_t \sum_{i \in I} \bar{\varphi}^i_{t-1} \leq I \gamma(\bar{p}_t, \bar{\kappa}_t) = 0
\]
which is also a contradiction. Hence, we get that
\[
\forall t \in 0, \ldots, T, \quad \sum_{i \in I} (\bar{c}^i_t - \omega^i_t) - \xi_t = 0. \tag{1.5.10}
\]

If
\[
\bar{p}_t \left( \sum_{i \in I} \bar{\varphi}^i_t + \xi_t \right) - \bar{\kappa}_t \sum_{i \in I} \bar{\varphi}^i_{t-1} < 0,
\]
from (1.5.8), $\bar{\kappa}_t = 0$ and $\bar{p}_t < 0$. But in this case, from Claim 1.5.3, $\forall i \in I, \bar{c}^i_t = \rho$ and (1.5.10) is violated. If it was the case that
\[
\bar{p}_t \left( \sum_{i \in I} \bar{\varphi}^i_t + \xi_t \right) - \bar{\kappa}_t \sum_{i \in I} \bar{\varphi}^i_{t-1} > 0,
\]
from (1.5.8), $\bar{\kappa}_t = 1$ and $\gamma(\bar{p}_t, \bar{\kappa}_t) = 0$. From the “modified” budget constraints and (1.5.10) we have
\[
0 < \bar{p}_t \left[ \sum_{i \in I} (\bar{c}^i_t - \omega^i_t) - \xi_t \right] + \bar{p}_t(\sum_{i \in I} \bar{\varphi}^i_t + \xi_t) - \bar{\kappa}_t \sum_{i \in I} \bar{\varphi}^i_{t-1} \leq I \gamma(\bar{p}_t, \bar{\kappa}_t) = 0,
\]
which is a contradiction. Therefore,
\[
\bar{p}_t \left( \sum_{i \in I} \bar{\varphi}^i_t + \xi_t \right) - \bar{\kappa}_t \sum_{i \in I} \bar{\varphi}^i_{t-1} = 0. \tag{1.5.11}
\]
This implies $\bar{p}_t \geq 0, \forall t \in \{0, \ldots, T\}$. If $\bar{p}_t = 0$, then $\forall i \in I, \bar{c}^i_t = \rho$: a contradiction with (1.5.10). Thus $\bar{p}_t > 0, \forall t \in \{0, \ldots, T\}$. The “modified” budget constraints should hold with equality and summing over $i$ we get that $\gamma(\bar{p}_t, \bar{\kappa}_t) = 0$.

Observe that we have
\[
\bar{p}_0 \left( \sum_{i \in I} \bar{\varphi}^i_0 + \xi_t \right) = \bar{\kappa}_0 \sum_{i \in I} \bar{\varphi}^i_{-1}.
\]

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This implies $\bar{\kappa}_0 > 0$. Similarly for $t \in \{1, \ldots, T\}$ we have

$$\bar{p}_t \xi_t \leq \hat{p}_t \left( \sum_{i \in I} \bar{\varphi}_i^t + \xi_t \right) = \bar{\kappa}_t \sum_{i \in I} \bar{\varphi}_i^{t-1}$$

which implies $\bar{\kappa}_t > 0, \sum_{i \in I} \bar{\varphi}_i^{t-1} > 0$.

Under the imposed assumptions on the instantaneous utility functions $u_i$, it is straightforward to show that $\forall t \in \{0, \ldots, T\}, c^t_i > 0$ and $\bar{\varphi}_T^t = 0$.

The proof is complete.

Remark 1.5.1. We can normalize prices such that $p_t + \kappa_t = 1$ for all $t \in \{0, \ldots, T\}$.

**Step 3:** Existence of equilibrium in the auxiliary economy $\hat{E}^T$.

**Proposition 1.5.2.** The unbounded economy $\hat{E}^T$ has an equilibrium denoted $(\hat{p}, \hat{\kappa}, (\hat{c}^t_i, \hat{\varphi}^t_i)_{i \in I})$.

**Proof.** We know that for any $\rho > M$, the bounded economy $\bar{E}^T(\rho)$ attains an equilibrium $(p(\rho), \kappa(\rho), (c^t(\rho), \varphi^t(\rho))_{i \in I})$. We know that $\forall i \in I, \forall t \in \{0, \ldots, T\}$, $c^t_i(\rho) \leq M < \rho$. If there exists a $\rho > M$ such that at equilibrium we have $\varphi^t_i(\rho) < \rho$ for all $i$, all $t \in \{0, \ldots, T\}$, then it is straightforward to show that $(p(\rho), \kappa(\rho), (c^t(\rho), \varphi^t(\rho))_{i \in I})$ is an equilibrium for the economy $\hat{E}^T$.

Let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence converging to $+\infty$ with $n$. We claim that there exists $\nu$ and an equilibrium $(p(\rho_\nu), \kappa(\rho_\nu), (c^t(\rho_\nu), \varphi^t(\rho_\nu))_{i \in I})$ such that $\varphi^t_i(\rho_\nu) < \rho_\nu$ for all $i \in I$, for all $t \in \{0, \ldots, T\}$.

Assume that it is not true. Then, for any $n$ there exist $i \in I$ and a period $t \in \{0, \ldots, T\}$ such that $\varphi^t_i(\rho_n) \geq \rho_n$. This implies that $\sum_{t=0}^T \sum_i \varphi^t_i(\rho_n) \rightarrow +\infty$ when $n \rightarrow +\infty$. Without loss of generality we can assume that the corresponding sequences of equilibrium allocations and prices $(p(\rho_n), \kappa(\rho_n), (c^t(\rho_n), \varphi(\rho_n))_{i \in I})_{n \in \mathbb{N}}$ converge to some $(\hat{p}, \hat{\kappa}, (\hat{c}^t_i, \hat{\varphi}^t_i)_{i \in I})$.

Given any $n \in \mathbb{N}$, we have the following first order conditions in the corresponding bounded economy: $\forall i \in I, \forall t \in \{0, \ldots, T\}$,

$$\beta^t_i u^t_i(c^t_i(\rho_n)) = \lambda^t_i(\rho_n)p_t(\rho_n);$$

$$-\lambda^t_i(\rho_n)p_t(\rho_n) + \lambda^t_{i+1}(\rho_n)\kappa_{i+1}(\rho_n) + \zeta^t_i(\rho_n) - \eta^t_i(\rho_n) = 0;$$

$$\lambda^t_i(\rho_n) \geq 0, \lambda^t_i(\rho_n)[p_t(\rho_n)c^t_i(\rho_n) + p_t(\rho_n)\varphi^t_i(\rho_n) - p_t(\rho_n)\omega^t_i - \kappa_t(\rho_n)\varphi^t_{i-1}(\rho_n)] = 0;$$

$$\zeta^t_i(\rho_n) \geq 0, \eta^t_i(\rho_n) \geq 0, \zeta^t_i(\rho_n)\varphi^t_i(\rho_n) = 0, \eta^t_i(\rho_n)(\varphi^t_i(\rho_n) - \rho_n) = 0.$$  

For every period $t$, $\lambda^t_i(\rho_n)$ is the multiplier associated to the budget constraint, $\zeta^t_i(\rho_n)$ is the multiplier associated to the constraint $\varphi^t_i(\rho_n) \geq 0$ while $\eta^t_i(\rho_n)$ is the multiplier associated to the constraint $\varphi^t_i(\rho_n) \leq \rho_n$.

---

9Equilibrium prices $(\hat{p}, \hat{\kappa})$ as well as equilibrium allocations $(\hat{c}, \hat{\varphi})$ depend on the choice of the terminal date $T$. At this stage it is not necessary to make this dependence explicit.

10Observe that we cannot exclude that $\varphi^t_i(\rho_n) \rightarrow +\infty$ for some $i \in I$ and some $t \in \{0, \ldots, T\}$. 

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The proof follows from a series of claims and is a bit more complicated than the proof of Theorem 4 in Becker et al. (2012), which deals with the transition to the unbounded economy for the finite horizon case in their model. The reason for this is that in their model they can choose bounds that are large enough, such that the equilibrium lies in the interior of the feasible set. Given this, the authors can construct the following contradiction. They assume that there exists an equilibrium outside the feasible set. By taking a convex combination of the two equilibria, there must exist a second equilibrium inside the feasible set that dominates the original equilibrium inside the feasible set and thus contradicts the fact that it is an equilibrium. In our model, we cannot guarantee that the equilibrium does not lie on the boundary and thus we cannot apply their strategy. Instead, we have to show that by increasing the bounds and essentially going over into an unbounded economy, the equilibrium will converge to a particular point in space which ultimately is inside the feasible set once the bounds are large enough. This equilibrium than clearly constitutes an equilibrium for the unbounded economy as mentioned earlier.

Claim 1.5.5. We cannot have $\forall t \in \{0, \ldots, T\}, \hat{p}_t = 0$.

Proof. The budget constraints (1.5.1) in any bounded economy hold with equality. Therefore,

$$\forall t \in \{0, \ldots, T - 1\}, \quad p_t(p_n)c^t_i(p_n) + p_t(p_n)\varphi^t_i(p_n) = p_t(p_n)\omega^t_i + \kappa_t(p_n)\varphi^t_{t-1}(p_n),$$

(1.5.16)

and

$$p_T(p_n)c^t_T(p_n) = p_T(p_n)\omega^t_T + \kappa_T(p_n)\varphi^t_{T-1}(p_n)$$

If $\hat{p}_T = 0$, then $\hat{c}_T = 1$ and $\hat{\varphi}_{T-1} = 0$. If $\forall t \in \{0, \ldots, T\}, \hat{p}_t = 0$, it follows by backward induction that $\forall i \in I, \varphi^t_{-1} = 0$: a contradiction.

Claim 1.5.6. We have $\hat{p}_T > 0$, $\hat{c}_T > 0$ and $\forall i \in I, \hat{c}_T \geq \omega^t_T, \sum_{i \in I} \hat{\varphi}_{T-1} \in (0, +\infty)$.

Proof. From Claim (1.5.5) there exists $\tau \in \{0, \ldots, T\}$ such that $\hat{p}_\tau > 0$. Let $\tau$ coincide with the larger period $t$ such that $\hat{p}_t > 0$. Using the same argument as for the proof of the previous claim, it follows that $\hat{p}_t = 0$, $\hat{c}_t = 1$ and $\forall i \in I, \hat{\varphi}_{t-1} = 0$ for $t \in \{\tau + 1, \ldots, T\}$. From equation (1.5.16) we conclude that $\forall i \in I, \hat{c}_\tau \geq \omega^\tau_T > \omega^\tau_T - \epsilon > 0$. Using (1.5.12)-(1.5.15), there exists $N$ large enough, such that $\forall i \in I, \forall n > N, \eta^t_i(p_n) = 0$ and

$$\beta^\tau_t u^t_i(\omega^\tau_T - \epsilon) \geq \beta^\tau_t u^t_i(c^\tau_t(p_n)) = \lambda^\tau_t(p_n)p_T(p_n) \geq \lambda^\tau_{\tau+1}(p_n)\kappa_{\tau+1}(p_n).$$

(1.5.17)

We also know that for any period $t \in \{0, \ldots, T\}$, the sequence of equilibrium consumption profiles $(c^t_i(p_n))_{i \in I}$ are uniformly bounded from above by $M$. It follows
that $\forall i \in I, \forall n > N,$
\[
\beta_t^{T+1} u_i^t(M) < \beta_t^{T+1} u_i^t(c_{i+1}(\rho_n)) = \lambda_{i+1}(\rho_n)p_{i+1}(\rho_n).
\] (1.5.18)

If $\hat{p}_{T+1} = 0,$ from (1.5.18), $\lambda_i^t(\rho_n) \rightarrow +\infty$: But since $\kappa_{T+1}(\rho_n) \rightarrow 1$ we get a contradiction to (1.5.17). Therefore, we must have $\tau = T.$ Hence, $\hat{p}_T > 0$ and $\forall i \in I, \hat{\varphi}_T \geq \omega_T^i.$ It follows from (1.5.12) that $\forall i \in I, \hat{\lambda}_T^i < +\infty.$

We know that
\[
p_T(\rho_n)\xi_T = \kappa_T(\rho_n)\sum_{i \in I} \varphi_{i-1}(\rho_n).
\]

If $\hat{\kappa}_T = 0,$ then $\sum_{i \in I} \varphi_{i-1}(\rho_n) \rightarrow +\infty.$ Therefore, there exists $i \in I$ such that $\varphi_i^T(\rho_n) \rightarrow +\infty.$ For this agent, there exists $N$ large enough such that $\forall n > N,$ $\hat{\lambda}_{i-1}^i(\rho_n) = 0$ and
\[
\beta_i^{T-1} u_i^t(M) \leq \beta_i^{T-1} u_i^t(c_{i-1}(\rho_n)) = \lambda_{i-1}(\rho_n)p_{i-1}(\rho_n) \leq \lambda_T^i(\rho_n)\kappa_T(\rho_n).
\]

But this implies that $\lambda_T^i(\rho_n) \rightarrow +\infty$: a contradiction. Hence, $\hat{\kappa}_T > 0$ and $\sum_{i \in I} \hat{\varphi}_{i-1} \in (0, +\infty).$

\textbf{Claim 1.5.7.} For all $t \in \{0, ..., T-1\}, \hat{p}_t > 0, \hat{\kappa}_t > 0$ and $\forall i \in I, \hat{\varphi}_t > 0,$ $\sum_{i \in I} \hat{\varphi}_{i-1} \in (0, +\infty).$

\textbf{Proof.} We prove the claim for $t = T-1.$ If $\hat{p}_{T-1} = 0,$ then $\hat{\kappa}_{T-1} = 1.$ Using the budget constraints (for the sequence of bounded economies) we conclude that $\forall i \in I, \varphi_i^{T-2}(\rho_n) \rightarrow 0.$ That is, for any $\varepsilon > 0,$ there exists $N$ large enough, such that, $\forall n > N,$ $\varphi_i^{T-2}(\rho_n) < \varepsilon.$ Since for any $n > N,$ $p_{T-2}(\rho_n) > 0$ (see Claim 1.5.4), we get that, $c_{T-2}(\rho_n) > \omega_{T-2}^i - \varepsilon.$ It follows that $\forall i \in I,$ there exists $\alpha > 0$ such that $\inf_{\{n : n > N\}} c_{T-2}(\rho_n) > \alpha.$ It follows that $\forall i \in I, \forall n > N,$ $\eta_{T-2}^i(\rho_n) = 0$ and
\[
\beta_i^{T-2} u_i^t(\alpha) \geq \lambda_{T-2}^i(\rho_n)p_{T-2}(\rho_n) \geq \lambda_{T-1}^i(\rho_n)\kappa_{T-1}(\rho_n).
\] (1.5.19)

We also have that $\forall i \in I, \forall n > N$
\[
\beta_i^{T-1} u_i^t(M) < \beta_i^{T-1} u_i^t(c_{i-1}(\rho_n)) = \lambda_{i-1}(\rho_n)p_{i-1}(\rho_n).
\] (1.5.20)

If $\hat{p}_{T-1} = 0,$ from (1.5.20), $\lambda_{T-1}^i(\rho_n) \rightarrow +\infty.$ But since $\kappa_{T-1}(\rho_n) \rightarrow 1$ we get a contradiction to (1.5.19). Therefore, we must have $\hat{p}_{T-1} > 0.$ If there exists $i \in I$ such that $\hat{\varphi}_{i-1} > 0,$ then
\[
\hat{\lambda}_{i-1}^i \hat{p}_{T-1} \leq \hat{\lambda}_T^i \hat{\kappa}_T \leq \hat{\lambda}_T^i < +\infty.
\]

where the last inequality follows from the proof of Claim 1.5.6. From (1.5.12) we get that $\varphi_{i-1} > 0.$ If $\varphi_{i-1} = 0,$ then $\varphi_{i-1} \geq \omega_{T-1} > 0$ (since $\hat{p}_{T-1} > 0$). From (1.5.12)
we also have \( \hat{\lambda}_{t-1} < +\infty \).

The proof of \( \hat{\kappa}_{T-1} > 0 \) and \( \sum_{i \in I} \hat{\varphi}_{i-2} \in (0, +\infty) \) is similar to the one used to show that \( \hat{\kappa}_T > 0 \) and \( \sum_{i \in I} \hat{\varphi}_{T-1} \in (0, +\infty) \).

Working by induction, we can prove the claim for \( t \in \{0, ..., T-2\} \).

Summing up, we have proved that \( \lim_{n \to +\infty} \sum_{t=0}^{T} \sum_{i \in I} \varphi^i_{t}(\rho_n) < +\infty \). We conclude there exists \( \nu \) and an equilibrium \((p(\rho), \kappa(\rho), (c^i(\rho), \varphi^i(\rho))_{i \in I})\) with \( \varphi^i(\rho) < \rho \) for all \( i, \) all \( t \in \{0, ..., T\} \). This equilibrium is an equilibrium of the economy \( \hat{E}^T \).

**Step 4:** From equilibrium in economy \( \hat{E}^T \) to equilibrium with zero wealth constraints.

Let \((\hat{p}, \hat{\kappa}, (\hat{c}^i, \hat{\varphi}^i)_{i \in I})\) denote the equilibrium in the auxiliary economy \( \hat{E}^T \) in which for every \( i \in I \), \( \hat{\varphi}^i_{-1} = \varphi^i_{-1} \). Let \((p, q) \in \Pi^T \) be a sequence of prices and \((c^i, \theta^i)_{i \in I} \in \prod_{i \in I} (\mathbb{C}^i \times \Theta^i) \) where

\[
\forall i \in I, \quad \Theta^{i,T} = \left\{ \theta^i \in \mathbb{R}_+^\infty : \forall t > T, \quad \theta^i_t = 0 \right\},
\]

be a consumption-wealth profile such that \( \forall t \in \{0, ..., T\} \),

\[
p_t = \hat{p}_t, \quad q_t = \hat{p}_t \frac{\hat{p}_{t+1}}{\hat{\kappa}_{t+1}}, \quad c^i_t = \hat{c}^i_t, \quad \theta^i_t = \frac{\hat{\varphi}^i_t \hat{p}_t}{q_t}, \quad \tag{1.5.21}
\]

Fix \( r_0 = \hat{\kappa}_0 - \rho_0 \xi_0 \) and denote

\[
\forall i \in I, \quad \theta^i_{-1} = \left[ \frac{r_0}{\rho_0} + \xi_0 \right] x^i_{-1}.
\]

Consider next the sequence of prices \((r_t)_{t \geq 1} \) defined (implicitly) as follows

\[
\forall t \in \{1, ..., T\}, \quad \sum_{i \in I} \theta^i_{t-1} = \frac{r_t}{p_t} + \xi_t \quad \tag{1.5.22}
\]

and \( r_t = 0 \) for \( t > T \).

For any period \( t \in \{0, ..., T\} \)

\[
\hat{p}_t \hat{c}^i_t + \hat{p}_t \hat{\varphi}^i_t = \hat{p}_t \omega^i_t + \hat{\kappa}_t \hat{\varphi}^i_{t-1},
\]

therefore summing over \( i \) gives that \( r_t \geq 0 \) for all \( t \in \{0, ..., T\} \). Observe that \( \forall i \in I, \)

\[
p_0 \theta^i_{-1} = \hat{\kappa}_0 \varphi^i_{-1} \text{ and }
\]

\[
\forall t \in \{0, ..., T\}, \quad p_t c^i_t + q_t \theta^i_t = p_t \omega^i_t + p_t \theta^i_{t-1}. \quad \tag{1.5.23}
\]

By construction, given the prices \((p, q, r)\), the consumption-wealth profile \((c^i, \theta^i)\) gives the highest utility among the budgetary feasible consumption-wealth profiles.
Moreover, summing (1.5.23) over \(i\) gives

\[
\forall t \in \{0, ..., T\}, \quad \sum_{i \in I} \theta_{t-1}^i = \frac{q_t}{p_t} \sum_{i \in I} \theta_t + \xi_t.
\]

From (1.5.22) we conclude that

\[
\forall t \in \{0, ..., T\}, \quad \frac{r_t}{p_t} = \frac{q_t}{p_t} \left[ \frac{r_{t+1}}{p_{t+1}} + \xi_{t+1} \right].
\]

It follows that the price sequences \((p, q, r)\) together with the consumption-wealth profiles \((c^i, \theta^i)_{i \in I}\) satisfy conditions (a)-(d) of Definition 1.3.1 (for any period \(t \in \{0, ..., T\}\)) and therefore constitute a \(T\)-period equilibrium with zero wealth constraints.

### 1.5.2 Existence in the infinite horizon case

Given any \(T > 0\), let \((\tilde{p}^T, \tilde{\varphi}^T, (\tilde{c}^i, \tilde{\varphi}^i)_{i \in I})\) be an equilibrium of the truncated auxiliary economy \(\tilde{\mathcal{E}}^T\). The sequence \((\tilde{p}^T, \tilde{\varphi}^T, (\tilde{c}^i, \tilde{\varphi}^i)_{i \in I})_T\) belongs to a compact set for the product topology, so it has a convergent subsequence. Assume that such a subsequence converges to some \((\bar{p}, \bar{\varphi}, (\bar{c}^i, \bar{\varphi}^i)_{i \in I})\). We will show that \((\bar{c}^i)_{T}^+\) also converges to some \(\bar{c}^i\) with \(\bar{c}^i < +\infty\). Moreover, \((\bar{p}, \bar{\varphi}, (\bar{c}^i, \bar{\varphi}^i)_{i \in I})\) constitutes an equilibrium for the infinite horizon auxiliary economy \(\tilde{\mathcal{E}}^\infty\).

**Theorem 1.5.1.** \((\bar{p}, \bar{\varphi}, (\bar{c}^i, \bar{\varphi}^i)_{i \in I})\) is an equilibrium for the economy \(\tilde{\mathcal{E}}^\infty\)

**Proof.** Recall that for any \(T > 0\)

(i) \(\forall t \in \{0, ..., T\}, \tilde{p}^T_t > 0, \tilde{\varphi}^T_t > 0\) with \(\tilde{p}^T_T = 1\) and \(\tilde{c}^i_T > 0, \forall i \in I\);

(ii) \(\forall t \in \{0, ..., T-1\}, \sum_{i \in I} \tilde{\varphi}^{i, T}_t > 0\) and \(\sum_{i \in I} \tilde{\varphi}^{i, T}_t = 0\);

(iii) \(\forall t \in \{0, ..., T\}, \sum_{i \in I} \tilde{c}^{i, T}_t = \sum_{i \in I} \omega^i_t + \xi_t \leq \bar{M}, \quad \text{where} \quad \bar{M} = \sum_{i \in I} \sup_t (\omega^i_t + \xi_t);

(iv) \(\forall t \in \{0, ..., T\}, \tilde{p}^T_t \left( \sum_{i \in I} \tilde{\varphi}^{i, T}_t + \tilde{\xi}_t \right) = \tilde{R}^T_t \sum_{i \in I} \tilde{\varphi}^{i, T}_{t-1};\)

(v) \(\forall t \in \{0, ..., T\}, \forall i \in I, \beta^i_t u^i_t(c^{i, T}_t) = \lambda^{i, T}_t \tilde{p}^T_t \tilde{\varphi}^{i, T}_t \) and \(\lambda^{i, T}_{t-1} \tilde{p}^T_{t-1} \tilde{\varphi}^{i, T}_{t-1} \geq \lambda^{i, T}_t \tilde{\varphi}^{i, T}_t;\)

(vi) \(\forall t \in \{0, ..., T\}, \forall i \in I, \lambda^{i, T}_{t-1} \tilde{p}^T_{t-1} \tilde{\varphi}^{i, T}_{t-1} = \lambda^{i, T}_t \tilde{\varphi}^{i, T}_t \).

The proof follows from a series of claims.

**Claim 1.5.8.** For any \(t \in T, \bar{c}^i_t > 0\) and \(\sum_{i \in I} \bar{c}^i_t = \sum_{i \in I} \omega^i_t + \xi_t.\)

**Proof.** Indeed, let \(\alpha = \inf_{t, i} \omega^i_t.\) We have for any \(\tau \geq 0, \) for any \(T > \tau\)
\[
\sum_{t=\tau}^{T} \big( \hat{\lambda}^{i,T}_{t} \hat{\pi}^T_{t} c^{i}_{t} \big) = \sum_{t=\tau}^{T} \big( \hat{\lambda}^{i,T}_{t} \hat{p}^{T}_{t} \omega^{i}_{t} + \hat{\lambda}^{i,T}_{t} \varphi^{i}_{\tau-1} \big) \\
\geq \sum_{t=\tau}^{T} \hat{\lambda}^{i,T}_{t} \hat{p}^{T}_{t} \omega^{i}_{t} \geq \alpha \sum_{t=\tau}^{T} \hat{\lambda}^{i,T}_{t} \hat{p}^{T}_{t}.
\]

Using the concavity of \( u \), we get

\[
\frac{\beta_{t}^{T}}{1 - \beta_{t}^{i}} u_{i}(M) \geq \sum_{t=\tau}^{T} \beta_{t}^{i} (u_{i}(\hat{c}^{i}_{t}) - u_{i}(0)) \\
\geq \sum_{t=\tau}^{T} \beta_{t}^{i} u_{i}^{i}(\hat{c}^{i}_{t}) \hat{c}^{i}_{t} \\
= \sum_{t=\tau}^{T} \hat{\lambda}^{i,T}_{t} \hat{p}^{T}_{t} \hat{c}^{i}_{t} \geq \alpha \sum_{t=\tau}^{T} \hat{\lambda}^{i,T}_{t} \hat{p}^{T}_{t}.
\]

Given a \( T > 0 \) we define the sequence \( \hat{\pi}^{i,T}_{t} \) as follows

\[
\hat{\pi}^{i,T}_{t} = \begin{cases} 
\hat{\lambda}^{i,T}_{t} \hat{p}^{T}_{t} & \text{if } t \leq T \\
0 & \text{otherwise }
\end{cases}
\]

It follows that for any \( T > 0 \)

\[
\sum_{t=0}^{\infty} \hat{\pi}^{i,T}_{t} \leq \frac{u^{i}(M)}{1 - \beta_{t}^{i}} \alpha
\]

and that for any \( \varepsilon > 0 \) there exists \( \tau \) such that for any \( s \geq \tau \), any \( T \)

\[
\sum_{t=s}^{\infty} \hat{\pi}^{i,T}_{t} \leq \varepsilon.
\]

The sequence \( (\hat{\pi}^{i,T})_{T} \) is in a \( \sigma(\ell_{1}, \ell_{\infty}) \)-compact set of \( \ell_{1} \), where \( \sigma(\ell_{1}, \ell_{\infty}) \) is the weak topology on \( \ell_{1} \) and its dual \( \ell_{\infty} \). The weak topology is a polar topology containing the finest open sets. Given the definition of a dual space, \( \ell_{\infty} \) is the collection of all linear functionals of the type \( \varphi : \ell_{1} \to \mathbb{R} \). The weak topology is then the finest topology, i.e. the topology containing the least open sets, that makes all these linear functionals continuous. This implies that a sequence that converges in \( \ell_{1} \), will also converge in \( \ell_{\infty} \) with respect the weak topology \( \sigma(\ell_{1}, \ell_{\infty}) \). We can therefore suppose that for any \( i \in I, \hat{\pi}^{i,T} \to \pi^{i} \in \ell_{1} \) for \( \sigma(\ell_{1}, \ell_{\infty}) \) and therefore that for any \( i \in I, \hat{\pi}^{i,T} \to \pi^{i} \in \ell_{\infty} \). In particular, using (v), for any \( i \in I, \) for any \( t \in T \) we have

\[
\beta_{t}^{i} u_{i}(\hat{c}^{i}_{t}) \to \beta_{t}^{i} u_{i}(\hat{c}^{i}) < +\infty.
\]

\( ^{11} \)This follows from Dunford-Pettis Property, see Chapter 8 in Aliprantis and Border (1999).
This implies $\tau^i_t > 0$. Given any $T > 0$ we have
\[
\forall t \in \{0, ..., T\}, \sum_{i \in I} \hat{c}^iT_t = \sum_{i \in I} \omega^i_t + \xi_t,
\]
in which case we get
\[
\forall t \in T, \sum_{i \in I} \tau^i_t = \sum_{i \in I} \omega^i_t + \xi_t.
\]

Claim 1.5.9. For any $t \in T$, $p_t > 0$, $\kappa_t > 0$, $\varphi^i_t < +\infty$ and $\sum_{i \in I} \varphi^i_t > 0$.

Proof. We prove the claim by induction. Consider the period $t = 0$. If $\hat{p}^T_0 \to 0$ as $T \to +\infty$, then $\hat{\varphi}^i_0 \to +\infty$. Recall that $W = \max_{i \in I} ||\omega^i|| + ||\xi||$ and
\[
\sum_{t=1}^T \beta^i_t u_t(W) \geq \sum_{t=1}^T \beta^i_t (u_t(\hat{c}^iT_t) - u_t(0))
\geq \sum_{t=1}^T \hat{\lambda}^i_t \hat{p}^T_t \varphi^i_t
= \sum_{t=1}^T \hat{\lambda}^i_t \hat{p}^T_t \omega^i_t + \hat{\lambda}^i_t \hat{\kappa}^T_t \varphi^i_0
\geq \hat{\lambda}^i_0 \hat{\kappa}^T_0 \varphi^i_0 = \hat{\lambda}^i_0 \hat{p}^T_0 \varphi^i_0.
\]
Since $\hat{\lambda}^i_0 \hat{p}^T_0 = u^i_t(\hat{c}^iT_0) \to u^i_t(\tau^i_0) \in (0, +\infty)$, $\varphi^i_0$ is uniformly bounded from above yielding a contradiction. Hence, $\hat{p}^T_0 > 0$. This implies $\varphi^i_T \to \varphi^i < +\infty$. From (iv) we have
\[
\hat{p}^T_0 \xi_t \leq \hat{\kappa}^T_0 \sum_{i \in I} \varphi^i_{t-1}
\]
in which case we obtain $\hat{\kappa}_0 > 0$.

Now consider the period $t = 1$. If $\hat{p}_1 = 0$, from (ii) and (v), $\hat{\kappa}^T_1 \to 1$ and $\hat{\lambda}^i_1 \to +\infty$. This gives a contradiction since from (v) we have
\[
+\infty > u^i_t(\tau^i_0) \geq \lim_{T} \hat{\lambda}^i_T \hat{\kappa}^T_1 = +\infty.
\]
Hence $\hat{p}_1 > 0$.

If $\hat{\kappa}_1 = 0$, from (iv), $\sum_{i \in I} \hat{\varphi}^i_0 \to +\infty$. This gives a contradiction since for any $i \in I$, $\hat{\varphi}^i_0 \to \varphi^i_0 < +\infty$. Hence, $\hat{\kappa}_1 > 0$. Since $\hat{p}_1 > 0$, we have $\varphi^i_1 \to \varphi^i_1 < +\infty$. Moreover, from (iv)
\[
\hat{p}^T_1 \xi_t \leq \hat{\kappa}^T_1 \sum_{i \in I} \hat{\varphi}^i_0.
\]

\[\text{12This is because}\]
\[
\hat{p}_0 \tau^i_0 + \hat{p}_0 \varphi^i_0 = \hat{p}_0 \varphi^i_0 + \kappa_0 \varphi^i_{-1}.
\]
\[\text{with } \kappa_0 = 1 \text{ and } \varphi^i_{-1} > 0.\]
Taking the limit as $T \to +\infty$ implies that $\sum_{i \in I} \bar{\varphi}_0 > 0$.

By induction, it follows that for any $t \in T$, $p_t > 0$, $\bar{\varphi}_t > 0$, $\bar{\varphi}_t^i < +\infty$, $\forall i \in I$ and $\sum_{i \in I} \bar{\varphi}_t^i > 0$.

Claim 1.5.10. For any $i \in I$, $(\bar{c}_t^i, \bar{\varphi}_t^i)$ is agent $i$’s optimal choice given the prices $(\bar{p}, \bar{\varphi})$.

Proof. First, observe that for any $i \in I$, for any $t \in T$ we have

$$\hat{\lambda}_t^i \to \lambda_t^i = \frac{\beta_t^i u_i(\bar{c}_t^i)}{\bar{p}_t}.$$ 

Moreover, from (v) and (vi)

$$\bar{\lambda}_t^i \bar{p}_t \geq \bar{\lambda}_{t+1}^i \bar{\varphi}_{t+1}^i$$

$$\bar{\lambda}_t^i \bar{p}_t \bar{\varphi}_t^i = \bar{\lambda}_{t+1}^i \bar{\varphi}_{t+1}^i.$$ 

Observe also that for any $\varepsilon > 0$ there exists a period $\tau$ such that for any $t \geq \tau$ we have

$$\forall T > t, \sum_{s=t}^{T} \beta^*_s u_i(W) \leq \varepsilon.$$ 

Since

$$\sum_{s=t}^{T} \beta^*_s u_i(W) \geq \sum_{s=t}^{T} \beta^*_s [u_i(\bar{c}_t^s) - u_i(0)]$$

$$\geq \sum_{s=t}^{T} \hat{\lambda}_s^i \bar{c}_s^i \bar{p}_s^i$$

$$= \sum_{s=t}^{T} \hat{\lambda}_s^i \bar{\varphi}_s^i \bar{p}_s^i \bar{\varphi}_s^i + \hat{\lambda}_t^i \bar{\varphi}_t^i \bar{\varphi}_{t-1}$$

$$\geq \hat{\lambda}_t^i \bar{\varphi}_t^i \bar{\varphi}_{t-1} = \hat{\lambda}_{t-1}^i \bar{p}_{t-1}^i \bar{\varphi}_{t-1}^i.$$ 

we can conclude that there exists a period $\tau$ such that for any $t \geq \tau$ we have

$$\forall T > t, \hat{\lambda}_t^i \bar{\varphi}_t^i \bar{\varphi}_{t-1} \leq \varepsilon.$$ 

Therefore, given any $\varepsilon > 0$, for $t$ large enough we have

$$\bar{\lambda}_t^i \bar{p}_t \bar{\varphi}_t^i \leq \varepsilon.$$ 

Let $(c^i, \varphi^i) \in \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty$ satisfy

$$\forall t \in T, \bar{p}_t c_t^i + \bar{p}_t \varphi_t^i \leq \bar{p}_t o_t^i + \bar{\varphi}_t \varphi_{t-1}^i.$$

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These constraints are equivalent to

$$\forall t \in \mathcal{T}, \quad \lambda_i \overline{p}_t c^i_t + \lambda_i \overline{p}_t \varphi^i_t \leq \lambda_i \overline{p}_t \omega^i_t + \lambda_i \overline{\kappa}_t \varphi^i_{t-1}.$$ 

We also have

$$\forall t \in \mathcal{T}, \quad \lambda_i \overline{p}_t c^i_t + \lambda_i \overline{p}_t \varphi^i_t = \lambda_i \overline{p}_t \omega^i_t + \lambda_i \overline{\kappa}_t \varphi^i_{t-1}.$$ 

Let

$$\Delta_{t-1} = \sum_{s=0}^{t-1} \beta^*_i u_i(c^i_s) - \sum_{s=0}^{t-1} \beta^*_i u_i(c^i_s).$$

We have

$$\Delta_{t-1} \geq \sum_{s=0}^{t-1} \beta^*_i u_i(c^i_s) (c^i_s - c^i_s) = \sum_{s=0}^{t-1} \lambda_i \overline{p}_s (c^i_s - c^i_s) \geq -\lambda_i \overline{p}_{t-1} \overline{\varphi}_{t-1}.$$ 

For any \( t \) large enough, we have \( \Delta_{t-1} \geq -\varepsilon \). Hence, for any \( \varepsilon > 0 \)

$$\sum_{s=0}^{\infty} \beta^*_i u_i(c^i_s) \geq \sum_{s=0}^{\infty} \beta^*_i u_i(c^i_s) - \varepsilon.$$ 

Finally, it is easy to check that \( \forall t \in \mathcal{T} \)

$$\sum_{i \in I} c^i_t = \sum_{i \in I} \omega^i_t + \xi_t;$$

$$\overline{p}_t \left( \sum_{i \in I} \overline{\varphi}^i_t + \xi_t \right) = \overline{\kappa}_t \sum_{i \in I} \overline{\varphi}^i_{t-1};$$

$$\sum_{i \in I} \overline{\varphi}^i_{t-1} > 0.$$ 

This concludes the proof of claim 1.5.10. \( \square \)

The proof of theorem follows from claim 1.5.8, 1.5.9 and 1.5.10. \( \square \)

### 1.6 Concluding remarks

One feature that makes General Equilibrium model with collateral constraints appealing is that, similar constraints can be observed in actual markets. As such, they provide a natural and intuitive method of imposing a Transversality Condition and reducing the threat of default. The main idea is to use a natural limit on debts that for example excludes default at equilibrium as in our case, and in addition restricts agents choices in such a way that Ponzi Schemes cannot arise.

We provided an equilibrium existence result for an exchange economy with collateral constraints. This result provides the foundations for similar models that use
collateral requirements in Macroeconomic theory (see Kiyotaki and Moore, 1997; Cordoba and Ripoll, 2004). The bubble example we provided shows that collateral constraints can give rise to bubbles. A particularity of our example is that the asset containing the bubble is not fiat money, but is a physical asset paying positive dividends.

The connection we make between Arrow-Debreu markets and sequential markets is helpful for models that need to rely on the techniques and concepts of Arrow-Debreu theory.

As a natural extension to our model one might consider an economy in which the long-lived asset is replaced by a production factor. In this line of research Becker et al. (2012) have shown that no bubbles can occur in a Ramsey model with heterogeneous discounting, elastic labour and borrowing constraint. Their results rely on the fact that, in Ramsey models with heterogeneous discounting the most patient agent will end up with all the capital as time approaches infinity.

The question then remains whether one would get bubbles in an economy where agents have individual production functions as in Kiyotaki and Moore (1997), since under the Inada conditions agent would always hold some positive amount of the production factor.
CHAPTER 2

Debt sustainability with one-sided exclusion

2.1 Introduction

There exists a vast literature on consumer default that focuses mainly on consumption and asset price fluctuations. In these types of literature the punishment for defaulting can consist in a seizure of a collateral, a complete exclusion from financial markets or partial exclusion from financial markets. The severity of the punishment may also vary in the length of the exclusion. Borrowing and lending takes place by trading a financial asset which is in zero net supply. Kehoe and Levine (1993) show that in a stochastic endowment economy there exists equilibria for sufficiently patient agents in which borrowing occurs despite an option to default. The model they consider is an Arrow-Debreu economy in which punishment for default consist in an complete (two-sided) exclusion from financial markets. Their definition of equilibrium includes a participation constraint that guarantees that no agent who engages in trade has an incentive to default.

The findings of Kehoe and Levine (1993) are in line with the findings of Kocherlakota (1996), who shows that for sufficient patient agent a “Folk Theorem” similar to Kimball (1988) can be achieved leading to an equilibrium identical to the one under full commitment in which agents have no incentive to default.

In a more recent paper Hellwig and Lorenzoni (2009) consider a general equilibrium economy in which agents are excluded permanently from savings after default. They find that equilibria with trading can exist for sufficient low interest rate that guarantee agents willingness to repay their debts.

Azariadis and Kaas (2013) consider a full exclusion from the credit market and show that in these type of stochastic exchange economies the length of the punishment for default affects the efficiency of equilibria. They show that the longer
the punishment, the more likely it is that there exists an equilibrium in which there is trade in the financial asset without default.

A model in which default occurs at equilibrium can be found in Chatterjee et al. (2007). The authors match key economic aggregates of U.S. data by considering a general equilibrium model with default. The punishment consist in a seizure of the households’ assets up to a limit, an exclusion from savings in the defaulting period and a temporary exclusion from borrowing.

In a similar fashion Braido (2008) constructs a model with default at equilibrium. The punishment for default consists in his model in a tightening of borrowing constraints. The novelty of his paper is that agents are not excluded from financial trade, but by defaulting agents will gradually tightening there borrowing constraints.

A separate part of the literature focuses on sovereign debt. In this type of literature the punishment consists mostly of a loss in reputation and an exclusion from future borrowing. The reason for this is that when negotiating with countries, creditors cannot demand a collateral and it is also difficult to exclude a defaulting country from saving. In some of the literature on sovereign debt, default is also followed by an external penalty that can be seen as a retaliation from the creditor countries, which could consists in an imposition of extra tariffs on international trade or cancellation of trade agreements. In a similar type of model Eaton and Gersovitz (1981) show that with an extra penalty on borrowing without collateral and partial exclusion from financial markets, borrowing can be sustained, in contrast to Bulow and Rogoff (1989) where borrowing can never be sustained on the basis of reputation alone. A crucial condition for the no trade result by Bulow and Rogoff (1989) is that countries can continue to participate in financial markets as creditors. Cole and Kehoe (1995) show that in a model in which no side can commit to honouring ones obligations, borrowing can be sustained, since a default can lead to a full exclusion from the financial market. The literature on sovereign debt is tightly related to the one on private debt, since countries are modelled as representative consumer. The issue of unilateral commitment is also highlighted by Hellwig and Lorenzoni (2009). Moreover, since in Bulow and Rogoff (1989) agents can always save at a global interest rate, an equilibrium with low interest rates cannot arise.

Our model is in the spirit of Cole and Kehoe (1995) and Hellwig and Lorenzoni (2009) with respect to the punishment we impose on default. We consider a deterministic General Equilibrium model in which agent start with an endowment of a commodity that can either be used for consumption or can be invested in a productive activity that yields a positive payoff in the next period. The commodity itself cannot be directly stored. Compared to a stochastic endowment economy, the future aggregate endowment of the commodity depends entirely on the production choices of the agents. To simplify our analysis we study an economy with a finite set of consumers that have a common discount factor and can invest in the same
productive activity.

We first consider an economy in which investment is unbounded. By a non-arbitrage argument follows then instantly that if we allow for borrowing and lending the return on savings must be equal to the ones of the productive activity. In an economy with full commitment this implies that we have for each equilibrium without borrowing an infinite amount of equilibria with borrowing. Once we introduce the option to default, all borrowing equilibria disappear, since the lending party can guarantee itself at least the same amount of future consumption through investing and thus has no incentive to lend. Analogously, the borrowing party can guarantee itself the same amount of consumption smoothing through investing and does not suffer disutility from being excluded from financial markets. As a result there cannot exist an equilibrium with borrowing.

In the second part we study the case in which the maximum amount that can be invested is bounded from above. In this version of the model agents that start with a relative high endowment of the commodity might want to lend to other agents in order to carry over more consumption into future periods. This implies that if we consider an economy with full commitment there can exist equilibria where agents lend and borrow to each other, and in which the consumption plans can only be achieved with borrowing. We show that with a partial exclusion constraint agents have always an incentive to default and the autarky equilibrium is the only outcome for these type of economies.

Apart from the non stochastic environment, the model differs from most of the literature with respect to the punishment we impose on default, since we use a partial exclusion constraint as in Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009).

The fact that we are in a non stochastic environment simplifies the analysis significantly, but allows us to focus our interest on efficient resource sharing. Our no trade result implies that uncertainty is a crucial condition for an equilibrium with trade and without default.

2.2 The model

2.2.1 Agents and commodities

There is a single non durable good at every period $t \in T = \{0, 1, 2, \ldots \}$. There is a finite set $I = \{1, 2, \ldots, I \}$ of agents. Agents can use the good for consumption or production. Each agent chooses a consumption sequence $c^i = (c^i_t)_{t \in T}$ where $c^i_t \in \mathbb{R}_+$. The utility function is assumed to be time-additively separable, i.e.

$$U^i (c^i) = \sum_{t=0}^{\infty} \beta^t u^i (c^i_t)$$  \hspace{1cm} (2.2.1)
where \( u^i : \mathbb{R}_+ \to [0, +\infty) \) is the instantaneous utility function at period \( t \) and \( \beta \in (0, 1) \) is the common discount factor. All agents discount future utility at the same rate. We impose the following assumption on the utility functions:

(A.1) \( \forall i, u^i \) is strictly increasing, strictly concave, continuous, \( u^i (0) = 0 \), differentiable with Inada condition

\[
\lim_{c^i_t \to 0} \frac{\partial U^i (c^i_t)}{\partial c^i_t} = +\infty; \quad \forall i \in I, \forall t \in T
\]

The continuation utility in any period \( s \) is denoted as

\[
U^i_s (c^i_t) = \sum_{t=s}^{\infty} \beta^{t-s} u^i (c^i_t)
\]

Agents are equally productive but the productivity varies across time. Let \( A_t \) be the sequence of productivity factors across time where \( A_{t+1} \) is the productivity in period \( t+1 \). By investing an amount \( x^i_{t+1} > 0 \) of the good in period \( t \) an agent will receive an output \( A_{t+1} x^i_{t+1} \) in units of the good in period \( t+1 \). The sequence of investment decisions for agent \( i \) will be denoted by \( x^i_t = (x^i_{t+1})_{t \in T} \), where \( x^i_t \in \mathcal{X} \subseteq \mathbb{R}_+ \). \( \mathcal{X} \) is a convex subset of \( \mathbb{R}_+ \). In the course of this analysis we will consider a setup in which investment choices are only bounded from below, that is \( \mathcal{X} = \mathbb{R}_+ \) and one in which there exists an upper bound on investment, i.e. \( \mathcal{X} \) is closed and convex. In the case where an upper bound exists we will assume it is given by 1 such that \( \mathcal{X} = [0, 1] \subset \mathbb{R}_+ \).

2.2.2 Financial markets

There exists a short-lived security available for trade at each period \( t \) paying one unit of the consumption good at period \( t+1 \). This security can be short sold and is provided in zero net supply, i.e.

\[
\forall t \in T \quad \sum b^i_{t+1} = 0 \quad (2.2.2)
\]

Let \( q_t = (q_{t+1})_{t \in T} \) be the asset price sequence where \( q_{t+1} \in \mathbb{R}_+ \) represents the asset price at period \( t \). Let \( b^i_{t+1} \) denote the net financial position for this security at the end of period \( t \), i.e. the beginning of period \( t+1 \). An agent that short sells this security in period \( t \), that is, if \( b^i_{t+1} < 0 \), will receive \( -q_{t+1} b^i_{t+1} \) units of the good in period \( t \) and promises to repay \( -b^i_{t+1} \) units of the good in period \( t+1 \). An agent that buys the security in period \( t \), i.e. \( b^i_{t+1} > 0 \), will pay \( q_{t+1} b^i_{t+1} \) units of the good at period \( t \) and will receive \( b^i_{t+1} \) units in period \( t+1 \). The security holding sequence for agent \( i \) is denoted by \( b^i = (b^i_{t+1})_{t \in T} \).

The amount of short-lived security an agent can short sale is observable and is subject to an upper bound, i.e. \( b^i_{t+1} \) is subject to a lower bound. The bound is
endogenous and follows from the institutional environment in the economy. The institution cannot force agents to repay the debt fully. If agent \( i \) has been short in period \( t \), she/he should deliver the promise \( b_{i+1}^t \) in period \( t + 1 \). However agent \( i \) may decide to default and choose to deliver a quantity \( d_{i+1}^t = 0 \). Lenders keep track of the borrowers asset position and the provision of credit never exceeds \( A_{t+1}x_{t+1}^i \). That is, each agent \( i \) faces the following borrowing constraint

\[
\forall t \in T \quad b_{i+1}^t \geq -A_{t+1}x_{t+1}^i
\]

This constraint ensures that every borrower can repay in full amount its debt at any point in time.

An agent that defaults will be punished. In this model the punishment consists in a partial exclusion from the financial market. Partial exclusion implies that after a default an agent cannot borrow at any future period but only save. Specifically, if an agent \( i \) defaults in period \( s \), then the borrowing constraint is given by

\[
\forall t \geq s \quad b_{i+1}^t \geq 0
\]

We will impose the following assumptions on \( x_0^i \) and \( b_0^i \):

(A.2) All agents start with a positive net position in the assets, i.e. \( A_0x_0^i + b_0^i > 0 \), \( \forall i \in I \).

(A.3) All agents start without debt, i.e. \( b_0^i = 0 \), \( \forall i \in I \).

Assumption (A.2) implies that agents can afford some positive consumption in the first period without borrowing. Assumption (A.3) is self explanatory.

2.2.3 Budget constraints

Let \( A \) be the space of sequences \( a = (a_t)_{t \in T} \) with

\[
a_t = (c_t, x_{t+1}^i, b_{t+1}^i) \in \mathbb{R}_+ \times A \times \mathbb{R}
\]

Agent \( i \)'s choice \( a^i = (c^i, x^i, b^i) \in A \) must satisfy the following constraints:

- budget constraint

\[
\forall t \in T \quad c^i_t + x_{t+1}^i + q_{t+1}b_{t+1}^i \leq A_t x^i_t + b^i_t \tag{2.2.3}
\]

- borrowing constraint

\[
\forall t \in T \quad b_{t+1}^i \geq -A_{t+1}x_{t+1}^i \tag{2.2.4}
\]
In case that there exists a period $s \in T$ in which the agent defaulted, the borrowing (2.2.4) constraint is replaced by a partial exclusion constraint

$$\forall t \geq s \in T \quad b_{t+1}^i \geq 0$$  \hspace{1cm} (2.2.5)

The partial exclusion constraint implies that an agent is not allowed to short sell the financial security and thus effectively borrow.

### 2.2.4 The equilibrium concept

Denote by $Q$ the set of sequences of prices $q$ satisfying

$$\forall t \in T \quad q_t \in \mathbb{R}_+$$

Given a sequence of prices $q \in Q$ let $B^i(q)$ denote the set of plans $a \in A$ satisfying constraints (2.2.3) and (2.2.4).

**Definition 2.2.1.** A competitive equilibrium for the economy with full commitment

$$E_{fc} = (U^i, \beta, x^i_0, b^i_0)_{i \in I}$$

is a price sequence $q \in Q$ and a family of allocations $(a^i)_{i \in I}$, with $a^i \in A$ such that

(A) for every agent $i$, the plan $a^i$ is optimal among the budget feasible plans, i.e. $a^i$ maximises (2.2.1) and $a^i \in B^i(q)$

(B) the financial market clears at every period, i.e.

$$\forall t \in T \quad \sum_{i \in I} b^i_{t+1} = 0$$

**Lemma 2.2.1.** $q_{t+1} > 0$, $\forall t \in T$.

**Proof.** Assume there exists a period $t$ in which $q_{t+1} = 0$. In this case $b^i_{t+1} < \infty$ is not optimal for any agent $i$. This can be seen by considering an alternative strategy $\hat{b}^i_{t+1} = b^i_{t+1} + \epsilon$, $\epsilon > 0$. The new strategy is feasible since $q_{t+1} = 0$, and by applying this strategy the agent could increase consumption in period $t+1$ by $\epsilon$. Since this holds for all agents $i$, it follows $\sum_{i \in I} b^i_{t+1} > 0$, which contradicts the market clearing condition (2.2.2). $\square$
For an equilibrium price sequence $q$, with $q_{t+1} > 0$, $\forall t$, any agent $i \in I$ solves the following optimisation problem

$$\max_{c^i,b^i} U^i(c^i) = \sum_{t=0}^{\infty} \beta^t u^i(c^i_t)$$

s.t. $\forall t \in T$ : $c^i_t + x^i_{t+1} + q_{t+1}b^i_{t+1} \leq A_t x^i_t + b^i_t$

$\forall t \in T$ : $b^i_{t+1} = -A_{t+1} x^i_{t+1}$

Let $V^i_0(q)$ be the maximum value an agent $i$ can achieve for the associated equilibrium price sequence $q$. The maximum continuation utility for an agent $i$ from period $t$ onwards will be denoted as $V^i_t(q)$.

When allowing for default any agent $i$ that chooses to default in a period $s$, fails to deliver $b^i_s$ and delivers instead $d^i_s = 0$. The agent is forbidden to short sell the financial security and has to choose a continuation consumption sequence. Let $c^{i,s}$, $b^{i,s}$ be the alternative consumption and asset holding choices for agent $i$ after default in period $s$. The continuation utility is given by $U^{i,d}_s(c^{i,s})$. The maximisation problem after default for any agent $i$ is given by

$$\max_{c^{i,s},b^{i,s}} U^{i,d}_s(c^{i,s}) = \sum_{t=s}^{\infty} \beta^{t-s} u^i(c^{i,s}_t)$$

s.t. $t \geq s$ : $c^i_t + x^i_{t+1} + q_{t+1}b^i_{t+1} \leq A_t x^i_t + b^i_t$ ; $b^i_s = 0$

$\forall t \geq s$ : $b^i_{t+1} \geq 0$

Let $V^{i,d}_s(q)$ be the maximum continuation utility an agent $i$ can achieve when defaulting in period $s$ associated with a price vector $q$, where $q$ is an equilibrium price vector for the economy with full commitment.

**Definition 2.2.2.** A competitive equilibrium for the economy without full commitment

$E_{nc} = (U^i, \beta, x^i_0, b^i_0)_{i \in I}$

is a price sequence $q \in Q$ and a family of allocations $(a^i)_{i \in I}$, with $a^i \in A$ such that

(A) for every agent $i$, the plan $a^i$ is optimal among the budget feasible plans, i.e. $a^i$ maximises (2.2.1) and $a^i \in B^i(q)$

(B) the financial market clears at every period, i.e.

$$\forall t \in T : \sum_{i \in I} b^i_{t+1} = 0$$
(C) no agent has an incentive to default at any point in time

\[ V^i_t(q) \geq V^{i,d}_t(q); \quad \forall t \in T; \forall i \in I \]

Throughout the paper we will define the Arrow-Debreu price at time 0 for one unit of consumption at time \( t \) as

\[ p_0 = 1, \quad p_{t+1} = p_t q_{t+1}, \quad \forall t \geq 0 \]

We can think of it as the spot price at time 0 for one unit of consumption at time \( t + 1 \). By a non-arbitrage argument we can find the spot price at time \( s \) for one unit of consumption at time \( t + 1 \) as

\[ p_{t+1}^s = \frac{p_{t+1}}{p_s} \]

where the non-arbitrage condition for spot prices is given by

\[ p_{t+1} = p_s p_{t+1}^s = p_1 p_2 \cdots p_{t+1} \]

\[ = q_1 \cdots q_{t+1} \]

2.2.5 Characterisation of the equilibrium when investment is unbounded from above

In this case \( \mathcal{X} = \mathbb{R}_+ \). The optimisation problem of any agent \( i \) can be characterised by the following Lagrangian

\[ \mathcal{L} (c^i, x^i, b^i) = \sum_{t=0}^{\infty} \beta^t u^i (c^i_t) + \sum_{t=0}^{\infty} \lambda^i_t (A_t x^i_t + b^i_t - q_{t+1} b^i_{t+1} - x^i_{t+1} - c^i_t) + \]

\[ \sum_{t=0}^{\infty} \mu^i_{t+1} x^i_{t+1} + \sum_{t=0}^{\infty} \zeta^i_{t+1} (A_{t+1} x^i_{t+1} + b^i_{t+1}) \]

\((\lambda^i_t)_{t=0}^{\infty}\) is the sequence of Lagrange multipliers associated with the budget constraint. \((\mu^i_{t+1})_{t=0}^{\infty}\) is the sequence of Lagrange multipliers associated with the non-negativity constraint of \( x^i_{t+1} \). \((\zeta^i_{t+1})_{t=0}^{\infty}\) is the sequence of Lagrange multipliers associated with the borrowing constraint. Given the Inada condition and assumption (A.2) it follows
immediately \( c_i^t > 0, \forall t, \forall i \). The FOC are given by

\[
\frac{\partial L (c^t, x^t, b^t)}{\partial c_i^t} = \beta_t \frac{\partial u^i (c_i^t)}{\partial c_i^t} - \lambda_i^t = 0
\]

\[
\frac{\partial L (c^t, x^t, b^t)}{\partial x_i^t} = -\lambda_i^t + \lambda_{i+1}^t A_{i+1} + \mu_{i+1}^t + A_{i+1} \zeta_{i+1}^t = 0
\] (2.2.6)

\[
\frac{\partial L (c^t, x^t, b^t)}{\partial b_i^t} = -\lambda_i^t q_{t+1} + \lambda_{i+1}^t + \zeta_i^t = 0
\] (2.2.7)

\[
\frac{\partial L (c^t, x^t, b^t)}{\partial \lambda_i^t} = A_{i+1} x_i^t + b_i^t \geq 0
\] (2.2.8)

\[
\frac{\partial L (c^t, x^t, b^t)}{\partial \zeta_i^t} = x_{i+1}^t \geq 0;
\] (2.2.9)

\[
\frac{\partial L (c^t, x^t, b^t)}{\partial \mu_i^t} = x_i^t + 1 \geq 0;
\] (2.2.10)

Since the utility function is strictly increasing with respect to consumption, it follows immediately that the budget constraint is binding and \( \lambda_i^t > 0, \forall t, \forall i \).

From the FOC (2.2.7), we have

\[
q_{t+1} \geq \frac{\lambda_{i+1}^t}{\lambda_i^t}, \forall t \geq 0, \forall i
\]

\[
q_{t+1} = \frac{\lambda_{i+1}^t}{\lambda_i^t} \text{ if } \zeta_i^t = 0
\]

\[
q_1 \ldots q_{t+1} = p_{t+1} \geq \frac{\lambda_1^t}{\lambda_0^t} \frac{\lambda_2^t}{\lambda_1^t} \ldots \frac{\lambda_{t+1}^t}{\lambda_t^t} = \frac{\lambda_{t+1}^t}{\lambda_0^t}, \forall t \geq 0, \forall i
\]

To derive the last line, the non-arbitrage condition for spot prices was used.

In the following part of this chapter it will be shown that an economy without full commitment \( E_{nc} \) does not have any equilibrium in which trade occurs. All the results derived from now on apply to economy \( E_{nc} \). Although, some of the results derived below may hold even for an economy with full commitment, we will not mention it since the full commitment case is well known and is only used for comparison reasons at a later stage.

**Lemma 2.2.2.** Let \((a^i_{i \in I}, q)\) be an equilibrium. Then

\[
1 - q_{t+1} A_{t+1} \geq 0, \forall t \geq 0
\]

**Proof.** From the market clearing condition (2.2.2) follows that there must exist in
every period at least one agent, with $b_{t+1} > 0$. Since $c_{t+1} > 0$ it follows that there exists in every period at least one agent for which $A_t x_t^i + b_t^i > 0$. We fix a period $t + 1$ and denote by $j$ the agent with $A_{t+1} x_{t+1}^j + b_{t+1}^j > 0$. This implies $\zeta_{t+1}^j = 0$. From the FOC (2.2.6) follows then

$$q_{t+1} = \frac{\lambda_{t+1}^j}{\lambda_{t}^j} \leq \frac{1}{A_{t+1}}$$

And this concludes the proof.

Lemma 2.2.2 implies that an agent $i$ that borrows $b_{t+1}^i = -1$ units of consumption and invests it in the productive activity will not receive more than what she/he has to repay next period. In an economy with unbounded investment this is equivalent to a non-arbitrage condition, which does not permit an agent to increase its consumption in the next period to an infinite amount. To see this, suppose $b_{t+1}^i < 0$ and let $x_{t+1}^i = -q_{t+1} b_{t+1}^i > 0$. The borrowing constraint (2.2.4) is then given by

$$b_{t+1}^i \geq A_{t+1} q_{t+1} b_{t+1}^i$$

$$1 \leq A_{t+1} q_{t+1}, \quad \text{since } b_{t+1}^i < 0$$

For $1 - A_{t+1} q_{t+1} < 0$, the borrowing constraint is fulfilled for any $b_{t+1}^i < 0$, and since the condition holds with strict inequality we have for $b_{t+1}^i \to -\infty$, $A_{t+1} x_{t+1}^i + b_{t+1}^i = -A_{t+1} q_{t+1} b_{t+1}^i + b_{t+1}^i \to \infty$. Given the budget constraint (2.2.3) we could have then $c_{t+1}^i \to \infty$. This example shows that if Lemma (2.2.2) does not hold, we would have no equilibrium since all agents would prefer to borrow and thus the market clearing condition for the financial asset (2.2.2) would not be fulfilled.

The next Lemma derives a similar non-arbitrage condition for the case where agents saves.

**Lemma 2.2.3.** Let $((a_i^i)_{i \in \mathcal{I}}, q)$ be an equilibrium. Then:

$$A_t x_t^i + b_t^i > 0, \quad \forall t \geq 1, \quad \forall i$$

Moreover, we have

$$1 - q_{t+1} A_{t+1} = 0, \quad \forall t \geq 0$$

and

$$p_{t+1} = \frac{\lambda_{t+1}^i}{\lambda_0^i}, \quad \forall i$$

$$= \frac{1}{\prod_{\tau=1}^{t+1} A_{\tau}}$$
Proof. We first prove

\[ A_{t+1}x_{t+1}^i + b_{t+1}^i = 0 \Rightarrow A_t x_t^i + b_t^i > 0, A_{t+2} x_{t+2}^i + b_{t+2}^i > 0 \]

Indeed, we have in this case \( b_{t+1}^i = -A_{t+1}x_{t+1}^i \). Inserting this equality in the budget constraint at period \( t \) we get

\[
A_t x_t^i + b_t^i = c_t^i + x_{t+1}^i + q_{t+1} \left[ -A_{t+1} x_{t+1}^i \right] \\
= c_t^i + x_{t+1}^i \left[ 1 - q_{t+1} A_{t+1} \right] > c_t^i > 0
\]

where the first inequality follows from Lemma 2.2.2 and from the fact that \( x_{t+1}^i \geq 0 \).

Now, assume \( A_{t+2} x_{t+2}^i + b_{t+2}^i = 0 \), from the result we just obtained above, it should follow \( A_{t+1} x_{t+1}^i + b_{t+1}^i > 0 \), but this contradicts our initial assumption. So it follows \( A_{t+2} x_{t+2}^i + b_{t+2}^i > 0 \) or equivalently \( b_{t+2}^i > -A_{t+2} x_{t+2}^i \).

Inserting this inequality in the budget constraint at period \( t + 1 \) implies

\[
0 = A_{t+1} x_{t+1}^i + b_{t+1}^i = c_{t+1}^i + x_{t+2}^i + q_{t+2} b_{t+2}^i \\
A_{t+1} x_{t+1}^i + b_{t+1}^i > c_{t+1}^i + x_{t+2}^i \left[ 1 - q_{t+2} A_{t+2} \right] \geq c_{t+1} > 0
\]

which is a contradiction. Therefore \( A_{t+1} x_{t+1}^i + b_{t+1}^i > 0, \forall t \geq 0, \forall i \).

As a consequence it follows from the FOC \( c_{t+1}^i = 0, \forall t \geq 0, \forall i \). Thus \( \frac{\lambda_{t+1}^i}{\lambda_t^i} = q_{t+1}, \forall t \geq 0, \forall i \). By summing across the budget constraints we get

\[
A_{t+1} \sum_{i=1}^{I} x_{t+1}^i = \sum_{i=1}^{I} c_{t+1}^i + \sum_{i=1}^{I} x_{t+1}^i \geq \sum_{i=1}^{I} c_{t+1}^i > 0
\]

Thus in every period there must exist at least one agent that invest such that \( A_{t+1} \sum_{i=1}^{I} x_{t+1}^i > 0 \). Let this agent be denoted by \( j \). From the non-negativity constraint for \( x_{t+1}^i \), follows then \( \mu_{t+1}^j = 0 \). Inserting this result in FOC (2.2.6) we get

\[
\frac{\lambda_{t+1}^j}{\lambda_t^j} = \frac{1}{A_{t+1}}
\]

It follows then directly from the definition of \( p_{t+1} \), that

\[
p_{t+1} = \frac{\lambda_{t+1}^j}{\lambda_0^j}, \forall i \\
= \frac{1}{\prod_{r=1}^{t+1} A_r}
\]

This concludes the proof. \( \square \)

From Lemma 2.2.3 follows \( A_t x_t^i + b_t^i > 0, \forall t \geq 1, \forall i \) and \( A_{t+1} = \frac{1}{q_{t+1}}, \forall t \in T \).

The case \( A_{t+1} > \frac{1}{q_{t+1}} \) has been already discussed. For \( A_{t+1} < \frac{1}{q_{t+1}} \) it would follow
that the return on saving is higher than the return on investing in the productive activity. Given that \( A_t x_t^i + b_t^i > 0 \), an agent would invest nothing in the productive activity, if this second non-arbitrage condition does not hold. In such a case we would have \( b_t^i > 0 \), \( \forall i \) and the financial market clearing condition (2.2.2) would not be fulfilled.

Lemma 2.2.2 and 2.2.3 imply that the two assets have the same return at equilibrium and thus the financial assets is redundant. Intuitively, should there be trade in the financial asset, i.e. \( \exists i, \exists t: b_{t+1}^i < 0 \), then once we allow for the option to default, agents should stop trading in the financial asset, because the borrower has always an incentive to default. This is demonstrated by the next proposition.

**Proposition 2.2.1.** If \( ((a^i)_{i \in I}, q) \) is an equilibrium, then \( b_t^i = 0, \forall t \geq 1, \forall i \). This implies the only feasible equilibrium in this economy is a no trade equilibrium.

**Proof.** Consider an allocation \( ((a^i)_{i \in I}, q)_{t \in T} \) in which there exist an agent \( j \) and a period \( s-1 \) such that \( b_s^j < 0 \). We claim that given the option of default there exists an alternative allocation for agent \( j \), \( \tilde{a}^j = (\tilde{c}^j_t, \tilde{x}^j_{t+1}, \tilde{b}^j_{t+1})_{t \in T} \) that is feasible and yields a higher utility, \( U^j(\tilde{c}^j) > U^j(c^j) \). Indeed, consider this alternative consumption, borrowing and investment path for agent \( j \) that is given by

\[
\begin{align*}
\tilde{c}^j_t &= c^j_t; \quad \forall t < s \\
\tilde{c}^j_t &= c^j_t - b^j_t > c^j_t; \quad t = s \\
\tilde{c}^j_t &= c^j_t; \quad \forall t > s \\
\tilde{b}^j_t &= b^j_t; \quad \forall t \leq s \\
\tilde{b}^j_t &= 0; \quad \forall t > s \\
\tilde{x}^j_t &= x^j_t; \quad \forall t \leq s \\
\tilde{x}^j_t &= x^j_t + \frac{1}{A_t} b_t; \quad \forall t > s
\end{align*}
\]

For all periods \( t < s \), the allocation is unchanged, thus it must be shown that the allocation is budget feasible for \( t \geq s \). In period \( s \), given the default decision the agent will deliver \( d = 0 \), the budget constraint is given by

\[
\begin{align*}
\tilde{c}^j_s + \tilde{x}^j_{s+1} + q_{s+1} \tilde{b}^j_{s+1} &\leq A_s \tilde{x}^j_s + d \\
c^j_s - b^j_s + x^j_{s+1} + \frac{1}{A_{s+1}} b_{s+1} &\leq A_s x^j_s \\
c^j_s + x^j_{s+1} + \frac{1}{A_{s+1}} b_{s+1} &\leq A_s x^j_s + b^j_s
\end{align*}
\]

We recall that the constraint in the old allocation was given by

\[
c^j_s + x^j_{s+1} + q_{s+1} b^j_{s+1} \leq A_s x^j_s + b^j_s
\]
From Lemma 2.2.3 follows \( \frac{1}{A_{t+1}} = q_{t+1} \) and that the inequality was fulfilled in the old equilibrium it holds in the new one, too. To determine whether the new allocation is feasible for \( t > s \) we write the budget constraint for \( t > s \) as

\[
\begin{align*}
\tilde{c}_t^j + \tilde{x}_{t+1}^j + q_{t+1} \tilde{b}_{t+1}^j &\leq A_t \tilde{x}_t^j + \tilde{b}_t^j \\
c_t^j + x_{t+1}^j + \frac{1}{A_{t+1}} b_{t+1}^j &\leq A_t \left( x_t^j + \frac{1}{A_t} b_t^j \right) \\
c_t^j + x_{t+1}^j + q_{t+1} b_{t+1}^j &\leq A_t x_t^j + b_t^j
\end{align*}
\]

Again, given that this inequality was satisfied under the old allocation it is satisfied also under the new. Thus the consumption stream is feasible. To check that \( \tilde{c}^j \) is a consumption stream which yields a higher utility, we note that \( \forall t \in T \setminus \{s\} \), \( \tilde{c}_t^j = c_t^j \) and for \( t = s \), \( \tilde{c}_t^j > c_t^j \). Since the utility function is strictly increasing in the consumption arguments it follows \( U^j (\tilde{c}^j) > U^j (c^j) \). From this follows that there exists a period \( s \) in which \( V_{s}^j (q) < V_{s}^{j,d} (q) \) and the proof is concluded.

The case with unbounded investment is not surprising and very intuitive. The economic argument for the no trade result is based on the fact that the financial asset is redundant in an economy with full commitment. This implies we have multiple equilibria. In other words, given any equilibrium where there exists an agent \( i \) that borrows in a period \( t \), we can find an alternative equilibrium without borrowing, in which all consumption sequences of the agents are unchanged. So once we introduce the option to default, an agent can make use of this option without having to sacrifice on future consumption streams, which is what the trading strategy in Proposition 2.2.1 demonstrates.

However, the next section shows that in the case where investments are bounded from above the financial assets is not redundant anymore and the proof of the no trade result is more elaborated.

### 2.2.6 Characterisation of the equilibrium when investment is bounded from above

In this case \( X = [0,1] \). Consider the following problem \( P^i (q) \) which corresponds to agent’s \( i \) optimisation problem

\[
\max_{c^i, x^i, b^i} \sum_{t=0}^{\infty} \beta^t u^i (c_t^i)
\]

s.t.\( \forall t \geq 0 \), \( c_t^i + x_{t+1}^i + q_{t+1} b_{t+1}^i \leq A_t x_t^i + b_t^i \)

\( \forall t \geq 0 \), \( c_t^i \geq 0 \), \( x_{t+1}^i \in [0,1] \), \( b_{t+1}^i \geq -A_{t+1} x_{t+1}^i \)

\( x_0^i \geq 0 \), \( b_0 \) are given
An allocation will be called $P^i(q)$-feasible, if it fulfils the budget constraint, the borrowing constraint and the lower and upper bound on investment. An allocation is called $P^i(q)$-optimal if it is $P^i(q)$-feasible and if there does not exist an allocation that is $P^i(q)$-feasible and yields higher utility.

The Lagrangian associated with $P^i(q)$ is given by

$$\mathcal{L} \left( c^i, x^i, b^i \right) = \sum_{t=0}^{\infty} \beta^t u^i \left( c^i_t \right) + \sum_{t=0}^{\infty} \lambda^i_t \left( A_t x^i_t + b^i_t - q_{t+1} b^i_{t+1} - x^i_{t+1} - c^i_t \right) + \sum_{t=0}^{\infty} \mu^i_{t+1} x^i_{t+1} + \sum_{t=0}^{\infty} \nu^i_{t+1} \left( 1 - x^i_{t+1} \right) + \sum_{t=0}^{\infty} \zeta^i_{t+1} \left( A_{t+1} x^i_{t+1} + b^i_{t+1} \right)$$

$(\lambda^i_t)_{t=0}^{\infty}$ is the sequence of Lagrange multipliers associated with the budget constraint. $(\mu^i_{t+1})_{t=0}^{\infty}$ is the sequence of Lagrange multipliers associated with the non-negativity constraints of $x^i_{t+1}$. $(\nu^i_{t+1})_{t=0}^{\infty}$ is the sequence of Lagrange multipliers associated with the upper bound on $x^i_{t+1}$ and $(\zeta^i_{t+1})_{t=0}^{\infty}$ is the sequence of Lagrange multipliers associated with the borrowing constraint. The FOC are given by

$$\frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial c^i_t} = \beta^i \frac{\partial u^i \left( c^i_t \right)}{\partial c^i_t} - \lambda^i_t = 0$$

$$\frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial x^i_{t+1}} = -\lambda^i_t + \lambda^i_{t+1} A_t + \mu^i_{t+1} - \nu^i_{t+1} + A_{t+1} \zeta^i_{t+1} = 0 \quad (2.2.11)$$

$$\frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial b^i_{t+1}} = -\lambda^i_t q_{t+1} + \lambda^i_{t+1} + \zeta^i_{t+1} = 0 \quad (2.2.12)$$

$$\frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial \lambda^i_{t+1}} = A_t x^i_t + b^i_t - x^i_{t+1} - q_{t+1} b^i_{t+1} - c^i_t > 0; \quad \lambda^i_t \geq 0; \quad \lambda^i_{t+1} \left[ \frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial \lambda^i_{t+1}} \right] = 0$$

$$\frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial s^i_{t+1}} = A_{t+1} x^i_{t+1} + b^i_{t+1} > 0; \quad \zeta^i_{t+1} \geq 0; \quad \zeta^i_{t+1} \left[ \frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial \zeta^i_{t+1}} \right] = 0$$

$$\frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial \mu^i_{t+1}} = x^i_t \geq 0; \quad \mu^i_{t+1} \geq 0; \quad \mu^i_{t+1} \left[ \frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial \mu^i_{t+1}} \right] = 0 \quad (2.2.13)$$

$$\frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial \nu^i_{t+1}} = 1 - x^i_{t+1} \geq 0; \quad \nu^i_{t+1} \geq 0; \quad \nu^i_{t+1} \left[ \frac{\partial \mathcal{L} \left( c^i, x^i, b^i \right)}{\partial \nu^i_{t+1}} \right] = 0 \quad (2.2.14)$$

**Lemma 2.2.4.** Let $\left( (a^i)_{i \in I}, q \right)$ be an equilibrium. It follows $A_{t+1} x^i_{t+1} + b^i_{t+1} > 0 \Rightarrow A_t x^i_t + b^i_t > 0$.

**Proof.** For $A_{t+1} x^i_{t+1} + b^i_{t+1} > 0$ it follows $\zeta^i_{t+1} = 0$ and from FOC (2.2.12) $q_{t+1} = \frac{\lambda^i_{t+1}}{\lambda^i_t}$.

At this stage we will consider different cases.

**Case 1:**

$x^i_{t+1} = 0, \Rightarrow b^i_{t+1} > 0$. In this case the budget constraint in period $t$ can be
written as
\[ c^i_t + q_{t+1}b^i_{t+1} \leq A_t x^i_t + b^i_t \]

Since \( c^i_t > 0 \) and \( q_{t+1}b^i_{t+1} > 0 \) it follows \( A_{t+1}x^i_{t+1} + b^i_{t+1} > 0 \).

**Case 2:**

\[ 0 < x^i_{t+1} < 1, \Rightarrow b^i_{t+1} > -A_{t+1}x^i_{t+1}. \]

In this case it follows from FOC (2.2.13) and (2.2.14), \( \mu^i_{t+1}, \nu^i_{t+1} = 0 \). From FOC (2.2.11) follows \( \frac{\lambda^i_{t+1}}{\lambda^i_t} = \frac{1}{A_{t+1}} = q_{t+1} \). The condition \( A_{t+1}x^i_{t+1} + b^i_{t+1} > 0 \) can then be written
\[ x^i_{t+1} + \frac{1}{A_{t+1}}b^i_{t+1} > 0; \quad \text{since } A_{t+1} > 0 \]
\[ x^i_{t+1} + q_{t+1}b^i_{t+1} > 0 \]

The budget constraint in period \( t \) can then be written as
\[ 0 < c^i_t + x^i_{t+1} + q_{t+1}b^i_{t+1} \leq A_t x^i_t + b^i_t \]

where the first inequality follows from the fact that \( c^i_t > 0 \) and \( x^i_{t+1} + q_{t+1}b^i_{t+1} > 0 \).

**Case 3:**

\[ x^i_{t+1} = 1 \Rightarrow b^i_{t+1} > -A_{t+1}x^i_{t+1}. \]

In this case we have again \( x^i_{t+1} + \frac{1}{A_{t+1}}b^i_{t+1} > 0 \). From FOC (2.2.13) and (2.2.14) follows \( \mu^i_{t+1}, \nu^i_{t+1} \geq 0 \). We can then use FOC (2.2.11) to write
\[ \frac{\lambda^i_{t+1}}{\lambda^i_t} = \frac{\lambda^i_{t+1}}{\lambda^i_t + \nu^i_{t+1}} \leq \frac{\lambda^i_{t+1}}{\lambda^i_t} = q_{t+1}. \]

So it follows
\[ x^i_{t+1} + \frac{\lambda^i_{t+1}}{\lambda^i_t + \nu^i_{t+1}}b^i_{t+1} \leq x^i_{t+1} + \frac{\lambda^i_{t+1}}{\lambda^i_t}b^i_{t+1} \]
\[ 0 < x^i_{t+1} + \frac{1}{A_{t+1}}b^i_{t+1} \leq x^i_{t+1} + q_{t+1}b^i_{t+1} \]

The budget constraint in period \( t \) can then be written as
\[ 0 < c^i_t + x^i_{t+1} + q_{t+1}b^i_{t+1} \leq A_t x^i_t + b^i_t \]

and we can conclude \( A_t x^i_t + b^i_t > 0 \).

Considering all the cases we can conclude that \( A_{t+1}x^i_{t+1} + b^i_{t+1} > 0 \Rightarrow A_t x^i_t + b^i_t > 0. \]

**Corollary 2.2.1.** For any \( i \in \mathcal{I} \) and \( s \in \mathcal{T} \) such that \( A_s x^i_s + b^i_s = 0 \), follows \( A_t x^i_t + b^i_t = 0, \forall t > s \).

**Proof.** We will use a proof by induction. Assume there exists an agent \( i \) and a period \( s \) in which \( A_s x^i_s + b^i_s = 0 \), then if \( A_{s+1} x^i_{s+1} + b^i_{s+1} > 0 \) we have a contradiction to Lemma (2.2.4), so it follows \( A_s x^i_s + b^i_s = 0 \Rightarrow A_{s+1} x^i_{s+1} + b^i_{s+1} = 0. \)

Assume now that for a period \( \tau > s \) we have \( A_\tau x^i_\tau + b^i_\tau = 0 \), then if \( A_{\tau+1} x^i_{\tau+1} + b^i_{\tau+1} > 0 \), and
$b_{i+1} > 0$ we have again a contradiction to Lemma (2.2.4), and so by the induction principle it follows $A_s x_i^t + b_i^t = 0$ \implies $A_s x_i^t + b_i^t, \forall t > s$.

Lemma 2.2.4 implies that an agent that is unconstrained in borrowing in period $t$, must have been unconstrained in all previous periods. Given that in every period $t$ there must exist at least one agent that is unconstrained in borrowing, it follows that there exists at least one agent that will always be unconstrained. Corollary 2.2.1 implies that if the borrowing constraint for one agent binds in period $t$, it will bind in all future periods. We can now define the set of agents whose borrowing constraint will never bind as $I_1 \subseteq I$. As stated above, there exists at least one agent, that will never be borrowing constrained, so we have $I_1 \neq \emptyset$. Let $I_2 = I_1^c = I \setminus I_1$. The set $I_2$ is the set of agent for whom the borrowing constraint will bind at some point in time. Note that, it is possible that $I_1 = I$ and $I_2 = \emptyset$.

For $i \in I_1$ we can show that $\frac{\lambda_{i+1}^t}{\lambda_i^t} = p_{t+1}$.

**Lemma 2.2.5.** For all $i \in I_1$ follows $\frac{\lambda_{i+1}^t}{\lambda_i^t} = p_{t+1}$.

**Proof.** From the definition of $p_{t+1}$ we have

$$p_{t+1} = \prod_{s=1}^{t+1} q_s$$

For each agent $i \in I_1$ we have $\zeta_i^{t+1} = 0, \forall t$. By using the FOC we get

$$q_{t+1} = \frac{\lambda_{i+1}^t}{\lambda_i^t}, \quad \forall t, \forall i \in I_1$$

Inserting this result in the definition of $p_{t+1}$ gives us the desired result

$$p_{t+1} = \frac{\lambda_{i+1}^t}{\lambda_i^t}, \quad \forall t, \forall i \in I_1$$

The next Lemma will show that the present value of savings and returns from the productive activity must go to zero as time approaches infinity.

**Lemma 2.2.6.** Let for an arbitrary agent $i$, $a^i$ be $P^1(q)$-feasible. If there exists $M > 0$, $T$ s.t. for all $t \geq T$, we have $p_t(A_i x_i^t + b_i^t) \geq M$, then $a^i$ is **NOT** $P^1(q)$ optimal.
Proof. Let $0 < \epsilon < M$. Define
\[
\tilde{c}_i^t = c_i^t, \quad \tilde{b}_{i+1}^t = b_{i+1}^t, \quad \forall t \leq T - 1
\]
\[
\tilde{c}_T = c_T^i + \frac{\epsilon}{p_T}, \quad \tilde{b}_{T+1}^i = b_{T+1}^i - \frac{\epsilon}{p_{T+1}}
\]
\[
\tilde{c}_i^t = c_i^t, \quad \tilde{b}_{i+1}^t = b_{i+1}^t - \frac{\epsilon}{p_{i+1}}, \quad \forall t \geq T + 1
\]
\[
\tilde{x}_{i+1}^t = x_{i+1}^t, \quad \forall t \in T
\]
The allocation $a^i$ is $\mathcal{P}^i (q)$ feasible. Indeed, for $t \leq T - 1$:
\[
\tilde{c}_i^t + \tilde{x}_{i+1}^t + q_{i+1} \tilde{b}_{i+1} = \tilde{c}_i^t + x_{i+1}^t + q_{i+1} \tilde{b}_{i+1}
\]
\[
= A_t x_i^t + b_T^i
\]
\[
= A_t \tilde{x}_i^t + \tilde{b}_T^i
\]
For $t = T$:
\[
\tilde{c}_T + \tilde{x}_{T+1}^t + q_{T+1} \tilde{b}_{T+1} = \left( c_T^i + \frac{\epsilon}{p_T} \right) + x_{T+1}^t + q_{T+1} \left( b_{T+1}^i - \frac{\epsilon}{p_{T+1}} \right)
\]
\[
= \left( c_T^i + \frac{\epsilon}{p_T} \right) + x_{T+1}^t + q_{T+1} b_{T+1}^i - \frac{\epsilon}{p_T}
\]
\[
= c_T^i + x_{T+1}^t + q_{T+1} b_{T+1}^i
\]
\[
= A_t x_i^t + b_T^i = A_T \tilde{x}_T^i + \tilde{b}_T^i
\]
For $t > T$:
\[
\tilde{c}_i^t + \tilde{x}_{i+1}^t + q_{i+1} \tilde{b}_{i+1}^t = \tilde{c}_i^t + x_{i+1}^t + q_{i+1} \left( b_{i+1}^t - \frac{\epsilon}{p_{i+1}} \right)
\]
\[
= c_i^t + x_{i+1}^t + q_{i+1} b_{i+1}^t - \frac{\epsilon}{p_t}
\]
\[
= A_t x_i^t + b_T^i - \frac{\epsilon}{p_t} \geq \frac{M - \epsilon}{p_t} > 0
\]

\[\square\]

**Corollary 2.2.2.** If $a^i$ is $\mathcal{P}^i (q)$ optimal, then $\liminf_{t \to \infty} p_t (A_t x_i^t + b_T^i) = 0$.

**Proof.** From Lemma 2.2.6 we have
\[
\forall i \in I, \forall M > 0, \forall T, \exists t \geq T, \text{ s.t. } p_t (A_t x_i^t + b_T^i) \leq M
\]
This implies:
\[
\forall i \in I, \forall T, \inf_{t \geq T} p_t (A_t x_i^t + b_T^i) \leq M
\]
and hence
\[
\liminf_{t \to \infty} p_t (A_t x^i_t + b^i_t) = \liminf_{t \to \infty} p_t (A_t x^i_t + b^i_t) \leq M
\]

Let \( M \to 0 \). We get \( \liminf_{t \to \infty} (A_t x^i_t + b^i_t) \leq 0 \). Since \( A_t x^i_t + b^i_t \geq 0 \), we have \( \liminf_{t \to \infty} p_t (A_t x^i_t + b^i_t) = 0 \). \( \square \)

Lemma 2.2.6 and Corollary 2.2.2 are similar to the condition that in a finite period model, agents will consume everything in the last period and invest and save nothing. The difference is that since the model consists of infinite periods, we can not use an endpoint condition, but we must use a limit condition, i.e. a Transversality Condition (TVC).

The next Proposition demonstrates that this Transversality Condition together with the FOC yields the only optimal solutions of the problem \( P^i (q) \).

**Proposition 2.2.2** (TVC as necessary and sufficient condition for optimality). The allocation \( a^i \) is \( P^i (q) \)-optimal, if and only if, it satisfies the FOC and TVC:

\[
\text{(TVC)} \quad \liminf_{t \to \infty} p_t (A_t x^i_t + b^i_t) = 0
\]

**Proof.** Let \( a^i \) satisfy the FOC. After tedious computations (see Appendix 1.) we obtain

\[
\sum_{t=0}^{T} \beta^t \left[ u^i (c^i_t) - u^i (\hat{c}^i_t) \right] \geq - \lambda_{T+1} (A_{T+1} x^i_{T+1} + b^i_{T+1})
\]

where \( \hat{a}^i = (\hat{c}^i_t, \hat{x}^i_{T+1}, \hat{b}^i_{T+1}) \) is \( P^i (q) \)-feasible.

Since \(-\lambda^i q_{T+1} + \lambda^i_{T+1} + \zeta^i_{T+1} = 0 \) we have \( \lambda^i_{T+1} \geq \lambda^i_{t+1} \), for all \( t \geq 0 \). This implies

\[
\begin{align*}
\lambda^i_{T+1} & \geq \lambda^i_{T+1} \\
\lambda^i_{T-1} q_{T+1} & \geq \lambda^i_{T+1} \\
\vdots & \geq \vdots \\
\lambda^i_0 q_1 q_2 \ldots q_{T+1} & \geq \lambda^i_{T+1} \\
\text{or} \quad \lambda^i_{0} p_{T+1} & \geq \lambda^i_{T+1}
\end{align*}
\]

Hence

\[
\sum_{t=0}^{T} \beta^t \left[ u^i (c^i_t) - u^i (\hat{c}^i_t) \right] \geq - \lambda^i_{0} p_{T+1} (A_{T+1} x^i_{T+1} + b^i_{T+1})
\]

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Lemma 2.2.7. Let
\[ \lambda_t \text{ show that at equilibrium, } \lim \inf_{t \to \infty} \] converges to two different limit values. We continue by 
\[ A \] is not optimal.

Corollary 2.2.3. Let \( a^i \) be \( P^i (q) \)-feasible. If there exists \( M > 0 \), \( T \) s.t. for all 
\[ T \geq T, \text{ we have } p_i b_i^T \geq M, \text{ then } a^i \text{ is NOT optimal.} \]
Hence, if \( a^i \) is \( P^i (q) \)-optimal, then \( \lim \inf_{t \to \infty} p_i b_i^T \leq 0 \).

Proof. It is easy since \( p_i b_i^T \geq M \) implies \( p_i (A_t x_i^t + b_i^T) \geq M \). From Lemma 2.2.6 the 
analogue is not optimal.

The proof of Corollary 2.2.3 is straightforward since we have \( A_t x_i^t \geq 0, \forall t, \forall i \).
At this point we do not know if \( \lim \inf_{t \to \infty} p_i b_i^T = \lim \sup_{t \to \infty} p_i b_i^T \), since it could be 
that the borrowing sequence converges to two different limit values. We continue by 
showing that at equilibrium, \( \lim \inf_{t \to \infty} \lambda_i^T (A_t x_i^t + b_i^T) = 0 \).

Lemma 2.2.7. Let \( a^i \) be \( P^i (q) \)-feasible. If there exists \( M > 0 \), \( T \) s.t. for all \( t \geq T, \) 
we have \( \lambda_i^T (A_t x_i^t + b_i^T) \geq M \), then \( a^i \) is NOT \( P^i (q) \)-optimal.

Proof. Let \( 0 < \epsilon < M \). Define
\[ \tilde{c}_i = c_i, \quad \tilde{b}_i^{T+1} = b_i^{T+1}, \quad \forall t < T - 1 \]
\[ \tilde{c}_T = c_T + \frac{\epsilon}{\lambda_T}, \quad \tilde{b}_i^{T+1} = b_i^{T+1} - \frac{\epsilon}{\lambda_T^{T+1}} \]
\[ \tilde{c}_i = c_i, \quad \tilde{b}_i^{T+1} = b_i^{T+1} - \frac{\epsilon}{\lambda_T^{T+1}}, \quad \forall t \geq T + 1 \]
\[ \tilde{x}_i^{T+1} = x_i^{T+1}, \quad \forall t \geq 0. \]
The allocation \( a_i \) is \( P^i (q) \)-feasible. First observe that for \( t \geq T, \) we have \( \zeta_i^T = 0, \) 
since \( A_t x_i^t + b_i^T \geq M > 0. \)
We have, for \( t \leq T - 1 \):

\[
\begin{align*}
\tilde{c}_t^i + \tilde{x}_{t+1}^i + q_{t+1} \tilde{b}_{t+1}^i &= c_t^i + x_{t+1}^i + q_{t+1} b_{t+1}^i \\
&= A_t x_t^i + b_t^i \\
&= A_t \tilde{x}_t^i + \tilde{b}_t^i
\end{align*}
\]

For \( t = T \):

\[
\begin{align*}
\tilde{c}_T^i + \tilde{x}_{T+1}^i + q_{T+1} \tilde{b}_{T+1}^i &= \left( c_T^i + \frac{\epsilon}{\lambda_T^i} \right) + x_{T+1}^i + q_{T+1} \left( b_{T+1}^i - \frac{\epsilon}{\lambda_{T+1}^i} \right) \\
&= \left( c_T^i + \frac{\epsilon}{\lambda_T^i} \right) + x_{T+1}^i + q_{T+1} b_{T+1}^i - \frac{\epsilon}{\lambda_T^i} \\
&= c_T^i + x_{T+1}^i + q_{T+1} b_{T+1}^i \\
&= A_T x_T^i + b_T^i = A_T \tilde{x}_T^i + \tilde{b}_T^i
\end{align*}
\]

For \( t > T \):

\[
\begin{align*}
\tilde{c}_t^i + \tilde{x}_{t+1}^i + q_{t+1} \tilde{b}_{t+1}^i &= c_t^i + x_{t+1}^i + q_{t+1} \left( b_{t+1}^i - \frac{\epsilon}{\lambda_{t+1}^i} \right) \\
&= c_t^i + x_{t+1}^i + q_{t+1} b_{t+1}^i - \frac{\epsilon}{\lambda_t^i} \\
&= A_t x_t^i + b_t^i - \frac{\epsilon}{\lambda_t^i} \geq \frac{M - \epsilon}{\lambda_t^i} > 0.
\end{align*}
\]

Since \( \tilde{c}_t^i = c_t^i, \forall t \neq T \) and \( \tilde{c}_T^i > c_T^i \), the proof is over. \( \square \)

**Corollary 2.2.4.** If \( a^i \) is \( \mathcal{P}^i (q) \)-optimal, then \( \liminf_{t \to \infty} \lambda_t^i (A_t x_t^i + b_t^i) = 0. \)

**Proof.** From Lemma 2.2.7 we have

\[
\forall M > 0, \forall i, \forall T, \exists t \geq T, \text{ s.t. } \lambda_t^i (A_t x_t^i + b_t^i) \leq M
\]

This implies:

\[
\forall T, \inf_{t \geq T} \lambda_t^i (A_t x_t^i + b_t^i) \leq M
\]

and hence

\[
\liminf_{t \to \infty} \lambda_t^i (A_t x_t^i + b_t^i) = \lim_{T \to \infty} \inf_{t \geq T} \lambda_t^i (A_t x_t^i + b_t^i) \leq M
\]

Let \( M \to 0 \). We get \( \liminf_{t \to \infty} \lambda_t^i (A_t x_t^i + b_t^i) \leq 0. \) Since \( A_t x_t^i + b_t^i \geq 0 \), we have

\[
\liminf_{t \to \infty} \lambda_t^i (A_t x_t^i + b_t^i) = 0.
\]

Lemma 2.2.7 and Corollary 2.2.4 imply that the marginal utility of an extra unit of the consumption good at time \( t \) is equal to 0 as \( t \) approaches infinity. Since, we do not know if the sequence \( (\lambda_t^i (A_t x_t^i + b_t^i))_{t \in T} \) approaches a unique limit, the
Proposition 2.2.3 (TVC as necessary and sufficient condition for optimality). The allocation $a^i$ is $\mathcal{P}^i(q)$-optimal if, and only if, it satisfies the FOC and the TVC:

$$(TVC) \quad \liminf_{t \to \infty} \lambda_t^i (A_t x_t^i + b_t^i) = 0$$

Proof. Let $(c_t^i, x_{t+1}^i, b_{t+1}^i)$ satisfy the FOC. After tedious computations (see Appendix 2.5.1) we obtain

$$\sum_{t=0}^{T} \beta^t [u^i(c_t^i) - \hat{c}_t^i] \geq - \lambda_{T+1}^i (A_{T+1} x_{T+1}^i + b_{T+1}^i)$$

where $\hat{a}^i = (\hat{c}_t^i, \hat{x}_{t+1}^i, \hat{b}_{t+1}^i)_{t \in T}$ is $\mathcal{P}^i(q)$-feasible.

Hence

$$\lim_{T \to \infty} \sum_{t=0}^{T} \beta^t [u^i(c_t^i) - u^i(\hat{c}_t^i)] \geq \limsup_{T \to \infty} \{- \lambda_{T+1}^i (A_{T+1} x_{T+1}^i + b_{T+1}^i)\}$$

$$= - \liminf_{T \to \infty} \lambda_{T+1}^i (A_{T+1} x_{T+1}^i + b_{T+1}^i) = 0$$

Conversely, if $a^i$ is $\mathcal{P}^i(q)$-optimal then it satisfies the FOC and from Corollary 2.2.4 the TVC holds.

Note that, so far we can have an allocation $a^i$ which is $\mathcal{P}^i(q)$-optimal even when $\limsup_{t \to \infty} \lambda_t^i (A_t x_t^i + b_t^i) > 0$. The next Corollary shows that for every optimal allocation the marginal utility of one extra unit of savings, has a lower bound equals to 0 as time approaches infinity.

Corollary 2.2.5. Let $a^i$ be $\mathcal{P}^i(q)$-feasible. If there exists $M > 0$, $T$ s.t. for all $t \geq T$, we have $\lambda_t^i b_t^i \geq M$, then $a^i$ is NOT $\mathcal{P}^i(q)$-optimal.

Hence, if $a^i$ is $\mathcal{P}^i(q)$-optimal, then $\liminf_{t \to \infty} \lambda_t^i b_t^i \leq 0$.

Proof. It is easy since $\lambda_t^i b_t^i \geq M$ implies $\lambda_t^i (A_t x_t^i + b_t^i) \geq M$. From Lemma 2.2.6 follows $(c_t^i, x_{t+1}^i, b_{t+1}^i)_{t \in T}$ is not $\mathcal{P}^i(q)$-optimal.

So far we derived a Transversality Condition for initial optimisation problem of the agent in which she/he is allowed to borrow up to $-A_{t+1} x_{t+1}$ in every period $t$. We will now consider an alternative problem in which the agent is only allowed to save. This is interesting in so far that an agent that defaults, will solve this alternative problem from the defaulting period onwards.
We consider Problem $Q^i(q)$:

$$\max_{c^i, x^i, b^i} \sum_{t=0}^{\infty} \beta^t u^i \left( c^i \right)$$

s.t. $\forall t \geq 0$, $c^i_t + x^i_{t+1} + q_{t+1} b^i_{t+1} \leq A_t x^i_t + b^i_t$

$\forall t \geq 0$, $c^i_t \geq 0$, $x^i_{t+1} \in [0, 1]$, $b^i_{t+1} \geq 0$

$x^i_0 \geq 0$, $b^i_0$ are given

An allocation is $Q^i(q)$-feasible if it fulfils the budget constraint, the non-negativity constraint on borrowing, the non-negativity constraint on investment and upper bound on investment. An allocation is $Q^i(q)$-optimal if there does not exist among all $Q^i(q)$-feasible allocations an allocation that yields a higher utility. As for problem $P^i(q)$ we can derive similar limit conditions for the problem $Q^i(q)$.

**Lemma 2.2.8.** Let $a^i$ be $Q^i(q)$-feasible and $b^i_{t+1} \geq 0$, $\forall t \geq 0$. If there exists $M > 0$, $T$ s.t. for all $t \geq T$, we have $p_t b^i_t \geq M$, then $a^i$ is NOT $Q^i(q)$-optimal.

Hence, if $a^i$ is $Q^i(q)$-optimal, then $\lim_{t \to \infty} p_t b^i_t = 0$.

**Proof.** Define

$$\tilde{b}^i_{t+1} = b^i_{t+1}, \quad \forall t \leq T - 1$$

$$\tilde{c}^i_{t+1} = c^i_{t+1}, \quad \forall t \leq T - 1$$

$$\tilde{x}^i_t = x^i_t, \quad \forall t$$

$$\tilde{b}^i_{T+1} = b^i_{T+1} - \frac{\varepsilon}{p_{T+1}} \text{ with } 0 < \varepsilon < M$$

$$\tilde{c}^i_T = c^i_T + \frac{\varepsilon}{p_T}$$

$$\tilde{c}^i_t = c^i_t, \quad t \geq T + 1$$

$$\tilde{b}^i_{t+1} = b^i_{t+1} - \frac{\varepsilon}{p_{t+1}}, \quad \forall t \geq T + 1$$

We have $\tilde{b}^i_t \geq 0$, $\forall t$.

We claim that $\tilde{a}^i = \left( \tilde{c}^i_t, \tilde{x}^i_{t+1}, \tilde{b}^i_{t+1} \right)_{t \in T}$ is $Q^i(q)$-feasible. We have:

For $t \leq T - 1$,

$$\tilde{c}^i_t + \tilde{x}^i_{t+1} + q_{t+1} \tilde{b}^i_{t+1} = c^i_t + x^i_{t+1} + q_{t+1} b^i_{t+1}$$

$$= A_t x^i_t + b^i_t$$

$$= A_t \tilde{x}^i_t + \tilde{b}^i_t$$

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For $t = T$,
\[
\tilde{c}_T + \tilde{x}_{T+1}^i + q_{T+1} \tilde{b}_{T+1}^i = \left( c_T + \frac{\varepsilon}{p_T} \right) + x_{T+1}^i + q_{T+1} \left( b_{T+1}^i - \frac{\varepsilon}{p_{T+1}} \right)
\]
\[
= c_T^i + \frac{\varepsilon}{p_T} + x_{T+1}^i + q_{T+1} b_{T+1}^i - \frac{\varepsilon}{p_T}
\]
\[
= c_T^i + x_{T+1}^i + q_{T+1} b_{T+1}^i
\]
\[
= A_T x_T^i + b_T^i
\]
\[
= A_T \tilde{x}_T^i + \tilde{b}_T^i
\]

For $t \geq T - 1$,
\[
\tilde{c}_t + \tilde{x}_{t+1}^i + q_{t+1} \tilde{b}_{t+1}^i = c_t^i + x_{t+1}^i + q_{t+1} \left( b_{t+1}^i - \frac{\varepsilon}{p_{t+1}} \right)
\]
\[
= c_t^i + x_{t+1}^i + q_{t+1} b_{t+1}^i - \frac{\varepsilon}{p_t}
\]
\[
= A_t x_t^i + \left( b_t^i - \frac{\varepsilon}{p_t} \right)
\]
\[
= A_t \tilde{x}_t^i + \tilde{b}_t^i
\]

Since we have $\tilde{c}_t^i = c_t$, $\forall t \neq T$ and $\tilde{c}_T^i > c_T^i$, the proof is over. \qed

This shows that in the limit the infimum of the present value of savings tends to 0 as time approaches infinity, for problem $Q^i(q)$. Before we compare the two problems we will consider a simple case in which agent $i$ will always default. This simple case consists in a borrowing sequence in which there exits a period $\tau$ in which the present value $p_\tau b_\tau^i < \liminf_{t \to \infty} p_t b_t^i \leq \limsup_{t \to \infty} p_t b_t^i$. Economically, speaking this is the case of a plan in which there exists a period $\tau$ after which an agents starts repaying its debt.

We start by showing that if there exists a period $\tau$ such that $p_\tau b_\tau^i < \liminf_{t \to \infty} p_t b_t^i$, then then we have a case where the present value of the debts starts decreasing after the period $\tau$, i.e. the present value of the savings starts increasing for $b_{t+1}^i < 0$. In other words the agents starts repaying its debts.

**Lemma 2.2.9.** Let $a^i$ be $P^i(q)$-optimal. Assume
\[
p_\tau b_\tau^i < \liminf_{t \to \infty} p_t b_t^i
\]

Then
\[
\exists T, \ s.t. \ \forall t \geq T, p_t b_t^i < p_\tau b_\tau^i
\]

**Proof.** If not, $\forall T, \exists t \geq T, p_t b_t^i \leq p_\tau b_\tau^i$. This implies $\inf_{t \geq T} p_t b_t^i \leq p_\tau b_\tau^i$. Let $T \to \infty$. We get $\liminf_{t \to \infty} p_t b_t^i \leq p_\tau b_\tau^i$: contradiction. \qed
Lemma 2.2.10. Let \( a^i \) be \( P^i(q) \)-optimal. Assume \( p_i b^i_t < \liminf_{t \to \infty} p_t b^i_t \). Then there exists \( s \geq r, t > s, \) s.t. \( p_t b^i_t \geq p_s b^i_s, \) \( \forall t > s. \)

Moreover, \( \liminf_{t \to \infty} p_t b^i_t > p_s b^i_s. \)

Proof. From Lemma 2.2.9, there exists \( T > r \) such that \( \forall t > T, p_t b^i_t > p^*_b. \) We can choose \( T > r. \) Let \( s \) satisfy \( p_s b^i_s = \min \{ p_t b^i_t : r \leq t \leq T \}. \) Then, for \( t > T, \) we have \( p_t b^i_t > p_s b^i_s, \) \( \forall t \leq T, \) we have \( p_t b^i_t \geq p_s b^i_s. \)

It is trivial that \( \liminf_{t \to \infty} p_t b^i_t > p_s b^i_s \) since \( p_t b^i_t \geq p_s b^i_s. \)

Under this circumstances, we can show that an agent will default, since essentially at some point in time the agent is only repaying its debt. The following Proposition demonstrates that by defaulting the agent can increase its utility.

Proposition 2.2.4. Let \( a^i \) be \( P^i(q) \)-optimal. Assume \( p_i b^i_t < \liminf_{t \to \infty} p_t b^i_t. \) Then agent \( i \) will default.

Proof. First remember that \( \liminf_{t \to \infty} p_t b^i_t \leq 0. \) Thus \( b^i_t < 0. \) From Lemma 2.2.10 there exists \( s \geq r, \) s.t. \( p_t b^i_t \geq p_s b^i_s, \) \( \forall t \geq s \) and \( \liminf_{t \to \infty} p_t b^i_t > p_s b^i_s. \) Define for any \( t \geq s 

\[
\tilde{b}^i_t = b^i_t - \frac{p_t b^i_s}{p_t} \\
\tilde{x}^i_{t+1} = x^i_{t+1} \\
\tilde{c}^i_t = c^i_t
\]

We have \( \tilde{b}^i_t = 0 \) and \( \tilde{b}^i_t \geq 0, \forall t \geq s. \)

Consider the program \( Q^i_s(q): \)

\[
\max_{(\tilde{c}^i_t, \tilde{x}^i_{t+1}, \tilde{b}^i_{t+1})} \sum_{t=s}^{\infty} \beta^t u^i (\tilde{c}^i_t) \\
\forall t \geq s, \quad \tilde{c}^i_t + \tilde{x}^i_{t+1} + q_{t} \tilde{b}^i_{t+1} \leq A_s \tilde{x}^i_{t} + \tilde{b}^i_t \\
\forall t \geq s, \quad \tilde{c}^i_t \geq 0, \quad 1 = \tilde{x}^i_{t+1} \geq 0, \quad \tilde{b}^i_{t+1} \geq 0, \quad \tilde{x}^i_{t+1} = x^i_{t+1} \\
\tilde{b}^i_s = 0
\]

The plan \( a^i \) is \( Q^i_s(q) \)-feasible. Indeed:

For \( t = s \)

\[
\tilde{c}^i_s + \tilde{x}^i_{s+1} + q_{s+1} \tilde{b}^i_{s+1} = c^i_s + x^i_{s+1} + q_{s+1} \left( b^i_{s+1} - \frac{p_s b^i_s}{p_{s+1}} \right) \\
= c^i_s + x^i_{s+1} + q_{s+1} b^i_{s+1} - \frac{p_s b^i_s}{p_s} \\
= A_s x^i_s + b^i_s - b^i_s \\
= A_s x^i_s
\]

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For \( t > s \)

\[
\tilde{c}_t^i + \tilde{x}_{t+1}^i + q_{t+1} \tilde{b}_{t+1}^i = c_t^i + x_{t+1}^i + q_{t+1} \left( b_{t+1}^i - \frac{p_s b_t^i}{p_{t+1}} \right) \\
= c_t^i + x_{t+1}^i + q_{t+1} b_{t+1}^i - \frac{p_s b_t^i}{p_{t+1}} \\
= A_t x_t^i + b_t^i - \frac{p_s b_t^i}{p_{t+1}} \\
= A_t \tilde{x}_t^i + \tilde{b}_t^i
\]

However, this plan is not \( Q_s^i(q) \)-optimal since \( \liminf_{t \to \infty} p_t b_t^i = \liminf_{t \to \infty} p_t b_t^i - p_s b_s > 0 \) (see Lemma 2.2.8). Therefore there exists another plan \( \tilde{a}_s^i = (\tilde{c}_t^i, \tilde{x}_{t+1}^i, \tilde{b}_{t+1}^i) \) \( t \geq s \) which is \( Q_s^i(q) \)-optimal. In particular,

\[
\sum_{t=s}^{\infty} \beta^t u^i (\tilde{c}_t^i) > \sum_{t=s}^{\infty} \beta^t u^i (\tilde{c}_t^i) = \sum_{t=s}^{\infty} \beta^t u^i (c_t^i)
\]

This implies that agent \( i \) will default at date \( s \). \( \square \)

We can generalise the result above to any equilibrium allocation in which agents borrow and \( \liminf_{t \to \infty} p_t b_t^i = 0 \), as shown in the next Proposition.

**Proposition 2.2.5.** (i) Let \( ((a_t^i)_{i \in I}, q) \) be an equilibrium. Assume there exists an agent \( i \) such that \((b_t^i)_{t \in T}\) has a subsequence which is non positive, different from zero and satisfies \( \liminf_{t \to \infty} p_t b_t^i = 0 \). In this case agent \( i \) will default.

(ii) Therefore, an equilibrium \((c_t^i, x_{t+1}^i, b_{t+1}^i, q_t)_{t \in T}\) which satisfies \( \liminf_{t \to \infty} p_t b_t^i = 0 \) for any \( i \) is without incentive to default if, and only if, it is no-trade, i.e. \( b_t^i = 0, \forall t \geq 0, \forall i \).

**Proof.** (i) If the non positive sequence is different is different from zero then \( p_r b_r^i < 0 = \liminf_{t \to \infty} p_t b_t^i \) for some \( r \). Apply Proposition 2.2.4.

(ii) Let \( ((a_t^i)_{i \in I}, q) \) be an equilibrium without incentive to default. It follows from (i) that \( b_t^i = 0 \) if \( b_t^i \) is non positive. This property holds for any \( i \). Since \( \sum_i b_t^i = 0 \) if \( b_t^i = 0 \) is non negative. \( \square \)

Note that, in the economy with full commitment it is possible to have an equilibrium in which agents borrow and repay their debts fully, or let the present value of their debts decrease gradually until \( \liminf_{t \to \infty} p_t b_t^i = 0 \). But by Proposition 2.2.5, this equilibrium cannot exist once we introduce the option to default, since there exists at least one agent who will default.

Let \( M_t = (\prod_{s=1}^{t} A_s) (\sum_i x_0^i) \). We add the following assumption:

**A.4** \( \forall i, \sum_{t=0}^{\infty} \beta^t u^i (M_t) < \infty \)

\( M_t \) is the maximum amount of the consumption good than can be achieved by the economy in each period \( t \). In other words, we imagine that each agent \( i \) consumes
0 units in every period, and the whole amount of the consumption good is invested in every period. We assume that even if it were possible to consume this amount in each period, no agent could achieve an infinite amount of utility.

With this additional assumption we are able to show that \( \lim_{t \to \infty} p_{t+1} [A_{t+1}x_{t+1}^i + b_{t+1}] = 0 \). This is done in the following two Lemmas.

**Lemma 2.2.11.** Assume assumptions (A.1), (A.2), (A.3), (A.4) hold. Let \((a^i)_{i \in I}, q \) be an equilibrium. Then \( \forall i, \sum_{t=0}^{\infty} \lambda^i_t c^i_t < \infty \).

**Proof.** We have:

\[
\sum_{t=0}^{\infty} \beta^t u^i (M_t) \geq \sum_{t=0}^{\infty} \beta^t [u^i (c^i_t) - u^i (0)] \\
\geq \sum_{t=0}^{\infty} \beta^t \frac{\partial u^i (c^i_t)}{\partial c^i_t} c^i_t \\
= \sum_{t=0}^{\infty} \lambda^i_t c^i_t
\]

\[\square\]

**Lemma 2.2.12.** Assume assumptions (A.1), (A.2) hold. Let \((a^i)_{i \in I}, q \) be an equilibrium. Then

\( \forall i, \lim_{t \to \infty} p_{t+1} [A_{t+1}x_{t+1}^i + b_{t+1}^i] = 0 \)

**Proof.** We have for all \( i \in I_1 \), for all \( t \):

\[\lambda^i_t c^i_t + \lambda^i_t x^i_{t+1} + \lambda^i_t g_{t+1} b^i_{t+1} = \lambda^i_t A_t [x^i_t + b^i_t]\]

Given that \( g_{t+1} = \frac{1}{A_{t+1}} = \frac{\lambda^i_{t+1}}{\lambda^i_t}, \forall i \in I_1 \), adding the constraints from \( t = 0 \) to \( t = T \) and taking into account the FOC, we get

\[\sum_{t=0}^{T} \lambda^i_t c^i_t + \lambda^i_{T+1} [A_{T+1}x^i_{T+1} + b^i_{T+1}] = \lambda^i_0 (A_0 x^i_0 + b^i_0)\]

Since \( \sum_{t=0}^{\infty} \lambda^i_t c^i_t < \infty \) (Proposition 2.2.5), the limit, when \( T \to \infty \), of \( \lambda^i_{T+1} [A_{T+1}x^i_{T+1} + b^i_{T+1}] \) exists. From Corollary 2.2.4, we obtain

\[\lim_{t \to \infty} \lambda^i_{T+1} [A_{T+1}x^i_{T+1} + b^i_{T+1}] = 0\]

From Lemma 2.2.5 this equality is equivalent to

\( \forall i \in I_1, \lim_{t \to \infty} \lambda_0 p_{t+1} [A_{t+1}x^i_{t+1} + b^i_{t+1}] = 0 \iff \lim_{t \to \infty} p_{t+1} [A_{t+1}x^i_{t+1} + b^i_{t+1}] = 0 \)
For \( i \in \mathcal{I}_2 \) we get the same condition since for all \( i \in \mathcal{I}_2 \), there exists a period \( s \) such that

\[
A_t x^i_t + b^i_t = 0, \quad \forall t > s
\]

So it follows for all \( i \in \mathcal{I} \)

\[
\lim_{t \to \infty} \lambda_{t+1} \left[ A_{t+1} x^i_{t+1} + b^i_{t+1} \right] = 0
\]

We now come to our main result:

**Theorem 2.2.1.** Assume assumptions (A.1), (A.2), (A.3), (A.4) hold. Let \((a^i_{i \in \mathcal{I}}, q)\) be an equilibrium. It is without incentive to default if and only if, it is no-trade.

**Proof.** From Lemma 2.2.12 we have

\[
\lim_{t \to \infty} \left( \lambda^i_{t} x^i_{t+1} + \lambda^i_{t} q_{t+1} b^i_{t+1} \right) = 0, \quad \forall i
\]

or

\[
\lim_{t \to \infty} \left( p_t x^j_{t+1} + p_{t+1} b^j_{t+1} \right) = 0, \quad \forall i
\]

Summing over \( i \) we get:

\[
\lim_{t \to \infty} p_t \left( \sum_{i \in \mathcal{I}} x^i_t \right) = 0 \Rightarrow \liminf_{t \to \infty} p_t x^i_{t+1} = 0, \quad \forall i.
\]

Consider an arbitrary agent \( j \in \mathcal{I} \). Let the subsequence \( \{t_k\}^\infty_{k=1} \) satisfy \( \limsup_{t \to \infty} p_t x^j_t = \lim_{k \to \infty} p_{t_k} x^j_{t_k} \). Since

\[
\forall i, p_{t_k} x^i_{t_k} \leq \frac{M_t}{\prod_{s=1}^{t_k} A_s} \leq \sum_{i \in \mathcal{I}} x^i_0
\]

we can assume, by taking eventually a subsequence, that \( \forall i \neq j, p_{t_k} x^i_{t_k} \to l_i, l_i > -\infty, l_i < +\infty \), i.e. the subsequence converges to a finite limit. Hence \( \sum_{i \in \mathcal{I}} \lim_{k \to \infty} p_{t_k} x^i_{t_k} = 0 \). In particular, \( \limsup_{t \to \infty} p_t x^j_{t+1} = 0 \). Since \( \liminf_{t \to \infty} p_t x^j_{t+1} = 0 \), we actually have \( \lim_{t \to \infty} p_t x^j_{t+1} = 0 \). Since this will hold for any arbitrary agent \( j \) it follows that \( \lim_{t \to \infty} p_t x^i_{t+1} = 0, \forall i \). This implies \( \lim_{t \to \infty} p_{t+1} b^i_{t+1} = 0, \forall i \). We apply Proposition 2.2.5, statement (ii) to conclude the proof.

From the discussion above follows that the only feasible equilibrium is the autarky equilibrium. The intuition for the no trade result of this part is the following. We have shown that the present value of debt must converge to zero for each optimal
trading sequence. An agent that accumulates some debt in a finite period has the option to default, repay the debt or roll it over such that its present value converges to zero. By a non arbitrage argument we must conclude that for the creditor it makes no difference whether the debt is repaid or rolled over, i.e. the discounted sum of interest payments should have the same present value as repaying the debt in full now or any finite repayment schedule. The creditor is strictly worse off in terms of present value for the case the borrower defaults. This implies that for the borrower the present value of defaulting is higher than repaying or rolling over the debt. This present value can be invested in the productive activity and thus improve the future utility of the defaulting agents.

2.3 An example with full commitment

In this section we present an example with full commitment which highlights the benefits of trade when investment is bounded. In this example there are two agents with the same discount factor $\beta^t = 3/4, \forall t \in \mathcal{T}, \forall i \in \{1,2\}$ and the same utility function given

$$U^i(c^i) = \sum_{t=0}^{\infty} \beta^t \ln c^i_t; \forall i \in \{1,2\}$$

The other fundamentals are given by $A_{t+1} = 2, \forall t \in \mathcal{T}$, $x^1_0 = 16/6$, $x^2_0 = 8/6$ and $b^1_0 = 0$, $b^2_0 = 0$. The equilibrium allocation with full commitment is given by

$$c^1_t = \frac{7}{6}; c^2_t = \frac{5}{6}; \forall t \in \mathcal{T}$$
$$x^1_{t+1} = 1; x^2_{t+1} = 1; \forall t \in \mathcal{T}$$
$$b^1_{t+1} = \frac{2}{3}; b^2_{t+1} = -\frac{2}{3}; \forall t \in \mathcal{T}$$
$$q_t = \frac{3}{4}; \forall t \in \mathcal{T}$$

The Lagrange multipliers are given by

$$\lambda^1_t = \beta^t \left( \frac{6}{7} \right); \lambda^2_t = \beta^t \left( \frac{6}{5} \right); \forall t \in \mathcal{T}$$
$$\zeta^1_{t+1} = 0; \zeta^2_{t+1} = 0; \forall t \in \mathcal{T}$$
$$\mu^1_{t+1} = 0; \mu^2_{t+1} = 0; \forall t \in \mathcal{T}$$
$$\nu^1_{t+1} = \beta^t \left( \frac{3}{7} \right); \nu^2_{t+1} = \beta^t \left( \frac{3}{5} \right); \forall t \in \mathcal{T}$$
In this equilibrium with full commitment the individual utilities are

\[ U^1(c^1) = \frac{\ln \left( \frac{7}{6} \right)}{4} \approx 0.62 \]

\[ U^2(c^2) = \frac{\ln \left( \frac{5}{6} \right)}{4} \approx -0.73 \]

To compare this solution with the autarky solution, we state the autarky optimisation problem. In autarky we have \( b^i_t = 0, \forall t \in \mathcal{T}, \forall i \in \{1, 2\} \). The Lagrangian for an individual \( i \) are then given by

\[
\mathcal{L}(c^i, x^i) = \sum_{t=0}^{\infty} \beta^t u^i(c^i_t) + \sum_{t=0}^{\infty} \lambda^i_t (A_t x^i_t - x^i_{t+1} - c^i_t) + \sum_{t=0}^{\infty} \mu^i_{t+1} x^i_t + \sum_{t=0}^{\infty} \nu^i_{t+1} (1 - x^i_{t+1})
\]

The FOC are given by

\[
\frac{\partial \mathcal{L}(c^i, x^i)}{\partial c^i_t} = \beta^t - \lambda^i_t = 0
\]

\[
\frac{\partial \mathcal{L}(c^i, x^i)}{\partial x^i_{t+1}} = -\lambda^i_t + \lambda^i_{t+1} A_t + \mu^i_{t+1} - \nu^i_{t+1} = 0
\]

\[
\frac{\partial \mathcal{L}(c^i, x^i)}{\partial \lambda^i_t} = A_t x^i_t - x^i_{t+1} - c^i_t = 0;
\]

\[
\lambda^i_t \geq 0; \quad \lambda^i_{t+1} \geq 0; \quad \lambda^i_t \left[ \frac{\partial \mathcal{L}(c^i, x^i)}{\lambda^i_t} \right] = 0
\]

\[
\frac{\partial \mathcal{L}(c^i, x^i)}{\partial \mu^i_{t+1}} = x^i_{t+1} \geq 0; \quad \mu^i_{t+1} \geq 0; \quad \mu^i_{t+1} \left[ \frac{\partial \mathcal{L}(c^i, x^i)}{\mu^i_{t+1}} \right] = 0
\]

\[
\frac{\partial \mathcal{L}(c^i, x^i)}{\partial \nu^i_{t+1}} = 1 - x^i_{t+1} \geq 0; \quad \nu^i_{t+1} \geq 0; \quad \nu^i_{t+1} \left[ \frac{\partial \mathcal{L}(c^i, x^i)}{\nu^i_{t+1}} \right] \geq 0
\]

Under the given fundamentals the equilibrium is

\[
c^1_0 = \frac{10}{6}; \quad c^2_0 = \frac{10}{21}
\]

\[
c^1 = 1; \quad c^2 = \frac{5}{7}
\]

\[
c^1_t = 1; \quad c^2_t = 1; \quad \forall t \geq 2
\]

\[
x^1_{t+1} = 1; \quad x^2_{t+1} = \frac{12}{14}
\]

\[
x^1_t = 1; \quad x^2_{t+1} = 1; \quad \forall t \geq 1
\]
The Lagrange multipliers are given by

\[
\begin{align*}
\lambda_0^1 &= \frac{6}{10}, & \lambda_0^2 &= \frac{21}{10}, \\
\lambda_1^1 &= \frac{3}{4}, & \lambda_1^2 &= \frac{21}{20}, \\
\lambda_t^1 \left(\frac{3}{4}\right)^t, & \lambda_t^2 = \left(\frac{3}{4}\right)^t; & \forall t \geq 2 \\
\mu_{t+1}^1 &= 0; & \mu_{t+1}^2 &= 0; & \forall t \in T \\
\nu_1^1 &= \frac{9}{10}, & \nu_1^2 &= 0, \\
\nu_1^2 &= \frac{3}{8}, & \nu_2^2 &= \frac{3}{40}, \\
\nu_{t+1}^1 &= \left(\frac{3}{4}\right)^t \frac{1}{2}, & \nu_{t+1}^2 &= \left(\frac{3}{4}\right)^t \frac{1}{2}; & \forall t \geq 2
\end{align*}
\]

The utilities in the autarky equilibrium are given by

\[
\begin{align*}
U_1^1 (c^1) &= \ln \left(\frac{10}{6}\right) \approx 0.51 \\
U_1^2 (c^2) &= \ln \left(\frac{10}{21}\right) + \ln \left(\frac{5}{7}\right) \approx -1.08
\end{align*}
\]

By comparing the utilities under autarky and full commitment we see that borrowing and lending benefits both agents and the new allocation is Pareto improving. Our no trade results has then important welfare implications for models in which investment is bounded.

### 2.4 Concluding remarks

We have proven a no trade result in the spirit of Bulow and Rogoff (1989) for a General Equilibrium economy, where the future endowment of the consumption good depends on the investment decisions of the agents and does not follow a random process.

Our main finding is that even in an economy with bounded investment in which agents benefit from trade, a partial exclusion constraint is a too weak punishment to guarantee repayment incentives.

The main limitation of our model is the fact that we are considering a model in which agents have the same productivity. A natural extension would be to consider different productivity factors or stochastic productivity factors. The fact that we are in a non stochastic environment diminishes the benefits and thus weakens the punishment we impose, when excluding defaulting agents from future borrowing.

Given the no-trade result, the question arises what motivates borrowing between countries or agents when there is only one-sided exclusion from capital markets. A
frequent explanation offered by the literature is the introduction of a reputation loss in case of default. One can think of this being an exogenous cost or a direct reduction of utility associated with the decision to default. Trade could also be motivated by the presence of uncertainty. The idea is that, in the future there could arise a state where an agent faces an extremely low endowment stream. In such a case it might be possible that saving alone is not enough to insure against such a bad state. For future work, the model could be extended along both lines, and important insights can be gained by observing where the above proof for the no-trade result fails if trade takes place at equilibrium.

Finally we would like to highlight an area where our results might be of interest. As discussed before we expect that uncertainty can improve the equilibrium in the sense agents have more incentives to trade and less incentives to default. This could be of a major interest when considering loans in underdeveloped countries, where the lack of property rights leads to the impossibility of collateral backed loans.
2.5 Appendix Chapter 2

2.5.1 Discussion of TVC in chapter 2

Consider and allocation \((\hat{c}_t, \hat{x}_{t+1}, \hat{b}_{t+1})\)\(_{t=0}^{\infty}\) that is \(P^i(q)\)-feasible and satisfies the FOC. Multiplying each budget constraint (2.2.3) for period \(t\) by \(\lambda^i_t\) and summing up to a period \(T\) we get

\[
\sum_{t=0}^{T} \lambda^i_t \hat{c}_t^i = \sum_{t=0}^{T} \hat{\lambda}^i_t \left( A_t \hat{x}_t + \hat{b}_t \right) - \sum_{t=0}^{T} \hat{\lambda}^i_t \hat{x}_{t+1} - \sum_{t=0}^{T} \hat{\lambda}^i_t \hat{b}_{t+1} \\
= \hat{\lambda}^0_0 \left( A_0 \hat{x}_0 + \hat{b}_0 \right) + \sum_{t=0}^{T-1} \hat{\lambda}^i_{t+1} \left( A_{t+1} \hat{x}_{t+1} + \hat{b}_{t+1} \right) - \sum_{t=0}^{T} \hat{\lambda}^i_t \hat{x}_{t+1} - \sum_{t=0}^{T} \hat{\lambda}^i_t \hat{b}_{t+1}
\]

By replacing \(\hat{\lambda}^i_t \hat{b}_{t+1}\) in the last term with FOC (2.2.12) we can write

\[
\sum_{t=0}^{T} \lambda^i_t \hat{c}_t^i = \hat{\lambda}^0_0 \left( A_0 \hat{x}_0 + \hat{b}_0 \right) + \sum_{t=0}^{T-1} \hat{\lambda}^i_{t+1} \left( A_{t+1} \hat{x}_{t+1} + \hat{b}_{t+1} \right) - \sum_{t=0}^{T} \hat{\lambda}^i_t \hat{x}_{t+1} - \sum_{t=0}^{T} \left( \hat{\lambda}^i_t + \hat{\zeta}^i_{t+1} \right) \hat{b}_{t+1}
\]

By replacing \(\hat{\lambda}^i_t\) in the third term with FOC (2.2.11) we can write

\[
\sum_{t=0}^{T} \lambda^i_t \hat{c}_t^i = \hat{\lambda}^0_0 \left( A_0 \hat{x}_0 + \hat{b}_0 \right) + \sum_{t=0}^{T-1} \hat{\lambda}^i_{t+1} A_{t+1} \hat{x}_{t+1} - \\
\sum_{t=0}^{T} \left( \hat{\lambda}^i_{t+1} A_{t+1} + \hat{\mu}_{t+1} - \hat{\nu}_{t+1} + \hat{\zeta}_{t+1} A_{t+1} \right) \hat{x}_{t+1} - \sum_{t=0}^{T} \hat{\zeta}_{t+1} \hat{b}_{t+1} - \hat{\lambda}^i_{T+1} \hat{b}_{T+1}
\]

\[
\sum_{t=0}^{T} \lambda^i_t \hat{c}_t^i = \hat{\lambda}^0_0 \left( A_0 \hat{x}_0 + \hat{b}_0 \right) - \sum_{t=0}^{T} \hat{\lambda}^i_{t+1} \hat{b}_{t+1} + \sum_{t=0}^{T} \hat{\nu}_{t+1} \hat{x}_{t+1} - \\
\sum_{t=0}^{T} \hat{\zeta}_{t+1} \left( A_{t+1} \hat{x}_{t+1} + \hat{b}_{t+1} \right) - \hat{\lambda}^i_{T+1} \left( A_{T+1} \hat{x}_{T+1} + \hat{b}_{T+1} \right)
\]

By adding and subtracting \(\sum_{t=0}^{T} \hat{\nu}_{t+1}\) to the above equation and using FOC \(\hat{\zeta}_{t+1} \left( A_{t+1} \hat{x}_{t+1} + \hat{b}_{t+1} \right) = 0\) and the FOC (2.2.13), (2.2.14) we can write

\[
\sum_{t=0}^{T} \lambda^i_t \hat{c}_t^i = \hat{\lambda}^0_0 \left( A_0 \hat{x}_0 + \hat{b}_0 \right) + \sum_{t=0}^{T} \hat{\nu}_{t+1} - \hat{\lambda}^i_{T+1} \left( A_{T+1} \hat{x}_{T+1} + \hat{b}_{T+1} \right)
\] (2.5.1)
Consider now a second sequence \( (c^i_t, x^i_{t+1}, b^i_{t+1}) \) that satisfies the FOC. We can write

\[
\sum_{t=0}^{T} \left( \lambda^i_t c^i_t - \hat{\lambda}^i_t \hat{c}^i_t \right) = \lambda^i_0 \left( A_0 x^i_0 + b^i_0 \right) - \hat{\lambda}^i_0 \left( A_0 \hat{x}^i_0 + \hat{b}^i_0 \right) + \sum_{t=0}^{T} \nu^i_{t+1} - \sum_{t=0}^{T} \nu^i_{t+1} = \lambda^i_0 \left( A_0 x^i_0 + b^i_0 \right)
\]

(2.5.2)

\[
\lambda^i_{t+1} \left( A_{T+1} x^i_{t+1} + b^i_{T+1} \right) + \hat{\lambda}^i_{T+1} \left( A_{T+1} \hat{x}^i_{T+1} + \hat{b}^i_{T+1} \right)
\]

(2.5.3)

By combining FOC (2.2.11), FOC (2.2.12) and FOC (2.2.13) it is possible to write

\[
\nu^i_{t+1} x^i_{t+1} = (-\lambda^i_t + \lambda^i_t q_{t+1} A_{t+1}) x^i_{t+1}
\]

\[
\nu^i_{t+1} = \lambda^i_t (q_{t+1} A_{t+1} - 1)
\]

Equation (2.5.1) that can be written as

\[
\sum_{t=0}^{T} \hat{\lambda}^i_t c^i_t = \hat{\lambda}^i_0 \left( A_0 \hat{x}^i_0 + \hat{b}^i_0 \right) + \sum_{t=0}^{T} \hat{\lambda}^i_t \left( q_{t+1} A_{t+1} - 1 \right) - \hat{\lambda}^i_{T+1} \left( A_{T+1} \hat{x}^i_{T+1} + \hat{b}^i_{T+1} \right)
\]

(2.5.4)

(2.5.5)

By applying FOC (2.2.12) recursively it is possible to write

\[
\lambda_{t+1} = \lambda^i_t q_{t+1} - \zeta^i_{t+1}
\]

\[
= \lambda^i_t q_{t+1} - \zeta^i_{t+1}
\]

\[
= \lambda^i_{t-1} q_{t+1} - \zeta^i_{t+1}
\]

\[
= \lambda^i_{t-2} q_{t+1} - \zeta^i_{t+1}
\]

\[
\vdots
\]

\[
= \lambda^i_0 \prod_{s=0}^{t+1} q_s - \sum_{s=0}^{t+1} \left( \zeta^i_{t+1-s} \prod_{r=0}^{s} q_r \right)
\]

where \( q_0 = 1 \). Note that, \( \lambda^i_t \geq 0, \forall t \), so it follows

\[
\lambda^i_0 \prod_{s=0}^{t+1} q_s \geq \sum_{s=0}^{t+1} \left( \zeta^i_{t+1-s} \prod_{r=0}^{s} q_r \right)
\]

(2.5.6)

Equation (2.5.5) can be written as

\[
\sum_{t=0}^{T} \hat{\lambda}^i_t c^i_t = \hat{\lambda}^i_0 \left( A_0 \hat{x}^i_0 + \hat{b}^i_0 \right) + \\
\sum_{t=0}^{T} \left[ \left( \lambda^i_0 \prod_{s=0}^{t} q_s - \sum_{s=0}^{t} \left( \zeta^i_{t+1-s} \prod_{r=0}^{s} q_r \right) \right) \left( q_{t+1} A_{t+1} - 1 \right) \right] - \\
\hat{\lambda}^i_{T+1} \left( A_{T+1} \hat{x}^i_{T+1} + \hat{b}^i_{T+1} \right)
\]

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By using the FOC $\lambda_i^t = \beta_i^t \frac{\partial u^i(c_t)}{\partial c_t}$ and the concavity of $u^i(c_t)$ it is possible to write

$$\sum_{t=0}^{T} \beta_i^t u^i(\hat{c}_t^i) \geq \sum_{t=0}^{T} \left[ \lambda_0^i \prod_{s=0}^{t} q_s - \sum_{s=0}^{t} \left( \zeta_{t-s}^i \prod_{r=0}^{s} q_r \right) (q_{t+1} A_{t+1}^i - 1) \right] - \hat{\lambda}_{T+1}^i \left( A_{T+1} \hat{x}_{T+1}^i + \hat{b}_{T+1}^i \right)$$

(2.5.7)

Consider now a second allocation that maximises the total utility up to a period $T$, i.e. we can think of it as if $\beta_i^t = 0$, $\forall t > T$. We must have $\lambda_{T+1} = 0$ and $\hat{b}_{T+1} = -A_{T+1} \hat{x}_{T+1}$. In other words, since we are maximising the utility up to a period $T$, we can use the end point condition for a finite time optimisation problem that states that in the final period $T$ the agent borrows up to its limit, since she/he will repay the debt in the periods that are not counted in the utility. Given that the allocation $\hat{a}^i$ solves a finite time maximisation problem the FOC are still necessary conditions and must hold. Consider now the allocation $a^i$. Given inequality (2.5.6) we can choose a sequence such that

$$\lambda_0^i = \hat{\lambda}_0^i$$

$$b_{T+1} \geq \hat{b}_{T+1}$$

By inserting equation (2.5.7) into (2.5.6), using the concavity of $u^i$, inequality (2.5.6) and Corollary (2.2.1) follows then

$$\sum_{t=0}^{T} \beta_i^t \left[ u^i(c_t^i) - u^i(\hat{c}_t^i) \right] \geq -\lambda_{T+1}^i \left( A_{T+1} x_{T+1} + b_{T+1} \right)$$
3.1 Introduction

According to Fama and French (2004) the Capital Asset Pricing Model of Sharpe (1964) and Lintner (1965) is still widely used and of interest. The simple and intuitive results of the model are as appealing to practitioners as they are to academic scholars. But the model is also widely criticised because of its strong assumptions and poor empirical record.

From a General Equilibrium point of view, the model is not satisfying since it holds only under strong assumptions on the utility functions, see Berk (1997), and/or strong conditions on the distribution of returns, see Ross (1978). From an empirical point of view, the Sharp-Lintner version of the CAPM is rejected by most tests. For an excellent exposition of the tests, see Fama and French (2004).

An interesting critique of the CAPM tests is presented by Roll (1977). Roll argues that testing the Sharp-Lintner version of the CAPM poses several difficulties because the market portfolio is not known. The major argument is that most tests reject the CAPM because the portfolio used to approximate the market portfolio, the so-called market proxy, is not close to the true market portfolio resulting in a misspecification of the CAPM. The importance of the market proxy when testing the CAPM rests in the fact that a market portfolio has to be identified in relation to which the CAPM is constructed. Given that it is not agreed upon what constitutes an asset or not (consider for example human capital), and given that not all data can be readily available, any portfolio which is chosen is essentially a market proxy.

Fama and French (2004, p.41) consider the problem of choosing a suitable market proxy one of the major reasons why the CAPM has been rejected in major CAPM tests. In this chapter we use a complete artificial economy to compare the estimation
results based on the true market portfolio with the ones based on a market proxy. The economy is completely artificial in the sense that we model and fix every aspect of it and as such, we know exactly whether the CAPM holds or not and we know exactly the distribution of returns and what the market portfolio is. For our estimation process we generate a sequence of observed returns and use these sequence to test the CAPM in this model. In this sense the approach we adopt is similar to a Monte Carlo simulation. Our paper shows that for a particular class of economies a market proxy does give reliable test results. But in this class of economies it is very unlikely for the CAPM to be rejected even when it does not hold. We argue that even if one were to know what the market portfolio is, estimating the CAPM would require a very high amount of observations until the asymptotic properties of the estimators are relevant. The artificial economy we propose will be presented in two different versions. The first has agents with quadratic utilities and markets are complete. In this case the CAPM will always hold. The second economy consists of agents with logarithmic utilities and complete markets. In this economy the CAPM does not hold in general. After calculating the equilibrium prices we will generate a sequence of observed returns based on the probabilities specified in the model. We then conduct an Ordinary Least Square estimation on the observed returns similar to Jensen (1968). In a second test we omit a collection of assets from the portfolio that is used in the estimation in order to test the CAPM with a market proxy. Given that the CAPM does hold when agents have quadratic utility we use this type of economy to test whether using a market proxy leads to a rejection of the CAPM. The economy with logarithmic utility in which the CAPM does not hold, is used to test what impact a market proxy has in case the CAPM does not hold.

Our major finding is that using a market proxy gives us a good test and estimation results. Good in the sense that, the estimation results are not far from the ones based on the true market portfolio. A second finding is that even using a sequence of 5000 observations it is still unlikely to reject the CAPM for the economy where it is known not to hold.

The main limitation of our paper is that we cannot calibrate our model on real data and as such we cannot answer the question whether the CAPM holds or not. The relevance of our results lies in the fact that we have full control of the data generated in the economy and as such we do not face the usual problems that models based on real data face, e.x. autocorrelation, structural breaks and so on. Thus the estimation results depend only on the specification of the market proxy and we can use simple statistical methods in our analysis.

The paper is organised as follows. In the first part we describe the model. In the second part we describe the estimation process and the results. The last part contains concluding remarks.
3.2 The model

There are two periods in this economy, indexed by $t \in T = \{0, 1\}$. There exists one consumption good at time $t = 0$. The consumption good is not storable and thus can only be consumed in the period where it is available. In the second period $t = 1$, there is a new endowment of this consumption good. The new endowment of the consumption good is produced from assets. There are $M$ states of the world indexed by $m \in \mathcal{M} = \{1, 2, \ldots, M\}$. Every asset yields a specific amount of the consumption good in every future state of the world. There are $N = M$ different assets indexed by $n \in \mathcal{N} = \{1, 2, \ldots, N\}$. Assets differ in their payoffs. These payoffs can be represented by vectors with $M = N$ entries. The whole structure of assets payoffs can be represented by a $M \times N$ or $N \times N$ payoff matrix given by

$$
\Pi = \begin{bmatrix}
\pi_{1,1} & \cdots & \pi_{1,N} \\
\vdots & \ddots & \vdots \\
\pi_{M,1} & \cdots & \pi_{M,N}
\end{bmatrix}
$$

where $\pi_{m,n}$ denotes the payoff of asset $n \in \mathcal{N}$ in state $m \in \mathcal{M}$. It will be assumed that the matrix $\Pi$ is of full rank. This implies that we have a complete market structure.

Assets can be split in any amount and can be short sold. An agent can exchange shares in assets for units of the consumption good in period $t = 0$.

An alternative way of thinking about this economy, is one in which the consumption good consists of fruits and the assets are trees. An agent exchanges fruits at $t = 0$ for a claim on the future production of a tree, which in our economy is represented by an asset. Give random influences the trees’ fruit production may vary in the second period.

One of the assets is denoted as the risk free asset. For example, let asset $N$ be the risk free asset, such that it pays the fixed amount $\bar{\pi}$ for each state $m \in \mathcal{M}$. The payoff matrix can then be written as

$$
\Pi = \begin{bmatrix}
\pi_{1,1} & \pi_{1,2} & \cdots & \bar{\pi} \\
\pi_{2,1} & \pi_{2,2} & \cdots & \bar{\pi} \\
\vdots & \vdots & \ddots & \vdots \\
\pi_{M,1} & \pi_{M,2} & \cdots & \bar{\pi}
\end{bmatrix}
$$

The initial endowments of the consumption good and of the assets in the whole
economy are specified by an endowment vector with \((N + 1)\) entries given by

\[
E = \begin{bmatrix}
\varepsilon_0 \\
\varepsilon_1 \\
\vdots \\
\varepsilon_N
\end{bmatrix}; \quad E \in \mathbb{R}^{M+1}
\]

where \(\varepsilon_0\) indicates the total endowment of the consumption good in period \(t = 0\) and \(\varepsilon_n\) indicates the total initial endowment of asset \(n \in \mathcal{N}\). The endowments satisfy the following assumptions

**(A.1)** The consumption good and all assets are given in positive aggregate endowment, i.e.

\[
\varepsilon_0 > 0 \\
\varepsilon_n > 0; \quad \forall n \in \mathcal{N}
\]

**(A.2)** The individual endowments of consumption good and assets are non-negative, i.e.

\[
e_{i,0} \geq 0; \quad \forall i \in \mathcal{I}
\]
\[
e_{i,n} \geq 0; \quad \forall i \in \mathcal{I}, \forall n \in \mathcal{N}
\]

**(A.3)** Each agent holds at least some positive endowment of the consumption good or of some asset, i.e.

\[
e_{i,0} + \sum_{n=1}^{N} e_{i,n} > 0; \quad \forall i \in \mathcal{I}
\]

Assumption (A.1) is self-explanatory and implies that none of the assets is a financial asset given in zero net supply. Assumption (A.2) is convenient since together with Assumption (A.3) it guarantees that an agent with logarithmic utility can achieve an allocation for which the utility function is defined.

The price of the consumption good is given by \(q_0\), the prices of the assets are given by \(q_n\). The price vector can be written as

\[
Q = \begin{bmatrix}
q_0 \\
q_1 \\
\vdots \\
q_N
\end{bmatrix}; \quad Q \in \mathbb{R}^{M+1}
\]

The states of the world occur with given probabilities specified by a probability
where \( p_m \) indicates the probability with which state \( m \) occurs in period \( t = 1 \). The probability vector satisfies the standard conditions \( p_m > 0, \forall m \in \mathcal{M} \) and \( \sum_{m=1}^{M} p_m = 1 \). Depending on which state is realised and on the choices the agent has made, a consumption stream is realised.

There are \( I \) agents in the economy indexed by \( i \in \mathcal{I} = \{1, 2, \ldots, I\} \). An agent chooses a level of consumption in period \( t = 0 \), and then chooses a vector of possible consumptions of which only one element will be realised. The consumption vector can be written as

\[
C_i = \begin{bmatrix} c_{i,0} \\ c_{i,1} \\ \vdots \\ c_{i,M} \end{bmatrix}; \quad C_i \in \mathbb{R}^{M+1}
\]

where \( c_{i,0} \) indicates the consumption in period \( t = 0 \) and \( c_{i,m} \) indicates the consumption if the state of the world \( m \) is realized in period \( t = 1 \). Agents choose their consumption by choosing how much they want to consume in period \( t = 0 \) and what assets they want to hold at the given prices. The choices of asset holdings for an agent \( i \) are given by

\[
S_i = \begin{bmatrix} s_{i,1} \\ s_{i,2} \\ \vdots \\ s_{i,N} \end{bmatrix}; \quad S_i \in \mathbb{R}^{M}
\]

where \( s_{i,n} \) denotes agent \( i \)'s holding of asset \( n \) in period \( t = 1 \).

Given the information that is available the consumption levels can be computed by a map \( \Gamma_O : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^{N \times M} \rightarrow \mathbb{R}_+^{N+1} \) such that \( \Gamma_O (c_{i,0}, S_i, \Pi) = C_i \). The map \( \Gamma_O \) is given by the set of equations

\[
c_{i,0} = c_{i,0} \quad (is \ a \ choice \ variable)
\]

\[
c_{i,m} = \sum_{n=1}^{N} s_{i,n} \pi_{m,n}; \quad \forall m \in \mathcal{M}
\]

Equation (3.2.1) highlights the fact that agents do not choose consumption in period

\[
\text{Equation (3.2.1) highlights the fact that agents do not choose consumption in period}
\]
\( t = 1 \) directly, but choose the assets that they want to hold and then consume the payoffs received from these assets.

Let the total amount of the consumption good in period \( t = 0 \) be given by

\[
g_0 = \varepsilon_0
\]

Let the total amount of the consumption good in period \( t = 1 \) and for each state \( m \in \mathcal{M} \) be given by

\[
g_m = \sum_{n=1}^{N} \varepsilon_n \pi_{m,n}
\]

We denote by \( \bar{g} \) the maximum amount of the consumption good available in each period and state, i.e.

\[
\bar{g} = \max \{g_0, \ldots, g_M\}
\]

The utility function of agent \( i \) is of the form

\[
EU_i (C_i) = u_i (c_{i,0}) + \sum_{m \in \mathcal{M}} p_m u_i (c_{i,m})
\]

We will impose the following Assumption on the utility functions

(A.4) For all \( i \in \mathcal{I} \)

\[
\left. \frac{\partial u_i (c)}{\partial c} \right|_{0 \leq c \leq \bar{g}} > 0
\]

During the course of our analysis we will consider a version of the model in which agent have a quadratic utility function of the form

\[
EU_i (C_i) = \gamma_i + (\delta_i c_{i,0} - \kappa_i c_{i,0}^2) + \sum_{m=1}^{M} p_m (\delta_i c_{i,m} - \kappa_i c_{i,m}^2)
\]

where \( \gamma_i, \delta_i \) and \( \kappa_i \) are individual parameters of the utility function. Assumption (A.4) is then needed to ensure that none of the agents reaches or surpasses her/his bliss point.

We will also consider a version of the model in which agents have logarithmic utility functions given by

\[
EU_i (C_i) = \chi_i + \ln (c_{i,0} + \theta_i) + \sum_{m=1}^{M} p_m \ln (c_{i,m} + \theta_i)
\]

where \( \chi_i \) and \( \theta_i \) are individual parameters of the utility function.
The initial endowments of an agent $i$ are given by the vector

$$E_i = \begin{bmatrix} e_{i,0} \\ e_{i,1} \\ \vdots \\ e_{i,N} \end{bmatrix}; \quad E_i \in \mathbb{R}^{M+1}$$

where $e_{i,0}$ is agent $i$’s initial endowment of the consumption good and $e_{i,n}$ is agent $i$’s endowment of asset $n \in \mathcal{N}$. Obviously the sum of the endowments of the agents must sum up to the total endowments, that is

$$\sum_{i=1}^{I} e_{i,0} = \varepsilon_0$$

$$\sum_{i=1}^{I} e_{i,n} = \varepsilon_n; \quad \forall n \in \mathcal{N}$$

The budget restriction of an agent $i$ is then given by

$$q_0 e_{i,0} + \sum_{n=1}^{N} q_n e_{i,n} - q_0 c_{i,0} - \sum_{n=1}^{N} q_n s_{i,n} \geq 0 \quad (3.2.3)$$

Given that no agent will reach her/his bliss point, the budget restriction will always bind at equilibrium. This follows from Assumption (A.4) which implies that more consumption yields always a positive amount of utility and thus it can never be optimal to leave consumption idle. The market clearing condition for the consumption good in period $t = 0$ is given by

$$\sum_{i=1}^{I} c_{i,0} = \sum_{i=1}^{I} e_{i,0} = \varepsilon_0 \quad (3.2.4)$$

The market clearing condition for the asset markets in period $t = 1$ can be written as

$$\sum_{i=1}^{I} s_{i,n} = \sum_{i=1}^{I} e_{i,n} = \varepsilon_n; \quad \forall n \in \mathcal{N} \quad (3.2.5)$$

Let $\mathcal{F}_O = \{\Pi, P, (EU_i)_{i \in \mathcal{I}}, (E_i)_{i \in \mathcal{I}}\}$ denote the fundamentals of this economy. An allocation in this economy is defined by $\mathcal{A}_O = \{(c_{i,0})_{i \in \mathcal{I}}, (S_i)_{i \in \mathcal{I}}, Q\}$, i.e. an allocation is defined by the collection of consumption choices at $t = 0$ of each agent $i$, the collection of security choice vectors of each agent $i$ and the price vector.

**Definition 3.2.1** (Competitive Equilibrium). A competitive equilibrium for the economy with fundamentals $\mathcal{F}_O$ is defined by the allocation $\mathcal{A}_O^*$ that fulfils the following two conditions
(a) every agent maximises its utility function $EU_i$ with respect to the budget constraint, i.e. each agent $i \in I$ maximises (3.2.2) with respect to (3.2.3)

(b) markets clear, i.e. equation (3.2.4) and (3.2.5) are fulfilled for all $n \in N$

**Proposition 3.2.1.** There does not exist an equilibrium with $q_0 \leq 0$, i.e. at equilibrium $q_0 > 0$

*Proof.* The proof follows from the fact that the the derivative $\frac{\partial EU_i(c_i)}{\partial c_{i,0}} > 0, \forall i$. For $q_0 \leq 0$ any agent $i$ can afford $c_{i,0} = \varepsilon_0$. Since the utility function is strictly increasing with respect to $c_{i,0} \in [0, \bar{g}]$ any agent would prefer to consume $c_{i,0} \geq \varepsilon_0$ and given that $q_0 \leq 0$, any amount $c_{i,0} > 0$ is affordable. This implies that for $\varepsilon_0 < +\infty$ the market clearing conditions can never be fulfilled.\]

From this proposition follows that $q_0$ can be normalised to 1 and thus all prices can be measured in units of the consumption good.

### 3.2.1 Normalisation to an orthonormal basis

In the model proposed in this chapter, the market is complete. Mathematically, this means that the price vectors span the state-space and thus it is possible to rewrite the model to one containing only Arrow securities. This is done by switching to an orthonormal basis. This is very useful when simulating the economy since, the equilibrium solution in the Arrow security models has a very simple form and can be easily converted to a solution for the original economy. In the following it will be shown how the solution for the Arrow securities economy is derived.

Consider the payoff matrix $\Pi$. The matrix has to be of full rank, otherwise the assets are not linear independent and thus at least one asset is redundant. Let $\vec{\pi}_n$ denote the payoff vector of asset $n$. $\vec{\pi}_n$ is the $n$-th column of matrix $\Pi$. Each payoff vector can be treated as a basis vector in an $N$-dimensional Euclidean space $\mathbb{R}^N$. This basis can be converted into an orthonormal basis that is equivalent to a payoff matrix given by

$$\Pi_A = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}$$

Each column in the matrix $\Pi_A$ is a basis vector in the orthonormal basis and will be denoted by $\vec{\pi}_{A,n}$. The matrix $\Pi_A$ is the identity matrix. To convert to the orthonormal basis the inverse of $\Pi$ must be calculated such that

$$\Pi \cdot \Pi^{-1} = \Pi_A$$
Let the inverse be given by

$$\Pi^{-1} = \begin{bmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,N} \\ x_{2,1} & x_{2,2} & \ddots & x_{2,N} \\ \vdots & \ddots & \ddots & \vdots \\ x_{N,1} & x_{N,2} & \ldots & x_{N,N} \end{bmatrix}$$

To find the elements of this inverse we can apply the Gauss-Jordan method. Geometrically, this implies deriving a linear combination of the original basis vectors such that for each column vector $j \in \{1, \ldots, N\}$ of the matrix $\Pi^{-1}$ the following equation is fulfilled

$$x_{1,j} \bar{\pi}_1 + x_{2,j} \bar{\pi}_2 + \ldots + x_{N,j} \bar{\pi}_N = \bar{\pi}_{A,j}$$

where $\bar{\pi}_{A,j}$ is the $j$-th column vector of the identity matrix. Economically, this implies that the original assets are combined into new assets such that each asset pays out exactly 1 unit of consumption in only one state of world. In other words, the original assets are combined into Arrow Securities. Once the inverse of the payoff matrix is obtained, the endowments of Arrow Securities in the economy have to be derived. Since 1 Arrow Security pays exactly 1 unit of the consumption good in only 1 state of the world, the endowment of an Arrow Security that pays 1 unit of consumption in state $m$ is given by the total amount of the consumption good available in state $m$ in the original economy. Let the endowment vector of Arrow Securities be given

$$\mathbf{E}_A = \begin{bmatrix} \varepsilon_{A,1} \\ \varepsilon_{A,2} \\ \vdots \\ \varepsilon_{A,M} \end{bmatrix}; \quad \mathbf{E}_A \in \mathbb{R}^M$$

Each element $m \in \mathcal{M}$ is given by

$$\varepsilon_{A,m} = \sum_{n=1}^{N} \varepsilon_n \pi_{m,n}$$

The individual endowment vector of Arrow Securities of an agent $i$ is given by

$$\mathbf{E}_{A,i} = \begin{bmatrix} e_{i,0} \\ e_{A,i,1} \\ \vdots \\ e_{A,i,m} \end{bmatrix}; \quad \mathbf{E}_{A,i} \in \mathbb{R}^{M+1}$$

where $e_{i,0}$ is the original endowment of the consumption good in period $t = 0$, which
is not affected by the conversion of the basis. Each element \( m \in \mathcal{M} \) of the individual endowment vector of Arrow Securities is given by

\[
e_{A,i,m} = \sum_{n=1}^{N} e_{i,n} \pi_{m,n} \quad (3.2.6)
\]

To convert the optimal security choices from the original economy we let \( A_i \) be the choice of Arrow Securities holdings of an agent \( i \). A vector \( A_i \) is given by

\[
A_i = \begin{bmatrix} a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,M} \end{bmatrix}; \quad A_i \in \mathbb{R}^M
\]

Let \( C_{A,i} \) be the consumption vector in the Arrow Securities economy for agent \( i \). For each agent \( i \), \( C_{A,i} \) is given by

\[
C_{A,i} = \begin{bmatrix} c_{i,0} \\ a_{i,1} \\ \vdots \\ a_{i,M} \end{bmatrix}; \quad \mathbb{R}^{M+1}
\]

Given the choice vector \( S_i \) of the original economy, each element of \( A_i \) can be calculated by a map \( \Gamma_A : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^{N \times M} \rightarrow \mathbb{R}_+^{N+1} \) such that \( \Gamma_A (c_{i,0}, S_i, \Pi) = C_{A,i} \). The map \( \Gamma_A \) is given by

\[
c_{i,0} = c_{i,0}
\]

\[
a_{i,m} = \sum_{n=1}^{N} s_{i,n} \pi_{m,n} \quad (3.2.1')
\]

This implies that the Arrow Securities choices are the same as the consumption choices for the solution of the original economy, i.e.

\[
a_{i,m} = c_{i,m}; \quad \forall m \in \mathcal{M}
\]

Note, at this stage we have not shown that the equilibrium in the Arrow Securities economy is the same as the equilibrium in the original economy. So far, we converted the original maximisation problem into a problem that consists of choosing the optimal consumption allocation directly.

It remains to convert the price vector from the original economy to the one of the Arrow Securities economy. Given that an Arrow Security can be seen as a portfolio of original assets that pays 1 unit of consumption in only 1 state of the world, calculating the price of the Arrow Security corresponds to calculating the price of
this portfolio for given prices of the original securities. The portfolio weights for each Arrow Security are given by the column vectors of $\Pi^{-1}$. So for a given price vector $Q$ the corresponding price vector of the Arrow Securities economy $Q_A$ can be found by calculating $\forall m \in M$

$$q_{A,m} = \sum_{n=1}^{N} q_n x_{n,m}$$

It is also possible to go from a given price vector of Arrow Securities to a price vector in the original economy. In this case the price of an Arrow Security corresponds to the price of 1 unit of consumption in state $m$. Thus the price of an asset in the original economy is the value of its consumption and is given by

$$q_n = \sum_{m=1}^{M} q_{A,m} \pi_{m,n} \quad (3.2.7)$$

In other words, given the price vector of Arrow Securities, the original asset can be seen as a portfolio of Arrow Securities with portfolio weights given by the column vectors of $\Pi$.

The market clearing conditions in the Arrow Securities economy are given by

$$\sum_{i=1}^{I} c_{i,0} = \sum_{i=1}^{I} e_{i,0} \quad (3.2.4)$$

$$\sum_{i=1}^{I} c_{i,m} = \sum_{i=1}^{I} a_{i,m}; \quad \forall m \in M$$

Let $\mathcal{F}_A = \{\Pi_A, P, (EU_i)_{i \in \mathcal{I}}, (E_{A,i})_{i \in \mathcal{I}}, Q_A\}$ denote the fundamentals of the economy with Arrow Securities and $\mathcal{A}_A = \{(c_{i,0})_{i \in \mathcal{I}}, (A_i)_{i \in \mathcal{I}}, Q_A\}$ an allocation in this economy.

**Proposition 3.2.2.** For each economy with fundamentals given by $\mathcal{F}_O$ and an equilibrium allocation given by $\mathcal{A}_O^*$ there exist an equivalent allocation $\mathcal{A}_A^*$ which is an equilibrium in the economy with fundamentals $\mathcal{F}_A$, such that the consumption streams are equivalent, i.e. $C_i = C_{A,i}$ for all $i \in \mathcal{I}$. And where the individual consumption streams $C_i$, $C_{A,i}$ follow from $\Gamma_O (c_{i,0}, S_i, \Pi) = C_i$ and $\Gamma_A (c_{i,0}, S_i, \Pi) = C_{A,i}$.

**Proof.** Let the set of feasible consumption allocations in the original economy for an agent $i$ be given by $\mathcal{B}_{i,O}$. The set of feasible consumption allocations in the Arrow Securities economy for an agent $i$ will be given by $\mathcal{B}_{i,A}$. The proof will proceed as follows. We will show that for each $i$, $\mathcal{B}_{i,O} = \mathcal{B}_{i,A}$. Since the utility functions did not change it follows that $\arg \max_{C_i \in \mathcal{B}_{i,O}} EU_i = \arg \max_{C_{A,i} \in \mathcal{B}_{i,A}} EU_i$ and thus $C_i = C_{A,i}$. It remains then to show that the market clearing conditions are identical, too.
To determine the budget set for the consumption vector let $\tilde{\Pi}_O$ be defined as

$$
\tilde{\Pi}_O = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & \pi_{1,1} & \ddots & \pi_{1,N} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \pi_{m,n} & \ldots & \pi_{M,N}
\end{bmatrix}
$$

In this matrix the consumption good at time $t = 0$ is treated as an asset that pays 1 unit of consumption at time $t = 0$ and 0 in period $t = 1$ for each state $m \in \mathcal{M}$. Let $\tilde{S}_{i,O}$ be given by

$$
\tilde{S}_{i,O} = \begin{bmatrix}
c_{i,0} \\
s_{i,1} \\
\vdots \\
s_{i,N}
\end{bmatrix}; \quad \tilde{S}_{i,O} \in \mathbb{R}^{M+1}
$$

Given this notation the budget constraint (3.2.3) for the original economy can be written in matrix notation as

$$
Q^\top E_i - Q^\top \tilde{\Pi}_O \tilde{S}_{i,O} \succeq 0
$$

The mapping $\Gamma_O$ can then be written as

$$
\Gamma_O \left( \tilde{S}_{i,O}, \tilde{\Pi}_O \right) = \tilde{\Pi}_O \tilde{S}_i = C_i \quad (3.2.8)
$$

Thus the budget restriction can be written as

$$
Q^\top E_i - Q^\top \tilde{\Pi}_O^{-1} C_i \succeq 0
$$

The budget restriction for the Arrow Securities economy is given by

$$
q_0 e_{i,0} + \sum_{n=1}^N q_{A,n} e_{A,i,n} - q_0 c_{i,0} - \sum_{n=1}^N q_{A,n} a_{i,n} \geq 0
$$

In matrix notation this can be written as

$$
Q_A^\top E_{A,i} - Q_A^\top C_{A,i} \succeq 0
$$

Equations (3.2.6) and (3.2.7) can be written as matrix equations of the form

$$
E_{A,i} = \tilde{\Pi}_O E_i \\
Q = Q_A \tilde{\Pi}_O \rightarrow Q_A^\top = Q^\top \tilde{\Pi}_O^{-1}
$$

(3.2.9)
Inserting this equations into the budget constraint for the Arrow Securities economy yields

\[
Q^\top \tilde{\Pi}_O^{-1} \tilde{\Pi}_O E_i - Q^\top \tilde{\Pi}_O^{-1} C_{A,i} \geq 0 \\
Q^\top E_i - Q^\top \tilde{\Pi}_O^{-1} C_{A,i} \geq 0
\]

So, by defining the budget sets as

\[
B_{i,O} = \left\{ C_i \in \mathbb{R}^{M+1} : Q^\top E_i - Q^\top \tilde{\Pi}_O^{-1} C_i \geq 0 \right\} \\
B_{i,A} = \left\{ C_{A,i} \in \mathbb{R}^{M+1} : Q^\top E_i - Q^\top \tilde{\Pi}_O^{-1} C_{A,i} \geq 0 \right\}
\]

it is obvious that the two set are identical. Since the utility functions in both economies are the same it follows that

\[
\begin{align*}
\arg\max_{C_i \in B_{i,O}} E U_i &= \arg\max_{C_{A,i} \in B_{i,A}} E U_i \\
C_i &= C_{A,i}
\end{align*}
\]

It remains to show that the market-clearing conditions do not change, such that a market clears in one economy if and only if it clears in the other economy. The market clearing conditions (3.2.5) in the original economy can be rewritten in matrix notation as

\[
\tilde{S}_1 + \ldots + \tilde{S}_I - E_1 - \ldots - E_I = 0
\]

where \( \tilde{0} \) is the zero vector. Analogously, in the Arrow Securities economy, the market clearing conditions can be written as

\[
C_{A,1} + \ldots + C_{A,i} - E_{A,1} - \ldots - E_{A,I} = 0
\]

Given that it has been shown that \( C_{A,i} = C_i \) it is possible to use equation (3.2.8) and (3.2.9) to write

\[
\tilde{\Pi}_O \tilde{S}_1 + \ldots + \tilde{\Pi}_O \tilde{S}_I - \tilde{\Pi}_O E_1 - \ldots - \tilde{\Pi}_O E_I = \tilde{0}
\]

By multiplying this equation by \( \tilde{\Pi}_O^{-1} \) it is possible to write

\[
\begin{align*}
\tilde{\Pi}_O^{-1} \tilde{\Pi}_O \tilde{S}_1 + \ldots + \tilde{\Pi}_O^{-1} \tilde{\Pi}_O \tilde{S}_I - \tilde{\Pi}_O^{-1} \tilde{\Pi}_O E_1 - \ldots - \tilde{\Pi}_O^{-1} \tilde{\Pi}_O E_I &= \tilde{\Pi}_O^{-1} \tilde{0} \\
\tilde{S}_1 + \ldots + \tilde{S}_I - E_1 - \ldots - E_I &= \tilde{0}
\end{align*}
\]

So it follows that

\[
\tilde{S}_1 + \ldots + \tilde{S}_I - E_1 - \ldots - E_I = C_{A,1} + \ldots + C_{A,i} - E_{A,1} - \ldots - E_{A,I} = \tilde{0}
\]
and the proof is concluded.

3.2.2 The maximisation problem in the orthonormal basis for the quadratic utility model

In this section we will derive the equilibrium for the model with quadratic utilities given by

\[ EU_i (C_i) = \gamma_i + (\delta_i c_{i,0} - \kappa_i c_{i,0}^2) + \sum_{m=1}^{M} p_m (\delta_i c_{i,m} - \kappa_i c_{i,m}^2); \quad i \in I \]

For any agent \(i\) the Lagrangian including FOC is given by

\[
L_i (C_A,i, \lambda_i) = \gamma_i + \delta_i c_{i,0} - \kappa_i c_{i,0}^2 + \sum_{m=1}^{M} p_m (\delta_i c_{i,m} - \kappa_i c_{i,m}^2) + \lambda_i \left( e_{i,0} + \sum_{m=1}^{M} q_{A,m} e_{A,i,m} - c_{i,0} - \sum_{m=1}^{M} q_{A,m} c_{i,m} \right)
\]

\[
\frac{\partial L (C_A,i, \lambda_i)}{\partial c_{i,0}} = \delta_i - 2\kappa_i c_{i,0} - \lambda_i = 0 \quad (3.2.10)
\]

\[
\frac{\partial L (C_A,i, \lambda_i)}{\partial c_{i,m}} = p_m (\delta_i - 2\kappa_i c_{i,m}) - q_{A,m} \lambda_i = 0; \quad \forall m \in M \quad (3.2.11)
\]

\[
\frac{\partial L (C_A,i, \lambda_i)}{\partial \lambda_i} = e_{i,0} + \sum_{m=1}^{M} q_{A,m} e_{A,i,m} - c_{i,0} - \sum_{m=1}^{M} q_{A,m} c_{i,m} = 0 \quad (3.2.12)
\]

\[
\lambda_i \geq 0; \quad \lambda_i \frac{\partial L (C_A,i, \lambda_i)}{\partial \lambda_i} = 0
\]

Combing (3.2.10) with (3.2.11) yields the following system of equations

\[
c_{i,m} = \frac{q_{A,m}}{p_m} c_{i,0} + \left( 1 - \frac{q_{A,m}}{p_m} \right) \frac{\delta_i}{2\kappa_i}; \quad \forall m \in M \quad (3.2.13)
\]

Inserting each of these \(M\) functions into (3.2.12) leads to the demand function for the consumption good at \(t = 0\)

\[
c_{i,0} = \frac{e_{i,0} + \sum_{m=1}^{M} q_{A,m} e_{A,i,m} - \sum_{m=1}^{M} \left[ \left( q_{A,m} - \frac{q_{A,m}^2}{p_m} \right) \frac{\delta_i}{2\kappa_i} \right]}{1 + \sum_{m=1}^{M} \frac{q_{A,m}^2}{p_m}} \quad (3.2.14)
\]

Given the demand function for the consumption good at \(t = 0\), the demand functions for the Arrow Securities can be derived by inserting (3.2.14) into the system of equations (3.2.13).

To solve for the Arrow Security prices we consider the market clearing condition
for the consumption good at \( t = 0 \)

\[
\sum_{i \in I} c_{i,0} = \sum_{i \in I} e_{i,0} = \varepsilon_0
\]

The market clearing conditions for the Arrow Securities are given by the following set of equations

\[
\sum_{i \in I} c_{i,m} = \sum_{i \in I} e_{A,i,m} = \varepsilon_{A,m} = \frac{q_{A,m}}{p_m} \varepsilon_0 + \left(1 - \frac{q_{A,m}}{p_m}\right) \sum_{i \in I} \frac{\delta_i}{2\kappa_i}; \quad \forall m \in \mathcal{M}
\]

The prices can be found by solving each of these equations for \( q_{A,m} \)

\[
q_{A,m} = p_m \frac{\left(\varepsilon_{A,m} - \sum_{i \in I} \frac{\delta_i}{2\kappa_i}\right)}{\varepsilon_0 - \sum_{i \in I} \frac{\delta_i}{2\kappa_i}}; \quad \forall m \in \mathcal{M}
\] (3.2.15)

Given the simplicity of equation (3.2.15) it is possible to simulate an equilibrium of a large economy by computing first the Arrow Security prices, then inserting the results in (3.2.14) and computing the demand for \( c_{i,0} \). Finally, we insert prices and \( c_{i,0} \) into the system of equations (3.2.13) to compute the demand for Arrow Securities. Once the solution for the Arrow Security economy is obtained we can convert this solution to the original economy and verify that in this case the CAPM always holds.

3.2.3 The maximisation problem in the orthonormal basis for the logarithmic utility model

The equilibrium allocation with logarithmic utilities is computed in a similar way as the one with quadratic utilities. We choose a logarithmic utility function for the case where the CAPM does not hold, because the solution in the Arrow securities economy for this utility function is simple and convenient for the simulation. The utility functions are of the form

\[
EU_i(C_i) = \chi_i + \ln (c_{i,0} + \theta_i) + \sum_{m=1}^{M} p_m \ln (c_{i,m} + \theta_i)
\]
The Lagrangian including FOC for any agent $i$’s maximisation problem is given by

$$
\mathcal{L}_i(C_i, \lambda_i) = \chi_i + \ln (c_{i,0} + \theta_i) + \sum_{m=1}^{M} p_m \ln (c_{i,m} + \theta_i) + 
\lambda_i \left( e_{i,0} + \sum_{m=1}^{M} q_{A,m} e_{A,i,m} - c_{i,0} - \sum_{m=1}^{M} q_{A,m} c_{i,m} \right)
$$

$$
\frac{\partial \mathcal{L}_i(C_i, \lambda_i)}{\partial c_{i,0}} = \frac{1}{c_{i,0} + \theta_i} - \lambda_i = 0 \quad \text{(3.2.16)}
$$

$$
\frac{\partial \mathcal{L}_i(C_i, \lambda_i)}{\partial c_{i,m}} = \frac{p_m}{c_{i,m} + \theta_i} - q_m \lambda_i = 0; \quad \forall m \in \mathcal{M} \quad \text{(3.2.17)}
$$

$$
\frac{\partial \mathcal{L}_i(C_i, \lambda_i)}{\partial \lambda_i} = e_{i,0} + \sum_{m=1}^{M} q_{A,m} e_{A,i,m} - c_{i,0} - \sum_{m=1}^{M} q_{A,m} c_{i,m} = 0 \quad \text{(3.2.18)}
$$

$$
\lambda_i \geq 0; \quad \lambda_i \frac{\partial \mathcal{L}(C_{A,i}, \lambda_i)}{\partial \lambda_i} = 0
$$

Combining (3.2.16) with (3.2.17) yields the following system of equations

$$
c_{i,m} = \frac{p_m}{q_{A,m}} c_{i,0} + \left( \frac{p_m}{q_{A,m}} - 1 \right) \theta_i; \quad \forall m \in \mathcal{M} \quad (3.2.19)
$$

Inserting each of this $M$ functions into (3.2.18) leads to the demand function for the consumption good at $t = 0$

$$
c_{i,0} = e_{i,0} + \sum_{m=1}^{M} q_{A,m} e_{A,i,m} + \theta_i \left( \sum_{m=1}^{M} q_{A,m} - 1 \right) \frac{2}{\sum_{i \in I} \theta_i}; \quad \forall m \in \mathcal{M} \quad (3.2.20)
$$

The demand functions of Arrow Securities can then be derived by inserting (3.2.20) into the system of equations (3.2.19).

By using the market clearing condition for the consumption good at $t = 0$, the market clearing conditions for the Arrow Securities can be written as

$$
\sum_{i \in I} c_{i,m} = \sum_{i \in I} e_{A,i,m} = e_{A,m} = \frac{p_m}{q_{A,m}} e_0 + \left( \frac{p_m}{q_{A,m}} - 1 \right) \sum_{i \in I} \theta_i; \quad \forall m \in \mathcal{M}
$$

Solving for the prices yields

$$
q_{A,m} = p_m \left( e_0 + \sum_{i \in I} \theta_i \right) \frac{1}{e_{A,m} + \sum_{i \in I} \theta_i}; \quad \forall m \in \mathcal{M} \quad (3.2.21)
$$

To simulate this economy, we will first derive the prices. The solution is then inserted into (3.2.20) and the demand for $c_{i,0}$ is derived. Last, we insert our result into (3.2.19) and derive the demand for Arrow Securities.

Once the equilibrium in the Arrow Securities economy is derived we can convert our result to the original economy. In the log utility model the CAPM does not
always hold, in contrast to the case in the quadratic utility economy.

3.3 The Estimation process

Our estimation process consists in simulating an artificial economy as specified above. We will calculate the equilibrium prices and then generate a sequence of observed returns based on the probabilities specified in the fundamentals of the economy. With the sequence of observed returns we will conduct and OLS estimation of the CAPM, by using the true market portfolio and a proxy for the market portfolio. This process will be performed for economies with logarithmic utility where the CAPM holds and for economies with quadratic utilities.

The next three subsections describe the simulation process, the econometric model and present the results of the estimation process.

3.3.1 The Simulation

We fix the number of asset and states of the world $M = N$. Given that the solution of the equilibrium price vector depends only on the aggregate endowments, we will consider only $I = 2$ agents. The payoff matrix $\Pi$, the probability vector $P$ and the endowments $E_1, E_2$ and $e_{1,0}, e_{2,0}$ are generated randomly.

To increase the spread between the return of the market portfolio and the return of the risk free asset we will add the following matrix to the randomly generated payoff matrix $\Pi$

$$T = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
\tau & \tau & \ldots & 0 \\
2\tau & 2\tau & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
(M-1)\tau & (M-1)\tau & \ldots & 0 \\
\end{bmatrix}$$

where $\tau$ is a positive scalar and matrix $T$ is of dimension $M \times N$. Note that since matrix $T$ is of rank 1 the new payoff matrix $\Pi_T$, given by

$$\Pi_T = \Pi + T$$

is of full rank, if $\Pi$ is of full rank. The economy including matrix $T$ can be interpreted as an economy in which state $m = 1$ is the worst state and state $m = M$ is the best. It does not necessary mean that each asset pays the most in state $M$, but in general it will be true.

To avoid unrealistic returns we will also scale the endowment of the consumption good and the risk-free asset. Note that, since returns are measured in units of the
consumption good, the initial endowment of the consumption good will determine whether the returns on the assets are negative or positive.

Without loss of generality, we will consider only economies with non-negative payoffs in which there exists a risk free asset and in which the payoff matrix has full rank, i.e.

$$
\pi_{m,n} \geq 0; \quad \forall n \in N, \forall m \in M
$$

$$
\pi_{m,N} = \bar{\pi} > 0; \quad \forall m \in M
$$

$$
\text{det } \Pi \neq 0
$$

The generated probability vector satisfies the standard assumption and the generated endowments satisfy assumption (A.1)-(A.3).

To construct the market portfolio we consider the equilibrium prices and construct the portfolio weights according to the following formula

$$
\omega_n = \frac{q_n \epsilon_n}{\sum_{n=1}^{N-1} q_n \epsilon_n}; \quad 1 \leq n \leq N - 1
$$

The weights for the proxy of the market portfolio are constructed by ignoring the first $k$ assets. The weights of the market proxy are calculated according to

$$
\tilde{\omega}_n = \frac{\omega_n}{\sum_{n=2}^{N-1} \omega_n}; \quad k + 1 \leq n \leq N - 1
$$

Given the equilibrium prices we can construct the equilibrium returns as shown in the following matrix

$$
R = \begin{bmatrix}
\frac{\pi_{1,1}}{q_1} - 1 & \frac{\pi_{1,2}}{q_2} - 1 & \ldots & \frac{\bar{\pi}}{q_N} - 1 \\
\frac{\pi_{2,1}}{q_1} - 1 & \frac{\pi_{2,2}}{q_2} - 1 & \ldots & \frac{\bar{\pi}}{q_N} - 1 \\
\vdots & \ddots & \ddots & \vdots \\
\frac{\pi_{M,1}}{q_1} - 1 & \frac{\pi_{M,2}}{q_2} - 1 & \ldots & \frac{\bar{\pi}}{q_N} - 1
\end{bmatrix}
$$

With the equilibrium returns we can construct the CAPM as shown in the Appendix. We will then proceed by generating a sequence of observed returns and estimating the CAPM with an OLS estimation. The economy is not a dynamic economy, in the sense that we will use the result of the two period model for constructing the sequence of observed returns, i.e. we will simulate the economy only once and extract the equilibrium prices. Judd et al. (2003) show that in an economy with shocks the optimal portfolio hold by agents is time and shock independent. In our case we can see that the optimal solution depends only on the aggregate endowments, thus considering a dynamic economy would not change the results significantly unless we change the aggregate endowments.
3.3.2 The econometric model

The Sharp-Lintner Model predicts that given that, the market portfolio is mean-variance efficient and given that the capital market line is tangent to the minimum variance frontier, for each asset $n \in \mathcal{N}$ the following equation holds

$$ r_n = r_f + \beta_n (r_{MP} - r_f) \quad (3.3.1) $$

$$ \beta_n = \frac{\sigma_{MP,n}}{\sigma_{MP,MP}} $$

where $r_f$ is the risk free return, which corresponds to the return of asset $N$ in our economy. $r_n$ and $r_{MP}$ are the expected return of asset $n$ and of the market portfolio, respectively. Merton (1972) demonstrates the derivation of equation (3.3.1) through a variance minimisation problem. $\beta_n$ is economically speaking the degree to which asset $n$ is correlated with the market portfolio and how risky it is compared to the market.

For our estimation we rewrite (3.3.1) as

$$ r_n - r_f = \beta_n (r_{MP} - r_f) \quad (3.3.2) $$

as demonstrated by Jensen (1968), where $(r_n - r_f)$ is the risk-premium of asset $n$ over the risk free asset and $(r_{MP} - r_f)$ is the risk-premium of the market portfolio over the risk free asset.

We will denote by $J$ the length of the sequence of observed returns that will be generated according to the probability distribution specified in the model. To construct our data a pseudo random number generator is used to generate a sequence of $\{m_j\}_{j=1}^J$ states of the world such that $m_j \in \mathcal{M}, \forall j \in \{1, \ldots, J\}$. We use then our equilibrium prices to generate for each asset $n \in \mathcal{N}$ a sequence of observed returns $\{r_{n,j}\}_{j=1}^J$. With the observed returns we can construct our data set consisting of $N - 1$ dependent variable vectors $Y_n = (y_{n,1}, y_{n,2}, \ldots, y_{n,J})$ and one independent variable vector $X = (x_1, x_2, \ldots, x_J)$. The elements of these vectors are given by

$$ y_{n,j} = r_{n,j} - r_f; \quad \forall j \in \{1, \ldots, J\} $$

$$ x_{n,j} = r_{MP,j} - r_f; \quad \forall j \in \{1, \ldots, J\} $$

We then proceed in the estimation of (3.3.2) by using an OLS estimation. The econometric equation is given by

$$ y_{n,j} = \alpha_n + \beta_n x_{j} + u_{n,j} $$

where $\alpha_n$ is Jensen’s Alpha for asset $n$. $\alpha_n$ has the economic interpretation of excess return of asset $n$. When the CAPM holds the return of an asset depends on the risk this asset has compared to the market portfolio. In other words the $\beta_n$. An
\( \alpha_n > 0 \) implies that the asset pays a higher return compared to the amount of risk it carries. In our case a positive \( \alpha_n \) implies that the CAPM does not hold for this asset.

For the estimation where we use a market portfolio proxy we replace our dependent variable by

\[
\tilde{x}_j = r_{pr,j} - r_f; \quad \forall j \in \{1, \ldots, J\}
\]

where \( r_{pr,j} \) is the return of the market portfolio proxy when state \( j \) is observed. To derive the econometric model for the market proxy we define by \( \mathcal{N}_1 \subset \mathcal{N} \) all assets that are included in the market proxy. The econometric equation is then given by

\[
y_{n,j} = \alpha'_{n,j} + \beta'_{n,j} \tilde{x}_j + u'_j; \quad \forall n \in \mathcal{N}_1
\]

After deriving our estimates we test for the significance of our estimates. To test the significance of the estimated parameters we use a 2 tailed student t-test at a 5% significance level. For the model where we use the correct market portfolio we will test for each \( a_n \) the following null-hypothesis

\[
H_0: \hat{\alpha}_n = 0; \quad H_1: \hat{\alpha}_n \neq 0
\]

For the \( \hat{\beta}_n \)-estimators we test the following null-hypothesis

\[
H_0: \hat{\beta}_n = \beta_n; \quad H_1: \hat{\beta}_n \neq \beta_n
\]

where we use the simulation results to estimate the true value for \( \beta_n \).

### 3.4 Simulation results

In this section we report the result of our simulation calibrations. To avoid examples with extremely high or low returns we will adjust the payoffs, scale the endowments of the consumption good and of the risk free asset, and adjust the parameters of the utility function. The test results are very sensitive to the size of the returns and the size of Jensen’s \( \alpha \). The examples below try to match realistic values, although we encountered problems in keeping the spread between the return of the market portfolio and the risk free return above 2% in the case where agents have quadratic utilities.

Since we are considering large economies, we will summarise the main properties of our economy by by showing key values. We will denote by \( r_{MP} \) the expected return of the market portfolio at equilibrium. The mean of the observed returns will be denoted as \( \bar{r}_{MP} \). In a similar way we will denote the equilibrium expected return of the market proxy as \( r_{PR} \) and the mean of the observed returns of the market
proxy will be given by $\bar{r}_{PR}$. Recall that, the true values of Jensen’s $\alpha$ is given by

$$\alpha_n = r_n - r_N - \beta_n (r_{MP} - r_N); \quad \forall n \in \mathcal{N}$$

We define by $A$ the vector containing the absolute $\alpha$-values, i.e.

$$A = \begin{bmatrix} |\alpha_1| \\ |\alpha_2| \\ \vdots \\ |\alpha_N| \end{bmatrix}$$

We will denote by $\alpha_{\text{max}}$ the maximum element of vector $A$ and by $\alpha_{\text{min}}$ the minimum element of $A$. Given the large amount of assets considered, rounding errors will occur. For the quadratic utility economy $\alpha_{\text{max}}$ and $\alpha_{\text{min}}$ capture the size of this rounding errors. For the logarithmic utility economy $\alpha_{\text{max}}$ and $\alpha_{\text{min}}$ are different from 0, because the CAPM does not hold. In this case the size of $\alpha_{\text{max}}$ and $\alpha_{\text{min}}$ highlight how close the true value is to zero, including rounding errors. Of crucial interest will be the amount of assets for which the null hypothesis is rejected. We denote by $H_{0,\alpha}$ the number of assets for which we reject the null for the estimators $\hat{\alpha}_n$ and by $H_{0,\beta}$ the number of assets for which the null is rejected for the estimators $\hat{\beta}_n$. The relative amount of rejection for each set of estimators will be denoted by $\tilde{H}_{0,\alpha}$ and $\tilde{H}_{0,\beta}$. When we use the market portfolio in our independent variable, this values are given by

$$\tilde{H}_{0,\alpha} = \frac{H_{0,\alpha}}{N - 1}$$
$$\tilde{H}_{0,\beta} = \frac{H_{0,\beta}}{N - 1}$$

When we use the market proxy the values are given by

$$\tilde{H}_{0,\alpha} = \frac{H_{0,\alpha}}{N - 1 - k}$$
$$\tilde{H}_{0,\beta} = \frac{H_{0,\beta}}{N - 1 - k}$$

where $k$ is the amount of assets that are excluded in the market proxy.

### 3.4.1 Results for the quadratic utility

The first example we consider follows below.

Table 3.2 reports the results of our estimation in which we use the market portfolio as the independent variable. Table 3.3 reports the results of the estimation in which the market proxy was used as the independent variable.

From the definition of the significance level follows that there is a probability that
we will reject the null hypothesis even when it is true. Given that under quadratic utility the CAPM holds, the rejection rate of our 5% t-test should be around 5%. This should hold for the tests performed on the \( \hat{\alpha}_n \)-estimators and on the tests performed on the \( \hat{\beta}_n \)-estimators. In our example we can see that the true \( \alpha \) values are almost zero, except for rounding errors in the range of \((-1.4e^{-15}, +6.5e^{-16})\). In the estimation where we used the market portfolio, \( H_{0,\alpha}^r \) in table 3.2 reports the amount of assets for which the null hypothesis was rejected on the set of estimators \( \{\hat{\alpha}_n\}_{n=1}^{249} \). Out of 249 assets that are contained in the market portfolio the null hypothesis on the respective estimator \( \hat{\alpha}_n \) was rejected for 6 assets, which corresponds to around 2.41% of the assets. For the set of estimators \( \{\hat{\beta}_n\}_{n=1}^{249} \) the amount of assets for which we reject the null hypothesis is given by \( H_{0,\beta}^r \) in table 3.2. We see that the null hypothesis was rejected for 10 assets which corresponds to around 4% of the assets.

When using the market proxy the amount of assets for which the tests are performed is given by 199. The sets of estimators are given by \( \{\hat{\alpha}_n\}_{n=51}^{249} \) and \( \{\hat{\beta}_n\}_{n=51}^{249} \). \( H_{0,\alpha}^r \) in table 3.3 shows the amount of assets for which we reject the null hypothesis.
on the set of estimators $\{\hat{\alpha}_n\}_{n=51}^{249}$. For this set of estimators the null hypothesis is rejected for 7 assets out of 199, which corresponds to around 3.5% of the assets. $H^r_{0,\alpha}$ show the amount of assets for which we reject the null hypothesis on the set of estimators $\{\hat{\beta}_n\}_{n=51}^{249}$. Out of 199 assets we reject the null hypothesis for 7 assets which is around 3.5% of the assets.

Given that in both cases the rejection rates are close to the 5% significance level, it is very unlikely that we would have rejected the CAPM for this data set. Even for the case where we used the market proxy the rejection rates are not high enough to allow us to reject the CAPM. Based on this test results we would draw the correct conclusion which is not to reject the CAPM.

### 3.4.2 Results for the logarithmic utility

We consider now an example with logarithmic utility. The example is specified in table 3.4.

<table>
<thead>
<tr>
<th>no. of assets $N = M$</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>no. of observations $J$</td>
<td>365</td>
</tr>
<tr>
<td>no. of assets omitted in the market proxy $k$</td>
<td>50</td>
</tr>
<tr>
<td>size of the market proxy relative to market portfolio $\frac{M-1-k}{M-1}$</td>
<td>0.7992</td>
</tr>
<tr>
<td>$r_N$</td>
<td>0.002368</td>
</tr>
</tbody>
</table>

Table 3.4: Example 2

Table 3.5 and table 3.6 summarise the results for the market portfolio and the market proxy, respectively.

<table>
<thead>
<tr>
<th>$r_{MP}$</th>
<th>0.090639</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{r}_{MP}$</td>
<td>0.082722</td>
</tr>
<tr>
<td>$\alpha_{min}$</td>
<td>-0.0057955</td>
</tr>
<tr>
<td>$\alpha_{max}$</td>
<td>0.005024</td>
</tr>
<tr>
<td>$H^r_{0,\alpha}$, $\left( \tilde{H}^r_{0,\alpha} \right)$</td>
<td>10, (0.0401606)</td>
</tr>
<tr>
<td>$H^r_{0,\beta}$, $\left( \tilde{H}^r_{0,\beta} \right)$</td>
<td>21, (0.0843373)</td>
</tr>
</tbody>
</table>

Table 3.5: Results example 2: market portfolio

When considering an economy with logarithmic utility the CAPM does not hold in general. We can see from table 3.5 that the true $\alpha$ values are not equal to zero. Comparing them with the errors from table 3.2 we conclude that these are not rounding errors. In this case the rejection rates should be considerably higher than 5%.

The test results on the $\hat{\beta}_n$-estimators are more difficult to interpret and to predict. Although the CAPM does not hold in this case, it is not excluded that the risk-premium of the market portfolio $r_{MP} - r_N$ can have some explanation power for the
individual risk premium of the assets. For this reason we do not know what the true value is.

When using the market portfolio as independent variable the sets of estimators are given by \( \{ \hat{\alpha}_n \}_{n=1}^{249} \) and \( \{ \hat{\beta}_n \}_{n=1}^{249} \). \( H_{0,\alpha}^r \) in table 3.5 reports the amount of assets for which we reject the null hypothesis out of the 249 \( \hat{\alpha}_n \)-estimators. In this case the null hypothesis is rejected for 10 assets out of 249, which corresponds to around 4% of the assets. For the 249 \( \hat{\beta}_n \)-estimators the amount of assets for which the null is rejected is reported by \( H_{0,\beta}^r \) in table 3.5. We see that the null is rejected for 21 assets out of 249, which is around 8.4% of the assets.

When using the market proxy as independent variable the amount of assets for which we reject the null on the set of estimators \( \{ \hat{\alpha}_n \}_{n=1}^{249} \) is given by \( H_{0,\alpha}^r \) in table 3.6. Out of 199 assets the null is rejected for 7 assets, which corresponds to around 3.5% of the assets. The amount of assets for which we reject the null hypothesis on the set of estimators \( \{ \hat{\beta}_n \}_{n=1}^{249} \) is given by \( H_{0,\beta}^r \) in table 3.6. Out of 199 assets the null hypothesis is rejected for 19, i.e. around 9.5% of the assets.

The example above shows that the market proxy does still give estimates that are close to the ones based on the market portfolio. But although the CAPM does not hold, the rejection rates for the tests performed on the \( \hat{\alpha}_n \)-estimators are lower than 5%. Based on the test results on the \( \hat{\alpha}_n \)-estimators it is unlikely that we would reject the CAPM even if in this case the CAPM does not hold.

To clarify why we are not able to reject the CAPM based solely on the test of the \( \hat{\alpha} \) estimators, we need to study the asymptotic properties of our model. This is done in the next section.

### 3.5 Asymptotic behaviour of the logarithmic model

We use a similar economy as in example 1 and example 2, but increase the number of observations \( J \) for each asset to show that we can get a significant rejection rate for high values. The question is whether these amount of observations are feasible in reality. Table 3.7 specifies our third example.

The results for the market portfolio and the market proxy are reported in table
no. of assets \( N = M \) 250

no. of observations \( J \) 2002

no. of assets omitted in the market proxy \( k \) 50

size of the market proxy relative to market portfolio \( \frac{M-1-k}{M-1} \) 0.7992

\[ r_N \] -0.022718

Table 3.7: Example 3

3.8 and table 3.9, respectively.

| \( r_{MP} \) | 0.063564 |
| \( \bar{r}_{MP} \) | 0.074579 |
| \( \alpha_{min} \) | -0.0048623 |
| \( \alpha_{max} \) | 0.0038018 |
| \( H_{0,\alpha}^r, \left( \tilde{H}_{0,\alpha}^r \right) \) | 18, (0.0722892) |
| \( H_{0,\beta}^r, \left( \tilde{H}_{0,\beta}^r \right) \) | 11, (0.0441767) |

Table 3.8: Results example 3: market portfolio

| \( r_{PR} \) | 0.063468 |
| \( \bar{r}_{PR} \) | 0.074778 |
| \( \alpha_{min} \) | -0.0048515 |
| \( \alpha_{max} \) | 0.0038301 |
| \( H_{0,\alpha}^r, \left( \tilde{H}_{0,\alpha}^r \right) \) | 14, (0.0703518) |
| \( H_{0,\beta}^r, \left( \tilde{H}_{0,\beta}^r \right) \) | 10, (0.0502513) |

Table 3.9: Results example 3: market proxy

By comparing \( H_{0,\alpha}^r \) from table 3.8 with \( H_{0,\alpha}^r \) from table 3.9 we see that the test results on the \( \hat{\alpha}_n \)-estimators are still close. When using the market portfolio as the independent variable the null hypothesis on the \( \hat{\alpha}_n \)-estimators is rejected for 18 assets, i.e. around 7% of the assets. When using the market proxy we reject the null for 14 assets out of 199, which is around 7% of the assets, too. In this example we included 2002 observations for each asset. If these were daily observations this would correspond to almost 5 years and 6 months of observations. The rejection rates are still close to the 5% significance level, but they are both above the 5% significance level. In this case we would reject the CAPM, which would be the correct decision. But considering the amount of observations we have included, we find that the rejection rate is surprisingly low.

In the next example we increase the number observations to an amount that allows us to reject the CAPM, convincingly.
\begin{align*}
\text{no. of assets } N &= M = 250 \\
\text{no. of observations } J &= 50002 \\
\text{no. of assets omitted in the market proxy } k &= 50 \\
\text{size of the market proxy relative to market portfolio } \frac{M-1-k}{M-1} &= 0.7992 \\
\bar{r}_N &= -0.028398
\end{align*}

Table 3.10: Example 4

The results for the market portfolio and the market proxy are shown in table 3.11 and table 3.12, respectively.

\begin{table}[h]
\centering
\begin{tabular}{ll}
\hline
\rowcolor{mygray}
\textit{r}_{\text{MP}} & 0.05557 \\
\textit{\bar{r}}_{\text{MP}} & 0.05608 \\
\alpha_{\text{min}} & -0.004123 \\
\alpha_{\text{max}} & 0.00407 \\
\hline
\textit{H}_{0,\alpha}^r, \left(\tilde{H}_{0,\alpha}^r\right) & 51, (0.204819) \\
\textit{H}_{0,\beta}^r, \left(\tilde{H}_{0,\beta}^r\right) & 7, (0.028112) \\
\hline
\end{tabular}
\caption{Table 3.11: Results example 4: market portfolio}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{ll}
\hline
\rowcolor{mygray}
\textit{r}_{\text{PR}} & 0.055789 \\
\textit{\bar{r}}_{\text{PR}} & 0.056309 \\
\alpha_{\text{min}} & -0.0033618 \\
\alpha_{\text{max}} & 0.0039242 \\
\hline
\textit{H}_{0,\alpha}^r, \left(\tilde{H}_{0,\alpha}^r\right) & 35, (0.175879) \\
\textit{H}_{0,\beta}^r, \left(\tilde{H}_{0,\beta}^r\right) & 7, (0.035176) \\
\hline
\end{tabular}
\caption{Table 3.12: Results example 4: market proxy}
\end{table}

Note that, if these were daily observations we would have included around 136 years of observations for each asset. The rejections rates for the test on the \( \alpha \) values are in this cases far from the 5\% significance level, but the results of the market proxy are still close to the results of the market portfolio. \( H_{0,\alpha}^r \) in table 3.11 shows that we reject the null on the \( \hat{\alpha}_n \)-estimators for 51 assets out of 199, when the market portfolio is the independent variable. This is a rejection rate of around 20\%. When the market proxy is the independent variable, \( H_{0,\alpha}^r \) shows that we reject the null hypothesis for 35 assets out of 199. This is a rejection rate of around 17\%. In both cases we would draw the correct conclusion and reject the CAPM.

In all the examples we have shown, the tests performed on the estimators where the market proxy has been used as the independent variable would lead as to the same conclusion as the tests performed on the estimators where the true market
portfolio has been used. From our analysis we conclude that in our economies, the market portfolio does give reliable results.

Surprisingly we find it hard to reject the CAPM when it does not hold. This raises the question whether it is feasible to test the CAPM based on real data. Considering that the most convincing result would correspond to more than 136 years of daily observations, it can be expected that by this time a structural break would have occurred.

It has also to be noted that the rejection rates on the $\hat{\beta}_n$-estimators are never far from the 5% significance level. This suggests that even when the CAPM does not hold, there exists a linear relation between the risk-premium of the market portfolio and the risk-premium of the individual assets. But since we have no theoretical model at hand, we can not comment further on this result.

### 3.6 Reducing the accuracy of the market proxy

In this section we will reduces the accuracy of the market portfolio by omitting more than half of the assets contained in the market portfolio. The example for the quadratic utility function follows below.

<table>
<thead>
<tr>
<th>no. of assets $N = M$</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>no. of observations $J$</td>
<td>365</td>
</tr>
<tr>
<td>no. of assets omitted in the market proxy $k$</td>
<td>125</td>
</tr>
<tr>
<td>size of the market proxy relative to market portfolio $\frac{M-1-k}{M-1}$</td>
<td>0.4980</td>
</tr>
<tr>
<td>$r_N$</td>
<td>0.060731</td>
</tr>
</tbody>
</table>

Table 3.13: Example 5

The estimation results for the market portfolio and the market proxy follow in table 3.14 and 3.15.

<table>
<thead>
<tr>
<th>$r_{MP}$</th>
<th>0.079668</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{r}_{MP}$</td>
<td>0.075548</td>
</tr>
<tr>
<td>$\alpha_{min}$</td>
<td>$-1.1102 \exp(-0.015)$</td>
</tr>
<tr>
<td>$\alpha_{max}$</td>
<td>$9.2981 \exp(-0.016)$</td>
</tr>
<tr>
<td>$H_{r,\alpha}^r$, $\left(\tilde{H}_{0,\alpha}^r\right)$</td>
<td>12, (0.0481928)</td>
</tr>
<tr>
<td>$H_{r,\beta}^r$, $\left(\tilde{H}_{0,\beta}^r\right)$</td>
<td>13, (0.0522088)</td>
</tr>
</tbody>
</table>

Table 3.14: Results example 5: market portfolio

We can see that the $\alpha$ values increase in absolute terms and also the rejection rates increase slightly. The rejection rates with the true market portfolio are around 5%, the rejection rates with the market proxy are slightly higher. But given that
Table 3.15: Results example 5: market proxy

we ignored more than half of the assets that are contained in the market portfolio
the rejection rates are still surprisingly low.

We will now show a similar example for the case where the CAPM does not hold.

<table>
<thead>
<tr>
<th>no. of assets</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>no. of observations</td>
<td>365</td>
</tr>
<tr>
<td>no. of assets omitted in the market proxy</td>
<td>125</td>
</tr>
<tr>
<td>size of the market proxy relative to market portfolio</td>
<td>$\frac{M-1-k}{M-1}$</td>
</tr>
<tr>
<td>$r_N$</td>
<td>0.0013782</td>
</tr>
</tbody>
</table>

Table 3.16: Example 6

The next two tables show the estimation results based on the market portfolio
and the ones based on the market proxy.

Table 3.17: Results example 6: market portfolio

Table 3.18: Results example 6: market proxy

This last examples confirms the results that we have so far collected. Even when
the CAPM does not hold a misspecification of the market proxy has a minimal
effect on the results. In this case the $\alpha$ values do not even increase significantly. The rejection rates increase only slightly. Given the low rejection rates we would reach the wrong conclusion in this case and not reject the CAPM.

### 3.7 Concluding remarks

We have shown that in general the market proxy provides accurate estimates of the model. Roll’s critique remains still valid, but our model raises the question whether noise in the data would not have a greater impact on the test results. This can be also be seen in our second argument, where we have shown that a high amount of observations is required to get significant rejection rates. Our examples suggest that there could exist external factors that drive the rejection rates in real world tests of the CAPM.

The limitations, of our analysis are in so far that we did not impose a particular structure on the assets that are not included in the market proxy. It can be expected that, if the assets we exclude from the market proxy have a very strong negative correlation with the remaining assets contained in the market proxy, our results might change. A second limitation is that we added some autocorrelation in our payoffs, this has been done to achieve a realistic spread between risk free return and return of the market portfolio. The problem of doing so is that the asset are in a way very similar and thus the impact of excluding one asset might be lower.

A third limitation is that with this type of models it is difficult to calibrate the returns to similar values observed in actual data. On the other hand, the values used here might be closer to the true values, since they contain less noise than actual data.

Despite the limitations mentioned here, we would like to mention a possible extension of our framework, that due to its scope is not included in this chapter. The results presented here show that other factors than the market proxy may have a big impact on the results of the CAPM tests. One of these factors might be structural breaks. When structural breaks occur, the statistician is using data that is based on two (or more) different data generating processes. Thus, it is very likely that the test results lead to a rejection because part of the data does simply not reflect the model that is assumed to be true. In our framework it is possible to introduce structural breaks and compare the results with the current ones to assess whether structural breaks or the misspecification of the market portfolio have a bigger impact. This can be of interest for future CAPM tests insofar as it puts the problem of the market proxy in relation with the problem of structural breaks and can give us a better understanding of the magnitude of both problems.
3.8 Appendix Chapter 3

3.8.1 Computing the CAPM

We compute the equilibrium of the quadratic utility economy as described in Section 2.3 and for the logarithmic utility economy as described in Section 2.4. For an economy with fundamentals $F$ and an equilibrium allocation $A^*_O$ the return matrix is given by

$$R = \begin{bmatrix}
    r_{1,1} & r_{1,2} & \ldots & r_{1,N} \\
    r_{2,1} & r_{2,2} & \ddots & r_{2,N} \\
    \vdots & \ddots & \ddots & \vdots \\
    r_{M,1} & r_{M,2} & \ldots & r_{M,N}
\end{bmatrix} = \begin{bmatrix}
    \frac{\pi_{1,1}}{q_1} - 1 & \frac{\pi_{1,2}}{q_2} - 1 & \ldots & \frac{\bar{r}}{q_N} - 1 \\
    \frac{\pi_{2,1}}{q_1} - 1 & \frac{\pi_{2,2}}{q_2} - 1 & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    \frac{\pi_{M,1}}{q_1} - 1 & \frac{\pi_{M,2}}{q_2} - 1 & \ldots & \frac{\bar{r}}{q_N} - 1
\end{bmatrix}$$

The average return vector for each asset $n \in \mathcal{N}$ is given by

$$\bar{r} = \begin{bmatrix}
    \bar{r}_1 \\
    \bar{r}_2 \\
    \vdots \\
    \bar{r}_N
\end{bmatrix} = \begin{bmatrix}
    \left( \sum_{m=1}^M p_m \pi_{m,1} \right) \frac{q_1}{q_1} - 1 \\
    \left( \sum_{m=1}^M p_m \pi_{m,2} \right) \frac{q_2}{q_2} - 1 \\
    \vdots \\
    \frac{\bar{r}}{q_N} - 1
\end{bmatrix}$$

The variance covariance matrix can be computed as

$$\Sigma = \begin{bmatrix}
    \sigma_{1,1} & \sigma_{1,2} & \ldots & \sigma_{1,N} \\
    \sigma_{2,1} & \sigma_{2,2} & \ddots & \sigma_{2,N} \\
    \vdots & \ddots & \ddots & \vdots \\
    \sigma_{N,1} & \sigma_{N,2} & \ldots & \sigma_{N,N}
\end{bmatrix}$$

where $\sigma_{n,n} = \sigma_n^2$ is the variance of returns of asset $n$ which is equivalent to the covariance of return of asset $n$ with itself. The formula for the covariances is given by

$$\sigma_{k,l} = \sum_{m=1}^M p_m (r_{m,k} - \bar{r}_k) (r_{m,l} - \bar{r}_l); \quad \forall k, l \in \mathcal{N}$$

By using the aggregate endowment vector $E$ we can compute the portfolio share of the market portfolio weights

$$\omega_n = \frac{q_n \varepsilon_n}{\sum_{n=1}^{N-1} q_n \varepsilon_n}; \quad 1 \leq n \leq N - 1$$

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The vector of market portfolio weights will be denoted by

$$\vec{\omega} = \begin{bmatrix} \frac{q_1}{\sum_{n=1}^{N-1} q_n \varepsilon_n} \\ \frac{q_2}{\sum_{n=1}^{N-1} q_n \varepsilon_n} \\ \vdots \\ \frac{q_{N-1} \varepsilon_{N-1}}{\sum_{n=1}^{N-1} q_{N-1} \varepsilon_{N-1}} \end{bmatrix}$$

The return and variance of the market portfolio are given by

$$r_{MP} = \sum_{n=1}^{N-1} \omega_n \bar{r}_n$$

$$\sigma_{MP,MP} = \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \omega_k \omega_l \sigma_{k,l}$$

The covariance between the market portfolio and an asset $n$ is given by

$$\sigma_{MP,n} = \sum_{k=1}^{N-1} \omega_k \sigma_{n,k}$$

We can now construct the SML equation

$$r_n = r_f + \beta_n (r_{MP} - r_f) \quad (3.3.1)$$


**URL:** [http://ideas.repec.org/p/inu/caeprp/2012-001.html](http://ideas.repec.org/p/inu/caeprp/2012-001.html)


URL: http://gallica.bnf.fr/ark:/12148/bpt6k111752b