

Essays on Taxation, Economic Growth and Fluctuations

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Abstract

The first chapter studies the stabilizing role of fiscal policy. We particularly emphasize the role played by the assumption on the use of public expenditure. We find that if wasteful government expenditure is assumed in a two sector growth model then progressive taxation prevails as a stabilizer of belief-driven fluctuations. In an one-sector model, we observe that when a lump-sum transfer is introduced a stabilizing role for regressive income taxation emerges. In addition, we show that with a lump-sum transfer it turns out that the labour and capital income taxes serve in a different way to dampen belief-driven fluctuations. This difference between labour and capital income taxes can also be observed in a growth model with productive government spending.

The second chapter demonstrates the optimality of intertemporally-regressive taxation in a Barro-type growth model with decreasing-returns-to-scale and a public consumption good. The constant tax rate that achieves the first-best steady-state decentralizes an excessive rate of growth with lower welfare than the first-best. We find that the use of regressive taxation moves the growth rate closer to that of the first-best, and the optimum tax matches the first-best growth path to a first-order approximation.

In the third chapter, we introduce quasi-hyperbolic preferences into the growth model with productive public spending. First, we compare the competitive equilibrium with the first-best equilibrium in a centrally-planned economy. Second, we explore second-best taxation using the Ramsey fiscal policy approach with three different degrees of commitment: full commitment, complete absence of commitment, and one-period partial commitment.

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# 1 Introduction to the Thesis

The theory of fiscal policy involves so many aspects of economics. Among them, this thesis particularly pays attention to two aspects of fiscal policy. In a micro-economic perspective, it concerns taxation or public finance issues which seek an efficient resource allocation that is not reached by the competitive equilibrium due to various market failures. When there is no market failure, the competitive equilibrium has no problem in attaining a Pareto-optimal resource allocation without the need to resort to government intervention. However, with market failure, the fundamental welfare theorem does not apply so the competitive market equilibrium loses its Pareto-optimality. For this reason, the government should correct the inefficient competitive outcome with various fiscal instruments including distortionary taxation, public goods, and sometimes debts. The optimal composition of these instruments to regain efficiency in the "Laissez-faire" economy is the main interest in this part.

In a macro-economic view, fiscal policy concerns the stabilization of economic fluctuations. Traditionally the usefulness of fiscal policy as a macroeconomic management tool has been challenged for various reasons. Discretionary fiscal intervention, in particular, has long been criticized in that it has not contributed to economic stability or has even caused more instability (Fatas and Mihov, 2003) in some cases mainly due to the time-lag in the decision-making process. On the contrary, automatic stabilizers which are operated by the counter-cyclical rule embedded in various fiscal instruments such as a progressive income tax or unemployment benefits are relatively free from criticism. In fact, it has recently been shown that automatic stabilizers have been working successfully in the most of the developed economies where the balanced-budget rule is managed in a medium-run (Fatas and Mihov, 2012). In these countries, automatic stabilizers are operated by counter-cyclical budget balance as the tax rate is progressive and government spending is maintained at a stable level. Gali (1994) earlier suggested a possibility in a RBC model that automatic stabilizers built into distortionary taxes with a continuous balanced-budget rule can destabilize the economy by providing pro-

cyclical public spending, but it turned out that empirical data does not support the predictions of the model.

This thesis consists of three distinct chapters ranging across these two issues. The first chapter deals with the stabilizing role of the fiscal policy. Traditional studies about stabilizing fiscal policy have been concerned with the economic fluctuations resulting from the intrinsic shocks or shocks that influence economic fundamentals such as endowments, preferences, or production technologies. In contrast, the business cycle that this chapter aims to tackle is a type of economic fluctuation that arises endogenously due to the existence of a sunspot equilibrium. In the seminal work of Cass and Shell (1983), sunspot equilibrium refers to the situation where the purely extrinsic uncertainty can have real effects on economic outcomes. In a sunspot equilibrium, even if uncertainty about all fundamental variables is excluded, there still remains the volatility in the economy which can be interpreted as the contribution of extrinsic uncertainty, whether it be market psychology such as waves of pessimism or optimism or animal spirits as Keynes named it.

In the context of dynamic general equilibrium, the existence of indeterminate equilibrium provides the mechanism where the sunspot fluctuations occur. In addition to its contribution on the variety of the source of economic fluctuations, indeterminacy in the dynamic economic model provides a plausible explanation for the stylized fact that economies that have seemingly identical economic fundamentals follow different growth paths. Palivos et al (2003), for example, paid attention to this aspect of indeterminacy and showed that fiscal policy can be used as a selection device among multiple equilibria.

For the study of the Keynesian fiscal response to address the business cycles arising from indeterminacy, we follow the tradition of standard real business cycles (RBC) models suggested by Benhabib and Farmer (1994, 1996)<sup>1</sup>. In this strand, an increasing-returns-to-scale technology is critical in the mechanism which induces indeterminacy.

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<sup>1</sup>A monetary economy based on Woodford (1986) and Grandmont et al (1998) has also commonly used in literature.

The policy recommendations from this RBC literature, however, are mixed. In Guo and Lansing (1998), they argue that progressive income tax can stabilize sunspot fluctuations in the one-sector growth model with increasing returns. This result is also confirmed in Schmitt-Grohe and Uribe (1997) where regressive income tax causes indeterminacy. On the contrary, Guo and Harrison (2001) drew the conclusion that progressive taxation makes an economy more susceptible to indeterminacy in a two-sector growth model. In a case where government productive public spending is the source of indeterminacy as in Guo and Harrison (2008) and Kamiguchi and Tamai (2011), Chen and Guo (2010) also show that regressive taxation can reduce the probability of indeterminacy.

The first chapter starts with an effort to settle the contradictory results in one-sector and two-sector growth model. We take notice of the difference in assumptions on the use of public expenditure. Guo and Lansing (1998) assume that public expenditure does not provide any service to the private sector. In contrast, the tax revenue in Guo and Harrison (2001) is assumed to be repaid as lump-sum transfer to the agents. We find that if we maintain the wasteful government expenditure assumption in the two sector growth model of Benhabib and Farmer (1998) progressive taxation prevails as a stabilizer of belief-driven fluctuations.

We also revisit the one-sector growth model of Benhabib and Farmer (1994), observing that when wasteful government expenditure assumption in Guo and Lansing (1998) is changed stabilizing role for regressive income taxation emerges. In addition, we show that with a lump-sum transfer it turns out that the labour and capital income taxes serve in a different way to dampen belief-driven fluctuations. This difference between labour and capital income tax can also be observed in a growth model with productive government spending in Guo and Harrison (1998) and Chen and Guo (2010).

The second and third chapters study two types of market failures and the optimal taxation policy to address them. In the second chapter, we explore optimal taxation in a growth model with productive government spending and public consumption goods.

The existence of public goods in our model causes the competitive economy not to obtain a Pareto-optimal economic outcome, so the objective of this chapter is to find an optimal tax mix to implement the first-best outcome of a central planner in a decentralized economy.

Most of the literature concerning growth models with productive public spending starting from Barro (1990) assumes that the contribution of the public expenditure is sufficiently large as to create a balanced growth path. This assumption has great merit in that the economy has no transition dynamics so that the comparison of welfare is straightforward. In Barro (1990), it is shown that it is optimal that productive government spending should be financed with a lump-sum tax, with income being left untaxed. This result generally holds when the labour is endogenized (Turnovsky, 2000) or the model is extended to an open economy (Turnovsky, 1996c and 1999) or public consumption is introduced in the utility function (Turnovsky, 1996a). Since Futagami et al (1993), some literature has considered productive public spending as a stock variable rather than a flow variable as in Barro (1990) and it has been shown that the existence of a balanced-growth path and the optimality of non-income tax result also hold. The role of distortionary income tax emerges when the government fails to set the level of public spending at the optimum or factors such as congestion and adjustment costs are considered. But, even when an income tax is needed, a constant tax rate is enough to implement the first-best in the competitive economy.

The model in the second chapter basically follows Barro (1990) except for the decreasing-returns-to-scale technology with respect to private capital and productive public spending. We also include an endogenous labour choice and public consumption in consumer utility. We characterize the first-best optimum chosen by a central planner and then try to construct a tax policy that replicates this optimum in a competitive economy. Due to decreasing returns, our economy shows transitional dynamics and we show that constant income tax rate cannot implement the first-best outcome along the growth path. So we introduce a non-linear income tax and find that an intertempo-

rally regressive income tax schedule is required to generate a growth path close to the first-best optimum.

The third chapter studies time-inconsistent preferences as a source of market failure. Economic models typically assume that the agents discount their future utility in an exponential fashion, which conveniently precludes the occurrence of preference reversals. However, a large literature studying behavioural anomalies such as Ainslie (1992) and Lowenstien and Prelec (1992) and even our real-life experience indicate that the actual discount function is more concave than an exponential curve, so that it produces dynamic inconsistency. The topic of dynamic inconsistency in economics has a long tradition as the first analysis can be traced back to David Hume and Adam Smith (Palacios-Huerta, 2003) and the contribution of Strotz (1956) and the increasingly sophisticated analysis of his model by Pollak (1968), Peleg and Yarri (1973), and Goldman (1980) have been provided a novel approach to the issue where the decision-making process of an agent with time-inconsistent preferences is interpreted as an intrapersonal game among different temporal selves.

Time-inconsistency is captured in our model by quasi-hyperbolic preferences following Phelps and Pollak (1968) and Laibson (1996). Laibson (1996) shows that quasi-hyperbolic preferences induce a non-Pareto optimal competitive equilibrium so that it creates a market failure. In a standard Ramsey growth model, Krusell et al (2002) also show that the competitive equilibrium is not efficient. In addition, they point out that if the central planner inherits the quasi-hyperbolic preferences the cost of time-inconsistency is even larger, so that the first-best chosen by this planner leads to a worse-off outcome than competitive equilibrium.

Extending Krusell et al (2002), the third chapter embraces quasi-hyperbolic preferences into a growth model with productive public spending. Since there exists externality from public spending, the result of Krusell et al (2002) that laissez-faire equilibrium outperforms the planned equilibrium does not hold. So we focus on the difference that quasi-hyperbolic preferences make in the optimal tax policy.

First, we compare the competitive equilibrium with the first-best choice in a centrally-planned economy. When a lump-sum tax is available the tax policy that can implement the first-best in a decentralized economy has a zero capital income tax rate, which is not the case in a standard Ramsey growth model. We also observe that quasi-hyperbolic preferences result in failure of the first-best choice to attain Pareto-efficiency. A better resource allocation than the first-best choice can be accomplished with a negative capital income tax rate (a subsidy to capital income).

Second, we explore second-best taxation from the Ramsey fiscal policy approach with three different degrees of commitment, when a lump-sum tax is not available. Under full commitment, when labour supply is inelastic, it is optimal for the government to levy a negative capital tax rate, while a positive capital tax rate which should be larger for more impatient consumers is the second-best choice when labour supply is endogenous. In the complete absence of commitment, we observe a continuum of optimal tax policies in the fixed labour case, and a bad equilibrium where the government confiscates all of capital income so that no agent saves in the elastic labour case. Finally, under one-period partial commitment, we find that a zero capital income tax rate is restored in the inelastic labour case and a positive capital tax rate, which is higher than the full commitment case, should be levied in the elastic labour case.

## 2 Fiscal Policy and Macroeconomic Stability

### 2.1 Indeterminacy in a Two-sector Growth Model and Progressive Taxation

#### 2.1.1 Introduction

Benhabib and Farmer (1994, 1996) have shown that the introduction of an increasing-returns-to-scale technology in a standard real-business-cycles model could allow the indeterminacy of equilibrium to arise under empirically plausible parameter values. These studies have provided the foundation for studying the role of Keynesian-type policy interventions to address the economic fluctuations which would solely be caused by the existence of indeterminacy. In the one-sector growth model with increasing returns of Benhabib-Farmer (1994), Guo and Lansing (1998) and Guo (1999) have claimed that progressive income taxation could effectively be used to eliminate the indeterminacy and restore saddle-path stability. On the contrary, in the two-sector model with sector-specific externalities of Guo and Farmer (1996), the conclusion of the study of Guo and Harrison (2001) gives the opposite stabilizing tax policy implication to the one-sector case. That is, in a two-sector model, when the externality is sufficiently large, then a regressive income tax schedule could work as a stabilizer and more progressive taxation just leads the economy to be more susceptible to the indeterminacy. The study of Sims (2005) was also based on the two-sector model of Benhabib and Farmer (1996) and in this case he considered different tax schemes which are levied separately on labour and capital income. What Sims (2005) found in his study is that only progressive taxation could be effective in reducing the likelihood of indeterminacy, which quite contrasts with the result of Guo and Harrison (2001).

So it is very natural to question what makes the results of these two literatures different even though they are basically based on the same model. That is what this work aims to do. There are three differences in the model descriptions. Firstly, as

mentioned above, while Guo and Harrison (2001) considered a tax on total income, Sims (2005) assumed that the tax is levied separately on wages and capital income. Secondly, Guo and Harrison (2001) assume a different externality for each sector, but in Sims (2005), the sectoral externality is assumed to be equal. Finally, as is clarified later in Guo and Harrison (2012), Guo and Harrison (2001) assume that the tax revenue is repaid to households in the form of a lump-sum transfer while Sims (2005) maintains the wasteful government spending assumption in Guo and Lansing (1998).

To find out whether these three factors could explain the difference in results, I will apply the approach described in Sims (2005) to the case of a tax on total income with different sectoral externalities and see if it can provide the same result as Guo and Harrison (2001). The results show that the assumption about how the government's tax revenue is used plays a very critical role in choosing an appropriate tax schedule to dampen belief-driven fluctuations in the economy. When the tax revenue is used to finance government spending which does not contribute to either production or household utility, only progressive income tax can be used to secure the saddle-path stability of the economy. By contrast, when the tax revenue is repaid as a lump-sum transfer, the stabilizing role of the regressive income taxation emerges when sector-specific externalities are sufficiently large as concluded in Guo and Harrison (2001). It turns out that whether the income tax is levied separately on labour and capital income and whether the sectoral externality is assumed to be the same for the consumption good sector and the investment good sector do not affect the conclusion.

The rest of this section is organized as follows. The next part will describe the basic model, most of which is based on the model suggested by Benhabib and Farmer (1996). The third part will characterize the steady-state equilibrium and define the stability condition of the steady-state equilibrium with the determinant and trace of the Jacobian matrix. This part will also briefly address what would happen to the stability condition when the sectoral externalities are assumed to be different and we will also present the results of numerical exercises to see the effect of changes in the parameter values on

the stability of the dynamic system. In addition, the macroeconomic dynamics under the assumption of a lump-sum transfer will be compared with the case of wasteful government spending. Concluding remarks will be presented in the last sub-section.

### 2.1.2 Model

**Private Technology** In Benhabib and Farmer (1996), it is assumed that there are two sectors producing consumer goods, ‘ $C$ ’, and investment goods, ‘ $I$ ’. Each sector faces its own constant-returns-to-scale technology

$$C = A(\mu_L L)^a (\mu_K K)^b, \quad I = B(\{1 - \mu_L\}L)^a (\{1 - \mu_K\}K)^b, \quad a + b = 1, \quad (1)$$

where  $K$  and  $L$  represent economy-wide capital and labour input and  $\mu_K$  and  $\mu_L$  denote the fractions of capital and labour used to produce consumer goods. Now normalize the price of investment goods to 1 and denote the price of consumer goods by  $p$ . The first-order conditions for profit maximization in each sector are

$$\frac{pbC}{\mu_K K} = \frac{bI}{\{1 - \mu_K\}K} = r, \quad \frac{paC}{\mu_L L} = \frac{aI}{\{1 - \mu_L\}L} = w, \quad (2)$$

where  $r$  and  $w$  represent economy-wide factor prices for capital and labour. From (2), we can notice the fact that  $\mu_K = \mu_L = \mu$ . Then (1) can be rewritten as

$$I + \left(\frac{B}{A}\right)C = I + pC = BL^a K^b \equiv Y. \quad (3)$$

Equation (3) describes the production possibilities frontier, the ‘ppf’, which has a linear form for a private firm (Benhabib and Farmer, 1996).

**Social Technology** Scaling coefficients for each sector,  $A$  and  $B$ , are defined as

$$A = (\bar{\mu}_L \bar{L})^{a\theta} (\bar{\mu}_K \bar{K})^{b\theta} \bar{L}^{a\sigma} \bar{K}^{b\gamma}, \quad (4)$$

$$B = (\{1 - \bar{\mu}_L\} \bar{L})^{a\theta} (\{1 - \bar{\mu}_K\} \bar{K})^{b\theta} \bar{L}^{a\sigma} \bar{K}^{b\gamma}, \quad (5)$$

where variables with a bar denote economy-wide averages, which are assumed as given to each private firm. In this form, externalities arise from two sources: an aggregate labour and capital externality represented by  $\sigma$  and  $\gamma$ , and a sector-specific externality by  $\theta$ . If  $\theta$  equals zero, i.e. there is no sector-specific externality, the model corresponds to one sector growth model in Benhabib and Farmer (1994). Since identical firms are assumed, it holds that  $\bar{\mu}_K = \bar{\mu}_L = \mu$  and  $\bar{K} = K$ ,  $\bar{L} = L$ .

$$C = \mu^{(1+\theta)} L^{\alpha(1+\theta+\sigma)} K^{b(1+\theta+\gamma)} = \mu^\nu L^\alpha K^\beta, \quad (6)$$

$$I = \{1 - \mu\}^{(1+\theta)} L^{\alpha(1+\theta+\sigma)} K^{b(1+\theta+\gamma)} = \{1 - \mu\}^\nu L^\alpha K^\beta, \quad (7)$$

where  $\nu = 1 + \theta$ ,  $\alpha = a(1 + \theta + \sigma)$  and  $\beta = b(1 + \theta + \gamma)$ . Previous results lead to the social production possibilities frontier, the social ‘ppf’ as

$$C^{\frac{1}{\nu}} + I^{\frac{1}{\nu}} = L^{\frac{\alpha}{\nu}} K^{\frac{\beta}{\nu}}.$$

(Benhabib and Farmer, 1996)

**Government Sector** Following Guo and Lansing (1998), the government is assumed to finance its expenditure with an income tax schedules,  $\tau$  that has the form

$$\tau = 1 - \psi \left( \frac{\bar{Y}}{Y} \right)^\phi,$$

where  $\bar{Y}$  represents the steady-state level of output for the economy. The parameter  $\psi$  shows the level of the income tax schedule which has income tax rate,  $1 - \psi$  at the steady-state of the economy. The progressivity of the tax schedule depends on the value of parameter  $\phi$ . The progressivity of the income tax schedule can be explained by the relationship between the average income tax rate ( $\tau$ ) and the marginal income tax rate

$(\tau_m)$  which is defined as

$$\tau_m = \frac{\partial(\tau Y)}{\partial Y} = 1 - \psi(1 - \phi)\left(\frac{\bar{Y}}{Y}\right)^\phi = \tau + \phi\psi\left(\frac{\bar{Y}}{Y}\right)^\phi.$$

If  $\phi$  is greater than zero, the income tax can be said to be progressive since the marginal tax rate is greater than the average tax rate. When  $\phi = 0$ , the income tax has a constant tax schedule. For negative values of  $\phi$ , the income tax rate shows a regressive property as it decreases as the tax base expands. Both  $\tau$  and  $\tau_m$  are assumed to be less than 1 to exclude the case where the government confiscates all of the income or there is no incentive for the consumer to save and work. In addition, the marginal after-tax interest rate,  $(1 - \tau_m)r$ , is assumed to be decreasing in  $K$ . Summarising these restrictions, we impose a boundary on  $\psi$  and  $\phi$  as

$$0 < \psi \leq 1, \quad \frac{\beta - 1}{\beta} < \phi < 1.$$

Since the government is assumed to be implementing a balanced-budget rule, government expenditure,  $G$ , is

$$G = \tau Y. \tag{8}$$

This government spending does not provide any services to private production or utility.

**Consumer Problem** The representative consumer is assumed to choose a consumption and labour bundle for each time to solve the present value utility maximization problem

$$\max \int_0^\infty U(C, L)e^{-\rho t} dt, \tag{9}$$

where  $\rho$  is discount rate reflecting time-preference of the consumer. The utility function,  $U(C, L)$  is given by

$$U(C, L) = \ln C - \frac{L^{1+\chi}}{1+\chi}, \tag{10}$$

where  $C$  and  $L$  are the consumption and labour supply choice of the individual consumer and  $\chi$  denotes the inverse of the labour supply elasticity. Each consumer earns his income,

$$wL + rK = pC + I = Y, \quad (11)$$

in return for providing capital and labour to the production process. Since the government has its own demand for goods from government expenditure,  $G$ , we should consider the composition of this government spending. For convenience, we assume that the government only demands the investment good. Then the investment good production should cover the government expenditure,  $G$  and the demand from the private sector for the capital accumulation,  $\dot{K} + \delta K$  where  $\delta$  is depreciation rate so that

$$I = \dot{K} + \delta K + G. \quad (12)$$

Using (11), (12) and (8), the budget constraint of the consumer can be expressed as

$$\dot{K} = (1 - \tau)BL^a K^b - pC - \delta K.$$

We can construct the Hamiltonian as

$$H(C, L, K, \Lambda) = \ln C - (1 + \chi)^{-1} L^{1+\chi} + \Lambda \{(1 - \tau)BL^a K^b - pC - \delta K\},$$

where  $\Lambda$  is the co-state variable associated with the law of motion for capital. The first-order conditions for this problem can be written as

$$\frac{\partial H}{\partial C} = 0 \iff \frac{1}{C} = p\Lambda, \quad (13)$$

$$\frac{\partial H}{\partial L} = 0 \iff L^\chi = \Lambda(1 - \phi)\psi\left(\frac{\bar{Y}}{Y}\right)^\phi aBL^{a-1}K^b, \quad (14)$$

$$\frac{\partial H}{\partial K} = \rho\Lambda - \dot{\Lambda} \iff \frac{\dot{\Lambda}}{\Lambda} = (\rho + \delta) - (1 - \phi)\psi\left(\frac{\bar{Y}}{Y}\right)^\phi bBL^a K^{b-1}, \quad (15)$$

$$\dot{K} = \psi \bar{Y}^\phi Y^{1-\phi} - pC - \delta K, \quad (16)$$

with transversality condition,  $\lim_{t \rightarrow \infty} e^{-\rho t} \Lambda K = 0$ .

### 2.1.3 Analysis of Dynamics

**Equilibrium and Stability Condition** In a symmetric equilibrium, every consumer makes the same choice, so that  $\bar{K} = K$ ,  $\bar{L} = L$  and  $\bar{\mu} = \mu$ . Then we define a new variable  $S = 1/\mu$  á la Benhabib and Farmer (1996), and from (2) and (3), it can be shown that  $BL^\alpha K^\beta = pC + I = S/\Lambda$ . Using this, equation (15) and (16) can be rewritten as

$$\frac{\dot{\Lambda}}{\Lambda} = (\rho + \delta) - \frac{(1 - \phi)\psi(\bar{Y})^\phi b S^{1-\phi}}{\Lambda^{1-\phi} K}, \quad (17)$$

$$\frac{\dot{K}}{K} = \frac{\psi(\bar{Y})^\phi S^{1-\phi}}{\Lambda^{1-\phi} K} - \frac{1}{\Lambda K} - \delta. \quad (18)$$

Letting  $\lambda = \log \Lambda$ ,  $k = \log K$ , and  $s = \log S$ , these two equations can be expressed as

$$\dot{\lambda} = (\rho + \delta) - (1 - \phi)\psi(\bar{Y})^\phi b e^{(1-\phi)s - (1-\phi)\lambda - k}, \quad (19)$$

$$\dot{k} = \psi(\bar{Y})^\phi e^{(1-\phi)s - (1-\phi)\lambda - k} - e^{-\lambda - k} - \delta. \quad (20)$$

If the variable  $s$  could be defined as a function of  $\lambda$  and  $k$ , the dynamic system described above could be understood as a two-dimensional system of differential equations with respect to  $\lambda$  and  $k$ . From (6),  $S$  can be newly defined as

$$S = \frac{L^{\frac{\alpha}{v}} K^{\frac{\beta}{v}}}{C^{\frac{1}{v}}}, \quad (21)$$

and  $A$  and  $B$  in (4) and (5) can be rewritten using  $S$  as

$$A = \frac{L^{\alpha-a} K^{\beta-b}}{S^{v-1}} = \frac{1}{(S-1)^{v-1}} B.$$

Then, from (13) and (14),  $C$  can be expressed using  $S$  and  $\Lambda$  as

$$C = \frac{1}{(S-1)^{v-1}\Lambda},$$

$$L^{\chi+1} = a(1-\phi)\psi(\bar{Y})^\phi S^{1-\phi}\Lambda^\phi$$

Using these two expressions, equation (21) can be transformed to be a function of  $S$ ,  $\Lambda$  and  $K$  as

$$(S-1)^{1-v}S^{v-\frac{(1-\phi)\alpha}{1+\chi}} = [a(1-\phi)\psi(Y)^\phi]^{\frac{\alpha}{1+\chi}}\Lambda^{\frac{\phi\alpha}{1+\chi}+1}K^\beta. \quad (22)$$

From the implicit function (22), we can define the partial derivatives of  $s(\lambda, k)$  with respect to  $\lambda$  and  $k$ , denoted as  $s_\lambda$  and  $s_k$  by

$$\frac{\partial s}{\partial \lambda} = s_\lambda = \frac{\frac{\phi\alpha}{1+\chi} + 1}{v - \frac{(1-\phi)\alpha}{1+\chi} + (1-v)\frac{S}{S-1}},$$

$$\frac{\partial s}{\partial k} = s_k = \frac{\beta}{v - \frac{(1-\phi)\alpha}{1+\chi} + (1-v)\frac{S}{S-1}} = \left( \frac{\beta}{\frac{\phi\alpha}{1+\chi} + 1} \right) s_\lambda.$$

Now the Jacobian,  $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$ , for the dynamic system (19) and (20) can be derived as

$$J_{11} = (1-\phi)^2\psi(\bar{Y})^\phi b(1-s_\lambda)e^{(1-\phi)s-(1-\phi)\lambda-k},$$

$$J_{12} = (1-\phi)\psi(\bar{Y})^\phi b[1 - (1-\phi)s_k]e^{(1-\phi)s-(1-\phi)\lambda-k},$$

$$J_{21} = e^{-\lambda-k} - (1-\phi)\psi(\bar{Y})^\phi(1-s_\lambda)e^{(1-\phi)s-(1-\phi)\lambda-k},$$

$$J_{22} = e^{-\lambda-k} - \psi(\bar{Y})^\phi[1 - (1-\phi)s_k]e^{(1-\phi)s-(1-\phi)\lambda-k}.$$

At the steady-state,  $\bar{Y} = B\tilde{L}^a\tilde{K}^b = \tilde{S}/\tilde{\Lambda} = e^{\tilde{s}-\tilde{\lambda}}$ , where variables with a tilde represent steady-state values. Putting the steady-state value of  $\bar{Y}$  into equations (17)

and (18), three conditions can be derived:

$$e^{\tilde{s}-\tilde{\lambda}-\tilde{k}} = \frac{\rho + \delta}{b\psi(1 - \phi)},$$

$$e^{-\tilde{\lambda}-\tilde{k}} = \frac{\rho + \delta\{1 - b(1 - \phi)\}}{b(1 - \phi)},$$

$$e^{\tilde{s}} = \tilde{S} = \frac{\rho + \delta}{\psi[\rho + \delta\{1 - b(1 - \phi)\}]}.$$

The previous results lead us to a Jacobian matrix evaluated at the steady-state:

$$J_{11}^{s.s} = (1 - \phi)(1 - \tilde{s}_\lambda)(\rho + \delta),$$

$$J_{12}^{s.s} = [1 - (1 - \phi)\widehat{\beta}\tilde{s}_\lambda](\rho + \delta),$$

$$J_{21}^{s.s} = \frac{[1 - (1 - \phi)(1 - \tilde{s}_\lambda)]}{b(1 - \phi)}(\rho + \delta) - \delta,$$

$$J_{22}^{s.s} = \frac{(\rho + \delta)}{b}\widehat{\beta}\tilde{s}_\lambda - \delta,$$

where  $\tilde{s}_\lambda = \frac{\frac{\phi\alpha}{1+\chi}+1}{v-\frac{(1-\phi)\alpha}{1+\chi}+(1-v)\frac{\tilde{S}}{\tilde{S}-1}}$ ,  $\tilde{s}_k = \frac{\beta}{v-\frac{(1-\phi)\alpha}{1+\chi}+(1-v)\frac{\tilde{S}}{\tilde{S}-1}}$ , and  $\widehat{\beta} = \frac{\beta}{\frac{\phi\alpha}{1+\chi}+1}$ . This Jacobian now yields a trace and a determinant given by

$$Tr(J) = \frac{\rho + \delta}{b} \left[ \left\{ \widehat{\beta} - b(1 - \phi) \right\} \tilde{s}_\lambda + \frac{b\{(1 - \phi)\rho - \phi\delta\}}{\rho + \delta} \right], \quad (23)$$

$$Det(J) = \frac{(\rho + \delta)^2}{b} \left[ \frac{1}{1 - \phi} - \frac{b\delta}{\rho + \delta} \right] \left[ (1 - \phi)(\widehat{\beta} - 1)\tilde{s}_\lambda - \phi \right]. \quad (24)$$

It can be easily checked that when  $\phi = 0$  and  $\psi = 1$ , i.e. there is no tax,

$$\tilde{s}_\lambda = \frac{1}{v - \frac{\alpha}{1+\chi} + (1-v)\frac{\tilde{S}}{\tilde{S}-1}}, \quad \tilde{s}_k = \frac{\beta}{v - \frac{\alpha}{1+\chi} + (1-v)\frac{\tilde{S}}{\tilde{S}-1}} = \beta\tilde{s}_\lambda,$$

$$Tr(J) = \frac{\rho + \delta}{b} \left[ (\beta - b)\tilde{s}_\lambda + \frac{b\rho}{\rho + \delta} \right], \quad Det(J) = \frac{(\rho + \delta)^2}{b} \left[ 1 - \frac{b\delta}{\rho + \delta} \right] (\beta - 1)\tilde{s}_\lambda,$$

which exactly correspond to the results from Benhabib and Farmer (1996).

The trace and the determinant of the Jacobian matrix defined in (23) and (24) determine the stability properties of the dynamic system described by (19) and (20). Firstly, since the system has one pre-determined variable,  $k_t$ , the necessary and sufficient condition for the system to have a saddle-path equilibrium is that the two eigenvalues of the Jacobian should have opposite signs to each other, which corresponds to the condition that the determinant should have a negative value. The system will show multiple equilibria if and only if both eigenvalues have negative real parts so that the trace would be negative with the determinant being positive. The final case is that with no equilibrium path, where any deviation from the steady-state would result in divergence. This arises if and only if both the determinant and the trace are positive.

**A Case with Different Sectoral Externalities** The model discussed so far has assumed that the consumer good sector and the investment good sector have the same sectoral externality,  $\theta$ . However, Harrison (2001) discovered that indeterminacy could arise even when only the investment good sector has a certain amount of externality. If the assumption that the sector has a different externality factors denoted as  $\theta_c$  and  $\theta_I$ , is added to the previous model, the scaling coefficient in (4) and (5) can be redefined as

$$A = (\bar{\mu}_L \bar{L})^{a\theta_C} (\bar{\mu}_K \bar{K})^{b\theta_C} \bar{L}^{a\sigma} \bar{K}^{b\gamma},$$

$$B = (\{1 - \bar{\mu}_L\} \bar{L})^{a\theta_I} (\{1 - \bar{\mu}_K\} \bar{K})^{b\theta_I} \bar{L}^{a\sigma} \bar{K}^{b\gamma},$$

and the production functions for the consumer and investment sectors would be written as

$$C = \mu^{(1+\theta_C)} L^{a(1+\theta_C+\sigma)} K^{b(1+\theta_C+\gamma)} = \mu^{v_C} L^{\alpha_C} K^{\beta_C}, \quad (25)$$

$$I = \{1 - \mu\}^{(1+\theta_I)} L^{a(1+\theta_I+\sigma)} K^{b(1+\theta_I+\gamma)} = \{1 - \mu\}^{v_I} L^{\alpha_I} K^{\beta_I}, \quad (26)$$

where  $v_c = 1 + \theta_C$ ,  $v_I = 1 + \theta_I$ ,  $\alpha_C = a(1 + \theta_C + \sigma)$ ,  $\beta_C = b(1 + \theta_C + \gamma)$ ,  $\alpha_I = a(1 + \theta_I + \sigma)$ ,  $\beta_C = b(1 + \theta_I + \gamma)$ . From (25),  $S(1/\mu)$  can be redefined as

$$S = \frac{L^{\frac{\alpha_C}{v_C}} K^{\frac{\beta_C}{v_C}}}{C^{\frac{1}{v_C}}}.$$

Using the fact that  $A = \mu^{v_C-1} L^{\alpha_C-a} K^{\beta_C-b}$  and  $B = \{1 - \mu\}^{v_I-1} L^{\alpha_I-a} K^{\beta_I-b}$ ,

$$p = \frac{B}{A} = (S - 1)^{v_I-1} S^{v_C-v_I} L^{\alpha_I-\alpha_C} K^{\beta_I-\beta_C}. \quad (27)$$

Since the consumer still faces the same utility maximization problem as defined in (9), equations (13) and (14) still hold in this case, leading to the results

$$C = \frac{1}{(S - 1)^{v_I-1} S^{v_C-v_I} L^{\alpha_I-\alpha_C} K^{\beta_I-\beta_C} \Lambda},$$

$$L^{x+1} = a(1 - \phi)\psi(\bar{Y})^\phi S^{1-\phi} \Lambda^\phi.$$

With these results, equation (27) can be transformed as an implicit function of  $S$ ,  $\Lambda$  and  $K$  as

$$(S - 1)^{1-v_I} S^{v_I-\frac{(1-\phi)\alpha_I}{x+1}} = [a(1 - \phi)\psi(Y)^\phi]^{\frac{\alpha_I}{1+x}} \Lambda^{\frac{\phi\alpha_I}{1+x}+1} K^{\beta_I}.$$

From this function, the partial derivatives of  $s(\lambda, k)$  with respect to  $\lambda$  and  $k$  calculated at the steady-state can be newly defined as

$$\hat{s}_\lambda = \frac{\frac{\phi\alpha_I}{1+x} + 1}{v_I - \frac{(1-\phi)\alpha_I}{1+x} + (1 - v_I)\frac{\tilde{S}}{\tilde{S}-1}}, \quad (28)$$

$$\hat{s}_k = \frac{\beta}{v_I - \frac{(1-\phi)\alpha_I}{1+x} + (1 - v_I)\frac{\tilde{S}}{\tilde{S}-1}} = \left( \frac{\beta}{\frac{\phi\alpha_I}{1+x} + 1} \right) \hat{s}_\lambda. \quad (29)$$

From (28) and (29), it is noticeable that these values of the partial derivatives only depend on the degree of externality of the investment good sector. Therefore, even though it is assumed that  $\theta_c \neq \theta_I$ , this dynamic system has exactly the same dynamic

properties to the case with same sectoral externalities only if  $\theta = \theta_I$ . This independency from the consumer sector externality corresponds to the result of Harrison (2001), where it was pointed out that this feature would result from the logarithmic utility function assumption.

**Results from Benchmark Parameterization** In this sub-section, we will present some results from a numerical exercise using the calibrated parameters in Benhabib and Farmer (1996), where the capital share of national income,  $b$ , is set at 0.3, so the share of labour,  $a$ , is chosen to be 0.7. The discount factor,  $\rho$  and the depreciation rate of capital,  $\delta$  are fixed at 0.05 and 0.1. The aggregate externality levels,  $\sigma$  and  $\gamma$ , are both set equal to zero. As a start, the first numerical experiment is implemented for a case with no tax. The result is shown in Figure 1-(a) which displays the same area of indeterminacy in the  $\chi - \theta$  space as Benhabib and Farmer (1996, p433). Figure 1-(b) is the result of a case with a flat income tax schedule. The graph is drawn in the same  $\chi - \theta$  space as Figure 1-(a) and it can easily be noted that the area in which the indeterminacy can arise is significantly reduced by the introduction of flat income tax. The consideration of a progressive feature in the income tax is shown in Figure 1-(c) which shows that progressive taxation can reduce the parameter space for the indeterminacy even further. On the contrary, the economy is more likely to show indeterminacy with the regressive income tax schedule as shown in Figure 1-(d).

To check the effect of the flat tax scheme on stability, Figure 1-(e) is drawn in the  $\chi - \psi$  space under the fixed sectoral externality,  $\theta = 0.108$ , which is the empirically calibrated value for the U.S economy suggested in Guo and Harrison (2001). Even though it is very unlikely that the indeterminacy occurs with this level of externality, it can be seen that the economy can be moved from an indeterminate steady-state to a determinate one by a flat income tax schedule. Finally Figure 1-(f) shows the relationship between the degree of progressivity and indeterminacy. The parameter values for  $\psi$  and  $\chi$  are deliberately chosen to be 0.8 and 0.25 for the purpose of comparison with the result of Guo and Harrison (2001, p83). According to Figure 1-(f), unlike the

conclusion of Guo and Harrison (2001), a more regressive income tax schedule makes the economy more susceptible to the indeterminacy.

**A Case with Transfers** The previous analysis is based on the assumption that government spending does not provide any service to the private sector as in Guo and Lansing (1998). If we assume that tax revenue is repaid as lump-sum transfer to the agents as in Guo and Harrison (2001), the dynamic system should be written as

$$\dot{\lambda} = (\rho + \delta) - (1 - \phi)\psi(\bar{Y})^\phi b e^{(1-\phi)s - (1-\phi)\lambda - k},$$

$$\dot{k} = e^{s-\lambda-k} - e^{-\lambda-k} - \delta.$$

The Jacobian,  $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$  of this system can be derived as

$$J_{11} = (1 - \phi)^2 \psi(\bar{Y})^\phi b (1 - s_\lambda) e^{(1-\phi)s - (1-\phi)\lambda - k},$$

$$J_{12} = (1 - \phi)\psi(\bar{Y})^\phi b [1 - (1 - \phi)s_k] e^{(1-\phi)s - (1-\phi)\lambda - k},$$

$$J_{21} = e^{-\lambda-k} - (1 - s_\lambda) e^{s-\lambda-k},$$

$$J_{22} = e^{-\lambda-k} - (1 - s_k) e^{s-\lambda-k}.$$

At a steady-state, these three results hold:

$$e^{\bar{s} - \bar{\lambda} - \bar{k}} = \frac{\rho + \delta}{b\psi(1 - \phi)},$$

$$e^{-\bar{\lambda} - \bar{k}} = \frac{\rho + \delta \{1 - b(1 - \phi)\psi\}}{b(1 - \phi)\psi},$$

$$e^{\bar{s}} = \bar{S} = \frac{\rho + \delta}{\rho + \delta \{1 - b(1 - \phi)\psi\}}.$$

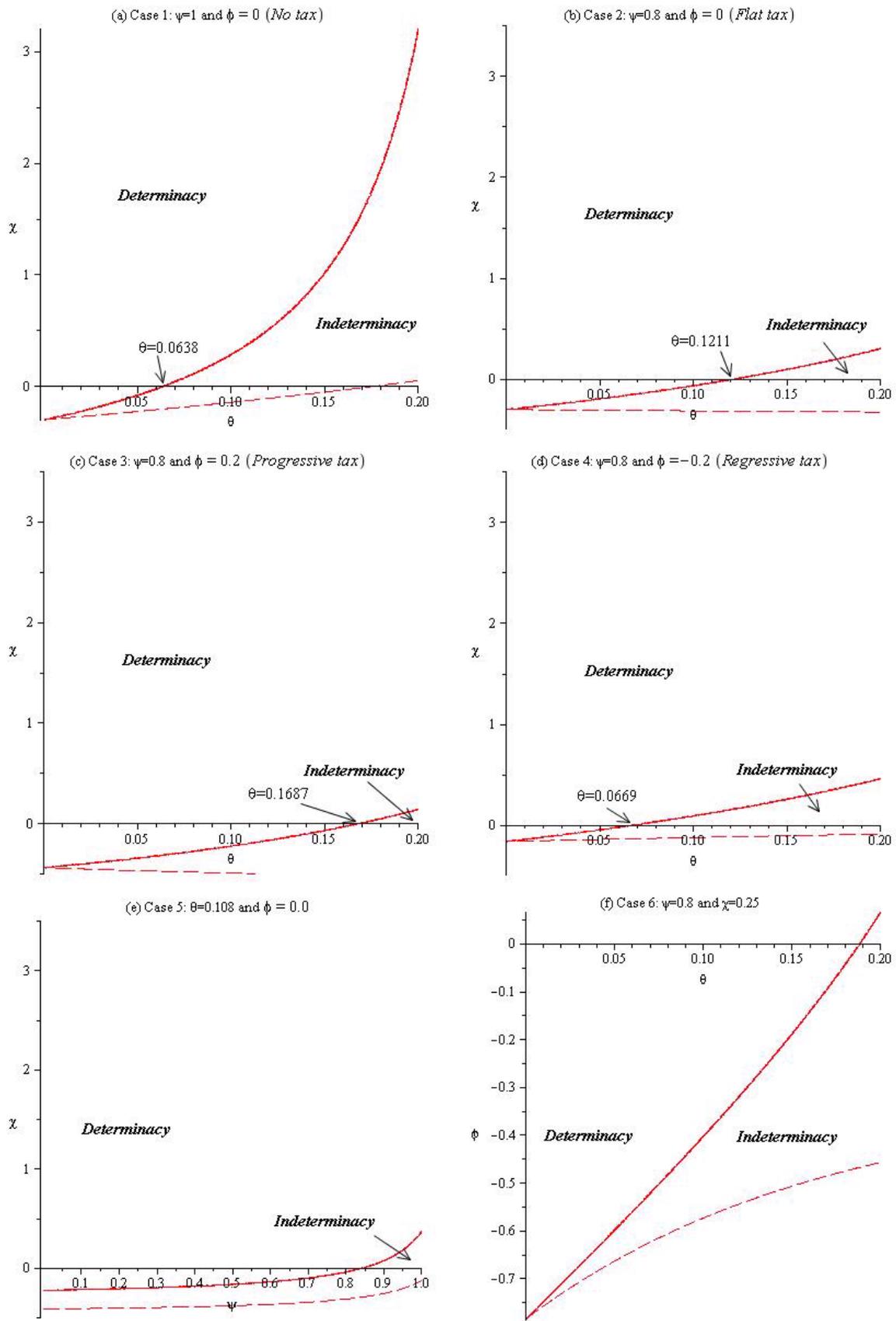


Figure 1: Results from Benchmark Parameterization

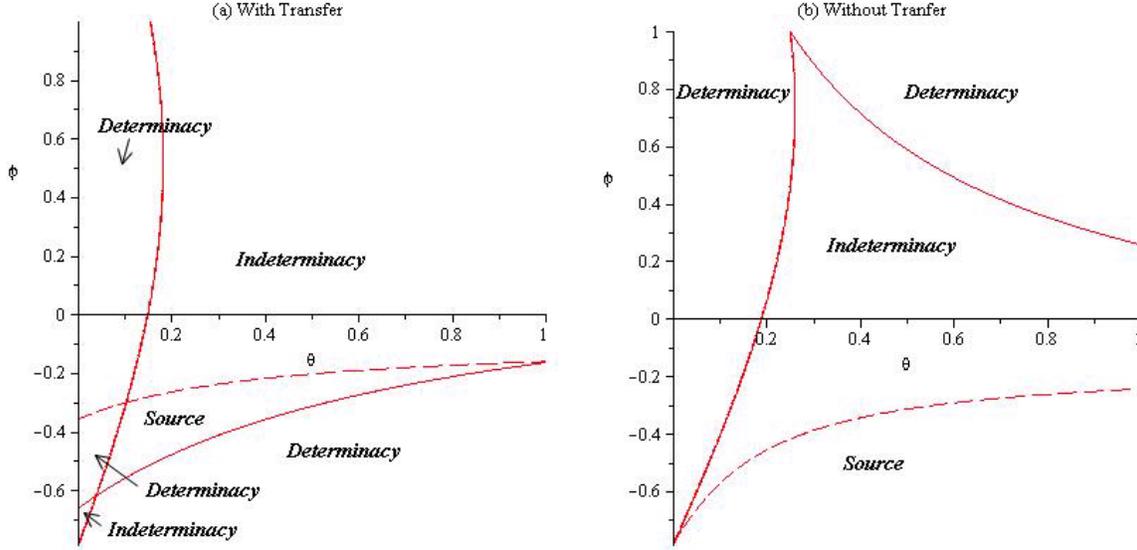


Figure 2: Macroeconomic Dynamics with or without transfer

where variables with a bar denote the steady-state value. Using these facts, the Jacobian matrix evaluated at the steady-state can be obtained as

$$J_{11}^{s.s} = (1 - \phi)(1 - \bar{s}_\lambda)(\rho + \delta),$$

$$J_{12}^{s.s} = [1 - (1 - \phi)\widehat{\beta}\bar{s}_\lambda](\rho + \delta),$$

$$J_{21}^{s.s} = \frac{\bar{s}_\lambda(\rho + \delta)}{b(1 - \phi)\psi} - \delta,$$

$$J_{22}^{s.s} = \frac{(\rho + \delta)}{b(1 - \phi)\psi}\widehat{\beta}\bar{s}_\lambda - \delta,$$

where  $\bar{s}_\lambda = \frac{\frac{\phi\alpha}{1+\chi}+1}{v-\frac{(1-\phi)\alpha}{1+\chi}+(1-v)\frac{\bar{s}}{\bar{s}-1}}$ ,  $\widehat{\beta} = \frac{\beta}{\frac{\phi\alpha}{1+\chi}+1}$ .

Figure 2 displays the comparison of the stability of the steady-state in the presence of lump-sum transfers with the case of wasteful government spending. With lump-sum transfers, Figure 2-(a) shows that when the degree of the externality ( $\theta$ ) is sufficiently large determinacy can be obtained under a regressive tax schedule, which reproduces the assertions from Guo and Harrison (2001). On the contrary, with the wasteful government spending, only progressive taxation can secure determinacy regardless of the magnitude of the externality as can be seen in Figure 2-(b).

#### **2.1.4 Concluding Remarks**

This section, as opposed to Guo and Harrison (2001), shows that progressive taxation can work as a stabilizer of economic fluctuations caused by indeterminacy in a two-sector growth model with sector-specific externalities. It also shows that this benefit of progressive taxation in a two sector model is possible when income tax is just levied on total income, rather than levied separately on labour and capital income as in Sims (2005). And the assumption of different sector-specific externalities turns out to make no difference to the results. One intriguing observation here is that unlike in the one-sector case, a flat tax can change the macroeconomic dynamics. As for the reason why Guo and Harrison (2001) were in favour of regressive taxation as sunspot stabilizer, we show that the assumption that the tax revenue is repaid as a lump-sum transfer in Guo and Harrison (2001) is very critical for reaching that conclusion. In our model, tax revenue finances government spending which is wasteful.

## **2.2 Indeterminacy in One-sector Growth model and Progressive Taxation: Revisit**

### **2.2.1 Introduction**

In a one-sector growth model with increasing returns to scale, it is regarded as a well-established conclusion that progressive income tax can mitigate business cycle fluctuations driven by the self-fulfilling expectations. The usefulness of this classic automatic stabilizer even for belief-driven fluctuations has been repeatedly assured in many papers such as Guo and Lansing (1998), Guo (1999), Schmit-Grohe and Uribe (1997) and Christiano and Harrison (1998). Most of these papers supporting progressive taxation, however, share the assumption that the government levies the income tax to finance government expenditure but this expenditure is simply wasteful as it does not serve the private sector at all. As we observed in the previous section, what we assume about the use of tax revenue can make a very significant difference to conclusions about whether

a progressive or a regressive income tax schedule should be adopted to suppress the likelihood of indeterminacy.

In this section, we revisit the Benhabib-Farmer one-sector (1994) growth model and reinvestigate the role of income tax progressivity in eliminating the indeterminate equilibrium of an economy. The model we use here differs from Guo and Lansing (1998) in only one aspect. We amend the wasteful government expenditure assumption and assume that the government provides a lump-sum transfer with its tax revenue. With this change, we find that the conclusion of Guo and Lansing (1998) is changed significantly. Not only progressive taxation but also sufficiently regressive taxation can effectively stabilize sunspot fluctuations.

When we consider separate income taxes for labour and capital income as in Guo (1999), we find that the change in the assumption about the use of tax revenue can also give us an insight about the role of the capital income tax which is not mentioned in the previous literature. In Guo (1999), only progressive labour income taxation can prevent the indeterminacy from arising, while there is no room for capital income tax to be used. On the contrary, with lump-sum transfers, it turns out that the capital income tax should be regressive to ensure the saddle-path stability.

The remainder of this section proceeds as follows. The second sub-section investigates the one-sector growth model with increasing-returns-to-scale (Benhabib and Farmer, 1994) with income tax on total income and lump-sum transfers. In the third sub-section, the income tax is assumed to be levied separately on labour and capital income. The last sub-section concludes.

### **2.2.2 One-sector Growth Model with Tax on Total Income**

Since the Benhabib-Farmer two-sector model (1996) embraces the one-sector model as a special case, we construct our model from the set-up introduced in the previous section. If we assume that there is only one good in the economy in the model in the previous section, the private technology can be described with a Cobb-Douglas

production function,

$$Y = BL^a K^b, \quad a + b = 1.$$

There is no sectoral externality so that  $\theta = 0$ , so if we suppose that  $\sigma = \gamma$ , the scaling coefficient,  $B$ , is defined here as

$$B = \bar{L}^{a\sigma} \bar{K}^{b\sigma},$$

where  $\bar{L}$  and  $\bar{K}$  are economy-wide average levels of labour and capital. In a symmetric equilibrium, it holds that  $\bar{L} = L$  and  $\bar{K} = K$ . Then the aggregate production function is given by

$$Y = L^\alpha K^\beta, \tag{30}$$

where  $\alpha = a(1 + \sigma)$  and  $\beta = b(1 + \sigma)$ .

The government's tax policy is the same as in the previous section. The government chooses the income tax rate,  $\tau$  given by

$$\tau = 1 - \psi \left( \frac{\bar{Y}}{Y} \right)^\phi,$$

where the level and slope parameters,  $\psi$  and  $\phi$  satisfies the condition that  $0 < \psi \leq 1$ ,  $\frac{\beta-1}{\beta} < \phi < 1$ .

### A Case with Wasteful Government Spending

We start this section by reviewing the result of Guo and Lansing (1998) where the government implements a balanced-budget with the budget constraint,  $G = \tau Y$ . Government spending,  $G$  is assumed to be wasteful. With the utility function in (10), the representative consumer still faces the same optimization problem as (9) and has the same first-order conditions from (13) to (16). As shown in Guo and Lansing (1998), the dynamic system derived from (15) and (16) can be rewritten with the logarithmic transformation of variables,  $k = \log K$ ,  $y = \log Y$  and  $c = \log C$ , as

$$\dot{c} = b(1 - \phi)\psi \bar{Y}^\phi e^{(1-\phi)y-k} - (\rho + \delta),$$

$$\dot{k} = \psi \bar{Y}^\phi e^{(1-\phi)y-k} - e^{c-k} - \delta.$$

From (30) and (14),

$$y = \alpha l + \beta k, \quad (31)$$

$$(\chi + 1)l = \log \left( a(1 - \phi)\psi \bar{Y}^\phi \right) + (1 - \phi)y - c, \quad (32)$$

where  $l = \log L$ . Using these facts, the dynamic system can be rewritten as

$$\dot{c} = b(1 - \phi)\psi \bar{Y}^\phi e^{\lambda_0 + \lambda_1 k + \lambda_2 c} - (\rho + \delta), \quad (33)$$

$$\dot{k} = \psi \bar{Y}^\phi e^{\lambda_0 + \lambda_1 k + \lambda_2 c} - e^{c-k} - \delta, \quad (34)$$

where

$$\lambda_0 = \frac{\alpha(1 - \phi)}{(\chi + 1) - \alpha(1 - \phi)} \log \left( a(1 - \phi)\psi \bar{Y}^\phi \right),$$

$$\lambda_1 = \frac{(\chi + 1)(\beta(1 - \phi) - 1) + \alpha(1 - \phi)}{(\chi + 1) - \alpha(1 - \phi)},$$

$$\lambda_2 = -\frac{\alpha(1 - \phi)}{(\chi + 1) - \alpha(1 - \phi)}.$$

From (33) and (34) we can define the steady-state values as

$$e^{\bar{y}-\bar{k}} = \frac{(\rho + \delta)}{b(1 - \phi)\psi},$$

$$e^{\bar{c}-\bar{k}} = \frac{(\rho + \delta)}{b(1 - \phi)} - \delta,$$

where variables with a bar denote steady-state values. Combining these results, the Jacobian matrix of the system in (33) and (34) evaluated at the steady-state can be derived as

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

where

$$J_{11} = \lambda_2(\rho + \delta),$$

$$\begin{aligned}
J_{12} &= \lambda_1(\rho + \delta), \\
J_{21} &= \frac{(\rho + \delta)}{b(1 - \phi)}(\lambda_2 - 1) + \delta, \\
J_{22} &= \frac{(\rho + \delta)}{b(1 - \phi)}(\lambda_1 + 1) - \delta.
\end{aligned}$$

If we maintain the imposition of a lower bound on  $\phi$ ,  $\beta(1 - \phi) - 1 < 0$  which is a condition that guarantees the after-tax return on capital is strictly decreasing in  $K$ , the trace ( $T$ ) and the determinant ( $D$ ) of this Jacobian give us a necessary and sufficient condition for the model to exhibit deterministic equilibrium as

$$\alpha(1 - \phi) - 1 < \chi. \quad (35)$$

So, when an economy shows indeterminacy, saddle-path stability can be restored by adding more progressivity (larger  $\phi$ ) to the income tax schedule.

### A Case with Lump-sum Transfers

In Guo and Lansing (1998), government tax revenue, or public purchases of the good, is assumed to be ‘*thrown into the sea*’ without providing any services to the economy. Now following Guo and Harrison (2001, 2013), we assume that tax revenue is returned to the household as a lump-sum transfer ( $T$ ). While the consumer’s budget constraint is changed, the first-order conditions remain the same as long as the transfer is taken as given by the consumer. The aggregate resource constraint is changed from (16) to

$$\dot{K} = Y - C - \delta K. \quad (36)$$

(15) and (36) constitute a dynamic system after logarithmic transformation given by

$$\begin{aligned}
\dot{c} &= b(1 - \phi)\psi\bar{Y}^{\phi}e^{(1-\phi)y-k} - (\rho + \delta), \\
\dot{k} &= e^{y-k} - e^{c-k} - \delta.
\end{aligned}$$

Using (31) and (32), the dynamic system can be rewritten as

$$\dot{c} = b(1 - \phi)\psi\bar{Y}^\phi e^{\lambda_0 + \lambda_1^c k + \lambda_2^c c} - (\rho + \delta), \quad (37)$$

$$\dot{k} = e^{\lambda_0 + \lambda_1^k k + \lambda_2^k c} - e^{c-k} - \delta, \quad (38)$$

where

$$\lambda_0 = \frac{\alpha(1 - \phi)}{(\chi + 1) - \alpha(1 - \phi)} \log \left( a(1 - \phi)\psi\bar{Y}^\phi \right),$$

$$\lambda_1^c = \frac{(\chi + 1)(\beta(1 - \phi) - 1) + \alpha(1 - \phi)}{(\chi + 1) - \alpha(1 - \phi)},$$

$$\lambda_1^k = \frac{(\beta - 1)(\chi + 1) + \alpha(1 - \phi)}{(\chi + 1) - \alpha(1 - \phi)},$$

$$\lambda_2^c = -\frac{\alpha(1 - \phi)}{(\chi + 1) - \alpha(1 - \phi)},$$

$$\lambda_2^k = -\frac{\alpha}{(\chi + 1) - \alpha(1 - \phi)}.$$

It can be noticed that at the steady-state,

$$\bar{Y}^\phi e^{(1-\phi)y-k} = \left(\frac{\bar{Y}}{Y}\right)^\phi \frac{Y}{K} = \frac{Y}{K} = e^{y-k},$$

and from (37) and (38), we can define the steady-state values as

$$\bar{Y}^\phi e^{\lambda_0 + \lambda_1^c \bar{k} + \lambda_2^c \bar{c}} = e^{\lambda_0 + \lambda_1^k \bar{k} + \lambda_2^k \bar{c}} = \frac{(\rho + \delta)}{b(1 - \phi)\psi},$$

$$e^{\bar{c} - \bar{k}} = \frac{(\rho + \delta)}{b(1 - \phi)\psi} - \delta.$$

Combining these results, the Jacobian matrix of the system in (37) and (38) evaluated at the steady-state can be derived as

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

where

$$J_{11} = \lambda_2^c(\rho + \delta),$$

$$J_{12} = \lambda_1^c(\rho + \delta),$$

$$J_{21} = \frac{(\rho + \delta)}{b(1 - \phi)\psi}(\lambda_2^k - 1) + \delta,$$

$$J_{22} = \frac{(\rho + \delta)}{b(1 - \phi)\psi}(\lambda_1^k + 1) - \delta.$$

The determinant of this Jacobian matrix is

$$D = -\frac{(\rho + \delta)^2}{b(1 - \phi)\psi} \frac{\alpha\phi}{\chi + 1 - \alpha(1 - \phi)} + \left[ \frac{(\rho + \delta)^2}{b(1 - \phi)\psi} - \delta(\rho + \delta) \right] \frac{(\chi + 1)(\beta(1 - \phi) - 1)}{\chi + 1 - \alpha(1 - \phi)}.$$

Suppose that there is indeterminacy with a flat tax schedule so that  $D > 0$  when  $\phi = 0$ . If we add progressivity (positive  $\phi$ ) large enough to satisfy the condition that  $\chi + 1 - \alpha(1 - \phi) > 0$ , the sign of the determinant is changed from positive to negative so that saddle-path stability is restored. Therefore, (35) is still a sufficient condition for determinacy. However, it is not a necessary condition since the determinant also has a negative value when a sufficient regressivity is imposed on the income tax schedule even though  $\chi + 1 - \alpha(1 - \phi) < 0$ .

Figure 3 displays the value of the determinant and the trace of the Jacobian matrix under various levels of income tax progressivity<sup>2</sup>. In the case with wasteful government spending, Figure 3-(a) shows that only positive  $\phi$  (progressive taxation) can guarantee the negative determinant which is the necessary and sufficient condition for saddle-path stability as Guo and Lansing (1998) stated. On the side of negative  $\phi$ , the negative determinant value can be observed only when the degree of regressivity is beyond the lower bound of  $\phi$ . On the contrary, under the consideration of the lump-sum transfer, Figure 3-(b) shows that not only progressive taxation but also regressive taxation can ensure a determinate equilibrium. The negative determinant value is observed in both

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<sup>2</sup>For the other parameter values, it is assumed that  $a = 0.7$ ,  $b = 0.3$ ,  $\rho = 0.05$ ,  $\delta = 0.1$ ,  $\chi = 0.05$ ,  $\psi = 0.8$  and  $\sigma = 0.6$ .

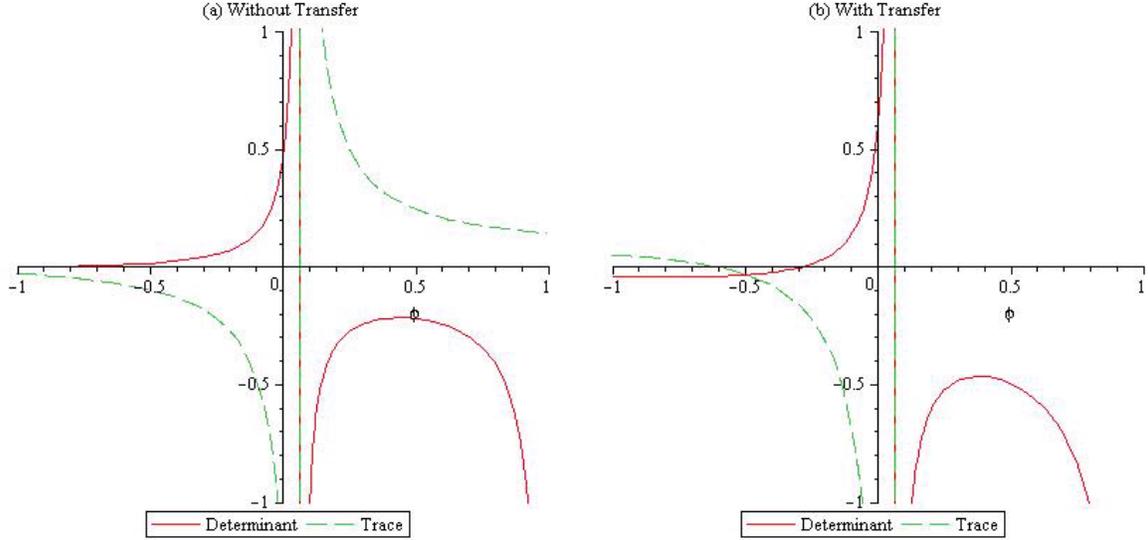


Figure 3: Use of Tax Revenue and Income Tax Progressivity

sides. The magnitude of the regressivity required for determinacy is fairly mild as it exceeds the lower bound of  $\phi$ .

### 2.2.3 One-sector Growth Model with Separate Tax on Labour and Capital Income

Guo (1999) introduced separate labour and capital income taxes with different progressivities into the one-sector growth model of Benhabib-Farmer (1994). The tax rates for labour income ( $\tau_l$ ) and capital income ( $\tau_k$ ) are defined as

$$\tau_l = 1 - \psi_l \left( \frac{\bar{wL}}{w_t L_t} \right)^{\phi_l}, \quad \tau_k = 1 - \psi_k \left( \frac{\bar{rK}}{r_t K_t} \right)^{\phi_k}, \quad (39)$$

where the parameters  $\psi_i$  and  $\phi_i$  represent the level and slope of the tax schedules and variables with a bar denote the steady-state level which is taken as given by the representative consumer. To ensure the existence of the steady-state,  $\phi_k$  is assumed to be within the boundary where the after-tax marginal return on capital is decreasing in  $K$ , which gives the condition that  $\phi_k \in \left( \frac{\beta-1}{\beta}, 1 \right)$ . Again, we consider two cases in this sub-section. The first case assumes that government spending does not contribute to either production or household utility. In the second case, the government repays the

tax revenue to the household as a lump-sum transfer.

### A Case with the Wasteful Government Spending

Under the tax schedule defined in (39), the current-value Hamiltonian for the problem can be constructed as

$$H(C_t, L_t, K_t, \Lambda_t) = \ln C_t - \frac{L_t^{1+\chi}}{1+\chi} + \Lambda_t \{ \psi_l (\overline{wL})^{\phi_l} (w_t L_t)^{1-\phi_l} + \psi_k (\overline{rK})^{\phi_k} (r_t K_t)^{1-\phi_k} - \delta K_t - C_t \}.$$

The first-order conditions for this problem are

$$\frac{1}{C_t} = \Lambda_t, \quad (40a)$$

$$L_t^\chi = \Lambda_t \psi_l (1 - \phi_l) \left( \frac{\overline{wL}}{w_t L_t} \right)^{\phi_l} w_t, \quad (41)$$

$$\psi_k (1 - \phi_k) \left( \frac{\overline{rK}}{r_t K_t} \right)^{\phi_k} r_t - \delta = \rho \Lambda_t - \dot{\Lambda}_t, \quad (42)$$

with transversality condition  $\lim_{t \rightarrow \infty} e^{-\rho t} \Lambda_t K_t = 0$ . Since  $w_t L_t = aY$  and  $r_t K_t = bY$ , it holds that

$$\frac{\overline{wL}}{w_t L_t} = \frac{\overline{rK}}{r_t K_t} = \frac{\bar{Y}}{Y}.$$

Using this fact, (41) and (42) can be expressed as

$$L_t^{\chi+1} = \Lambda_t a \psi_l (1 - \phi_l) \bar{Y}^{\phi_l} Y_t^{1-\phi_l} \quad (43)$$

$$\Lambda_t \left[ b \psi_k (1 - \phi_k) \bar{Y}^{\phi_k} \frac{Y_t^{1-\phi_k}}{K_t} - \delta \right] = \rho \Lambda_t - \dot{\Lambda}_t. \quad (44)$$

From these results, a system of two-dimensional differential equations can be identified as

$$\frac{\dot{C}_t}{C_t} = b \psi_k (1 - \phi_k) \bar{Y}^{\phi_k} \frac{Y_t^{1-\phi_k}}{K_t} - (\delta + \rho), \quad (45)$$

$$\frac{\dot{K}_t}{K_t} = a \psi_l \bar{Y}^{\phi_l} \frac{Y_t^{1-\phi_l}}{K_t} + b \psi_k \bar{Y}^{\phi_k} \frac{Y_t^{1-\phi_k}}{K_t} - \frac{C_t}{K_t} - \delta. \quad (46)$$

In (45) and (46), the ratios of steady-states variables, capital ( $\bar{K}$ ), consumption ( $\bar{C}$ )

and output ( $\bar{Y}$ ) are defined by

$$\frac{\bar{Y}}{\bar{K}} = \frac{(\delta + \rho)}{b\psi_k(1 - \phi_k)},$$

$$\frac{\bar{C}}{\bar{K}} = \frac{a\psi_l + b\psi_k}{b\psi_k(1 - \phi_k)}(\delta + \rho) - \delta.$$

With the logarithm transformation, the system of (45) and (46) can be described as

$$\dot{k}_t = a\psi_l \bar{Y}^{\phi_l} e^{(1-\phi_l)y_t - k_t} + b\psi_k \bar{Y}^{\phi_k} e^{(1-\phi_k)y_t - k_t} - e^{c_t - k_t} - \delta, \quad (48)$$

$$\dot{c}_t = b\psi_k(1 - \phi_k) \bar{Y}^{\phi_k} e^{(1-\phi_k)y_t - k_t} - (\delta + \rho). \quad (49)$$

Using the fact that  $y = \alpha l + \beta k$  and  $l = \frac{\ln[a\psi_l(1-\phi_l)\bar{Y}^{\phi_l}]}{\chi+1} + \frac{1-\phi_l}{\chi+1}y - \frac{1}{\chi+1}c$  from (43), it can be obtained that

$$y = \frac{\alpha \ln [a\psi_l(1 - \phi_l)\bar{Y}^{\phi_l}]}{\chi + 1 - \alpha(1 - \phi_l)} - \frac{\alpha}{\chi + 1 - \alpha(1 - \phi_l)}c + \frac{\beta(\chi + 1)}{\chi + 1 - \alpha(1 - \phi_l)}k. \quad (50)$$

By putting (50) into the system of (48) and (49), the dynamic system can finally be defined with logarithmically transformed variables as

$$\dot{k}_t = a\psi_l \bar{Y}^{\phi_l} e^{\mu_0 + \mu_1 k_t + \mu_2 c_t} + b\psi_k \bar{Y}^{\phi_k} e^{v_0 + v_1 k_t + v_2 c_t} - e^{c_t - k_t} - \delta, \quad (51)$$

$$\dot{c}_t = b\psi_k(1 - \phi_k) \bar{Y}^{\phi_k} e^{v_0 + v_1 k_t + v_2 c_t} - (\delta + \rho), \quad (52)$$

where

$$\begin{aligned} \mu_0 &= \frac{\alpha(1 - \phi_l) \ln [a\psi_l(1 - \phi_l)\bar{Y}^{\phi_l}]}{\chi + 1 - \alpha(1 - \phi_l)}, \\ \mu_1 &= \frac{(\chi + 1)(\beta(1 - \phi_l) - 1) + \alpha(1 - \phi_l)}{\chi + 1 - \alpha(1 - \phi_l)}, \\ \mu_2 &= -\frac{\alpha(1 - \phi_l)}{\chi + 1 - \alpha(1 - \phi_l)}, \end{aligned}$$

$$v_0 = \frac{\alpha(1 - \phi_k) \ln \left[ a\psi_l(1 - \phi_l)\bar{Y}^{\phi_l} \right]}{\chi + 1 - \alpha(1 - \phi_l)},$$

$$v_1 = \frac{(\chi + 1)(\beta(1 - \phi_k) - 1) + \alpha(1 - \phi_l)}{\chi + 1 - \alpha(1 - \phi_l)},$$

$$v_2 = -\frac{\alpha(1 - \phi_k)}{\chi + 1 - \alpha(1 - \phi_l)}.$$

Then the Jacobian matrix ( $J$ ) of the dynamic system of (51) and (52) evaluated at the steady-state has the following form :

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

$$J_{11} = \frac{(\delta + \rho)}{b\psi_k(1 - \phi_k)} ((\mu_1 + 1)a\psi_l + (v_1 + 1)b\psi_k) - \delta,$$

$$J_{12} = \frac{(\delta + \rho)}{b\psi_k(1 - \phi_k)} ((\mu_2 - 1)a\psi_l + (v_2 - 1)b\psi_k) + \delta,$$

$$J_{21} = v_1(\delta + \rho),$$

$$J_{22} = v_2(\delta + \rho).$$

Figure 4 displays the dynamic properties of the steady-state captured by the determinant and the trace values of the Jacobian matrix under various progressivities of the capital and labour income taxes. The parameter values except the progressivity parameters,  $\phi_k$  and  $\phi_l$  are set to show indeterminacy when both income taxes are flat. The level of the labour and capital income taxes,  $\psi_k$  and  $\psi_l$  are the same here. Figure 4-(a) shows the case where the labour income tax is flat ( $\phi_l = 0$ ) and the capital income tax can have various progressivity,  $\phi_k$ . As it shows, if the government can only vary the progressivity of the capital income tax, neither progressive nor regressive capital income tax can help to restore saddle-path stability. On the contrary, in a case where the government implements a flat capital income taxation ( $\phi_k = 0$ ) and changes the

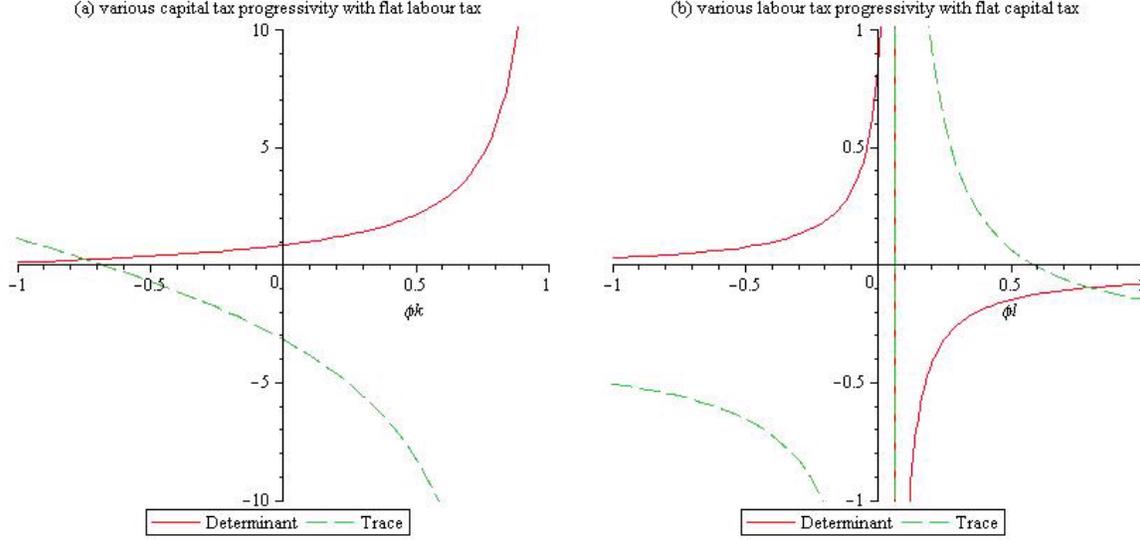


Figure 4: Income tax progressivity and stability: without transfer

progressivity of the labour income tax,  $\phi_l$ , Figure 4-(b) shows that the determinate equilibrium can be secured by progressive labour income taxation. This result is a mere reiteration of the findings in Guo (1999).

### Analysis of Dynamics in a Case with a Lump-sum Transfer

If we consider the lump-sum transfer ( $TR_t$ ) in the household's budget constraint as

$$\dot{K} = \psi_l(\overline{wL})^{\phi_l} (w_t L_t)^{1-\phi_l} + \psi_k(\overline{rK})^{\phi_k} (r_t K_t)^{1-\phi_k} - \delta K_t - C_t + TR_t,$$

the corresponding dynamic system can be identified as

$$\frac{\dot{C}_t}{C_t} = b\psi_k(1 - \phi_k)\overline{Y}^{\phi_k} \frac{Y_t^{1-\phi_k}}{K_t} - (\delta + \rho), \quad (54)$$

$$\frac{\dot{K}_t}{K_t} = \frac{Y_t}{K_t} - \frac{C_t}{K_t} - \delta. \quad (55)$$

From (54) to (55), the ratio variables of steady-states capital ( $\bar{K}$ ), consumption ( $\bar{C}$ ) and output ( $\bar{Y}$ ) are defined by

$$\frac{\bar{Y}}{\bar{K}} = \frac{(\delta + \rho)}{b\psi_k(1 - \phi_k)},$$

$$\frac{\bar{C}}{\bar{K}} = \frac{(\delta + \rho)}{b\psi_k(1 - \phi_k)} - \delta.$$

In logarithm terms, the dynamic system of (54) to (55) is summarized as

$$\dot{k}_t = e^{\mu_0 + \mu_1 k_t + \mu_2 c_t} - e^{c_t - k_t} - \delta, \quad (57)$$

$$\dot{c}_t = b\psi_k(1 - \phi_k)\bar{Y}^{\phi_k} e^{v_1 + v_2 k_t + v_2 c_t} - (\delta + \rho), \quad (58)$$

where

$$\mu_0 = \frac{\alpha \ln \left[ a\psi_l(1 - \phi_l)\bar{Y}^{\phi_l} \right]}{\chi + 1 - \alpha(1 - \phi_l)},$$

$$\mu_1 = \frac{(\beta - 1)(\chi + 1) + \alpha(1 - \phi_l)}{\chi + 1 - \alpha(1 - \phi_l)},$$

$$\mu_2 = -\frac{\alpha}{\chi + 1 - \alpha(1 - \phi_l)},$$

$$v_0 = \frac{\alpha(1 - \phi_k) \ln \left[ a\psi_l(1 - \phi_l)\bar{Y}^{\phi_l} \right]}{\chi + 1 - \alpha(1 - \phi_l)},$$

$$v_1 = \frac{(\chi + 1)(\beta(1 - \phi_k) - 1) + \alpha(1 - \phi_l)}{\chi + 1 - \alpha(1 - \phi_l)},$$

$$v_2 = -\frac{\alpha(1 - \phi_k)}{\chi + 1 - \alpha(1 - \phi_l)}.$$

Then the Jacobian matrix ( $J$ ) of the dynamic system of (57) to (58) evaluated at the steady-state has the following form :

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

$$J_{11} = \frac{(\delta + \rho)}{b\psi_k(1 - \phi_k)}(\mu_1 + 1) - \delta,$$

$$J_{12} = \frac{(\delta + \rho)}{b\psi_k(1 - \phi_k)}(\mu_2 - 1) - \delta,$$

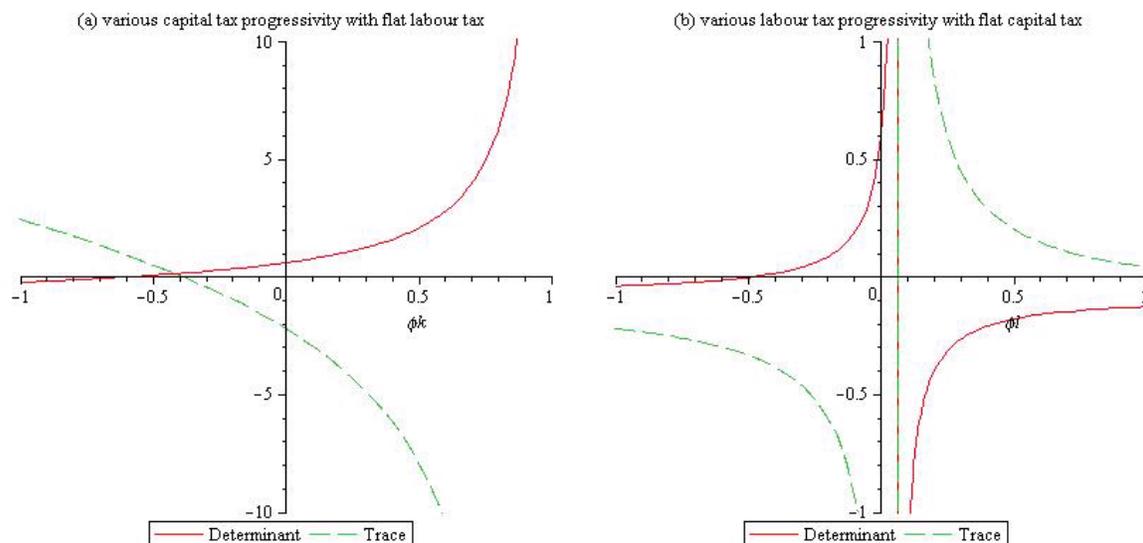


Figure 5: Income tax progressivity and stability: with transfer

$$J_{21} = v_1(\delta + \rho),$$

$$J_{22} = v_2(\delta + \rho).$$

Figure 5 compares the stabilizing role between the labour and capital income tax and is drawn under the same conditions as Figure 4 except for the assumption of the existence of a lump-sum transfer. In Figure 5-(a), variable capital income tax progressivity is imposed with a flat labour income tax. Unlike the case with wasteful government spending, sufficiently regressive capital income tax (negative  $\phi_k$ ) emerges as an effective stabilizing tax instrument with the lump-sum transfer. A negative determinant value is observed in the negative area of  $\phi_k$  above the lower bound,  $\frac{\beta-1}{\beta}$ . For the labour income tax, progressive taxation can still work as a stabilizer as shown in Figure 5-(b). In addition, a sufficiently large negative  $\phi_l$  can also restore the determinacy of the stationary equilibrium which is not observed in the case with wasteful government spending.

#### 2.2.4 Toward the Determinacy : Intuition

In Benhabib-Farmer (1994), the mechanism that induces indeterminacy is explained using the non-standard labour demand curve which is steeper than the labour supply curve. To understand how indeterminacy arises, recall that the optimizing behaviour

of the consumer leads to equation (15) which is expressed as

$$\frac{\dot{\Lambda}}{\Lambda} = (\rho + \delta) - (1 - \tau_m)r, \quad (59)$$

where the co-state variable,  $\Lambda$ , is the shadow price of investment and  $(1 - \tau_m)r$  is the after-tax marginal return on capital. Now consider a situation where the consumers at the stationary equilibrium somehow believe that  $\Lambda$  should be higher than the current level. This belief works in a way that consumers reallocate their income from consumption to investment. The reduction in consumption simultaneously moves the labour supply curve downward. Since the slope of the labour demand curve is greater than the supply curve under the sufficiently large externality, consumers work less which decreases the aggregate level of employment. The increase in capital stock and leisure decrease the after-tax marginal return on capital and  $\Lambda$  actually should appreciate to equate both sides of the equation, (59), which validates the initial belief of the consumers. However, the contraction in employment leads to decline in output and a consequential decrease in the capital stock which moves the labour demand curve downward, increasing employment. This process reverts the economy back to its initial state as the fall in capital stock and increase in the hours worked increase the after-tax marginal return on investment and as a result the shadow price of investment,  $\Lambda$ , depreciates this time to satisfy the equation (59).

Since the unusual labour demand curve is the core factor causing indeterminacy, Guo and Lansing (1998) seek to find a stabilizing fiscal rule with an emphasis on whether it can correct the labour demand curve, restoring the standard slope which is less steep than the labour supply curve. In the first-order condition for the optimal choice of labour, (14), the labour demand and the labour supply schedule can be broken into components as

$$\underbrace{CL^X}_{\text{supply}} = \underbrace{(1 - \tau_m)w}_{\text{after-tax wage}} = \underbrace{(1 - \phi)\psi\bar{Y}^\phi L^{\alpha(1-\phi)-1} K^\beta}_{\text{demand}}.$$

Taking logarithms and letting  $w_m$  represent the logarithm of the after-tax marginal

wage, we obtain

$$c + \chi l = w_m = \ln(1 - \phi)\psi\bar{Y}^\phi + (\alpha(1 - \phi) - 1)l + \beta k,$$

where the slope of the labour demand curve is given by  $\alpha(1 - \phi) - 1$  and the slope of the labour supply curve is  $\chi$ . Thus, the necessary and sufficient condition for determinacy, (35) suggested by Guo and Lansing (1998) simply states that a  $\phi$  sufficiently large enough to secure a labour demand curve that is less steep than the labour supply curve can eliminate indeterminacy. When the labour income tax and the capital income tax are levied separately, it can be noted from (41) that the slope of the labour demand curve is given by  $\alpha(1 - \phi_l) - 1$ , so that only the progressive labour income tax can stabilize the belief-driven fluctuations through the labour demand curve channel. Capital income tax cannot serve in this respect. Under the standard labour demand curve, when the consumers decrease consumption and increase investment based on the belief in a higher shadow price of the investment,  $\Lambda$ , employment does not contract but expands, leading to a further expansion of the capital accumulation. This increase in capital moves the labour demand curve upward, decreasing the level of employment and since the after-tax marginal return on capital falls in this condition, the appreciation of the shadow price of capital should continue in order to satisfy the equation, (59). This explosive over-accumulation process of capital cannot be sustained, eventually violating the transversality condition, thus the consumer's behaviour acting on this belief cannot construct a proper equilibrium path.

As for how the regressive income taxation eliminates the indeterminacy, the process where a contraction in employment driven by non-standard labour demand curve eventually leads to the contraction in capital stock should still be paid attention to. We can conjecture two channels through which indeterminacy is corrected. Firstly, regressive taxation can provide enough incentive for consumers to save to overturn the negative effect of decreasing employment on total output. Under regressive taxation, regardless of the decrease in employment, the output still increases due to sufficient investment,

enabling capital accumulation to keep expanding. However, when government spending is wasteful, however large the incentive offered, consumers cannot save enough to offset the effect of decreasing employment because the tax payment just undermines the consumers' income base to save. On the contrary, when the government's tax revenue is repaid to the consumer as a lump-sum transfer, a reasonable level of regressive taxation can motivate consumers who now face a larger income-base to save enough. Since this process acts on the consumer's saving behaviour, when there are separate labour and capital income taxes, only a regressive capital income taxation can work for this purpose.

The second channel is involved with the labour demand curve. A regressive income tax makes the slope of the labour demand curve even more steep than the labour supply curve. Then the decrease in employment driven by the same amount of reduction in consumption becomes smaller, alleviating the negative effect on output. If the consumer can secure enough income to save by the lump-sum transfer provision, this process can provide a sufficient mechanism to exclude a contraction in capital stock which drives the indeterminacy. However, under the wasteful government spending, a change in the elasticity of labour supply with respect to consumption is not enough to create an explosive capital accumulation process. As a result, when there is a lump-sum transfer, a determinate equilibrium can be secured by either a progressive or regressive labour income taxation as shown in Figure 5-(b). If the income tax is levied on total income, these two channels work at the same time, reducing the degree of regressivity required to eliminate the indeterminacy. In Figure 3-(b), a negative determinant is observed from a much milder degree of regressivity than in Figure 5 where the labour and the capital income tax are separately levied.

### **2.2.5 Concluding Remarks**

In this section, we reviewed Guo and Lansing (1998) and Guo (1999) with a different assumption about the use of the tax revenue. The well-established role of progressive

taxation which corrects the steeper slope of the labour demand curve relative to the labour supply curve can be confirmed here again irrespective of how the tax revenue is used. What we find is that the assumption that the tax revenue is not wasted as government spending but is repaid to the consumers as a lump-sum transfer enables a regressive income taxation to emerge as a stabilizer of the belief-driven fluctuation caused by the indeterminacy. The key difference is on the income-base of the consumer. With wasteful government spending, the consumer cannot save enough to overcome the effect of the non-standard labour demand curve, however large the regressivity is. By contrast, as the government's lump-sum transfer provides a strong income-base, consumers can invest a sufficient amount of capital in response to the regressive income taxation, more than making up for the shortfall in the output due to a fall in the employment.

In addition, we show that with lump-sum transfer it is clear that labour and capital income taxes serve in a different way to dampen the belief-driven fluctuations. For the labour income tax, both a progressive and regressive taxation can be used as a stabilizer. On the contrary, the capital income tax should be regressive in order for it to be used to restore the determinacy.

## **2.3 Indeterminacy and Income Tax Progressivity in a Growth Model with Productive Government Spending**

### **2.3.1 Introduction**

In the previous section, we investigated the stabilizing role of fiscal policy, the income tax schedule in particular, in a growth model with an increasing-returns-to-scale technology sustained by a production externality. However, several works have shown that macroeconomic-instability can be induced by some fiscal policy rules themselves. In Schmitt-Grohe and Uribe (1997), it is shown that indeterminacy can arise even under constant-returns-to-scale technology if government expenditure is fixed and the income

tax rate is determined by a balanced-budget rule. On the contrary, as Guo and Harrison (2004) point out, if the balanced-budget rule is implemented by a constant income tax rate and endogenous public spending, the economy always exhibits saddle-path stability.

Another example where the government's fiscal policy works as a source of indeterminacy is shown in Guo and Harrison (2008). They find that when government spending is productive or utility-generating, macroeconomic instability can be observed irrespective of whether the government's ability to change tax rates is restricted or not. In Chen and Guo (2010), the issue of indeterminacy is also addressed in the growth model with productive public expenditure first suggested by Barro (1990) where government spending is introduced as a complimentary input in the production technology. They showed that the externality due to the existence of productive public expenditure could act pretty much like the externality assumed in Benhabib and Farmer (1994) leading to an indeterminate steady-state equilibrium. As for fiscal response, they found some cases where the economy could be more susceptible to equilibrium indeterminacy when the income tax schedule is more progressive which is in sharp contrast to Guo and Lansing (1998).

In the previous section, we saw the case where labour and capital income taxes could have different roles in stabilizing the belief-driven fluctuations. This section aims to check this possibility again in the growth model of Chen and Guo (2010). While they consider a single income tax schedule for labour and capital income, this section introduces a separate income tax schedule for each type of income. Under the wasteful public expenditure case, studies of the separate taxation case like Guo (1999) and Sims (2005) provide the same conclusion on the effectiveness of progressive taxation as stabilizer. By contrast, quantitative analysis of this study shows that labour and capital income tax can be required to have the opposite direction of the progressivity parameter in order to eliminate indeterminacy of macroeconomic dynamics. It turns out that progressive labour income taxation can be effective as an automatic stabilizer

while capital tax should be regressive for the same purpose.

### 2.3.2 Model Description

The economy is assumed to be populated by a representative infinitely-lived household. Following Chen and Guo (2010), the life-time utility of the household to be maximized is defined as

$$U = \int_0^{\infty} (\ln C_t - BL_t)e^{-\rho t} dt, \quad B > 0,$$

where  $\rho$  is discount rate and  $C_t$  and  $L_t$  are consumption and hours worked respectively. The household earns income by providing labour and capital to the market with factor prices,  $w_t$  and  $r_t$ . Then the budget constraint for the household is given by

$$\dot{K}_t = (1 - \tau_l)w_t L_t + (1 - \tau_k)r_t K_t - \delta K_t - C_t,$$

where  $K_t$  is the level of capital stock and  $\delta$  is depreciation rate.  $\tau_l$  and  $\tau_k$  are labour and capital income tax rates respectively.

On the production side, a competitive firm produces a good with a technology defined by

$$Y_t = AK_t^\alpha L_t^{1-\alpha} G_t^\beta, \quad 0 < \alpha, \beta, \alpha + \beta < 1,$$

where  $G_t$  is the public service provided by the government and financed by labour and capital income tax, say,  $G_t = \tau_l w_t L_t + \tau_k r_t K_t$ . Under the assumption of a competitive market where the level of  $G_t$  is taken as given,  $w_t$  and  $r_t$  are simply

$$w_t = (1 - \alpha) \frac{Y_t}{L_t}, \quad r_t = \alpha \frac{Y_t}{K_t} \quad (60)$$

Following Guo (1999), the income tax rates,  $\tau_l$  and  $\tau_k$  are assumed to take the form

$$\tau_l = 1 - \psi_l \left( \frac{\overline{wL}}{w_t L_t} \right)^{\phi_l}, \quad \tau_k = 1 - \psi_k \left( \frac{\overline{rK}}{r_t K_t} \right)^{\phi_k}, \quad (61)$$

where  $\overline{wL}$  and  $\overline{rK}$  are the steady-state labour and capital income level respectively, which are taken as given by the household. In (61),  $\psi_i$  and  $\phi_i$  is the level and slope of each tax schedule respectively. When  $\phi$  is positive (negative), the tax rate increases (decreases) with the income, which is called progressive (regressive) income taxation. When  $\phi = 0$ , the tax rate is fixed at all level of income, which is a flat income taxation. Notice that  $w_t L_t = (1 - \alpha)Y_t$  and  $r_t K_t = \alpha Y_t$ . Then the income tax schedule in (61) can be expressed as

$$\tau_l = 1 - \psi_l \left(\frac{\overline{Y}}{Y_t}\right)^{\phi_l}, \quad \tau_k = 1 - \psi_k \left(\frac{\overline{Y}}{Y_t}\right)^{\phi_k}, \quad (62)$$

where  $\overline{Y}$  is the steady-state output level. From (62), the marginal tax rate of labour ( $\tau_{ml} = \frac{\partial \tau_l w_t L_t}{\partial w_t L_t}$ ) and capital income ( $\tau_{mk} = \frac{\partial \tau_k r_t K_t}{\partial r_t K_t}$ ) can be defined as  $\tau_{ml} = 1 - \psi_l (1 - \phi_l) \left(\frac{\overline{Y}}{Y_t}\right)^{\phi_l}$ ,  $\tau_{mk} = 1 - \psi_k (1 - \phi_k) \left(\frac{\overline{Y}}{Y_t}\right)^{\phi_k}$ . It can be observed here that when  $\phi_l$  or  $\phi_k$  is positive (negative), the average income tax rate is less (greater) than the marginal income tax rate. For convenience, it is assumed that  $\tau_l, \tau_k \in (0, 1)$  which implies  $\psi_l, \psi_k \in (0, 1)$ . The marginal tax rate is also assumed to be less than one, which provides the condition that  $\phi_l, \phi_k < 1$ .

### 2.3.3 Equilibrium

The current-value Hamiltonian for the problem can be constructed as

$$H(C_t, L_t, K_t, \Lambda_t) = \ln C_t - B L_t + \Lambda_t \{ \psi_l (\overline{wL})^{\phi_l} (w_t L_t)^{1-\phi_l} + \psi_k (\overline{rK})^{\phi_k} (r_t K_t)^{1-\phi_k} - \delta K_t - C_t \}.$$

The first-order conditions for this problem are

$$\frac{1}{C_t} = \Lambda_t,$$

$$B = \Lambda_t \psi_l (1 - \phi_l) \left(\frac{\overline{wL}}{w_t L_t}\right)^{\phi_l} w_t,$$

$$\psi_k (1 - \phi_k) \left(\frac{\overline{rK}}{r_t K_t}\right)^{\phi_k} r_t - \delta = \rho \Lambda_t - \dot{\Lambda}_t,$$

with transversality condition  $\lim_{t \rightarrow \infty} e^{-\rho t} \Lambda_t K_t = 0$ . Combining these conditions with (60) and (62), it can be obtained that

$$L_t = \frac{1}{B} \Lambda_t (1 - \alpha) \psi_l (1 - \phi_l) \bar{Y}^{\phi_l} Y_t^{1 - \phi_l}, \quad (63)$$

$$\Lambda_t \left[ \alpha \psi_k (1 - \phi_k) \bar{Y}^{\phi_k} \frac{Y_t^{1 - \phi_k}}{K_t} - \delta \right] = \rho \Lambda_t - \dot{\Lambda}_t. \quad (64)$$

From these results, a system of two-dimensional differential equations can be identified as

$$\frac{\dot{C}_t}{C_t} = \alpha \psi_k (1 - \phi_k) \bar{Y}^{\phi_k} \frac{Y_t^{1 - \phi_k}}{K_t} - (\delta + \rho), \quad (65)$$

$$\frac{\dot{K}_t}{K_t} = \{1 - \tau_l (1 - \alpha) - \tau_k \alpha\} \frac{Y_t}{K_t} - \frac{C_t}{K_t} - \delta. \quad (66)$$

Notice that

$$\begin{aligned} 1 - \tau_l (1 - \alpha) - \tau_k \alpha &= 1 - [1 - \psi_l (\frac{\bar{Y}}{Y_t})^{\phi_l}] (1 - \alpha) - [1 - \psi_k (\frac{\bar{Y}}{Y_t})^{\phi_k}] \alpha \\ &= (1 - \alpha) \psi_l (\frac{\bar{Y}}{Y_t})^{\phi_l} + \alpha \psi_k (\frac{\bar{Y}}{Y_t})^{\phi_k}. \end{aligned}$$

So (66) can be written

$$\frac{\dot{K}_t}{K_t} = (1 - \alpha) \psi_l \bar{Y}^{\phi_l} \frac{Y_t^{1 - \phi_l}}{K_t} + \alpha \psi_k \bar{Y}^{\phi_k} \frac{Y_t^{1 - \phi_k}}{K_t} - \frac{C_t}{K_t} - \delta. \quad (67)$$

From (65) and (67), the ratio variables of steady-states capital ( $\bar{K}$ ), consumption ( $\bar{C}$ ) and output ( $\bar{Y}$ ) are defined by

$$\begin{aligned} \frac{\bar{Y}}{\bar{K}} &= x_1 = \frac{(\delta + \rho)}{\alpha \psi_k (1 - \phi_k)}, \\ \frac{\bar{C}}{\bar{K}} &= x_2 = \frac{\Psi}{\alpha \psi_k (1 - \phi_k)} (\delta + \rho) - \delta, \\ \frac{\bar{Y}}{\bar{C}} &= \frac{x_1}{x_2}. \end{aligned}$$

where  $\Psi = (1 - \alpha)\psi_l + \alpha\psi_k$  which is the average level of labour and capital income tax.

The steady-state hours worked ( $\bar{L}$ ) can be derived from (63) as

$$\bar{L} = \frac{1}{B}(1 - \alpha)\psi_l(1 - \phi_l)\frac{x_1}{x_2}.$$

Finally, the steady-state capital ( $\bar{K}$ ) and consumption ( $\bar{C}$ ) have values given by

$$\bar{K} = \left( Ax_1^{\beta-1}(1 - \Psi)^\beta \bar{L}^{1-\alpha} \right)^{1/(1-\alpha-\beta)}, \bar{C} = x_2 \bar{K}$$

In logarithm terms, the system of (65) and (67) can be described as

$$\dot{k}_t = (1 - \alpha)\psi_l \bar{Y}^{\phi_l} e^{(1-\phi_l)y_t - k_t} + \alpha\psi_k \bar{Y}^{\phi_k} e^{(1-\phi_k)y_t - k_t} - e^{c_t - k_t} - \delta,$$

$$\dot{c}_t = \alpha\psi_k(1 - \phi_k) \bar{Y}^{\phi_k} e^{(1-\phi_k)y_t - k_t} - (\delta + \rho).$$

Then the Jacobian matrix ( $J$ ) of this dynamic system evaluated at the steady steady-state has the following form :

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

$$J_{11} = \left[ \frac{\alpha(1 - \phi_l)}{\Gamma} - 1 \right] (1 - \alpha)\psi_l x_1 + \left[ \frac{\alpha(1 - \phi_k)}{\Gamma} - 1 \right] \alpha\psi_k x_1 + x_2,$$

$$J_{12} = -\frac{(1 - \alpha)(1 - \phi_l)}{\Gamma} (1 - \alpha)\psi_l x_1 - \frac{(1 - \alpha)(1 - \phi_k)}{\Gamma} \alpha\psi_k x_1 - x_2,$$

$$J_{21} = \left[ \frac{\alpha(1 - \phi_l)}{\Gamma} - 1 \right] \alpha\psi_k (1 - \phi_k) x_1,$$

$$J_{22} = -\frac{(1 - \alpha)(1 - \phi_k)^2}{\Gamma} \alpha\psi_k x_1.$$

where

$$\Gamma = 1 - \beta \frac{1 - (1 - \alpha)(1 - \phi_l)\psi_l - \alpha(1 - \phi_k)\psi_k}{1 - (1 - \alpha)\psi_l - \alpha\psi_k} - (1 - \alpha)(1 - \phi_l).$$

When  $\psi_l = \psi_k = \psi$  and  $\phi_l = \phi_k = \phi$ , this corresponds to the case where the capital and income taxes have the same schedule. Under this uniform income tax schedule, it can easily be checked that all elements of Jacobian have the same values as those in Chen and Guo (2010).

Since consumption ( $c_t$ ) is the only non-predetermined variable in the dynamic system, the stability of the steady-state of this system can be checked with the eigenvalues of Jacobian,  $J$ . For a saddle-path equilibrium to exist, two eigenvalues must have values of opposite sign, which corresponds to the condition that the determinant value is negative (Saddle). When both eigenvalues have negative real parts so that the determinant is positive and the trace is negative, the equilibrium steady-state is locally indeterminate (Sink). In another case where the determinant and the trace are positive due to positive real parts in both eigenvalues, the steady-state is unstable since any small deviation from it leads to permanent divergence.

If we assume that  $\psi_k = \psi_l = \psi$  for convenience, the determinant of the Jacobian above is

$$Det = \left[ \left( \left( \frac{\alpha(1-\phi_l)}{\Gamma} - 1 \right) (1-\alpha) + \frac{\alpha^2(1-\phi_k)}{\Gamma} \right) (\phi_k - \phi_l)\psi(1-\alpha)x_1 + \Omega x_2 \right] \frac{\psi\alpha(1-\phi_k)}{\Gamma} x_1,$$

where

$$\Omega = \alpha(1-\phi_l) + (1-\alpha)(\phi_k - \phi_l) - (1-\beta) \frac{1 - (1-\alpha)(1-\phi_l)\psi_l - \alpha(1-\phi_k)\psi_k}{1 - (1-\alpha)\psi_l - \alpha\psi_k}.$$

If  $\phi_k = \phi_l = \phi$  as in Chen and Guo (2010), the determinant has a very concise form as

$$Det = \frac{\Omega}{\Gamma} \psi \alpha (1-\phi) x_1 x_2,$$

where

$$\begin{aligned}\Gamma &= 1 - \beta \frac{1 - (1 - \phi)\psi}{1 - \psi} - (1 - \alpha)(1 - \phi), \\ \Omega &= \alpha(1 - \phi) - (1 - \beta \frac{1 - (1 - \phi)\psi}{1 - \psi}).\end{aligned}$$

Given  $\alpha > 0$ ,  $\psi < 1$ ,  $x_1, x_2, (1 - \phi) > 0$  with an assumption that  $\Omega < 0$ , the necessary and sufficient condition for determinacy, *i.e.* the negative determinant in Chen and Guo (2010) is

$$\Gamma = 1 - \beta \frac{1 - (1 - \phi)\psi}{1 - \psi} - (1 - \alpha)(1 - \phi) > 0. \quad (70)$$

Since the slope of labour demand curve in this model is given by

$$\frac{(1 - \alpha)(1 - \phi)}{1 - \beta \frac{1 - (1 - \phi)\psi}{1 - \psi}} - 1,$$

and that of labour supply curve is zero, condition, (70) simply requires that the labour demand curve should be less steep than the labour supply curve. Therefore, as Chen and Guo (2010) pointed out, the externality provided by the productive government spending acts in the exactly same way as the externality in Benhabib and Farmer (1994) does, in generating indeterminacy.

When there are separate income taxes for the capital and labour income, the condition for determinacy is not as clear-cut as that of Chen and Guo (2010). However, as long as the non-standard labour demand curve is a source of indeterminacy, we can have a quick glance over the stabilizing role of the labour and capital income tax by checking the way that the tax schedule of each tax works in the slope of labour demand curve. In our model, the condition for the labour demand curve having a less steep slope than the labour supply curve is given by

$$\frac{(1 - \alpha)(1 - \phi_l)}{1 - \beta \frac{1 - (1 - \alpha)(1 - \phi_l)\psi_l - \alpha(1 - \phi_k)\psi_k}{1 - (1 - \alpha)\psi_l - \alpha\psi_k}} - 1 < 0. \quad (71)$$

Unlike the case of the Benhabib and Farmer (1994) growth model that we have seen in the previous section, the progressivity of capital income tax,  $\phi_k$  is involved in the slope of labour demand curve. And  $\phi_k$  should be less progressive to satisfy condition, (71). On the contrary, for the progressivity of the labour income tax,  $\phi_l$ , the condition, (71) is more likely to be met when  $\phi_l$  is either sufficiently progressive or regressive.

### 2.3.4 Numerial Analysis

Having identified the condition for checking the stability of the dynamic system, this section will explore how the stability is affected by changes in some parameter values. The basic parameter values used in the numerical analysis are  $\alpha$  (capital share of national income): 0.3,  $\rho$  (discount rate): 0.05 and  $\delta$  (depreciation rate): 0.1. Most of these parameters follow Chen and Guo (2010) or Benhabib and Farmer (1996). As for the parameter  $\beta$ , the degree of externality from the productive public expenditure, two values, 0.2 ( $<\alpha$ ) and 0.4 ( $>\alpha$ ) are chosen. Variation in the value of  $\beta$  also reflects the results from previous empirical research that show a wide range of estimates of  $\beta$  as noted in Chen and Guo (2010). For the level of income tax,  $\psi_k$  and  $\psi_l$ , the two values of 0.6 and 0.8 are used. The smaller (larger) value is the case where the level of income tax is larger (smaller) than the capital share of national income,  $\alpha$ .

**Capital Income Tax Progressivity with linear Labour Income Tax** Figure 6 shows the change in values of determinant and trace of  $J$  as the progressivity of capital income tax schedule ( $\phi_k$ ) changes. The labour income tax schedule is assumed to be flat here.

When  $\beta$  is low ( $\beta = 0.2$ ), the externality from the productive public expenditure is not large enough to create indeterminacy in equilibrium when both capital and labour income tax have a flat tax schedule as determinant value is negative in the origin in Figure 6-(i) and (ii). This saddle-path stability can be supported with regressive and modest progressivity. As the progressivity of capital income tax gets larger, the property of the dynamics starts to change, firstly leading to unstable steady-state equilibrium

( $T > 0, D > 0$ ) and then reaching indeterminate equilibrium ( $T < 0, D < 0$ ) with significantly large progressivity. When the level of the income tax ( $1 - \psi_l, 1 - \psi_k$ ) is large, it can be observed in Figure 6-(ii) that the chance of indeterminacy emerging is excluded with  $\phi_k$  less than zero.

When  $\beta$  is large ( $\beta = 0.4$ ), a flat income tax schedule for both capital and labour income creates indeterminacy as a positive determinant and a negative trace show up at the origin in Figure 6-(iii) and (iv). With a relatively low level of the income tax ( $\psi_l = \psi_k = 0.8$ ), the recovery of a saddle-path equilibrium can be made by either a mild regressive or an extremely progressive capital tax schedule (Figure 6-iii). When the level of income tax is relatively high ( $\psi_l = \psi_k = 0.6$ ), regressive capital taxation is required for the determinate equilibrium (Figure 6-iv).

The effectiveness of less progressive or regressive capital income tax as a sunspot stabilizer is confirmed again in Figure 7 which shows how the dynamic property of equilibrium steady-state changes as capital income tax progressivity and labour income tax level vary. The labour income tax is still assumed to have a flat schedule with  $\phi_l = 0$ . Under a low  $\beta$ , a less progressive capital income tax guarantees determinacy in the system (Figure 7-i). With a high  $\beta$ , the progressive capital income tax has a very high chance to lead to indeterminacy as shown in area IV in Figure 7-(ii), while in the case where the labour income tax level ( $1 - \psi_l$ ) is relatively low, a large degree of progressivity can also be used to eliminate the indeterminacy like area III in Figure 7-(ii). In Figure 7-(ii), it can also be observed that a mildly regressive capital income tax can stabilize belief-driven fluctuations irrespective of the degree of labour income tax level. Considering the fact that if ever there is a case for more progressive capital income tax acting as a stabilizer the degree of progressivity required for it is too high to be implemented in practice, a general conclusion can be made that the result of Chen and Guo (2012) still holds in the case of capital income taxation.

**Labour Income Tax Progressivity with linear Labour Income Tax** Figure 8 displays the change in the values of determinant and trace of  $J$  as the progressivity of

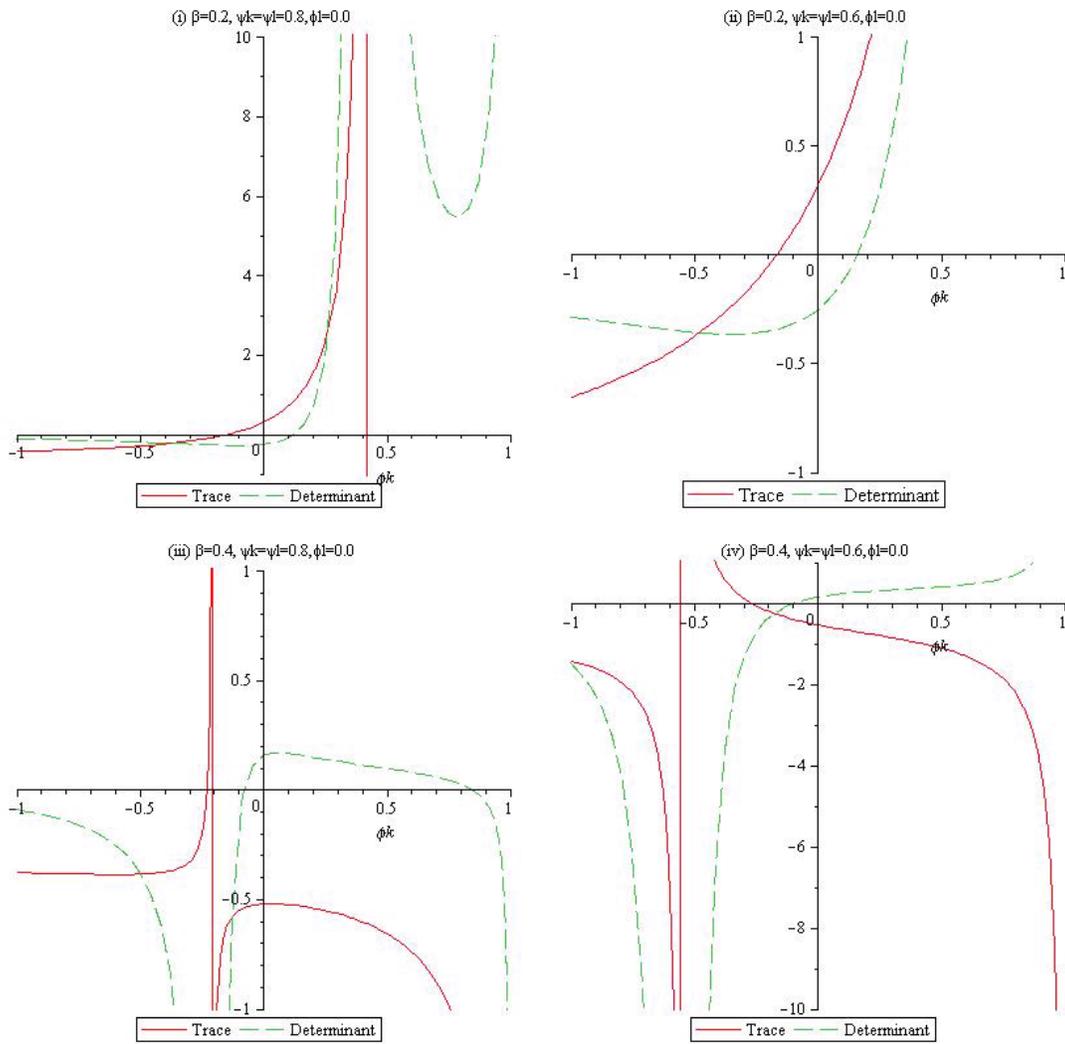


Figure 6: Capital Income Tax Progressivity and Macroeconomic Dynamics

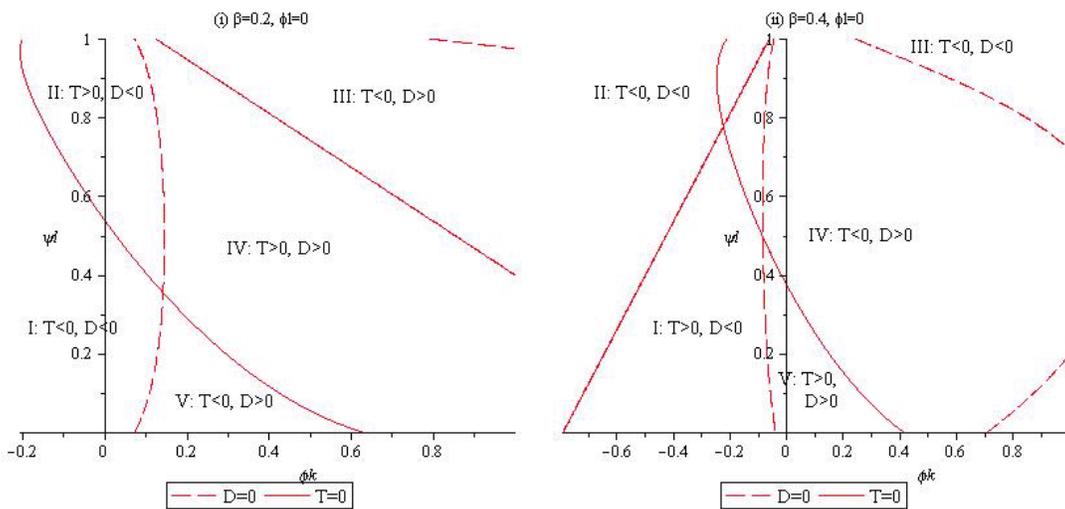


Figure 7: Level of Labour Income Tax and Capital Income Tax Progressivity

labour income tax schedule( $\phi_l$ ) changes. The capital income tax schedule is assumed to be flat in this section.

When  $\beta$  is low ( $\beta = 0.2$ ), indeterminacy does not emerge under a flat income tax schedule as the determinant shows a negative value. However, contrary to the case of the capital income tax, saddle-path stability is getting compromised as the labour income tax becomes more regressive (Figure 8-i and ii). As the progressivity of the labour income tax is getting smaller, the dynamic property of the equilibrium steady-state firstly turns to "Source" ( $T > 0, D > 0$ ) and then to "Sink" ( $T < 0, D < 0$ ) with significantly large regressivity.

When  $\beta$  is large ( $\beta = 0.4$ ), a flat income tax schedule for both capital and labour income makes indeterminacy show up as mentioned in the previous section. In both cases with high and low capital income tax level, the recovery of a saddle-path equilibrium can be made only by adding progressivity to the labour income tax (Figure 8-iii and iv). The level of the capital tax does not make a significant difference in results as the degree of progressivity of regressivity required for determinacy does not vary much with changes in the capital income tax level (Figure 9).

**Combined Income Tax Progressivity** Figure 10 displays the dynamic property of steady-state equilibrium under various combination of labour and capital income tax progressivity. It can be noted that an indeterminate equilibrium emerges when parameter values of  $\phi_k$  and  $\phi_l$  are in the area of IV in Figure 10-(i), (ii), V in Figure 10-(iii) and VI in Figure 10-(iv). In practice, the income tax schedule is neither severely progressive nor regressive. For example, Chen and Guo (2012) estimated the progressivity of the U.S income tax between 0.0634 and 0.1679. Considering this, the combination of labour and capital income progressivity causing indeterminacy in the case of low  $\beta$  (Figure 10-i and ii) can be regarded as implausible. When  $\beta$  is high, indeterminacy emerges with mild levels of labour and capital income tax progressivity (Figure 10-iii and iv).

When the level of the income tax is high (Figure 10-iv), determinacy can be restored

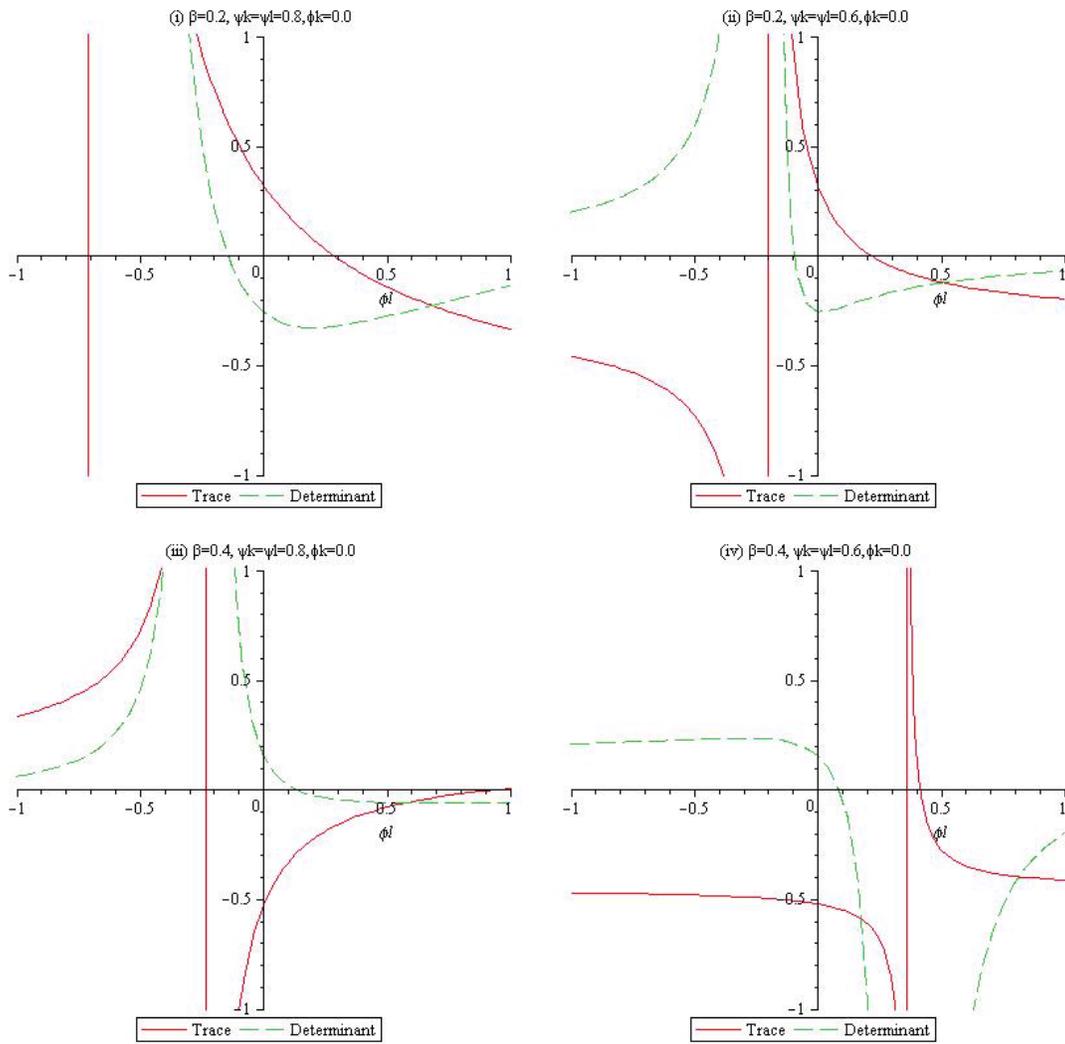


Figure 8: Labour Income Tax Progressivity and Macroeconomic Dynamics

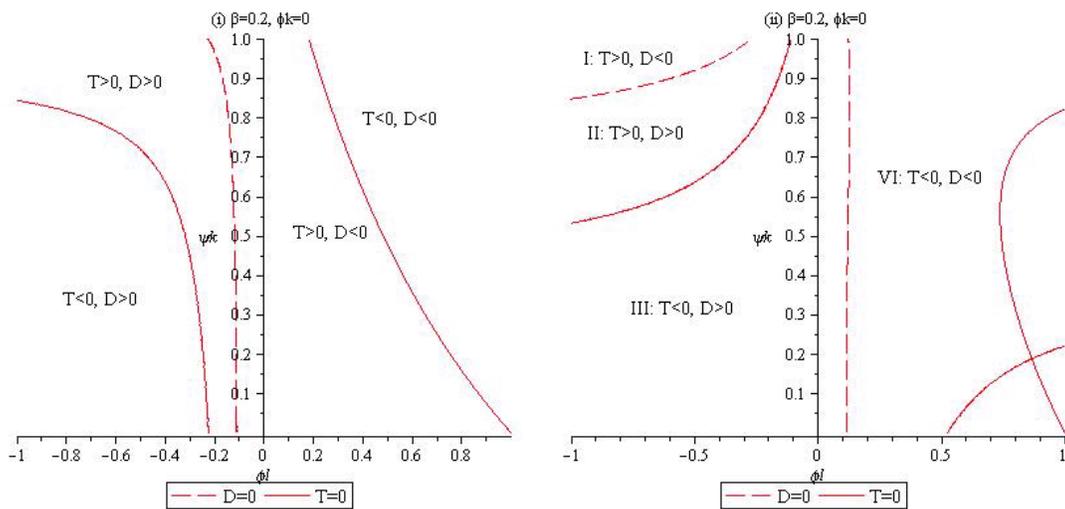


Figure 9: Level of Capital Income Tax and Labour Income Tax Progressivity

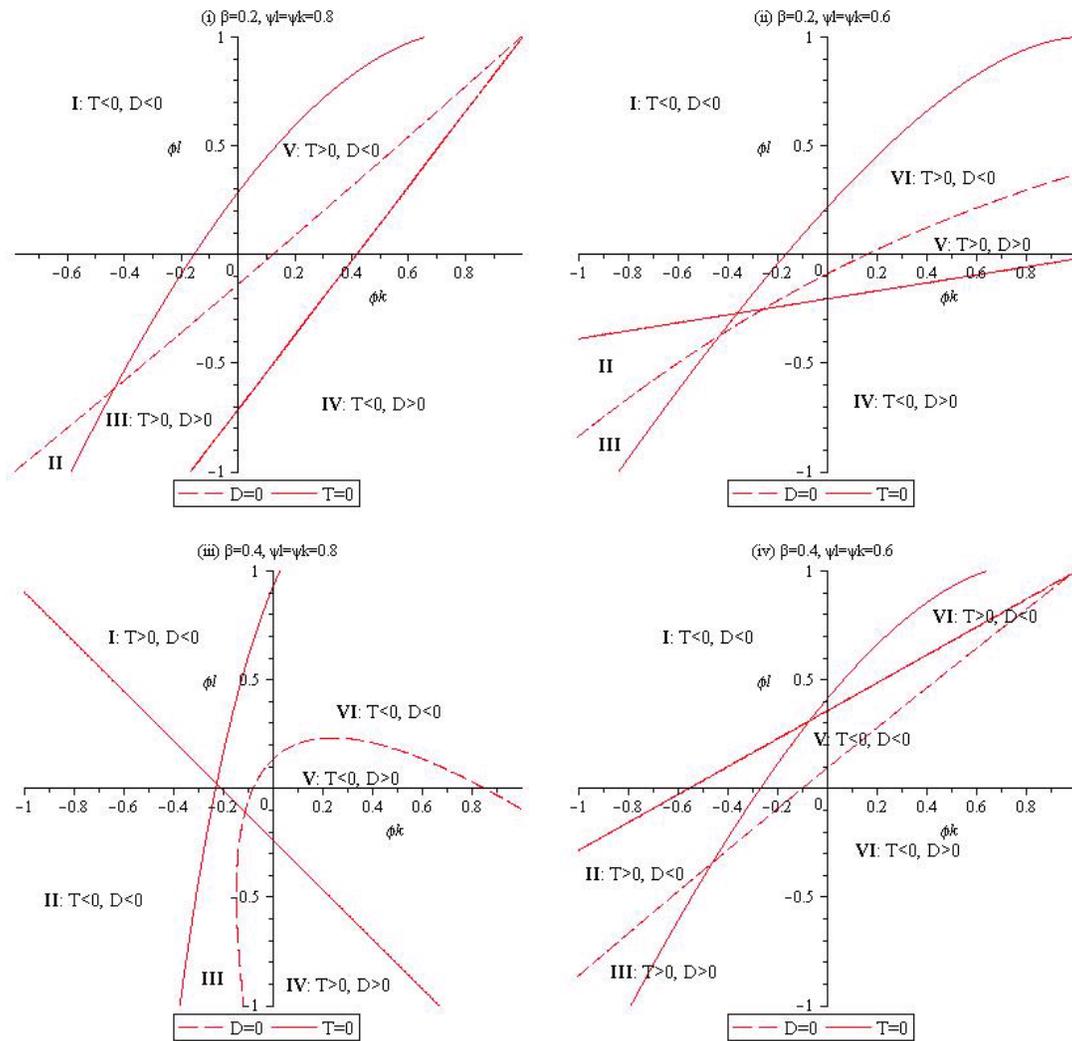


Figure 10: Income Tax Progressivity and Macroeconomic Dynamics

by adding more progressivity to the labour income tax or reducing the progressivity of the capital income tax or implementing both tax reforms at the same time. These three options work well to eliminate indeterminacy under the low income tax rate. However, in Figure 10-(iii), an additional option for achieving determinacy can be captured as the saddle-path equilibrium can also be obtained by a significantly large progressivity of capital income tax. In fact, the lower level of income tax is, it turns out that the effectiveness of progressive capital tax as a stabilizer stands out more.

### 2.3.5 Concluding Remarks

This chapter explores how the progressivity of the income tax schedule affects the dynamic property of equilibrium in a growth model with productive government spending. Chen and Guo (2010) already showed that less progressive taxation could work as a sunspot stabilizer in this type of growth model and this chapter tried to see if that result still holds when separate income tax schedules for labour and capital income are introduced. It turned out that in the case of capital income tax, regressive or less progressive taxation could be more effective in eliminating indeterminacy in line with Chen and Guo (2010). However, as for the labour income tax, it was shown that an economy with a less progressive tax schedule could be more susceptible to indeterminacy.

# 3 Public Input and Public Consumption : The Optimality of Regressive Taxation

## 3.1 Introduction

It is a widely accepted notion that the provision of productive public services by government such as infrastructure, education and public R&D spending can enhance the capability of an economy for steady growth. This possible contribution of public spending to economic growth was highlighted in Barro (1990) where productive public expenditure works as an engine for endogenously sustained long-term growth. By introducing government expenditure simply into a production function as an input factor, the economy in Barro (1990) generates a balanced-growth path.

Meanwhile, since Barro-type growth model involves an externality associated with public expenditure, much literature such as Turnovsky (1996a, 1996b, 2000) has studied the optimal tax schedule which can correct a suboptimal outcome chosen by a private agent into the socially most desirable one. The general conclusion of these works can be summarized as the existence of a constant income tax rate that implements the first-best optimum in a competitive economy<sup>3</sup>.

The findings mentioned above, though, require the strict assumption that the production function shows constant-returns-to-scale technology with respect to private capital and productive public expenditure. In other words, if the production function has the Cobb-Douglas form  $Y_t = AK_t^\alpha G_t^\beta$  where  $Y_t$ ,  $K_t$  and  $G_t$  are output, private capital and productive public expenditure respectively, a balanced-growth path exists only when  $\alpha + \beta = 1$ . However, none of empirical research that has tried to estimate the coefficient  $\beta$  has succeeded in showing that it is sufficiently large to enable constant-returns-to-scale (de Haan and Romp, 2007). The generally accepted estimate for the U.S production function from 1949 to 1985 is  $\alpha = 0.35$ . Alongside this estimate, Aschauer (1989) suggested that  $\beta$  has a value of 0.39. Even though it is larger than the

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<sup>3</sup>Results from major literatures on this issue was surveyed in Irmen and Kuehnel (2009).

productivity coefficient of private capital and has been regarded as too high by the later studies reported in de Haan and Romp (2007), it still implies  $\alpha + \beta < 1$ .

This paper explores the case where the production function has decreasing-returns-to scale with respect to private capital and productive government expenditure. The first-best optimum chosen by a central planner is characterized and then a tax policy that replicates this optimum in a competitive economy is constructed. Since the economy being considered here has a steady-state value rather than a balanced-growth path, the first-best optimum has a form of growth path converging to that steady-state from a given initial state of economy. Therefore, the optimality of the income tax schedule will be judged by how close the growth path it produces in the competitive economy is to the entire first-best growth path. The first part of this section deals with a model with inelastic labour. The major finding is that an intertemporally regressive income tax schedule is required to generate a growth path close to the first-best optimum. It also turns out that this conclusion holds in general when the model includes endogenous labour choice which we consider in the second part of this section.

## 3.2 A Growth Model with Inelastic Labour

### 3.2.1 Basic Model Description

The economy is assumed to consist of a large and fixed number of identical consumers, each with an infinite life. A representative consumer chooses consumption to maximize discounted lifetime utility given by

$$U = \int_0^{\infty} e^{-\rho t} \cdot u(C_t, H_t) dt,$$

where  $\rho$  is the discount rate,  $C_t$  is private consumption and  $H_t$  is the public consumption service provided by the government. Leisure has no influence on consumer utility, so labour services are provided inelastically. For convenience, the utility func-

tion,  $u(C_t, H_t)$  is assumed to have the logarithm form,  $\ln C_t + b \ln H_t$ , where  $b (\geq 0)$  represents the relative weight of public consumption in utility.

The production function follows Barro (1990) with public expenditure serving as an input to private production,

$$Y_t = f(K_t, G_t) = AK_t^\alpha G_t^\beta, \quad 0 < \alpha, \beta, \alpha + \beta < 1, \quad (72)$$

where  $K_t$  is the capital stock and  $G_t$  is the level of productive government expenditure. The model differs from Barro by assuming that the production function has decreasing-returns-to-scale with respect to private capital and public expenditure. Government expenditure is set as a fixed fraction of output

$$H_t = \theta_h \cdot Y_t, \quad G_t = \theta_g \cdot Y_t, \quad \theta_h + \theta_g < 1 \quad (73)$$

For simplicity, public expenditure is not subject to any congestion, i.e. it is a pure public good.

### 3.2.2 Centrally Planned Economy

Suppose that a benevolent social planner chooses the optimal resource allocation to maximize utility given the production technology and the government expenditure ratio. The global resource constraint for the planner's problem is

$$\dot{K}_t = Y_t - C_t - H_t - G_t = (1 - \theta_h - \theta_g)Y_t - C_t, \quad (74)$$

where the dot represents the time derivative. For tractability, private capital is assumed to depreciate fully.

Using the fact that the production function can be rewritten as  $Y_t = A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} K_t^{\frac{\alpha}{1-\beta}}$  from (72) and (73), a well-defined two-dimensional dynamic system can be derived as

follows:

$$\frac{\dot{K}_t}{K_t} = (1 - \theta_h - \theta_g) A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} K_t^{\frac{\alpha+\beta-1}{1-\beta}} - \frac{C_t}{K_t}, \quad (75)$$

$$\frac{\dot{C}_t}{C_t} = \frac{\alpha}{1-\beta} (1 - \theta_h - \theta_g) A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} K_t^{\frac{\alpha+\beta-1}{1-\beta}} + b \frac{\alpha}{1-\beta} \frac{C_t}{K_t} - \rho. \quad (76)$$

with transversality condition,  $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_t K_t = 0$ .

The steady-state economy  $(K^*, C^*)$  satisfying the condition  $\frac{\dot{K}_t}{K_t} = \frac{\dot{C}_t}{C_t} = 0$  can be found as

$$K^* = \left[ \frac{(1-\beta)\rho}{\alpha(1+b)(1-\theta_h-\theta_g)A^{\frac{1}{1-\beta}}\theta_g^{\frac{\beta}{1-\beta}}} \right]^{\frac{1-\beta}{\alpha+\beta-1}} \quad (77)$$

$$C^* = \frac{(1-\beta)\rho}{\alpha(1+b)} K^*. \quad (78)$$

**Proposition 1** *In the centrally planned economy, there exists a saddle path converging to the uniquely defined steady-state,  $(K^*, C^*)$ .*

**Proof.** The Jacobian matrix ( $J$ ) of the dynamic system (75) and (76) is

$$J = \begin{pmatrix} \frac{\rho}{1+b} & -\frac{\rho(1-\beta)}{\alpha(1+b)} \\ \frac{\rho}{1+b} \left[ \frac{\alpha-(1+b)(1-\beta)}{1-\beta} \right] & \frac{b\rho}{1+b} \end{pmatrix}.$$

The determinant of  $J$  is always negative. ■

### 3.2.3 Decentralized Economy

Now assume the representative agent takes the level of public consumption ( $H_t$ ) and productive public expenditure ( $G_t$ ) as given in her intertemporal decision. For simplicity, the number of agents is normalized to one. The agent maximizes utility subject to the budget constraint

$$\dot{K}_t = (1 - \tau_y)Y_t - (1 + \tau_c)C_t - T_t,$$

where  $\tau_y$  is the income tax rate and  $\tau_c$  is the consumption tax rate.  $T_t$  is a lump-sum tax or transfer to the consumer. To give variability to the income tax schedule, we

adopt the tax schedule of Guo (1997) and Guo and Lansing (1998):

$$\tau_y = 1 - \psi \left( \frac{\bar{Y}}{Y_t} \right)^\phi, \quad (79)$$

where the parameter  $\psi (> 0)$  is the level of the income tax and  $\phi$  represents the slope of tax schedule.  $\bar{Y}$  is the steady-state income level which is taken as given by the consumer. A positive (negative)  $\phi$  means that tax rate rises (falls) as income level increases, i.e. the income tax schedule is progressive (regressive). The government sets public expenditure as a fixed portion of output as in (73) and implements a balanced budget so

$$T_t = (\theta_h + \theta_g - \tau_y)Y_t - \tau_c C_t.$$

The solution for the optimization problem generates the dynamic system for the competitive economy

$$\frac{\dot{K}_t}{K_t} = (1 - \theta_h - \theta_g) \frac{Y_t}{K_t} - \frac{C_t}{K_t} \quad (80)$$

$$\frac{\dot{C}_t}{C_t} = \alpha \psi (1 - \phi) \bar{Y}^\phi \frac{Y_t^{1-\phi}}{K_t} - \rho, \quad (81)$$

The steady-state of this system  $(K_c^*, C_c^*)$  is defined by

$$K_c^* = \left[ \frac{\rho}{\alpha \psi (1 - \phi) A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}}} \right]^{\frac{1-\beta}{\alpha+\beta-1}} \quad (82)$$

$$C_c^* = \frac{(1 - \theta_h - \theta_g) \rho}{\alpha \psi (1 - \phi)} K_c^*. \quad (83)$$

**Proposition 2** *In a decentralized economy, there exists a saddle path converging to the steady-state,  $(K_c^*, C_c^*)$  when  $\frac{\alpha+\beta-1}{\alpha} < \phi < 1$ .*

**Proof.** The Jacobian  $(J^C)$  of (80) and (81) is

$$J^C = \begin{pmatrix} \frac{\rho(1-\theta_h-\theta_g)}{(1-\beta)\psi(1-\phi)} & -\frac{\rho(1-\theta_h-\theta_g)}{\alpha\psi(1-\phi)} \\ \frac{\rho[(1-\phi)\alpha+\beta-1]}{1-\beta} & 0 \end{pmatrix}.$$

The determinant of  $J^C$  is negative when  $\frac{\alpha+\beta-1}{\alpha} < \phi < 1$ . ■

The steady-state  $(K_c^*, C_c^*)$  can have different dynamic properties according to the value of  $\phi$ . Too much regressivity ( $\phi < \frac{\alpha+\beta-1}{\alpha}$ ) makes the steady-state a source which means there is no equilibrium path leading to it. If  $\phi$  is greater than one, the system shows indeterminacy. The saddle-path equilibrium on which this paper focuses exists only when  $\frac{\alpha+\beta-1}{\alpha} < \phi < 1$ .

### 3.2.4 Optimal Taxation

**Constant Income Tax Rate** First, suppose that  $\phi = 0$ , i.e. the marginal tax rate is constant. It is straightforward to find the income tax rate that attains the steady-state of the centrally planned economy by comparing (77) and (78) with (82) and (83). The long-term optimal income tax rate is defined as

$$\tau_y^* = 1 - \psi^* = 1 - \frac{(1+b)(1-\theta_h-\theta_g)}{1-\beta}. \quad (84)$$

If there is no public consumption service, i.e.  $b = 0$ , the optimal income tax rate defined in (84) holds throughout the entire growth path, since the dynamic system in (75) and (76) is exactly the same as that in (80) and (81).

When  $b \neq 0$ , a difference in transition dynamics can be observed. This is illustrated in figure 11, which plots growth paths from a log-linear approximation for an example.<sup>4</sup>

Figures 11-(a) and 11-(b) show that in the competitive economy with the constant income tax rate,  $\tau_y^*$  the consumer starts with more saving and less consumption than the optimal level chosen by the central planner. The competitive economy therefore reaches the steady-state faster. The reason for this can be found in figure 11-(c) which shows the social marginal return to capital ( $r$ ) and the net private after-tax return on

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<sup>4</sup>The parameter values are  $A = 1, \alpha = 0.3, \beta = 0.3, \rho = 0.04, b = 1, \theta_g = 0.08$  and  $\theta_h = 0.14$ . These values generally follow Turnovsky (2000) but the value of  $b$  is chosen to highlight the difference more clearly. These parameter values will be used throughout this chapter when the growth paths are displayed.

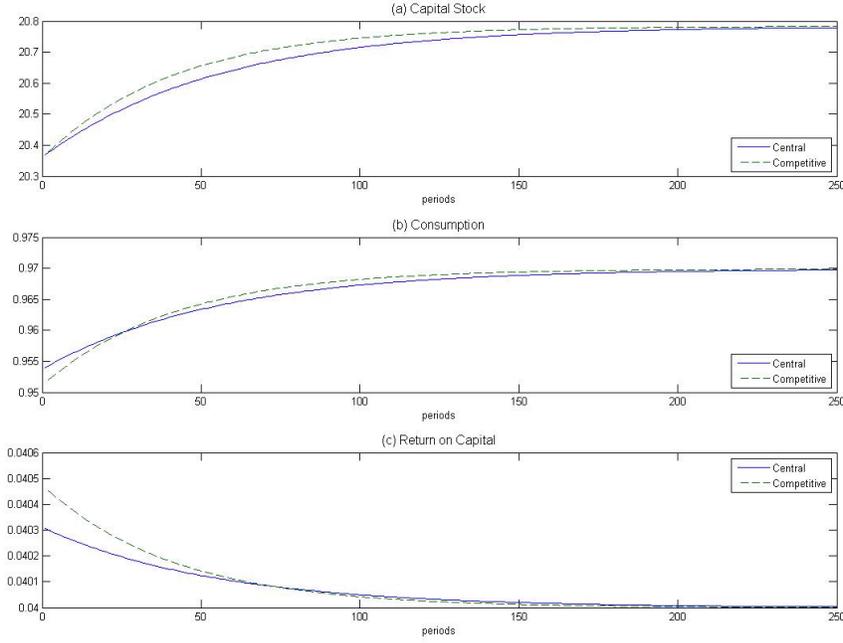


Figure 11: Growth Paths of Centrally-planned and Competitive Economy

capital ( $r_C$ ) defined as

$$r = \frac{\alpha}{1 - \beta} (1 - \theta_h - \theta_g) \frac{Y_t}{K_t} + b \frac{\alpha}{1 - \beta} \frac{C_t}{K_t},$$

$$r_C = \alpha (1 - \tau_y) \frac{Y_t}{K_t}.$$

The difference between these two rates of return results from the externality of the public expenditure and public consumption services, which is ignored in the saving decision of a consumer in the competitive economy. If these two rates are equal the externality is internalized into the decision-making process of the consumer. As is clear in figure 11-(c), a constant income tax fails to achieve the equality:  $r_C$  is greater than  $r$  at the initial point but the two rates become close as the economy converges to the steady-state. The formal comparison of growth rates is given in proposition 3.

**Proposition 3** *With a constant income tax rate,  $\tau_y^*$ , the negative eigenvalue of the Jacobian matrix in the competitive economy is always less than that in the centrally*

*planned economy.*

**Proof.** With a constant income tax rate,  $\tau_y^*$ ,  $J^C$  becomes

$$J^C = \begin{pmatrix} \frac{\rho}{1+b} & -\frac{\rho(1-\beta)}{\alpha(1+b)} \\ \frac{\rho[\alpha+\beta-1]}{1-\beta} & 0 \end{pmatrix}.$$

It is clear that  $D(J) = D(J^C)$  and  $T(J) = \rho > \frac{\rho}{1+b} = T(J^C)$ . Define the characteristic polynomials of  $p(J)$  and  $p(J^C)$  as

$$p(J) = \mu^2 - T(J)\mu + D(J), \quad p(J^C) = v^2 - T(J^C)v + D(J^C).$$

Since  $D(J) = D(J^C) < 0$ , each polynomial has a negative root which we denote by  $\mu_1$  and  $v_1$  respectively. Then

$$v_1^2 - T(J)v_1 + D(J) > \mu^2 - T(J)\mu + D(J) = 0.$$

Therefore,  $v_1 < \mu_1$  or  $|v_1| > |\mu_1|$ . ■

The negative eigenvalue provides an approximation of the rate at which the deviation from the steady-state is reduced, so is generally called "the speed of convergence to the steady-state". Hence proposition 3 shows that the competitive economy with a constant income tax rate implementing the first-best steady-state in the long-term grows faster than the centrally planned economy. The level of welfare for the constant tax rate is therefore lower than the first-best level.

**Non-linear Income Taxation** When  $\phi \neq 0$ , the comparison of the steady-states of the two economies provides the condition for the parameters of the income tax schedule,  $\psi$  and  $\phi$ , to implement the first-best steady-state in a competitive economy:

$$\psi^*(1 - \phi^*) = \frac{(1+b)(1 - \theta_h - \theta_g)}{1 - \beta}, \quad (85)$$

Under a tax schedule with  $(\psi^*, \phi^*)$  satisfying (85), the following proposition applies.

**Proposition 4** (i) *The level of the income tax schedule,  $\psi^*$ , has no effect on the transition dynamics.*

(ii) *The negative eigenvalue of Jacobian matrix in the competitive economy decreases as  $\phi^*$  increases.*

**Proof.** Evaluated at  $(\psi^*, \phi^*)$ ,  $J^C = \begin{pmatrix} \frac{\rho}{1+b} & -\frac{\rho(1-\beta)}{\alpha(1+b)} \\ \frac{\rho[(1-\phi^*)\alpha+\beta-1]}{1-\beta} & 0 \end{pmatrix}$ . This proves (i). The characteristic polynomial of  $J^C$  is

$$\mu^2 - \frac{\rho}{1+b}\mu + \frac{\rho^2[(1-\phi^*)\alpha+\beta-1]}{\alpha(1+b)} = 0,$$

from which the two eigenvalues can be defined as

$$\mu_1, \mu_2 = \frac{1}{2} \left( \frac{\rho}{1+b} \pm \sqrt{\left(\frac{\rho}{1+b}\right)^2 - \frac{4\rho^2[(1-\phi^*)\alpha+\beta-1]}{\alpha(1+b)}} \right), \quad \mu_1 < \mu_2. \quad (86)$$

Since  $\frac{\alpha+\beta-1}{\alpha} < \phi^* < 1$ , there are two real eigenvalues with different sign, i.e.  $\mu_1 < 0$  and  $\mu_2 > 0$ . From (86), (ii) follows directly. ■

Proposition 4 implies that the speed of convergence to the same steady-state is faster for higher  $\phi^*$ . Since the competitive economy grows faster than the centrally-planned economy when  $\phi = 0$ , it follows that the regressivity (negative  $\phi$ ) moves the growth path closer to the first-best choice. This is illustrated in figure 12. If the negative eigenvalues of the Jacobian matrix in the centrally-planned and competitive economies are the same, the approximations of the growth path around the steady-state for both economies are also the same, so they have a common steady-state and an equal speed of convergence. The existence of a regressive tax system that achieves this outcome is proved in proposition 5.

**Proposition 5** *There exists a unique value  $\phi^* < 0$  such that the negative eigenvalues of the Jacobian matrix in the centrally-planned and competitive economies are equal.*

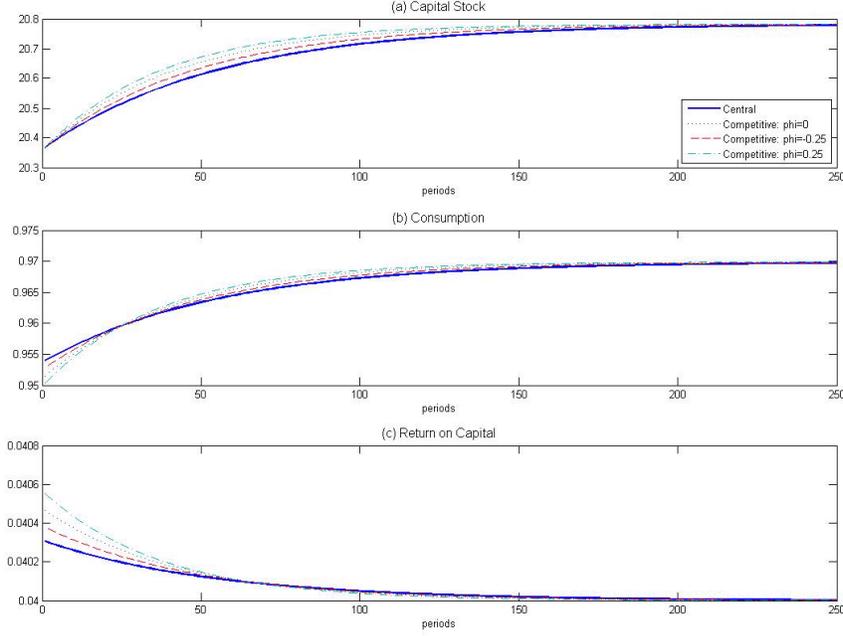


Figure 12: Growth Paths for Various Progressivity

**Proof.** Define  $f(\phi)$  by

$$f(\phi) = \rho - \sqrt{\rho^2 - \frac{4\rho^2(\alpha + \beta - 1)}{\alpha(1+b)}} - \frac{\rho}{1+b} + \sqrt{\left(\frac{\rho}{1+b}\right)^2 - \frac{4\rho^2[(1-\phi)\alpha + \beta - 1]}{\alpha(1+b)}}$$

Since  $\alpha + \beta - 1 < 0$ , it is easily checked that (i)  $f(0) > 0$  and (ii)  $f$  is monotonically increasing in  $\phi$ . Now define  $\tilde{\phi} < 0$  such that  $(1 - \tilde{\phi})\alpha + \beta - 1 = 0$  and observe  $f(\tilde{\phi}) < 0$ . The continuity of  $f(\phi)$  then establishes the existence of  $\phi^* < 0$  such that  $f(\phi^*) = 0$ . ■

### 3.3 A Growth Model with Endogenous Labour

In this section, we will consider the case where labour supply is endogenously chosen. The labour-leisure decision is now incorporated into the model by adding labour effort

( $L_t$ ) of an agent into the utility function<sup>5</sup> and the production function as

$$u(C_t, H_t, L_t) = \ln C_t + b \ln H_t - \frac{1}{1 + \chi} L_t^{1+\chi},$$

$$Y_t = AK_t^\alpha L_t^{1-\alpha} G_t^\beta, \quad 0 < \alpha, \beta < \alpha + \beta, \quad (87)$$

where  $\chi$  is the inverse of the labour supply elasticity. The public consumption service ( $H_t$ ) and productive public capital ( $G_t$ ) are provided by the rule defined in (73).

### 3.3.1 Centrally Planned Economy

While labour is introduced into utility and production sides, the global resource constraint faced by a benevolent social planner is the same as the one in (74). Combining (73) and (87), the production function can be defined as

$$Y_t = A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} K_t^{\frac{\alpha}{1-\beta}} L_t^{\frac{1-\alpha}{1-\beta}}.$$

Then the current-value Hamiltonian for the planner's problem,  $H(C_t, K_t, L_t, \lambda_t)$  can be described as

$$H(C_t, K_t, L_t, \lambda_t) = \ln C_t + b \ln(\theta_h A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} K_t^{\frac{\alpha}{1-\beta}} L_t^{\frac{1-\alpha}{1-\beta}})$$

$$- \frac{1}{1 + \chi} L_t^{1+\chi} + \lambda_t \left[ (1 - \theta_h - \theta_g) A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} K_t^{\frac{\alpha}{1-\beta}} L_t^{\frac{1-\alpha}{1-\beta}} - C_t \right],$$

where  $\lambda_t$  is the co-state variable. The first-order conditions for this Hamiltonian are

$$\frac{1}{C_t} = \lambda_t, \quad (88)$$

$$L_t^{\chi+1} = \frac{b(1-\alpha)}{1-\beta} + \lambda_t(1-\theta_h-\theta_g)\frac{1-\alpha}{1-\beta}Y_t, \quad (89)$$

$$b\frac{\alpha}{1-\beta}\frac{1}{K_t} + \lambda_t\frac{\alpha}{1-\beta}(1-\theta_h-\theta_g)\frac{Y_t}{K_t} = \rho\lambda_t - \dot{\lambda}_t, \quad (90)$$

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<sup>5</sup>This definition of utility function combines the framework of Barro (1990) and Benhabib and Farmer (1994).

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_t K_t = 0, \quad (91)$$

which leads us to the dynamic system

$$\frac{\dot{K}_t}{K_t} = (1 - \theta_h - \theta_g) \frac{Y_t}{K_t} - \frac{C_t}{K_t}, \quad (92)$$

$$\frac{\dot{C}_t}{C_t} = \frac{\alpha}{1 - \beta} (1 - \theta_h - \theta_g) \frac{Y_t}{K_t} + b \frac{\alpha}{1 - \beta} \frac{C_t}{K_t} - \rho. \quad (93)$$

Since  $\frac{\dot{K}_t}{K_t} = \frac{\dot{C}_t}{C_t} = 0$  at the steady-state, the ratio of steady-state variables,  $Y^*, K^*$  and  $C^*$  can be found from the system above as

$$\frac{Y^*}{K^*} = \frac{(1 - \beta)\rho}{\alpha(1 + b)(1 - \theta_h - \theta_g)},$$

$$\frac{C^*}{K^*} = \frac{(1 - \beta)\rho}{\alpha(1 + b)},$$

$$\frac{Y^*}{C^*} = \frac{1}{1 - \theta_h - \theta_g}.$$

Combining these results and the first-order conditions (88) and (89), the steady-state value of all variables can be defined as

$$L^* = \left[ \frac{(1 + b)(1 - \alpha)}{1 - \beta} \right]^{\frac{1}{1+x}} \quad (94)$$

$$K^* = \left[ \frac{(1 - \beta)\rho}{\alpha(1 + b)(1 - \theta_h - \theta_g) A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}}} \right]^{\frac{1-\beta}{\alpha+\beta-1}} (L^*)^{\frac{\alpha-1}{\alpha+\beta-1}} \quad (95)$$

$$C^* = \frac{(1 - \beta)\rho}{\alpha(1 + b)} K^*. \quad (96)$$

Now to look into the stability of this steady-state economy, we can rewrite the dynamic system of (92) and (93) using logarithmic variables,  $k_t = \ln K_t$ ,  $c_t = \ln C_t$  and  $l_t = \ln L_t$  as

$$\dot{k}_t = (1 - \theta_h - \theta_g) A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} e^{\frac{\alpha+\beta-1}{1-\beta} k_t + \frac{1-\alpha}{1-\beta} l_t} - e^{c_t - k_t}$$

$$\dot{c}_t = \frac{\alpha}{1-\beta}(1-\theta_h-\theta_g)A^{\frac{1}{1-\beta}}\theta_g^{\frac{\beta}{1-\beta}}e^{\frac{\alpha+\beta-1}{1-\beta}k_t+\frac{1-\alpha}{1-\beta}l_t} + b\frac{\alpha}{1-\beta}e^{c_t-k_t} - \rho.$$

From (88) and (89), we notice that  $L_t$  can be expressed as a function of  $C_t$  and  $K_t$  in an implicit form as

$$L_t^{\chi+1} = \frac{b(1-\alpha)}{1-\beta} + (1-\theta_h-\theta_g)\frac{1-\alpha}{1-\beta}\frac{A^{\frac{1}{1-\beta}}\theta_g^{\frac{\beta}{1-\beta}}K_t^{\frac{\alpha}{1-\beta}}L_t^{\frac{1-\alpha}{1-\beta}}}{C_t}.$$

Using this function, the partial derivatives of  $l_t$  with respect to  $c_t$  and  $k_t$  evaluated at the steady-state can be defined as

$$l_c = \frac{\partial l_t}{\partial c_t} = -\frac{1-\beta}{(1-\beta)(1+\chi)(1+b) - (1-\alpha)}$$

$$l_k = \frac{\partial l_t}{\partial k_t} = \frac{\alpha}{(1-\beta)(1+\chi)(1+b) - (1-\alpha)}.$$

Previous results can be used to derive the Jacobian Matrix,  $J = (J_{ij})_{2 \times 2}$  evaluated at the steady-state, which has the elements

$$J_{11} = \frac{\rho(1-\beta)(1+\chi)}{(1-\beta)(1+\chi)(1+b) - (1-\alpha)}$$

$$J_{12} = -\frac{(1-\beta)\rho}{\alpha} \left[ \frac{(1-\beta)(1+\chi)}{(1-\beta)(1+\chi)(1+b) - (1-\alpha)} \right]$$

$$J_{21} = \frac{\rho[(\alpha - (1+b)(1-\beta))(1+\chi) + (1-\alpha)]}{(1-\beta)(1+\chi)(1+b) - (1-\alpha)}$$

$$J_{22} = -\frac{\rho[-b(1-\beta)(1+\chi) + (1-\alpha)]}{(1-\beta)(1+\chi)(1+b) - (1-\alpha)}.$$

While each element seems to show much complexity, it can be checked that the value of trace ( $T$ ) and determinant ( $D$ ) for the Jacobian just collapse to

$$T = \rho,$$

$$D = \frac{\rho^2(\alpha + \beta - 1)(1-\beta)(1+\chi)}{\alpha[(1-\beta)(1+\chi)(1+b) - (1-\alpha)]}.$$

The discussion of the stability of this system will be put off until later when we compare the results of centrally planned economy and a competitive economy.

### 3.3.2 Decentralized Economy

For the representative agent in a competitive market, the budget constraint is

$$\dot{K}_t = (1 - \tau_k)r_t K_t + (1 - \tau_w)w_t L_t - (1 + \tau_c)C_t - T_t,$$

where  $r_t$  is the rate of return on private capital and  $w_t$  is the wage rate for a labour unit.  $\tau_k, \tau_w$  and  $\tau_c$  are tax rates for capital gain, wage and consumption respectively and  $T_t$  represents the lump-sum transfer to balance the government budget by the rule defined by

$$T_t = (\theta_g + \theta_h)Y_t - \tau_k r_t K_t - \tau_w w_t L_t - \tau_c C_t.$$

The current value Hamiltonian for the agent in a competitive market and its first-order conditions can be described as

$$H(C_t, K_t, \lambda_t) = \ln C_t + b \ln H_t - \frac{1}{1 + \chi} L_t^{1 + \chi} + \lambda_t [(1 - \tau_k)r_t K_t + (1 - \tau_w)w_t L_t - (1 + \tau_c)C_t - T_t]$$

$$\frac{1}{C_t} = (1 + \tau_c)\lambda_t \quad (97)$$

$$L_t^\chi = \lambda_t(1 - \tau_w)w_t \quad (98)$$

$$\lambda_t(1 - \tau_k)r_t = \rho\lambda_t - \dot{\lambda}_t, \quad (99)$$

with the transversality condition,  $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_t K_t = 0$ .

From the assumption of the competitive market,  $r_t$  and  $w_t$  are determined in the factor market according to their marginal production conditions:

$$r_t = \alpha A K_t^{\alpha - 1} L_t^{1 - \alpha} G_t^\beta \quad (100)$$

$$w_t = (1 - \alpha)AK_t^\alpha L_t^{-\alpha} G_t^\beta. \quad (101)$$

Combining the fact that  $Y_t = A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} K_t^{\frac{\alpha}{1-\beta}} L_t^{\frac{1-\alpha}{1-\beta}}$  with the first-order conditions and factor costs, a dynamic system can be defined as

$$\frac{\dot{K}_t}{K_t} = (1 - \theta_h - \theta_g) \frac{Y_t}{K_t} - \frac{C_t}{K_t} \quad (102)$$

$$\frac{\dot{C}_t}{C_t} = \alpha(1 - \tau_k) \frac{Y_t}{K_t} - \rho, \quad (103)$$

and this system has a steady-state,  $(L^*, K^*, C^*)$ :

$$L_C^* = \left[ \frac{(1 - \tau_w)(1 - \alpha)}{(1 + \tau_c)(1 - \theta_h - \theta_g)} \right]^{\frac{1}{1+\chi}} \quad (104)$$

$$K_C^* = \left[ \frac{\rho}{\alpha(1 - \tau_k)A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}}} \right]^{\frac{1-\beta}{\alpha+\beta-1}} (L^*)^{\frac{\alpha-1}{\alpha+\beta-1}} \quad (105)$$

$$C_C^* = \frac{(1 - \theta_h - \theta_g) \rho}{\alpha(1 - \tau_k)} K_C^*. \quad (106)$$

From (97), (98) and (101), we can know that  $L_t^{\chi+1} = \frac{(1-\tau_w)(1-\alpha)}{1-\tau_c} A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} K_t^{\frac{\alpha}{1-\beta}} L_t^{\frac{1-\alpha}{1-\beta}} C_t^{-1}$  and consequently it can be obtained that  $\ln(L_t) = \frac{1-\beta}{(\chi+1)(1-\beta)-(1-\alpha)} \ln \left[ \frac{(1-\tau_w)(1-\alpha)}{1+\tau_c} A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} \right] + \frac{\alpha}{(\chi+1)(1-\beta)-(1-\alpha)} \ln K_t - \frac{1-\beta}{(\chi+1)(1-\beta)-(1-\alpha)} \ln C_t$ . Then the dynamic system in (102) and (103) can be rewritten with logarithm variables,  $k_t = \ln K_t$ ,  $c_t = \ln C_t$  and  $l_t = \ln L_t$ :

$$\dot{k}_t = (1 - \theta_h - \theta_g) e^{\mu_0 + \mu_1 k_t + \mu_2 c_t} - e^{c_t - k_t}$$

$$\dot{c}_t = \alpha(1 - \tau_k) e^{\mu_0 + \mu_1 k_t + \mu_2 c_t} - \rho,$$

where

$$\mu_0 = \ln A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} + \frac{1-\alpha}{(\chi+1)(1-\beta)-(1-\alpha)} \ln \left[ \frac{(1-\tau_w)(1-\alpha)}{1+\tau_c} A^{\frac{1}{1-\beta}} \theta_g^{\frac{\beta}{1-\beta}} \right],$$

$$\mu_1 = \frac{(\alpha+\beta-1)(\chi+1)+(1-\alpha)}{(\chi+1)(1-\beta)-(1-\alpha)},$$

$$\mu_2 = -\frac{1-\alpha}{(\chi+1)(1-\beta)-(1-\alpha)}.$$

Notice that at the steady-state  $e^{\mu_0 + \mu_1 k^* + \mu_2 c^*} = \frac{Y^*}{K^*} = \frac{\rho}{\alpha(1-\tau_k)}$  and  $e^{c^* - k^*} = \frac{C^*}{K^*} =$

$\frac{(1-\theta_h-\theta_g)\rho}{\alpha(1-\tau_k)}$ . Using these result, the Jacobian matrix for the decentralized economy,  $J^C = (J_{ij}^C)_{2 \times 2}$  can be defined as

$$J_{11} = \frac{(1-\theta_h-\theta_g)\rho}{\alpha(1-\tau_k)}(\mu_1+1),$$

$$J_{12} = \frac{(1-\theta_h-\theta_g)\rho}{\alpha(1-\tau_k)}(\mu_2-1),$$

$$J_{21} = \mu_1\rho,$$

$$J_{22} = \mu_2\rho.$$

Finally, the trace ( $T$ ) and determinant ( $D$ ) of the Jacobian can be calculated as

$$T = \frac{\rho}{[(\chi+1)(1-\beta)-(1-\alpha)]} \left[ \frac{(1-\theta_h-\theta_g)(1+\chi)}{(1-\tau_k)} - (1-\alpha) \right],$$

$$D = \frac{\rho^2(1-\theta_h-\theta_g)}{\alpha(1-\tau_k)} \frac{(\alpha+\beta-1)(1+\chi)}{[(\chi+1)(1-\beta)-(1-\alpha)]}$$

The stability of this system will be discussed in the following section.

### 3.3.3 Optimal Taxation

#### (a) Constant Income Tax Rate

In the same manner in the case with inelastic labour, tax rates,  $(\tau_w^*, \tau_k^*, \tau_c^*)$  to implement the first-best steady-state in a competitive economy can be defined from the comparison of (94)-(96) with (104)-(106) by

$$\frac{1-\tau_w^*}{1+\tau_c^*} = 1-\tau_k^* = \frac{(1+b)(1-\theta_h-\theta_g)}{1-\beta}. \quad (107)$$

However, in this case we cannot always guarantee that the economy will reach this steady-state using these tax-rates since the stability of this steady-state could be different according to the parameter values. It is possible that even though a centrally-planned economy and a competitive economy have the same steady-state, the stability

of it could be different for the two cases. To check this fact, compare the values of trace and determinant of a centrally-planned economy with those of a competitive economy under the tax-rates satisfying (107):

|             | Central   | Competitive   |
|-------------|---|---|
| Trace       | $\rho$  | $\frac{\rho}{1+b} \frac{[(1-\beta)(1+\chi)-(1-\alpha)(1+b)]}{[(1-\beta)(1+\chi)-(1-\alpha)]}$ |
| Determinant | $\frac{\rho^2(\alpha+\beta-1)(1-\beta)(1+\chi)}{\alpha[(1-\beta)(1+\chi)(1+b)-(1-\alpha)]}$ | $\frac{\rho^2(\alpha+\beta-1)(1-\beta)(1+\chi)}{\alpha(1+b)[(1-\beta)(1+\chi)-(1-\alpha)]}$   |

Since the dynamic system we are dealing with here has one predetermined variable, the economy would show saddle-path stability and a unique growth path to a steady-state if and only if the determinant of Jacobian has a negative value or the Jacobian has two eigenvalues of opposite sign. If the determinant is positive and the trace is negative, the steady-state is locally indeterminate and the final case with positive determinant and trace would mean that the steady-state is unstable, where any small deviation from the steady-state would make the economy diverge from the steady-state. Here we have three cases to look into.

**Proposition 6** *With the tax rate given by (107),*

(i) *The centrally planned economy and the competitive economy have a unique saddle-path converging to the same-state if and only if  $(1-\beta)(1+\chi)-(1-\alpha) > 0$  so  $(1-\beta)(1+\chi)(1+b)-(1-\alpha) > 0$ .*

(ii) *A saddle-path can only be defined in the centrally-planned economy and the steady-state in the competitive economy is unstable if and only if  $(1-\beta)(1+\chi)-(1-\alpha) < 0$  but  $(1-\beta)(1+\chi)(1+b)-(1-\alpha) > 0$ .*

(iii) *The steady-state is unstable both in the centrally-planned economy and the competitive economy if and only if  $(1-\beta)(1+\chi)(1+b)-(1-\alpha) < 0$  so  $(1-\beta)(1+\chi)-(1-\alpha) < 0$ .*

In the first case, both economies converge to the same steady-state, but the paths are different if  $b \neq 0$ . As we have observed in the case of inelastic labour, the capital

stock and the consumption level would show a faster growing path converging to the steady-state than the centrally-planned economy since it can be shown that the speed of convergence of capital and consumption would be always greater in the competitive economy<sup>6</sup>. However, output may show a different growth pattern from capital and consumption because of the endogenous labour choice. As shown in Figure 13, under the constant income tax rate given by (107), the competitive agent chooses a higher level of labour than the first-best choice of the central planner at the initial point and decreases labour supply at a faster speed. In the competitive economy, output is decreasing while it grows in the centrally-planned economy.

In the second case, the planned equilibrium and the competitive equilibrium also show the same steady-state. However, this case means that the constant income tax rate defined in (107) cannot implement the first-best steady-state in a competitive economy unless the economy is put on that steady-state from the start. In the final case, it is impossible in both economies to construct a growth path converging to the common steady-state when the economy starts from a state away from the steady-state.

## (b) Non-linear Income Taxation : Uniform Income Tax

In this section, we consider the income tax schedule given by (79). For simplicity, it is assumed that income tax rate is set equal for capital and labour income, i.e.  $\tau_w = \tau_k = \tau_y$ . The current value Hamiltonian for the agent in a competitive market and its first order conditions are given by

$$H(C_t, L_t, K_t, \lambda_t) = \ln C_t + b \ln H_t - \frac{1}{1 + \chi} L_t + \lambda_t \left[ \psi \bar{Y}^\phi Y_t^{1-\phi} - (1 + \tau_c) C_t - T_t \right],$$

---

<sup>6</sup>From the table, we can check that  $T(J) > T(J^{CP})$  and  $D(J) > D(J^{CP})$ . So the characteristic polynomials of  $p(J)$  and  $p(J^{CP})$  can be defined as

$$p(J) = \mu^2 - T(J)\mu + D(J), \quad p(J^{CP}) = v^2 - T(J^{CP})v + D(J^{CP}).$$

Since  $D(J), D(J^{CP}) < 0$ , two polynomials have an negative root denoted by  $\mu_1$  and  $v_1$  respectively. Then

$$v_1^2 - T(J)v_1 + D(J) > v_1^2 - T(J^{CP})v_1 + D(J^{CP}) = 0,$$

since  $T(J) > T(J^{CP})$ ,  $D(J) > D(J^{CP})$  and  $v_1$  is negative. Therefore, we can conclude that  $v_1 < \mu_1$  or  $|v_1| > |\mu_1|$ .

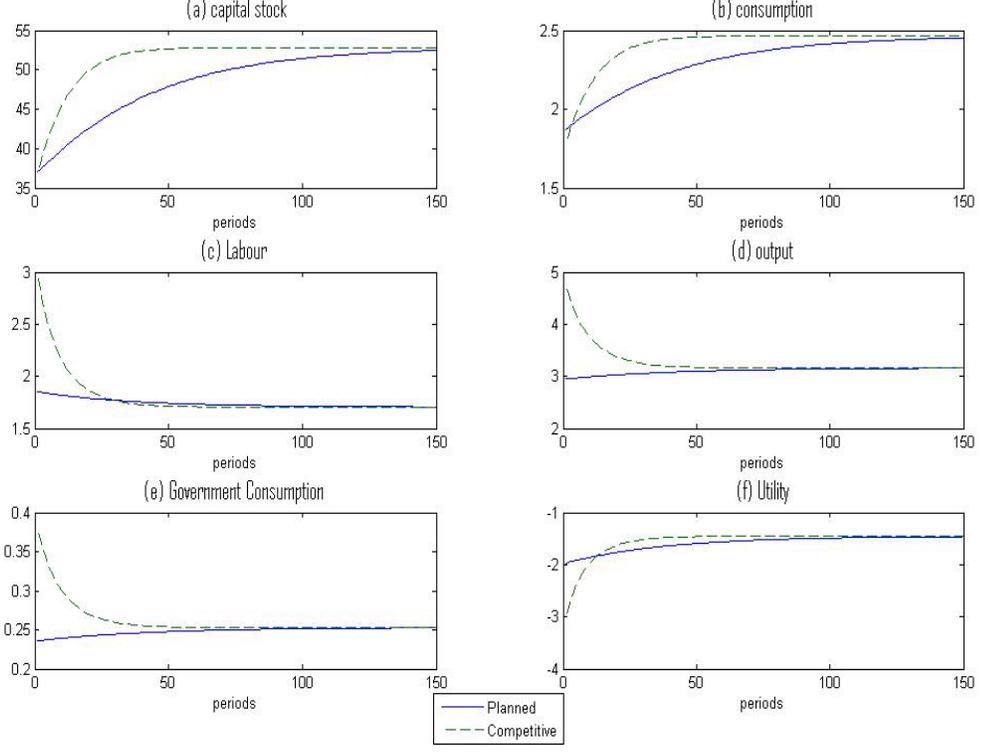


Figure 13: Growth Paths of Centrally-planned and Competitive Economy

$$\frac{1}{C_t} = (1 + \tau_c)\lambda_t, \quad (108)$$

$$L_t^{\chi+1} = \lambda_t \left[ \psi(1 - \phi)(1 - \alpha)\bar{Y}^\phi Y_t^{1-\phi} \right], \quad (109)$$

$$\psi(1 - \phi)\alpha\bar{Y}^\phi Y_t^{1-\phi} = \rho - \frac{\dot{\lambda}_t}{\lambda_t}, \quad (110)$$

with the transversality condition,  $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_t K_t = 0$ .

These results lead us to the dynamic system,

$$\frac{\dot{K}_t}{K_t} = (1 - \theta_h - \theta_g) \frac{Y_t}{K_t} - \frac{C_t}{K_t}, \quad (111)$$

$$\frac{\dot{C}_t}{C_t} = \psi(1 - \phi)\alpha\bar{Y}^\phi \frac{Y_t^{1-\phi}}{K_t} - \rho, \quad (112)$$

and the steady-state values,

$$L^* = \left[ \frac{\psi(1 - \phi)(1 - \alpha)}{(1 + \tau_c)(1 - \theta_h - \theta_g)} \right]^{\frac{1}{\chi+1}},$$

$$K^* = \left[ \frac{\rho}{\alpha\psi(1-\phi)A^{\frac{1}{1-\beta}}\theta_g^{\frac{\beta}{1-\beta}}} \right]^{\frac{1-\beta}{\alpha+\beta-1}} (L^*)^{\frac{1-\alpha}{\alpha+\beta-1}},$$

$$C^* = \frac{\rho(1-\theta_h-\theta_g)}{\alpha\psi(1-\phi)} K^*.$$

By comparing these steady-states with those of a central planner, the tax schedule to implement the first-best steady-state in a competitive economy can be defined as

$$\psi^*(1-\phi)^* = \frac{(1+b)(1-\theta_h-\theta_g)}{1-\beta}, \quad \tau_c = 0. \quad (113)$$

To check the stability of the system, the system of (111) and (112) should be rewritten with logarithmic terms,  $k_t = \ln K_t$ ,  $c_t = \ln C_t$  and  $l_t = \ln L_t$  as

$$\dot{k}_t = (1-\theta_h-\theta_g)e^{y_t-k_t} - e^{c_t-k_t} \quad (114)$$

$$\dot{c}_t = \psi(1-\phi)\alpha\bar{Y}^\phi e^{(1-\phi)y_t-k_t} - \rho. \quad (115)$$

From (108) and (109), we obtain the fact that  $l_t = \frac{1}{(1+\chi)} \log \frac{\psi(1-\phi)(1-\alpha)\bar{Y}^\phi}{(1+\tau_c)} + \frac{1-\phi}{1+\chi}y_t - \frac{1}{1+\chi}c_t$  and as a result, it holds that

$$\frac{\partial y_t}{\partial k_t} = v_1 = \frac{\alpha(\chi+1)}{(1-\beta)(1+\chi) - (1-\alpha)(1-\phi)},$$

$$\frac{\partial y_t}{\partial c_t} = v_2 = -\frac{1-\alpha}{(1-\beta)(1+\chi) - (1-\alpha)(1-\phi)}.$$

Combining these results with the fact that  $\frac{Y^*}{K^*} = \frac{\rho}{\psi(1-\phi)\alpha}$  and  $\frac{C^*}{K^*} = \frac{\rho(1-\theta_h-\theta_g)}{\psi(1-\phi)\alpha}$ , the elements of the Jacobian of the system (114) and (115) evaluated at the steady-state,  $J^{CP} = (J_{ij}^{CP})_{2 \times 2}$  can finally be given as

$$J_{11}^{CP} = \frac{(1-\theta_h-\theta_g)\rho}{\alpha\psi(1-\phi)} v_1,$$

$$J_{12}^{CP} = \frac{(1-\theta_h-\theta_g)\rho}{\alpha\psi(1-\phi)} (v_2 - 1),$$

$$J_{21}^{CP} = \rho[(1 - \phi)v_1 - 1],$$

$$J_{22}^{CP} = \rho(1 - \phi)v_2,$$

with the trace ( $T$ ) and determinant ( $D$ ):

$$T = \frac{\rho}{[(\chi + 1)(1 - \beta) - (1 - \alpha)(1 - \phi)]} \left[ \frac{(1 - \theta_h - \theta_g)(1 + \chi)}{\psi(1 - \phi)} - (1 - \phi)(1 - \alpha) \right],$$

$$D = \frac{\rho^2(1 - \theta_h - \theta_g)}{\alpha\psi(1 - \phi)} \left[ \frac{((1 - \phi)\alpha + \beta - 1)(1 + \chi) - \phi(1 - \alpha)}{(\chi + 1)(1 - \beta) - (1 - \alpha)(1 - \phi)} \right].$$

Under the tax schedule defined by (113), the values of  $T$  and  $D$  would be as follows:

|             | Central   | Competitive  |
|-------------|---|--|
| Trace       | $\rho$  | $\frac{\rho}{1+b} \frac{[(1-\beta)(1+\chi) - (1-\phi)(1-\alpha)(1+b)]}{[(1-\beta)(1+\chi) - (1-\phi)(1-\alpha)]}$                |
| Determinant | $\frac{\rho^2(\alpha+\beta-1)(1-\beta)(1+\chi)}{\alpha[(1-\beta)(1+\chi)(1+b) - (1-\alpha)]}$ | $\frac{\rho^2(1-\beta)[((1-\phi)\alpha+\beta-1)(1+\chi) - \phi(1-\alpha)]}{\alpha(1+b)[(1-\beta)(1+\chi) - (1-\alpha)(1-\phi)]}$ |

While it can be noticed that the degree of progressivity or regressivity,  $\phi$ , is now involved with the value of trace and determinant of Jacobian of a competitive economy, it is not straightforward to see the effect  $\phi$  has on those values since it influences both the denominator and the numerator. To make the points clear, figure 14 displays an example of the three cases in proposition 6. With other parameter values except  $\beta$  set equal to those in Turvosky (2000)<sup>7</sup>, three values of  $\beta$ , 0.4, 0.5 and 0.6 now create the situations representing case 1, 2 and 3 respectively.

Figure 14-(a) shows a case where the competitive and the centrally-planned equilibrium have saddle-path stability. When  $\phi = 0$ , the determinant and the trace in the competitive equilibrium are smaller than those in the centrally-planned economy, so that the absolute value of the negative eigenvalue of the competitive Jacobian is greater than that of the centrally-planned Jacobian as we have shown earlier. The negative eigenvalue of Jacobian approximates the pattern of growth in capital and con-

<sup>7</sup>Parameter values are  $A = 1, \alpha = 0.3, \rho = 0.04, \chi = 0.3, b = 1.0, \theta_g = 0.08$  and  $\theta_h = 0.14$ . These parameter values will be used throughout this chapter when we display the determinant and the trace.

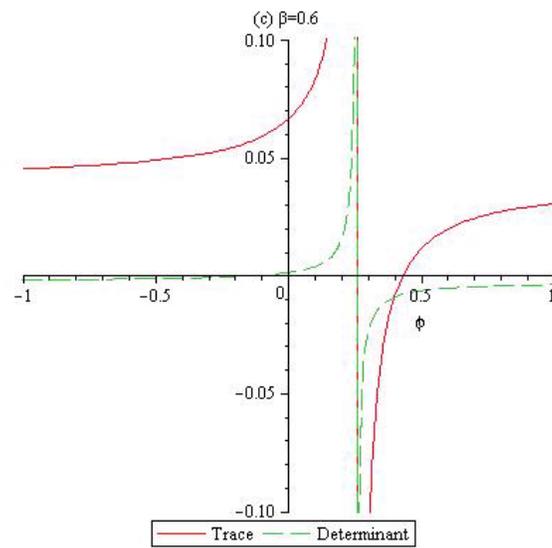
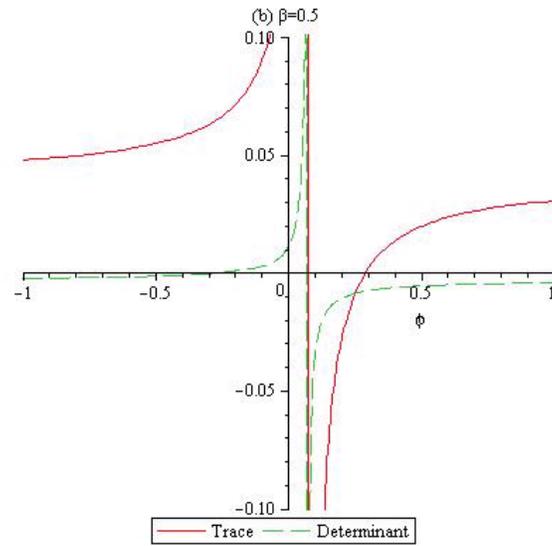
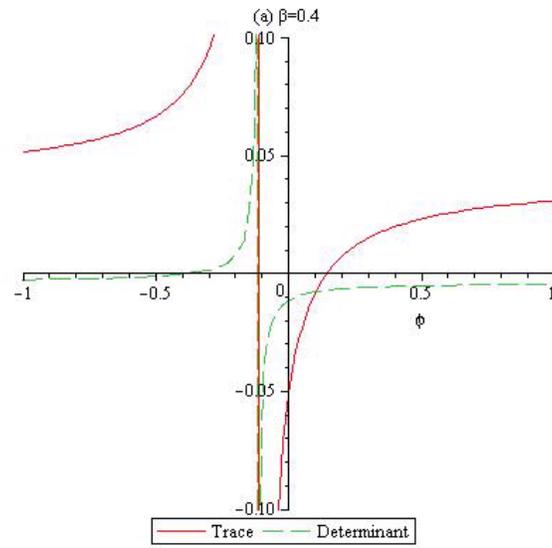


Figure 14: Trace and Determinant under the Various Progressivity

sumption by giving a rate of the convergence speed at which the deviation from the steady-state is reduced. In the case with fixed labour supply, the trajectory of capital governs the evolution of all variables including output and consumption since capital is the only production input. However, when labour is elastic, the negative eigenvalue alone is not enough to describe the evolution of other variables except capital since the level of capital and consumption endogenously defines the labour choice through equation, (109). For this reason, in addition to the negative eigenvalue, we should consider the stabilizing constant which defines the choice rule of initial consumption with respect to initial capital to assure that the economy is put on the determinate saddle-growth path<sup>8</sup>. When the income tax is flat ( $\phi = 0$ ), it turns out that the stabilizing constant in the competitive economy is larger than the one in the centrally-planned economy. Since the stabilizing constant represents the ratio of initial distance to steady-state of consumption over the distance of initial capital from the steady-state, the competitive economy starts from a lower level consumption and converges to the steady-state faster than the centrally-planned economy as we can observe in Figure 13. In this sense, the difference in the negative eigenvalue and the stabilizing constant between the competitive and the centrally-planned economy can capture how close to the social optimum the growth path in the competitive economy is.

In Figure 14-(a), saddle-path stability is maintained for positive  $\phi$ . It can be noted that the determinant and the trace are increasing in  $\phi$  for  $\phi > 0$ , but they are smaller than those of the centrally-planned equilibrium for all  $0 < \phi < 1$ . Thus, the absolute value of the negative eigenvalue of Jacobian is decreasing in  $\phi$  for  $\phi > 0$ . For the stabilizing constant, it starts from a larger value in the competitive economy than the planned economy when  $\phi = 0$  and it also decreases in  $\phi$  for  $\phi > 0$ . Thus, the competitive economy with a larger  $\phi$  shows a closer growth pattern to the first-best growth. For

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<sup>8</sup>In the first-order log-linear approximation, the initial consumption,  $c_0$  is chosen by the rule:

$$\ln c_0 - \ln c_{ss} = \text{stabilizing constant} \times (\ln k_0 - \ln k_{ss}),$$

where  $c_{ss}$  and  $k_{ss}$  are the steady-state values of the consumption and the capital respectively and  $k_0$  is the initial level of capital. For more details, see Novales et al (2008).

negative  $\phi$ , the dynamic property of the steady-state changes at  $\phi = \frac{(\alpha+\beta-1)(1+\chi)}{\alpha\chi+1}$  and  $-\frac{(1-\beta)(1+\chi)-(1-\alpha)}{(1-\alpha)}$ . The determinacy is observed in  $-\frac{(1-\beta)(1+\chi)-(1-\alpha)}{(1-\alpha)} < \phi < 0$  or  $\phi < \frac{(\alpha+\beta-1)(1+\chi)}{\alpha\chi+1}$ . When  $-\frac{(1-\beta)(1+\chi)-(1-\alpha)}{(1-\alpha)} < \phi < 0$ , the saddle-path in the competitive equilibrium is growing even faster than the competitive economy under a constant income tax rate, so that the growth path is getting more distant from the first-best optimum. On the contrary, when  $\phi$  is smaller than  $\frac{(\alpha+\beta-1)(1+\chi)}{\alpha\chi+1}$ , the determinant and the trace are larger than those of the centrally-planned economy and they are decreasing as the regressivity gets larger. The competitive economy under the sufficiently negative  $\phi$  shows a slower growth pattern than the planned economy and the absolute value of the negative eigenvalue is getting larger as  $\phi$  decreases. When  $\phi < \frac{(\alpha+\beta-1)(1+\chi)}{\alpha\chi+1}$ , the stabilizing constant also starts from a smaller value than the first-best optimum and increases as the regressivity gets larger. Thus, the growth path is getting closer to the first-best optimum as  $\phi$  decreases.

Both progressive and regressive tax schedules are welfare-improving in the sense that they can replicate the first-best steady-state with a growth path closer to the optimal one than a constant tax schedule. However, the effectiveness of each tax schedule can be very different as we can see in Figure 15. In Figure 15-(a), we display the change in the growth path as the progressivity ( $\phi$ ) increases. It can be observed that the growth path with a higher progressivity is closer to the centrally-planned growth path, but even with very high progressivity, the competitive equilibrium fails to replicate the first-best optimum completely. On the contrary, Figure 15-(b) shows that the competitive equilibrium creates a faster growing economy as the regressivity is getting larger. Since the centrally-planned growth path is observed in the middle of the competitive growth path with  $\phi = -1.0$  and  $-2.0$ , we can conjecture that there exist  $\phi^* \in (-2.0, -1.0)$  that enables the first-best growth path to be implemented in the decentralized economy. In fact, under the parameter values used in Figure 15 it can be checked that the negative eigenvalue of the Jacobian in the competitive equilibrium is monotonically decreasing as  $\phi$  decreases and it is equal to the negative eigenvalue of the centrally-planned economy

(-0.0262) when  $\phi^* \approx -1.32$ . And at this value of  $\phi^*$ , the stabilizing constant in the competitive economy also coincides with the one in the centrally-planned economy (0.7739). For positive  $\phi$ , as  $\phi$  goes to 1, the negative eigenvalue and the stabilizing constant converge to -0.0483 and 0.8406 respectively. To sum up, regressive taxation is superior to progressive taxation in creating a more desirable growth path.

In Figure 14-(b), the steady-state in the competitive equilibrium with a constant income tax schedule ( $\phi = 0$ ) becomes a source since both the determinant and the trace are positive, while the centrally-planned economy shows saddle-path stability. In this case, the negative determinant which is the necessary and sufficient condition for determinacy can be restored with either a progressive or a regressive income tax schedule, thus the socially-desirable steady-state can be reached through a uniquely defined growth path in the decentralized economy. However, as we have seen in the first case, regressive taxation effectively replicates the first-best growth path, while progressive taxation shows limited power.

Finally, Figure 14-(c) displays the macroeconomic dynamics when the production externality,  $\beta$  is high enough to make the steady-state in both the centrally-planned and the competitive economy become a source. Even when the central-planner cannot define an equilibrium path converging to the steady-state, a converging saddle-path can be implemented in the decentralized economy if the income tax schedule is either sufficiently progressive or regressive.

### 3.3.4 Non-linear Income Taxation : Separate Taxation on Capital and

#### Labour Income

In this section, we explore the case where the tax schedules of capital and labour income can be different. Following Guo (1999), non-linear tax schedules for capital and labour income are defined by

$$\tau_k = 1 - \psi_k \left( \frac{\overline{r_t K_t}}{r_t K_t} \right)^{\phi_k}, \quad \tau_l = 1 - \psi_k \left( \frac{\overline{w_t L_t}}{w_t L_t} \right)^{\phi_l},$$

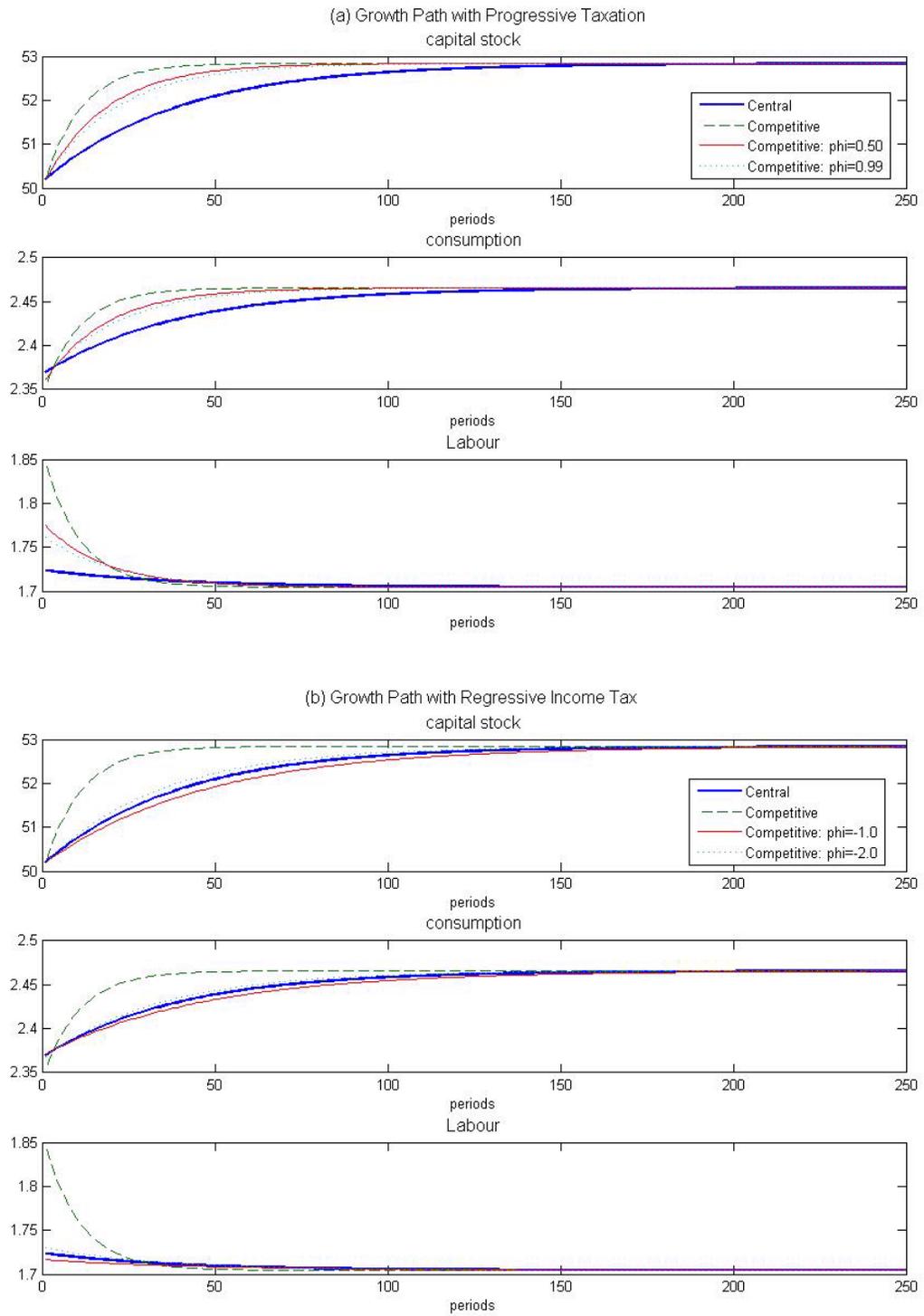


Figure 15: Growth Path under the Various Progressivity

where the parameters  $\psi_k$  and  $\psi_l$  are the level of the capital and the labour income tax and  $\phi_k$  and  $\phi_l$  represent the slopes of the tax schedules.  $\overline{r_t K_t}$  and  $\overline{w_t L_t}$  are the steady-state level of the capital and the labour income respectively which is taken as given by the consumer. With these tax schedules, the current-value Hamiltonian can be constructed as

$$H(C_t, L_t, K_t, \lambda_t) = \ln C_t + b \ln H_t - \frac{1}{1 + \chi} L_t + \lambda_t [\psi_k (\overline{r_t K_t})^{\phi_k} (r_t K_t)^{1 - \phi_k} + \psi_l (\overline{w_t L_t})^{\phi_l} (w_t L_t)^{1 - \phi_l} + (1 + \tau_c) C_t - T_t].$$

Then first-order conditions are

$$\frac{1}{C_t} = (1 + \tau_c) \lambda_t \quad (116)$$

$$L_t^x = \lambda_t \left[ \psi_l (1 - \phi_l) w_t \left( \frac{\overline{w_t L_t}}{w_t L_t} \right)^{\phi_l} \right] \quad (117)$$

$$\psi_k (1 - \phi_k) r_t \left( \frac{\overline{r_t K_t}}{r_t K_t} \right)^{\phi_k} = \rho - \frac{\dot{\lambda}_t}{\lambda_t} \quad (118)$$

$$\lim_{t \rightarrow \infty} \lambda_t K_t = 0.$$

Now notice that in a competitive market, it holds that  $w_t = (1 - \alpha) \frac{Y_t}{L_t}$  and  $r_t = \alpha \frac{Y_t}{K_t}$ , so  $\frac{\overline{w_t L_t}}{w_t L_t} = \frac{\overline{r_t K_t}}{r_t K_t} = \frac{\overline{Y_t}}{Y_t}$ . Put these results into (117) and (118) and we can obtain

$$L_t^{x+1} = \lambda_t \left[ \psi_l (1 - \phi_l) (1 - \alpha) \overline{Y}^{\phi_l} Y_t^{1 - \phi_l} \right] \quad (119)$$

$$\psi_k (1 - \phi_k) \alpha \overline{Y}^{\phi_k} \frac{Y_t^{1 - \phi_k}}{K_t} = \rho - \frac{\dot{\lambda}_t}{\lambda_t}.$$

The dynamic system then can be defined as

$$\frac{\dot{K}_t}{K_t} = (1 - \theta_h - \theta_g) \frac{Y_t}{K_t} - \frac{C_t}{K_t} \quad (120)$$

$$\frac{\dot{C}_t}{C_t} = \psi_k (1 - \phi_k) \alpha \overline{Y}^{\phi_k} \frac{Y_t^{1 - \phi_k}}{K_t} - \rho. \quad (121)$$

This system provides us the steady-state values

$$L^* = \left[ \frac{\psi_l(1 - \phi_l)(1 - \alpha)}{(1 + \tau_c)(1 - \theta_h - \theta_g)} \right]^{\frac{1}{\chi+1}},$$

$$K^* = \left[ \frac{\rho}{\alpha\psi_k(1 - \phi_k)A^{\frac{1}{1-\beta}}\theta_g^{\frac{\beta}{1-\beta}}} \right]^{\frac{1-\beta}{\alpha+\beta-1}} (L^*)^{\frac{1-\alpha}{\alpha+\beta-1}},$$

$$C^* = \frac{\rho(1 - \theta_h - \theta_g)}{\alpha\psi_k(1 - \phi_k)} K^*.$$

By comparing these steady-states with those of a central planner, the tax schedule to implement the first-best steady-state in a competitive economy can be defined as

$$\psi_l^*(1 - \phi_l^*) = \psi_k^*(1 - \phi_k^*) = \frac{(1 + b)(1 - \theta_h - \theta_g)}{1 - \beta}, \quad (122)$$

when we set  $\tau_c = 0$  for convenience. The expression of the system of (120) and (121) in terms of logarithmic variables,  $k_t = \ln K_t$ ,  $c_t = \ln C_t$  and  $l_t = \ln L_t$  is

$$\dot{k}_t = (1 - \theta_h - \theta_g)e^{y_t - k_t} - e^{c_t - k_t} \quad (123)$$

$$\dot{c}_t = \psi_k(1 - \phi_k)\alpha\bar{Y}^{\phi_k}e^{(1-\phi_k)y_t - k_t} - \rho. \quad (124)$$

Notice that (116) and (119) are implying that  $l_t = \frac{1}{(1+\chi)} \log \frac{\psi_l(1-\phi_l)(1-\alpha)\bar{Y}^{\phi_l}}{(1+\tau_c)} + \frac{1-\phi_l}{1+\chi} y_t - \frac{1}{1+\chi} c_t$  and as a result

$$\frac{\partial y_t}{\partial k_t} = v_1 = \frac{\alpha(\chi + 1)}{(1 - \beta)(1 + \chi) - (1 - \alpha)(1 - \phi_l)},$$

$$\frac{\partial y_t}{\partial c_t} = v_2 = -\frac{1 - \alpha}{(1 - \beta)(1 + \chi) - (1 - \alpha)(1 - \phi_l)}.$$

Using these results, the elements of Jacobian of the system (123) and (124),  $J^{CP} = (J_{ij}^{CP})_{2 \times 2}$  can finally defined as

$$J_{11}^{CP} = \frac{(1 - \theta_h - \theta_g)\rho}{\alpha\psi_k(1 - \phi_k)}v_1,$$

$$J_{12}^{CP} = \frac{(1 - \theta_h - \theta_g)\rho}{\alpha\psi_k(1 - \phi_k)}(v_2 - 1),$$

$$J_{21}^{CP} = \rho[(1 - \phi_k)v_1 - 1],$$

$$J_{22}^{CP} = \rho(1 - \phi_k)v_2,$$

with the trace ( $T$ ) and determinant ( $D$ ):

$$T = \frac{\rho}{[(\chi + 1)(1 - \beta) - (1 - \alpha)(1 - \phi_l)]} \left[ \frac{(1 - \theta_h - \theta_g)(1 + \chi)}{\psi_k(1 - \phi_k)} - (1 - \phi_k)(1 - \alpha) \right],$$

$$D = \frac{\rho^2(1 - \theta_h - \theta_g)}{\alpha\psi_k(1 - \phi_k)} \left[ \frac{((1 - \phi_k)\alpha + \beta - 1)(1 + \chi) - \phi_l(1 - \alpha)}{(\chi + 1)(1 - \beta) - (1 - \alpha)(1 - \phi_l)} \right].$$

Under the tax schedules defined by (122), the values of  $T$  and  $D$  would be as follows:

|             | Central   | Competitive  |
|-------------|---|--|
| Trace       | $\rho$  | $\frac{\rho}{1+b} \frac{[(1-\beta)(1+\chi) - (1-\phi_k)(1-\alpha)(1+b)]}{[(1-\beta)(1+\chi) - (1-\phi_l)(1-\alpha)]}$                      |
| Determinant | $\frac{\rho^2(\alpha+\beta-1)(1-\beta)(1+\chi)}{\alpha[(1-\beta)(1+\chi)(1+b) - (1-\alpha)]}$ | $\frac{\rho^2(1-\beta)[((1-\phi_k)\alpha + \beta - 1)(1+\chi) - \phi_l(1-\alpha)]}{\alpha(1+b)[(1-\beta)(1+\chi) - (1-\alpha)(1-\phi_l)]}$ |

From the table, we can notice that  $\phi_k$  and  $\phi_l$  change the value of the trace and the determinant in different ways since  $\phi_k$  does not appear in the denominator of the values while  $\phi_l$  is involved with both denominator and numerator. In the same manner with the uniform income tax case, in Figure 16 we display some examples representing each case in proposition 6. The first column of Figure 16 shows the dynamic property of each case when the capital income tax is assumed to be flat and only the progressivity of the labour income tax can change. In the second column, the labour income tax is flat and the capital income tax can vary in the progressivity.

The role of progressivity of the labour income tax is similar to that of the tax levied

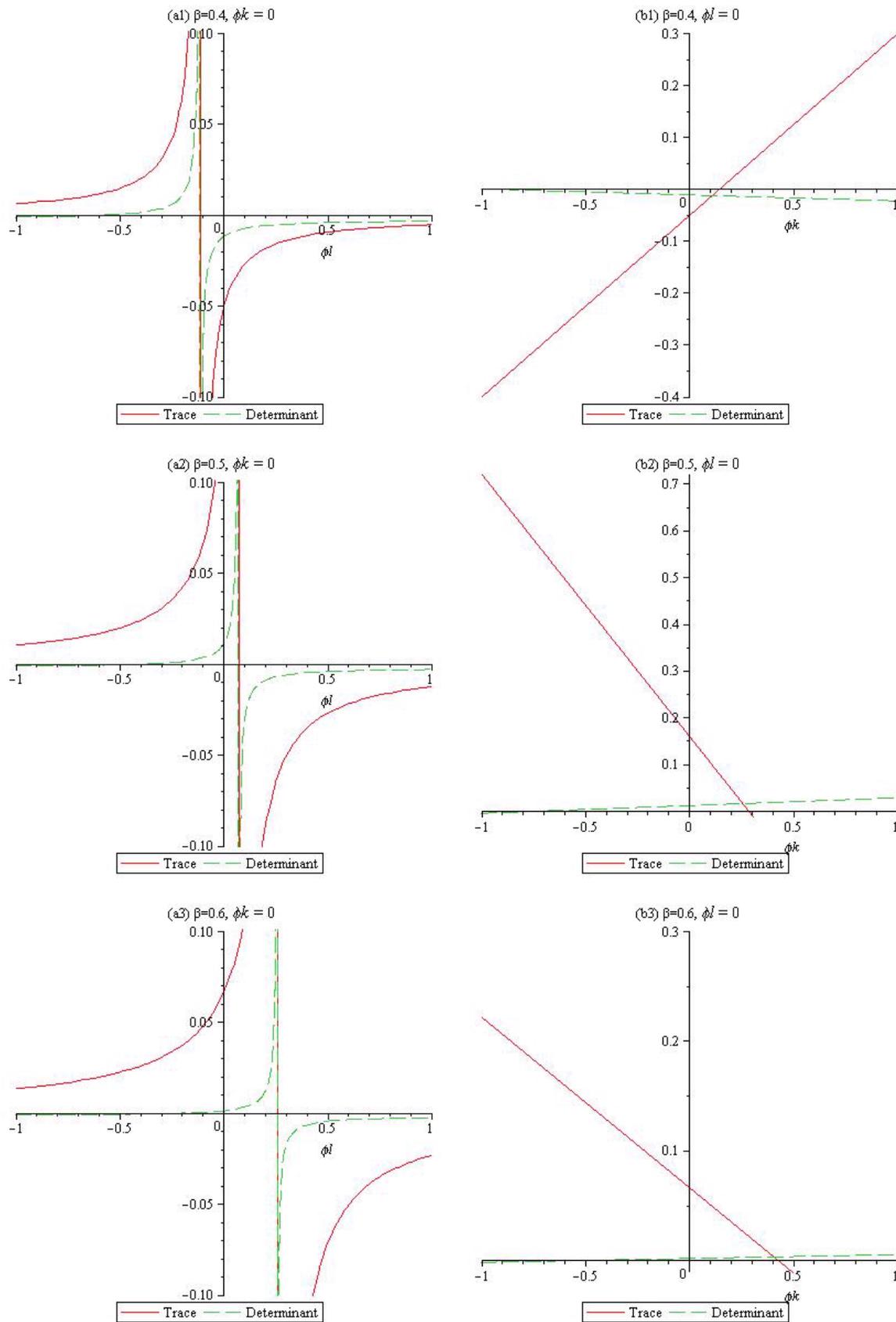


Figure 16: Trace and Determinant for Separate Income Tax

on the total income. When both the centrally-planned economy and the competitive economy have saddle-path stability (Case1), the competitive economy with a flat labour income tax shows a larger absolute value of the negative eigenvalue of the Jacobian than the centrally-planned economy and for  $\phi_l > 0$  the negative eigenvalue increases in  $\phi_l$ , getting closer to the one in the planned economy. In Figure 17-(a) where the growth paths for progressive labour income tax are displayed, it is observed that the growth path for the capital with a larger  $\phi_l$  is closer to the planner's optimal growth path. However, for the growth path of consumption a larger progressivity does not create a closer path to the first-best optimum as the initial consumption level with a larger progressivity ( $\phi_l = 0.99$ ) turns out to be farther apart from the first-best level of initial consumption. This occurs because the stabilizing constant increases as the progressivity is getting larger when  $\phi_l$  is positive. When the labour income tax is regressive, the saddle-path stability is attained for  $\phi_l < \frac{(\alpha+\beta-1)(1+\chi)}{1-\alpha}$  and  $-\frac{(1-\beta)(1+\chi)-(1-\alpha)}{1-\alpha} < \phi_l < 0$ . Figure 17-(b) suggests that there also exists  $\phi_l^* \in (-2.0, -1.0)$  that provides the same negative eigenvalue to the centrally-planned economy as the planned growth path is placed between the growth paths with  $\phi_l = -1.0$  and  $-2.0$ . When  $\phi_l^* \approx -1.31$ , the negative eigenvalue of the Jacobian in both competitive and planned economy is  $-0.0262$ . Although  $\phi_l^*$  succeeds in creating the first-best growth path of the capital in the decentralized economy, it does not automatically guarantee the implementation of the first-best growth path of consumption since the stabilizing constant is still far off from the one in the centrally-planned economy. At  $\phi_l^* = -1.31$ , the stabilizing constant in the competitive economy is 0.9435 and it is much smaller in the centrally-planned economy (0.7739). Therefore, when only labour income tax is allowed to vary in progressivity, neither progressive nor regressive taxation can be straightforwardly welfare-improving compared to a constant labour income tax. When the centrally-planned economy has a unique saddle-path converging to the steady-state, but the steady-state in the competitive economy with a constant tax rate becomes a source (Case 2), both a progressive or a regressive labour income tax schedule can restore

saddle-path stability in the decentralized economy. But the superiority of regressive taxation over progressive taxation in implementing the first-best optimum cannot be asserted in a way that we do in the previous section. When both economies show the source property (Case 3), both a progressive or a regressive labour income tax can create a saddle-path converging to the steady-state of the central planner.

For the capital income tax, the way the trace and the determinant change according to progressivity is quite different. When the production externality is small (Figure 16-b1), the trace is monotonically increasing in  $\phi_k$  while the determinant moves in the opposite direction. Since the pace with which the trace is increasing is far greater than that of the decreasing determinant, the economy with more progressive capital income tax shows a slower growth pattern in the capital than the economy with the constant income tax as we can see in Figure 18-(a). The growth path in capital created by larger  $\phi_k$  is closer to the first-best optimum. However, for consumption, a more progressive capital income tax fails to create a more desirable growth path as the distance in stabilizing constant between the competitive economy and the centrally-planned economy is getting larger as  $\phi_k$  increases. The role of regressive taxation in securing a determinate equilibrium revives as  $\beta$  gets larger. In Figure 16-(b2) and 16-(b3), the unstable equilibrium prevails for the positive  $\phi_k$  and saddle-path stability can be guaranteed only when  $\phi_k$  is sufficiently large. It can be checked that the negative eigenvalue of the Jacobian is monotonically decreasing in  $\phi_k$  for  $\phi_k < \frac{\alpha+\beta-1}{\alpha}$ , so that the speed of the convergence is greater for the smaller  $\phi_k$ . Thus, the growth path in capital approaches the first-best one as the regressivity in the capital income tax increases. However, the centrally-planned consumption path cannot be implemented with a regressive capital income tax neither, since the stabilizing constant is smaller for a larger capital income tax regressivity. In Figure 18-(b), when  $\phi_k = -1.0$ , the stabilizing constant is 0.5907 while it is 0.5776 when  $\phi_k = -2.0$ . Compared to the stabilizing constant in the centrally-planned economy, 0.7739, the initial level of consumption in the competitive economy with a larger regressivity is farther from the first-best choice than

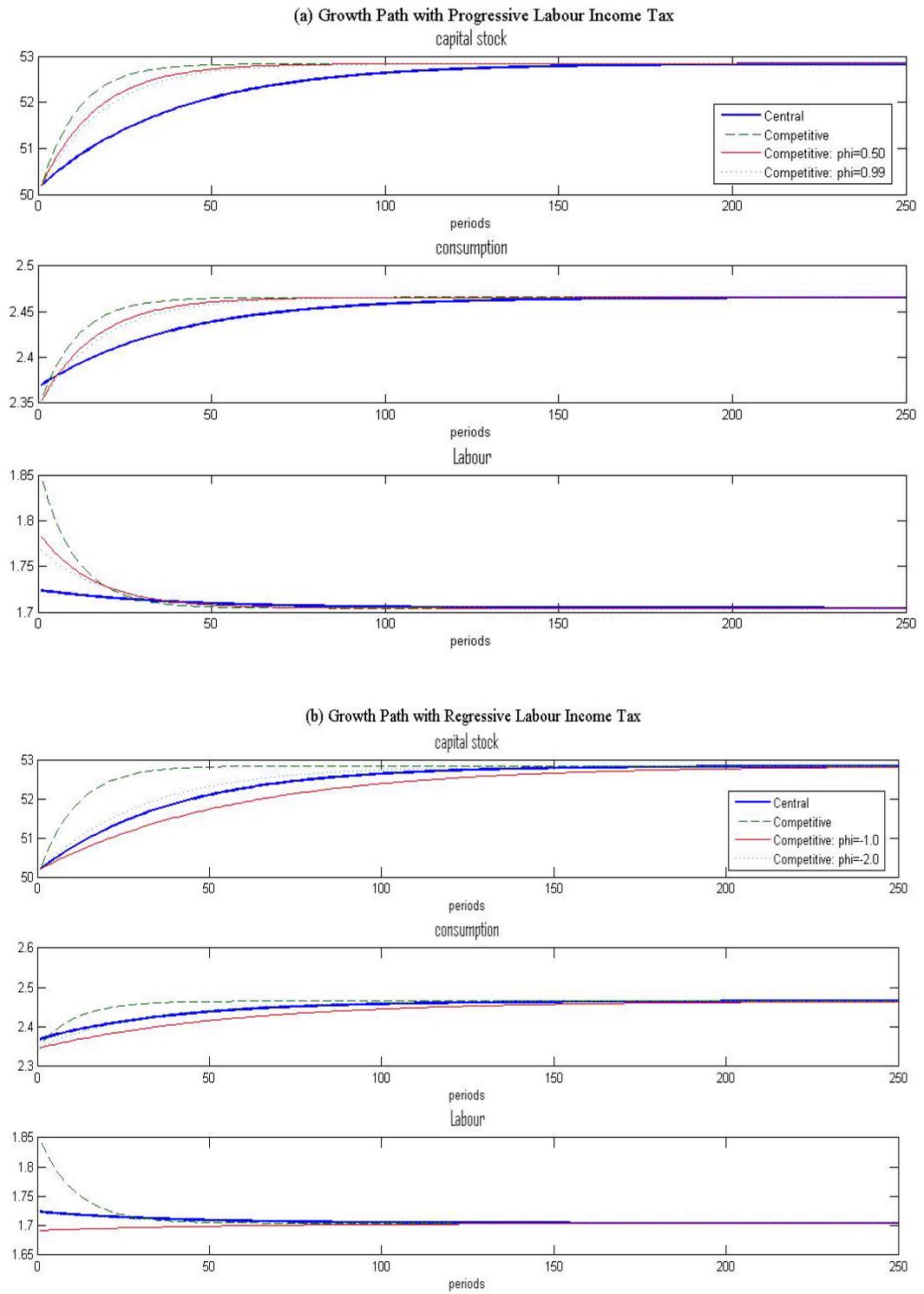


Figure 17: Labour Income Tax Progressivity and Growth Path

the one in a smaller regressivity. To sum up, when the change in progressivity is allowed only in the capital income tax, the first-best growth path cannot be decentralized in the competitive economy.

### 3.4 Conclusion

This section studied the optimal income tax structure in a Barro-type growth model with decreasing-returns-to-scale technology and a public consumption good. When the externality from productive public spending is large enough to create a balanced-growth path, the central planner can internalize the externality in a decentralized economy with a combination of constant income tax and lump-sum tax. If there is no public consumption, a constant income tax is still effective in decreasing-returns-to-scale technology, but we find that the inclusion of the public consumption can make a difference. In our model, a constant income tax rate that implements the first-best optimum steady-state in the long-run does not replicate the first-best growth path converging to the steady-state irrespective of whether the labour supply is fixed or endogenously chosen. When the labour supply is fixed, the decentralized economy grows too quickly and a progressive tax worsens the excessively rapid growth. In contrast, intertemporally regressive income tax moves the decentralized growth path closer to the optimal path, and a unique level of regressivity exists for which the first-best optimum is decentralized to a first-order approximation.

When the labour supply is endogenous, both progressive and regressive income taxation can implement a welfare-improving growth path compared to the competitive equilibrium with the constant income tax rate. However, regressive taxation turns out to be more effective in replicating the central planner's equilibrium since it can create approximately the same growth-path to the first-best while the progressive cannot. When the production externality is large, the competitive economy cannot define any equilibrium path to the first-best steady-state with a constant income tax. A regressive tax schedule can restore saddle-stability in the competitive economy and the higher is

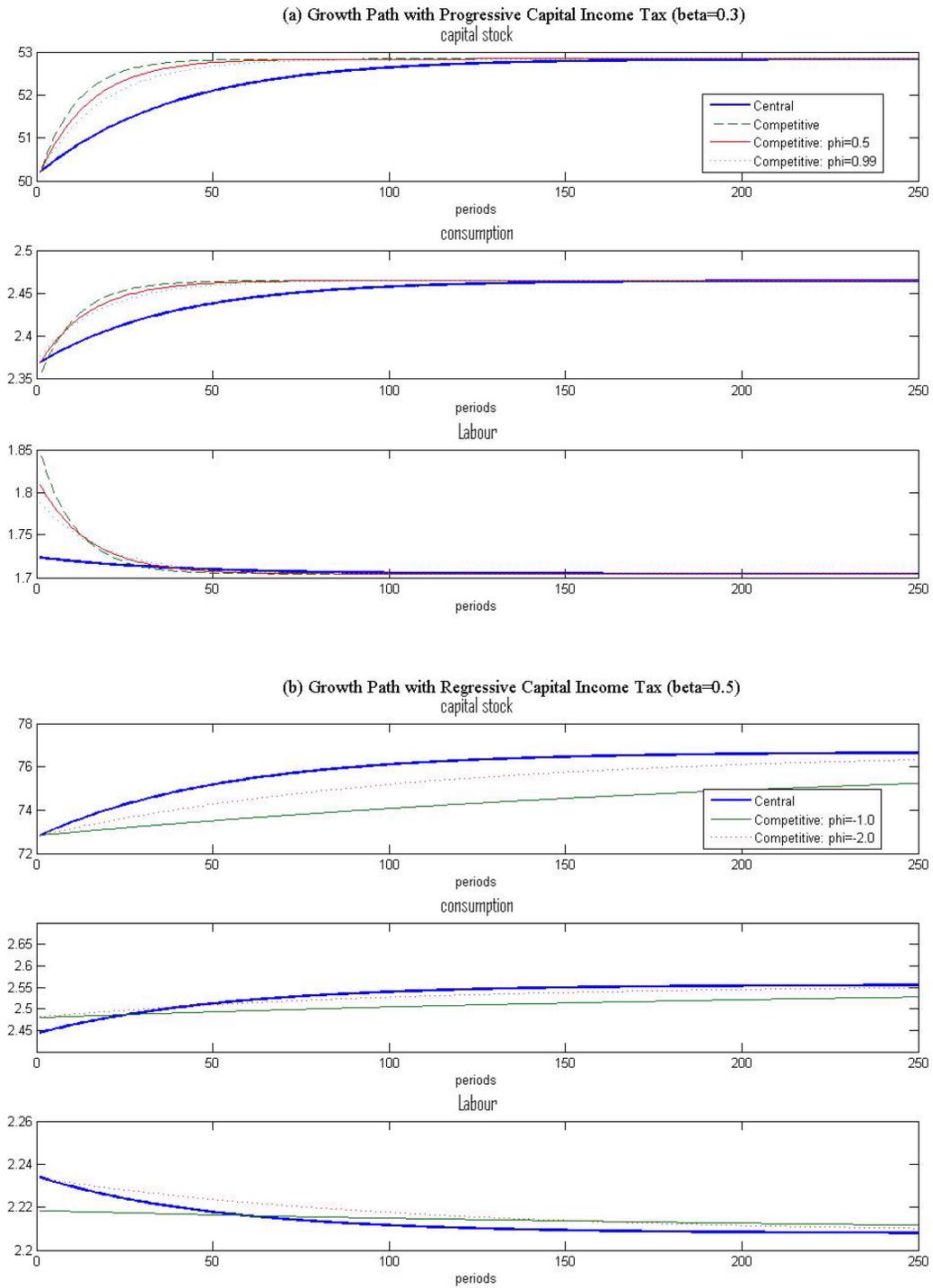


Figure 18: Capital Income Tax Progressivity and Growth Path

the regressivity, the closer the saddle-growth path to the first-best.

# 4 Quasi-hyperbolic Preferences and Optimal Taxation in a Growth Model with Productive Public Spending

## 4.1 Introduction

Many of the standard economic models that include an intertemporal utility maximization problem normally assume that consumers have exponential discount functions. A consumer with this form of discount function assigns a weight to the utility of each period in a manner that decreases exponentially so that the relative importance of today's utility with respect to tomorrow's is evaluated equally to that of utility of period  $t$  with respect to utility of the next period  $t + 1$ . This assumption not only gives a model more tractability and implies a crucial condition for the existence of equilibrium, but also gives the very convenient property that once a life-time consumption path is chosen by a consumer, the same consumption path is chosen regardless of the timing of decision making. What consumers have to do is to solve the optimization problem at the beginning and just follow the consumption path onwards. Since they do not need to think or worry about revision of their first choice, the assumption of an exponential discount function precludes a situation where inconsistency issues arise.

The problem is, however, that the rationale for this assumption has not been properly verified. Rather, the argument against the exponential discount function has been gaining more ground since the results of experimental research like Ainslie (1992) suggest that the actual discount function can be closer to a hyperbola than exponential. A consumer having hyperbolic preferences now applies a higher discount rate between today and tomorrow than any other far off adjacent periods,  $t$  and  $t + 1$ , and consequently shows a tendency to postpone an action leading to immediate disutility to a later date.

Strotz (1955) studied a dynamic utility maximization model with a general form of

discount function embracing the exponential function as a special case. He showed that except for a very special case where the discount function happens to be exponential, the optimal plan of the present date always turns out to be not optimal for a consumer in later dates as they need to revise their plan in every period (or at every moment in continuous time). To address this dynamic inconsistency, Strotz suggested two alternatives. The first is to contrive a commitment scheme which prevents all future selves from deviating from the initial plan by imposing a penalty for disobedience. While this approach can be widely observed in practice such as joining a pension scheme which has penalty terms for early withdrawal, the efficacy of this strategy is very difficult to be guaranteed especially under circumstances where the financial market can provide various borrowing options for which the commitment scheme can be used as collateral (Barro, 1999). An individual may hesitate to set up a strict precommitment device simply because of future risk and uncertainty (Strotz, 1955). The other alternative is to assume more sophistication in the decision-making process. The agent gives up any plan that will be rejected by his future selves and finds the plan which his selves will follow by embracing the untrustworthiness of the future selves. As we will see from the next sections, the solution from this strategy of consistent planning can be found by applying backward-induction. Since this sophisticated reasoning is assumed for all the selves the initial plan by the current agent is realized all through his life.

Assuming agents take a consistent planning strategy, Phelps and Pollak (1968) firstly studied the welfare implications of the competitive equilibrium under quasi-hyperbolic preferences and found out that the marginal upward deviation from the equilibrium saving rate can make the economy better off. While they did not explore the specific optimal saving rate, the result shows that the equilibrium is not Pareto-optimal at least and the economy is undersaving at equilibrium. This non-Pareto optimality of competitive equilibrium under quasi-hyperbolic preferences was studied further by Liabson (1996) which defined specific saving rates that Pareto-dominate the equilibrium saving rate and suggested that public policy commands those saving rates in the com-

petitive economy. Krusell et al (2002) introduced quasi-hyperbolic consumers into a standard Ramsey growth model and found out that quasi-hyperbolic preferences break the fundamental welfare theorem that states the Pareto-optimality of a competitive equilibrium. Furthermore, they showed an interesting result that competitive equilibrium outperforms a planned economy so that any policy intervention of the government commanding the planner's equilibrium is not desirable.

Taking one step further from Krusell et al (2002), this chapter embraces quasi-hyperbolic preferences into a growth model with productive public spending. In Krusell et al (2002), the role of tax policy is only restricted to correcting the agent's saving behaviour. In this chapter, government spending financed by taxes provides a complementary input factor to the production function of the private sector as in Barro (1990). The focus should be different since the competitive equilibrium cannot be Pareto-optimal in its nature when there is externality that cannot be fully internalized by a competitive agent. Thus, our main interest in this chapter is in the difference in the optimal tax policy resulting from quasi-hyperbolic preferences.

To address this issue, we take two different approaches to defining the optimal tax policy. First, we consider a fictitious planning problem where the central planner shares the quasi-hyperbolic preferences with a representative agent and we investigate tax policy that obtains the planner's equilibrium in a decentralized economy. When a lump-sum tax is available the tax policy that can implement the first-best in a decentralized economy has a zero capital income tax rate, which is not the case in a standard Ramsey growth model. We also observe that quasi-hyperbolic preferences result in a failure of the first-best choice to attain Pareto-efficiency. A better resource allocation than the first-best choice can be accomplished with negative capital income tax rates or a subsidy to capital income. Second, we explore second-best taxation from the Ramsey fiscal policy approach with three different degrees of commitment, when a lump-sum tax is not available. Under full commitment, when labour supply is inelastic, it is optimal for the government to levy a negative capital tax rate, while a positive capital tax rate

which should be larger for more impatient consumers is the second-best choice when labour supply is endogenous. In the complete absence of commitment, we observe a continuum of optimal tax policies in the fixed labour case and a bad equilibrium where the government confiscates all of capital income so that no agent saves in the elastic labour case. Finally, under one-period partial commitment, we find that a zero capital income tax rate is restored in the inelastic labour case and a positive capital tax rate, which is higher than the full commitment case, should be levied in the elastic labour case.

The remainder of this chapter is organized as follows. Section 2 reviews the two basic growth models with agents that have the quasi-hyperbolic preferences of Laibson (1996) and Krusell et al (2002). We intend this part to be an introduction of our backward induction approach that we take to obtain solutions throughout this chapter. In section 3 we deal with a Barro-type growth model with quasi-hyperbolic preferences and study the tax policy implications of quasi-hyperbolic preferences. The final section finishes the chapter with conclusion.

## 4.2 Quasi-hyperbolic Preferences and Saving Behaviour: Review

### 4.2.1 A General Overview of Laibson(1996) Model

**Quasi-hyperbolic preferences and dynamic inconsistency** The discount functions suggested in behavioural experiments are close to a form of hyperbola generalized by the function  $f(t) = (1 + \alpha t)^{-\gamma/\alpha}$  at time  $t$  with  $\alpha, \gamma > 0$  (Lowenstein and Prelec, 1992). In this hyperbolic discount function, the discount rate declines as time goes further away from the present. The most intriguing property of this function is that the instantaneous discount rate at time  $t$  is defined by  $\gamma/(1 + \alpha t)$ , falling as  $t$  moves forward, which is in contrast with the exponential discount function which has a constant instantaneous discount rate. Barro (1999) used a more general discount function in

continuous time which was given by  $\exp[-\rho \cdot t - \phi(t)]$  at time  $t$  where  $\phi(t)$  is a function satisfying  $\phi'(t) \geq 0$ ,  $\phi''(t) \leq 0$  and  $\phi'(t) \rightarrow 0$  as  $t \rightarrow \infty$ . The instantaneous discount rate for this function is  $\rho + \phi'(t)$  which is not constant and higher in the near term than in the distant future.

Laibson (1996) adopted a special case of the discount function named as "quasi-hyperbolic or quasi-geometric" which has a value of 1 for  $t = 0$  and  $\beta\delta^t$  for  $t \geq 1$  in discrete time. This function preserves most of the tractability of the standard exponential function and practically mimics the qualitative property of the hyperbolic function at the same time. It also encompasses exponential function as its special case when  $\beta = 1$ .

Under this assumption of quasi-hyperbolic preferences, Laibson (1996) took a game-theoretic approach where an individual is modelled as a collection of temporal selves who make a decision on the consumption level in their respective periods. The self at each period  $t$  inherits wealth ( $W_t^{t-1}$ ) from the previous self at time  $(t - 1)$  and chooses a flow of current and future consumption ( $c_t^t$ ) maximizing discounted lifetime utility. If we confine our time horizon to a finite  $T$  periods, the objective function which self  $t$  wants to maximize is

$$U_t = u(c_t^t) + \sum_{i=1}^{T-t} \beta\delta^i u(c_{t+i}^t), \quad (125)$$

where  $\beta$  and  $\delta$  are discount parameters and the utility function,  $u(c_t)$  is assumed to be simple logarithm utility function,  $\log(c_t)$  for simplicity. The budget constraint is simply written as

$$W_{t+1}^t = R(W_t^t - c_t^t), \quad 0 \leq c_t^t \leq W_t^t, \quad c_T^t = W_T^t, \quad (126)$$

where  $R$  is a fixed rate of gross return on wealth and the initial level of wealth ( $W_0^0$ ) is given. Suppose that the decision is made at time 0. The first-order conditions for

maximization problem (125) subject to (126) faced by self of time 0 are

$$\begin{aligned}\frac{c_1^0}{c_0^0} &= \beta\delta R, \\ \frac{c_{t+1}^0}{c_t^0} &= \delta R \text{ for } t = 1 \dots T - 1\end{aligned}$$

From (126), we can notice that  $c_t^0 = W_t^0 - W_{t+1}^0/R$  from  $t = 0$  to  $t - 1$  and  $c_T^0 = W_T^0$ .

Then

$$\begin{aligned}W_t^0 - \frac{W_{t+1}^0}{R} &= \beta\delta^t R^t (W_0^0 - \frac{W_1^0}{R}) \text{ for } t = 1 \dots T - 1, \\ W_T^0 &= \beta\delta^T R^T (W_0^0 - \frac{W_1^0}{R}).\end{aligned}$$

Recursively the level of wealth at time  $t$  can be derived as

$$W_t^0 = \left[ \beta\delta^t R^t \sum_{i=0}^{T-t} \delta^i \right] (W_0^0 - \frac{W_1^0}{R}).$$

Then finally we can get to the series for  $W_t^0$ , the level of wealth at time  $t$  chosen by the self at time 0, all expressed with the current wealth endowment,  $W_0^0$ :

$$W_t^0 = \frac{\beta\delta^t R^t \sum_{i=0}^{T-t} \delta^i}{1 + \beta\delta \sum_{i=0}^{T-1} \delta^i} W_0^0.$$

From the perspective of the self at time 0, it would be optimal that his solution sequences,  $\{c_t^0\}_{t=0}^T$  and  $\{W_t^0\}_{t=0}^T$  are realized by future selves. However, in this model, we do not assume any commitment mechanism through which the self of time  $t$  has the ability to force future selves to stick to his choices. Then the role of self 0 just ends once he consumes  $c_0^0$  and passes the remaining wealth,  $R(W_0^0 - c_0^0)$  to the next period.

At time 1, self 1 should choose his own consumption and wealth sequences taking his initial endowment,  $W_1^0 = R(W_0^0 - c_0^0)$  as given. The optimizing procedure of self 0 can be applied to the case of self 1 in the same way, so the level of wealth ( $W_t^1$ ) at time

$t$  chosen by self 1 can be expressed with  $W_1^0$ , his initial wealth as

$$W_t^1 = \frac{\beta\delta^{t-1}R^{t-1}\sum_{i=0}^{T-t}\delta^i}{1 + \beta\delta\sum_{i=0}^{T-2}\delta^i}W_1^0.$$

At time 2 and onwards, selves at each time period,  $p$  repeat this process again and then the general decision rule for the level of wealth at time  $t$  chosen by the self at time  $p$ ,  $W_t^p$  is defined as

$$W_t^p = \frac{\beta\delta^{t-p}R^{t-p}\sum_{i=0}^{T-t}\delta^i}{1 + \beta\delta\sum_{i=0}^{T-(p+1)}\delta^i}W_p^{p-1}, \quad t = (p+1)\dots T,$$

where  $W_p^{p-1}$  is the initial wealth for the self at  $p$ . Selves at each period,  $p$ , define their wealth sequence throughout all the remaining life-time in this manner, but their plan cannot actually be realized except for the choice of the level of wealth for the next period,  $p+1$ . Therefore, the realized sequence for the level of wealth,  $\{W_t\}_{t=0}^T$  can be expressed as

$$W_{t+1} = \frac{\beta\delta R\sum_{i=0}^{T-(t+1)}\delta^i}{1 + \beta\delta\sum_{i=0}^{T-(t+1)}\delta^i}W_t,$$

where  $W_0 = W_0^0$ ,  $W_t = W_t^{t-1}$ . Notice that self of  $(t-1)$  chose the optimal level of wealth at time  $(t+1)$ ,  $W_{t+1}^{t-1}$  as

$$W_{t+1}^{t-1} = \beta\delta^2 R^2 \frac{\sum_{i=0}^{T-(t+1)}\delta^i}{1 + \beta\delta\sum_{i=0}^{T-t}\delta^i}W_{t-1}^{t-2}, \quad (127)$$

but the realized wealth at time  $(t+1)$  implemented by self of time  $t$  is

$$W_{t+1} = W_{t+1}^t = \frac{\beta\delta R\sum_{i=0}^{T-(t+1)}\delta^i}{1 + \beta\delta\sum_{i=0}^{T-(t+1)}\delta^i} \cdot \frac{\beta\delta R\sum_{i=0}^{T-t}\delta^i}{1 + \beta\delta\sum_{i=0}^{T-t}\delta^i}W_{t-1}^{t-2}.$$

It can be observed that

$$W_{t+1}^t = W_{t+1}^{t-1} \cdot \frac{\beta\sum_{i=0}^{T-t}\delta^i}{1 + \beta\delta\sum_{i=0}^{T-(t+1)}\delta^i}. \quad (128)$$

When  $0 < \beta < 1$ ,  $W_{t+1}^t$  is always smaller than  $W_{t+1}^{t-1}$  and  $W_{t+1}^t = W_{t+1}^{t-1}$  holds if and only

if  $\beta = 1^9$ . Since the case with  $\beta = 1$  represents an exponential discount function, the result (128) confirms that there is no inconsistency issue when standard exponential preferences are assumed. When  $\beta$  is less than one, every self at  $t$  wants to relish more pleasure from the current consumption and pass disutility from saving onto future selves. Thus, the realized savings are always smaller than those expected by previous selves.

So far consumers have been assumed to be ‘naive’ in a sense that they just repeat the revision process every period even though they observe that future selves do not abide by their optimal choices. Let us suppose now that the present self is ‘sophisticated’ enough to anticipate the future self’s saving behaviour and considers this prediction as one of constraints on his life-time utility maximization problem. The addition of consumer sophistication enables us to understand this model as a game between present self and future selves as intended earlier in Phelps and Pollak (1968). Then the backward induction method can be used to find a sub-game perfect equilibrium. This strategy where a consumer seeks to find a future plan that he actually will follow was named ‘consistent planning’ in Strotz (1956) and Pollak (1968). As the name implies literally, intertemporal inconsistency is not an issue any more, because the planned consumption and saving by self of  $t$  are actually implemented by his future selves.

Returning back to the optimization problem of the present self at time 0, the self of time 0 can reason now that the self of time  $T - 1$  would choose his consumption level ( $c_{T-1}$ ) to maximize lifetime utility  $U_{T-1} = u(c_{T-1}^{T-1}) + \beta\delta u(c_T^{T-1})$  with budget constraint  $W_T^{T-1} = R(W_{T-1}^{T-2} - c_{T-1}^{T-1})$  and  $c_T^{T-1} = W_T^{T-1}$ .  $W_{T-1}^{T-2}$  is taken as given to the self of  $T - 1$  from his previous self. The decision rule for  $W_T^{T-1}$  is

$$W_T^{T-1} = \frac{R\beta\delta}{1 + \beta\delta} W_{T-1}^{T-2}. \quad (129)$$

The current self also knows that the self of time  $T - 2$  considers in his consumption

$$\frac{9}{1 + \beta\delta} \frac{\beta \sum_{i=0}^{T-t} \delta^i}{\sum_{i=0}^{T-(t+1)} \delta^i} = \frac{\beta \sum_{i=0}^{T-t} \delta^i}{1 + \beta(\sum_{i=0}^{T-t} \delta^i - 1)} = \frac{\beta \sum_{i=0}^{T-t} \delta^i}{(1-\beta) + \beta \sum_{i=0}^{T-t} \delta^i} < 1 \text{ when } \beta < 1 \text{ and } \frac{\beta \sum_{i=0}^{T-t} \delta^i}{(1-\beta) + \beta \sum_{i=0}^{T-t} \delta^i} = 1 \text{ when } \beta = 1.$$

decision the fact that his next self, the self of time  $T - 1$  is going to spend and save according to (129). Solving the problem of self of  $T - 2$ ,

$$\max U_{T-2} = u(c_{T-2}^{T-2}) + \beta\delta u(c_{T-1}^{T-2}) + \beta\delta^2 u(c_T^{T-2})$$

s.t.  $W_{T-1}^{T-2} = R(W_{T-2}^{T-3} - c_{T-2}^{T-2})$ ,  $c_{T-1}^{T-2} = \frac{1}{1+\beta\delta} W_{T-1}^{T-2}$ ,  $c_T^{T-2} = \frac{\beta\delta R}{1+\beta\delta} W_{T-1}^{T-2}$  with an inherited value of  $W_{T-2}^{T-3}$ , we can get the decision rule for  $W_{T-1}^{T-2}$  as

$$W_{T-1}^{T-2} = \frac{R\beta\delta(1+\delta)}{1+\beta\delta(1+\delta)} W_{T-2}^{T-3}.$$

Repeating this procedure, a general decision rule for  $\{W_t\}_{t=0}^T$  can be described as

$$W_{t+1} = \frac{\beta\delta R \sum_{i=0}^{T-(t+1)} \delta^i}{1+\beta\delta \sum_{i=0}^{T-(t+1)} \delta^i} W_t,$$

which turns out to be the same with the result from the naive consumer case in (127). The identity of the results from ‘naive’ and ‘sophisticated’ consumer obviously stems from the logarithmic utility assumption as pointed out in O’Donoghue and Rabin (1999).

From the results we have derived so far, we can compare the saving behaviours of the two types of consumers. The first case which will be called ‘commitment choice’ assumes that the consumer can commit to their initial choice. The other is the case where the consumer is naive or sophisticated, which will be called ‘equilibrium choice’. If we define saving ratio as  $s_t = (W_t - c_t)/W_t$ , the saving ratios of ‘commitment’ ( $s_t^c$ ) and ‘equilibrium’ ( $s_t^e$ ) are

$$\begin{aligned} \text{commitment : } s_0^c &= \frac{\beta\delta \sum_{i=0}^{T-1} \delta^i}{1+\beta\delta \sum_{i=0}^{T-1} \delta^i} \\ s_t^c &= \frac{\sum_{i=0}^{T-t} \delta^{i-1}}{\sum_{i=0}^{T-t} \delta^i} \quad \text{for } 1 \leq t \leq T \\ \text{equilibrium : } s_t^e &= \frac{\beta\delta \sum_{i=0}^{T-(t+1)} \delta^i}{1+\beta\delta \sum_{i=0}^{T-(t+1)} \delta^i} \quad \text{for } 0 \leq t \leq T. \end{aligned} \tag{130}$$

As long as  $\beta$  is smaller than one, it is clear that  $s_t^c > s_t^e$  except for the current time,

i.e.  $s_0^c = s_0^e$ . To sum up, the consumer with quasi-hyperbolic preferences always ends up saving less in the future than his initial plan. Now we face a more fundamental question: is it better from a welfare perspective to force the consumer to stick to his initial choice by providing a commitment mechanism? However, a welfare comparison between commitment and equilibrium choice is very troublesome, since any choice fails to be unanimously supported by all selves. Definitely the commitment saving-ratio at time  $t$ ,  $s_t^c$  is welcomed by all selves except the self of time  $t$  who prefers  $s_t^e$  and the choice of  $s_t^e$  by the self of time  $t$  can make himself better-off while making all other period selves worse-off. For this reason, a Pareto-dominance relation cannot be defined between these two choices.

To address this problem, some literature such as O'Donoghue and Rabin (1999) assume that a social planner is not as myopic or impatient as an individual consumer and has a long-term perspective criterion which is normally represented by standard exponential preferences, i.e.  $\beta = 1$ . Then from this social planner's point of view, all periods selves except the current consumer are undersaving, which provides grounds for policy intervention to correct the consumer's behaviour. However, this approach is problematic since it is difficult to justify the difference in preference between a planner and an individual consumer. For example, the self at time  $T - 1$  voluntarily chooses to save  $s_t^e$  of his income which is smaller than what the planner expects him to do, while being fully conscious of the fact that his undersaving now will reduce what he can consume in the last period,  $T$ . Then there is no proper rationale for the planner insisting his preferred saving ratio against the will of an individual consumer. The alternative which is adopted, for example, in Phelps and Pollak (1968) and Laibson (1996) is to explore the infinite-time behaviour in (130). The discussion of this matter continues in the next section.

**Infinite-time behaviour and policy implication** In the infinite-period horizon with  $T = \infty$ , it is straightforward in (130) to show that the equilibrium saving ratio,

$s_t^e$  converges to a unique saving ratio,  $s^e$  :

$$s^e = \frac{\beta\delta}{1 - \delta + \beta\delta}. \quad (131)$$

Now that (131) is implying that all selves will choose the same saving ratio,  $s^e$ , the question to be asked is now whether or not making all selves perturb their saving ratio from  $s^e$  can make all better-off. According to Phelps and Pollak (1968) and Laibson (1996), the answer is yes. Following the reasoning of Laibson (1996), suppose that the current self can pick a life-time saving ratio ( $s^*$ ) and all future selves should follow it. Then  $s^*$  satisfies

$$s^* = \arg \max_s \ln [(1 - s)W] + \beta \sum_{i=1}^{\infty} \delta^i \ln [(1 - s)s^i R^i W], \quad (132)$$

where  $W$  is the initial wealth of the current self. Solving problem (132), it can be obtained that

$$s^* = \frac{\beta\delta}{(1 - \delta)^2 + \beta\delta(1 - \delta)} > s^e,$$

for  $0 < \delta, \beta < 1$ . It is clear, then, that making all selves including the current self stick to the saving ratio  $s^*$  provides more life-time utility to the current self even though he should sacrifice some of the current consumption. All future selves will agree with this current self's action because they can inherit more wealth from the previous self. Therefore, as Laibson (1996) pointed out,  $s^*$  Pareto-dominates  $s^e$ . In addition, he also showed that the saving ratio from a standard exponential preference case can Pareto-dominate  $s^e$  in (131) if  $\beta$  is sufficiently close to unity. So it can be said that as long as a consumer's impatience is weak enough a social planner has a good reason to believe that exponential preferences could give, if not an optimal, at least a better guideline for the economy.

Liabson (1996) suggests two policy interventions to address this undersaving behaviour of impatient consumers. The first is to restrict the liquidity of saving by requiring advance notification to spend saving on consumption. Liabson shows that

just an one-period delay of the timing of consumption can restore the saving rate of the economy with exponential preferences. The other option for commanding the desired saving rate is for the government to implement an interest subsidy and a penalty to over-consumption at the same time. Liabson takes his argument on a negative capital income tax rate or interest subsidy for optimality as an example where the Chamley-Judd zero capital taxation result does not hold. These two policy interventions can be justified because they can provide more welfare than the competitive equilibrium without intervention. However, we should notice that there is still an implicit assumption behind these results. Whoever implements these interventions, all the reasoning in Liabson (1996) assumes that the central planner or the government should commit to its policy. If this commitment assumption is relaxed, say the planner or the government is as impatient as an individual consumer, policy intervention can lead to a worse outcome as shown in Krusell et al (2002). We will discuss this case in the next section.

#### 4.2.2 Quasi-hyperbolic Preference in a Standard Ramsey Growth Model

In this section, we will review the growth model with quasi-hyperbolic preferences in Krusell et al (2002). The model is no more than a standard Ramsey growth model except for the assumption that consumers have quasi-hyperbolic preferences which implies the following form of consumers' lifetime utility,  $U_t$  :

$$U_t = u(c_t) + \sum_{i=1}^{T-t} \beta \delta^i u(c_{t+i}), \quad 0 < \delta, \beta < 1,$$

where the utility function is restricted to logarithmic function,  $u(c_t) = \ln(c_t)$  again. Consumers provide capital ( $k_t$ ) and fixed amount of labour effort to firms which have a Cobb-Douglas production technology,  $f(k_t) = Ak_t^\alpha$ . The cost of capital and labour in perfect competition is the marginal product of each input so that it holds

$$r_t = \alpha Ak_t^{\alpha-1} \text{ and } w_t = (1 - \alpha)Ak_t^\alpha. \quad (133)$$

Consumers take these prices as given and aim to maximize their life-time utility subject to their resource constraint,

$$k_{t+1} = r_t k_t + w_t - c_t,$$

where full depreciation of capital is assumed.

## Competitive Equilibrium and Planner's Choice

**(a) Competitive Equilibrium** A sophisticated consumer in this model follows exactly the same steps as the consumer in the previous section. For a finite life-time,  $T$ , the consumer at time  $t$  predicts the action of his future self at time  $T$  and derives the whole sequence of consumption from  $t$  to  $T$  recursively. This backward induction gives us the following proposition.

**Proposition 7** (*Krusell et al, 2002*) *For our economy, the competitive equilibrium time path for capital is*

$$k_{t+1} = \frac{\beta\delta}{1 - \delta + \beta\delta} \alpha A k_t^\alpha. \quad (134)$$

**Proof.** Firstly, the current self knows that the self of  $T - 1$  chooses his consumption and saving by solving

$$\begin{aligned} \max_{k_T} U_{T-1} &= u(c_{T-1}) + \beta\delta u(c_T) \text{ s.t} \\ c_{T-1} &= r_{T-1} k_{T-1} + w_{T-1} - k_T \\ c_T &= r_T k_T + w_T. \end{aligned}$$

The solution gives a prediction for  $k_T$  as

$$k_T = \frac{\beta\delta}{1 + \beta\delta} (r_{T-1} k_{T-1} + w_{T-1}) - \frac{1}{1 + \beta\delta} \frac{w_T}{r_T}. \quad (135)$$

This decision rule for  $k_T$  is considered as one of constraints in the self of  $T - 2$  who solves the problem,

$$\begin{aligned}
\max_{k_{T-1}} U_{T-2} &= u(c_{T-2}) + \beta\delta u(c_{T-1}) + \beta\delta^2 u(c_T) \text{ s.t.} \\
c_t &= r_t k_t + w_t - k_{t+1} \text{ for } t = T-2, T-1 \\
c_T &= r_T k_T + w_T \text{ and (135).}
\end{aligned}$$

The decision rule of the self of  $T-2$  for  $k_{T-1}$  can be obtained as

$$k_{T-1} = \frac{\beta\delta + \beta\delta^2}{1 + \beta\delta + \beta\delta^2} (r_{T-2}k_{T-2} + w_{T-2}) - \frac{1}{1 + \beta\delta + \beta\delta^2} \left( \frac{w_{T-1}}{r_{T-1}} + \frac{w_T}{r_{T-1}r_T} \right).$$

This process is repeated by all the future selves from  $T-3$ , and finally the current self at  $t$  can define his law of motion in capital as

$$k_{t+1} = \frac{\beta \sum_{i=1}^{T-t} \delta^i}{1 + \beta \sum_{i=1}^{T-t} \delta^i} (r_t k_t + w_t) - \frac{1}{1 + \beta \sum_{i=1}^{T-t} \delta^i} \sum_{i=t+1}^T \frac{w_i}{\prod_{j=t+1}^i r_j} \quad (136)$$

As  $T \rightarrow \infty$ , the first term of right-hand side in (136) converges to  $\frac{\beta\delta}{1-\delta+\beta\delta}$ .

To obtain the infinite behaviour of the second term of right-hand side in (136), it is necessary to guess that the consumer would have the law of motion of capital  $k_{t+1} = s\alpha A k_t^\alpha$  in the infinite-period horizon. Combining this guess with  $r_t$  and  $w_t$  in (133), we can get the simplified form of  $\frac{w_i}{\prod_{j=t+1}^i r_j}$  and  $\sum_{i=t+1}^T \frac{w_i}{\prod_{j=t+1}^i r_j}$  as

$$\begin{aligned}
\frac{w_i}{\prod_{j=t+1}^i r_j} &= \frac{1-\alpha}{\alpha} s^{i-(t+1)} k_{t+1}, \\
\sum_{i=t+1}^{\infty} \frac{w_i}{\prod_{j=t+1}^i r_j} &= \frac{1-\alpha}{\alpha(1-s)} k_{t+1}.
\end{aligned} \quad (137)$$

By inserting (137) into (136), we can get to the law of motion in capital in infinite time in Proposition 1. ■

**(b) Planner's Choice** In a growth model with standard exponential preferences, it is well known that the competitive equilibrium can also solve the planner's problem

where the planner has the same preference as an individual consumer. That is, when  $\beta = 1$  in (134), the decision rule,  $k_{t+1} = \delta\alpha Ak_t^\alpha$  can be obtained from the planner's problem. However, Krusell et al (2002) showed that under quasi-hyperbolic preferences this welfare theorem does not hold. To see this point, let us describe the planner's problem as

$$\begin{aligned} \max_{\{c_i\}_{i=t}^T} U_t &= u(c_t) + \sum_{i=1}^{T-t} \beta\delta^i u(c_{t+1}) \\ \text{s.t } c_t &= Ak_t^\alpha - k_{t+1} \text{ for } t = t \dots T - 1 \\ c_T &= Ak_T^\alpha. \end{aligned}$$

Since this planner shares the preference structure with an individual consumer, the same recursive planning is also applied here and gives an equilibrium for a central planner.

**Proposition 8** (Krusell et al, 2002) *In our economy, the central planner's equilibrium time path for capital is*

$$k_{t+1} = \frac{\alpha\beta\delta}{1 - \alpha\delta + \alpha\beta\delta} Ak_t^\alpha.$$

**Proof.** At  $T - 1$ , the planner faces the following problem:

$$\begin{aligned} \max_{k_T} U_{T-1} &= u(c_{T-1}) + \beta\delta u(c_T) \text{ s.t} \\ c_{T-1} &= Ak_{T-1}^\alpha - k_T \\ c_T &= Ak_T^\alpha. \end{aligned}$$

The solution gives a prediction for  $k_T$  as

$$k_T = \frac{\beta\alpha\delta}{1 + \beta\alpha\delta} Ak_{T-1}^\alpha \tag{138}$$

At time  $T - 2$ , the planner solves the problem

$$\begin{aligned} \max_{k_{T-1}} U_{T-2} &= u(c_{T-2}) + \beta\delta u(c_{T-1}) + \beta\delta^2 u(c_T) \text{ s.t.} \\ c_t &= Ak_t^\alpha - k_{t+1} \text{ for } t = T - 2, T - 1 \\ c_T &= Ak_T^\alpha \text{ and (138).} \end{aligned}$$

The decision rule of the self of  $T - 2$  for  $k_{T-1}$  can be obtained as

$$k_{T-1} = \frac{\beta\alpha\delta + \beta\alpha^2\delta^2}{1 + \beta\alpha\delta + \beta\alpha^2\delta^2} Ak_{T-2}^\alpha$$

The backward reasoning finally leads us to the law of motion in capital as

$$k_{t+1} = \frac{\beta \sum_{i=1}^{T-t} \alpha^i \delta^i}{1 + \beta \sum_{i=1}^{T-t} \alpha^i \delta^i} Ak_t^\alpha.$$

As  $T \rightarrow \infty$ , the result of proposition 2 can be obtained. ■

**(c) Welfare comparison of two equilibrium and policy implication** From

proposition 1 and 2, let the saving ratio of the competitive equilibrium and the planner's equilibrium be  $s^{CE}$  and  $s^P$  respectively. It is straightforward to see that  $s^{CE} > s^P$ . Now following Krusell et al (2002), the value function of the current self,  $V_0(k)$  can be defined as

$$V_0(k) = u(f(k) - k') + \beta\delta V(k'),$$

where  $V(k) = u(f(k) - k') + \delta V(k')$  and  $k, k'$  are the capital level of the current and the next period respectively. Then we can find a stationary saving ratio that maximizes  $V_0(k)$ , which is different from  $s^{CE}$  and  $s^P$ .

**Proposition 9** (Krusell et al, 2002) *In our economy, there exists  $s^*$  such that*

$$s^* = \arg \max_s \{V_0(k) = u(f(k) - k') + \beta\delta V(k')\},$$

and  $s^* > s^{CE} > s^P$ . Therefore the competitive equilibrium provides higher welfare than the planner's solution.

**Proof.** This proof is a simple summary of that in Krusell et al (2002). If we suppose that a consumer saves at a fixed rate such that  $k' = sAk^\alpha$ , we can write  $V(k)$  as

$$V(k) = \sum_{t=0}^{\infty} \delta^t u((1-s)Ak_t^\alpha),$$

where  $u(c) = \ln c$  and  $k_0 = k$ . Then we can obtain an explicit form of  $V_0(k)$  as

$$V_0(k) = \ln(1-s)Ak^\alpha + \beta\delta \left[ \frac{\ln(1-s)}{1-\delta} + \frac{\alpha\delta \ln s}{(1-\delta\alpha)(1-\delta)} + \frac{\ln A}{(1-\delta\alpha)(1-\delta)} + \frac{\alpha \ln s Ak^\alpha}{1-\delta\alpha} \right].$$

Then it is straightforward to obtain

$$s^* = \frac{\alpha\beta\delta}{(1-\alpha\delta)(1-\delta+\beta\delta) + \alpha\beta\delta}, \quad (139)$$

and check  $s^* > s^{CE} > s^P$ . Finally since  $V_0(k)$  is monotonically increasing for  $s < s^*$ ,  $s^{CE}$  implies more welfare than  $s^P$ . ■

Normally, a central planner is assumed to have the ability to catch what a competitive consumer would miss. The planner here knows that his action will affect the return to savings of future periods while competitive agents act just as price-takers, so that the expected marginal product of capital perceived by the planner should be lower than one of the competitive consumers. Then the planner saves less than the competitive equilibrium, which results in a worse performance in welfare in a situation where the competitive equilibrium already shows undersaving behaviour. As a consequence, if we think of the planner as the government with various tax instruments whose objective is to maximize the life-time utility of the current electorate who has quasi-hyperbolic preferences, its taxation plan should include a disincentive to save, e.g. a positive tax rate on capital income in Krusell et al (2002) and so its tax intervention guides the economy in the wrong direction.

### 4.3 Quasi-hyperbolic Preferences in a Growth Model with Public Spending: Barro meets Laibson

In this section we embrace quasi-hyperbolic preferences into a growth model with productive public spending. The government spending provides a complementary input factor to the production function of the private sector as in Barro (1990). Since the competitive equilibrium cannot be Pareto-optimal in its nature when there exists an externality, our main interest in this chapter is in the difference in the optimal tax policy resulting from quasi-hyperbolic preferences. First, we compare the competitive equilibrium with the first-best choice in a centrally-planned economy. Second, we explore second-best taxation from the Ramsey fiscal policy approach with three different degrees of commitment: full commitment, complete absence of commitment, and one-period partial commitment when a lump-sum tax is not available.

#### 4.3.1 Model Description

The model assumes a representative consumer who obtains utility from his life-time consumption and leisure. We adopt the quasi-hyperbolic preference structure given by

$$U_t = u(c_t, l_t) + \beta \sum_{i=1}^{\infty} \delta^i u(c_{t+i}, l_{t+i}).$$

We assume that the consumer's utility function is

$$u(c, l) = \lambda \ln c + (1 - \lambda) \ln(1 - l),$$

where  $c$  is consumption and  $l$  is labour effort.

The aggregate resource constraint in our economy is assumed to be

$$c_t + k_{t+1} = (1 - \theta)y_t,$$

where  $y$  is total production and  $\theta$  is the ratio of productive government spending with

respect to the total output. The productive government spending ( $g_t = \theta y_t$ ) works as a complementary input factor in the production process where the production technology is defined as

$$y_t = Ak_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha},$$

where  $\alpha$  is assumed to be between zero and one.

We assume that the government collects tax revenue to finance government spending with a lump-sum tax, and capital and labour income taxes. Thus, its budget constraint reads

$$g_t = \tau_k r_t k_t + \tau_l w_t l_t + M_t = \theta y_t,$$

where  $\tau_k$  and  $\tau_l$  are the constant capital and labour income tax rates respectively, and  $M_t$  is the lump-sum tax. Under the assumption of perfect competition in product markets, we can apply marginal-product pricing of input factors to capital and labour:

$$r_t = A\alpha k_t^{\alpha-1} l_t^{1-\alpha} g_t^{1-\alpha} \text{ and } w_t = A(1-\alpha)k_t^{\alpha-1} l_t^{-\alpha} g_t^{1-\alpha}.$$

### 4.3.2 Competitive Equilibrium

In this section, we will search for an equilibrium of a representative consumer in the competitive market. The process to obtain a closed-form equilibrium is the same as that we adopted in the standard Ramsey growth model. The representative agent can internalize his inconsistency in the optimization process by applying a backward induction technique. The following proposition describes the saving-rule for the representative consumer who takes factor prices and all tax variables as exogenous. The first proposition describes the consumer's behaviour in a finite-time horizon.

**Proposition 10** *In a finite-time period,  $T$ , the saving rule for the representative agent*

is

$$k_{t+1} = \frac{\beta \sum_{i=1}^{T-t} \delta^i}{\lambda + \beta \sum_{i=1}^{T-t} \delta^i} ((1-\tau_k)r_t k_t + (1-\tau_l)w_t l_t - M_t) - \frac{\lambda}{\lambda + \beta \sum_{i=1}^{T-t} \delta^i} \sum_{i=t+1}^T \left( \frac{(1-\tau_l)w_i - M_i}{\prod_{j=t+1}^i (1-\tau_k)^{i-t} r_j} \right). \quad (140)$$

**Proof.** See appendix. ■

Using (140), we can derive the consumer's saving and labour choice behaviour in an infinite time setting. The result is described in the following proposition.

**Proposition 11** *In the infinite-period horizon, the competitive equilibrium values for capital and labour are*

$$k_{t+1} = \frac{\alpha\beta\delta}{1-\delta+\beta\delta} (1-\tau_k) A k_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha}, \quad (141)$$

$$l_t = l_{CE} = \frac{\lambda(1-\alpha)(1-\tau_l)(1-\delta+\beta\delta)}{(1-\alpha)(1-\tau_l)(1-\delta+\beta\delta) + (1-\lambda)[(1-\delta)\alpha(1-\tau_k) - \Omega(1-\delta+\beta\delta)]}, \quad (142)$$

where  $\Omega = \theta - \alpha\tau_k - (1-\alpha)\tau_l$ . At the equilibrium, the economy grows at the constant rate

$$\gamma_{CE} = \frac{\alpha\beta\delta}{1-\delta+\beta\delta} (1-\tau_k) \theta^{(1-\alpha)/\alpha} A^{1/\alpha} l_{CE}^{(1-\alpha)/\alpha}.$$

**Proof.** See appendix. ■

If we assume that there is no productive government spending, (141) and (142) correspond to the saving rule and labour choice in the standard Ramsey growth model which was shown earlier in Krusell et al (2000). Because of our log-utility function specification, the labour income tax rate does not involve the saving function which shows the same form as the inelastic labour case where  $\lambda = 1$ . As is typical in Barro-type growth models, the economy shows a balanced growth path without any transition dynamics. Output, capital and consumption grow at a constant rate, so that the lump-tax rate ( $M_t/y_t$ ) is also constant.

### 4.3.3 Planner's Equilibrium and Pareto Optimality

In this section, we assume that there is a central planner who is a representative of the consumer and is able to fully dictate savings and labour choice decision. The central planner is assumed to be given a fixed ratio of government spending to output,  $\theta$ . Since this central planner recognizes the positive externality from the productive public spending and the consequent higher return on capital and labour than a competitive agent, it is evident that the economy governed by the central planner shows higher saving rates than a competitive equilibrium. We assume that the central planner inherits quasi-hyperbolic preferences from the representative consumer, so that he also resolves his inconsistency problem by applying the consistent planning technique. Backward induction of the central planner himself provides us the following two propositions.

**Proposition 12** *In a finite-time period,  $T$ , the saving rule for the central planner is*

$$k_{t+1} = \frac{\beta \sum_{i=1}^{T-t} \delta^i}{1 + \beta \sum_{i=1}^{T-t} \delta^i} (1 - \theta) A^{1/\alpha} \theta^{(1-\alpha)/\alpha} k_t.$$

**Proof.** See appendix. ■

**Proposition 13** *In the infinite-period horizon, the equilibrium choices of the central planner for capital and labour are*

$$k_{t+1} = \frac{\beta \delta}{1 - \delta + \beta \delta} (1 - \theta) A k_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha}, \quad (143)$$

$$l_t = l_P = \frac{\lambda(1 - \alpha)(1 - \delta + \beta \delta)}{\lambda(1 - \delta + \beta \delta)(1 - \alpha) + (1 - \lambda)(1 - \delta)\alpha}. \quad (144)$$

*At the equilibrium, the economy grows at the constant rate*

$$\gamma_P = \frac{\beta \delta}{1 - \delta + \beta \delta} (1 - \theta) \theta^{(1-\alpha)/\alpha} A^{1/\alpha} l_P^{(1-\alpha)/\alpha}. \quad (145)$$

**Proof.** See appendix. ■

(143) and (144) describes the best resource allocation rule for the central planner for a given  $\theta$ . In a standard exponential preferences case, this solution for the central planner serves as a Pareto-efficient allocation. However, because of the quasi-hyperbolic preferences structure, the central-planner fails to obtain Pareto-optimality as we already observed in the standard Ramsey growth model case in Krusell et al (2000, 2002). This fact can be verified by simply showing that there is another stationary saving ratio or growth rate that can make selves in all periods better off. Of course, the existence of that kind of saving ratio requires a commitment scheme which prevents all future selves of the central planner from deviating from the chosen saving ratio. In this sense, the welfare loss from choosing the saving rule, (143) can be interpreted as the cost of a lack of commitment. The following proposition shows a closed-form rule for Pareto-optimal saving and labour choice in this economy. Let  $s$  denote the saving ratio to disposable income such that  $k_{t+1} = s(1 - \theta)\theta^{(1-\alpha)/\alpha}A^{1/\alpha}l_t^{(1-\alpha)/\alpha}k_t$ . Then the equilibrium saving ratio of the central planner is  $s^P = \frac{\beta\delta}{1-\delta+\beta\delta}$ .

**Proposition 14** *There exists a stationary saving ratio ( $s^*$ ) and labour choice ( $l^*$ ) that Pareto-dominates the central planner's choice,  $s^P$  and  $l^P$  such that*

$$s^* = \arg \max_s V_0(k),$$

where  $V_0(k) = u(k, k', l) + \beta\delta V(k')$  and  $V(k) = u(k, k', l) + \delta V(k')$ . The closed-form of  $s^*, l^*$  and  $\gamma^*$  are

$$s^* = \frac{\beta\delta}{(1-\delta)(1-\delta+\beta\delta) + \beta\delta}, \quad (146)$$

$$l^* = \frac{(1-\alpha)\lambda[(1-\delta)(1-\delta+\beta\delta) + \beta\delta]}{(1-\alpha)\lambda[(1-\delta)(1-\delta+\beta\delta) + \beta\delta] + \alpha(1-\lambda)(1-\delta)(1-\delta+\beta\delta)},$$

$$\gamma^* = \frac{\beta\delta}{(1-\delta)(1-\delta+\beta\delta) + \beta\delta} (1-\theta)\theta^{(1-\alpha)/\alpha}A^{1/\alpha} (l^*)^{(1-\alpha)/\alpha}. \quad (147)$$

**Proof.** See appendix. ■

It is clear that  $s^* > s^P$ . Since  $V_0(k)$  is monotonically increasing for  $s < s^*$ , the central planner is under-saving at his equilibrium.  $s^P$  cannot be Pareto-efficient in the

sense that a marginal upward deviation from  $s^P$  can make all of the central planner's selves including the current-period self better off. An increased  $V_0(k)$  definitely means that it is a better choice for the current central planner and more saving by the current planner is always beneficial to all of future selves because they can inherit more initial wealth from their predecessors, while  $s^*$  satisfies Pareto-optimality since any deviation from it does harm to at least one self. For  $s < s^*$ , all future selves would not be happy and for  $s > s^*$ , there is no motivation for the current consumer to choose that saving ratio by sacrificing his consumption. Comparing (146) with (139), the optimal saving rule for the standard Ramsey growth model, we can notice that (146) is a special case of (139) where  $\alpha = 1$ , which is not surprising since it is well known that Barro-type growth model shows the same growth dynamics as the AK economy.

#### 4.3.4 Optimal Taxation

In the previous section, we have identified the saving and labour choice behaviour in a competitive economy and a centrally-planned economy. With exponential preferences, we can define the optimal tax plan as one that can implement the central-planner's first-best resource allocation which is Pareto-optimal. But we have shown that the central planner cannot achieve Pareto-optimality, so the tax plan defined in that way does not deserve to be called optimal. Therefore, we need to find another tax schedule that can achieve true optimality in a competitive economy.

Before we see the specific tax schedule, we should notice that  $s^P$  and  $s^*$  are constrained allocation rules where the central planner takes the share of public spending,  $\theta$  as given. Only if the central planner is allowed to also choose  $\theta$  at its optimal level, can an allocation rule in that situation be called the first-best or Pareto-optimal choice. Barro (1990) showed earlier that when the economy with a Cobb-Douglas production technology is in a position of balanced growth the utility maximizing public spending ratio to output ( $\theta_P$ ) coincides with the one that maximizes the growth rate,  $\gamma_p$ . Since labour choice is independent from the choice of  $\theta$ , Barro's argument can be applied to

our model. From the previous propositions, the optimal public spending ratio to output can be straightforwardly defined as in the following statement.

**Proposition 15** *The optimal ratio of public spending to output ( $\theta$ ) for the central planner is*

$$\theta_P = 1 - \alpha.$$

**Proof.** Proof is straightforward from (145) and (147). ■

Now we can define the capital and labour tax rates that can achieve the outcome of the central-planner or the optimal resource allocation for the central planner in a competitive equilibrium with the optimal ratio of productive public spending,  $\theta_P = 1 - \alpha$ .

**Proposition 16** (i) *Capital and labour income tax rates that can implement the central planner's equilibrium choice in a competitive equilibrium are*

$$\tau_k^P = 0 \text{ and } \tau_l^P = 1 \text{ when } \lambda = 1,$$

$$\tau_k^P = 0 \text{ and } \tau_l^P = 0 \text{ when } 0 < \lambda < 1.$$

(ii) *Capital and labour income tax rates that can implement the central planner's optimal choice in a competitive equilibrium are*

$$\tau_k^* = \frac{-\delta(1-\delta)(1-\beta)}{(1-\delta)(1-\delta+\beta\delta)+\beta\delta} < 0 \text{ and } \tau_l^* = 1 - \frac{\alpha}{1-\alpha}\tau_k^* > 1 \text{ when } \lambda = 1, \quad (148)$$

$$\tau_k^* = \frac{-\delta(1-\delta)(1-\beta)}{(1-\delta)(1-\delta+\beta\delta)+\beta\delta} < 0 \text{ and } \tau_l^* = 0 \text{ when } 0 < \lambda < 1.$$

**Proof.** Proof is straightforward. ■

In the first statement, the labour tax part is in fact trivial. When the utility function does not contain leisure, the labour income tax works in the exactly same way that a lump-sum tax does. In that case, full confiscation of labour income is chosen to attain first-best outcome. In the case of elastic labour supply, the result implies that uniform

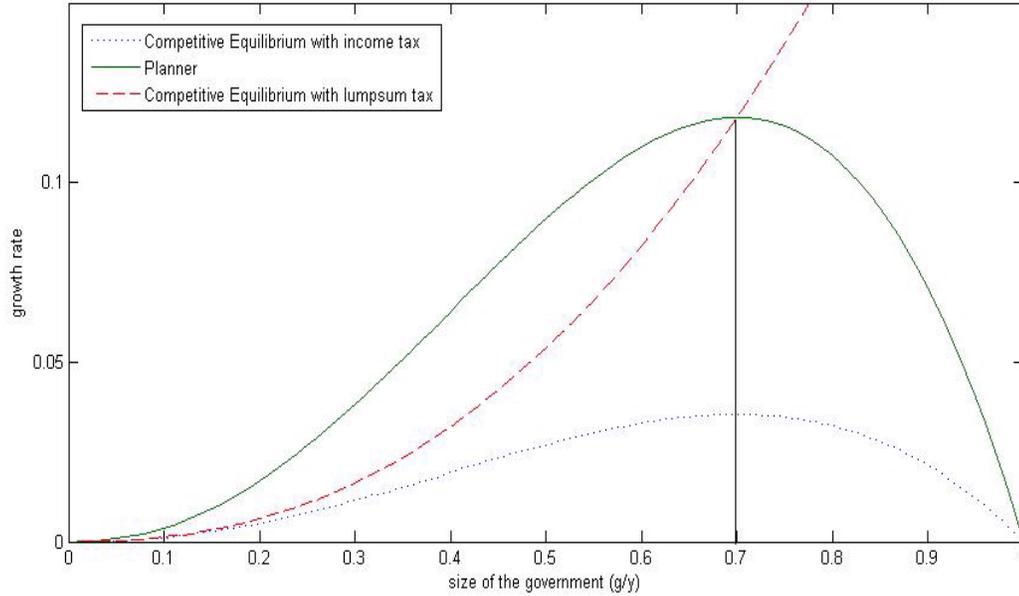


Figure 19: Growth Rates in Three Environments

tax rates for capital and labour income are required for replicating the central planner's choice. However, if an additional distortionary tax instrument is added, for example, a consumption tax, the labour income tax does not need to be zero, while zero-capital tax rate result still holds.

Figure 19 summarizes the first statement. It shows the growth rate for a fixed size of public spending to total output in three environments: central planner's growth rate (solid line), growth rate in a decentralized economy where public spending is financed with uniform income tax (dotted line) and growth rate in a decentralized economy where public spending is financed with a lump-sum tax (dashed line). The parameter values are  $\alpha = 0.3, \beta = 0.5, \delta = 0.95$ . The central planner's economy shows its maximum growth rate at  $\theta = 0.7$  which is equals to  $1 - \alpha$  and this growth rate can be accomplished by financing public spending only with a lump-sum tax.

The first statement in the proposition shows a different result to Krusell et al (2002) where a non-zero capital tax is required for implementation of the central planner's choice in a competitive equilibrium. As long as the central planner suffers from inconsistency arising from quasi-hyperbolic preferences, the central planner's equilibrium is

still the first-best outcome attainable in this economy regardless of whether it is optimal or not. In a standard Ramsey growth model, quasi-hyperbolic preferences result in a worse-off outcome which is achieved by a positive capital tax rate. However, in a Barro-type growth model, quasi-hyperbolic preferences do not require a non-zero capital tax rate to replicate the central-planner's first-best outcome.

Since the first-best outcome achieved by zero-distortionary taxes is not optimal for the economy, we can define optimal income tax rates to attain a Pareto-optimal resource allocation, which is described in the second statement of the proposition. It says that a negative capital income tax or a subsidy to capital income is optimal for any value of  $\lambda$ . When  $\lambda = 1$ , it seems rather problematic that the optimal labour income tax rate is greater than unity. If labour income tax rate cannot exceed 1, the Pareto-optimal outcome cannot be obtained since the capital income tax should be left untaxed. In this case, an introduction of another tax instrument such as a consumption tax can help restore the Pareto-optimality since it also plays a role of lump-sum tax when the labour supply is fixed.

It is rather surprising that the optimal labour income tax rate is still zero when labour supply is endogenous. In a competitive economy, the agent's undersaving results from two facts. Firstly, the agent fails to internalize the positive externality from the productive public spending and secondly, quasi-hyperbolic preferences force the agent to choose to save less. The first part of undersaving is corrected by financing public spending only with a lump-sum tax and the second part of undersaving can be corrected by increasing the lump-sum tax further and providing a subsidy to capital income while labour income remains untaxed.

Figure 20 summarizes the second statement in the proposition. We add one more case, the optimal growth rate for the central planner (Bold solid line) to the first figure with the same parameters. The optimal growth rate is also maximized when the size of government is equal to  $(1 - \alpha) = 0.7$ , but this growth rate cannot be implemented in a decentralized economy with a lump-sum tax only. At  $\theta = 0.7$ , the lump-sum tax

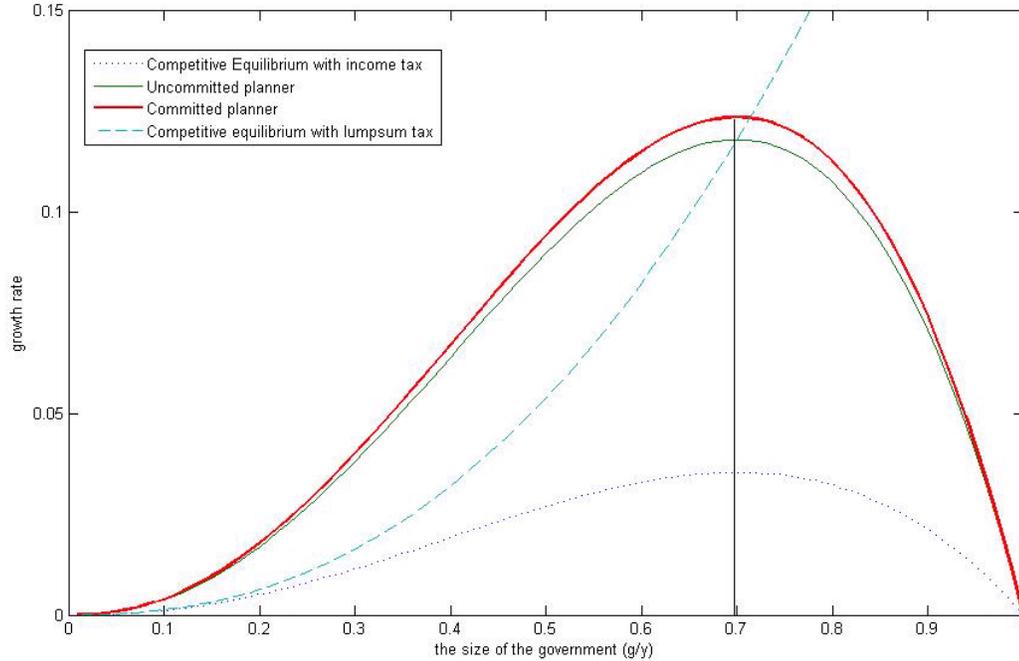


Figure 20: Growth Rates in Four Environments

generates a closer growth rate than the income tax; therefore the lump-sum tax would still be preferred here. The optimal tax strategy is then to increase the lump-sum tax and provide a subsidy to capital income with the excess of lump-tax revenue over the optimal size of the government until the growth rate coincides with the optimal growth rate.

#### 4.3.5 Ramsey Fiscal Policy and Consistent Taxation

In the previous section, we have seen how optimal tax rates can be defined from the perspective of a central planner, meaning the policy maker is assumed to be able to employ non-distortionary tax instruments, e.g. a lump-sum tax. In this section, we take a different but more realistic approach, the so called "Ramsey approach" where a government with only distortionary tax instruments has to determine the optimal tax schedule subject to the agents' behaviour and the resource constraint in a competitive economy.

Since the consumer has quasi-hyperbolic preferences, the government in each period

should also experience an inconsistency problem in their optimal tax plan. That is, the optimal tax schedule for the next period set by the current government should always be revised by the next period's government. As a consequence, the equilibrium tax schedule should be defined under the constraint of lack of commitment. We call this tax schedule consistent taxation.

**Second-best Ramsey Fiscal Policy** Before we discuss consistent taxation, we start here by describing the second-best Ramsey policy where all of the governments throughout time follow a pre-determined tax schedule. The government's problem in this case is

$$\max_{\tau_l, \tau_k} U = \ln c_t + (1 - \lambda) \ln(1 - l_t) + \beta \sum_{i=t+1}^{\infty} \delta^i [\ln c_i + (1 - \lambda) \ln(1 - l_i)]. \quad (149)$$

This optimization problem has four constraints: a budget constraint for an agent (150), capital (151) and labour (152) choice rules for an agent in the decentralized economy and the government's balanced-budget constraint (153).

$$c_t = (1 - \tau_l)w_t l_t + (1 - \tau_k)r_t k_t - k_{t+1}, \quad (150)$$

$$k_{t+1} = \frac{\alpha\beta\delta}{1 - \delta + \beta\delta} (1 - \tau_k) A k_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha}, \quad (151)$$

$$l_t = \frac{\lambda(1 - \tau_l)(1 - \alpha)(1 - \delta + \beta\delta)}{(1 - \tau_l)(1 - \alpha)(1 - \delta + \beta\delta) + (1 - \lambda)(1 - \delta)(1 - \tau_k)\alpha}. \quad (152)$$

$$g_t = \tau_k r_t k_t + \tau_l w_t l_t. \quad (153)$$

By embedding all of these constraints into the objective function, we can simplify the life-time utility in (149) as

$$U(\theta_k, \theta_l) = \lambda U^c(\theta_k, \theta_l) + (1 - \lambda) U^l(\theta_k, \theta_l), \quad (154)$$

where  $\theta_k = \alpha(1 - \tau_k)$ ,  $\theta_l = (1 - \alpha)(1 - \tau_l)$  and  $U^c(\theta_k, \theta_l)$ ,  $U^l(\theta_k, \theta_l)$  are defined by

$$U^c = \frac{1 - \delta + \beta\delta}{1 - \delta} \ln \left( \theta_l + \frac{1 - \delta}{1 - \delta + \beta\delta} \theta_k \right) + \frac{\beta\delta}{(1 - \delta)^2} \ln \theta_k +$$

$$\frac{1 - \alpha}{\alpha} \left( \frac{1 - \delta + \beta\delta}{1 - \delta} + \frac{\beta\delta}{(1 - \delta)^2} \right) \ln \frac{(1 - \theta_l - \theta_k) \theta_l}{(1 - \delta + \beta\delta)\theta_l + (1 - \lambda)(1 - \delta)\theta_k} + \Theta,$$

$$U^l = \frac{1 - \delta + \beta\delta}{1 - \delta} \ln \frac{(1 - \delta + \beta\delta)\theta_l + (1 - \delta)\theta_k}{(1 - \delta + \beta\delta)\theta_l + (1 - \lambda)(1 - \delta)\theta_k} + \Lambda.$$

$\Theta$  and  $\Lambda$  are collections of terms not including choice variables,  $\theta_k$  and  $\theta_l$  (For details, see appendix). The first-order conditions for maximization of (154) are

$$\begin{aligned} \frac{\partial U}{\partial \theta_k} &= \lambda \frac{\partial U^c}{\partial \theta_k} + (1 - \lambda) \frac{\partial U^l}{\partial \theta_k} = 0, \\ \frac{\partial U}{\partial \theta_l} &= \lambda \frac{\partial U^c}{\partial \theta_l} + (1 - \lambda) \frac{\partial U^l}{\partial \theta_l} = 0, \end{aligned} \quad (155)$$

where

$$\begin{aligned} \frac{\partial U^c}{\partial \theta_k} &= \frac{1 - \delta + \beta\delta}{(1 - \delta + \beta\delta)\theta_l + (1 - \delta)\theta_k} + \frac{\beta\delta}{(1 - \delta)^2\theta_k} \\ &\quad - \frac{1 - \alpha}{\alpha} \left( \frac{1 - \delta + \beta\delta}{1 - \delta} + \frac{\beta\delta}{(1 - \delta)^2} \right) \left( \frac{1}{1 - \theta_l - \theta_k} + \frac{(1 - \lambda)(1 - \delta)}{(1 - \delta + \beta\delta)\theta_l + (1 - \lambda)(1 - \delta)\theta_k} \right), \\ \frac{\partial U^l}{\partial \theta_k} &= \frac{1 - \delta + \beta\delta}{1 - \delta} \left( \frac{1 - \delta}{(1 - \delta + \beta\delta)\theta_l + (1 - \delta)\theta_k} - \frac{(1 - \lambda)(1 - \delta)}{(1 - \delta + \beta\delta)\theta_l + (1 - \lambda)(1 - \delta)\theta_k} \right), \\ \frac{\partial U^c}{\partial \theta_l} &= \frac{1 - \delta + \beta\delta}{1 - \delta} \frac{1 - \delta + \beta\delta}{(1 - \delta + \beta\delta)\theta_l + (1 - \delta)\theta_k} \\ &\quad - \frac{1 - \alpha}{\alpha} \left( \frac{1 - \delta + \beta\delta}{1 - \delta} + \frac{\beta\delta}{(1 - \delta)^2} \right) \left( \frac{1}{1 - \theta_l - \theta_k} - \frac{1}{\theta_l} + \frac{1 - \delta + \beta\delta}{(1 - \delta + \beta\delta)\theta_l + (1 - \lambda)(1 - \delta)\theta_k} \right), \\ \frac{\partial U^l}{\partial \theta_l} &= \frac{1 - \delta + \beta\delta}{1 - \delta} \left( \frac{1 - \delta + \beta\delta}{(1 - \delta + \beta\delta)\theta_l + (1 - \delta)\theta_k} - \frac{1 - \delta + \beta\delta}{(1 - \delta + \beta\delta)\theta_l + (1 - \lambda)(1 - \delta)\theta_k} \right). \end{aligned}$$

When labour supply is inelastic ( $\lambda = 1$ ), these F.O.C provide us a clear view on the second-best Ramsey policy. The optimal tax rates in that case are

$$\theta_k = \frac{\alpha(1 - \delta + \beta\delta)}{(1 - \delta)(1 - \delta + \beta\delta) + \beta\delta} \iff \tau_k = -\frac{\delta(1 - \delta)(1 - \beta)}{(1 - \delta)(1 - \delta + \beta\delta) + \beta\delta} < 0,$$

$$\theta_l = -\frac{\alpha\delta(1 - \delta)(1 - \beta)}{(1 - \delta)(1 - \delta + \beta\delta) + \beta\delta} \iff \tau_l = \frac{(1 - \alpha)[(1 - \delta)(1 - \delta + \beta\delta) + \beta\delta] + \alpha\delta(1 - \delta)(1 - \beta)}{(1 - \alpha)[(1 - \delta)(1 - \delta + \beta\delta) + \beta\delta]} > 1,$$

which coincide with the optimal tax rates for the central planner defined in (148). This is not surprising since the tax imposed on inelastic labour effort has the property of a lump-sum tax and we are assuming commitment capability for the central planner and the government in each case.

On the other hand, when  $0 < \lambda < 1$ , the government's second-best fiscal policy cannot be the same as the central planner's optimal tax schedule as long as its tax instruments are restricted to distortionary income taxes. On top of that, it is no longer possible to find closed-forms for  $\theta_k$  and  $\theta_l$  satisfying the first-order conditions. In Table 1, we present some numerical examples of second-best taxation. From the results, it is noticeable that the optimal tax mix of capital and labour income tax rates always satisfies the condition that the size of the government to output,  $\theta = \alpha\tau_k + (1 - \alpha)\tau_l$  coincides with  $(1 - \alpha)$ . Thus, the government decision is about how it can finance that portion of public spending from capital and labour income tax. When the lump-sum tax is dropped from the list of available tax instruments, any change in the capital income tax rate should entail a corresponding change in the labour tax rate which can maintain the optimal size of the government spending,  $\theta$ . When there is an increase in  $\tau_k$  and a corresponding decrease in  $\tau_l$ , it has mixed consequences on the life-time utility of consumers. In a positive way, it decreases the saving ratio so that consumers can consume more portion of their income. Also the new tax mix makes consumers choose to work more, increasing total output of the economy. On the other hand, the negative effect is that less saving decreases total output of the economy and consumers lose some utility from making more labour effort. Therefore, the second-best tax rates in Table 1 are the tax rates that equalize the marginal cost to the marginal benefit of the tax policy change.

The results in Table 1 shows that the more impatient the consumer (the smaller the value of  $\beta$ ) is, the higher the second-best capital tax rate is. The labour tax rate should be correspondingly lower for smaller  $\beta$ . This tax policy response to a change in  $\beta$  can be explained in terms of the intertemporal utility trade-off. Suppose that the agent gets more impatient (smaller  $\beta$ ). Then the relative weight that the agent put on the

| $\beta$ | $\alpha = 0.3$  |          |                 |          | $\alpha = 0.7$  |          |                 |          |
|---------|-----------------|----------|-----------------|----------|-----------------|----------|-----------------|----------|
|         | $\lambda = 0.7$ |          | $\lambda = 0.3$ |          | $\lambda = 0.7$ |          | $\lambda = 0.3$ |          |
|         | $\tau_k$        | $\tau_l$ | $\tau_k$        | $\tau_l$ | $\tau_k$        | $\tau_l$ | $\tau_k$        | $\tau_l$ |
| 1.0     | 0.1840          | 0.9212   | 0.2555          | 0.8905   | 0.0867          | 0.7976   | 0.1178          | 0.7251   |
| 0.9     | 0.1911          | 0.9181   | 0.2652          | 0.8863   | 0.0894          | 0.7914   | 0.1216          | 0.7162   |
| 0.8     | 0.1993          | 0.9146   | 0.2763          | 0.8816   | 0.0925          | 0.7841   | 0.1261          | 0.7059   |
| 0.7     | 0.2090          | 0.9104   | 0.2893          | 0.8760   | 0.0963          | 0.7754   | 0.1312          | 0.6938   |
| 0.6     | 0.2207          | 0.9054   | 0.3048          | 0.8694   | 0.1008          | 0.7649   | 0.1374          | 0.6794   |
| 0.5     | 0.2351          | 0.8993   | 0.3237          | 0.8613   | 0.1064          | 0.7517   | 0.1450          | 0.6617   |
| 0.4     | 0.2535          | 0.8913   | 0.3475          | 0.8511   | 0.1137          | 0.7347   | 0.1546          | 0.6392   |
| 0.3     | 0.2785          | 0.8807   | 0.3791          | 0.8375   | 0.1237          | 0.7114   | 0.1675          | 0.6092   |
| 0.2     | 0.3151          | 0.8650   | 0.4242          | 0.8182   | 0.1386          | 0.6767   | 0.1860          | 0.5661   |
| 0.1     | 0.3780          | 0.8380   | 0.4984          | 0.7864   | 0.1645          | 0.6161   | 0.2166          | 0.4946   |

Table 1: Second-best Income Tax Rates

current-period utility compared with future periods increases, so that the government needs to amend its tax policy to increase the current period's utility of the agent. (156) shows the utility gain from the current period consumption and labour effort,

$$u_0 = \lambda \ln \left( \theta_l + \frac{1-\delta}{1-\delta+\beta\delta} \theta_k \right) + \lambda \frac{1-\alpha}{\alpha} (\ln \theta_l - \ln(\theta_l(1-\delta+\beta\delta) + (1-\lambda)(1-\delta)\theta_k)) \quad (156)$$

$$+ (1-\lambda)(\ln(\theta_l(1-\delta+\beta\delta) + (1-\delta)\theta_k) - \ln(\theta_l(1-\delta+\beta\delta) + (1-\lambda)(1-\delta)\theta_k)) + \Delta,$$

where  $\Delta$  is the collection of terms not including choice variables. Since the optimal tax mix should satisfy  $\theta_k + \theta_l = \alpha$ , a change in  $\theta_l$ ,  $d\theta_l$  should be balanced by the change in  $\theta_k$ ,  $d\theta_k$  such that  $d\theta_k = -d\theta_l$ . Then the marginal utility of the current period with respect to tax rate changes,  $du_0$  can be expressed only with  $d\theta_l$  as shown in (157). It can be checked that  $du_0$  is positive for all the second best tax rates shown in Table 1, which means that the marginal increase in  $\theta_l$  (marginal decrease in  $\tau_l$ ) provides more utility in the current period. Thus, the optimal  $\tau_l$  should decrease while  $\tau_k$  increases as  $\beta$  gets smaller.

$$du_0 = \left( \begin{array}{c} \frac{\beta\delta}{\theta_l(1-\delta+\beta\delta)+(1-\delta)\theta_k} + \lambda \frac{1-\alpha}{\alpha} \frac{1}{\theta_l} - \\ (\lambda \frac{1-\alpha}{\alpha} + (1-\lambda)) \frac{\lambda(1-\delta)+\beta\delta}{\theta_l(1-\delta+\beta\delta)+(1-\lambda)(1-\delta)\theta_k} \end{array} \right) d\theta_l \quad (157)$$

This result suggests a different implication for capital income tax from the previ-

ous literature. Laibson (1996) and Krusell et al (2002) recommend that the government mitigate the self-control problem stemming from quasi-hyperbolic preferences by subsidizing the agent's saving behaviour. This recommendation still holds as long as lump-sum tax is an available option as we can see in Proposition 10. However, once the lump-sum tax is dropped, the government should respond to the agent's impatience by increasing the capital income tax.

This observation also raises an important question: should the consumer's short-run impatience,  $\beta$ , be distinctively treated with a normal discount factor,  $\delta$ ? As pointed out in the previous literature, it is not easy to distinguish the saving behaviour of a quasi-hyperbolic consumer from that of an exponential consumer. In our model, the equilibrium with quasi-hyperbolic preferences parameter,  $(\beta, \delta)$  is identical to the equilibrium of a standard growth model with a discount rate,  $\hat{\delta} = \frac{\beta\delta}{1-\delta+\beta\delta}$ . From the planner's view, the two equilibria should be considered completely different, since the former is undersaving, while the latter is saving at its Pareto-optimal level. For the government taking a Ramsey taxation approach, the distinction between these two cases may also lead to a difference in tax policy. Table 2 shows an example of how a failure to capture quasi-hyperbolic preferences influences the government's taxation decision. What we can observe in Table 2 is that when the government views this economy as a standard exponential world with discount factor,  $\hat{\delta}$  instead of perceiving true quasi-hyperbolic preferences, it levies a higher tax on capital income, which discourages capital accumulation. The equilibrium under false exponential preferences has lower saving than the equilibrium under true quasi-hyperbolic preferences. Therefore, a proper understanding of consumers' preference structure is critical in diagnosing and correcting undersaving in the economy in the Ramsey taxation approach.

**Consistent Taxation** The discussion of the second-best tax policy in the previous section was based on the assumption of the government's commitment capability. In this section, we will consider the tax policy in the situation where the government lacks future commitment. In this case, the government should experience inconsistency in

| Quasi-hyperbolic Preferences [True] |         |          |          | Exponential Preferences [False] |          |          |
|-------------------------------------|---------|----------|----------|---------------------------------|----------|----------|
| $\delta$                            | $\beta$ | $\tau_k$ | $\tau_l$ | $\hat{\delta}$                  | $\tau_k$ | $\tau_l$ |
|                                     | 1.0     | 0.1840   | 0.9212   | 0.9500                          | 0.1840   | 0.9212   |
|                                     | 0.9     | 0.1911   | 0.9181   | 0.9448                          | 0.1931   | 0.9172   |
|                                     | 0.8     | 0.1993   | 0.9146   | 0.9383                          | 0.2038   | 0.9127   |
|                                     | 0.7     | 0.2090   | 0.9104   | 0.9301                          | 0.2165   | 0.9072   |
|                                     | 0.6     | 0.2207   | 0.9054   | 0.9194                          | 0.2320   | 0.9006   |
| 0.95                                | 0.5     | 0.2351   | 0.8993   | 0.9048                          | 0.2514   | 0.8923   |
|                                     | 0.4     | 0.2535   | 0.8913   | 0.8837                          | 0.2767   | 0.8814   |
|                                     | 0.3     | 0.2785   | 0.8807   | 0.8507                          | 0.3119   | 0.8663   |
|                                     | 0.2     | 0.3151   | 0.8650   | 0.7917                          | 0.3660   | 0.8431   |
|                                     | 0.1     | 0.3780   | 0.8380   | 0.6552                          | 0.4675   | 0.7997   |

\*  $\alpha = 0.3, \lambda = 0.7$

Table 2: Observational Equivalence and Second-best Taxation

its tax policy. That is, the tax plan regarded as the best for the current government should be revised by the government in the next period since that plan is not able to serve as the best plan for any government other than the current one.

The fundamental strategy to obtain a time-inconsistent tax policy in this section is based on the approach suggested by Krusell et al (2000, 2002). Suppose that the current government at time 0 faces given sequences of capital and labour tax rates for all future periods, all of which are assumed to be constant such that

$$\{(\tau_t^k, \tau_t^l)\}_{t=1}^{\infty} = \{(\tau_k, \tau_l)\}.$$

Then the current government's optimal tax choice for the current period, 0,  $(\tilde{\tau}_k, \tilde{\tau}_l)$  should satisfy the following conditions:

- (i)  $(\tilde{\tau}_k, \tilde{\tau}_l)$  solves

$$\begin{aligned} \max_{(\tilde{\tau}_k, \tilde{\tau}_l)} U_0(\tilde{\tau}_k, \tau_k, \tilde{\tau}_l, \tau_l) &= \ln c_0(\tilde{\tau}_k, \tau_k, \tilde{\tau}_l, \tau_l) + \ln l_0(\tilde{\tau}_k, \tau_k, \tilde{\tau}_l, \tau_l) + \\ &\beta \sum_{t=1}^{\infty} \delta^t [\ln c_t(\tilde{\tau}_k, \tau_k, \tilde{\tau}_l, \tau_l) + \ln l_t(\tilde{\tau}_k, \tau_k, \tilde{\tau}_l, \tau_l)]. \end{aligned} \quad (158)$$

- (ii)  $c_t(\tilde{\tau}_k, \tau_k, \tilde{\tau}_l, \tau_l)$  and  $l_t(\tilde{\tau}_k, \tau_k, \tilde{\tau}_l, \tau_l)$  should be consistent with competitive equilibrium.

(iii) When  $(\tilde{\tau}_k, \tilde{\tau}_l) = (\tau_k, \tau_l)$ , we can call it a time-consistent tax policy equilibrium in the sense that for all governments throughout all future periods there would be no incentive to deviate from those tax rates.

We can find an optimal,  $(\tilde{\tau}_k, \tilde{\tau}_l)$  from the Ramsey approach where a consumer's competitive behaviour, global resource constraint, and government's balanced budget constraint are all embedded into the objective function, (158). Thus, firstly we need to identify a competitive equilibrium when consumers face tax rates,  $(\tilde{\tau}_k, \tilde{\tau}_l)$  at the present and  $(\tau_k, \tau_l)$  in the future, which is described in the following proposition.

**Proposition 17** *With the current period tax rates,  $(\tilde{\tau}_k, \tilde{\tau}_l)$  and tax rates,  $(\tau_k, \tau_l)$  for all future periods, the competitive equilibrium for a representative consumer is*

$$k_1 = s_0 \alpha (1 - \tau_k) [(1 - \alpha)(1 - \tilde{\tau}_l) + \alpha(1 - \tilde{\tau}_k)] A k_0^\alpha l_0^{1-\alpha} g_0^{1-\alpha}, \quad (159)$$

$$k_{t+1} = \frac{\alpha \beta \delta}{1 - \delta + \beta \delta} (1 - \tau_k) A k_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha} \text{ for } t \geq 1,$$

$$l_0 = \frac{\lambda(1 - \alpha)(1 - \tilde{\tau}_l)}{(1 - \alpha)(1 - \tilde{\tau}_l) + (1 - \lambda)\alpha(1 - \tilde{\tau}_k - s_0((1 - \alpha)(1 - \tilde{\tau}_l) + \alpha(1 - \tilde{\tau}_k)))},$$

$$l_t = \frac{\lambda(1 - \alpha)(1 - \tau_l)(1 - \delta + \beta \delta)}{(1 - \alpha)(1 - \tau_l)(1 - \delta + \beta \delta) + \alpha(1 - \lambda)(1 - \delta)(1 - \tau_k)} \text{ for } t \geq 1,$$

$$\text{where } s_0 = \frac{1 - \tau_k}{(1 - \alpha)(1 - \tau_l) + \alpha(1 - \tau_k)} \frac{\beta \delta}{1 - \delta + \beta \delta}.$$

**Proof.** See appendix. ■

Since we identified an agent's response to the government's fiscal policy in a decentralized economy, the government's objective function in (158) can be rewritten with respect to its choice variables,  $\tilde{\tau}_k$  and  $\tilde{\tau}_l$  as

$$\begin{aligned} \max_{\tilde{\theta}_k, \tilde{\theta}_l} U_0 &= \lambda \left( 1 + \frac{\beta \delta}{1 - \delta} \right) \left( \begin{aligned} &\ln(\tilde{\theta}_l + \tilde{\theta}_k) + \frac{1 - \alpha}{\alpha} (\ln(1 - \tilde{\theta}_l - \tilde{\theta}_k) + \\ &\ln(\frac{\tilde{\theta}_l}{\tilde{\theta}_l + (1 - \lambda)\tilde{\theta}_k - s_0 \alpha (1 - \lambda)(\tilde{\theta}_l + \tilde{\theta}_k)}) \end{aligned} \right) \quad (160) \\ &+ (1 - \lambda) \ln\left(\frac{\tilde{\theta}_l + \tilde{\theta}_k}{\tilde{\theta}_l + (1 - \lambda)\tilde{\theta}_k - s_0 \alpha (1 - \lambda)(\tilde{\theta}_l + \tilde{\theta}_k)}\right) + \Delta, \end{aligned}$$

where  $\tilde{\theta}_k = \alpha(1 - \tilde{\tau}_k)$ ,  $\tilde{\theta}_l = (1 - \alpha)(1 - \tilde{\tau}_l)$ ,  $s_0 = \frac{1 - \tau_k}{(1 - \alpha)(1 - \tau_l) + \alpha(1 - \tau_k)} \frac{\beta \delta}{1 - \delta + \beta \delta}$  and  $\Delta$  is a

collection of terms not including  $\tilde{\tau}_k$  and  $\tilde{\tau}_l$  (See appendix for details). The first-order conditions for (160) can be written as

$$\begin{aligned} \frac{\partial U_0}{\partial \tilde{\theta}_k} &= \lambda \frac{1-\delta+\beta\delta}{1-\delta} \left( \frac{1}{\tilde{\theta}_l + \tilde{\theta}_k} + \frac{1-\alpha}{\alpha} \left( -\frac{1}{1-\tilde{\theta}_l - \tilde{\theta}_k} - \frac{(1-\lambda)(1-s_0\alpha)}{(1-s_0\alpha(1-\lambda))\tilde{\theta}_l + (1-\lambda)(1-s_0\alpha)\tilde{\theta}_k} \right) \right) \\ &\quad + (1-\lambda) \left( \frac{1}{\tilde{\theta}_l + \tilde{\theta}_k} - \frac{(1-\lambda)(1-s_0\alpha)}{(1-s_0\alpha(1-\lambda))\tilde{\theta}_l + (1-\lambda)(1-s_0\alpha)\tilde{\theta}_k} \right), \end{aligned}$$

$$\begin{aligned} \frac{\partial U_0}{\partial \tilde{\theta}_l} &= \lambda \frac{1-\delta+\beta\delta}{1-\delta} \left( \frac{1}{\tilde{\theta}_l + \tilde{\theta}_k} + \frac{1-\alpha}{\alpha} \left( -\frac{1}{1-\tilde{\theta}_l - \tilde{\theta}_k} + \frac{1}{\tilde{\theta}_l} - \frac{1-s_0\alpha(1-\lambda)}{(1-s_0\alpha(1-\lambda))\tilde{\theta}_l + (1-\lambda)(1-s_0\alpha)\tilde{\theta}_k} \right) \right) \\ &\quad + (1-\lambda) \left( \frac{1}{\tilde{\theta}_l + \tilde{\theta}_k} - \frac{1-s_0\alpha(1-\lambda)}{(1-s_0\alpha(1-\lambda))\tilde{\theta}_l + (1-\lambda)(1-s_0\alpha)\tilde{\theta}_k} \right). \end{aligned}$$

From these F.O.C, we can observe some straightforward results in the following proposition.

**Proposition 18** (i) *When  $\lambda = 1$ , at the equilibrium there is a set of consistent tax rates,  $\tau$ :*

$$\tau = \{(\tau_k, \tau_l) | \alpha(1 - \tau_k) + (1 - \alpha)(1 - \tau_l) = \alpha\}.$$

(ii) *When  $0 < \lambda < 1$ , at the equilibrium the consistent tax rates,  $(\tau_k, \tau_l)$  can be defined as*

$$\tau_k = 1, \tau_l = 1 - \frac{\alpha}{1 - \alpha}.$$

**Proof.** (i) When  $\lambda = 1$ , the two F.O.C provide identical equations,

$$\frac{1}{\tilde{\theta}_l + \tilde{\theta}_k} = \frac{1-\alpha}{\alpha} \left( \frac{1}{1-\tilde{\theta}_l - \tilde{\theta}_k} \right).$$

(ii) When  $0 < \lambda < 1$ , there is no interior solution since there is an implicit assumption that  $\tilde{\theta}_l$  and  $\tilde{\theta}_k$  should not be negative. Suppose that the corner solution is  $\tilde{\theta}_k = 0$ . If we construct the Lagrangian by adding  $\mu \cdot \tilde{\theta}_k$  to the objective function, (160), F.O.C would be changed into

$$\frac{\partial L}{\partial \tilde{\theta}_k} = \frac{\partial U_0}{\partial \tilde{\theta}_k} + \mu = 0 \text{ and } \frac{\partial L}{\partial \tilde{\theta}_l} = \frac{\partial U_0}{\partial \tilde{\theta}_l} = 0.$$

When  $\tilde{\theta}_k = 0$ ,  $\frac{\partial U_0}{\partial \tilde{\theta}_l} = 0$  at  $\tilde{\theta}_l = \alpha$ . And we can check that at  $(\tilde{\theta}_k, \tilde{\theta}_l) = (0, \alpha)$ ,  $-\frac{\partial U_0}{\partial \tilde{\theta}_k} = \mu > 0$ . Therefore,  $(\tilde{\theta}_k, \tilde{\theta}_l) = (0, \alpha)$  is the solution. ■

When labour supply is inelastic or  $\lambda = 1$ , the proposition states that any combination of labour and capital income tax rates is well qualified as an equilibrium tax policy as long as that tax mix can finance the optimal level of productive public spending. In the expression of the objective function in (160), the future tax rates,  $\tau_k$  and  $\tau_l$  have no marginal effect when  $\lambda = 1$ . Thus, the current government is indifferent to future tax rates. And in (159), the saving ratio of the current period agent only depends on future tax rates. As a result, the current government only cares about assuring that the after-tax disposable income of the current period is maximized. This condition is accomplished when the level of productive public spending is  $(1 - \alpha)$  regardless of the tax policy composition used to finance it. It is also interesting to compare this result with the optimal taxation from a perspective of the planner. As mentioned earlier, when commitment is assumed, we can reach the same conclusion about optimal taxation whether we take a central planner's view or Ramsey taxation approach. In both cases, it is optimal to implement a negative capital income tax rate and finance the productive public spending plus subsidy to capital income with a labour income tax. However, without commitment, the Ramsey taxation approach cannot pinpoint the optimal taxation that a central planner can find.

When labour supply is endogenous,  $\lambda \neq 1$ , the consistent taxation approach leads to an equilibrium in which governments in all periods choose a tax plan where capital income is fully confiscated while maintaining the required optimal level of public spending. It is very obvious that all of the future governments will have no incentive to deviate its policy from this tax plan simply because they will have no income to tax at all. Under this equilibrium tax plan, the consumer in the current period does not save so that there is no production from the next period. However, even though it meets the condition of consistency, this tax plan should not actually be optimal for any government considering that zero consumption in logarithmic utility means negative

infinity. A very infinitesimally lower capital income tax that can give some positive saving is better than this worst equilibrium. The problem is that even for the capital income tax rate as low as that myopic governments are still likely to be tempted by the chance to enhance the welfare from their perspective by increasing the portion of public spending financed by the capital income tax. This untrustworthiness of future governments is a fundamental reason why the worst outcome could happen when the government lacks commitment to its tax policy.

**Optimal Taxation with One-period Partial Commitment** In the previous section, we have found that the lack of commitment could lead to either uncertainty in tax policy (inelastic labour supply case) or a bad equilibrium where there is no saving at all (elastic labour supply case). In this section, we amend the assumption about the commitment capability of the government. Suppose that the current government can choose tax policy for the next period, while it should follow the tax policy set by the previous government. In the same manner, the new government in the next period will implement the tax policy which was decided by the current government and it will choose tax policy for the next period. This assumption matches real practice better concerning the time-lag of tax policy. Tax policy should be enforced at law by the principle of no taxation without representation. Thus, it would take some time before the decision on the change in tax policy is eventually put into practice after the legislation process during which the old tax policy is still valid.

The strategy to find an equilibrium is the same as the previous section. Suppose that there are constant tax rates,  $(\tau_k, \tau_l)$ , for all time periods. The current government is also given  $(\tau_k, \tau_l)$  as its tax policy by the previous government and wishes to choose optimal tax rates for the next period,  $(\tilde{\tau}_k, \tilde{\tau}_l)$ . If there is no incentive for the current government to deviate from  $(\tau_k, \tau_l)$ , so that  $(\tau_k, \tau_l) = (\tilde{\tau}_k, \tilde{\tau}_l)$ , we define it as an equilibrium tax policy. Before we set the optimization problem for the current government, we need to identify a competitive equilibrium which should be included in constraints of the government's problem.

**Proposition 19** Under the tax policy given as  $\{\tau_k^t\}_{t=0}^\infty = \{\tau_k, \tilde{\tau}_k, \tau_k, \tau_k, \dots\}$  and  $\{\tau_l^t\}_{t=0}^\infty = \{\tau_l, \tilde{\tau}_l, \tau_l, \tau_l, \dots\}$ , the competitive equilibrium for a representative agent is

$$k_1 = \frac{\tilde{\theta}_k \beta \delta}{\mu \tilde{\theta}_l + v \tilde{\theta}_k} (\theta_l + \theta_k) A k_0^\alpha l_0^{1-\alpha} g_0^{1-\alpha}, \quad (161)$$

$$k_2 = \frac{\beta \delta}{1 - \delta + \beta \delta} \frac{\theta_k}{\theta_l + \theta_k} (\tilde{\theta}_l + \tilde{\theta}_k) A k_1^\alpha l_1^{1-\alpha} g_1^{1-\alpha}, \quad (162)$$

$$k_{t+1} = \frac{\beta \delta}{1 - \delta + \beta \delta} \theta_k A k_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha} \text{ for } t \geq 2, \quad (163)$$

$$l_0 = \frac{\lambda \theta_k}{\theta_l + (1 - \lambda)(\theta_k - s)}, \quad (164)$$

$$l_1 = \frac{\lambda \tilde{\theta}_l}{(1 - (1 - \lambda)s_0 \alpha) \tilde{\theta}_l + (1 - \lambda)(1 - s_0 \alpha) \tilde{\theta}_k}, \quad (165)$$

$$l_t = \bar{l} = \frac{\lambda \theta_l (1 - \delta + \beta \delta)}{\theta_l (1 - \delta + \beta \delta) + \theta_k (1 - \lambda)(1 - \delta)} \text{ for } t \geq 2, \quad (166)$$

where  $\theta_l = (1 - \alpha)(1 - \tau_l)$ ,  $\theta_k = \alpha(1 - \tau_k)$ ,  $\tilde{\theta}_l = (1 - \alpha)(1 - \tilde{\tau}_l)$ ,  $\tilde{\theta}_k = \alpha(1 - \tilde{\tau}_k)$ ,  $\mu = (1 - \delta)(1 - (1 - \lambda)s_0 \alpha) + \beta \delta \Theta$ ,  $v = (1 - \delta)(1 - (1 - \lambda)s_0 \alpha) + \beta \delta(1 + \Theta)$ ,  $\Theta = \frac{(1 - \delta + \beta \delta)\theta_l + (1 - \lambda)(1 - \delta)\theta_k}{\lambda(1 - \delta + \beta \delta)(\theta_l + \theta_k)}$ ,  $s = \frac{\tilde{\theta}_k \beta \delta}{\mu \tilde{\theta}_l + v \tilde{\theta}_k} (\theta_l + \theta_k)$  and  $s_0 \alpha = \frac{\theta_k}{\theta_l + \theta_k} \frac{\beta \delta}{1 - \delta + \beta \delta}$ .

**Proof.** See appendix. ■

Using these competitive equilibrium conditions, the government's problem can be defined as

$$\max_{\tilde{\theta}_k, \tilde{\theta}_l} U_0 = \lambda U^c + (1 - \lambda) \ln U^l,$$

where  $U^c$  and  $U^l$  are given by

$$U^c = \frac{1 - \delta + \beta \delta}{1 - \delta} \ln(\tilde{\theta}_l + \tilde{\theta}_k) - \frac{1 - \delta + \beta \delta}{1 - \delta} \ln(\mu \tilde{\theta}_l + v \tilde{\theta}_k) + \frac{\beta \delta}{1 - \delta} \ln \tilde{\theta}_k + \quad (167)$$

$$\frac{\beta \delta}{1 - \delta} \frac{1 - \alpha}{\alpha} \ln(1 - \tilde{\theta}_l - \tilde{\theta}_k) + \frac{1 - \delta + \beta \delta}{1 - \delta} \frac{1 - \alpha}{\alpha} \ln l_0 + \frac{\beta \delta}{1 - \delta} \frac{1 - \alpha}{\alpha} \ln l_1 + X,$$

$$U^l = \ln(1 - l_0) + \beta \delta \ln(1 - l_1) + Y, \quad (168)$$

subject to global budget constraint, government balance-budget constraint and com-

petitive equilibrium from (161) to (166) (See appendix for details). The first-order conditions are then derived as

$$\begin{aligned}\frac{\partial U_0}{\partial \tilde{\theta}_l} &= \lambda \frac{\partial U^c}{\partial \tilde{\theta}_l} + (1 - \lambda) \ln \frac{\partial U^l}{\partial \tilde{\theta}_l}, \\ \frac{\partial U_0}{\partial \tilde{\theta}_k} &= \lambda \frac{\partial U^c}{\partial \tilde{\theta}_k} + (1 - \lambda) \ln \frac{\partial U^l}{\partial \tilde{\theta}_k},\end{aligned}$$

where

$$\begin{aligned}\frac{\partial U^c}{\partial \tilde{\theta}_l} &= \frac{1 - \delta + \beta\delta}{1 - \delta} \frac{1}{\tilde{\theta}_l + \tilde{\theta}_k} - \frac{1 - \delta + \beta\delta}{1 - \delta} \frac{\mu}{\mu\tilde{\theta}_l + v\tilde{\theta}_k} + \\ &\quad \frac{1 - \delta + \beta\delta}{1 - \delta} \frac{1 - \alpha}{\alpha} \left( \frac{\mu}{\mu\tilde{\theta}_l + v\tilde{\theta}_k} - \frac{(\mu\theta_l + (1 - \lambda)\mu\theta_k)}{\Psi} \right) - \\ &\quad \frac{\beta\delta}{1 - \delta} \frac{1 - \alpha}{\alpha} \frac{1}{1 - \tilde{\theta}_l - \tilde{\theta}_k} + \frac{\beta\delta}{1 - \delta} \frac{1 - \alpha}{\alpha} \left( \frac{1}{\tilde{\theta}_l} - \frac{(1 - (1 - \lambda)\alpha s_0)}{\Phi} \right),\end{aligned}$$

$$\begin{aligned}\frac{\partial U^c}{\partial \tilde{\theta}_k} &= \frac{1 - \delta + \beta\delta}{1 - \delta} \frac{1}{\tilde{\theta}_l + \tilde{\theta}_k} - \frac{1 - \delta + \beta\delta}{1 - \delta} \frac{v}{\mu\tilde{\theta}_l + v\tilde{\theta}_k} \\ &\quad + \frac{1 - \delta + \beta\delta}{1 - \delta} \frac{1 - \alpha}{\alpha} \left( \frac{v}{\mu\tilde{\theta}_l + v\tilde{\theta}_k} - \frac{((v - (1 - \lambda)\beta\delta)\theta_l + (1 - \lambda)(v - \beta\delta)\theta_k)}{\Psi} \right) \\ &\quad + \frac{\beta\delta}{1 - \delta} \frac{1}{\tilde{\theta}_k} - \frac{\beta\delta}{1 - \delta} \frac{1 - \alpha}{\alpha} \frac{1}{1 - \tilde{\theta}_l - \tilde{\theta}_k} + \frac{\beta\delta}{1 - \delta} \frac{1 - \alpha}{\alpha} \left( -\frac{(1 - \lambda)(1 - \alpha s_0)}{\Phi} \right),\end{aligned}$$

$$\frac{\partial U^l}{\partial \tilde{\theta}_l} = (1 + \beta\delta) \frac{1}{\tilde{\theta}_l + \tilde{\theta}_k} - \frac{(\mu\theta_l + (1 - \lambda)\mu\theta_k)}{\Psi} - \beta\delta \frac{(1 - (1 - \lambda)\alpha s_0)}{\Phi}$$

$$\frac{\partial U^l}{\partial \tilde{\theta}_k} = (1 + \beta\delta) \frac{1}{\tilde{\theta}_l + \tilde{\theta}_k} - \frac{(v - (1 - \lambda)\beta\delta)\theta_l + (1 - \lambda)(v - \beta\delta)\theta_k}{\Psi} - \beta\delta \frac{(1 - \lambda)(1 - \alpha s_0)}{\Phi},$$

where  $\Psi = \tilde{\theta}_l(\mu\theta_l + (1 - \lambda)\mu\theta_k) + \tilde{\theta}_k((v - (1 - \lambda)\beta\delta)\theta_l + (1 - \lambda)(v - \beta\delta)\theta_k)$  and  $\Phi = (1 - (1 - \lambda)\alpha s_0)\tilde{\theta}_l + (1 - \lambda)(1 - \alpha s_0)\tilde{\theta}_k$ .

When the labour supply is inelastic ( $\lambda = 1$ ), the equilibrium derived from the F.O.C is very simple as we can see in the following proposition.

**Proposition 20** *When  $\lambda = 1$ , the equilibrium tax policy for the current government,  $(\tilde{\tau}_k, \tilde{\tau}_l)$  is*

$$\tilde{\tau}_k = 0 \text{ and } \tilde{\tau}_l = 1.$$

**Proof.** From F.O.C above, the decision rule of  $(\tilde{\theta}_k, \tilde{\theta}_l)$  of the current government can be defined as

$$\tilde{\theta}_k = \alpha + \tilde{\theta}_l, \quad \tilde{\theta}_l = -\frac{\alpha\beta\delta}{1-\delta} \frac{\theta_l}{\theta_k + \theta_l}.$$

Therefore, for  $(\tilde{\theta}_k, \tilde{\theta}_l) = (\theta_k, \theta_l)$  at equilibrium,  $(\tilde{\theta}_k, \tilde{\theta}_l)$  should be  $(\alpha, 0)$ , which corresponds to tax rates,  $(\tilde{\tau}_k, \tilde{\tau}_l) = (0, 1)$ . ■

It is very interesting to see the fact that the optimal tax rates stated in proposition 14 actually coincide with the tax rates in proposition 10, which can obtain the central planner's first-best choice in the decentralized economy. Proposition 14 implies that even though it is assumed that the central planner is totally uncommitted, the implementation of the best resource allocation of the central planner in the competitive economy implicitly requires a guarantee that the government of the next period should commit to the tax policy chosen by the government of the previous government.

When the labour supply is endogenous ( $\lambda \neq 1$ ), it is not possible to define a closed-form of the decision rule. In Table 3, we present equilibrium tax plans under one-period commitment with the same parameter values used in Table 1. We saw in the previous section an autarky equilibrium where the government confiscates all of capital income arising without commitment. Partial commitment being restored, we can notice that a fairly low capital income tax rate appears at equilibrium while the level is still larger than the second-best capital tax rate. It is also noticeable that the influence of impatience parameter,  $\beta$ , is pretty much weakened now. We can observe again that as  $\beta$  gets lower equilibrium capital income tax rate increases while labour income tax rate decreases, however the change in tax rates is insignificantly small compared with Table 1.

## 4.4 Conclusion

In this chapter, we consider consumers with quasi-hyperbolic preferences in a growth model with productive public spending and explore the characteristics of the optimal

| $\beta$ | $\alpha = 0.3$  |          |                 |          | $\alpha = 0.7$  |          |                 |          |
|---------|-----------------|----------|-----------------|----------|-----------------|----------|-----------------|----------|
|         | $\lambda = 0.7$ |          | $\lambda = 0.3$ |          | $\lambda = 0.7$ |          | $\lambda = 0.3$ |          |
|         | $\tau_k$        | $\tau_l$ | $\tau_k$        | $\tau_l$ | $\tau_k$        | $\tau_l$ | $\tau_k$        | $\tau_l$ |
| 1.0     | 0.4415          | 0.8108   | 0.5875          | 0.7482   | 0.2370          | 0.4471   | 0.2799          | 0.3469   |
| 0.9     | 0.4425          | 0.8104   | 0.5877          | 0.7481   | 0.2383          | 0.4440   | 0.2806          | 0.3452   |
| 0.8     | 0.4437          | 0.8099   | 0.5881          | 0.7480   | 0.2399          | 0.4402   | 0.2815          | 0.3431   |
| 0.7     | 0.4451          | 0.8092   | 0.5886          | 0.7477   | 0.2437          | 0.4314   | 0.2825          | 0.3405   |
| 0.6     | 0.4470          | 0.8084   | 0.5893          | 0.7475   | 0.2457          | 0.4267   | 0.2840          | 0.3374   |
| 0.5     | 0.4495          | 0.8074   | 0.5898          | 0.7472   | 0.2493          | 0.4182   | 0.2879          | 0.3283   |
| 0.4     | 0.4529          | 0.8059   | 0.5914          | 0.7466   | 0.2543          | 0.4067   | 0.2941          | 0.3137   |
| 0.3     | 0.4580          | 0.8037   | 0.5932          | 0.7458   | 0.2615          | 0.3898   | 0.2960          | 0.3094   |
| 0.2     | 0.4665          | 0.8001   | 0.5961          | 0.7445   | 0.2733          | 0.3622   | 0.3043          | 0.2900   |
| 0.1     | 0.4831          | 0.7930   | 0.6019          | 0.7420   | 0.2972          | 0.3066   | 0.3232          | 0.2459   |

Table 3: Income Tax Rates under One-period Commitment

rates. Our first question is whether quasi-hyperbolic preferences can make the same difference to the optimal tax rates compared to standard exponential preferences as the results in Krusell et al (2002) for the Ramsey growth model. To explore this issue, we assume an omniscient central planner who can govern the resource allocation of the economy. According to Krusell et al (2002), this planner who shares quasi-hyperbolic preferences with a representative agent, ends up implementing a non-zero capital income tax that does harm to the economy in a standard Ramsey growth model. In our model, however, we find out that the first-best tax plan that obtains a planner's equilibrium in a decentralized economy still coincides with the Chamley-Judd zero capital income tax result. As long as a lump-sum tax is available, the planner will not use capital income taxation to finance the productive public spending. Of course, the Pareto-optimality of the planner's first-best equilibrium does not hold in our model either, since undersaving behaviour is inevitable when there is dynamic inconsistency due to quasi-hyperbolic preferences. Thus, a subsidy to capital income as an incentive to save is an option that makes the economy better off, but this policy will be considered only when the government can commit to the policy in the future.

We also investigate Ramsey taxation when the government has only two distortionary tax instruments, a capital income tax and a labour income tax. In the central planner's approach, we do not restrict the availability of a lump-sum tax, so that it is

assumed to be possible that the planner or government can finance the desirable level of public spending without incurring any distortion of agents' saving and labour decisions. In a very extreme case where there is no lump-sum tax available, the government should adjust distortions from taxes up to the level where its objective, the maximization of its electorates' life-time utility, is obtained as well as the optimal level of public spending can be provided. Under this circumstance, the policy implications of quasi-hyperbolic preferences on optimal taxation could be very different.

Firstly, the mitigation of the capital income tax burden to increase capital accumulation may not be the correct measure to respond to increased impatience of consumers. Rather it is a more desirable action for the government to increase the portion of public spending financed from the capital income tax and to encourage the consumers to work more by decreasing the labour income tax. Secondly, the time-consistent tax schedule for the government without commitment is the autarchy equilibrium where the capital income tax rate is one and the agent does not save at all. Thirdly, we observe a much more moderate capital tax rate at the equilibrium when we allow just one-period commitment power to the government.

Also, there are a couple of interesting observations in the case that labour supply is fixed. A lack of commitment power makes the government indifferent to tax plans as long as they can finance a desired level of public spending. Finally, without commitment, a first-best tax policy from the perspective of the planner's approach cannot be implemented by a government who follows the Ramsey taxation approach. It turns out that the planner's first-best tax plan is the one that is equilibrium for the government with one-period partial commitment.

## 5 Conclusion of the Thesis

In the thesis, we have presented three distinct works on the subject of fiscal policy. The first chapter studies the stabilizing role of income taxation. Our main focus in this chapter is to settle the different policy implications existing in the early literature. We start from the two-sector growth model of Benhabib and Farmer (1996) where the difference is the most striking and conclude that the assumption about how the government's tax revenue is used plays a very critical role in choosing an appropriate tax schedule to dampen belief-driven fluctuations in the economy. When the tax revenue is used to finance government spending which does not contribute to either production or household utility, only progressive income tax can be used to secure the saddle-path stability of the economy. The result of Guo and Harrison (2001) that the stabilizing role of regressive taxation emerges for sufficiently large sector-specific externalities can be attributed to the assumption that the tax revenue is repaid as a lump-sum transfer.

The assumption on the use of tax revenue also makes a significant difference in the one-sector growth model of Benhabib and Farmer (1994). The effectiveness of progressive taxation as an automatic stabilizer of sunspot fluctuations in Guo and Lansing (1998) has been regarded as an established conclusion in the Benhabib and Farmer (1994) model. However, we show that when a lump-sum transfer rather than wasteful government spending as in Guo and Lansing (1998) is assumed regressive income tax can also stabilize the belief-driven fluctuations. Furthermore, it turns out that the capital income tax that is ignored in Guo (1999) can have a stabilizing effect when it is sufficiently regressive. The difference in the stabilizing role between labour and capital income tax stands out more in a growth model with productive government spending in Chen and Guo (2010) as it is shown that less progressive capital income tax is more effective in eliminating indeterminacy while for the labour income tax larger progressivity can reduce the probability of indeterminacy.

Even though it needs to be checked if these results hold under a wider set of functional forms and parameter values, this chapter contributes to the existing literature

by firstly taking notice of the importance of the assumption on the use of tax revenue and identifying different roles of labour and capital income tax in addressing the belief-driven business cycle in RBC models<sup>10</sup>.

The second chapter studies the optimal income tax structure in a Barro-type growth model with decreasing-returns-to-scale technology and a public consumption good. We characterize the first-best optimum growth path chosen by a central planner and then try to construct a tax policy that replicates this entire optimal growth path in a competitive economy. The non-linearity of the income tax schedule is required as a constant income tax rate that implements the first-best steady-state in the long-run does not replicate the growth path converging to the steady-state. The main finding in this chapter is the optimality of a regressive tax schedule. When labour supply is fixed, the decentralized economy grows too quickly and a progressive tax worsens the excessively rapid growth. In contrast, an intertemporally regressive income tax moves the decentralized growth path closer to the optimal path, and a unique level of regressivity exists for which the first-best optimum is decentralized to a first-order approximation. When labour supply is endogenous, both the progressive and the regressive income taxation can implement a welfare-improving growth path than the competitive equilibrium with the constant income tax rate. However, regressive taxation turns out to be more effective in replicating the central planner's equilibrium since it can create an approximately same growth-path to the first-best while the progressive cannot.

It should be noted that the optimality of regressive taxation pretty much relies on the model and the parameter values, so the result of this chapter should be interpreted as a specific case where the regressive income tax can effectively replicate the growth path. It should also be pointed out that even if the regressive income tax is optimal, it does not convey any redistributive implication. Our model considers an identical agent, so that there is no need for redistribution policy. Rather, the result of this chapter suggests that the the income tax rate should be intertemporally designed to

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<sup>10</sup>In the Woodford (1989) and Grandmont et al (1998) framework, separate taxation for labour and capital income is considered in Lloyd-Braga and Modesto (2012) and Gokan (2013). Gokan (2013) confirms that labour and capital income taxes have different impacts on the indeterminacy.

decrease as the economy grows.

The third chapter explores optimal taxation in a growth model with productive public spending when consumers have quasi-hyperbolic preferences. This chapter takes two different approaches to defining the optimal tax policy. First, we consider a fictitious planning problem where the central planner shares the quasi-hyperbolic preferences with a representative agent and we investigate tax policy that obtains the planner's equilibrium in a decentralized economy. To guarantee the implementation of the first-best outcome, we allow the government to use lump-sum tax or transfer. In a standard Ramsey growth model, Krusell et al (2002) earlier showed that a quasi-hyperbolic central planner uses a positive income tax to implement the best equilibrium from his point of view, which leads the economy to a worse equilibrium than the competitive equilibrium. On the contrary, when the productive government spending creates a balanced-growth path in the economy, this chapter shows that the tax policy that can implement the first-best optimum chosen by a central planner has a zero capital income tax rate. It is also observed that quasi-hyperbolic preferences result in a failure of the first-best choice to attain Pareto-efficiency. Negative capital income tax rates or a subsidy to capital income can accomplish a better resource allocation than the first-best choice.

Second, we investigate second-best taxation from the Ramsey fiscal approach where a government with only distortionary tax instruments has to determine the optimal tax schedule subject to the agents' behaviour and the resource constraint in a competitive economy. When we assume that all of the governments throughout time follow a pre-determined tax schedule, this approach interestingly suggests that the government should respond to the agent's impatience by increasing the capital income tax. This result contrasts to the conclusion in most of the existing literature concerning quasi-hyperbolic preferences that the recommended policy response to an increased impatience is more subsidy to capital income financed by non-distortinary lump-sum tax to attract more capital accumulation to correct "under-saving" resulting from quasi-hyperbolic preferences. We also show that in the case of a complete lack of commitment the

time-consistent tax schedule can only induce a worst equilibrium where the government confiscates all of the capital income so that the agent does not save at all.

The analysis in this chapter is restricted to the log-utility and Cobb-Douglas technology since we pursue manageable closed-form solutions from recursive backward induction and also want to exclude an indeterminate equilibria. When we depart from this model set-up, the solutions should be approached approximately by numerical methods as in Krusell et al (2002) and we leave this extension to the future research.

# Appendix

## A Appendix for Chapter 4

**Proof of proposition 10** With constant tax rates,  $\tau_k$  and  $\tau_l$ , we define  $R_t = (1 - \tau_k)r_t$  and  $W_t = (1 - \tau_l)w_t$ . At the final period,  $T$ , the consumer does not save, so only labour choice is asked for him. The problem can be described as

$$\begin{aligned} \max_{l_T} U_T &= \lambda \ln(c_T) + (1 - \lambda) \ln(1 - l_T) \text{ s.t.} \\ c_T &= R_T k_T + W_T l_T - M_T \text{ with a given } k_T. \end{aligned}$$

The optimal choice of  $l_T$  is

$$l_T = 1 - (1 - \lambda) \left( 1 + \frac{R_T k_T - M_T}{W_T} \right). \quad (169)$$

At time  $T - 1$ , the representative agent knows that his future self will choose  $l_T$  by (169) and chooses his saving and labour keeping it in mind. That is, the agent at  $T - 1$  describes his life-time utility maximization problem as

$$\begin{aligned} \max_{k_T, l_{T-1}} U_{T-1} &= u(c_{T-1}, l_{T-1}) + \beta \delta u(c_T, l_T) \text{ s.t.} \\ c_{T-1} &= R_{T-1} k_{T-1} + W_{T-1} l_{T-1} - M_{T-1} - k_T, \\ c_T &= R_T k_T + W_T l_T - M_T, \\ l_T &= 1 - (1 - \lambda) \left( 1 + \frac{R_T k_T - M_T}{W_T} \right) \end{aligned}$$

The first-order conditions are

$$\frac{\partial U_{T-1}}{\partial k_T} = -\frac{\lambda}{R_{T-1} k_{T-1} + W_{T-1} l_{T-1} - M_{T-1} - k_T} + \beta \delta \underbrace{\frac{R_T + W_T \frac{\partial l_T}{\partial k_T}}{R_T k_T + W_T l_T - M_T}}_A - \beta \delta (1 - \lambda) \underbrace{\frac{\frac{\partial l_T}{\partial k_T}}{1 - l_T}}_B = 0, \quad (170)$$

$$\frac{\partial U_{T-1}}{\partial l_{T-1}} = \frac{\lambda W_{T-1}}{R_{T-1} k_{T-1} + W_{T-1} l_{T-1} - M_{T-1} - k_T} - \frac{1 - \lambda}{1 - l_{T-1}} = 0. \quad (171)$$

From (169),  $\frac{\partial l_T}{\partial k_T} = -(1 - \lambda) \frac{R_T}{W_T}$ .

$$\begin{aligned} A &= \frac{\lambda R_T}{R_T k_T + W_T l_T - M_T}, \\ B &= -\frac{R_T}{R_T k_T + W_T l_T - M_T}. \end{aligned}$$

Thus, (170) can be simplified into

$$\frac{\lambda}{R_{T-1}k_{T-1} + W_{T-1}l_{T-1} - M_{T-1} - k_T} = \beta\delta \frac{R_T}{R_T k_T + W_T l_T - M_T}. \quad (172)$$

From (171) and (172), the decision rules of capital and labour for the agent at  $T - 1$  are

$$k_T = \frac{\beta\delta}{\lambda + \beta\delta} (R_{T-1}k_{T-1} + W_{T-1}l_{T-1} - M_{T-1}) - \frac{\lambda}{\lambda + \beta\delta} \left( \frac{W_T - M_T}{R_T} \right), \quad (173)$$

$$l_{T-1} = \lambda - (1 - \lambda) \frac{R_{T-1}k_{T-1} - M_{T-1} - k_T}{W_{T-1}}. \quad (174)$$

At time  $T - 2$ , the decision rules of the agent at  $T - 1$  are integrated as constraints for the consumer's problem at  $T - 2$ . Thus, the optimization problem can be defined as

$$\max_{k_{T-1}} U_{T-2} = u(c_{T-2}, l_{T-2}) + \beta\delta u(c_{T-1}, l_{T-1}) + \beta\delta^2 u(c_T, l_T) \text{ s.t} \quad (175)$$

$$c_t = R_t k_t + W_t l_t - M_t - k_{t+1} \text{ for } t = T - 2, T - 1$$

$$c_T = c_T = R_T k_T + W_T l_T - M_T$$

$$\text{with (169), (173) and (174)}. \quad (176)$$

The first-order conditons are

$$\frac{\partial U_{T-1}}{\partial k_{T-1}} = - \frac{\lambda}{R_{T-2}k_{T-2} + W_{T-1}l_{T-2} - M_{T-2} - k_{T-1}} + \quad (177)$$

$$\beta\delta \left( \frac{\lambda \left( \overbrace{R_{T-1} + W_{T-1} \frac{\partial l_{T-1}}{\partial k_{T-1}} - \frac{\partial k_T}{\partial k_{T-1}}}^A \right)}{\underbrace{R_{T-1}k_{T-1} + W_{T-1}l_{T-1} - M_{T-1} - k_T}_B} - \frac{(1 - \lambda) \frac{\partial l_{T-1}}{\partial k_{T-1}}}{\underbrace{1 - l_{T-1}}_C} \right) + \quad (178)$$

$$\beta\delta^2 \left( \frac{\lambda \left( \overbrace{R_T \frac{\partial k_T}{\partial k_{T-1}} + W_T \frac{\partial l_T}{\partial k_{T-1}}}^D \right)}{\underbrace{R_T k_T + W_T l_T - M_T}_E} - \frac{(1 - \lambda) \frac{\partial l_T}{\partial k_{T-1}}}{\underbrace{1 - l_{T-1}}_F} \right),$$

$$\frac{\partial U_{T-1}}{\partial k_{T-1}} = \frac{\lambda W_{T-2}}{R_{T-2}k_{T-2} + W_{T-2}l_{T-2} - M_{T-2} - k_{T-1}} - \frac{1 - \lambda}{1 - l_{T-2}}. \quad (179)$$

Notice that when (174) is put into (173),

$$\begin{aligned} k_T &= \frac{\beta\delta}{\lambda + \beta\delta} (R_{T-1}k_{T-1} + W_{T-1}l_{T-1} - M_{T-1}) - \frac{\lambda}{\lambda + \beta\delta} \left( \frac{W_T - M_T}{R_T} \right) \\ &= \frac{\beta\delta}{1 + \beta\delta} (R_{T-1}k_{T-1} + W_{T-1} - M_{T-1}) - \frac{1}{1 + \beta\delta} \left( \frac{W_T - M_T}{R_T} \right). \end{aligned}$$

Using this fact,

$$\begin{aligned} A &= R_{T-1} - (1 - \lambda) \left( R_{T-1} - \frac{\partial k_T}{\partial k_{T-1}} \right) - \frac{\partial k_T}{\partial k_{T-1}} \\ &= \lambda \left( R_{T-1} - \frac{\partial k_T}{\partial k_{T-1}} \right) = \lambda \left( R_{T-1} - \frac{\beta\delta}{1 + \beta\delta} R_{T-1} \right) = \frac{\lambda}{1 + \beta\delta} R_{T-1}, \end{aligned}$$

$$B = \lambda(R_{T-1}k_{T-1} + W_{T-1} - M_{T-1} - k_T),$$

$$\begin{aligned} C &= \frac{(1 - \lambda) \left( -(1 - \lambda) \frac{R_{T-1} - \frac{\partial k_T}{\partial k_{T-1}}}{W_{T-1}} \right)}{1 - \lambda + (1 - \lambda) \frac{R_{T-1}k_{T-1} - M_{T-1} - k_T}{W_{T-1}}} \\ &= \frac{-(1 - \lambda) \frac{1}{1 + \beta\delta} R_{T-1}}{R_{T-1}k_{T-1} + W_{T-1} - M_{T-1} - k_T}, \end{aligned}$$

$$\begin{aligned} D &= \frac{\beta\delta}{1 + \beta\delta} R_T R_{T-1} - \frac{\beta\delta}{1 + \beta\delta} (1 - \lambda) R_T R_{T-1} \\ &= \frac{\lambda\beta\delta}{1 + \beta\delta} R_T R_{T-1}, \end{aligned}$$

$$\begin{aligned} E &= R_T k_T + W_T \left( 1 - (1 - \lambda) \left( 1 + \frac{R_T k_T - M_T}{W_T} \right) \right) - M_T \\ &= R_T k_T + W_T - (1 - \lambda) W_T - (1 - \lambda) (R_T k_T - M_T) - M_T \\ &= \lambda (R_T k_T + W_T - M_T), \end{aligned}$$

$$F = \frac{-(1 - \lambda) \frac{\beta\delta}{1 + \beta\delta} R_T R_{T-1}}{R_T k_T + W_T - M_T}.$$

Combining results from A to F, we can obtain

$$\begin{aligned} \frac{\lambda}{R_{T-2}k_{T-2} + W_{T-1}l_{T-2} - M_{T-2} - k_{T-1}} &= \frac{\beta\delta}{1 + \beta\delta} \frac{R_{T-1}}{R_{T-1}k_{T-1} + W_{T-1} - M_{T-1} - k_T} \\ &\quad + \beta\delta^2 \frac{\beta\delta}{1 + \beta\delta} \frac{R_T R_{T-1}}{R_T k_T + W_T - M_T}. \end{aligned}$$

Notice that

$$\begin{aligned} R_{T-1}k_{T-1} + W_{T-1} - M_{T-1} - k_T &= \frac{1}{1 + \beta\delta} (R_{T-1}k_{T-1} + W_{T-1} - M_{T-1} + \frac{W_T - M_T}{R_T}), \\ R_T k_T + W_T - M_T &= \frac{\beta\delta}{1 + \beta\delta} R_T (R_{T-1}k_{T-1} + W_{T-1} - M_{T-1} + \frac{W_T - M_T}{R_T}). \end{aligned}$$

Then, (177) can finally be simplified as

$$\frac{\lambda}{R_{T-2}k_{T-2} + W_{T-1}l_{T-2} - M_{T-2} - k_{T-1}} = \frac{(\beta\delta + \beta\delta^2) R_{T-1}}{R_{T-1}k_{T-1} + W_{T-1} - M_{T-1} + \frac{W_T - M_T}{R_T}}. \quad (180)$$

From (180) and (179), the decision rules for the agent at time  $T - 2$  can be derived as

$$k_{T-1} = \frac{\beta\delta + \beta\delta^2}{\lambda + \beta\delta + \beta\delta^2} (R_{T-2}k_{T-2} + W_{T-1}l_{T-2} - M_{T-2}) - \frac{\lambda}{\lambda + \beta\delta + \beta\delta^2} \left( \frac{W_{T-1} - M_{T-1}}{R_{T-1}} + \frac{W_T - M_T}{R_T R_{T-1}} \right),$$

Iteration of this process to time  $t$  gives us the law of motion of capital in proposition 10.

**Proof of proposition 11** We find out the raw of motion of capital in a finite-period horizon

as

$$k_{t+1} = \frac{\beta \sum_{i=1}^{T-t} \delta^i}{\lambda + \beta \sum_{i=1}^{T-t} \delta^i} ((1 - \tau_k)r_t k_t + (1 - \tau_l)w_t l_t - M_t) - \frac{\lambda}{\lambda + \beta \sum_{i=1}^{T-t} \delta^i} \sum_{i=t+1}^T \left( \frac{(1 - \tau_l)w_i - M_i}{\prod_{j=t+1}^i (1 - \tau_k)^{i-t} r_j} \right). \quad (181)$$

Conjecture that the saving rule in an infinite-time period has a form of  $k_{t+1} = s\alpha A k_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha}$  and the labour choice is constant at  $\bar{l}$ , which will be verified later. Using  $r_t = A\alpha k_t^{\alpha-1} l_t^{1-\alpha} g_t^{1-\alpha}$ ,  $w_t = A(1 - \alpha)k_t^{\alpha-1} l_t^{1-\alpha} g_t^{1-\alpha}$  and  $M_t = (\theta - \alpha\tau_k - (1 - \alpha)\tau_l) y_t = \Omega y_t$

$$\begin{aligned} \frac{(1 - \tau_l)w_i - M_{t+i}}{(1 - \tau_k)^{i-t} \prod_{j=t+1}^i r_j} &= \frac{1}{(1 - \tau_k)^{i-t}} \left( \frac{1 - \alpha}{\alpha} \frac{1 - \tau_l}{\bar{l}} - \frac{\Omega}{\alpha} \right) s^{i-(t+1)} k_{t+1}, \\ \sum_{i=t+1}^{\infty} \frac{(1 - \tau_l)w_i - M_{t+i}}{(1 - \tau_k)^{i-t} \prod_{j=t+1}^i r_j} &= \left( \frac{1 - \alpha}{\alpha} \frac{1 - \tau_l}{\bar{l}} - \frac{\Omega}{\alpha} \right) k_{t+1} \sum_{i=t+1}^{\infty} \frac{s^{i-(t+1)}}{(1 - \tau_k)^{i-t}} \\ &= \left( \frac{1 - \alpha}{\alpha} \frac{1 - \tau_l}{\bar{l}} - \frac{\Omega}{\alpha} \right) \frac{1}{1 - \tau_k - s} k_{t+1}. \end{aligned}$$

Except for the last period, the labour choice rule is

$$l_t = \lambda - (1 - \lambda) \frac{R_t k_t - M_t - k_{t+1}}{W_t},$$

so that  $\bar{l}$  can be derived as

$$\begin{aligned} l_t &= \lambda - (1 - \lambda) \frac{(1 - \tau_k)\alpha - \Omega - s\alpha}{(1 - \alpha)(1 - \tau_l)} l_t \\ \bar{l} &= \frac{\lambda(1 - \alpha)(1 - \tau_l)}{(1 - \alpha)(1 - \tau_l) + (1 - \lambda)[(1 - \tau_k - s)\alpha - \Omega]} \end{aligned}$$

As  $T \rightarrow \infty$ , (181) is converging to

$$\begin{aligned} k_{t+1} &= \frac{\beta\delta}{\lambda(1 - \delta) + \beta\delta} ((1 - \tau_k)r_t k_t + (1 - \tau_l)w_t l_t - M_t) - \\ &\quad \frac{1 - \delta}{\lambda(1 - \delta) + \beta\delta} \frac{(1 - \alpha)(1 - \tau_l) + (1 - \lambda)(1 - \tau_k - s)\alpha - \Omega}{\alpha(1 - \tau_k - s)} k_{t+1} \end{aligned}$$

Since  $(1 - \tau_k)r_t k_t + (1 - \tau_l)w_t l_t - M_t = (1 - \theta)A k_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha}$ ,

$$k_{t+1} = \frac{\beta\delta(1 - \tau_k - s)(1 - \theta)}{(1 - \delta + \beta\delta)(1 - \tau_k - s)\alpha + (1 - \delta)(1 - \alpha)(1 - \tau_l) - (1 - \delta)\Omega} \alpha A k_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha}.$$

Therefore,  $s^*$  should satisfy the following equation:

$$s^* = \frac{\beta\delta(1 - \tau_k - s^*)(1 - \theta)}{(1 - \delta + \beta\delta)(1 - \tau_k - s^*)\alpha + (1 - \delta)(1 - \alpha)(1 - \tau_l) - (1 - \delta)\Omega}. \quad (182)$$

Finally we can obtain an equilibrium  $s^*$  as

$$s^* = \frac{\beta\delta}{1 - \delta + \beta\delta} (1 - \tau_k).$$

**Proof of proposition 12 and 13** The central planner's problem at time  $t$  is described as

$$\begin{aligned} U_t &= u(c_t, l_t) + \beta \sum_{i=1}^{T-t} \delta^i u(c_{t+i}, l_{t+i}), \quad 0 < \delta, \beta < 1, \\ k_{t+1} &= (1 - \theta)y_t - c_t \text{ for } t = t \dots T - 1 \\ c_T &= (1 - \theta)y_T \end{aligned}$$

Since  $g_t = \theta y_t$ ,

$$\begin{aligned} y_t &= A k_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha} \\ &= A^{1/\alpha} \theta^{(1-\alpha)/\alpha} l_t^{(1-\alpha)/\alpha} k_t. \end{aligned}$$

At the final period, the central planner only chooses the labour supply choice as

$$l_T = \lambda.$$

At time  $T - 1$

$$\begin{aligned} \max_{k_T} U_{T-1} &= u(c_{T-1}, l_{T-1}) + \beta \delta u(c_T, l_T) \text{ s.t.} \\ c_{T-1} &= (1 - \theta) A^{1/\alpha} \theta^{(1-\alpha)/\alpha} l_{T-1}^{(1-\alpha)/\alpha} k_{T-1} - k_T \\ c_T &= (1 - \theta) A^{1/\alpha} \theta^{(1-\alpha)/\alpha} l_T^{(1-\alpha)/\alpha} k_T \end{aligned} \quad (183)$$

The first-order conditions are

$$\frac{1}{(1 - \theta) A^{1/\alpha} \theta^{(1-\alpha)/\alpha} l_{T-1}^{(1-\alpha)/\alpha} k_{T-1} - k_T} = \frac{\beta \delta}{k_T}, \quad (184)$$

$$\frac{1 - \alpha}{\alpha} \frac{\lambda (1 - \theta) A^{1/\alpha} \theta^{(1-\alpha)/\alpha} l_{T-1}^{(1-\alpha)/\alpha - 1} k_{T-1}}{(1 - \theta) A^{1/\alpha} \theta^{(1-\alpha)/\alpha} l_{T-1}^{(1-\alpha)/\alpha} k_{T-1} - k_T} = \frac{1 - \lambda}{1 - l_{T-1}} \quad (185)$$

From (184), the saving rule is

$$k_T = \frac{\beta \delta}{1 + \beta \delta} (1 - \theta) A^{1/\alpha} \theta^{(1-\alpha)/\alpha} l_{T-1}^{(1-\alpha)/\alpha} k_{T-1}, \quad (186)$$

and by putting this result into (185),

$$l_{T-1} = \frac{\lambda (1 - \alpha) (1 + \beta \delta)}{\lambda (1 - \alpha) (1 + \beta \delta) + \alpha (1 - \lambda)}. \quad (187)$$

At time  $T - 2$

$$\begin{aligned} \max_{k_{T-1}} U_{T-2} &= u(c_{T-2}, l_{T-2}) + \beta \delta u(c_{T-1}, l_{T-1}) + \beta \delta^2 u(c_T, l_T) \text{ s.t.} \\ c_t &= (1 - \theta) A^{1/\alpha} \theta^{(1-\alpha)/\alpha} l_t^{(1-\alpha)/\alpha} k_t - k_{t+1} \text{ for } t = T - 2, T - 1 \\ c_T &= (1 - \theta) A^{1/\alpha} \theta^{(1-\alpha)/\alpha} l_T^{(1-\alpha)/\alpha} k_T \text{ and (186), (187)} \end{aligned} \quad (188)$$

The first-order conditions are

$$\frac{1}{(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_{T-2}^{(1-\alpha)/\alpha}k_{T-2}-k_{T-1}} = \underbrace{\beta\delta\frac{(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_{T-1}^{(1-\alpha)/\alpha}-\frac{\partial k_T}{\partial k_{T-1}}}{(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_{T-1}^{(1-\alpha)/\alpha}k_{T-1}-k_T}}_{1/k_{T-1}} + \underbrace{\beta\delta^2\frac{(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_T^{(1-\alpha)/\alpha}\frac{\partial k_T}{\partial k_{T-1}}}{(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_T^{(1-\alpha)/\alpha}k_T}}_{1/k_{T-1}}, \quad (189)$$

$$\frac{1-\alpha}{\alpha}\frac{\lambda(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_{T-2}^{(1-\alpha)/\alpha-1}k_{T-2}}{(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_{T-2}^{(1-\alpha)/\alpha}k_{T-2}-k_{T-1}} = \frac{1-\lambda}{1-l_{T-2}}. \quad (190)$$

(185) can be simplified as

$$\frac{1}{(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_{T-2}^{(1-\alpha)/\alpha}k_{T-2}-k_{T-1}} = (\beta\delta + \beta\delta^2)\frac{1}{k_{T-1}},$$

and the saving rule for the central planner at time  $T-2$  can be defined as

$$k_{T-1} = \frac{\beta\delta + \beta\delta^2}{1 + \beta\delta + \beta\delta^2}(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_{T-2}^{(1-\alpha)/\alpha}k_{T-2}, \quad (191)$$

and by putting this result into (190)

$$l_{T-2} = \frac{\lambda(1-\alpha)(1 + \beta\delta + \beta\delta^2)}{\lambda(1-\alpha)(1 + \beta\delta + \beta\delta^2) + \alpha(1-\lambda)}.$$

By repeating this process until the current period,  $t$ , we can obtain the saving and labour choice rules as

$$k_{t+1} = \frac{\beta\sum_{i=1}^{T-t}\delta^i}{1 + \beta\sum_{i=1}^{T-t}\delta^i}(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_t^{(1-\alpha)/\alpha}k_t$$

$$l_t = \frac{\lambda(1-\alpha)(1 + \beta\sum_{i=1}^{T-t}\delta^i)}{\lambda(1-\alpha)(1 + \beta\sum_{i=1}^{T-t}\delta^i) + \alpha(1-\lambda)}$$

As  $T \rightarrow \infty$

$$k_{t+1} = \frac{\beta\delta}{1 - \delta + \beta\delta}(1-\theta)A^{1/\alpha}\theta^{(1-\alpha)/\alpha}l_t^{(1-\alpha)/\alpha}k_t,$$

$$l_t = l_P = \frac{\lambda(1-\alpha)(1 - \delta + \beta\delta)}{\lambda(1-\alpha)(1 - \delta + \beta\delta) + \alpha(1-\lambda)(1 - \delta)}.$$

**Proof of proposition 14** Define a value-function for the current consumer at time 0 as

$$V_0(k) = \lambda \ln((1-\theta)f(k, l) - k') + (1-\lambda) \ln(1-l) + \beta\delta V(k'),$$

where

$$V(k) = \lambda \ln((1 - \theta)f(k, l) - k') + (1 - \lambda) \ln(1 - l) + \delta V(k').$$

Let a stationary saving rate be  $s$ , such that  $k_{t+1} = s(1 - \theta)\theta^{(1-\alpha)/\alpha} A^{1/\alpha} l_t^{(1-\alpha)/\alpha} k_t$ . Since  $f(k, l) = (1 - \theta)\theta^{(1-\alpha)/\alpha} A^{1/\alpha} l^{(1-\alpha)/\alpha} k$ ,

$$\begin{aligned} \ln c_t &= \ln(1 - s)(1 - \theta)\theta^{(1-\alpha)/\alpha} A^{1/\alpha} + \frac{1 - \alpha}{\alpha} \ln l_t + \ln k_t \\ \ln k_t &= \ln s(1 - \theta)\theta^{(1-\alpha)/\alpha} A^{1/\alpha} + \frac{1 - \alpha}{\alpha} \ln l_t + \ln k_{t-1} \\ l_t = \bar{l} &= \frac{(1 - \alpha)\lambda}{(1 - \alpha)\lambda + \alpha(1 - \lambda)(1 - s)}. \end{aligned} \quad (192)$$

Combining these facts,  $V(k)$  can be expressed with respect to given parameter values and the saving ratio,  $s$  as

$$\begin{aligned} V(k) &= \sum_{t=1}^{\infty} \delta^t [\lambda \ln c_t + (1 - \lambda)(1 - l_t)] \\ &= \frac{\lambda\delta}{1 - \delta} \left( \ln(1 - s)(1 - \theta)\theta^{(1-\alpha)/\alpha} A^{1/\alpha} + \frac{1 - \alpha}{\alpha} \ln \bar{l} \right) \\ &\quad + \frac{\lambda\delta}{(1 - \delta)^2} \left( \ln s(1 - \theta)\theta^{(1-\alpha)/\alpha} A^{1/\alpha} + \frac{1 - \alpha}{\alpha} \ln \bar{l} \right) \\ &\quad + \frac{\lambda\delta}{1 - \delta} \ln k_0 + \frac{(1 - \lambda)\delta}{1 - \delta} \ln \frac{\alpha(1 - \lambda)(1 - s)}{(1 - \alpha)\lambda + \alpha(1 - \lambda)(1 - s)} \end{aligned}$$

Finally, we can obtain the expression of  $V_0(k)$  as

$$\begin{aligned} V_0(k) &= \left( \frac{1 - \delta + \beta\delta}{1 - \delta} \right) \ln(1 - s) + \frac{\lambda\beta\delta}{(1 - \delta)^2} \ln s - \\ &\quad \left( \frac{\lambda(1 - \alpha)}{\alpha(1 - \delta)} \left( 1 - \delta + \beta\delta + \frac{\beta\delta}{1 - \delta} \right) + (1 - \lambda) \left( \frac{1 - \delta + \beta\delta}{1 - \delta} \right) \right) \ln \left( \frac{(1 - \alpha)\lambda +}{\alpha(1 - \lambda)(1 - s)} \right) + \Gamma, \end{aligned}$$

where  $\Gamma$  is the collection of terms not including  $s$ .

It can be shown that there exists a unique  $s^*$  such that

$$s^* = \arg \max_s V_0(k),$$

and a closed-form of  $s^*$  is

$$s^* = \frac{\beta\delta}{(1 - \delta)(1 - \delta + \beta\delta) + \beta\delta}. \quad (193)$$

By putting (193) into (192), the labour choice corresponding to this saving ratio in proposition can be found.

**Proof of Proposition 17** Backward induction gives a current agent the saving decision

rule as

$$k_1 = \frac{\beta\delta}{\lambda(1-\delta) + \beta\delta} ((1 - \tilde{\tau}_k)r_0k_0 + (1 - \tilde{\tau}_l)w_0l_0) - \frac{\lambda(1-\delta)}{\lambda(1-\delta) + \beta\delta} \sum_{i=1}^{\infty} \frac{(1-\tau_l)w_i}{\prod_{j=1}^i (1-\tau_k)r_j}, \quad (194)$$

$$l_0 = \lambda - (1-\lambda) \frac{(1-\tilde{\tau}_k)r_0k_0 - k_1}{(1-\tilde{\tau}_l)w_0}. \quad (195)$$

Let  $s$  be a saving rule for all future selves such that  $k_{t+1} = s\alpha Ak_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha}$ . Then we already know that  $s = \frac{\beta\delta}{1-\delta+\beta\delta}(1-\tau_k)$  and the labour choice for all future selves would be  $\bar{l} = \frac{\lambda(1-\alpha)(1-\tau_l)(1-\delta+\beta\delta)}{(1-\alpha)(1-\tau_l)(1-\delta+\beta\delta) + \alpha(1-\lambda)(1-\delta)(1-\tau_k)}$ . The second term of the right-hand side in (194) can be rewritten as

$$\begin{aligned} \frac{\lambda(1-\delta)}{\lambda(1-\delta) + \beta\delta} \sum_{i=1}^{\infty} \frac{(1-\tau_l)w_i}{\prod_{j=1}^i (1-\tau_k)r_j} k_1 &= \frac{\lambda(1-\delta)}{\lambda(1-\delta) + \beta\delta} \frac{1-\alpha}{\alpha} \frac{1-\tau_l}{1-\tau_k-s} \frac{1}{\bar{l}} k_1 \\ &= \frac{(1-\alpha)(1-\tau_l)(1-\delta+\beta\delta) + \alpha(1-\lambda)(1-\delta)(1-\tau_k)}{\alpha(1-\tau_k)(\lambda(1-\delta) + \beta\delta)} k_1. \end{aligned}$$

Using this result, we can obtain the current period saving and labour choice rule in the proposition.

**Proof of Proposition 19** Let  $k_{t+1} = s_2\alpha k_t^\alpha l_t^{1-\alpha} g_t^{1-\alpha}$  for  $t \geq 2$  and  $k_2 = s_1\alpha k_1^\alpha l_1^{1-\alpha} g_1^{1-\alpha}$ .

From proposition 5 and 11, we know that

$$s_1 = s_0 [(1-\alpha)(1-\tilde{\tau}_l) + \alpha(1-\tilde{\tau}_k)],$$

$$s_2 = \frac{\beta\delta}{1-\delta+\beta\delta} (1-\tau_k),$$

where  $s_0 = \frac{1-\tau_k}{(1-\alpha)(1-\tau_l) + \alpha(1-\tau_k)} \frac{\beta\delta}{1-\delta+\beta\delta}$ . For the labour choice, it is known that

$$l_1 = \frac{\lambda(1-\alpha)(1-\tilde{\tau}_l)}{(1-\alpha)(1-\tilde{\tau}_l) + (1-\lambda)\alpha(1-\tilde{\tau}_k - s_1)}, \quad (196)$$

$$l_t = \bar{l} = \frac{\lambda(1-\alpha)(1-\tau_l)(1-\delta+\beta\delta)}{(1-\alpha)(1-\tau_l)(1-\delta+\beta\delta) + \alpha(1-\lambda)(1-\delta)(1-\tau_k)} \text{ for } t \geq 2. \quad (197)$$

Competitive equilibrium in a finite-time period,  $T$  can be written as

$$k_1 = \frac{\beta \sum_{i=1}^T \delta^i}{\lambda + \beta \sum_{i=1}^T \delta^i} (R_0k_0 + W_0l_0) - \frac{\lambda}{\lambda + \beta \sum_{i=1}^T \delta^i} \sum_{i=1}^T \frac{W_i}{\prod_{j=1}^i R_j}. \quad (198)$$

The first term in the right-hand side is converging to  $\frac{\beta\delta}{\lambda(1-\delta)+\beta\delta} (R_0k_0 + W_0l_0)$ . For the second term of the right-hand side, notice that

$$\sum_{i=1}^T \frac{W_i}{\Pi_{j=1}^i R_j} = \frac{W_1}{R_1} + \sum_{i=2}^T \frac{W_i}{\Pi_{j=2}^i R_j}.$$

Since  $\frac{W_1}{R_1} = \frac{(1-\tilde{\tau}_l)w_1}{(1-\tilde{\tau}_k)r_1} = \frac{(1-\tilde{\tau}_l)}{(1-\tilde{\tau}_k)} \frac{1-\alpha}{\alpha} \frac{k_1}{l_1}$  and  $\frac{W_i}{\Pi_{j=2}^i R_j} = \frac{(1-\tau_l)}{(1-\tau_k)^{i-1}(1-\tilde{\tau}_k)} \frac{1-\alpha}{\alpha} \frac{s_2^{i-2}s_1k_1}{l}$ ,

$$\sum_{i=1}^{\infty} \frac{W_i}{\Pi_{j=t+1}^i R_j} = \frac{(1-\tilde{\tau}_l)}{(1-\tilde{\tau}_k)} \frac{1-\alpha}{\alpha} \frac{k_1}{l_1} + \frac{1-\alpha}{\alpha} \frac{(1-\tau_l)}{(1-\tilde{\tau}_k)} \frac{s_1}{1-\tau_k-s_2} \frac{k_1}{l}. \quad (199)$$

By putting (196) and (197) into (199), (198) can be simplified in the infinite-time period as the equation in proposition 13.

**Derivation of (154)** For convenience, let  $(1-\alpha)(1-\tau_l) = \theta_l$ ,  $\alpha(1-\tau_k) = \theta_k$ , making constraints from (150) to (152) simpler:

$$c_t = \left( \theta_l + \frac{1-\delta}{1-\delta+\beta\delta} \theta_k \right) (1-\theta_l-\theta_k)^{(1-\alpha)/\alpha} l_t^{(1-\alpha)/\alpha} k_t,$$

$$l_t = \frac{\lambda(1-\delta+\beta\delta)\theta_l}{(1-\delta+\beta\delta)\theta_l + (1-\lambda)(1-\delta)\theta_k},$$

$$k_{t+1} = \frac{\beta\delta}{1-\delta+\beta\delta} \theta_k (1-\theta_l-\theta_k)^{(1-\alpha)/\alpha} l_t^{(1-\alpha)/\alpha} k_t.$$

Then we can obtain

$$\begin{aligned} \ln c_t &= \ln \left( \theta_l + \frac{1-\delta}{1-\delta+\beta\delta} \theta_k \right) + \frac{1-\alpha}{\alpha} \ln(1-\theta_l-\theta_k) \\ &\quad + \frac{1-\alpha}{\alpha} \ln \frac{\lambda(1-\delta+\beta\delta)\theta_l}{(1-\delta+\beta\delta)\theta_l + (1-\lambda)(1-\delta)\theta_k} + \ln k_t. \end{aligned}$$

Let  $S = \sum_{t=1}^{\infty} \delta^t \ln k_t$ .

$$\begin{aligned} S &= \frac{\delta}{1-\delta} \left( \ln \frac{\beta\delta}{1-\delta+\beta\delta} \theta_k (1-\theta_l-\theta_k)^{(1-\alpha)/\alpha} + \frac{1-\alpha}{\alpha} \ln \frac{\lambda(1-\delta+\beta\delta)\theta_l}{(1-\delta+\beta\delta)\theta_l + (1-\lambda)(1-\delta)\theta_k} \right) + \delta \ln k_0 + \delta S \\ &= \frac{\delta}{(1-\delta)^2} \left( \ln \theta_k + \frac{1-\alpha}{\alpha} \ln(1-\theta_l-\theta_k) + \frac{1-\alpha}{\alpha} \ln \frac{(1-\delta+\beta\delta)\theta_l}{(1-\delta+\beta\delta)\theta_l + (1-\lambda)(1-\delta)\theta_k} \right) + \Gamma \end{aligned}$$

$U$  has two parts, one from consumption ( $U^c$ ) and the other from leisure ( $U^l$ ).

Since  $U^c = \ln c_t + \beta \sum_{i=t+1}^{\infty} \delta^{i-t} \ln c_i$ ,

$$U^c = \frac{1-\delta+\beta\delta}{1-\delta} \left( \ln \left( \theta_l + \frac{1-\delta}{1-\delta+\beta\delta} \theta_k \right) + \frac{1-\alpha}{\alpha} \ln (1-\theta_l-\theta_k) + \frac{1-\alpha}{\alpha} \ln \frac{(1-\delta+\beta\delta)\theta_l}{(1-\delta+\beta\delta)\theta_l+(1-\lambda)(1-\delta)\theta_k} \right) + \frac{\beta\delta}{(1-\delta)^2} \left( \ln \theta_k + \frac{1-\alpha}{\alpha} \ln (1-\theta_l-\theta_k) + \frac{1-\alpha}{\alpha} \ln \frac{\lambda(1-\delta+\beta\delta)\theta_l}{(1-\delta+\beta\delta)\theta_l+(1-\lambda)(1-\delta)\theta_k} \right) + \Gamma'$$

Since  $U^l = \ln(1-l_0) + \beta \sum_{i=t+1}^{\infty} \delta^{i-t} \ln(1-l_t)$ ,

$$U^l = \frac{1-\delta+\beta\delta}{1-\delta} \ln \frac{(1-\delta+\beta\delta)\theta_l + (1-\delta)\theta_k}{(1-\delta+\beta\delta)\theta_l + (1-\lambda)(1-\delta)\theta_k} + \Lambda.$$

**Derivation of (160)** The current government's problem can now be defined as

$$\max_{\theta_l, \theta_k} U_0 = u(c_0, 1-l_0) + \beta \sum_{t=1}^{\infty} \delta^t u(c_t, 1-l_t),$$

subject to

$$k_1 = s_0 \alpha (\tilde{\theta}_l + \tilde{\theta}_k) (1 - \tilde{\theta}_l - \tilde{\theta}_k)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} l_0^{\frac{1-\alpha}{\alpha}} k_0,$$

$$\begin{aligned} c_0 &= (\tilde{\theta}_l + \tilde{\theta}_k) (1 - \tilde{\theta}_l - \tilde{\theta}_k)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} l_0^{\frac{1-\alpha}{\alpha}} k_0 - k_1 \\ &= (1 - s_0 \alpha) (\tilde{\theta}_l + \tilde{\theta}_k) (1 - \tilde{\theta}_l - \tilde{\theta}_k)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} l_0^{\frac{1-\alpha}{\alpha}} k_0 \end{aligned}$$

$$l_0 = \frac{\lambda \tilde{\theta}_l}{\tilde{\theta}_l + (1-\lambda)\tilde{\theta}_k - s_0 \alpha (1-\lambda)(\tilde{\theta}_l + \tilde{\theta}_k)},$$

$$k_{t+1} = \frac{\beta\delta}{1-\delta+\beta\delta} \theta_k (1-\theta_l-\theta_k)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} l_t^{\frac{1-\alpha}{\alpha}} k_t \text{ for } t \geq 1,$$

$$c_t = \left( \theta_l + \frac{1-\delta}{1-\delta+\beta\delta} \theta_k \right) (1-\theta_l-\theta_k)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} l_t^{\frac{1-\alpha}{\alpha}} k_t \text{ for } t \geq 1,$$

$$l_t = \frac{\lambda \theta_l (1-\delta+\beta\delta)}{\theta_l (1-\delta+\beta\delta) + (1-\lambda)(1-\delta)\theta_k} \text{ for } t \geq 1,$$

where  $s_0 \alpha = \frac{\beta\delta}{1-\delta+\beta\delta} \frac{\theta_k}{\theta_l+\theta_k}$ .

Since for  $t \geq 1$

$$\begin{aligned}
\ln c_t &= \ln k_t + A_t \\
&= \ln k_1 + A'_t \\
&= \ln(\tilde{\theta}_l + \tilde{\theta}_k) + \frac{1-\alpha}{\alpha}(\ln(1 - \tilde{\theta}_l - \tilde{\theta}_k) + \ln l_0) + A''_t, \\
\ln k_t &= \ln k_{t-1} + B_t = \ln k_1 + B'_t,
\end{aligned}$$

where  $A_t, A'_t, A''_t$  and  $B_t, B'_t$  are collections of terms not including  $\tilde{\theta}_l, \tilde{\theta}_k$ . Using this fact

$$\begin{aligned}
U^c &= \ln c_0 + \beta \sum_{t=1}^{\infty} \delta^t \ln c_t \\
&= \left( \ln(\tilde{\theta}_l + \tilde{\theta}_k) + \frac{1-\alpha}{\alpha}(\ln(1 - \tilde{\theta}_l - \tilde{\theta}_k) + \ln l_0) \right) \left(1 + \frac{\beta\delta}{1-\delta}\right) + \Gamma. \\
&= \left( \begin{array}{l} \ln(\tilde{\theta}_l + \tilde{\theta}_k) + \frac{1-\alpha}{\alpha}(\ln(1 - \tilde{\theta}_l - \tilde{\theta}_k) + \\ \ln\left(\frac{\tilde{\theta}_l}{\tilde{\theta}_l + (1-\lambda)\tilde{\theta}_k - s_0\alpha(1-\lambda)(\tilde{\theta}_l + \tilde{\theta}_k)}\right) \end{array} \right) \left(1 + \frac{\beta\delta}{1-\delta}\right) + \Gamma,
\end{aligned}$$

$$\begin{aligned}
U^l &= \ln(1 - l_0) + \Theta \\
&= \ln\left(\frac{\tilde{\theta}_l + \tilde{\theta}_k}{\tilde{\theta}_l + (1-\lambda)\tilde{\theta}_k - s_0\alpha(1-\lambda)(\tilde{\theta}_l + \tilde{\theta}_k)}\right) + \Delta,
\end{aligned}$$

where  $\Gamma$  and  $\Delta$  are collections of terms not including  $\tilde{\theta}_l, \tilde{\theta}_k$ . Then we can obtain the expression of (160) from

$$U_0 = \lambda U^c + (1 - \lambda)U^l.$$

**Derivation of (167) and (168)** Notice that  $U^c = \ln c_0 + \beta \sum_{t=1}^{\infty} \delta^t \ln c_t$ . From the competitive equilibrium, (161) to (166),

$$\begin{aligned}
c_0 &= \left(\frac{\mu\tilde{\theta}_l + (v - \beta\delta)\tilde{\theta}_k}{\mu\tilde{\theta}_l + v\tilde{\theta}_k}\right)(\theta_l + \theta_k)(1 - \theta_l - \theta_k)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} l_0^{\frac{1-\alpha}{\alpha}} k_0 \\
c_1 &= \frac{(1 - \delta + \beta\delta)\theta_l + (1 - \delta)\theta_k}{(1 - \delta + \beta\delta)(\theta_l + \theta_k)} (\tilde{\theta}_l + \tilde{\theta}_k)(1 - \tilde{\theta}_l - \tilde{\theta}_k)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} l_1^{\frac{1-\alpha}{\alpha}} k_1, \\
c_t &= \frac{(1 - \delta + \beta\delta)\theta_l + (1 - \delta)\theta_k}{1 - \delta + \beta\delta} (1 - \theta_l - \theta_k)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} l_t^{\frac{1-\alpha}{\alpha}} k_t \text{ for } t \geq 2.
\end{aligned}$$

Then

$$\begin{aligned}
U^c &= \ln\left(\frac{\mu\tilde{\theta}_l + (v - \beta\delta)\tilde{\theta}_k}{\mu\tilde{\theta}_l + v\tilde{\theta}_k}\right) + \beta\delta \ln(\tilde{\theta}_l + \tilde{\theta}_k) + \frac{1 - \alpha}{\alpha} \ln(1 - \tilde{\theta}_l - \tilde{\theta}_k) + \\
&\quad \frac{1 - \alpha}{\alpha} (\ln l_0 + \beta\delta \ln l_1) + \beta \sum_{t=1}^{\infty} \delta^t \ln k_t + \Gamma,
\end{aligned} \tag{200}$$

where  $\Gamma$  is a collection of terms not including  $\tilde{\theta}_l$  and  $\tilde{\theta}_k$ . Since

$$\begin{aligned}
\delta \ln k_1 &= \delta \ln \tilde{\theta}_k - \delta \ln(\mu\tilde{\theta}_l + v\tilde{\theta}_k) + \delta \frac{1 - \alpha}{\alpha} \ln l_0 + \delta \ln k_0 + Z_1, \\
\delta^2 \ln k_2 &= \delta^2 \ln(\tilde{\theta}_l + \tilde{\theta}_k) + \delta^2 \frac{1 - \alpha}{\alpha} \ln(1 - \tilde{\theta}_l + \tilde{\theta}_k) + \delta^2 \frac{1 - \alpha}{\alpha} \ln l_1 + \delta^2 \ln k_1 + Z_2, \\
\delta^3 \ln k_t &= \delta^3 \ln k_{t-1} + Z_3 \text{ for } t \geq 3,
\end{aligned}$$

$$\begin{aligned}
\sum \delta^t \ln k_t &= \frac{\delta}{1 - \delta} \ln \tilde{\theta}_k - \frac{\delta}{1 - \delta} \ln(\mu\tilde{\theta}_l + v\tilde{\theta}_k) + \frac{\delta^2}{1 - \delta} \ln(\tilde{\theta}_l + \tilde{\theta}_k) \\
&\quad \frac{\delta}{1 - \delta} \frac{1 - \alpha}{\alpha} \ln l_0 + \frac{\delta^2}{1 - \delta} \frac{1 - \alpha}{\alpha} \ln l_1 + \Upsilon,
\end{aligned}$$

where  $\Upsilon$  is a collection of terms not including  $\tilde{\theta}_l$  and  $\tilde{\theta}_k$ . Plugging this expression into (200), we can obtain  $U^c$  in (167).

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