Riddling and invariance for discontinuous maps preserving Lebesgue measure*

Peter Ashwin, Xin-Chu Fu
School of Mathematical Sciences
University of Exeter
Exeter EX4 4QE, U.K.

John R. Terry
Department of Mathematics
University of Queensland
St. Lucia, Brisbane
Queensland, 4072, Australia

February 20, 2002

Abstract

In this paper we use the mixture of topological and measure-theoretic dynamical approaches to consider riddling of invariant sets for some discontinuous maps of compact regions of the plane that preserve two dimensional Lebesgue measure. We consider maps that are piecewise continuous and with invertible except on a closed zero measure set. We show that riddling is an invariant property that can be used to characterize invariant sets, and prove results that give a nontrivial decomposion of what we call partially riddled invariant sets into smaller invariant sets. For a particular example, a piecewise isometry that arises in signal processing (the overflow oscillation map) we present evidence that the closure of the set of trajectories that accumulate on the discontinuity is fully riddled. This supports a conjecture that there are typically an infinite number of periodic orbits for this system.

Keywords: Invariant Set, Riddling, Area-preserving Maps, Dynamical System, Piecewise Continuous.

*AMS Subject Classifications: 37B05, 37M99.

1 Introduction

As noted in the mid-1980s [9, 11] smooth area-preserving maps may possess chaotic regions that, to numerical approximation, have positive Lebesgue measure but are 'fat fractals' in the sense that their complements may be dense in phase space owing to the existence of dense sets of elliptic islands. In the terminology of [1, 6] we say these invariant sets are *riddled.*

In a recent paper [6] we develop the idea of riddling as a property of positive measure sets that are invariant under continuous mappings. This paper extends the ideas developed in [6] to show that they can be usefully applied to questions about invariant sets of a general class of discontinuous area preserving maps of the plane. We also give a new example to show that a partially riddled set may have an unriddled component that is not equal to its closure up to a zero measure set.

Measure theory in general (and ergodic theory in particular) are essential for a good understanding of invariant sets, and the ergodic theory of smooth maps is highly developed. Similarly, the theory of measure preserving transformations, iterated maps with no assumption other than the map is measurable is well-understood. The type of piecewise continuous map we consider in this paper sit between these two classes. The maps that satisfy (H1) preserve Lebesgue measure and are continuous on a full measure subset of phase space, but not everywhere.

Certain features of such maps that would be missed if they were viewed simply as measurable; these features involve using a combination of measure and topological notions. We use notions of almost invariant, almost open and almost closed and almost minimal sets in order to allow us to make simple, precise and intuitively enlightening statements about these sets, while still encapsulating the usual caveats about sets of zero measure.

For such maps trajectories may accumulate on discontinuities. Let D be the closure of the set of such trajectories. The Lebesgue measure of D, $\ell(D)$ may be zero, full or neither, depending on the map considered and may show a variety of topologies in its closure. The following example examines such a D for a simple planar map.

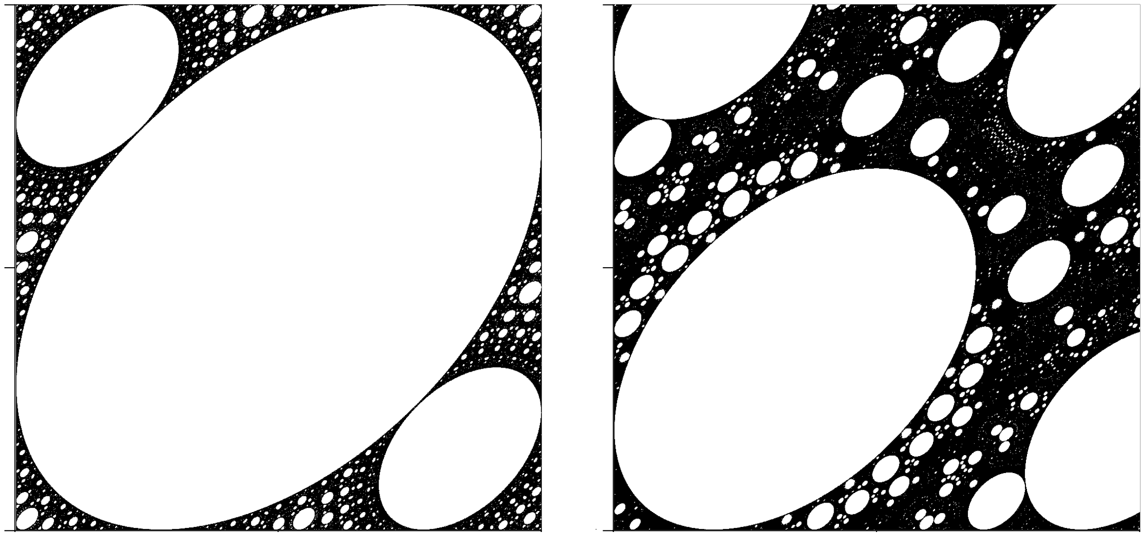
1.1 Riddling of D for the overflow oscillation map

Our example is an invertible piecewise isometry; such maps in two dimensions have been considered previously by Goetz and others [12, 13, 2, 3] or polygon exchange transformations (eg [15]), and there are several examples of such maps that arise in applications (See [7, 8, 10, 2, 3, 20]).

We consider the second order lossless digital filter map for two's complement overflow [17, 7, 3]; we refer to this as the *overflow oscillation map.* This is a map f on the phase space $M = [-1, 1) \times [-1, 1)$ defined by

$$f(x, y) = (y, g(bx + ay)). \tag{1}$$

The overflow is governed by the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x$ for $x \in [-1, 1)$ and $g(x+2) = g(x)$ for all x. For $a = 0.9$ and $b = -1$ the map preserves two dimensional



(a) (b)

Figure 1: (a) *Shows the region D of points that accumulate on the discontinuity for the map (1) for $a = 0.9$ in the square* $[-1, 1]^2$. (b) *Shows a detail of the bottom left corner of* (a)*: the square* $[-1, -\frac{7}{8}]^2$*. Numerical evidence in this paper suggests that the set D is fully riddled (see text).*

Lebesgue measure. Giving the phase space M the topology of the torus, there exists a single line of discontinuity at

$$\mathcal{D} = \{(-1, y) : y \in [-1, 1)\} \subseteq M.$$

Let D be the closure of the set of trajectories that accumulate on $\mathcal{D}$ and let $D^c = M \setminus D$. An approximation of the set D is illustrated in Figure 1 for the case $a = 0.9$. The paper [2] observes that all periodic points that do not hit $\mathcal{D}$ are in D^c. In fact D^c consists of a disjoint union of convex polygons that are mapped to themselves; we generalize this result in Theorem 3.2. In [3] it was conjectured that $\ell(D) > 0$ for typical values of a, i.e., for those that give rise to irrational roots of unity. In particular, Theorem 3.3 shows that if such an invariant set D is partially riddled then one can decompose this set into at least two positive measure invariant subsets, one of which is fully riddled and one of which is unriddled.

Note that on transforming the coordinates using a linear shear (see [8, 3]) one can view f as an irrational rearrangement on a rhombus with angle $\theta = \cos^{-1}(a/2)$. Figure 2 illustrates the piecewise isometry corresponding to the overflow map (1) in sheared coordinates [4].

1.2 The structure of the paper

The remainder of the paper is organized as follows: In Section 2.1 and the following subsection we adapt and generalize some definitions from [6] to this context and discuss almost open and almost closed sets. We give new examples of partial riddling that

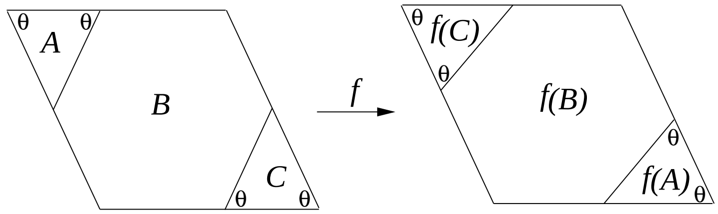


Figure 2: *The overflow oscillation map (1) is equivalent to this piecewise isometry after a linear change of coordinates. The letters A, B, C index the atoms of the partition on which the map acts simply as a rotation and translation in the plane. The angle θ is a parameter that determines both the isometries and the shape of the partition.*

demonstrate a distinction between what we term *proper* and *improper* partial riddling. For convenience we concentrate on two classes of maps; (H1) are maps that preserve Lebesgue measure and are invertible except on a zero measure set, and (H2) are piecewise isometries. Note that (H2) is a specialization of (H1). We discuss and relate the concepts of invariance, almost invariance and consequences of the Poincaré Recurrence Theorem in this context. Section 2.3 examines invariance of riddled and partially riddled sets for (H1) maps. In particular we show in Theorem 2.7 that a partially riddled almost invariant set cannot be almost minimal, i.e. there are closed almost invariant subsets that are neither empty nor equal to the whole set modulo a zero measure set.

Section 3 looks at the images and preimages of the discontinuity set for (H2) maps and in this context one can split the phase space into a disjoint union of sets depending on whether the trajectory through a point accumulates on the discontinuity set or not. In Theorem 3.2 we describe the structure of the set D and in Theorem 3.3 we show that if the set of points that accumulate on the discontinuity is partially riddled then there must be a positive measure set of initial conditions that do not have dense orbits in this set.

In Section 4 we return to the overflow oscillation map and give numerical evidence that the invariant set in Figure 1 is fully riddled. Finally Section 5 discusses the results and some open problems.

2 Piecewise continuous mappings

2.1 Definitions and Preliminaries

Throughout we assume that M is a region in $\mathbb{R}^n$, $\ell(.)$ is Lebesgue measure on M (with Borel σ-algebra $\mathcal{B}$) such that $\ell(M) > 0$. By region we mean a compact set that is the closure of its interior, with zero measure boundary. As in [6] we introduce the notion $=_0$ to mean equivalence of Borel sets up to zero measure, i.e. $U =_0 V$ means that $\ell(U \Delta V) = 0$, where $U \Delta V = (U \setminus V) \cup (V \setminus U)$ is the symmetric difference.

We say V is *almost open* if $V =_0 \mathrm{Int}(V)$, whereas V is *almost closed* if $V =_0 \overline{V}$. If V is a zero measure set, using the notion $=_0$ we may conveniently denote this by $V =_0 \emptyset$.

Suppose A is almost open. Since $A \Delta \mathrm{Int}(A) = A^c \Delta (\mathrm{Int}(A))^c$ we have $A^c =_0 \overline{A^c}$ and so A^c is almost closed. Conversely, if A is almost closed then its complement is almost open.

We use the following definitions for riddling of measurable sets in a way that generalizes those of Alexander *et al.* [1] and is equivalent to that in [6]. We write $B_\delta(x)$ to denote the δ-neighbourhood of a point x. Given any subset $V \subseteq M$ we define the set

$$\mathcal{L}(V) = \{x \in M \ : \ \ell(B_\delta(x) \cap V) > 0 \ \text{ for all } \delta > 0\}.$$

Observe that $\mathcal{L}(V) \subseteq \overline{V}$. Moreover, by Lebesgue density $V \subseteq_0 \mathcal{L}(V)$. It follows that if V is almost closed then $\mathcal{L}(V) =_0 V$.

Suppose that $V \subseteq M$ is a Borel set with positive measure. We define the *riddled component* of V to be

$$V_{rid} = \mathcal{L}(V) \cap \mathcal{L}(V^c) \cap V. \tag{2}$$

We refer to $V_{unrid} = V \setminus V_{rid}$ as the *unriddled component* of V.

If $V_{rid} =_0 V \neq_0 \emptyset$ then we say that the set V is *(fully) riddled*. If $V_{rid} =_0 \emptyset$ we say it is *unriddled*. If it is neither riddled nor unriddled, we say it is *partially riddled*. As shown in [6], V_{rid} is closed in the subset topology of V and V_{unrid} is open in the subset topology of V. We say that a partially riddled set V is *properly partially riddled* if it is partially riddled and V_{unrid} is almost closed.

Remark 2.1 *It is easy to construct examples of properly partially riddled sets. In fact, for any fully riddled set S_1 (such as a fat fractal), take an almost closed unriddled set S_2 with $S_1 \cap S_2 =_0 \emptyset$, and let $S = S_1 \cup S_2$, then S is a properly partially riddled set. The example of an attractor with a partially riddled basin given in [6] is an example of properly partially riddled set.*

Remark 2.2 *Construction of a partially riddled subset that is not properly partially riddled is a little more difficult. The simplest example in $[0,1]$ that we could find is constructed by considering a sequence $\epsilon_n > 0$ of numbers that sum to $\epsilon < 1/2$. Let $J_0 = [0,1]$ and $S_0 = [(1-\epsilon_0)/2, (1+\epsilon_0)/2]$ and inductively define the sets $J_n \supset S_n$ as follows* (See Figure 3). *For each interval in $J_n \setminus S_n$ we remove proper open subintervals of combined length at most ϵ_n at the nth step from the centre of each interval to give a new set J_{n+1} with triple the number of closed intervals. We pick new proper subintervals with combined length at most ϵ_n of the intervals of J_{n+1} which do not intersect S_n and construct S_{n+1} as the union of S_n and these subintervals. This gives a closed subset $J = \cap_n J_n$ of the interval such that $J_{unrid} = \cup_n Int(S_n)$ and so $\ell(J_{unrid}) \leq \epsilon$. Examining the complement we have $\ell([0,1] \setminus J) \geq 1 - \epsilon mal$, so J is partially riddled. Moreover, $\ell(\overline{J_{unrid}}) \geq 1 - \epsilon$, J_{unrid} is not almost closed and so J is not properly partially riddled.*

Suppose that $f : M \to M$ is measurable and let

$$B_f := \{y \in M, \#\{f^{-1}(y)\} \neq 1\}$$

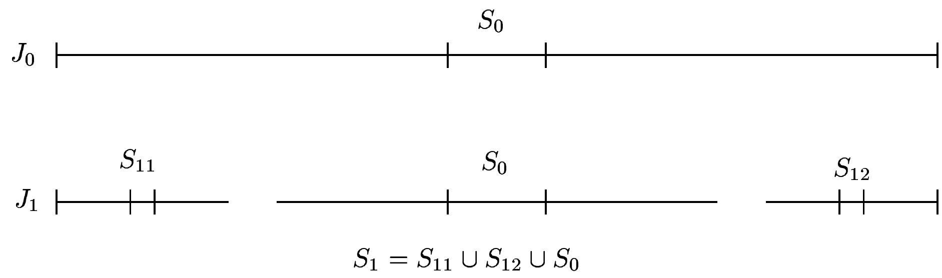


Figure 3: *Construction of a partially riddled subset in* $[0, 1]$ *that is not properly partially riddled (see Remark 2.2).*

where we use the convention that $\#\{f^{-1}(y)\} = 0$ if $y \notin f(M)$. As introduced in [6, 14], we say f is *almost invertible* if $B_f =_0 \emptyset$, i.e. if f is invertible on M except on a zero measure subset. If $f(A) \subseteq A \subseteq M$ we say f is *almost invertible on* A if and only if $B_F =_0 \emptyset$ where $F = f|_A$.

If $f(V) =_0 V$ we say V is *almost invariant* under f. In the stronger case that V is almost invariant and $f(V) \subseteq V$ we say V is *quasi-invariant* under f (as in [14]) (we include trivial cases with $V =_0 \emptyset$). In the case that $f(V) = V$ we say V is *invariant* under f.

Discontinuous maps and hypothesis (H1) We say a map (f, M) satisfies hypothesis **(H1)** if $f : M \to M$ is such that:

1. f is a Lebesgue nonsingular map that preserves Lebesgue measure.[1]
2. f is both continuous and invertible except on a closed zero measure set $\mathcal{D} \subset M$.

By hypothesis, such a map is almost invertible with $B_f \subseteq f(\mathcal{D})$. We refer to $\mathcal{D}$ as the *discontinuity set* of f, though our requirement that it is closed means that it may contain points of continuity.

Discontinuous maps and hypothesis (H2) A map (f, M) is said to satisfy **(H2)**, or be a *piecewise isometry* if it is almost invertible and there is a finite partition of the phase space $M = \bigcup_{k=0}^{n} \overline{M_k}$ into open convex cells $\{M_k\}_{k=1}^{n}$ such that $f|_{M_k} = f_k$ is an isometry for each k.

2.2 Invariance for (H1) maps

The assumptions (H1) imply invariance of Lebesgue measure under both forward and backwards iterates, and so we relate the notions of invariance, almost invariance and quasi-invariance for these maps.

Lemma 2.1 *Suppose* (f, M) *satisfies (H1) and* $A \in \mathcal{B}$.

[1]That is, $V =_0 \emptyset$ if and only if $f(V) =_0 \emptyset$ and $\ell(f^{-1}(V)) = \ell(V)$ for all $V \in \mathcal{B}$.

1. *$\ell(A)=\ell(f^n(A))$ for all $n\in\mathbb{Z}$.*

2. *If A is almost invariant then there is a subset $A_1\subseteq A$, $A_1=_0 A$ such that A_1 is quasi-invariant.*

3. *If A is quasi-invariant then there is a subset $A_1\subseteq A$, $A_1=_0 A$ such that A_1 is invariant and $f|_{A_1}$ is invertible.*

Proof:

1. We only need to show invariance under one forward iterate by the map f. Consider $A'=A\cap\mathcal{D}^c$ and note that $A'=_0 A$. Since M has compact and has finite measure, $f(\mathcal{D}^c)^c=_0\emptyset$. Then we have

$$A=_0 A'\subseteq f^{-1}(f(A))\subseteq A'\cup\mathcal{D}\cap f^{-1}(f(\mathcal{D}^c)^c)=_0 A$$

and hence $\ell(f(A))=\ell(A)$; the result follows.

2. For any almost invariant set A of f consider the subset

$$A_1=A\setminus\bigcup_{k\geq 1}f^{-k}(f(A)\setminus A).$$

then A_1 is forward invariant, and by 1., $f(A_1)=_0 A_1$.

3. Suppose now that $A\subseteq M$ is quasi-invariant and so f is almost invertible on A. Let $A_1=A\setminus B$ with

$$B=\bigcup_{k\geq 0}\bigcup_{l\geq 0}f^{-k}f^{l}(B_{f|_A}),$$

the set of all points that hit $B_{f|_A}$ under forward or backward iteration (or both). Then f is invertible on A_1 and $f(A_1)=A_1$. □

Now denote the orbit through x by $\mathrm{orb}(x)=\{f^n(x):n\in\mathbb{Z}\}$, the forward orbit $\mathrm{orb}_+(x)=\{f^n(x):n=0,1,2,\cdots\}$ and the ω-limit by

$$\omega(x)=\bigcap_{k\geq 0}\overline{\mathrm{orb}_+(f^k(x))}.$$

Observe that although the orbit is always a countable union of zero-dimensional points, the closure of the orbit can have arbitrarily high dimension. We now recall a classical result of Poincaré (see Petersen [19] for a proof). We apply this to maps satisfying (H1).

Proposition 2.2 (Poincaré Recurrence) *Suppose that f is a measure preserving tranformation and A is a subset with non-zero measure. Then there is a full measure set of $x\in A$ that are recurrent, ie. such that $x\in\omega(x)$.*

A consequence of this is that any attractors in the sense of Milnor [18] will be equal to their basins of attraction up to a set of zero measure. Namely, we have the following:

Lemma 2.3 *Suppose that $A \subseteq M$ is closed and f-invariant for (f, M) satisfying (H1). Then*

$$\{x \in M : \ \omega(x) \subset A\} =_0 A.$$

Proof: Let $B = \{x \in M : \ \omega(x) \subset A\}$; clearly $A \subseteq B$. Suppose $B \neq_0 A$, then by Poincaré recurrence, almost all points of $B \setminus A$ return infinitely often to $B \setminus A$ under iteration of f. So there exist points in B for whom some ω-limit points are outside A. This is a contradiction, and hence $B =_0 A$. □

Remark 2.3 *From Lemma 2.3, every closed f-invariant set A with $\ell(A) > 0$ is a weak attractor in the sense of [6].*

Note that although Poincaré recurrence holds for measure preserving maps that may be discontinuous everywhere, assumptions (H1) mean that ω-limit sets can have invariance properties in following sense.

Lemma 2.4 *If (f, M) satisfies (H1) then for any x, $\omega(x)$ is almost invariant.*

Proof: Note that if $\omega(x)$ is a zero measure set then it is trivially almost invariant. Hence we only need to consider $\omega(x)$ with positive measure. For any point of continuity y for the map, if $y \in \omega(x)$ then $f(y) \in \omega(x)$. To see this, consider any such x and y; we can find a sequence $\{n_k\}$ such that $d(f^{n_k}(x), y) \to 0$. Continuity means that $d(f^{n_k+1}(x), f(y)) \to 0$ and the result follows. □

2.3 Partial riddling and invariance

We now generalize a result of [6] to give an invariant decomposition of riddled invariant sets for f satisfying (H1).

Lemma 2.5 *Suppose that V is a partially riddled set that is almost invariant for (f, M) satisfying (H1). Then V_{rid} and V_{unrid} are almost invariant.*

Proof: Because the set of continuous points is open and has full measure, almost every $x \in V$ lies in some open (convex) $U_1 \subset M$ on which f is a homeomorphism onto its image $f(U_1) = U_2$. Consider any $\delta > 0$ such that $B_\delta(x) \subseteq U_1$ and $B_\delta(f(x)) \subset U_2$. By continuity of f on U_1 we can find an ε with $\delta > \varepsilon > 0$ such that $f(B_\varepsilon(x)) \subseteq B_\delta(f(x))$.

As $f : U_1 \to U_2$ is invertible we have $f(A \cap B) = f(A) \cap f(B)$ for any $A, B \subseteq U_1$. Now $f(V \cap U_1) = f(V \cap U_1) \cap U_2 =_0 V \cap U_2$ because V is almost invariant. Hence its complement in U_1 is almost invariant, i.e. $f(V^c \cap U_1) =_0 V^c \cap U_2$.

Suppose that $\ell(B_\delta(f(x)) \cap V^c) = 0$ and note that $\ell(f(B_\varepsilon(x) \cap V^c)) = \ell(f(B_\varepsilon(x)) \cap V^c) \leq \ell(B_\delta(f(x)) \cap V^c)$. Therefore $\ell(f(B_\varepsilon(x) \cap V^c)) = 0$ and since f is nonsingular and preserves measure we have $\ell(B_\varepsilon(x) \cap V^c) = 0$.

Considering $f^{-1} : U_2 \to U_1$ this is also a homeomorphism that preserves measure and so for any $\delta > 0$ such that $\ell(B_\delta(x) \cap V^c) = 0$ and $B_\delta(x) \subseteq U_1$ we can find an $\varepsilon > 0$ such that $\ell(B_\varepsilon(f(x) \cap V^c)) = 0$.

Hence, for almost all $x \in V$, $\ell(B_\delta(x) \cap V^c) = 0$ for some $\delta > 0$ if and only if $\ell(B_\epsilon(f(x)) \cap V^c) = 0$ for some $\epsilon > 0$. This is equivalent to saying that V_{rid} and V_{unrid} are almost invariant under f. □

Almost minimal sets Suppose that (f, M) satisfies (H1). The following definition generalizes the concept of a minimal Milnor attractor for such systems.

Definition 2.1 *Suppose that $S \subseteq M$ is compact, almost invariant, and not a zero measure set. We say S is an* almost minimal set *if for any T that is a compact, almost invariant subset of S we have either $T =_0 \emptyset$ or $T =_0 S$.*

We remark here that if A is almost minimal set this does not mean that $\ell|_A$ is ergodic; see for example [16] for an example from interval exchange maps. However, if S is an almost invariant set on which $\ell|_S$ is ergodic for f then S is almost minimal. Note that a compact almost invariant set need not have any almost minimal subsets, just as a Milnor attractor need not contain any minimal Milnor attractors; for example the identity map on a compact positive measure set contains no almost minimal subsets.

Lemma 2.6 *Suppose that $S \neq_0 \emptyset$ is compact and almost invariant, and for almost all $x \in S$, $S = \overline{orb(x)}$. Then S is almost minimal.*

Proof: First suppose that $S =_0 \overline{\mathrm{orb}(x)}$ for almost all $x \in S$. Let $S_1 \subseteq S$ be any almost invariant subset with positive measure. Note that there is an $x \in S_1$ such that $\overline{\mathrm{orb}(x)} = S$ and so $\overline{\mathrm{orb}(x)} \subseteq_0 S_1$. This implies $S_1 =_0 S$. □

Remark 2.4 *Note that the converse of Lemma 2.6 holds for continuous maps on dropping the 'almost'. However, we do not believe the converse is true without these changes. We do not believe that (H1) is an optimal assumption for Lemma 2.6 to hold.*

The following result relates to invariance of partially riddled sets.

Theorem 2.7 *Suppose that f satisfies (H1) and V is a partially riddled set:*

1. *If V is compact then it is not almost minimal.*

2. *If V is an almost closed f-invariant set that is properly partially riddled then the set of points in V with orbits dense in V has zero measure.*

Proof: We apply Lemma 2.5 to find a decomposition (up to full measure) $V =_0 V_1 \cup V_2$ such that V_1 is fully riddled and V_2 is unriddled, and each of these is invariant. For part 1, note that V_{rid} and V_{unrid} are both almost invariant. Therefore one can find full measure subsets V_1 and V_2 respectively that are invariant (however these are not necessarily closed).

For part 2, let $\tilde{V} = \{x \in V : \omega(x) =_0 V\}$; we now show that this set has zero measure. By the proof of part 1. we have full measure subsets $V_1 \subseteq V_{rid}$ and $V_2 \subseteq V_{unrid}$ that are invariant. Because V_{rid} is closed in V there is a closed C such that $V_{rid} = C \cap V$ so

$$\overline{V_{rid}} \subseteq C \cap \overline{V} =_0 C \cap V = V_{rid}$$

and hence V_{rid} is almost closed. Then we have

$$\overline{V_1} \subseteq \overline{V}_{rid} =_0 V_{rid} =_0 V_1$$

and so V_1, and similarly V_2 are almost closed. If $x \in V_k$ then $\omega(x) \subseteq \overline{V_k} =_0 V_k$, $k = 1, 2$. these imply that $x \notin \tilde{V}$ and hence

$$\tilde{V} \subset V \setminus (V_1 \cup V_2) =_0 \emptyset.$$

□

3 Properties of the discontinuity set for maps satisfying (H2)

For all maps f satisfying (H2) we define the set of points of (possible) discontinuity under iteration as $\mathcal{D} = \bigcup_{k=0}^{n} \partial M_k$ and the set of points of continuity as $\mathcal{C} = \bigcup_{k=0}^{n} M_k$. The set of points that avoid $\mathcal{D}$ on both forwards and backwards iteration under f we write as

$$\hat{M} = \bigcap_{i=-\infty}^{\infty} f^i(\mathcal{C}).$$

Note that $f(\hat{M}) \subseteq \hat{M}$; it is almost invariant and $\ell(\hat{M}) = \ell(M)$ because the complement of $\hat{M}$ is a countable union of zero measure sets. Define

$$D^{\pm} = \bigcup_{k \geq 0} \bigcup_{l \geq 0} f^{-k} f^{l}(\mathcal{D})$$

to be the set of points whose trajectories land on and/or originate on the discontinuity. Its closure

$$D = \overline{D^{\pm}} \tag{3}$$

may have positive measure even though $D^{\pm}$ is a measure zero set (cf. Figure 1). The properties of this set are the subject of a number of conjectures; see [3]. Recently it has been shown [4] that for the overflow map (1), D^c consists of a union of disks that possess no tangencies, for all except countably many values of θ.

Lemma 3.1 *Suppose (f, M) satisfies (H2). Then the set D defined above is almost invariant. Moreover, if S is an almost minimal set, then either $S \subseteq_0 D^c$ or $S \subseteq_0 D$.*

Proof: To see this, consider a point $x \in D$ where f is continuous (there is a full measure set where this holds). There is a sequence $x_n \in D^{\pm}$ with $x_n \to x$ as $n \to \infty$. Continuity at x means that $f(x_n) \to f(x)$ as $n \to \infty$ and invariance of $D^{\pm}$ implies that $f(x) \in D$. Hence D is almost invariant.

For the second statement, note that D is almost invariant and closed. Let $S_1 = S \cap D$; since f is a piecewise homeomorphism, $f(A \cap B) =_0 f(A) \cap f(B)$ for any subsets of M. So $S_1 \subseteq S$ is closed and almost invariant. Therefore either $S_1 =_0 \emptyset$ or $S_1 =_0 S$, i.e., either $S \subseteq_0 D^c$ or $S \subseteq_0 D$. □

Remark 3.1 *For (H2) maps, if $D =_0 \emptyset$ then the dynamics of the system is distal in the sense that for almost all $x, y \in D^c$ we have*

$$\liminf_{n \to \infty} |f^n(x) - f^n(y)| > 0.$$

In fact, $\forall x \notin D$, $\exists \delta > 0$ $\forall y \in B_\delta(x)$ with $y \neq x$, $\forall n \in \mathbb{Z}$,

$$|f^n(x) - f^n(y)| = |x - y| > 0.$$

Remark 3.2 *If (f, M) satisfies (H2) then for almost all $x \in M$ and all $N > 0$ there exists a $\delta > 0$ such that for any $x_1, x_2 \in B_\delta(x)$ we have*

$$d(f^k(x_1), f^k(x_2)) = d(x_1, x_2), \ \ |k| \leq N;$$

moreover, for all $x \in D^c$ there is a $\delta > 0$ such that

$$d(f^n(x_1), f^n(x_2)) = d(x_1, x_2)$$

for every $n \in \mathbb{Z}$ and $x_1, x_2 \in B_\delta(x)$. This can be noted by remarking that for $x \notin D^{\pm}$, $N > 0$, we can take

$$\delta = \min\{d(f^k(x), \mathcal{D}), |k| \leq N - 1\}.$$

For $x \in D^c$, we can take

$$\delta = \inf_n d(f^n(x), \mathcal{D}).$$

Irrational rearrangements Our final specialization is to consider maps f of regions $M \subset \mathbb{R}^2$ that satisfy (H2), where the isometry can be written

$$f_k(x) = e^{i\theta} x + b_k$$

such that $x \in \mathbf{C}$, where θ/π is an irrational number independent of k and $b_k \in \mathbf{C}$.

The following result for (H2) maps on subsets of the plane characterizes D^c and is essentially contained in [2, 3, 12, 13, 17] for specific examples. We say a trajectory is *quasiperiodic* if some iterate of the map is topologically conjugate to an irrational rotation on a circle.

Theorem 3.2 *If (f, M) satisfies (H2) with M a compact region in $\mathbb{R}^2$, then D^c is a countable disjoint union of invariant convex regions $\{P_n\}_{n\in\mathbb{N}}$,*

$$D^c = \bigcup_{n\in\mathbb{N}} P_n$$

such that all points in P_n are periodic or quasiperiodic. If in addition (f, M) is an irrational rearrangement then P_n is a disk with a periodic point at its center that is surrounded by nested invariant circles with quasiperiodic dynamics.

Proof: If $x \in D^c$ then $\inf_n d(f^n(x), \mathcal{D}) = \lambda > 0$, and so no point in $f^n(B_\lambda(x))$ hits the discontinuity at any point. Now consider the largest region N such that $y \in N$ implies that $f^n(y)$ and $f^n(x)$ are in the same M_{k_n} for all $n \in \mathbb{Z}$. This is a convex region that contains $B_\lambda(x)$ and so has positive measure. Now consider $f^p(N)$ and use the maximality of N we have either $f^p(N) = N$ or $f^p(N) \cap N = \emptyset$. Since M is compact we have that there must be a smallest $0 < p \leq \frac{\ell(M)}{\ell(N)}$ such that $f^p(N) = N$. Since $f^p|_N$ is now an isometry on N, so $f^p : N \to N$ is a reflection and/or solid rotation and/or translation. By Brouwer's Fixed Point Theorem, there exists an $a \in N$ such that $f^p(a) = a$. $\forall x \in N$, $|f^{kp}(x) - a| = |x - a|$, so x is a quasiperiodic or periodic point. Since each region N has positive measure, the number of such regions must be countable.

For the last part, since $f^p|_N$ is now just an irrational rotation leaving the centre of N fixed we have that N must be a disk with a period p point at its centre surrounded by invariant circles with rotation number $p\theta$. Thus each $x \in D^c$ lies in some invariant disk as described. $\square$

Remark 3.3 *As a consequence of the previous theorem, if (f, M) satisfies (H2) for M a compact region in $\mathbb{R}^2$ and S is an almost minimal set of f then $S \subseteq_0 D$. Note that all orbits in D^c are periodic or quasiperiodic and hence all ω-limit sets in D^c have zero measure.*

Remark 3.4 *Theorem 3.2 can be adapted to noninvertible piecewise isometries, as long as the piecewise rotation can be reduced to an invertible piecewise rotation on a bounded global attractor ([5]).*

The following Theorem relates riddling of D in this context to some dynamical properties.

Theorem 3.3

1. *If f is an irrational rearrangement with only finitely many periodic orbits in D^c then $D_{rid} =_0 \emptyset$ and $D_{unrid} \neq_0 \emptyset$.*
2. *On the other hand, if D is partially riddled then it is not almost minimal.*

Proof: If there are finitely many periodic orbits, because by Theorem 3.2, D^c is a finite union of disks and so D is an almost open region with positive measure. If D is partially riddled we apply Lemma 2.5 to show that it is not almost minimal. □

In the following section we present numerical evidence that D is in fact fully riddled, at least for typical parameter values for the lossless overflow oscillation map.

4 Numerical approximation of D for the overflow oscillation map

We now return to our motivating example of the overflow map (1). Under a linear change of coordinates this can be seen as an irrational rearrangement. We examine this mapping numerically by considering a finite grid approximation of M, $M_{(n)}$, a $2^n \times 2^n$ regular discretization of the phase space. We approximate the line of discontinuity $\mathcal{D}$ by $\mathcal{D}_{(n)}$, the set of points which lie on the left hand edge of $M_{(n)}$. We approximate $\Phi(a) := \ell(D)$ by on $M_{(n)}$ by $\Phi_{(n)}(a)$.

This approximation of D is computed as in [3], with the value of the parameter $a = 0.9$. We consider values of the resolution up to $n = 12$ and observe an apparent positive measure set of points belonging to D in each case. For increasing resolution, smaller and smaller periodic islands can be observed which suggest the conjecture that D is fully riddled. To approach this we approximate the riddled component numerically.

Definition 4.1 *Let V be a positive measure subset of $\mathbb{R}^n$. We define the* δ-riddled component *of V, $V_{rid}(\delta)$ to be the set of points $x \in V$ such that*

$$\ell(B_\delta(x) \cap V)\ell(B_\delta(x) \cap V^c) > 0.$$

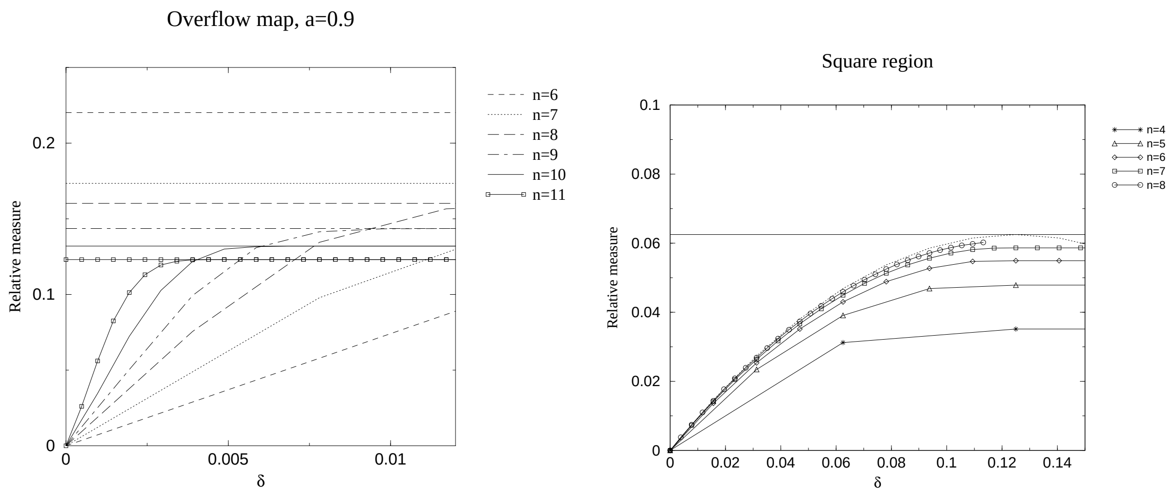
Note that

$$V_{rid} = \bigcap_{\delta>0} V_{rid}(\delta)$$

and hence the notation. We apply this notion of δ-riddling for the discretized phase space $M_{(n)}$ as follows. For each point a_{ij} belonging to D we consider a square matrix centred on the point a_{ij} of width k units, i.e. a discretized neighbourhood of the point a_{ij}. We then examine this submatrix to see if it contains any points belonging to D^c. If this was the case we assume that the point a_{ij} belonged to the $D_{rid}(n,k)$ (i.e. a discretized $D_{rid}(\delta)$ for a particular resolution n and neighbourhood k).

We then examine the proportion of points belonging to $D_{rid}(\delta)$ relative to the number of points belonging to $M_{(n)}$. For each resolution n considered, we repeat this procedure for increasing k until such a δ-neighbourhood about each point a_{ij} was achieved that all points belonging to D also belonged to the set $D_{rid}(\delta)$.

For the black set in Figure 1, the proportion of points that appear to belong to the 'riddled component' for equivalent neighbourhoods of D (*i. e.* doubling the size of k when increasing the resolution n) increases, as the resolution is increased. This is an indicator that D is riddled.



(a) (b)

Figure 4: (a): *Dependence of the measure of the δ-riddled component of the set D on the discretization n and δ. The relative measure is $\ell(D_{rid})/\ell(D)$. The horizontal lines indicate the measure of D computed at the increasing resolutions n. The y-value of this line converges to a positive constant (ie $\ell(D)$) as n becomes infinite. The curves starting at the origin correspond (from left to right) to decreasing n and give the measure of the δ-riddled subset of D at that level of approximation. Note that this curve hits the approximation of $\ell(D)$ at the first value of δ_1 such that there are no neighbourhoods of size δ_1 are contained in D. The curves show that on increasing the resolution of the approximation, the set appears riddled at smaller values of δ. This supports the conjecture that D is riddled.* (b): *In contrast with* (a) *we show the dependence of $\ell(S_{rid})/\ell(S)$ for a square S of area $1/16$ of the phase space on the discretization n and δ. The horizontal line indicates the measure of this square and approximations are computed at the increasing resolutions n. The solid curves starting at the origin correspond (from left to right) to decreasing n and give the measure of the δ-riddled subset of D at that level of approximation. The dotted curve is the analytically obtained curve for the measure of the δ-riddled component for $\delta < 0.125$. The curves appear to limit onto the analytical bound uniformly; the behaviour of the curves is very different from that in* (a).

Figure 4 (a) shows the numerically obtained approximations of the measure of the δ-riddled component of D (ie the approximation of $\ell(D_{rid}(\delta))$ as a function of δ for increasing resolution of approximation n. The fact that the curves increase to a constant value independent of δ as n is increased suggests that the set is riddled. Contrast this to Figure 4 (b) showing the same measurements for a square subset of M. In this case the curves accumulate on a well defined limit that can be easily analytically obtained. The linear behaviour of this curve for small δ corresponds to the boundary of the square being one dimensional. For unriddled sets with boundaries that are fractal with dimension greater than one this will manifest itself as an infinite slope of the curve at $\delta = 0$.

5 Discussion

In this paper we develop ideas of riddling and equivalence up to zero measure sets to characterize invariant sets of *discontinuous* area-preserving maps on $\mathbb{R}^2$ and give some examples of results that we believe shed light on their dynamics. Such maps provide, at least numerically, examples of dynamically defined riddled invariant sets other than the standard examples (eg [1, 6]) showing existence of riddled basins for attractors with invariant subspaces.

As we have demonstrated, in particular in Theorem 3.3 that this concept of riddling is illuminating for irrational rearrangements. We also show for irrational rearrangements that partial riddling or riddling of the discontinuity set implies an infinite number of periodic orbits. In general there are still very few general methods at ones disposal for such nowhere hyperbolic maps (all Lyapunov exponents are zero).

The discontinuity set D contains the most interesting dynamics in that all points in D^c are either periodic or quasiperiodic (Theorem 3.2). Although we have shown that D being partially riddled implies that it must have non-trivial decomposition into invariant sets (Theorem 3.3), we leave open the question of what dynamics will occur on D. For isometries that are pure translations, one can find non-trivial almost minimal sets (eg minimal interval exchanges), but for irrational rearrangements it is not clear how many almost minimal sets D may contain; numerical experiments suggest that there are typically minimal sets of large measure.

Acknowledgements: The authors would like to thank the EPSRC for support via grants GR/K77365 (PA and JT) and GR/M36335 (PA and XF). We also thank Arek Goetz and Miguel Mendes for several interesting conversations related to this work, and the referees for their perceptive comments.

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E-mail Addresses:

Peter Ashwin: p.ashwin@ex.ac.uk
Xin-Chu Fu: fu-xin-chu@ex.ac.uk
John R. Terry: jterry@ma.uq.edu.au

Proposed Running Head:

Riddling and Invariance for Discontinuous Maps

Corresponding Author:

Dr Peter Ashwin
School of Math Sci
University of Exeter
Exeter EX4 4QE, U.K.
E-mail: P.Ashwin@ex.ac.uk