A study of three fluid dynamical problems

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Abstract

In this thesis, three fluid dynamical problems are studied.

First in chapter 2 we investigate, via both theoretical and experimental methods, the swimming motion of a magnetotactic bacterium having the shape of a prolate spheroid in a viscous liquid under the influence of an imposed magnetic field. The emphasis of the study is placed on how the shape of the non-spherical magnetotactic bacterium, marked by the size of its eccentricity, affects the pattern of its swimming motion. It is revealed that the pattern/speed of a swimming spheroidal magnetotactic bacterium is highly sensitive not only to the direction of its magnetic moment but also to its shape.

Secondly, an important unanswered mathematical question in the theory of rotating fluids has been the completeness of the inviscid eigenfunctions which are usually referred to as inertial waves or inertial modes. In chapter 3 we provide for the first time a mathematical proof for the completeness of the inertial modes in a rotating annular channel by establishing the completeness relation, or Parseval’s equality, for any piecewise continuous, differentiable velocity of an incompressible fluid.

Thirdly, in chapter 4 we investigate, through both asymptotic analysis and direct numerical simulation, precessionally driven flow of a homogeneous fluid confined in a fluid-filled circular cylinder that rotates rapidly about its symmetry axis and precesses about a different axis that is fixed in space. A particular emphasis is placed on the spherical-like cylinder whose diameter is nearly the same as its length. An asymptotic analytical solution in closed form is derived in the mantle frame of reference for describing weakly precessing flow in the spherical-like cylinder at asymptotically small Ekman numbers. We also construct a three-dimensional finite element model, which is checked against the asymptotic solution, in attempting to elucidate the structure of the nonlinear flow.
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Chapter 1

Introduction

Fluid mechanics is a diverse field. People study the motion of fluid itself, or its multiple properties such as pressure, temperature, or magnetic field generated by the fluid. Also sometimes people study the properties of other objects which are immersed in fluid. The swimming motion of microorganisms is an interesting topic which has attracted lots of study. In this thesis, we study the swimming of magnetotactic bacteria (in short MTB). MTB are a group of bacteria discovered by Richard P. Blakemore in 1975. MTB can sense the Earth’s magnetic field and navigate itself through the oxic-anoxic transition zone in chemically stratified environments (e.g., Bazylinski, 2004; Blakemore, 1975). The most special property of MTB is the biomineralization of intracellular membrane-bound iron minerals named magnetosomes (magnetite or greigite). Magnetosomes are characterized by small grain-size distributions (30-120nm), different species-specific crystal morphology, chemical purity, and arrangement in single or multiple linear chains (e.g., Faivre, 2008; Schuler, 1999). Because of these intracellular magnetic minerals, MTB are of great interest in microbiology, biomineralization, advanced magnetic materials and bio-geosciences. Magnetotactic bacteria exist in a number of different shapes, but for all of them we should apply a low Reynolds number condition. Because the swimming motion comprises the translation and rotation of a bacterium, its shape – which determines both the size and direction of the viscous drag and torque acting on the bacterium – plays an essential role in understanding its dynamics. Previous study, say Nogueira and Lins de Barros (1995), simulated the movement of a spherical magnetotactic bacterium in water by solving a set of ordinary differential equations resulting from Newton’s second law together with the solutions of the spherical Stokes flow. They use three key assumptions in the investigation. First the shape of the bacteria is non-deformable so that the laws of rigid-body mechanics are applicable. Second the bacterium has the shape of a sphere for which the Stokes solution is simple and available. Thirdly the motion of a magnetotactic bacterium is propelled by a force acting at a fixed point on its membrane. We still use the first and the third assumption, but change the second one, that is to study the swimming motion
of magnetotactic bacteria that have the shape of an elongated prolate spheroid with arbitrary eccentricity $\mathcal{E}$. For this purpose we need two solutions of the Stokes flow, which are, a three-dimensional Stokes flow driven by a translating prolate spheroid of arbitrary eccentricity $\mathcal{E}$ at an arbitrary angle of attack $\gamma$, the angle between the direction of translation $\mathbf{v}$ and the symmetry axis, and a three-dimensional Stokes flow driven by a rotating spheroid of arbitrary eccentricity $\mathcal{E}$ with an arbitrary angle $\alpha$, the angle between the angular velocity $\Omega$ and the symmetry axis. One can then derive an expression for the corresponding drag and torque, which would be much more complicated. We use the solution from Kong et al (2012). The mechanical difference between a sphere and a spheroidal bacterium is that, for the spherical problem, the drag force is proportional to its swimming velocity and the viscous torque is also proportional to the angular velocity. These two properties do not exist in spheroidal case. As a result, the number of equations needed changed from 6 to 12.

The second problem studied in this thesis is about inertial waves. Inertial waves are ubiquitous in rotating fluid systems, ranging from geophysical and astrophysical problems to the fuel tank in spacecraft. They can be excited and maintained by many different mechanisms. The governing equations in this case are:

$$\frac{\partial \mathbf{u}}{\partial t} + 2\hat{z} \times \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0. \quad (1.1)$$

subject to the initial condition $\mathbf{u}(\mathbf{r}, t = 0) = \mathbf{u}_0$ and boundary condition $\mathbf{n} \cdot \mathbf{u} = 0$. In an eigenvalue style they could be written as

$$2\hat{z} \times \mathbf{u}_{mnk} + \nabla p_{mnk} = -i\omega_{mnk} \mathbf{u}_{mnk}.$$ 

The eigenfunctions $\mathbf{u}_{mnk}$ which depend on three parameters $m, n$ and $k$ are called inertial waves, or inertial modes. There are many aspects for inertial waves to be studied. We are mainly concerned with the completeness. The background of this problem is the classical Hilbert-Schmidt theorem (Renardy, 2004):

**Theorem:** Let $H$ be a Hilbert space and let $A \in \mathcal{L}(H)$ be a compact, self-adjoint operator. Then there is a sequence of nonzero real eigenvalues $\{\lambda_i\}_{i=1}^N$, such that $|\lambda_i|$ is monotone nonincreasing, and if $N = \infty$,

$$\lim_{i \to \infty} \lambda_i = 0.$$ 

Furthermore, if each eigenvalue of $A$ is repeated in the sequence according to its multiplicity, then there exists an orthonormal set $\{\phi_i\}_{i=1}^N$ of corresponding eigenfunctions, i.e.

$$A\phi_i = \lambda_i \phi_i.$$
Moreover, \( \{ \phi_i \}_{i=1}^N \) is a complete orthonormal basis for \( \mathcal{R}(A) \); and \( A \) can be represented by

\[
Au = \sum_{i=1}^{N} \lambda_i (\phi_i, u) \phi_i.
\]

As an example, suppose there are some orthogonal functions \( \{ \phi_n(x) \} \) in the interval \( x \in [a, b] \). If for every piecewise continuous function \( f(x) \) we have

\[
\lim_{N \to \infty} \int_a^b \left[ f(x) - \sum_{n=0}^{N} c_n \phi_n(x) \right]^2 dx = 0,
\]

then we say that \( \{ \phi_n(x) \} \) are complete (Debnath, 1999). A closely related problem is the Sturm-Liouville problem, which considers the second order ordinary differential equation

\[
-\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x) u(x) - \lambda w(x) u(x) = f(x)
\]

on the interval \((a, b)\) with appropriate boundary conditions. Many orthogonal function systems are solutions of certain Sturm-Liouville problems for certain functions \( p(x), q(x), w(x) \) and \( f(x) \). Examples include \( \{ \sin(nx), \cos(nx) \} \) over \([-\pi, \pi]\), the Legendre polynomials \( \{ P_n(x) \} \) over \([-1, 1]\), and \( \{ \sqrt{x} J_0(\alpha_n x) \} \) on \([0, 1]\), where \( J_0(z) \) is a Bessel function of the first kind and \( \alpha_n \) is its \( n \)th root. An expansion with the orthogonal systems \( \{ \sin(nx), \cos(nx) \} \) is the main topic in Fourier analysis. Under the assumption that above inertial modes are complete, an asymptotic solution could be derived to describe the flow of a viscous homogeneous fluid confined in an annular channel that rotates rapidly about its symmetry axis. Numerical analysis of the same problem has been carried out and there is an excellent agreement between the general asymptotic solution and the corresponding numerical solution. This mathematical approach has many advantages, but it hinges on the mathematical question of the completeness of the above inertial modes. In this thesis, chapter 3 represents the first attempt to answer the mathematical question of the completeness of the inviscid eigenfunctions by choosing the simple geometry of a rotating annular channel. Special methods are needed for the proof, instead of directly applying the Hilbert-Schmidt theorem - this is because the related operator in our problem is not self-adjoint.

In chapter 4, we study the precessional fluid contained in a cylinder container rotating about its symmetric axis, which has an additional precessing motion about
another axis fixed in the space. The governing equations are

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2 \left\{ \hat{\mathbf{z}} + \hat{\mathbf{o}} \left[ \hat{s} \sin \alpha_p \cos (\phi + t) - \hat{\phi} \sin \alpha_p \sin (\phi + t) + \hat{z} \cos \alpha_p \right] \right\} \times \mathbf{u} \\
= - \nabla p + E k \nabla^2 \mathbf{u} - 2 \hat{\mathbf{z}} \hat{\mathbf{o}} \mathbf{s} P_o \sin \alpha_p \cos (\phi + t),
\]

(1.2)

\[
\nabla \cdot \mathbf{u} = 0.
\]

(1.3)

These equations are difficult to solve, unless we use numerical methods, with the help of high performance supercomputers. Here we use the finite element method (FEM). For the mathematical theory of FEM, see Brenner (2008). Briefly speaking, FEM is based on the Galerkin method.

Suppose \( V \) is a Hilbert space. Consider the following problem

\[
\text{Find } u \in V \text{ such that } \forall v \in V, \ a(u, v) = f(v).
\]

(1.4)

Here \( a(u, v) \) is a bilinear form and \( f \) is a bounded linear functional on \( V \). To solve this problem, we use the Galerkin dimension reduction. Choose \( V_h \subset V \) to be a finite dimensional subspace of \( V \), and solve the following problem

\[
\text{Find } u_h \in V_h \text{ such that } \forall v_h \in V_h, \ a(u_h, v_h) = f(v_h).
\]

Suppose \( \{\phi_1, \phi_2, \cdots, \phi_N\} \) is a basis of \( V_h \). Then it is sufficient that

\[
a(u_h, \phi_i) = f(\phi_i), \quad i = 1, 2, \cdots, N.
\]

If we write \( u_h \) as

\[
u_h = \sum_{i=1}^{N} U_i \phi_i,
\]

then we can get the system of equations

\[
\sum_{j=1}^{N} a(\phi_j, \phi_i) U_j = f(\phi_i), \quad i = 1, 2, \cdots, N.
\]

These equations can be written in matrix form as

\[
A x = v.
\]

(1.5)

In our finite element code the equations (1.2)(1.3) are first transferred into a weak form as in (1.4), to get the linear system in (1.5), and then solve these linear equations by Krylov subspace (KSP) iteration methods.
When $Po$ is sufficiently small ($Po \ll 1$), the motion powered by the Poincaré force will also be small, that is $|u| = \epsilon \ll 1$. So (1.2) may be linearized by omitting all higher-order terms to get

$$\frac{\partial u}{\partial t} + 2\mathbf{z} \times u + \nabla p = Ek\nabla^2 u - 2\mathbf{z}Po \sin \alpha re^{i(\phi + t)},$$

$$\nabla \cdot u = 0.$$

If we consider the case when the Poincaré force resonates with the inertial wave mode $u_{111}$, then for these equations an analytical solution in closed form for $0 < Ek \ll 1$ becomes available.

Based on these numerical and theoretical solutions, we analyzed the properties of the rotating fluid, and investigated the structure of the nonlinear flow.
Chapter 2

The swimming motion of spheroidal magnetotactic bacteria

2.1 Introduction

The Earth has a large core of electrically conducting liquid metal where radial buoyancy forces, thermal or compositional, drive convective motions which, through magnetic induction, convert the mechanical energy of the fluid motions into the ohmic dissipation and maintain the geomagnetic field observed in the exterior of the Earth (e.g., Moffatt, 1978; Zhang and Schubert, 2000). It is the geomagnetic field that protects or influences a wide range of life, from human being to micro-scale organisms, on the planet Earth. A fascinating phenomenon in living microorganisms under the influence of the geomagnetic field is that the swimming motion of magnetotactic bacteria first discovered nearly four decades ago (Blakemore, 1975). The magnetotactic bacteria contain magnetic crystals of a narrow size between $35 - 120\, \text{nm}$ carrying permanent magnetization (See, for example, Bazylinski, 1995; Schüler and Frankel, 1999) and, hence, swim along the lines of the Earth’s magnetic field (See, for example, Faivre and Schüler, 2008). Reflecting the geomagnetic polarity, the bacteria in the Southern hemisphere possess the polarity of the magnetite crystals that have the opposite direction comparing to those found in the Northern hemisphere. Because of the magnetic moment, when the bacterium swims, there will be a magnetic torque exerted upon it by the geomagnetic field, so that it will tends to swim along the geomagnetic field line. In consequence, the magnetotactic bacteria are north-seeking in the northern hemisphere while south-seeking in the southern hemisphere (Blakemore et al., 1980; Kirschvink, 1980; Frankel, 1984).

Magnetotactic Bacteria exist in a number of different shapes: some have the shape of an elongated prolate spheroid (See, for example, Bazylinski and Frankel, 2004; Pan et al. 2009; Lin et al., 2012). Driven by rapid rotation of its helical flagellar filaments which generates torque, a magnetotactic bacterium swims in the form of helical fashion against the viscous drag and torque under the influence of
2. Swimming motion of spheroidal bacteria

an external magnetic field (See, for example, Jones and Aizawa, 1991). Since its swimming speed $U_0$ is very low and its characteristic length $c$ is extremely small, the motion of a bacterium in liquid is typically marked by a very small Reynolds number $Re$ (Purcell, 1977), which is defined as

$$Re = \frac{U_0 c \rho}{\mu},$$

where $\rho$ is the density of the liquid in which magnetotactic bacteria swim and $\mu$ is the dynamic viscosity. For example, the magnetotactic bacteria found in Lake Miyun near Beijing, China, which are used in our experimental study, have typically $U_0 = 10^{-4} \text{m/s}$ and $c = 10^{-6} \text{m}$, which give rise to $Re = 10^{-4}$ for water in room temperature. In consequence, the Stokes’ approximation (for example, see Batchelor, 1967), which neglects the inertial term in the Navier-Stokes equation in the limit $Re \to 0$, has been adopted for describing the motion of slowly swimming bacteria (Koiller et al., 1996). It should be noted that, because the swimming motion comprises the translation and rotation of a bacterium, its shape – which determines both the size and direction of the viscous drag and torque acting on the bacterium – plays an essential role in understanding its dynamics.

Complementary to the usual biological and chemical studies of magnetotactic bacteria, it is profitable to examine their swimming motion from a fluid dynamical point of view. In an important study of the swimming motion of magnetotactic bacteria, Nogueira and Lins de Barros (1995) simulated the movement of a spherical magnetotactic bacterium in water by solving a set of ordinary differential equations resulting from Newton’s second law together with solutions of the spherical Stokes flow. In order to make theoretical progress, they made three key assumptions in their investigation. First, the shape of the magnetotactic bacteria is non-deformable so that the laws of rigid body mechanics are applicable. Second, the bacterium has a shape of sphere for which the Stokes solution is simple and available (for example, see Batchelor, 1967). In this case, the drag force $D_\mu$ on a moving spherical bacterium is simply given by

$$D_\mu = -6\pi \mu r_0 v,$$  \hspace{1cm} (2.1)

where $r_0$ is the radius of the sphere and $v$ is its velocity, while the viscous torque $T_\mu$ on a rotating spherical bacterium is

$$T_\mu = -8\pi \mu r_0^3 \Omega,$$  \hspace{1cm} (2.2)

where $\Omega$ represents the angular velocity of its rotation. It is also significant to notice that the size of the drag force $D_\mu$ and the viscous torque $T_\mu$ for a spherical bacterium does not depend on the direction of its translation or rotation, which dramatically simplifies the relevant mathematical analysis of the problem. The third assumption
is that the motion of a magnetotactic bacterium is propelled by a force acting at a fixed point on its membrane. It is spherical geometry that allows Nogueira and Lins de Barros (1995) to derive a system of six ordinary differential equations that govern the motion of a swimming spherical magnetotactic bacterium.

The primary objective of this part is to study, via both theoretical and experimental methods, the swimming motion of magnetotactic bacteria that have a shape of an elongated prolate spheroid with arbitrary eccentricity \( E \) which, as depicted in Figure 2.1(a), may be mathematically described as

\[
\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1, \tag{2.3}
\]

where \( c^2 = a^2/(1 - E^2) \) with \( 0 < E < 1 \). A microscope image of magnetotactic bacterium that has spheroidal shape and is approximately described by (2.3) is displayed in Figure 2.7. It is of primary importance to notice that, in comparison with spherical geometry, elongated spheroidal geometry introduces not only complicated mathematics but also important new physics/dynamics for a better understanding of the swimming motion of non-spherical magnetotactic bacteria. Describing a swimming spheroidal bacterium requires three-dimensional solutions of the spheroidal Stokes flow in viscous and incompressible fluid. The following two solutions for the Stokes flow are needed: (i) a three-dimensional Stokes flow driven by a translating prolate spheroid of arbitrary eccentricity \( E \) at an arbitrary angle of attack \( \gamma \), the angle between the direction of translation \( \mathbf{v} \) and the symmetry axis \( z \) in Figure 2.1(a), and (ii) a three-dimensional Stokes flow driven by a rotating spheroid of arbitrary eccentricity \( E \) with arbitrary angle \( \alpha \), the angle between the angular velocity \( \Omega \) and the symmetry axis \( z \) in Figure 2.1(a). If the above two solutions are available, one can then derive an expression for the corresponding drag \( D_\mu \) and torque \( T_\mu \) which would be much more complicated than those given by (2.1) and (2.2). Motivated by the desire to understand the dynamics of slowly swimming magnetotactic bacterium that has the shape of an elongated prolate spheroid, we have made the first application of the Papkovich-Neuber formulation to the spheroidal Stokes problem and obtained the Papkovich-Neuber-type analytical solutions in prolate spheroidal coordinates for the three-dimensional spheroidal Stokes flow that is driven by either a translating spheroid at an arbitrary angle \( \gamma \) or a rotating spheroid with an arbitrary angle \( \alpha \) (Kong et al., 2012). The availability of the analytical solutions allows us to study the swimming motion of magnetotactic bacteria having a shape of nearly elongated prolate spheroid. Since the vectors \( \mathbf{v} \) and \( \Omega \) are not always collinear to the symmetry axis \( z \) in the helical motion of a magnetotactic bacterium, the two key characteristic angles, \( \alpha \) and \( \gamma \), must enter the dynamical problem of the swimming motion.

It should be pointed out that there exist, both physically and mathematically, fundamental differences between the problems of swimming spherical (\( E = 0 \) in (2.3)))
2. Swimming motion of spheroidal bacteria

Figure 2.1: Sketch of geometry of the problem as well as two cartesian coordinates, $(X, Y, Z)$ and $(x, y, z)$, used in our theoretical analysis. (a) A magnetotactic bacterium has a shape of an elongated prolate spheroid with arbitrary eccentricity $\mathcal{E}$ with the semi-axis $a$ and $c = a/\sqrt{1 - \mathcal{E}^2}$. Cartesian coordinates $(X, Y, Z)$ represent a reference of frame fixed in the laboratory while $(x, y, z)$ denote a reference of frame fixed in the body of the spheroidal bacterium. (b) The three Euler angles $(\theta, \psi, \phi)$ connect the two cartesian coordinates $(X, Y, Z)$ and $(x, y, z)$. 
and spheroidal \((0 < \varepsilon < 1 \text{ in } (2.3))\) magnetotactic bacterium. For the spherical problem, the drag force \(D_{\mu}\), because of its special geometry, is proportional to its swimming velocity \(v\), i.e., \(D_{\mu} \sim v\), in which \(D_{\mu}\) must be balanced by the propulsive force \(F\) (Nogueira and Lins de Barros, 1995). Note that the size of the drag force is independent of the direction of the velocity \(v\). Similarly, the viscous torque \(T_{\mu}\), because of its special geometry, is also proportional to the angular velocity \(\Omega\) of its rotation, \(T_{\mu} \sim \Omega\). Clearly, the size of the viscous torque is also independent of the direction of the angular velocity \(\Omega\). In consequence, only the six degree of freedom – its position in the laboratory and the three Euler angles for the associated angular velocity \(\Omega\) – are required to completely determine the swimming motion of a spherical magnetotactic bacterium. The simple picture and dynamics for spherical geometry are completely altered when it is extended to spheroidal geometry with \((0 < \varepsilon < 1 \text{ in } (2.3))\). This is owing mainly to the fact that the above linear relationships are replaced, as we will show, with the strongly nonlinear relationships in which the drag force \(D_{\mu}\) not only depends on the swimming velocity \(v\) but also on its direction relative to the axis of the bacterium while the viscous torque \(T_{\mu}\) not only depends on its rotation vector \(\Omega\) but also on its direction relative to the axis of the bacterium. In other words, we are, in contrast to the spherical model, no longer able to express the drag force \(|D_{\mu}|\) in terms of \(|v|\) alone and the viscous torque \(|T_{\mu}|\) in terms of the angular velocity \(|\Omega|\) alone. Physically, it means that the direction of both \(v\) and \(\Omega\) must enter the problem of dynamics for a swimming spheroidal bacterium. In consequence, we require 12 degrees of freedom to determine the swimming motion of a spheroidal magnetotactic bacterium: its position in the laboratory, the three components of its velocity vector, its angular rotating velocity and the associated three Euler angles. In other words, a system of twelve coupled equations, even in an inertia-less limit, is required to describe the motion of a swimming spheroidal magnetotactic bacterium.

It is worth mentioning that one can easily theoretically investigate how swimming magnetotactic bacteria sensitively depends on the degree of its prolation via using mathematically complex models. Experimentally, it is difficult to study the dependence of \(\varepsilon\) because it would require a collection of large number of different magnetotactic bacteria with different shapes. The primary aim of the present study is thus to derive a system of the twelve coupled nonlinear differential equations that can be used to describe the dynamics of swimming magnetotactic bacteria having non-spherical shape. With regard to its application by comparing solutions of the twelve differential equations to actual swimming trajectories, we shall only focus on one particular magnetotactic bacteria found in Lake Miyun near Beijing, China.

In what follows we shall begin in §2 by presenting the governing equations using the newly found formulas for the spheroidal drag and torque. This is followed in §3 by discussing the swimming motion of spheroidal magnetotactic bacteria with dif-
ferent eccentricities and magnetic moments and by comparing the theoretical swimming patterns to the trajectories of swimming motion observed in the laboratory experiments. The paper closes in §4 with a summary and some remarks.

2.2 Twelve Governing Differential Equations

Consider the swimming motion of a spheroidal magnetotactic bacterium having the magnetic moment vector \( \mathbf{m} \) with mass \( M \) in a uniform viscous fluid. Geometry of the magnetotactic bacterium, together with two cartesian coordinates \((x,y,z)\) and \((X,Y,Z)\), is sketched in Figure 2.1(a). Cartesian coordinates \((x,y,z)\) along with the corresponding unit vectors \((\hat{x},\hat{y},\hat{z})\) represent a reference of frame fixed in the bacterium’s body with \(z\) as its symmetry axis; this reference will be referred to as the body frame. The position of its center of mass is described by the position vector

\[
\mathbf{R} = X\hat{X} + Y\hat{Y} + Z\hat{Z},
\]

in the cartesian coordinates \((X,Y,Z)\) with the corresponding unit vectors \((\hat{X},\hat{Y},\hat{Z})\) which is fixed in the laboratory; this reference will be referred to as the laboratory frame. The magnetotactic bacterium swims under the combined action of a propel force \( \mathbf{F}_B \), a viscous drag \( \mathbf{D}_B \), a viscous torque \( \mathbf{T}_B \) and a torque by an externally imposed magnetic field \( \mathbf{B}_L \). Here a subscript \( B \) denotes the quantity measured in the body frame while a subscript \( L \) for the quantity in the laboratory frame. We shall assume that \( \frac{c^2\rho}{\mu} \) is small compared to the time-dependent forcing so that time-independent Stokes flow is appropriate. The complex mathematical analysis stems from spheroidal geometry as well as frequent transformation between the two coordinates.

Prior to deriving the 12 governing differential equations, let’s introduce the three different vectors for describing the swimming motion of a spheroidal bacterium: a translation vector \( \mathbf{v}_L \) of the mass center in the laboratory frame, a translation vector \( \mathbf{v}_B \) of the mass center in the body frame and a rotation vector \( \mathbf{\Omega} \) in the body frame (please note that a frame is different from a reference system). The velocity vector \( \mathbf{v}_B \), when \( \gamma \neq 0 \), is marked by the three quantities:

\[
|\mathbf{v}_B|, \quad \gamma = \cos^{-1} \left[ \frac{\hat{z} \cdot \mathbf{v}_B}{|\mathbf{v}_B|} \right], \quad \tau = \cos^{-1} \left[ \frac{\hat{x} \cdot \mathbf{v}_B}{\sin \gamma |\mathbf{v}_B|} \right], \tag{2.4}
\]

while there are also three quantities characterizing the vector \( \mathbf{\Omega} \):

\[
|\mathbf{\Omega}|, \quad \alpha = \cos^{-1} \left[ \frac{\hat{z} \cdot \mathbf{\Omega}}{|\mathbf{\Omega}|} \right], \quad \beta = \cos^{-1} \left[ \frac{\hat{x} \cdot \mathbf{\Omega}}{\sin \alpha |\mathbf{\Omega}|} \right], \tag{2.5}
\]

for \( \alpha \neq 0 \). The axisymmetric case \( \gamma = 0 \) (or \( \alpha = 0 \)) is specially treated. Moreover, the relative position between \((x,y,z)\) and \((X,Y,Z)\) is determined by the position
vector $\mathbf{R}$ together with the three Euler angles $(\theta, \phi, \psi)$ illustrated in Figure 2.1(b). There exist 12 degrees of freedom that completely determine the swimming motion of a spheroidal magnetotactic bacterium: its position in the laboratory $\mathbf{R} = (X, Y, Z)$, the three components of its velocity vector $\mathbf{v}_L$, its angular rotating velocity $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ and the three Euler angles $\theta, \phi$ and $\psi$.

We shall also make, similar to the study of Nogueira and Lins de Barros (1995), the following three assumptions: (i) the body of the magnetotactic bacterium is non-deformable and, hence, the equations of rigid-body dynamics are applicable; (ii) the interaction between different magnetotactic bacteria (see, for example, Ishikawa et al. 2007) on their swimming motion is weak and, hence, negligible; and (iii) the translation and rotation of the bacterium is powered by the rapid rotation of helical flagellar filaments at a fixed point $P$, as shown in Figure 2.1(a), in connection with a force $\mathbf{F}_B$ in the body frame:

$$\mathbf{F}_B = F_{12} [\cos(\omega_0 t)\mathbf{\hat{x}} + \sin(\omega_0 t)\mathbf{\hat{y}}] + F_3\mathbf{\hat{z}},$$

(2.6)

where $\omega_0$ is the frequency of flagellum rotation and $F_{12}, F_3$ and $\omega_0$ may be regarded as parameters of the problem. There exist, however, three major differences between our dynamical model and that of Nogueira and Lins de Barros (1995). First, our magnetotactic bacterium has the shape of an elongated prolate spheroid while theirs is spherical. It will be seen that the swimming motion of a magnetotactic bacterium is highly sensitive to its shape marked by the size of eccentricity $E$. Second, they assumed that $|\mathbf{D}_B/\mathbf{v}_B|$ on the bacterium, because of spherical geometry, is constant and, hence, leads to $\mathbf{v}_B \sim \mathbf{F}_B$. This is no longer valid as a result of spheroidal geometry: the size of $\mathbf{D}_B$ depends not only on $|\mathbf{v}_B|$ but also on the two angles, $\gamma$ and $\tau$ given in (2.4), in a strongly nonlinear way. Third, their model also assumed that $|\mathbf{T}_B/\Omega|$ on the bacterium is constant. In our model, $|\mathbf{T}_B|$ depends not only on $|\Omega|$ but also on the two angles, $\alpha$ and $\beta$ in (2.5), during its helical motion. In consequence, there exist 12 degrees of freedom in our spheroidal model while there are only 6 degrees of freedom in the spherical model.

Under the assumption of non-deformable and non-interactive bacteria, the laws of rigid body dynamics in conjunction with solutions of the spheroidal Stokes flow can be employed to study the swimming motion of a magnetotactic bacterium. Of the 12 coupled differential equations required to determine the 12 degrees of freedom, the first three equations are from Newton’s second law stating that the rate of change of the momentum must be equal to the sum of all external forces acting on it:

$$M \frac{d\mathbf{v}_L}{dt} = \mathbf{F}_L + \mathbf{D}_L,$$

(2.7)

where $\mathbf{F}_L$ and $\mathbf{D}_L$ are the propel and viscous forces in the laboratory frame. According to the recent analysis (Kong et al., 2012), the drag force vector $\mathbf{D}_B$ in the
2. Swimming motion of spheroidal bacteria

Figure 2.2: The normalized drag force, $|D_B|/(6|v_B|\pi\mu c_f)$ is plotted as a function of eccentricity $E$ for four different angles $\gamma$. The limit $E \to 0$ represents the spherical case with $c_f \xi_0 \to r_0$.

Body frame can be written as

$$D_B = 2\pi \mu |v_B| \left\{ -\frac{8 + 4(\xi_0^2 - 1)\left(-2 + \xi_0 \ln \frac{\xi_0 + 1}{\xi_0 - 1}\right)}{2\xi_0 - (\xi_0^2 - 3) \ln \frac{\xi_0 + 1}{\xi_0 - 1}} c_f \sin \gamma (\cos \tau \hat{x} + \sin \tau \hat{y}) ight. \\
+ \left. \frac{2\xi_0^2 - \xi_0(\xi_0^2 - 1) \ln \frac{\xi_0 + 1}{\xi_0 - 1}}{\xi_0^2 - \frac{\xi_0^2 - 3}{4} \ln \frac{\xi_0 + 1}{\xi_0 - 1}} c_f \cos \gamma \hat{z} \right\},$$

in which the velocity $v_B$ is in the body frame and can be written in the form

$$v_B = |v_B| \left( \sin \gamma \cos \tau \hat{x} + \sin \gamma \sin \tau \hat{y} + \cos \gamma \hat{z} \right).$$

Here $\xi = \xi_0 = 1/E$ denotes the bounding surface of the spheroid having the focal length $c_f$ in prolate spheroidal coordinates $(\xi, \eta, \zeta)$ whose relationship with the
2. Swimming motion of spheroidal bacteria

cartesian coordinates \((x, y, z)\) is given by

\[
\begin{align*}
    x &= c_f \sqrt{(\xi^2 - 1)(1 - \eta^2)} \cos \zeta, \\
    y &= c_f \sqrt{(\xi^2 - 1)(1 - \eta^2)} \sin \zeta, \\
    z &= c_f \xi \eta.
\end{align*}
\]

Dependence of the normalized drag force, \(|D_B|/(6|v_B|\pi \mu \xi_0 c_f)\), on the size of eccentricity \(\xi\) is shown on Figure 2.2 for four different angles \(\gamma\). However, it should be noted that the drag force vector \(D_L\) in (2.7) is measured in the laboratory frame. In order to compute \(D_L\) we introduce the three rotation matrices, \(R_x(\sigma), R_y(\sigma)\) and \(R_z(\sigma)\), defined as

\[
R_x(\sigma) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \sigma & \sin \sigma \\
0 & -\sin \sigma & \cos \sigma
\end{bmatrix},
\]

\[
R_y(\sigma) = \begin{bmatrix}
\cos \sigma & 0 & -\sin \sigma \\
0 & 1 & 0 \\
\sin \sigma & 0 & \cos \sigma
\end{bmatrix},
\]

\[
R_z(\sigma) = \begin{bmatrix}
\cos \sigma & \sin \sigma & 0 \\
-\sin \sigma & \cos \sigma & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

With the aid of the above rotation matrices, the first set of three ordinary differential equations can be written as

\[
M \frac{d\mathbf{v}_L}{dt} = + [R_z(-\phi)R_x(-\theta)R_z(-\psi)] [\mathbf{F}_B + \mathbf{D}_B].
\]

(2.10)

The second set of three ordinary differential equations is simply related to the position of the bacterium in the laboratory frame:

\[
\left( \mathbf{X} \frac{dX}{dt} + \mathbf{Y} \frac{dY}{dt} + \mathbf{Z} \frac{dZ}{dt} \right) = \mathbf{v}_L.
\]

(2.11)

The third set of three equations are related to the Euler equations

\[
\begin{align*}
\frac{d\theta}{dt} &= \Omega_x \cos(\psi) - \Omega_y \sin(\psi), \\
\frac{d\phi}{dt} &= \Omega_x \csc(\theta) \sin(\psi) + \Omega_y \csc(\theta) \cos(\psi), \\
\frac{d\psi}{dt} &= -\Omega_x \cot(\theta) \sin(\psi) - \Omega_y \cot(\theta) \cos(\psi) + \Omega_z,
\end{align*}
\]

(2.12) (2.13) (2.14)

where \(\Omega = (\Omega_x, \Omega_y, \Omega_z)\), which describes the rotation of the body frame with respect to the laboratory frame. The final set of three ordinary differential equations is
2. Swimming motion of spheroidal bacteria

derived from the rotational dynamics of the angular momentum \( \mathbf{L} \),

\[
\mathbf{L} = I_x \Omega_x \mathbf{\hat{x}} + I_y \Omega_y \mathbf{\hat{y}} + I_z \Omega_z \mathbf{\hat{z}},
\]

where \((I_x, I_y, I_z)\) denote the three principle moments of inertia of a non-deformable spheroidal bacterium. The rate of change of \( \mathbf{L} \) must be equal to the sum of all torques acting on the bacterium:

\[
\begin{align*}
(I_x \mathbf{\hat{x}} \frac{d\Omega_x}{dt} + I_y \mathbf{\hat{y}} \frac{d\Omega_y}{dt} + I_z \mathbf{\hat{z}} \frac{d\Omega_z}{dt}) + \Omega \times \mathbf{L} &= \mathbf{T}_p + \mathbf{T}_c + \mathbf{T}_M + \mathbf{T}_B, \\
\end{align*}
\]

(2.15)

where \( \mathbf{T}_p = -c \mathbf{\hat{z}} \times \mathbf{F}_B \), \( \mathbf{T}_c = -N \mathbf{\hat{z}} \) is related to the reaction couple of the flagellar rotation (Nogueira and Lins de Barros, 1995), \( \mathbf{T}_M = m \mathbf{\times B}_B \) is the magnetic torque in the body frame, and \( \mathbf{T}_B \) represents the viscous torque in the body frame. With some modification of what is derived by Kong et al. (2012), the viscous torque \( \mathbf{T}_B \) can be expressed as

\[
\begin{align*}
\mathbf{T}_B &= \frac{-8\pi}{-2\xi_0 + (\xi_0^2 + 1) \ln \frac{\xi_0 + 1}{\xi_0 - 1}} \\
&\times \left[ 2\xi_0(\xi_0^2 - 1) \tanh^{-1} \frac{1}{\xi_0} + \frac{-4 + 8\xi_0^2 - 3\xi_0(\xi_0^2 - 1) \ln \frac{\xi_0 + 1}{\xi_0 - 1}}{3} \right] \mathbf{\hat{z}} \\
&+ \frac{1}{3} \left[ \frac{32\pi c_f^3}{-2\xi_0 + (\xi_0^2 - 1) \ln \frac{\xi_0 + 1}{\xi_0 - 1}} \cos \alpha \right] \mathbf{\hat{z}}. \\
\end{align*}
\]

(2.16)

Note that \( \mathbf{D}_B \) in (3.30) and \( \mathbf{T}_B \) in (2.16) are a strongly nonlinear function of \( \mathcal{E} \) (or \( \xi_0 \)) and that \( \mathbf{T}_p \) and \( \mathbf{T}_c \) in our model are exactly the same as those used by Nogueira and Lins de Barros (1995). Dependence of the normalized torque, \( |\mathbf{T}_B|/|8\Omega|\pi \mu (\xi_0 c_f)^3| \), on the size of eccentricity \( \mathcal{E} \) is shown in Figure. 2.3 for four different angles \( \alpha \).

The swimming motion of a spherical magnetotactic bacterium is completely described by a mathematical solution to the 12 coupled ordinary differential equations: 3 for the velocity \( \mathbf{v}_L \) given by (2.10), 3 for the position vector \( \mathbf{R} \) by (2.11), 3 for the Euler angles \( (\theta, \phi, \psi) \) given by (2.12)–(2.14) and, finally, 3 for the rotation vector \( \Omega \) given by (2.15). They represent a mathematically tractable system that enables us to understand the dynamics of a swimming non-spherical magnetotactic bacterium.

Given an appropriate set of the model parameters with an initial condition, the 12 coupled differential equations can be numerically integrated to produce the trajectories of a swimming spheroidal bacterium having arbitrary size of eccentricity \( \mathcal{E} \).

It is noteworthy that the twelve coupled equations for our spheroidal model, in comparison to the six coupled equations for the spherical model (Nogueira and Lins de Barros, 1995), is not because of keeping swimmer inertia in dynamics. Let us
2.3 Swimming Motion of Magnetotactic Bacteria

2.3.1 Parameters and Solution Procedure

Evidently, many model parameters, geometric or physical, must be specified when integrating the 12 coupled ordinary differential equations. Several parameters may be regarded as being well known but some are poorly determined. For example, the typical length $c$ of magnetotactic bacteria and its magnetic moment $m$ can be approximately measured or deduced in a reasonably accurate way. Since the magnetic moment vector $m$ is not always parallel to the symmetry axis $z$, we shall assume that

$$m = m_0 (\hat{z} \cos \Phi + \hat{x} \sin \Phi),$$

where $m_0$ is the magnetic moment in the $z$ direction, and $\Phi$ is the angle between the magnetic moment and the symmetry axis. The magnetic moment vector $m$ is a function of the angle $\Phi$. The problem of spheroidal geometry involves nonlinear forces and equations.

Figure 2.3: The normalized drag force, $|T_B|/|8\Omega|\pi \mu (\xi_0 c_f)^3$ is plotted as a function of eccentricity $\mathcal{E}$ for four different angles $\alpha$. The limit $\mathcal{E} \to 0$ represents the spherical case with $c_f \xi_0 \to r_0$.

take the drag force $D_B$ as an example for illustration. A bacterium with a shape of sphere experiences the drag force $D_B$ that is linearly proportional to the velocity $v_B$ in the form $D_B \sim v_B$. As a consequence of this simple linear relationship, the unknown $v_B$ can be eliminated to reduce the number of the governing equations. It is fundamentally different in the problem of spheroidal geometry: the drag force vector $D_B$ is of the form $D_B = D(|v_B|, \tau, \gamma)$, a nonlinear function of unknowns $|v_B|, \tau$ and $\gamma$. Even though we may neglect swimmer inertia by setting $dL/dt = 0$, the vector equation (2.10) still gives rise to three nonlinear coupled scalar equations.
2. Swimming motion of spheroidal bacteria

where $0 \leq \Phi \leq 90^\circ$. In all the calculations reported in this paper, we shall take $c = 1.8 \times 10^{-6}m$, $m_0 = 2.20 \times 10^{-15}Am^2$ corresponding approximately to the magnetotactic bacteria used in our experimental study (Pan et al., 2009). The mass of a magnetotactic bacteria is estimated as $M = 10^{-10}kg$ – which may be overestimated. Since the effect of inertia is always small, the overestimated mass have no significant influences on the motion of magnetotactic bacteria.

For a given mass $M$, the moment of a bacterium’s inertia can be easily calculated according to

$$
I_x = \frac{c^2}{5}(2 - \mathcal{E}^2)M, \\
I_y = I_x, \\
I_z = \frac{2c^2}{5}(1 - \mathcal{E}^2)M,
$$

where the eccentricity $\mathcal{E}$ is a key geometric parameter of the problem. We know, however, very little about the details of the force $\mathbf{F}_B$ in connection with a bacterial flagellar motor.

As a consequence of non-spherical geometry and different coordinates, an extra care must be taken in the procedure of solving the 12 coupled nonlinear differential equations. The procedure underlies, in comparison to the previous spherical study, the complicated dynamics of a non-spherical magnetotactic bacterium. At each time step of the integration, a numerical solution to the 12 differential equations contains the 12 variables : the velocity $\mathbf{v}_L$ in the laboratory frame, the position vector $\mathbf{R}$ in the laboratory frame, the Euler angles $(\theta, \phi, \psi)$ and the rotation vector $\mathbf{\Omega}$ in the body frame. In order to obtain the viscous drag $\mathbf{D}_B$, given by (3.30), on the swimming bacterium in the body frame, we need first, at each time step, to transfer $\mathbf{v}_L$ in the laboratory frame to $\mathbf{v}_B$ in the body frame by using the three Euler angles

$$
\mathbf{v}_B = \mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi)\mathbf{v}_L.
$$

The vector $\mathbf{v}_B$ is then used to compute the two angles, $\gamma$ and $\tau$ according to (2.4). With $\gamma$, $\tau$ and $\mathbf{v}_B$, the drag force vector $\mathbf{D}_B$ in (2.10) can be obtained using the formula (3.30) at each time step of the numerical integration. A similar approach is also taken to find the viscous torque: we use (2.5) at each time step to calculate the two angles, $\alpha$ and $\beta$, which are then used to compute $\mathbf{T}_B$ in (2.16). Of course, an externally applied magnetic field $\mathbf{B}_L$ in the laboratory frame must be also transferred to the body frame by

$$
\mathbf{B}_B = [\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi)]\mathbf{B}_L
$$

at each time step of the numerical integration. The twelve nonlinearly coupled equations are numerically integrated using a Runge-Kutta scheme with a typical time step $\Delta = 5 \times 10^{-5}$. 

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Figure 2.4: Three different trajectories of the swimming motion computed from the 12 nonlinear coupled differential equations at three different values of eccentricity, $E = 0.1, 0.5, 0.6$ (as labeled in the figure) while other parameters remain fixed in the model.
Since the pattern of the swimming motion is largely determined by both the structure of an imposed magnetic field and the shape of a bacterium, the precise form of an initial condition for the numerical integration, which is usually arbitrarily prescribed in our numerical simulation, does not play an essential role.

The code used to compute the trajectory is listed below.

1. The file: exe.m

```matlab
format long

global pi B m radl eta M w alpha gama v3 nc F12 F3 A a I1 I2 I3

pi = 3.1415926;
M = 1e-10;
E = 0.6;
A = 1.2e-6;
a = A*(1-E^2);
I1 = 0.2*(A^2+a^2)*M;
I2 = I1;
I3 = 0.2*(a^2+a^2)*M;

radl = 1.2e-6;
eta = 1.0e-3;

alpha = 133;
gama = 125;
w = 300;
F12 = 8*pi*eta*radl^2*alpha;
F3 = 6*pi*eta*radl*v3;
v12 = 8/6*radl*alpha;
v3 = 1.25e-4;
nc = 8*pi*eta*radl^3*gama;

k = 1.38e-23;
T = 300;
m = 2.2e-15;
B = 2.5e-4;
theta0 = (2*k*T/m/B)^(0.5);
```
2. Swimming motion of spheroidal bacteria

\[ x_0 = [ \theta_0 0 0 0 0 0 0 0 0 0 0 0 0 v_12 0 v_3 ]; \]

\[
\text{options}=\text{odeset}('\text{reltol}',1e-10);
\]

\[
\text{tic}
[t,x]=\text{ode45}(@compute\_the\_traj,[0,0.4],x0,\text{options});
\text{toc}
\]

\[
\text{jubing} = \text{plot3}(x(:,7),x(:,8),x(:,9));
\]

\[
\text{canshu} = \text{get}(\text{jubing},'\text{Parent'});
\]

\[
\text{set}(\text{canshu},'\text{FontSize}',6);
\]

\[
\text{title('new\_trajectory','fontsize',10)}
\]

\[
\text{xlabel('X','fontsize',7)}
\]

\[
\text{ylabel('Y','fontsize',7)}
\]

\[
\text{zlabel('Z','fontsize',7)}
\]

\[
\text{axis equal}
\]

This file mainly defines the variables. \( M \) is the mass of the bacterium, \( E \) its eccentricity, \( A \) its semi-major axis, \( a \) its semi-minor axis. \( I_1, I_2, I_3 \) are the three principle moments of inertia. \( \alpha, \beta, \gamma \) are three parameters. \( \eta \) is the dynamical viscosity of water. \( w \) is the assumed angular velocity in the formula (2.6). \( F_1 \) and \( F_3 \) are the parameters in (2.6) denoting the strength of the force. \( v_1 \) and \( v_2 \) are the assumed initial values for the velocity. \( k \) is the Boltzmann constant. \( T \) is the assumed lab temperature (K). \( m \) is the magnetic moment of the magnetotactic bacterium. \( B \) is the imposed magnetic field. \( \theta_0 \) is the assumed initial value of the bacterium’s direction (we use a formula \( \sqrt{2kT/mB} \) which takes into account thermal disturbances. See Nogueira et al, 1995).

The command
\[
[t,x]=\text{ode45}(@\text{compute\_the\_traj},[0,0.4],x0,\text{options});
\]
calls the subroutine of differential equation solver “ode45” to solve the equations of our system listed in the file “compute\_the\_traj”.

2. The file: Compute\_The\_Traj.m

\[
\text{function } dx=\text{compute\_the\_traj}(t,x)
\]
2. Swimming motion of spheroidal bacteria

global pi B m eta M w nc F12 F3 A a I1 I2 I3

\[
D = \begin{bmatrix}
\cos((x(3))) \cos((x(2))) - \cos((x(1))) \sin((x(2))) \sin((x(3))) \\
-\sin((x(3))) \cos((x(2))) - \cos((x(1))) \sin((x(2))) \cos((x(3))) \\
\sin((x(1))) \sin((x(2))) - \cos((x(1))) \cos((x(2))) \cos((x(3)))
\end{bmatrix}, \ldots
\]

\[
d = \begin{bmatrix}
\sin((x(1))) \sin((x(2))), -\sin((x(1))) \cos((x(2))), \cos((x(1)))
\end{bmatrix};
\]

\[
% Spheroid
V = [x(10), x(11), x(12)];
W = (D * [x(4), x(5), x(6)])'';
r = [0, 0, 0];
u = zeros(3);
p = zeros(3, 3);

[FH, NH, u, p] = DragTorqCalc(V, W, d, r, A, a);  \% for spheroid
FH = eta * FH;
NH = eta * NH;
NH = (D \ NH')';

% Sphere
% FH = -6 * pi * eta * A * [x(10), x(11), x(12)];
% NH = -8 * pi * eta * A^3 * [x(4), x(5), x(6)];

Ff = (D * [F12 * cos(w * t), F12 * sin(w * t), F3'])';
NFf = (D \ (cross(-A * d, Ff)'))';
Nc = (nc) * [0, 0, -1];

% Circular B
\[ N_m = (D \times (m \times d, 10 \times B \times [0, \cos(w/15 \times t), \sin(w/15 \times t)])'))')'; \]

\[ N = N_{ff} + N_c + N_m + N_H; \]

\[ dx = \begin{bmatrix}
  x(4) \cdot \cos(x(3)) - x(5) \cdot \sin(x(3)) \\
  x(4) \cdot \csc(x(1)) \cdot \sin(x(3)) + x(5) \cdot \csc(x(1)) \cdot \cos(x(3)) \\
  -x(4) \cdot \cot(x(1)) \cdot \sin(x(3)) - x(5) \cdot \cot(x(1)) \cdot \cos(x(3)) + x(6) \\
  (1/I_1) \cdot (N(1) + x(5) \cdot x(6) \cdot (I_2 - I_3)) \\
  (1/I_2) \cdot (N(2) + x(6) \cdot x(4) \cdot (I_3 - I_1)) \\
  (1/I_3) \cdot (N(3) + x(4) \cdot x(5) \cdot (I_1 - I_2)) \\
  x(10) \\
  x(11) \\
  x(12) \\
  (1/M) \cdot (Ff(1) + FH(1)) \\
  (1/M) \cdot (Ff(2) + FH(2)) \\
  (1/M) \cdot (Ff(3) + FH(3)) 
\end{bmatrix}; \]

This file includes the 12 ordinary differential equations to solve. \( x(1), x(2), x(3) \) are the three Euler angles of the bacterium, \( x(4), x(5), x(6) \) its angular velocity, \( x(7), x(8), x(9) \) its position in the laboratory frame, and \( x(10), x(11), x(12) \) its translation velocity. \( D \) is the coordinate transformation matrix between the body frame and the laboratory frame. \( d \) is the orientation vector of the bacterium. \( r, u, p \) are three dummy vector parameters for the subroutine DragTorqCalc which are not used in this program. The subroutine DragTorqCalc calculates the drag force \( FH \) and torque \( NH \) exerted on the bacterium by the flow, given the velocity \( V \), the orientation \( W \), the semi-major and semi-minor axis \( A \) and \( a \). \( N_{ff} \) is the torque by the force (2.6). \( Nc \) is the reaction couple due to the couple that generates flagellar rotation (with frequency \( w \)). \( Nm \) is the torque by the magnetic field.

The subroutine DragTorqCalc is too long to include here. We put it in Appendix A.

### 2.3.2 Effects of the shape of magnetotactic bacteria

It is well known that magnetotactic bacteria possess permanent magnetization and, hence, swim along the lines of an external magnetic field. We know, however, very little about the effect of the shape of a magnetotactic bacterium on its swimming
2. Swimming motion of spheroidal bacteria

Figure 2.5: (a) The diameter of the helical trajectory $D_H$, (b) the diameter of the circular path $D_M$ and (c) the average speed $V_T$ are plotted as a function of eccentricity $E$ for three different strengths of the imposed magnetic field at $B_0 = 0.4 mT, 0.7 mT, 1.0 mT$ with $\Phi = 0$. 

![Image of Figure 2.5](image-url)
2. Swimming motion of spheroidal bacteria

A major objective of our theoretical study is to understand how its shape, marked by the size of its eccentricity $E$, affects the swimming pattern and speed. For achieving this objective, a large number of numerical solutions to the 12 coupled nonlinear equations have been obtained over a wide range of the model parameters, even though only a small number of the solutions are selected to highlight the shape effect in the swimming motion.

When an imposed magnetic field is rotating, the magnetotactic bacteria would swim along a nearly circular path in a helical fashion. In order to underpin/isolate the shape effect, we first fix the external magnetic field $B_L$ in the laboratory frame

$$B_L = 4.0 \times 10^{-4} T \left[ \cos\left(\frac{4\pi}{5} t\right) \hat{Y} + \sin\left(\frac{4\pi}{5} t\right) \hat{Z} \right],$$

while changing only the shape parameter $E$ in our numerical experiments. Moreover, we shall introduce the three quantities to describe a helical swimming motion in the rotating magnetic field (Pan et al., 2009): the diameter of the helical trajectory $D_H$, the diameter of the circular path $D_M$ and the average speed, $V_T$, of the swimming motion defined as

$$V_T = \frac{1}{T_0} \int_0^{T_0} |\mathbf{v}_L| \, dt,$$

where $T_0$ is the period of the imposed rotating magnetic field. Figure 2.4 depicts three trajectories of the swimming motion, representing three different solutions for the 12 coupled differential equations using three different values of eccentricity $E$ while all other parameters of the model remain fixed. The parameters in $F_B$ are chosen as $F_{12} = 4.62 \times 10^{-12}$ kg · m/s², $F_3 = 2.26 \times 10^{-12}$ kg · m/s², and $\omega_0 = 300$ (rad/s) together with $\Phi = 0$ and $N_c = 5.43 \times 10^{-18}$ kg · m²/s². A surprising result is that the pattern of swimming trajectory is strongly affected by the shape of the magnetotactic bacterium. When the shape of the bacterium is nearly spherical with $E = 0.1$, the helical motion is largely confined in the $YZ$-plane with a moderate circular diameter $D_M$. When the bacterium is in the shape of an elongated spheroid with $E = 0.5$, the helical motion is less confined in the $YZ$-plane with the circular diameter $D_M$ becoming much smaller, which is shown in Figure 2.4. When the shape of the bacterium is slightly elongated further to $E = 0.6$, the circular diameter $D_M$ suddenly increases to a much larger size. In short, the pattern of swimming trajectory – under the assumption that the flagellar power $F_B$ is fixed – is highly sensitive to the shape of magnetotactic bacteria in a strongly nonlinear way. This behavior should be perhaps anticipated as the viscous drag $D_B$ in (2.10) and the viscous torque $T_B$ in (2.16) are a strongly nonlinear function of $E$ (or $\xi_0$).

In an effort to further illustrate the shape influence on the swimming motion, we have also computed the swimming trajectories with different values of $E$ at various strengths of the imposed magnetic field. The results are summarized in Figure 2.5.
2. Swimming motion of spheroidal bacteria

Figure 2.6: The average speed $V_T$ is plotted as a function of eccentricity $\mathcal{E}$ for three different values of $B_0$ at $\Phi = 90^\circ$ with other parameters being the same as those in Figure 2.5.

which makes use of the imposed field

$$B_L = B_0 \left[ \cos\left(\frac{4\pi}{5}t\right)\hat{Y} + \sin\left(\frac{4\pi}{5}t\right)\hat{Z} \right]$$

at three different values of $B_0$. It reinforces the primary feature observed in Figure 2.4: the dynamics of the swimming motion is a strongly nonlinear function of the shape parameter. For spheroidal shape with moderate eccentricity $0 < \mathcal{E} < 0.5$, the swimming speed $V_T$ increases with increasing strength of the imposed magnetic field. For an highly elongated spheroid at $\mathcal{E} \approx 0.55$, as shown in Figure 2.5(b), however, this dependence becomes reversed: the swimming speed $V_T$ decreases with increasing strength of the imposed magnetic field. The swimming parameters $D_M$ and $D_H$ also exhibit a similar dependence on the shape parameter $\mathcal{E}$. In spherical geometry, the size of the drag force is independent of the direction of the velocity and the size of the viscous torque is also independent of the direction of the angular velocity. It follows that, when the shape parameter $\mathcal{E}$ is small, its dynamical behavior must be similar to that of spherical shape as the departure from spherical geometry can be treated as small perturbations. When eccentricity $\mathcal{E} \approx 0.5$, however, the spheroidal effect enters the leading order of its nonlinear dynamics and, thus, can be no longer regarded as small perturbations. This result confirms an expectation that the dynamics/pattern of the swimming motion, reflecting the nature of the highly nonlinear governing equations, is intricately determined by the competing influence of viscous (which sensitively depends on the size of $\mathcal{E}$) and magnetic (which depends on both the size and direction of $\mathbf{m}$) effects.

This complicated competition becomes more apparent in an unexpected exper-
imental observation that the swimming speed $V_T$ for some magnetotactic bacteria – whose magnetosome chain is inclined with the flagellar propulsion axis – decreases with increasing strength of the imposed magnetic field (Pan et al., 2009). In other words, the angle $\Phi$ (between the magnetic moment vector $m$ and the positive direction of the coordinates $oz$) in these magnetotactic bacteria is nonzero. To demonstrate how the angle $\Phi$ alters the dynamics of the swimming motion, we have repeated the similar computation by changing $\Phi = 0$ to $\Phi = 90^\circ$ while keeping other parameters unchanged. The results of the computation with $\Phi = 90^\circ$ are summarized in Figure 2.6, showing the average speed $V_T$ as function of eccentricity $E$ at three different values of $B_0$. An important feature is that, when comparing Figure 2.6 to Figure 2.5(c), the swimming speed $V_T$ at the same power $F_B$ decreases with increasing strength of the imposed magnetic field for all sizes of the shape parameter $E$. It is worth mentioning that, during the swimming motion, the magnetic moment vector $m$, as well as the symmetry axis $\hat{z}$, may be not parallel to the external magnetic field. As the magnetic torque increases, which has a different direction from that of the viscous effect, it can change the viscous effect and, hence, change the efficiency of a bacterium swimming at the same flagellar power $F_B$.

### 2.3.3 Swimming Motion: Theoretical vs. Experimental

In our experimental study of the swimming motion, the magnetotactic bacteria found in Lake Miyun near Beijing illustrated in Figure 2.7 are employed. The bacteria sample are put on a glass slide, which is put into a microscope. There is a CCD camera connected to the microscope, such that it could record the motion of the bacteria in the water sample. The records by the CCD are all black-white. As a key character, there is an external artificial magnetic field $B_L$ imposed on the system. Three different types of the magnetic field are used in the study: (i) steady, (ii) rotating and (iii) discontinuously changing. The corresponding numerical simulations are also carried out, unveiling that the laboratory trajectories of the swimming motion can be approximately reproduced by our numerical model for a spheroidal magnetotactic bacterium under the influence of the same applied magnetic field.

The first type of the imposed magnetic field is steady and time-independent in the laboratory frame,

$$B_L = -4.00 \times 10^{-4} T \hat{X}.$$  \hfill (2.20)

Figure 2.8 shows the swimming trajectories of a magnetotactic bacterium under the influence of the steady magnetic field (2.20). The top panel depicts a swimming trajectory that is recorded in the laboratory using charge-coupled device camera while the trajectory in the lower panel represents a numerical solution for the 12 coupled differential equations with the parameters $F_{12} = 2.41 \times 10^{-13} \text{ kg} \cdot \text{m/s}^2$, $F_3 = 1.41 \times 10^{-13} \text{ kg} \cdot \text{m/s}^2$, $\omega_0 = 94.24 \text{ rad/s}$, $N_c = 2.71 \times 10^{-19} \text{ kg} \cdot \text{m}^2/\text{s}^2$ and
2. Swimming motion of spheroidal bacteria

Figure 2.7: A microscope image of magnetotactic bacteria that is of spheroidal shape and has many bullet-shaped magnetite magnetosomes nearly parallel to the major axis of the spheroidal bacterium.

\[ \mathbf{m} = 1.80 \times 10^{-15} \hat{z} \, (Am^2) \]. The both trajectories, experimental and theoretical, describe the helical motion of a magnetotactic bacterium along the straight line of an imposed magnetic field (2.20).

In the second case, a time-dependent rotating magnetic field in the laboratory frame,

\[
\mathbf{B}_L = 4.00 \times 10^{-4} T \left[ \cos\left(\frac{4\pi}{5} t\right) \hat{Y} + \sin\left(\frac{4\pi}{5} t\right) \hat{Z} \right],
\]  

(2.21)
is imposed for both the experimental and numerical studies. Four different trajectories of the swimming motion recorded with a charge-coupled device camera are presented in the left panel of Figure 2.9. Different patterns observed under the same laboratory condition are interpreted as being different bacteria that have different powers of the flagellar motor. They can be therefore simulated by employing different parameter $F_B$ in our theoretical model. The right panel of Figure 2.9 shows the four trajectories computed from the 12 coupled differential equations with different powers $F_B$. For the trajectory with the largest radius of swimming path (at the top of the right panel in Figure 2.9) we take $F_{12} = 1.44 \times 10^{-13}$ kg·m/s², $F_3 = 2.12 \times 10^{-13}$ kg·m/s² and $\omega_0 = 103.67$ (rad/s) in our numerical simulation. For the trajectory with a slightly smaller radius of the circular path (at the middle of the right panel in Figure 2.9), the numerical simulation takes $F_{12} = 4.81 \times 10^{-13}$ kg·m/s², $F_3 = 2.26 \times 10^{-13}$ kg·m/s² and $\omega_0 = 235.62$ rad/s. For the two trajectories with the small radius (at the bottom of the right panel in Figure 2.9), we use $F_{12} = 1.44 \times 10^{-13}$ kg·m/s², $F_3 = 1.56 \times 10^{-13}$ kg·m/s² and $\omega_0 = 103.67$ rad/s in our numerical simulation. All the numerical trajectories have adopted $N_c = 5.43 \times 10^{-19}$ (kg·m²/s²), $\varepsilon = 0.2$ and $m = 1.80 \times 10^{-15}\hat{z}$ Am². It can be seen that the laboratory observations can be largely reproduced by the solutions of our theoretical model on the basis of the assumption that different magnetotactic bacteria can have different powers with their flagellar motors.

![Figure 2.9: The swimming trajectories of magnetotactic bacteria under the influence of a rotating magnetic field (2.21): the left panel is recorded in a laboratory experiment with a charge-coupled device camera while the right panel showing numerical solutions to the 12 coupled differential equations having different parameter $F_B$.](image)

In the third case, we first impose a time-dependent magnetic field in the labora-
Swimming motion of spheroidal bacteria

In the laboratory frame
\[ \mathbf{B}_L = 1.96 \times 10^{-4} T \left[ \cos\left(\frac{4\pi}{5}t\right) \hat{Y} + \sin\left(\frac{4\pi}{5}t\right) \hat{Z} \right] \] (2.22)

for \( 0 \leq t < t_0 = 1.873 \) s in both the experimental and numerical studies. At \( t = t_0 \), the imposed magnetic field is suddenly switched to a stationary field
\[ \mathbf{B}_L = 1.96 \times 10^{-4} T \hat{Y}. \] (2.23)

Figure 2.10 shows the trajectories of a magnetotactic bacterium under the influence of a suddenly changing magnetic field. In the left panel of Figure 2.10, the swimming trajectory, recorded with a charge-coupled device camera in the laboratory, switches suddenly from an uncompleted circular path to a nearly straight line following the variation of the applied magnetic field. In the corresponding numerical solution of the 12 coupled differential equations, which is illustrated in the right panel of Figure 2.10, we have adopted the same time-dependent magnetic field with \( F_{12} = 1.24 \times 10^{-13} \) kg \( \cdot \) m/s\(^2\) \( F_3 = 1.48 \times 10^{-13} \) kg \( \cdot \) m/s\(^2\) \( \omega_0 = 94.24 \) rad/s, \( N_C = 5.70 \times 10^{-19} \) kg \( \cdot \) m\(^2\)/s\(^2\) and \( \mathbf{m} = 1.80 \times 10^{-15} \hat{z} Am^2 \). It demonstrates again that the sudden change observed in the laboratory—from a circular path to a nearly straight line following the variation of \( \mathbf{B}_L \)—can be reproduced in our theoretical model.

Figure 2.10: The swimming trajectories of a magnetotactic bacterium under the influence of a time-dependent magnetic field described by (2.22) and (2.23): the left panel shows the experimental record using a charge-coupled device camera while the right panel from the corresponding numerical solution to the 12 coupled differential equations.

2.4 Summary and Some Remarks

We have studied, via both theoretical and experimental methods, the swimming motion of spheroidal magnetotactic bacteria in a viscous liquid under the influence of an imposed magnetic field. The emphasis of the study is placed on how
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the shape of a non-spherical magnetotactic bacterium, marked by the size of its spheroidal eccentricity, affects the pattern/speed of its swimming motion. A fully three-dimensional Stokes flow, driven by the translation and rotation of a swimming non-spherical bacterium, exerts the complicated viscous drag and torque on the motion of the bacterium. Under the assumptions that the body of the bacterium is non-deformable and that interactions between different bacteria are weak and negligible, we have derived a new system of the 12 coupled ordinary differential equations governing both the motion and orientation of a swimming spheroidal magnetotactic bacterium. We have revealed, for the first time, that the pattern and speed of the swimming motion are highly sensitive to the shape of a non-spherical bacterium. Using magnetotactic bacteria collected from Lake Miyun near Beijing, China, we have demonstrated that the theoretical swimming patterns described by solutions of the 12 coupled differential equations are largely similar to those observed in the laboratory experiments under the influence of external magnetic fields.

The present study examines the swimming motion of magnetotactic bacteria from a fluid dynamical point of view in association with the fully three-dimensional Stokes flow of a translational and rotational prolate spheroid of arbitrary eccentricity. Although the Stokes’ approximation neglecting the inertial term in the Navier-Stokes equation is appropriate for the slowly swimming motion of magnetotactic bacteria, the weakest link between our theoretical model and the real laboratory experiment is the two assumptions: (i) the body of a magnetotactic bacterium is non-deformable and (ii) the rapid rotation of helical flagellar filaments powering its swimming motion can be modeled simply by a prescribed force $F_B$. In the present model, the swimming pattern and speed are largely determined by the interplay between the viscous and magnetic torques on a non-deformable spheroidal body. The possible important physiological effects – for example, the power of helical flagellar filaments $F_B$ may depend on both the strength and direction of the external magnetic field and the shape of magnetotactic bacteria which sense the form of the magnetic field may become deformable – are completely neglected in our theoretical model.

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Chapter 3

The completeness of inertial wave modes in rotating annular channels

3.1 Introduction

Oscillatory motion, restored by the Coriolis force, can occur in a homogeneous fluid within a liquid-filled closed container that rotates at constant angular velocity (for example, Poincaré 1885, Greenspan 1968, Tilgner 2007a). This type of oscillatory motion, usually referred to as inertial waves, is ubiquitous in rotating fluid systems, ranging from geophysical and astrophysical problems to the fuel tank in spacecraft and liquid-filled ballistics which spin rapidly in flight. The inertial waves can be excited and maintained, for example, by thermal instabilities (e.g., Zhang, 1995; Busse and Simitev, 2004), by precession (e.g., Gans 1970, Zhang et al. 2010, Wei and Tilgner 2013, Triana et al. 2012), by tidal effects (e.g., Kerswell 1994), by planetary libration (e.g., Aldridge and Toomre 1969, Rieutord 1991, Noir et al. 2009, Zhang et al. 2012) and by differential rotation (e.g., Kelley et al. 2007, Tilgner 2007b). In rotating giant planets for which the tidal forcing frequencies are typically comparable to the spin frequency of the planets, it has been argued that inertial waves provide an effective mechanism for tidal dissipation (e.g., Ogilvie and Lin 2004). There also exist geomagnetic variations of non-axisymmetric magnetic flux, dominated by a single wavenumber at the core surface in the equatorial region, that are likely to be indicative of a magnetically modified equatorially trapped inertial wave (Finlay and Jackson 2003, Zhang 1993).

The present study is concerned with the completeness of inertial wave modes in an incompressible fluid confined in a closed container with bounding surface $S$ and outward normal $\hat{n}$ which rotates uniformly with angular velocity $\hat{z}\Omega_0$, where $\hat{z}$ is a unit vector. In the inviscid limit, an inertial wave (or an inertial mode) is usually described by its velocity $u_{mnk}$ satisfying the boundary condition $\hat{n} \cdot u_{mnk} = 0$ on $S$,
its frequency $\omega_{mnk}$ ($0 < |\omega_{mnk}| < 2$) and its pressure field $p_{mnk}$, where the subscripts $m, n, k$ denote three wavenumbers in the three different directions (Greenspan 1968). Suppose that explicit analytical solutions for the velocity of the inertial mode, $u_{mnk}$, exist and are normalized such that

$$\int_V u_{mnk}^* \cdot u_{mnk} \, dV = \int_V |u_{mnk}|^2 \, dV = 1,$$

where $u_{mnk}^*$ denotes the complex conjugate of $u_{mnk}$ and $V$ is the volume of the container. A fundamentally important question is whether the inertial modes $u_{mnk}$ (or the inviscid eigenfunctions) in rapidly rotating systems form a complete system of functions. If they are complete, an arbitrary flow velocity, $u$, in an incompressible fluid ($\nabla \cdot u = 0$) confined within the rotating container can always be approximated in the manner

$$\int_V \left| u - \sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{k=1}^{K} A_{mnk} u_{mnk} \right|^2 \, dV \to 0 \quad (3.1)$$

as $M \to \infty, N \to \infty$ and $K \to \infty$, where $A_{mnk}$ denote the expansion coefficients. We then state that the expansion (3.1) converges in the mean-square sense (or the $L^2$ sense) to an arbitrary velocity $u$ in $V$.

The completeness of the inertial modes offers an effective, powerful and straightforward way of solving many fluid dynamical problems in rapidly rotating systems. Suppose that the fluid motion $u$, driven by an external force $f(r, t)$, in the inviscid limit is piecewise continuous and differentiable, and is governed by

$$\frac{\partial u}{\partial t} + 2 \hat{z} \times u + \nabla p = f(r, t), \quad \nabla \cdot u = 0, \quad (3.2)$$

subject to the initial condition $u(r, t = 0) = u_0$. The mathematical problem of finding a solution to the partial differential equations (3.2) can be readily solved by making use of the expansions for $u$ and $p$. After a straightforward manipulation, a solution to (3.2) can be obtained by solving the system of ordinary differential equations

$$\frac{dA_{mnk}}{dt} - i\omega_{mnk} A_{mnk} = \int_V u_{mnk}^* \cdot f \, dV \quad (3.3)$$

integrating from the given initial condition $u = u_0$. In principle, there are no difficulties in carrying out all the integrations in (3.3) if explicit expressions for $u_{mnk}$ are available. For the steady problem, the solution $u$ to (3.2) satisfying the boundary condition is simply

$$u = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{k=1}^{K} \left( \frac{i \int_V u_{mnk}^* \cdot f \, dV}{\omega_{mnk}} \right) u_{mnk}(r). \quad (3.4)$$
For the nonlinear problem, this approach can be readily modified by including the term $\mathbf{u} \cdot \nabla \mathbf{u}$ on the right-hand side of (3.3). For the viscous problem, it can be also modified by including the effect of the viscous boundary layers (Greenspan 1968).

Under an assumption that the inertial wave modes $\mathbf{u}_{mnk}$ are complete, an asymptotic solution has been derived to describe the flow of a *viscous* homogeneous fluid confined in either a circular cylinder (Liao and Zhang 2012) or an annular channel (Zhang et al. 2010) that rotates rapidly about its symmetry axis and precesses slowly about a different axis that is fixed in space. By expanding the velocity $\mathbf{u}$ and pressure $p$ in terms of the inertial modes, there is a general asymptotic solution for an asymptotically small Ekman number describing the weakly precessing flow that is valid at or near or away from resonance (Zhang et al. 2010). Numerical analysis of the same problem was also carried out to demonstrate that an excellent agreement between the general asymptotic solution and the corresponding numerical solution is achieved at or near or away from resonance. It is of primary importance to notice that the fluid is viscous and the velocity of a precessing flow $\mathbf{u}$ is *piecewise continuous* and differentiable.

This mathematical approach has at least four advantages: (i) the inertial modes $\mathbf{u}_{mnk}$ have already accommodated some key dynamics of rotating flow in the weakly nonlinear regime, (ii) the incompressible condition $\nabla \cdot \mathbf{u} = 0$ is automatically satisfied, (iii) the normal boundary condition, $\hat{n} \cdot \mathbf{u} = 0$ on $S$, is also automatically satisfied and (iv) more significantly, the eigenfunctions $\mathbf{u}_{mnk}$, in contrast to other widely-used orthogonal functions such as Legendre polynomials, are directly associated with the differential operator on the left-hand side of (3.2a) and, therefore, the system (3.3) becomes decoupled. This highly effective approach, however, hinges on the mathematical question of the completeness of the inertial modes $\mathbf{u}_{mnk}$. Greenspan (1968) in his classical monograph stated that “In view of the rather peculiar features of Poincaré’s problem, attention must be directed to some of the unanswered mathematical questions. The most obvious of these concern the completeness of the inviscid eigenfunctions ...”. Whether a set of the inertial modes in a rotating system is complete represents a difficult, challenging mathematical problem.

Motivated by the wish to understand the dynamics of rotating flows taking place in the equatorial region or middle latitudes of planets and stars (Davies-Jones and Gilman 1971, Gilman 1973, Busse 2005) proposed the model of a rotating annular channel in which the gap-width of an annular channel is assumed to be sufficiently small in comparison with its radius so that the curvature effect can be neglected. There are two significant advantages in employing a channel-geometry model. First, the annular channel geometry is readily experimentally realizable. Second, the mathematical simplicity and clarity of planar geometry using cartesian coordinates remain unchanged in channel geometry, allowing a relatively simple analytical description of the inertial waves (Liao and Zhang 2009).
3. Completeness of inertial wave modes

This paper represents the first attempt to answer the mathematical question of the completeness of the inviscid eigenfunctions by choosing the simple geometry of a rotating annular channel. In what follows we shall begin in section 2 by discussing the rotating channel model. A detailed mathematical proof of the completeness for the axisymmetric problem is presented in section 3 and its extension to the general three-dimensional problem is given in section 4. The paper closes in section 5 with a summary and some remarks.

3.2 Convergence of Fourier series as an example

In order to make it more clear, we first briefly study the classical Fourier series, as an example for the following discussion of the special completeness problem. For a periodic function \( f(x) \), we can get its Fourier series over the interval \([-1, 1]\) with respect to the following functions

\[
\{1, \cos(k\pi t), \sin(k\pi t)\}_{k=1,2,3,...} \tag{3.5}
\]

for which the partial sum is

\[
s_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos(k\pi x) + b_k \sin(k\pi x)), \tag{3.6}
\]

where

\[
a_k = \int_{-1}^{1} f(t) \cos(k\pi t) dt, \quad k = 0, 1, 2, \ldots \tag{3.7}
\]

\[
b_k = \int_{-1}^{1} f(t) \sin(k\pi t) dt, \quad k = 1, 2, 3, \ldots \tag{3.8}
\]

Putting the expressions of \( a_k, b_k \) into (3.6), we can get its integration form

\[
s_N(x) = \int_{-1}^{1} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{N} \cos(k\pi(x - t)) \right\} dt
\]

\[
= \int_{-1}^{1} f(t) \frac{\sin((N + \frac{1}{2})(\pi(x - t)))}{2 \sin(\frac{\pi(x-t)}{2})} dt
\]

\[
= \int_{\pi x - \pi}^{\pi x + \pi} f(x - \frac{y}{\pi}) \frac{\sin((N + \frac{1}{2})y)}{2 \sin(\frac{y}{2})} \left(-\frac{dy}{\pi}\right)
\]

(substituting \( \pi(x - t) \) with \( y \), and now \( f \) for the variable \( y \) is periodic-2\( \pi \))

\[
(3.9)
\]

Here for each \( x \) the function \( f(-\frac{x}{\pi} + x) \) is a function of \( y \) of period 2\( \pi \). Further for the value \( \pi x \), we can find a largest integer \( k \) such that 2\( k \pi \) is no larger than \( \pi x \), so
we must have \( \pi x - 2k\pi < 2\pi \), and \( \pi x - \pi > 2k\pi + \pi \). Now we have

\[
\int_{\pi x + \pi}^{\pi x - \pi} f(x - y) \frac{\sin((N + \frac{1}{2})y)}{2 \sin(\frac{y}{2})} \left(-\frac{dy}{\pi}\right)
= \int_{\pi x + \pi}^{2k\pi + \pi} f(x - y) \frac{\sin((N + \frac{1}{2})y)}{2 \sin(\frac{y}{2})} \left(-\frac{dy}{\pi}\right) + \int_{2k\pi + \pi}^{\pi x - \pi} f(x - y) \frac{\sin((N + \frac{1}{2})y)}{2 \sin(\frac{y}{2})} \left(-\frac{dy}{\pi}\right)
= \int_{2k\pi + \pi}^{\pi x - \pi} f(x - y) \frac{\sin((N + \frac{1}{2})(y + 2\pi))}{2 \sin(\frac{y + 2\pi}{2})} \left(-\frac{dy}{\pi}\right) + \int_{2k\pi + \pi}^{\pi x - \pi} f(x - y) \frac{\sin((N + \frac{1}{2})y)}{2 \sin(\frac{y}{2})} \left(-\frac{dy}{\pi}\right)
= \int_{\pi x - \pi}^{-\pi} f(x - y) \frac{\sin((N + \frac{1}{2})(y))}{2 \sin(\frac{y}{2})} \left(-\frac{dy}{\pi}\right)
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - y) \frac{\sin((N + \frac{1}{2})(y))}{2 \sin(\frac{y}{2})} dy
= \frac{1}{2\pi} \int_{0}^{\pi} [f(x - \frac{y}{\pi}) + f(x + \frac{y}{\pi})] \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} dy. \tag{3.10}
\]

For this integration, we have the classical result, which is a weak version of the Jordan’s test,

\[ If \ f(x) \ is \ a \ periodic-2 \ function \ which \ is \ differentiable \ (and \ so \ continuous) \ everywhere, \ then \ s_N(x) \ converges \ to \ f(x) \ uniformly \ over \ [-1, 1]. \tag{3.11} \]

Here we list a proof from (Edwards, 1979). If we directly cite this theorem, then we will not need to prove it again in the following.

Denote \( g_x(y) = \frac{1}{\pi} [f(x - \frac{y}{\pi}) + f(x + \frac{y}{\pi})] \). It will be sufficient to show that uniformly for \( x \) we have

\[
\lim_{N \to +\infty} \frac{1}{\pi} \int_{0}^{\pi} g_x(y) \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} dy = g_x(0). \tag{3.12}
\]

Because

\[
\frac{1}{\pi} \int_{0}^{\pi} \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} dy
= \frac{1}{\pi} \int_{0}^{\pi} \left\{ 1 + 2 \sum_{k=1}^{N} \cos(ky) \right\} dy
= 1, \tag{3.13}
\]
we can write
\[ g_x(0) = g_x(0) \cdot \frac{1}{\pi} \int_0^\pi \sin((N + \frac{1}{2})y) \frac{1}{\sin(\frac{y}{2})} \, dy \]
\[ = \frac{1}{\pi} \int_0^\pi g_x(0) \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} \, dy, \] (3.14)
and only need to prove
\[ \lim_{N \to +\infty} \frac{1}{2\pi} \int_0^\pi (g_x(y) - g_x(0)) \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} \, dy = 0. \] (3.15)

Because \( g_x(y) \) is differentiable, there is a neighborhood interval of 0 where \( g_x(y) \) is monotone. Specially we suppose \( g_x(y) \) is nondecreasing over \([0, \delta_x]\). By the second mean value theorem of the integral calculus, we have
\[
\frac{1}{2\pi} \int_0^\pi (g_x(y) - g_x(0)) \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} \, dy
= \frac{1}{2\pi} \left( \int_0^{\delta_x} + \int_{\delta_x}^\pi \right) (g_x(y) - g_x(0)) \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} \, dy
= \frac{1}{2\pi} (g_x(\delta_x) - g_x(0)) \int_{\xi_{x,N}}^{\delta_x} \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} \, dy + \frac{1}{2\pi} \int_{\delta_x}^\pi (g_x(y) - g_x(0)) \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} \, dy
\] (3.16)
for some \( \xi_{x,N} \) in \([0, \delta_x]\).

Here lists the first and second mean value theorems.

The symbol \([a, b]\) is always an interval in \( \mathbb{R} \).

**Theorem** [First Mean-Value Theorem]: Let \( F, G \) be differentiable functions on \([a, b]\) with \( G \) increasing. Let \( m, M \) being respectively the minimum and maximum values of \( F \) on \([a, b]\). Then there exists \( \xi \in [a, b] \) such that
\[
\int_a^b F(x)G'(x) \, dx = F(\xi)(G(b) - G(a)).
\]

**Proof:** First we have
\[
m \leq F(x) \leq M, \quad \forall x \in [a, b],
\]
thus
\[
m(G(b) - G(a)) \leq \int_a^b F(x)G'(x) \, dx \leq M(G(b) - G(a)).
\]
If \( G(a) = G(b) \), we have \( G'(x) = 0, \forall x \in [a, b] \), thus \( \int_a^b F(x)G'(x) \, dx = f(\xi)(G(b) - G(a)) \).
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\( G(a) = 0 \), where \( \xi \) could be any \( \xi \in [a, b] \). If \( G(a) < G(b) \), we have

\[
m \leq \frac{\int_a^b F(x)G'(x)dx}{G(b) - G(a)} \leq M.
\]

By the ordinary mean value theorem for continuous function, there is a \( \xi \in [a, b] \) such that

\[
F(\xi) = \frac{\int_a^b F(x)G'(x)dx}{G(b) - G(a)}.
\]

**Theorem** [Second Mean-Value Theorem]: If \( F, G \) are differentiable functions over \([a, b]\) with \( F \) increasing. Then there is a point \( \xi \in [a, b] \) such that

\[
\int_a^b F(x)G'(x)dx = F(a)(G(\xi) - G(a)) + F(b)(G(b) - G(\xi)).
\]

**Proof:** Integrating by parts, we have

\[
\int_a^b F(x)G'(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b G(x)F'(x)dx.
\]

Applying the First Mean-Value Theorem to the integral \( \int_a^b G(x)F'(x)dx \) (here the roles of symbols \( F, G \) are exchanged with respect to the first theorem), we get

\[
\int_a^b F(x)G'(x)dx = F(b)G(b) - F(a)G(a) - G(\xi)(F(b) - F(a))
\]

\[
= F(a)(G(\xi) - G(a)) + F(b)(G(b) - G(\xi)),
\]

(3.17)

for some \( \xi \in [a, b] \).

Here to use the second theorem, we take \( F(t) = g_x(t) - g_x(0), G(t) = \int_0^t \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} dy \), and \([a, b] = [0, \delta_x]\). In this way we could get a \( \xi_{x,N} = \xi \). Specially we have

\[
G(b) - G(\xi) = \int_0^{\delta_x} \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} dy - \int_{\xi_{x,N}}^{\delta_x} \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} dy
\]

\[
= \int_{\xi_{x,N}}^{\delta_x} \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} dy,
\]

(3.18)

which is the integration in the first term in (3.16).

Thus

\[
\left| \frac{1}{2\pi} \int_0^\pi (g_x(y) - g_x(0)) \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} dy \right|
\]

\[
\leq \frac{1}{2\pi} (g_x(\delta_x) - g_x(0)) \left| \int_{\xi_{x,N}}^{\delta_x} \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} dy \right| + \frac{1}{2\pi} \left| \int_0^\pi (g_x(y) - g_x(0)) \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{y}{2})} dy \right|.
\]

(3.19)
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Now

\[
\left| \frac{1}{\pi} \int_{\xi_{x,N}}^{\delta_x} \frac{\sin((N + \frac{1}{2})y)}{2\sin(\frac{y}{2})} dy \right| = \left| \frac{1}{\pi} \int_{\xi_{x,N}}^{\delta_x} \left\{ \frac{\sin((N + \frac{1}{2})y)}{y} - \sin((N + \frac{1}{2})y) \left( \frac{1}{y} - \frac{1}{2\sin(\frac{y}{2})} \right) \right\} dy \right| \\
\leq \left| \frac{1}{\pi} \int_{\xi_{x,N}}^{(N + \frac{1}{2})\delta_x} \frac{\sin(t)}{t} dt \right| + \frac{1}{\pi} \int_{0}^{\pi} \left| \frac{1}{y} - \frac{1}{2\sin(\frac{y}{2})} \right| dy.
\]

(3.20)

For the first integration, we can see that the integrand function \( \frac{\sin(t)}{t} \) is alternately positive and negative over \( \mathbb{R}^+ \), and the period is always \( 2\pi \). For any interval \([ (N + \frac{1}{2})\xi_{x,N}, (N + \frac{1}{2})\delta_x ] \), if it’s not longer than \( 2\pi \), then the absolute value of the integration is smaller than a positive number, say \( C \), which is independent of the interval, because the integrand is absolutely bounded over \( \mathbb{R}^+ \). If the interval is longer than \( 2\pi \), then we can find a smallest number \( a \) such that \( a \geq (N + \frac{1}{2})\xi_{x,N} \) and \( a = 2k\pi \) for some integer \( k \), and also a largest number \( b \) such that \( b \leq (N + \frac{1}{2})\delta_x \) and \( b = 2m\pi \) for some integer \( m > k \). So the integration over \([ (N + \frac{1}{2})\xi_{x,N}, (N + \frac{1}{2})\delta_x ] \) is divided into three parts, that is, the integration over \([ (N + \frac{1}{2})\xi_{x,N}, 2k\pi ] \), \([ 2k\pi, 2m\pi ] \), and \([ 2m\pi, (N + \frac{1}{2})\delta_x ] \) respectively. The first and third part are bounded by \( C \). Now we show that the second integration is bounded independent of the length of the interval, that is, independent of \( k, m \).

In one period, we have

\[
\left| \int_{2n\pi}^{(2n+2)\pi} \frac{\sin(t)}{t} dt \right| = \left| -\cos(t) \right|_{2n\pi}^{(2n+2)\pi} - \int_{2n\pi}^{(2n+2)\pi} \frac{\cos(t)}{t^2} dt \leq \frac{1}{2n\pi} - \frac{1}{(2n + 1)\pi} - \int_{2n\pi}^{(2n+2)\pi} \frac{\cos(t)}{t^2} dt \leq \frac{1}{2n(n + 1)\pi} \leq \frac{1}{n^2}.
\]

(3.21)
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Thus, over the interval $[2k\pi, 2m\pi]$,

$$\left| \int_{2k\pi}^{2m\pi} \frac{\sin(t)}{t} dt \right| \leq \sum_{n=k}^{m-1} \frac{1}{n^2} \leq 2, \quad \forall \, k, \, m \in \mathbb{Z}^+. \quad (3.22)$$

So now, in the interval $[(N + \frac{1}{2})\xi_{x,N} (N + \frac{1}{2})\delta_x]$, we have

$$\left| \frac{1}{\pi} \int_{(N + \frac{1}{2})\xi_{x,N}}^{(N + \frac{1}{2})\delta_x} \frac{\sin(t)}{t} dt \right| \leq \left| \frac{1}{\pi} \int_{2k\pi}^{2m\pi} \frac{\sin(t)}{t} dt \right| + \left| \frac{1}{\pi} \int_{2k\pi}^{2m\pi} \frac{\sin(t)}{t} dt \right| + \left| \frac{1}{\pi} \int_{2m\pi}^{2k\pi} \frac{\sin(t)}{t} dt \right| \leq 1 + 2C. \quad (3.23)$$

For the second integration in (3.20), because

$$\lim_{y \to 0} \left( \frac{1}{y} - \frac{1}{2 \sin(y/2)} \right) = 0, \quad (3.24)$$

so the integrand is bounded over $[0, \pi]$ and thus its integration is bounded by a constant number independent of $x$ and $N$, say $B$.

Now we have found that in (3.20) both integrations are bounded independent of $x$ and $N$,

$$\left| \frac{1}{\pi} \int_{\xi_{x,N}}^{\delta_x} \sin((N + \frac{1}{2})y) \frac{1}{2 \sin(y/2)} dy \right| \leq 1 + 2C + B. \quad (3.25)$$

Given any small positive number $\epsilon$. First for each $x$ we assume $(1 + 2C + B) \cdot (g_x(\delta_x) - g_x(0)) \leq \epsilon$, and if not so we can choose smaller $\delta_x$ so as to meet this condition without losing the property that $g_x(y)$ is monotonically nondecreasing over $[0, \delta_x]$. Then we have obtained for (3.20)

$$\left| \frac{1}{2\pi} \int_0^\pi (g_x(y) - g_x(0)) \frac{\sin((N + \frac{1}{2})y)}{\sin(y/2)} dy \right| \leq \epsilon + \left| \frac{1}{2\pi} \int_{\delta_x}^{\pi} (g_x(y) - g_x(0)) \frac{\sin((N + \frac{1}{2})y)}{\sin(y/2)} dy \right|. \quad (3.26)$$
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For the second term,

\[
\frac{1}{2\pi} \int_{\delta_x}^{\pi} (g_x(y) - g_x(0)) \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{\pi}{2})} dy
\]

\[
= \left( \frac{g_x(y) - g_x(0)}{\sin(\frac{\pi}{2})} \right) \cdot \frac{-\cos((N + \frac{1}{2})y)}{N + \frac{1}{2}} \bigg|_{\delta_x}^{\pi} + \int_{\delta_x}^{\pi} \frac{\partial}{\partial y} \left( \frac{g_x(y) - g_x(0)}{\sin(\frac{\pi}{2})} \right) \cdot \frac{\cos((N + \frac{1}{2})y)}{N + \frac{1}{2}} dy,
\]

so it converges to zero when \( N \) turns to \(+\infty\).

Now we get

\[
\lim_{N \to +\infty} \left| \frac{1}{2\pi} \int_{0}^{\pi} (g_x(y) - g_x(0)) \frac{\sin((N + \frac{1}{2})y)}{\sin(\frac{\pi}{2})} dy \right| \leq \epsilon.
\]

Since \( \epsilon \) is free chosen, the convergence is thus established.

3.3 The geometry of the problem

Consider an incompressible fluid of kinematic viscosity \( \nu \) and density \( \rho \) confined in an annulus of inner radius \( r_id \) and outer radius \( r_od \) with depth \( d \), rotating uniformly about its vertical symmetry axis with a constant angular velocity \( z \Omega_0 \). The aspect ratio of the annulus in the cross section is denoted by \( \Gamma_y = (r_od - r_id)/d \). When the width of the annulus \( \Gamma_yd \) is much smaller than the outer radius \( r_od \), i.e., \( \Gamma_y/r_od \ll 1 \), one can make a local approximation by neglecting the curvature effect of the rotating annulus (Davies-Jones and Gilman 1971, Gilman 1973, Busse 2005). This small-gap approximation allows us to use simple cartesian coordinates: azimuthal coordinate \( x \), vertical coordinate \( z \) and inward radial coordinate \( y \), with unit vectors \( \hat{x}, \hat{y}, \hat{z} \).

In the inviscid limit, the eigenfunction \( (u_{mnk}, p_{mnk}, \omega_{mnk}) \) describing inertial waves or inertial oscillations is governed by the dimensionless equations (Liao and Zhang 2009)

\[
i\omega_{mnk} u_{mnk} + 2\hat{z} \times u_{mnk} + \nabla p_{mnk} = 0, \quad \nabla \cdot u_{mnk} = 0,
\]

subject to the boundary condition

\[
\hat{n} \cdot u_{mnk} = 0
\]

on the bounding surface, \( S \), of the container. In the subscript notation, \( m \) denotes the azimuthal wavenumber, \( n \) the vertical wavenumber and \( k \) the radial wavenumber. Moreover, we have employed depth \( d \) as the length scale and \( \Omega_0^{-1} \) as the unit of time in scaling the dimensional equations.

The boundary condition (3.30) is explicitly described as follows. On the top and
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bottom of the channel, we impose that

\[ \hat{z} \cdot \mathbf{u}_{mnk} = 0 \quad \text{at} \quad z = 0, 1. \]  

(3.31a)

On the two sidewalls of the channel, we have

\[ \hat{y} \cdot \mathbf{u}_{mnk} = 0 \quad \text{at} \quad y = 0, \Gamma_y. \]  

(3.31b)

The usual periodic condition is imposed in the azimuthal direction

\[ \mathbf{u}_{mnk}(x = 0) = \mathbf{u}_{mnk}(x = \Gamma_x), \]  

(3.31c)

where \( \Gamma_x \) denotes the aspect ratio in the azimuthal direction. More details about the model and geometry can be found in Davies-Jones and Gilman (1971), Gilman (1973), Busse (2005) or Liao and Zhang (2009). The primary concern of this study is with the mathematical question of the completeness of the inertial modes \( \mathbf{u}_{mnk} \) that satisfy both (3.29a,b) and (3.31a-c).

3.4 Completeness for the axisymmetric problem

Prior to presenting a mathematical proof of the completeness for the axisymmetric \((\partial/\partial x = 0)\) problem defined in the cross-section of the channel \(0 \leq y \leq \Gamma_y\) and \(0 \leq z \leq 1\), we shall first introduce the concept of Bessel’s inequality and Parseval’s equality for the axisymmetric inertial modes. Without loss of generality, we shall assume the aspect ratio \( \Gamma_y = 1 \), as it can be always arranged via a change of scale so that the axisymmetric velocity \( \mathbf{u}(y,z) \) and the inertial modes \( \mathbf{u}_{0nk}(y,z) \) (i.e., the azimuthal wavenumber \( m = 0 \)) are defined in the square domain \(0 \leq y \leq 1\) and \(0 \leq z \leq 1\).

All the axisymmetric solutions \( \mathbf{u}_{0nk} \) to (3.29a,b) and (3.30) satisfying (3.31a-c) can be written in the orthonormal form (Liao and Zhang 2009)

\[ \hat{x} \cdot \mathbf{u}_{0nk} = \sqrt{2} \sin (k \pi y) \cos (n \pi z), \]  

(3.32a)

\[ \hat{y} \cdot \mathbf{u}_{0nk} = \frac{i \sqrt{2} n}{\sqrt{n^2 + k^2}} \sin (k \pi y) \cos (n \pi z), \]  

(3.32b)

\[ \hat{z} \cdot \mathbf{u}_{0nk} = -\frac{i \sqrt{2} k}{\sqrt{n^2 + k^2}} \cos (k \pi y) \sin (n \pi z), \]  

(3.32c)

for \( k = 1, 2, 3, \ldots \) and \( n = 0, \pm 1, \pm 2, \pm 3, \ldots \), such that

\[ \int_0^1 \int_0^1 \mathbf{u}_{0nk} \cdot \mathbf{u}_{0n'k'}^* \, dy \, dz = 0 \text{ if } k \neq k' \text{ or } n \neq n', \]  

\[ \int_0^1 \int_0^1 \mathbf{u}_{0nk} \cdot \mathbf{u}_{0n'k'}^* \, dy \, dz = 1 \text{ if } k = k' \text{ and } n = n'. \]  

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In (3.32a-c), \( n \) can be both positive and negative: \( u_{0(-n)k} \) represents the complex conjugate of \( u_{0(+n)k} \), i.e., \( u_{0nk}^* = u_{0(-n)k} \). Note that the geostrophic-mode component is also included in (3.32a-c) as the special eigenfunction marked by \( n = 0 \).

Suppose that \( u(y,z) \) is an arbitrary axisymmetric (\( \partial / \partial x = 0 \)) velocity which is real, continuous and differentiable. It satisfies

\[
\int_0^1 \int_0^1 |u(y,z)|^2 \, dy \, dz \leq \text{finite}, \quad \nabla \cdot u = 0,
\]

and the no-slip boundary condition

\[
\hat{n} \cdot u = 0 \quad \text{and} \quad \hat{n} \times u = 0 \quad \text{on the four walls of the channel}.
\]

Define a series \( u_{NK} \) as

\[
u_{NK}(y,z) = \sum_{n=-N}^{N} \sum_{k=1}^{K} A_{0nk} u_{0nk}(y,z), \quad (3.33)\]

where the coefficient \( A_{0nk} \) is given by

\[
A_{0nk} = \int_0^1 \int_0^1 (u_{0nk}^* \cdot u) \, dy \, dz.
\]

We state that the inertial-mode expansion \( u_{NK} \) given by (3.33) converges in the mean-square sense to \( u \) if

\[
\int_0^1 \int_0^1 \left| u(y,z) - \sum_{n=-N}^{N} \sum_{k=1}^{K} A_{0nk} u_{0nk}(y,z) \right|^2 \, dy \, dz \to 0 \quad \text{as} \quad N \to \infty \quad \text{and} \quad K \to \infty.
\]

For finite \( N \) and \( K \), we may consider now the remainder in the partial sum

\[
\int_0^1 \int_0^1 \left| u(y,z) - \sum_{n=-N}^{N} \sum_{k=1}^{K} A_{0nk} u_{0nk}(y,z) \right|^2 \, dy \, dz \geq 0,
\]

leading to an inequality

\[
\int_0^1 \int_0^1 |u|^2 \, dy \, dz \geq \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} |A_{0nk}|^2, \quad (3.34)
\]

which is known as Bessel’s inequality. The infinite orthonormal set of the inertial modes \( u_{0nk}(y,z) \) is said to be complete if Bessel’s inequality (3.34) becomes an equality for all \( u(y,z) \):

\[
\int_0^1 \int_0^1 |u|^2 \, dy \, dz = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} |A_{0nk}|^2. \quad (3.35)
\]
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This is known as Parseval’s equality, or the completeness relation. The set of inertial modes \( v_0^{nk}(y, z) \) is complete if and only if Parseval’s equality (3.35) is true for all \( u \) (see, for example, Debnath and Mikusiński, 1999).

We shall provide a mathematical proof of the completeness by establishing that Parseval’s equality (3.35) holds for all \( u \). Let the axisymmetric, incompressible flow velocity \( u \) be in the form

\[
u(y, z) = u_x(y, z) \hat{x} + u_y(y, z) \hat{y} + u_z(y, z) \hat{z}.
\]

For convenience of writing, we introduce two sets of active variables, \((y_1, z_1)\) and \((y_2, z_2)\), such that \( A_0^{nk} \) in Parseval’s equality (3.35) can be written as

\[
|A_0^{nk}|^2 = \int_0^1 \int_0^1 u_0^{nk}(y_1, z_1) \cdot u(y_1, z_1) \, dy_1 \, dz_1 \int_0^1 \int_0^1 u_0^{nk}(y_2, z_2) \cdot u(y_2, z_2) \, dy_2 \, dz_2.
\]

With \( u_0^{nk} \) given by (3.32a-c), \( |A_0^{nk}|^2 \) is expressible as

\[
|A_0^{nk}|^2 = 2 \int_0^1 \int_0^1 \int_0^1 dy_1 \, dz_1 \, dy_2 \, dz_2 \\
\left[ u_x(y_1, z_1)u_x(y_2, z_2) \sin(k\pi y_1) \cos(n\pi z_1) \cos(k\pi y_2) \sin(n\pi z_2) \\
+ \frac{n^2u_y(y_1, z_1)u_y(y_2, z_2)}{n^2 + k^2} \sin(k\pi y_1) \cos(n\pi z_1) \sin(k\pi y_2) \cos(n\pi z_2) \\
+ \frac{k^2u_z(y_1, z_1)u_z(y_2, z_2)}{n^2 + k^2} \cos(k\pi y_1) \sin(n\pi z_1) \cos(k\pi y_2) \sin(n\pi z_2) \\
- \frac{kn u_y(y_1, z_1)u_z(y_2, z_2)}{n^2 + k^2} \sin(k\pi y_1) \cos(n\pi z_1) \sin(k\pi y_2) \cos(n\pi z_2) \\
- \frac{kn u_z(y_1, z_1)u_y(y_2, z_2)}{n^2 + k^2} \cos(k\pi y_1) \sin(n\pi z_1) \sin(k\pi y_2) \cos(n\pi z_2) \right].
\]

\[(3.36)\]

We notice that, when \( n \neq 0 \), the fourth term on the right-hand side of (3.36) can
be changed:

\[- \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dy_1 \, dz_1 \, dy_2 \, dz_2 \times \]
\[\left\{ \frac{k u_y(y_1, z_1) u_z(y_2, z_2)}{(n^2 + k^2)} \sin(k\pi y_1) \cos(n\pi z_1) \cos(k\pi y_2) \sin(n\pi z_2) \right\} \]

\[= - \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dy_1 \, dz_1 \, dy_2 \]
\[\left\{ \frac{k u_y(y_1, z_1)}{(n^2 + k^2)} \sin(k\pi y_1) \cos(n\pi z_1) \cos(k\pi y_2) \int_{0}^{1} u_z(y_2, z_2) \sin(n\pi z_2) \, dz_2 \right\} \]

\[= - \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dy_1 \, dz_1 \, dy_2 \]
\[\left\{ \frac{k u_y(y_1, z_1)}{(n^2 + k^2)} \sin(k\pi y_1) \cos(n\pi z_1) \cos(k\pi y_2) \times \right\}
\[\left\{ u_z(y_2, z_2) \cdot \left( -\frac{\cos(n\pi z_2)}{n\pi} \right) \right\}_{0}^{1} - \int_{0}^{1} \frac{\partial u_z(y_2, z_2)}{\partial z_2} \cdot \left( -\frac{\cos(n\pi z_2)}{n\pi} \right) \, dz_2 \]

\[= \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dy_1 \, dz_1 \, dy_2 \, dz_2 \]
\[\left\{ \frac{-k u_y(y_1, z_1)}{(n^2 + k^2)} \frac{\partial u_z(y_2, z_2)}{\partial z_2} \sin(k\pi y_1) \cos(n\pi z_1) \cos(k\pi y_2) \cos(n\pi z_2) \right\} \]

\[= \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dy_1 \, dz_1 \, dy_2 \, dz_2 \]
\[\left\{ \frac{k u_y(y_1, z_1)}{(n^2 + k^2)} \frac{\partial u_y(y_2, z_2)}{\partial y_2} \sin(k\pi y_1) \cos(n\pi z_1) \cos(k\pi y_2) \cos(n\pi z_2) \right\} \]

\[= \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dy_1 \, dz_1 \, dy_2 \, dz_2 \]
\[\left\{ \frac{k u_y(y_1, z_1)}{(n^2 + k^2)} \sin(k\pi y_1) \cos(n\pi z_1) \cos(n\pi z_2) \int_{0}^{1} \frac{\partial u_y(y_2, z_2)}{\partial y_2} \cos(k\pi y_2) \, dy_2 \right\} \]

\[= \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dy_1 \, dz_1 \, dy_2 \, dz_2 \]
\[\left\{ \frac{k u_y(y_1, z_1)}{(n^2 + k^2)} \sin(k\pi y_1) \cos(n\pi z_1) \cos(n\pi z_2) \times \right\}
\[\left\{ u_y(y_2, z_2) \cos(k\pi y_2) \right\}_{0}^{1} - \int_{0}^{1} u_y(y_2, z_2)(-k\pi) \sin(k\pi y_2) \, dy_2 \]

\[= \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dy_1 \, dz_1 \, dy_2 \, dz_2 \]
\[\left\{ \frac{k^2 u_y(y_1, z_1) u_y(y_2, z_2)}{(n^2 + k^2)} \sin(k\pi y_1) \cos(n\pi z_1) \sin(k\pi y_2) \cos(n\pi z_2) \right\} \]

(3.37a)

after making use of both the equation \( \nabla \cdot \mathbf{u} = 0 \) and the boundary condition \( \mathbf{n} \cdot \mathbf{u} = 0 \).
Likewise, the fifth term on the right-hand side of (3.36) can be also changed to
\[
- \int_0^1 \int_0^1 \int_0^1 dy_1 \ dz_1 \ dy_2 \ dz_2 \ k n u_z(y_1, z_1) u_y(y_2, z_2) (n^2 + k^2) \cos(k \pi y_1) \sin(n \pi z_1) \sin(k \pi y_2) \cos(n \pi z_2)
\]
\[
= \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy_1 \ dz_1 \ dy_2 \ dz_2 \ n^2 u_z(y_1, z_1) u_z(y_2, z_2) (n^2 + k^2) \cos(k \pi y_1) \sin(n \pi z_1) \cos(k \pi y_2) \sin(n \pi z_2). \quad (3.37b)
\]

In terms of (3.37a,b), the expression (3.36) for \(|A_{0nk}|^2\) is reduced to
\[
|A_{0nk}|^2 = 2 \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy_1 \ dz_1 \ dy_2 \ dz_2 \ \left[ u_x(y_1, z_1) u_x(y_2, z_2) \sin(k \pi y_1) \cos(n \pi z_1) \sin(k \pi y_2) \cos(n \pi z_2)
+ u_y(y_1, z_1) u_y(y_2, z_2) \sin(k \pi y_1) \cos(n \pi z_1) \sin(k \pi y_2) \cos(n \pi z_2)
+ u_z(y_1, z_1) u_z(y_2, z_2) \cos(k \pi y_1) \sin(n \pi z_1) \cos(k \pi y_2) \sin(n \pi z_2) \right]. \quad (3.38)
\]

This gives rise to a partial sum with finite \(N\) and \(K\) on the right-hand side of Parseval’s equality (3.35) in the form
\[
\sum_{n=-N}^{N} \sum_{k=1}^{K} |A_{0nk}|^2 = 2 \sum_{n=-N}^{N} \sum_{k=1}^{K} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy_1 \ dz_1 \ dy_2 \ dz_2 \ \left[ u_x(y_1, z_1) u_x(y_2, z_2) \sin(k \pi y_1) \cos(n \pi z_1) \sin(k \pi y_2) \cos(n \pi z_2)
+ u_y(y_1, z_1) u_y(y_2, z_2) \sin(k \pi y_1) \cos(n \pi z_1) \sin(k \pi y_2) \cos(n \pi z_2)
+ u_z(y_1, z_1) u_z(y_2, z_2) \cos(k \pi y_1) \sin(n \pi z_1) \cos(k \pi y_2) \sin(n \pi z_2) \right]
- 2 \sum_{k=1}^{K} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy_1 \ dz_1 \ dy_2 \ dz_2 u_y(y_1, z_1) u_y(y_2, z_2) \sin(k \pi y_1) \sin(k \pi y_2).
\quad (3.39)
\]

It is worth mentioning that the geostrophic component with \(n = 0\) appears only in the \(u_x(y_1, z_1) u_x(y_2, z_2)\) term. This is why we have to include the final \(u_y(y_1, z_1) u_y(y_2, z_2)\) term for \(n = 0\) as the final term on the right-hand side of (3.39). The next step of the mathematical proof is to show that the right-hand side of (3.39) becomes
\[
\int_0^1 \int_0^1 (u_x^2 + u_y^2 + u_z^2) \ dy \ dz \text{ in the limits } K \to \infty \text{ and } N \to \infty.
\]

By writing
\[
\sum_{k=1}^{K} \sin(k \pi y_1) \sin(k \pi y_2) = \sum_{k=1}^{K} \frac{e^{ik \pi y_1} - e^{-ik \pi y_1}}{2i} \frac{e^{ik \pi y_2} - e^{-ik \pi y_2}}{2i}
\quad (3.40)
\]
the formula for geometric progression. Thus we have:

\[
\sum_{k=1}^{K} \sin(k\pi y_1) \sin(k\pi y_2) = \frac{\sin\left((2K + 1)\frac{1}{2}\pi(y_1 - y_2)\right)}{4\sin\left(\frac{1}{2}\pi(y_1 - y_2)\right)} - \frac{\sin\left((2K + 1)\frac{1}{2}\pi(y_1 + y_2)\right)}{4\sin\left(\frac{1}{2}\pi(y_1 + y_2)\right)},
\]

(3.41a)

\[
\sum_{n=1}^{N} \cos(n\pi y_1) \cos(n\pi y_2) = \frac{\sin\left((2N + 1)\frac{1}{2}\pi(y_1 - y_2)\right)}{4\sin\left(\frac{1}{2}\pi(y_1 - y_2)\right)} + \frac{\sin\left((2N + 1)\frac{1}{2}\pi(y_1 + y_2)\right)}{4\sin\left(\frac{1}{2}\pi(y_1 + y_2)\right)} - \frac{1}{2},
\]

(3.41b)

\[
\sum_{n=-N}^{N} \cos(n\pi y_1) \cos(n\pi y_2) = \frac{\sin\left((2N + 1)\frac{1}{2}\pi(y_1 - y_2)\right)}{2\sin\left(\frac{1}{2}\pi(y_1 - y_2)\right)} + \frac{\sin\left((2N + 1)\frac{1}{2}\pi(y_1 + y_2)\right)}{2\sin\left(\frac{1}{2}\pi(y_1 + y_2)\right)}
\]

(3.41c)

for any positive integer \(K\) and \(N\). Upon using the above summations, the expression (3.39) can be changed to

\[
\sum_{n=-N}^{N} \sum_{k=1}^{K} |A_{0nk}|^2 = J_{1NK} + J_{2NK} + J_{3NK} + J_{4K},
\]

(3.42)

where

\[
J_{1NK} = \frac{1}{4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy_1 dz_1 dy_2 dz_2 u_x(y_1, z_1)u_x(y_2, z_2) \times \left\{ \frac{\sin\left((2K + 1)\frac{1}{2}\pi(y_1 - y_2)\right)}{\sin\left(\frac{1}{2}\pi(y_1 - y_2)\right)} - \frac{\sin\left((2K + 1)\frac{1}{2}\pi(y_1 + y_2)\right)}{\sin\left(\frac{1}{2}\pi(y_1 + y_2)\right)} \right\} \times \left\{ \frac{\sin\left((2N + 1)\frac{1}{2}\pi(z_1 - z_2)\right)}{\sin\left(\frac{1}{2}\pi(z_1 - z_2)\right)} + \frac{\sin\left((2N + 1)\frac{1}{2}\pi(z_1 + z_2)\right)}{\sin\left(\frac{1}{2}\pi(z_1 + z_2)\right)} \right\},
\]

\[
J_{2NK} = \frac{1}{4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy_1 dz_1 dy_2 dz_2 u_y(y_1, z_1)u_y(y_2, z_2) \times \left\{ \frac{\sin\left((2K + 1)\frac{1}{2}\pi(y_1 - y_2)\right)}{\sin\left(\frac{1}{2}\pi(y_1 - y_2)\right)} - \frac{\sin\left((2K + 1)\frac{1}{2}\pi(y_1 + y_2)\right)}{\sin\left(\frac{1}{2}\pi(y_1 + y_2)\right)} \right\} \times \left\{ \frac{\sin\left((2N + 1)\frac{1}{2}\pi(z_1 - z_2)\right)}{\sin\left(\frac{1}{2}\pi(z_1 - z_2)\right)} + \frac{\sin\left((2N + 1)\frac{1}{2}\pi(z_1 + z_2)\right)}{\sin\left(\frac{1}{2}\pi(z_1 + z_2)\right)} \right\}.
\]
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\[
J_{3NK} = \frac{1}{4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy_1 dy_2 dy_2 dz_1 \, u_z(y_1, z_1) u_z(y_2, z_2)
\times \left\{ \frac{\sin[(2K + 1)\frac{1}{2}\pi(y_1 - y_2)]}{\sin[\frac{1}{2}\pi(y_1 - y_2)]} + \frac{\sin[(2K + 1)\frac{1}{2}\pi(y_1 + y_2)]}{\sin[\frac{1}{2}\pi(y_1 + y_2)]} \right\}
\times \left\{ \frac{\sin[(2N + 1)\frac{1}{2}\pi(z_1 - z_2)]}{\sin[\frac{1}{2}\pi(z_1 - z_2)]} - \frac{\sin[(2N + 1)\frac{1}{2}\pi(z_1 + z_2)]}{\sin[\frac{1}{2}\pi(z_1 + z_2)]} \right\},
\]

\[
J_{4K} = \frac{1}{2} \int_0^1 \int_0^1 dy_1 dy_2 \, u_y(y_1, z_1) u_y(y_2, z_2)
\times \left\{ \frac{\sin[(2K + 1)\frac{1}{2}\pi(y_1 - y_2)]}{\sin[\frac{1}{2}\pi(y_1 - y_2)]} - \frac{\sin[(2K + 1)\frac{1}{2}\pi(y_1 + y_2)]}{\sin[\frac{1}{2}\pi(y_1 + y_2)]} \right\}.
\]

We are now in position to carry out the above four integrals. Since the procedure involved in all the integrations is quite similar, we shall take \(J_{1NK}\) as an example to illustrate the detail.

For the purpose of facilitating the analysis, we make an extension of \(u_x\) from \(0 < y_1 \leq 1\) to \(-1 \leq y_1 < 0\) by letting \(U_x(y_1, z_1) = u_x(y_1, z_1)\) in \(0 < y_1 \leq 1\) while \(U_x(y_1, z_1) = -u_x(-y_1, z_1)\) in \(-1 \leq y_1 < 0\). We also make a periodic extension of \(U_x(y_1, z_1)\) with a period of 2 by letting \(U_x(y_1, z_1) = U_x(y_1 + 2, z_1)\). Such extension allows the two integrals in \(J_{1NK}\) to be combined into a single integral, for example,

\[
\lim_{K \to \infty} \int_0^1 u_x(y_1, z_1) \left\{ \frac{\sin[(2K + 1)\frac{1}{2}\pi(y_1 - y_2)]}{2\sin[\frac{1}{2}\pi(y_1 - y_2)]} - \frac{\sin[(2K + 1)\frac{1}{2}\pi(y_1 + y_2)]}{2\sin[\frac{1}{2}\pi(y_1 + y_2)]} \right\} \, dy_1
= \lim_{K \to \infty} \int_{-1}^1 U_x(y_1, z_1) \left\{ \frac{\sin[(2K + 1)\frac{1}{2}\pi(y_1 - y_2)]}{2\sin[\frac{1}{2}\pi(y_1 - y_2)]} \right\} \, dy_1.
\]

Note that \(U_x(y_1, z_1)\) is arbitrary but continuous and piecewise differentiable. It becomes clear, by examining (3.42), that we would be able to derive the completeness relation, Parseval’s equality (3.35), if we can show that \(J_{4K} = 0\) in the limit \(K \to \infty\) and the integral relationship like

\[
\lim_{K \to \infty} \int_{-1}^1 U_x(y_1, z_1) \left\{ \frac{\sin[(2K + 1)\frac{1}{2}\pi(y_1 - y_2)]}{2\sin[\frac{1}{2}\pi(y_1 - y_2)]} \right\} \, dy_1 = U_x(y_2, z_1) \quad \text{(3.43)}
\]

For economy of writing, we introduce a new active variable \(\zeta = \pi(y_2 - y_1)\) with which (3.43) becomes

\[
\int_{-1}^1 U_x(y_1, z_1) \left\{ \frac{\sin[(2K + 1)\frac{1}{2}\pi(y_1 - y_2)]}{2\sin[\frac{1}{2}\pi(y_1 - y_2)]} \right\} \, dy_1
= \frac{1}{2\pi} \int_0^\pi \left[ U_x \left( y_2 - \frac{\zeta}{\pi}, z_1 \right) + U_x \left( y_2 + \frac{\zeta}{\pi}, z_1 \right) \right] \left\{ \frac{\sin[(K + 1/2)\zeta]}{\sin(\zeta/2)} \right\} \, d\zeta.
\]
Upon denoting
\[ F(y_2, z_1, \zeta) = \frac{1}{2} \left[ U_x \left( y_2 - \frac{\zeta}{\pi}, z_1 \right) + U_x \left( y_2 + \frac{\zeta}{\pi}, z_1 \right) \right] \]
and recognizing, for any \( K \), that
\[
\frac{1}{\pi} \int_0^\pi \left\{ \frac{\sin [(K + 1/2)\zeta]}{\sin (\zeta/2)} \right\} \, d\zeta = \frac{1}{\pi} \int_0^\pi \left[ 1 + 2 \sum_{k=1}^K \cos (k\zeta) \right] \, d\zeta = 1,
\]
we note that the integral relationship (3.43) is then equivalent to showing that
\[
\lim_{K \to \infty} \frac{1}{2\pi} \int_0^\pi [F(y_2, z_1, \zeta) - F(y_2, z_1, 0)] \left\{ \frac{\sin [(K + 1/2)\zeta]}{\sin (\zeta/2)} \right\} \, d\zeta = 0, \tag{3.44}
\]
which is singular at \( \zeta = 0 \) and needs to be treated carefully.

The condition that \( F \) is piecewise continuous and differentiable allows us to assume that \( F \) is non-decreasing in the neighborhood of \( \zeta = 0 \). For any given small \( \delta \), after applying the Second Mean Value Theorem for integration, the singular integral (3.44) is expressible as
\[
\frac{1}{2\pi} \int_0^\pi [F(y_2, z_1, \zeta) - F(y_2, z_1, 0)] \left\{ \frac{\sin [(K + 1/2)\zeta]}{\sin (\zeta/2)} \right\} \, d\zeta
= \frac{1}{2\pi} \left[ F(y_2, z_1, \delta) - F(y_2, z_1, 0) \right] \int_\xi^\delta \frac{\sin [(K + 1/2)\zeta]}{\sin (\zeta/2)} \, d\zeta
+ \frac{1}{2\pi} \int_\delta^\pi [F(y_2, z_1, \zeta) - F(y_2, z_1, 0)] \frac{\sin [(K + 1/2)\zeta]}{\sin (\zeta/2)} \, d\zeta, \tag{3.45}
\]
where \( 0 < \xi < \delta \). We first notice that the first integral on the right-hand side of (3.45) is bounded for any integer \( K \). This is because
\[
\frac{1}{\pi} \left| \int_\xi^\delta \frac{\sin [(K + 1/2)\zeta]}{2\sin (\zeta/2)} \, d\zeta \right|
= \frac{1}{\pi} \left| \int_\xi^\delta \frac{\sin [(K + 1/2)\zeta]}{\zeta} \, d\zeta \right| - \frac{1}{\pi} \int_\xi^\delta \sin [(K + 1/2)\zeta] \left[ \frac{1}{\zeta} - \frac{1}{2\sin (\zeta/2)} \right] \, d\zeta
\leq \frac{1}{\pi} \int_{(K+1/2)\xi}^{(K+1/2)\delta} \frac{\sin \zeta}{\zeta} \, d\zeta + \frac{1}{\pi} \int_0^\delta \left| \frac{1}{\zeta} - \frac{1}{2 \sin (\zeta/2)} \right| \, d\zeta.
\]
Since
\[
\lim_{\zeta \to 0} \left[ \frac{1}{\zeta} - \frac{1}{2 \sin (\zeta/2)} \right] = 0,
\]
implying that the function \( \{1/\zeta - 1/[2 \sin (\zeta/2)]\} \) in \( [0, \delta] \) is bounded, there exists a positive \( C_1 \) such that
\[
\int_0^\delta \left| \frac{1}{\zeta} - \frac{1}{2 \sin (\zeta/2)} \right| \, d\zeta < \pi C_1.
\]
Moreover, the interval \( [(K + 1/2)\xi, (K + 1/2)\delta] \) for any sufficiently large \( K \) can be
always divided into three parts:

\[ [(K + 1/2)\xi, 2k\pi], \quad [2k\pi, 2m\pi] \quad \text{and} \quad [2m\pi, (K + 1/2)\delta] \]

for some integer \( k \) and \( m \) such that

\[ 2k\pi - (K + 1/2)\xi \leq 2\pi \quad \text{and} \quad (K + 1/2)\delta - 2m\pi \leq 2\pi. \]

Therefore

\[
\left| \int_{(K+1/2)\xi}^{(K+1/2)\delta} \frac{\sin \hat{\zeta}}{\zeta} \, d\hat{\zeta} \right| \leq \left| \int_{(K+1/2)\xi}^{2k\pi} \frac{\sin \hat{\zeta}}{\zeta} \, d\hat{\zeta} \right| + \left| \int_{2k\pi}^{2m\pi} \frac{\sin \hat{\zeta}}{\zeta} \, d\hat{\zeta} \right| + \left| \int_{2m\pi}^{(K+1/2)\delta} \frac{\sin \hat{\zeta}}{\zeta} \, d\hat{\zeta} \right|.
\]

While the first and third integrals are obviously bounded

\[
\left| \int_{(K+1/2)\xi}^{2k\pi} \frac{\sin \hat{\zeta}}{\zeta} \, d\hat{\zeta} \right| \leq \pi C_2, \quad \left| \int_{2m\pi}^{(K+1/2)\delta} \frac{\sin \hat{\zeta}}{\zeta} \, d\hat{\zeta} \right| \leq \pi C_3,
\]

where \( C_2 \) and \( C_3 \) are some positive constants, the second integral is also bounded because

\[
\left| \int_{2k\pi}^{2m\pi} \frac{\sin \hat{\zeta}}{\zeta} \, d\hat{\zeta} \right| \leq \frac{\pi^2}{6}.
\]

This means that, for any large \( K \), we always have

\[
\left| \frac{1}{2\pi} \int_0^\pi \left[ F(y_2, z_1, \zeta) - F(y_2, z_1, 0) \right] \left\{ \frac{\sin [(K + 1/2)\zeta]}{\sin (\zeta/2)} \right\} \, d\zeta \right|
\]

\[
\leq |F(y_2, z_1, \delta) - F(y_2, z_1, 0)| \left[ C_1 + C_2 + C_3 + \frac{\pi}{6} \right].
\]

We also notice, through integration by parts, that the second integral on the right-hand side of (3.45) vanishes as \( K \to \infty \) since \( F \) is piecewise continuous and differentiable.

It follows that, for any small number \( \epsilon > 0 \), there exists a corresponding number \( \delta > 0 \) such that

\[
\lim_{K \to \infty} \frac{1}{2\pi} \int_0^\pi \left[ F(y_2, z_1, \zeta) - F(y_2, z_1, 0) \right] \left\{ \frac{\sin [(K + 1/2)\zeta]}{\sin (\zeta/2)} \right\} \, d\zeta - 0 \leq \epsilon.
\]

This implies (3.43) or that

\[
\lim_{K \to \infty} \int_0^1 u_x(y_1, z_1) \left\{ \frac{\sin [(2K + 1/2)\pi(y_1 - y_2)]}{2 \sin (\pi(y_1 - y_2))} - \frac{\sin [(2K + 1)\pi(y_1 + y_2)]}{2 \sin (\pi(y_1 + y_2))} \right\} \, dy_1
\]

\[
= u_x(y_2, z_1). \quad (3.46a)
\]
In a similar way, we can also show that

\[
\lim_{N \to \infty} \int_0^1 u_x(y_2, z_1) \left\{ \frac{\sin[(2N + 1) \frac{1}{2} \pi(z_1 - z_2)]}{2 \sin[\frac{1}{2} \pi(z_1 - z_2)]} + \frac{\sin[(2N + 1) \frac{1}{2} \pi(z_1 + z_2)]}{2 \sin[\frac{1}{2} \pi(z_1 + z_2)]} \right\} \, dz_1 = u_x(y_2, z_2)
\]

(3.46b)

without any difficulties. Applying the integral relations (3.46a,b) to \( J_{1NK} \) yields

\[
\lim_{K \to \infty} \lim_{N \to \infty} J_{1NK} = \int_0^1 \int_0^1 u_x(y_2, z_2) u_x(y_2, z_2) \, dy_2 \, dz_2.
\]

In the same manner, \( J_{2NK} \) can be also written as

\[
\lim_{K \to \infty} \lim_{N \to \infty} J_{2NK} = \int_0^1 \int_0^1 u_y(y_2, z_2) u_y(y_2, z_2) \, dy_2 \, dz_2.
\]

There is, however, an extra term in \( J_{3NK} \) after applying formulas similar to (3.46a,b),

\[
\lim_{K \to \infty} \lim_{N \to \infty} J_{3NK} = \int_0^1 \int_0^1 u_z(y_2, z_2) u_z(y_2, z_2) \, dy_2 \, dz_2
\]

\[
- \int_0^1 u_z(y_1, z_2) \, dy_1 \int_0^1 u_z(y_1, z_2) \, dy_1 = \oint_{C_z} [u_y(y_1, z_1) \, dz_1 - u_z(y_1, z_1) \, dy_1]
\]

\[
= \int_0^1 \int_0^1 \left( \frac{\partial u_y}{\partial y_1} + \frac{\partial u_z}{\partial z_1} \right) \, dy_1 \, dz_1 = 0,
\]

where \( C_z \) denotes the closed contour bounded by the domain \( z_2 \leq z_1 \leq 1 \) and \( 0 \leq y_1 \leq 1 \). It follows that

\[
\int_0^1 \int_0^1 u_z(y_2, z_2) \left( \int_0^1 u_z(y_1, z_2) \, dy_1 \right) \, dy_2 \, dz_2 = 0.
\]

The final term \( J_{4K} \) also vanishes in the limit \( K \to \infty \),

\[
\lim_{K \to \infty} J_{4K} = \int_0^1 \int_0^1 u_y(y_2, z_2) \left[ \int_0^1 u_y(y_2, z_1) \, dz_1 \right] \, dy_2 \, dz_2 = 0.
\]

This is because, for any \( y_2 \) with \( \dot{n} \cdot \mathbf{u} = 0 \) on the walls and \( \nabla \cdot \mathbf{u} = 0 \), we have

\[
\int_0^1 u_y(y_2, z_1) \, dz_1 = \oint_{C_y} [u_y(y_1, z_1) \, dz_1 - u_z(y_1, z_1) \, dy_1]
\]

\[
= \int_0^1 \int_0^1 \left( \frac{\partial u_y}{\partial y_1} + \frac{\partial u_z}{\partial z_1} \right) \, dy_1 \, dz_1 = 0,
\]

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where \( C_y \) denotes the closed contour bounded by the domain \( y_2 \leq y_1 \leq 1 \) and \( 0 \leq z_1 \leq 1 \).

In short, we have provided a mathematical proof that

\[
\sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} |A_{0nk}|^2 = \int_0^1 \int_0^1 (u_x^2 + u_y^2 + u_z^2) \, dy \, dz,
\]

which is the completeness relation, or Parseval’s equality, for the axisymmetric inertial modes \( u_{0nk}(y,z) \). We come to the conclusion that the inertial modes \( u_{0nk} \) (or the inviscid eigenfunctions) indeed form a complete system of functions: any piecewise continuous velocity \( u(y,z) \) – which obeys (i) \( \int_0^1 \int_0^1 |u|^2 \, dy \, dz < \infty \), (ii) \( \nabla \cdot u = 0 \) and (iii) \( |\partial u/\partial y| < \infty \) and \( |\partial u/\partial z| < \infty \) – can be approximated by the expansion (3.33) in the \( L^2 \) sense to any desired degree of accuracy.

3.5 Extension to the non-axisymmetric problem

An extension of the proof from the axisymmetric problem to the general three-dimensional problem defined in the domain \( 0 \leq x \leq \Gamma_x, 0 \leq y \leq \Gamma_y, 0 \leq z \leq 1 \) is straightforward but the analysis and manipulation are very lengthy and cumbersome. For mathematical convenience and without loss of generality, we shall assume the aspect ratio in the azimuthal direction \( \Gamma_x = 1 \) because it can be arranged via a change of scale that the inertial modes \( u_{mnk}(x,y,z) \), where \( m \) is an integer azimuthal wavenumber, are defined in the cubic domain \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) and \( 0 \leq z \leq 1 \). Since the analysis is exactly parallel to that for the axisymmetric problem, this section will be kept brief.

The general three-dimensional proof is algebraically much more complicated. The infinite orthonormal set of the general inertial modes \( u_{mnk}(x,y,z) \) (or the general eigenfunctions) consists of three subsets. The first subset describes the azimuthally propagating inertial waves

\[
\hat{x} \cdot u_{mnk} = \frac{\sqrt{2}}{n \sqrt{n^2 + 4m^2}} \left[ n^2 \sin(k\pi y) - 2mk\sigma_{mnk} \cos(k\pi y) \right] \cos(n\pi z)e^{i2\pi mx}, \tag{3.47a}
\]

\[
\hat{y} \cdot u_{mnk} = \frac{i\sqrt{2} \sigma_{mnk} \sqrt{n^2 + 4m^2}}{n} \sin(k\pi y) \cos(n\pi z)e^{i2\pi mx}, \tag{3.47b}
\]

\[
\hat{z} \cdot u_{mnk} = \frac{-i\sqrt{2}}{\sqrt{n^2 + 4m^2}} \left[ 2m \sin(k\pi y) + \sigma_{mnk} k \cos(k\pi y) \right] \sin(n\pi z)e^{i2\pi mx}, \tag{3.47c}
\]

for \( k = 1, 2, 3, \ldots, n = \pm1, \pm2, \pm3, \ldots \) and \( m = \pm1, \pm2, \pm3, \ldots \), where the general eigenvalue \( \sigma_{mnk} \) is

\[
\sigma_{mnk} = \frac{n}{\sqrt{n^2 + k^2 + 4m^2}};
\]
3. Completeness of inertial wave modes

The second subset is concerned with the non-axisymmetric geostrophic component

\[ \mathbf{x} \cdot \mathbf{u}_{m\ell k} = -\frac{\sqrt{2k}}{\sqrt{k^2 + 4m^2}} \cos(k\pi y)e^{i2\pi mx}, \quad (3.48a) \]

\[ \mathbf{y} \cdot \mathbf{u}_{m\ell k} = i\frac{\sqrt{2m}}{\sqrt{k^2 + 4m^2}} \sin(k\pi y)e^{i2\pi mx}, \quad (3.48b) \]

\[ \mathbf{z} \cdot \mathbf{u}_{m\ell k} = 0 \quad (3.48c) \]

for \( k = 1, 2, 3, \ldots, n = 0 \) and \( m = \pm 1, \pm 2, \pm 3, \ldots \); and the third subset is the axisymmetric mode \( \mathbf{u}_{0\ell k} \) given by \((3.32a-c)\). The general inertial modes \( \mathbf{u}_{m\ell k} \) are orthogonal and normalized such that

\[ \int_0^1 \int_0^1 \int_0^1 \mathbf{u}_{m\ell k} \cdot \mathbf{u}^*_{m'\ell'k'} \, dx \, dy \, dz = 0 \quad \text{if} \quad m \neq m' \quad \text{or} \quad k \neq k' \quad \text{or} \quad n \neq n', \]

\[ \int_0^1 \int_0^1 \int_0^1 \mathbf{u}_{m\ell k} \cdot \mathbf{u}^*_{m'\ell'k'} \, dx \, dy \, dz = 1 \quad \text{if} \quad m = m' \quad \text{and} \quad k = k' \quad \text{and} \quad n = n'. \]

Suppose that \( \mathbf{u}(x, y, z) \) is a three-dimensional, piecewise continuous velocity and obeys (i) \( \int_0^1 \int_0^1 \int_0^1 |\mathbf{u}|^2 \, dx \, dy \, dz < \infty \), (ii) \( \nabla \cdot \mathbf{u} = 0 \) and (iii) \( |\partial \mathbf{u}/\partial x| < \infty \), \( |\partial \mathbf{u}/\partial y| < \infty \) and \( |\partial \mathbf{u}/\partial z| < \infty \). We shall then prove the completeness relation

\[ \int_0^1 \int_0^1 \int_0^1 |\mathbf{u}|^2 \, dx \, dy \, dz = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} |A_{m\ell k}|^2, \quad (3.49) \]

for all possible \( \mathbf{u} \), where the coefficient \( A_{m\ell k} \) is defined as

\[ A_{m\ell k} = \int_0^1 \int_0^1 \int_0^1 (\mathbf{u}^*_{m\ell k} \cdot \mathbf{u}) \, dx \, dy \, dz. \quad (3.50) \]

The set of the general inertial modes \( \mathbf{u}_{m\ell k}(x, y, z) \) consisting of the three subsets is complete if and only if Parseval’s equality \( (3.49) \) is true for all \( \mathbf{u} \).
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We have

$$\sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{k=1}^{K} |A_{mnk}|^2$$

$$= \sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{k=1}^{K} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 dx_1 dy_1 dz_1 dx_2 dy_2 dz_2$$

$$\left\{ \frac{2u_x(x_1, y_1, z_1)u_x(x_2, y_2, z_2)}{n^2 + 4m^2} \left[ n^2 \sin(k\pi y_1) \sin(k\pi y_2) + \frac{4mk^2 \sigma_{mnk}^2}{n^2} \cos(k\pi y_1) \cos(k\pi y_2) \cos(n\pi z_1) \cos(n\pi z_2) \cos[2m\pi(x_2 - x_1)] \right] \right.$$  

$$+ \frac{2\sigma_{mnk}^2 (n^2 + 4m^2)}{n^2} u_y(x_1, y_1, z_1) u_y(x_2, y_2, z_2) \times \sin(k\pi y_1) \sin(k\pi y_2) \cos(n\pi z_1) \cos(n\pi z_2) \cos[2m\pi(x_2 - x_1)]$$  

$$+ \frac{2u_z(x_1, y_1, z_1)u_z(x_2, y_2, z_2)}{n^2 + 4m^2} \left[ 4m^2 \sin(k\pi y_1) \sin(k\pi y_2) + \frac{\sigma_{mnk}^2 k^2}{n^2} \cos(k\pi y_1) \cos(k\pi y_2) \right]$$

$$\times \sin(n\pi z_1) \sin(n\pi z_2) \cos[2m\pi(x_2 - x_1)]$$  

$$+ \frac{4mk\sigma_{mnk}^2}{n^2} u_x(x_1, y_1, z_1) u_y(x_2, y_2, z_2) \cos(k\pi y_1) \sin(k\pi y_2)$$

$$\times \cos(n\pi z_1) \cos(n\pi z_2) \sin[2m\pi(x_2 - x_1)]$$  

$$+ \frac{2u_z(x_1, y_1, z_1)u_z(x_2, y_2, z_2)}{n^2 + 4m^2} \left[ 2mn \sin(k\pi y_1) \sin(k\pi y_2) - \frac{2mk^2 \sigma_{mnk}^2}{n} \cos(k\pi y_1) \cos(k\pi y_2) \cos(n\pi z_1) \sin(n\pi z_2) \sin[2m\pi(x_2 - x_1)] \right]$$  

$$- \frac{4mk\sigma_{mnk}^2}{n^2} u_y(x_1, y_1, z_1) u_x(x_2, y_2, z_2) \sin(k\pi y_1) \cos(k\pi y_2)$$

$$\times \cos(n\pi z_1) \cos(n\pi z_2) \sin[2m\pi(x_2 - x_1)]$$  

$$- \frac{2k\sigma_{mnk}^2}{n} u_y(x_1, y_1, z_1) u_z(x_2, y_2, z_2) \sin(k\pi y_1) \cos(k\pi y_2)$$

$$\times \cos(n\pi z_1) \sin(n\pi z_2) \cos[2m\pi(x_2 - x_1)]$$  

$$+ \frac{2u_z(x_1, y_1, z_1)u_x(x_2, y_2, z_2)}{n^2 + 4m^2} \left\{ 2mn \sin(k\pi y_1) \sin(k\pi y_2) - \frac{2mk^2 \sigma_{mnk}^2}{n} \cos(k\pi y_1) \cos(k\pi y_2) \sin(n\pi z_1) \cos(n\pi z_2) \sin[2m\pi(x_2 - x_1)] \right\}.$$

After a lengthy and tedious manipulation that makes use of the boundary condition $\mathbf{n} \cdot \mathbf{u} = 0$ on $S$ and the condition $\nabla \cdot \mathbf{u} = 0$, we can show that
3. Completeness of inertial wave modes

\[
\sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{k=1}^{K} |A_{mnk}|^2
\]

\[
= \sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{k=1}^{K} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2
\]

\[
\left\{ 2u_x(x_1, y_1, z_1)u_x(x_2, y_2, z_2) \sin(k\pi y_1) \sin(k\pi y_2) \times \cos(n\pi y_1) \cos(n\pi y_2) \cos[2m\pi(x_2 - x_1)] + 2u_y(x_1, y_1, z_1)u_y(x_2, y_2, z_2) \sin(k\pi y_1) \sin(k\pi y_2) \times \cos(n\pi y_1) \cos(n\pi y_2) \cos[2m\pi(x_2 - x_1)] + 2u_z(x_1, y_1, z_1)u_z(x_2, y_2, z_2) \sin(k\pi y_1) \sin(k\pi y_2) \times \sin(n\pi y_1) \sin(n\pi y_2) \cos[2m\pi(x_2 - x_1)] + u_x(x_1, y_1, z_1) \frac{\partial u_y(x_2, y_2, z_2)}{\partial y_2} \frac{4m}{(n^2 + 4m^2)\pi} \cos[k\pi(y_1 + y_2)] \cos(n\pi z_1) \cos(n\pi z_2) \sin[2m\pi(x_2 - x_1)] + u_z(x_1, y_1, z_1) \frac{\partial u_y(x_2, y_2, z_2)}{\partial y_2} \frac{-2n}{(n^2 + 4m^2)\pi} \cos[k\pi(y_1 + y_2)] \cos(n\pi z_1) \cos(n\pi z_2) \cos[2m\pi(x_2 - x_1)] \right\}.
\]

Furthermore, for \( m = 0 \) we have

\[
|A_{0nk}|^2 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2
\]

\[
\left\{ 2u_x(x_1, y_1, z_1)u_x(x_2, y_2, z_2) \sin(k\pi y_1) \cos(n\pi z_1) \sin(k\pi y_2) \cos(n\pi z_2) + 2u_y(x_1, y_1, z_1)u_y(x_2, y_2, z_2) \sin(k\pi y_1) \cos(n\pi z_1) \sin(k\pi y_2) \cos(n\pi z_2) + 2u_z(x_1, y_1, z_1)u_z(x_2, y_2, z_2) \cos(k\pi y_1) \sin(n\pi z_1) \cos(k\pi y_2) \sin(n\pi z_2) \right\},
\]

and for \( n = 0 \) we have

\[
|A_{m0k}|^2 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2
\]

\[
\left\{ u_x(x_1, y_1, z_1)u_x(x_2, y_2, z_2) \cos(k\pi y_1) \cos(k\pi y_2) \cos[2m\pi(x_2 - x_1)] + u_y(x_1, y_1, z_1)u_y(x_2, y_2, z_2) \sin(k\pi y_1) \sin(k\pi y_2) \cos[2m\pi(x_2 - x_1)] \right\}.
\]

In all we have

\[
\sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{k=1}^{K} |A_{mnk}|^2 = J_{1Mnk} + J_{2Mnk} + J_{3Mnk} + J_{4Mnk} + J_{5nk} + J_{6kn}, \quad (3.52)
\]
where the first three non-zero integrals are

\[
J_{1MNK} = 2 \sum_{m=-M}^{M} \sum_{k=1}^{K} \sum_{n=-N}^{N} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2 \\
\times \left\{ u_x(x_1, y_1, z_1) u_x(x_2, y_2, z_2) \sin(k\pi y_1) \sin(k\pi y_2) \cos(n\pi z_1) \cos(n\pi z_2) \cos(2m\pi(x_2 - x_1)) \right\},
\]

\[
J_{2MNK} = 2 \sum_{m=-M}^{M} \sum_{k=1}^{K} \sum_{n=-N}^{N} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2 \\
\times \left\{ u_y(x_1, y_1, z_1) u_y(x_2, y_2, z_2) \sin(k\pi y_1) \sin(k\pi y_2) \cos(n\pi z_1) \cos(n\pi z_2) \cos(2m\pi(x_2 - x_1)) \right\},
\]

\[
J_{3MNK} = 2 \sum_{m=-M}^{M} \sum_{k=1}^{K} \sum_{n=-N}^{N} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2 \\
\times \left\{ u_z(x_1, y_1, z_1) u_z(x_2, y_2, z_2) \sin(k\pi y_1) \sin(k\pi y_2) \sin(n\pi z_1) \sin(n\pi z_2) \cos(2m\pi(x_2 - x_1)) \right\},
\]

while the other three integrals which vanish in the limit \(M \to \infty, N \to \infty\) and \(N \to \infty\) are

\[
J_{4MNK} = 2 \sum_{m=-M}^{M} \sum_{k=1}^{K} \sum_{n=-N}^{N} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2 \\
\left\{ 2mu_x(x_1, y_1, z_1) \frac{\partial u_y}{\partial y_2} \cos(k\pi(y_1 + y_2)) \cos(n\pi z_1) \cos(n\pi z_2) \cos(2m\pi(x_2 - x_1)) \right\},
\]

\[
J_{5MK} = 2 \sum_{m=-M}^{M} \sum_{k=1}^{K} \sum_{n=-N}^{N} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 \, dy_1 \, dx_2 \, dy_2 \\
\left\{ u_x(x_1, y_1, z_1) u_x(x_2, y_2, z_2) \cos(k\pi(y_1 + y_2)) \cos(2m\pi(x_2 - x_1)) \right\},
\]

\[
J_{6KN} = 2 \sum_{k=1}^{K} \sum_{n=-N}^{N} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dy_1 \, dz_1 \, dy_2 \, dz_2 \\
\left\{ u_z(x_1, y_1, z_1) u_z(x_2, y_2, z_2) \cos(k\pi(y_1 + y_2)) \sin(n\pi z_1) \sin(n\pi z_2) \right\}.
\]

In a manner similar to that in the axisymmetric problem, we can show that

\[
\lim_{M \to \infty} \lim_{N \to \infty} \lim_{K \to \infty} J_{4MNK} \to 0,
\]

\[
\lim_{M \to \infty} \lim_{K \to \infty} J_{5MK} \to 0,
\]

\[
\lim_{K \to \infty} \lim_{N \to \infty} J_{6KN} \to 0.
\]
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First we have

\[
\sum_{m=-M}^{M} \sum_{k=1}^{K} \sum_{n=-N}^{N} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2
\]

\[
\left\{ \frac{4m\mu_x(x_1, y_1, z_1)}{(n^2 + 4m^2)\pi} \frac{\partial u_y}{\partial y_2} \cos[k\pi(y_1 + y_2)] \cos(n\pi z_1) \cos(n\pi z_2) \sin[2m\pi(x_2 - x_1)] \right\}
\]

\[
= \sum_{m=-M}^{M} \sum_{k=1}^{K} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2
\]

\[
\times \left\{ \frac{4m\mu_x(x_1, y_1, z_1)}{(n^2 + 4m^2)\pi} \cos(n\pi z_1) \cos(n\pi z_2) \sin[2m\pi(x_2 - x_1)] \right\}
\]

\[
= \sum_{m=-M}^{M} \sum_{k=1}^{K} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2
\]

\[
\times \left\{ \frac{4m\mu_x(x_1, y_1, z_1)}{(n^2 + 4m^2)\pi} \cos(n\pi z_1) \cos(n\pi z_2) \sin[2m\pi(x_2 - x_1)] \right\}
\]

\[
= \sum_{m=-M}^{M} \sum_{n=-N}^{N} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2
\]

\[
\times \left\{ \frac{4m\mu_x(x_1, y_1, z_1)}{(n^2 + 4m^2)\pi} \cos(n\pi z_1) \cos(n\pi z_2) \sin[2m\pi(x_2 - x_1)] \right\}
\]

\[
\times \int_{0}^{1} dy_2 \frac{\partial u_y}{\partial y_2} \left( \frac{\sin(K + \frac{1}{2})\pi(y_1 + y_2)}{\sin \frac{\pi}{2}(y_1 + y_2)} - 1 \right). \tag{3.53}
\]

By (3.44) we know that

\[
\lim_{K \to \infty} \int_{0}^{1} dy_2 \frac{\partial u_y}{\partial y_2} \frac{\sin(K + \frac{1}{2})\pi(y_1 + y_2)}{\sin \frac{\pi}{2}(y_1 + y_2)} = 0.
\]

In addition we have the zero boundary condition for \( u_y \) when \( y_2 = 0 \) or \( 1 \). So (3.53) converges to zero.

For the same reason

\[
\sum_{m=-M}^{M} \sum_{k=1}^{K} \sum_{n=-N}^{N} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2
\]

\[
\left\{ -\frac{2m\mu_x(x_1, y_1, z_1)}{(n^2 + 4m^2)\pi} \frac{\partial u_y}{\partial y_2} \cos[k\pi(y_1 + y_2)] \sin(n\pi z_1) \cos(n\pi z_2) \cos[2m\pi(x_2 - x_1)] \right\}
\]
3. Completeness of inertial wave modes

also converges to zero when $K$ goes to infinity. Thus we have proved that

$$\lim_{M \to \infty} \lim_{N \to \infty} \lim_{K \to \infty} J_{4MNK} \to 0.$$ 

Next we have

$$2 \sum_{m=-M}^{M} \sum_{m \neq 0} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 dy_1 dx_2 dy_2$$

$$\{u_x(x_1, y_1, z_1)u_x(x_2, y_2, z_2)\cos[k\pi(y_1 + y_2)] \cos[2m\pi(x_2 - x_1)]\}$$

$$= \sum_{m=-M}^{M} \sum_{m \neq 0} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 dy_1 dx_2 dy_2 \{u_x(x_1, y_1, z_1)u_x(x_2, y_2, z_2)\cos[2m\pi(x_2 - x_1)]\}$$

$$\times \sum_{k=1}^{K} 2 \cos[k\pi(y_1 + y_2)]$$

$$= \sum_{m=-M}^{M} \sum_{m \neq 0} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 dy_1 dx_2 dy_2 \{u_x(x_1, y_1, z_1)u_x(x_2, y_2, z_2)\cos[2m\pi(x_2 - x_1)]\}$$

$$\times \left(\frac{\sin \left(\left(K + \frac{1}{2}\right)\pi(y_1 + y_2)\right)}{\sin \frac{\pi}{2}(y_1 + y_2)} - 1\right)$$

$$K \to +\infty \sum_{m=-M}^{M} \sum_{m \neq 0} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} dx_1 dy_1 dx_2 dy_2 \{u_x(x_1, y_1, z_1)u_x(x_2, y_2, z_2)\cos[2m\pi(x_2 - x_1)]\}$$

$$= 0. \quad (3.54)$$

The last equality is because the integration

$$\int_{0}^{1} \int_{0}^{1} u_x(x_2, y_2, z_2) dy_2 dz_2 \quad (3.55)$$

is a constant independent of $x_2$.

Thus we have

$$\lim_{M \to \infty} \lim_{N \to \infty} J_{5MK} \to 0.$$ 

For the same reason we have

$$\lim_{M \to \infty} \lim_{N \to \infty} J_{6MK} \to 0.$$
3. Completeness of inertial wave modes

It follows from (3.52), after applying integral relations such as (3.46a,b), that

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} |A_{mnk}|^2 = \int_0^1 \int_0^1 \int_0^1 \left( u_x^2 + u_y^2 + u_z^2 \right) dx_1 dy_1 dz_1$$

$$= \int_0^1 \int_0^1 \int_0^1 |u|^2 dx_1 dy_1 dz_1 \quad (3.56)$$

In other words, we have proved Parseval’s equality for the general inertial modes $u_{mnk}(x, y, z)$ (or the general eigenfunctions): Any piecewise continuous, differentiable velocity $u(x, y, z)$ satisfying $\nabla \cdot u = 0$ can be approximated by an expansion in the general inertial modes $u_{mnk}$ in the $L^2$ sense to any desired degree of accuracy.

3.6 Summary and remarks

As the first attempt to answer the mathematical question of the completeness of the inviscid eigenfunctions in rapidly rotating systems that was raised by Greenspan (1968) more than four decades ago, we have adopted the simple geometry of a rotating annular channel that uses cartesian coordinates and allows a relatively simple analytical description of all the inertial modes. We have concluded that the inviscid inertial modes $u_{mnk}$ indeed form a complete system of functions. In other words, any piecewise continuous flow $u$ that is incompressible

$$\nabla \cdot u = 0,$$

obeys

$$\int_0^1 \int_0^1 \int_0^1 |u|^2 dx dy dz < \infty,$$

and is differentiable with

$$|\partial u/\partial x| < \infty, \quad |\partial u/\partial y| < \infty, \quad |\partial u/\partial z| < \infty,$$

can be approximated by the inertial-mode expansion

$$u_{MNK}(x, y, z) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{k=1}^{K} A_{mnk} u_{mnk}(x, y, z)$$

to any desired degree of accuracy when $M, N$ and $K$ are sufficiently large.

It is essential to notice that the flow velocity $u$ that can be approximated by the inertial-mode expansion must be piecewise continuous and differentiable. For an inviscid rotating fluid, a spatial singularity (like the wave attractor) usually takes place in the fluid domain. It follows that the mathematical approach based on the completeness of inertial modes is generally not suitable for an inviscid rotating fluid whose velocity is marked by the existence of the spatial singularity. For a
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real rotating fluid such as precession and convection, however, the effect of viscosity
and nonlinearity regularizes the flow such that its velocity $u$—which is piecewise
continuous and differentiable—can be approximated by the inertial-mode expansion.

The explicit analytical expressions for all inertial modes are now available for
spheres (Zhang et al. 2001) and for oblate spheroids of arbitrary eccentricity (Zhang
et al. 2004). Although we have answered the mathematical question of the com-
pleteness of inertial modes for channel geometry, whether the inviscid eigenfunctions
are complete in other geometries, such as spherical and spheroidal, poses much more
difficult and challenging mathematical problems. We have made several attempts to
prove the completeness of the spherical inertial modes, which, unfortunately, were
unsuccessful because both the eigenfunction and eigenvalue expressions are too com-
licated. The mathematical question of the completeness of the inertial modes for
the geophysically and astrophysically relevant geometries remains to be answered.

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Chapter 4

Nonlinear flow in precessing circular cylinders

4.1 Introduction

With important direct or indirect applications to the stability properties of spinning spacecraft with fluid payloads and planetary dynamos, the problem of precessionally driven fluid motion in rotating circular cylinders has been studied for a long time (for example, Wood, 1966; Gans, 1970; Manasseh, 1992; Kobine, 1995; Meunier et al., 2008; Nore et al., 2011; Mouhali, 2012). The problem is concerned with a homogeneous fluid of viscosity $\nu$ confined in a fluid-filled circular cylinder of height $d$ and radius $\Gamma d$ that rotates rapidly with angular velocity $\Omega_0$ about its symmetry axis and precesses with angular velocity $\Omega_p$ about a different axis that is fixed in space. The problem is characterized by three key dimensionless parameters: the radius-height aspect $\Gamma$ of the cylinder, determining whether the precessional forcing can resonate directly with inertial wave modes, the Ekman number $Ek = \nu / |\Omega_0| d^2$ providing the measure of relative importance between the typical viscous force and the Coriolis force (Greenspan, 1968), and the Poincaré number $Po = |\Omega_p|/|\Omega_0|$ quantifying the strength of the precessional forcing (Wu and Roberts, 2009).

A particularly interesting cylinder is called the spherical-like cylinder with $\Gamma = 0.502559$ for which the diameter of the cylinder $2(\Gamma d) = 1.005d$ is nearly the same as its height $d$. The spherical-like cylinder has received considerable attention because it can produce the strongest response to precessional forcing (for example, Gans, 1970; Liao and Zhang, 2012). In the spherical-like cylinder, the precessional forcing resonates directly with the lowest-order inertial mode and, consequently, the resulting flow has the spatially simplest structure marked by the smallest viscous dissipation with its amplitude $|u|$ obeying the asymptotic scaling $|u| = O(Po / \sqrt{Ek})$ at $Ek \ll 1$.

The present study investigates the problem of fluid motion in the precessing spherical-like cylinder via asymptotic and numerical analysis. In the asymptotic
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analysis, upon assuming that the Poincaré force in the spherical-like cylinder resonates primarily with a single inertial mode $u_{111}$, we are able to derive an analytical solution in closed form in the mantle frame describing weakly precessing flow for $0 < Po \ll 1$ and $(Po/\sqrt{Ek}) \ll 1$. In the numerical analysis, it is well known that axial singularities at the symmetry axis of a circular cylinder often cause numerical difficulties when employing a spectral method. To avoid the axial problem, (Nore et al., 2011) adopted a spectral and finite-element hybrid method in which finite element representations are employed in the vertical/radial direction while the Fourier expansion is used in the azimuthal direction. We shall construct a fully three-dimensional finite element model with which numerical experiments will be carried out in attempting to elucidate the fluid’s structure.

In what follows we shall begin by presenting the mathematical formulation of the problem in §2. The finite element method is discussed in §3, and the derivation of an asymptotic solution for the spherical-like cylinder is presented in §4. The properties of nonlinear precessing flow are discussed in §4 while a brief summary and concluding remarks are given in §5.

4.2 Mathematical formulation of the problem

Consider a viscous, incompressible and homogeneous fluid occupying a circular cylinder of radius $\Gamma d$ and length $d$ with aspect ratio $\Gamma$. The cylinder rotates rapidly with an angular velocity $\Omega_0 = \Omega_0 \hat{z}$, where $\Omega_0$ is constant, about its axis of symmetry and precesses slowly with an angular velocity $\Omega_p$ that is fixed in an inertial frame at the angle $\alpha_p$, $0 < \alpha_p \leq \pi/2$. We shall adopt cylindrical polar coordinates $(s, \phi, z)$ with $s = 0$ representing the symmetry axis and $z = 0$ at the bottom surface and the corresponding unit vectors $(\hat{s}, \hat{\phi}, \hat{z})$. In these coordinates fixed in the container, the body or mantle frame of reference, the precession vector $\Omega_p$ is time-dependent,

$$\Omega_p = |\Omega_p| \left[ \dot{s} \sin \alpha_p \cos (\phi + \Omega_0 t) - \dot{\phi} \sin \alpha_p \sin (\phi + \Omega_0 t) + \dot{z} \cos \alpha_p \right],$$

and fluid motion driven by precession is described by the equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2 \left\{ \dot{z} \Omega_0 + |\Omega_p| \left[ \dot{s} \sin \alpha_p \cos (\phi + \Omega_0 t) - \dot{\phi} \sin \alpha_p \sin (\phi + \Omega_0 t) + \dot{z} \cos \alpha_p \right] \right\} \times \mathbf{u}$$

$$= -\nabla p + \nu \nabla^2 \mathbf{u} - 2 \dot{z} |\Omega_p| \Omega_0 s \sin \alpha_p \cos (\phi + \Omega_0 t),$$

$$\nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u}$ is the velocity of fluid motion and $p$ is the reduced pressure. The last term on the right-hand side of the equation of motion is known as the Poincaré forcing, which drives precessional flows against viscous dissipation.
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Since there exist two angular velocities $\Omega_0$ and $\Omega_p$ in the precession system, various units of time were adopted by different workers in scaling the equations (Wood, 1966; Manasseh, 1992; Kobine, 1995; Meunier et al., 2008). This study will adopt $\Omega_0^{-1}$ as the timescale. On employing the height $d$ as the length scale, $\Omega_0^{-1}$ as the unit of time and $\rho_0 d^2 \Omega_0^2$ as the unit of pressure, we obtain the dimensionless equations:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + 2 \left\{ \hat{z} + P_0 \left[ \hat{s} \sin \alpha_p \cos (\phi + t) - \dot{\phi} \sin \alpha_p \sin (\phi + t) + \hat{z} \cos \alpha_p \right] \right\} \times u = -\nabla p + E k \nabla^2 u - 2 \hat{z} s P_0 \sin \alpha_p \cos (\phi + t),$$

(4.1)

$$\nabla \cdot u = 0.$$

(4.2)

In the body frame of reference, the flow on the bounding surface $S$ of a precessing circular cylinder is at rest, imposing

$$\hat{z} \cdot u = 0 \text{ and } \hat{z} \times u = 0$$

on the bottom at $z = 0$ and the top at $z = 1$ and

$$\hat{s} \cdot u = 0 \text{ and } \hat{s} \times u = 0$$

on the sidewall at $s = \Gamma$.

When $P_0$ is sufficiently small ($P_0 \ll 1$) with the amplitude $|u| = \epsilon \ll 1$, (4.1) may be linearized by omitting the higher-order terms $u \cdot \nabla u$ and

$$\left| P_0 \left[ \hat{z} \cos \alpha_p + \hat{s} \sin \alpha_p \cos (\phi + t) - \dot{\phi} \sin \alpha_p \sin (\phi + t) \right] \times u \right| = O(P_0 \epsilon),$$

which results in the governing equations for a weakly precessing flow

$$\frac{\partial u}{\partial t} + 2 \hat{z} \times u + \nabla p = E k \nabla^2 u - 2 \hat{z} s P_0 \sin \alpha_p e^{i(\phi+t)},$$

(4.5)

$$\nabla \cdot u = 0,$$

(4.6)

where only the real part of the complex solution will be taken as the physical solution of the problem.

Equations (4.5)-(4.6) are derived under the assumption that $P_0$ and $E k$ are both sufficiently small together with $(P_0 / \sqrt{E k}) \ll 1$ at exact resonance. Since the Poincaré forcing, $2s \dot{z} P_0 \sin \alpha_p e^{i(\phi+t)}$, is described by the azimuthal wavenumber $m = 1$, the weakly precessing flow would be characterized by the same wavenumber, implying that the pressing flow in a cylinder must be three-dimensional. It is noteworthy that linearization can be also achieved by supposing that the angle $\alpha_p$ is sufficiently small. By letting $P_0$ be small with an arbitrary angle $\alpha_p$, however, we
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The weakly precessing problem defined by (4.5)-(4.6) subject to the boundary conditions (4.3)-(4.4) will be solved by an asymptotic method for $Ek \ll 1$ in the spherical-like cylinder and, then, the strongly precessing problem defined by (4.1)-(4.2) subject to the boundary conditions (4.3)-(4.4) will be solved by a finite element method for $Ek \ll 1$.

4.3 Numerical simulation using a finite element method

Direct nonlinear numerical simulation allows us to extend from the weakly precessing regime ($0 < Po/\sqrt{Ek} \ll 1$) to the strongly precessing regime ($Po/\sqrt{Ek} \geq O(1)$). Local numerical methods like finite element methods are particularly suitable for a precessing cylinder because it is well known that the standard spectral method or finite-difference methods, because of the existence of axial singularities, is numerically difficult for a cylindrical cavity. A three-dimensional tetrahedralization of the cylindrical cavity in our nonlinear finite element code can produce a finite element mesh that does not have axial singularities, a well-known numerical difficulty in the numerical analysis for cylindrical geometry. Moreover, the three-dimensional mesh is flexible enough to construct more nodes in the vicinity of the bounding surface of the cylindrical cavity for resolving the thin viscous boundary layer. A sketch of the finite element mesh is illustrated in Figure 4.1. The EBE (element-by-element) method used in this study is largely similar to that used for spheroidal geometry.

Figure 4.1: A schematic of the finite element mesh in a cylindrical cavity for nonlinear numerical simulation with the denser mesh in the vicinity of the cylindrical bounding surface $S$. 

...
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(Chan et al, 2010), except that the construction for the equations are changed.

After finishing the tetrahedralization for a cylindrical cavity, we then construct
the temporal discretization of the numerical precession model. Let \( T_f \) be a fixed
final time of a numerical simulation. We divide the time interval \([0, T_f]\) into \( M \)
equally spaced subintervals using the following nodal points

\[
0 = t_0 < t_1 < t_2 < \ldots < t_M = T_f,
\]

where \( t_n = n\Delta t \) for \( n = 0, 1, \ldots, M \). Let \( \mathbf{u}(\mathbf{r}, t) \) be a function continuous with re-
spect to \( t \) where \( \mathbf{r} \) is the position vector. Denote \( \mathbf{u}^n(\mathbf{r}) = \mathbf{u}(\mathbf{r}, t_n) \) for \( n = 0, 1, \ldots, M \).

An implicit time stepping scheme is employed for the time advancement of integra-
tion, in which an implicit second-order backward differentiation formula is adopted
for the time derivative

\[
\left( \frac{\partial \mathbf{u}}{\partial t} \right)^{n+1} = \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} + O(\Delta t^2)
\]

while the nonlinear term \( \mathbf{u} \cdot \nabla \mathbf{u} \) at \( t = t_{n+1} \) is approximated by the implicit scheme

\[
\mathbf{u}^{n+1} \cdot \nabla \mathbf{u}^{n+1} = (2\mathbf{u}^{n} - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{u}^{n+1} + O(\Delta t^2).
\]

The implicit temporal discretization of the full equations (4.1)-(4.2) produces

\[
\begin{align*}
\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} &+ (2\mathbf{u}^{n} - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{u}^{n+1} + 2 \left( \hat{\mathbf{z}} + P_0 \hat{\Omega}_p^{n+1} \right) \times \mathbf{u}^{n+1} \\
\nabla \cdot \mathbf{u}^{n+1} & = 0,
\end{align*}
\]

where

\[
\hat{\Omega}_p^{n+1} = \sin \alpha_p \left[ \hat{\mathbf{s}} \cos (\phi + t_{n+1}) - \hat{\mathbf{\phi}} \sin (\phi + t_{n+1}) \right] \hat{\mathbf{z}} + \hat{\mathbf{z}} \cos \alpha_p.
\]

They are solved, starting from an arbitrary initial condition, to find \( \mathbf{u}^{n+1}, P^{n+1} \) for
given \( \mathbf{u}^n \) and \( P^n \), subject to the boundary conditions (4.3)-(4.4), on modern parallel
computers. Although a fixed time-step approach is adopted in the current numerical
model, a variable time-step scheme can be readily implemented. It is noteworthy
that direct nonlinear simulation is numerically expensive at resonance. This is be-cause the viscous boundary layer plays an active role in controlling the dynamics
of a resonant precessing flow: it usually takes more than \( O(Ek^{-1/2}) \) dimensionless
time units, starting from an arbitrary initial condition, for a nonlinear simulation
to reach a final nonlinear equilibrium state.
4.3.1 An introduction of the finite element code

The code is written in Fortran. There is a main file including the main function and associated subroutines, with 11 assistant files including different parts of the program. The program runs over many processors simultaneously, with message passing interface (MPI) as the communication tool. We use a mesh generated by NETGEN (an open source software) to decompose the cylinder into many tetrahedron elements (see figure 4.1). Each processor deals with part of these elements. Suppose the total number of processors is \( NP \), the total number of elements in the mesh is \( Nel \), then we have

\[
Nel = \sum_{pe=1}^{NP} Nel_{pe},
\]

where \( Nel_{pe} \) is the number of elements treated by the \( pe^{th} \) processor.

The finite element method constructs a linear system for each element as follows.

Equation (4.7) could be rearranged as

\[
\begin{align*}
\frac{3u^{n+1}}{2\Delta t} + (2u^n - u^{n-1}) \cdot \nabla u^{n+1} + 2 \left( \hat{z} + P_0 \hat{\xi}_p^{n+1} \right) \times u^{n+1} - E_k \nabla^2 u^{n+1} + \nabla P^{n+1} \\
= -\frac{4u^n + u^{n-1}}{2\Delta t} - 2 P_0 [s \sin \alpha_p \cos (\phi + t_{n+1})] \hat{z},
\end{align*}
\]

which is in the form

\[
L(u^{n+1}, P^{n+1}) = f,
\]

where

\[
f = -\frac{4u^n + u^{n-1}}{2\Delta t} - 2 P_0 [s \sin \alpha_p \cos (\phi + t_{n+1})] \hat{z},
\]

and \( L \) denotes a linear operator. Suppose \( g \) is a test function. Multiply \( g \) in both sides of (4.9), and integrate over the cylinder \( \Omega \), we get (because the left hand side is a sum of several terms, and the operator \( L \) is linear, we can take one term as an example)

\[
\int_{\Omega} \frac{3}{2\Delta t} u^{n+1} \cdot g \, d\mu + \cdots = \int_{\Omega} f \cdot g \, d\mu.
\]

In every tetrahedron element, we take ten shape functions \( \{N_i, i = 1, 2, ..., 10\} \) as the basis of the function space of velocity, and four shape functions \( \{M_i, i = 1, ..., 4\} \) as the basis of the function space of pressure (The 10-4 combination comes from the
Taylor-Hood finite element as a stable pair. See Gerbeau, 2010):

\[
N_1 = L_1(2L_1 - 1), \quad N_2 = L_2(2L_2 - 1), \quad N_3 = L_3(2L_3 - 1), \quad N_4 = L_4(2L_4 - 1),
\]
\[
N_5 = 4L_1L_2, \quad N_6 = 4L_1L_3, \quad N_7 = 4L_1L_4,
\]
\[
N_8 = 4L_2L_3, \quad N_9 = 4L_2L_4, \quad N_{10} = 4L_3L_4;
\]
\[
M_1 = L_1, \quad M_2 = L_2, \quad M_3 = L_3, \quad M_4 = L_4.
\]

(4.11)

Here the \(L_i, i = 1, \ldots, 4\) are volume coordinates. For example, if the current tetrahedron element is labeled by its four nodes as \(ABCD\), then the volume coordinates of a point \(Q\) in the tetrahedron are defined as

\[
L_1 \equiv \frac{\text{Volume of } QBCD}{\text{Volume of } ABCD}, \quad L_2 \equiv \frac{\text{Volume of } QACD}{\text{Volume of } ABCD},
\]
\[
L_3 \equiv \frac{\text{Volume of } QABD}{\text{Volume of } ABCD}, \quad L_4 \equiv \frac{\text{Volume of } QABC}{\text{Volume of } ABCD}.
\]

Obviously we have \(\sum_{i=1}^{4} L_i = 1\). Now every velocity could be expressed as a linear sum of these basis,

\[
\mathbf{u}^{n+1} = (u^{n+1}, v^{n+1}, w^{n+1}) = \left( \sum_{i=1}^{10} U_i N_i, \sum_{i=1}^{10} V_i N_i, \sum_{i=1}^{10} W_i N_i \right),
\]
\[
\mathbf{g} = (g^x, g^y, g^z) = \left( \sum_{i=1}^{10} G^x_i N_i, \sum_{i=1}^{10} G^y_i N_i, \sum_{i=1}^{10} G^z_i N_i \right).
\]

Put them into (4.10). The left hand side is

\[
\int_{\Omega} \frac{3}{2\Delta t} \mathbf{u}^{n+1} : \mathbf{g} \, d\mu
\]
\[
= \int_{\Omega} \frac{3}{2\Delta t} (u^{n+1} g^x + v^{n+1} g^y + w^{n+1} g^z) \, d\mu
\]
\[
= \int_{\Omega} \frac{3}{2\Delta t} \left( \sum_{i=1}^{10} U_i N_i \sum_{i=1}^{10} G^x_i N_i + \sum_{i=1}^{10} V_i N_i \sum_{i=1}^{10} G^y_i N_i + \sum_{i=1}^{10} W_i N_i \sum_{i=1}^{10} G^z_i N_i \right) \, d\mu
\]
\[
= \sum_{i=1}^{10} \sum_{j=1}^{10} U_i G^x_j \int_{\Omega} \frac{3}{2\Delta t} N_i N_j \, d\mu +
\]
\[
\sum_{i=1}^{10} \sum_{j=1}^{10} V_i G^y_j \int_{\Omega} \frac{3}{2\Delta t} N_i N_j \, d\mu + \sum_{i=1}^{10} \sum_{j=1}^{10} W_i G^z_j \int_{\Omega} \frac{3}{2\Delta t} N_i N_j \, d\mu.
\]
The right hand side is

\[
\int_\Omega \mathbf{f} \cdot \mathbf{g} \, d\mu = \int_\Omega (f^x g^x + f^y g^y + f^z g^z) \, d\mu
\]

\[
= \int_\Omega (\sum_{j=1}^{10} G^x_j N_j + \sum_{j=1}^{10} G^y_j N_j + \sum_{j=1}^{10} G^z_j N_j) \, d\mu
\]

\[
= \sum_{j=1}^{10} G^x_j \int_\Omega f^x N_j \, d\mu + \sum_{j=1}^{10} G^y_j \int_\Omega f^y N_j \, d\mu + \sum_{j=1}^{10} G^z_j \int_\Omega f^z N_j \, d\mu.
\]

Because the test function \( \mathbf{g} \) is arbitrary, we can compare the coefficients of \( G^x_i, G^y_i, G^z_i, i = 1, 2, \ldots, 10 \) on both sides of the equation, and get

\[
\sum_{i=1}^{10} U_i \int_\Omega \frac{3}{2\Delta t} N_i N_j \, d\mu + \cdots = \int_\Omega f^x N_j \, d\mu \equiv f^x_i,
\]

\[
\sum_{i=1}^{10} V_i \int_\Omega \frac{3}{2\Delta t} N_i N_j \, d\mu + \cdots = \int_\Omega f^y N_j \, d\mu \equiv f^y_i,
\]

\[
\sum_{i=1}^{10} W_i \int_\Omega \frac{3}{2\Delta t} N_i N_j \, d\mu + \cdots = \int_\Omega f^z N_j \, d\mu \equiv f^z_i.
\]

In matrix form, this is

\[
\begin{bmatrix}
\int_\Omega \frac{3}{2\Delta t} N_1 N_1 \, d\mu & \cdots & \int_\Omega \frac{3}{2\Delta t} N_1 N_{10} \, d\mu \\
\vdots & \ddots & \vdots \\
\int_\Omega \frac{3}{2\Delta t} N_{10} N_1 \, d\mu & \cdots & \int_\Omega \frac{3}{2\Delta t} N_{10} N_{10} \, d\mu
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
\vdots \\
U_{10}
\end{bmatrix}
= 
\begin{bmatrix}
f^x_1 \\
f^x_2 \\
\vdots \\
f^x_{10}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\int_\Omega \frac{3}{2\Delta t} N_1 N_1 \, d\mu & \cdots & \int_\Omega \frac{3}{2\Delta t} N_1 N_{10} \, d\mu \\
\vdots & \ddots & \vdots \\
\int_\Omega \frac{3}{2\Delta t} N_{10} N_1 \, d\mu & \cdots & \int_\Omega \frac{3}{2\Delta t} N_{10} N_{10} \, d\mu
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_{10}
\end{bmatrix}
= 
\begin{bmatrix}
f^y_1 \\
f^y_2 \\
\vdots \\
f^y_{10}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\int_\Omega \frac{3}{2\Delta t} N_1 N_1 \, d\mu & \cdots & \int_\Omega \frac{3}{2\Delta t} N_1 N_{10} \, d\mu \\
\vdots & \ddots & \vdots \\
\int_\Omega \frac{3}{2\Delta t} N_{10} N_1 \, d\mu & \cdots & \int_\Omega \frac{3}{2\Delta t} N_{10} N_{10} \, d\mu
\end{bmatrix}
\begin{bmatrix}
W_1 \\
W_2 \\
\vdots \\
W_{10}
\end{bmatrix}
= 
\begin{bmatrix}
f^z_1 \\
f^z_2 \\
\vdots \\
f^z_{10}
\end{bmatrix}.
\]
Or we can make them into a whole equation

\[
\begin{bmatrix}
M_{uu} & 0 & 0 \\
0 & M_{vv} & 0 \\
0 & 0 & M_{ww}
\end{bmatrix}
\begin{bmatrix}
U_1 \\
\vdots \\
U_{10} \\
V_1 \\
\vdots \\
V_{10} \\
W_1 \\
\vdots \\
W_{10}
\end{bmatrix}
+ \cdots =
\begin{bmatrix}
f_{i_1} \\
\vdots \\
f_{i_{10}}
\end{bmatrix}
\]

where

\[
M_{uu} = M_{vv} = M_{ww} = \begin{bmatrix}
\int_{\Omega} \frac{3}{2 \Delta t} N_1 N_1 \, d\mu & \cdots & \int_{\Omega} \frac{3}{2 \Delta t} N_1 N_{10} \, d\mu \\
\vdots & \ddots & \vdots \\
\int_{\Omega} \frac{3}{2 \Delta t} N_{10} N_1 \, d\mu & \cdots & \int_{\Omega} \frac{3}{2 \Delta t} N_{10} N_{10} \, d\mu
\end{bmatrix}.
\]

The matrices generated by the second term \((2u^n - u^{n-1}) \cdot \nabla u^{n+1}\) are (here and following \(i, j = 1, 2, \ldots, 10\))

\[(Q_{uu})_{i,j} = (Q_{vv})_{i,j} = (Q_{ww})_{i,j} = \int_{\Omega} [(2u^n - u^{n-1}) \cdot \nabla N_j] N_i \, d\mu.
\]

The matrices generated by the third term \(2 \left( \hat{z} + Po \hat{\Omega}_p^{n+1} \right) \times u^{n+1}\) are

\[(S_{uv})_{i,j} = -\int_{\Omega} Po(\hat{\Omega}_p^{n+1})_z N_i N_j \, d\mu, \quad (S_{vu})_{i,j} = \int_{\Omega} Po(\hat{\Omega}_p^{n+1})_z N_i N_j \, d\mu,
\]

\[(S_{uw})_{i,j} = \int_{\Omega} Po(\hat{\Omega}_p^{n+1})_y N_i N_j \, d\mu, \quad (S_{wu})_{i,j} = -\int_{\Omega} Po(\hat{\Omega}_p^{n+1})_y N_i N_j \, d\mu,
\]

\[(S_{vw})_{i,j} = -\int_{\Omega} [1 + Po(\hat{\Omega}_p^{n+1})_z] N_i N_j \, d\mu, \quad (S_{wv})_{i,j} = \int_{\Omega} [1 + Po(\hat{\Omega}_p^{n+1})_z] N_i N_j \, d\mu.
\]

The matrices generated by the fourth term \(-Ek \nabla^2 u^{n+1}\) are

\[(S_{uu})_{i,j} = (S_{uv})_{i,j} = (S_{wu})_{i,j} = \int_{\Omega} Ek \left( \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right) \, d\mu.
\]

The matrices generated by the fifth term \(\nabla P^{n+1}\) are (\(\alpha = 1, \ldots, 4\))

\[(S_{up})_{i,\alpha} = \int_{\Omega} N_i \frac{\partial M_{\alpha}}{\partial x} \, d\mu, \quad (S_{vp})_{i,\alpha} = \int_{\Omega} N_i \frac{\partial M_{\alpha}}{\partial y} \, d\mu, \quad (S_{wp})_{i,\alpha} = \int_{\Omega} N_i \frac{\partial M_{\alpha}}{\partial z} \, d\mu.
\]

For the divergence free condition (4.8), multiplying a test function \(h = \sum_{i=1}^{4} H_i M_i\)

\[\int_{\Omega} \nabla \cdot (M_{uu} \mathbf{u} + M_{vv} \mathbf{v} + M_{ww} \mathbf{w} + M_{uv} \mathbf{u} + M_{vu} \mathbf{v} + M_{uw} \mathbf{u} + M_{wu} \mathbf{u} + M_{vw} \mathbf{v} + M_{wv} \mathbf{v}) \, d\Omega = 0.
\]
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in both sides we get the following matrices

\[(S_{pu})_{\alpha,j} = \int_\Omega M_\alpha \frac{\partial N_j}{\partial x} \, d\mu, \quad (S_{vp})_{\alpha,j} = \int_\Omega M_\alpha \frac{\partial N_j}{\partial y} \, d\mu, \quad (S_{pw})_{\alpha,j} = \int_\Omega M_\alpha \frac{\partial N_j}{\partial z} \, d\mu.\]

All these sub-matrices can be made into a whole one for the current element:

\[
\begin{bmatrix}
  S_{uu} & S_{uv} & S_{uw} & S_{up} \\
  S_{vu} & S_{vv} & S_{vw} & S_{vp} \\
  S_{wu} & S_{ww} & S_{ww} & S_{wp} \\
  S_{pu} & S_{pv} & S_{pw} & S_{pp} \\
\end{bmatrix}
\begin{bmatrix}
  U_1 \\
  \vdots \\
  U_{10} \\
  V_1 \\
  \vdots \\
  V_{10} \\
  W_1 \\
  \vdots \\
  W_{10} \\
  P_1 \\
  \vdots \\
  P_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
  f_1^x \\
  \vdots \\
  f_{10}^x \\
  f_1^y \\
  f_{10}^y \\
  f_1^z \\
  f_{10}^z \\
  0 \\
  0 \\
  0 \\
\end{bmatrix},
\]

where \(S_{uu} = S_{uu} + M_{uu} + Q_{uu}, S_{vv} = S_{vv} + M_{vv} + Q_{vv}, S_{ww} = S_{ww} + M_{ww} + Q_{ww}\). In short this is

\[A^e x^e = v^e.\]  \hspace{1cm} (4.12)

The superscript \(e\) means the current element.

Any value in the matrix \(A^e\), say \((M_{uu})_{1,1} = \int_\Omega \frac{3}{2\Delta t} N_1 N_1 \, d\mu\), is an integration. We estimate these integrals by Gauss points in the tetrahedron. That is

\[
\int_\Omega \frac{3}{2\Delta t} N_1 N_1 \, d\mu \approx \frac{3}{2\Delta t} \sum_{i=1}^{nip} w_i \cdot N_1(Q_i) N_1(Q_i),
\]

where \(nip\) is the number of Gauss points. In practice \(nip = 4\). \(N_1(Q_i)\) is the value of \(N_1\) at the \(i\)th Gauss point \(Q_i\). And \(w_i\) are the weights. For some terms, we first interpolate the value of the integrand by the shape functions (4.11) to the Gauss points, and then add them together.

Taken into consideration all elements, in fact we have to solve an assembled linear system

\[Ax = v.\]  \hspace{1cm} (4.13)

Our Fortran code solves this linear system by the BiCGstab(L) iterative solver (Slei-
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It is portable and does not require any support from other external numerical software packages. Further, we adopt the EBE parallelization strategy. For classical finite element method (Zienkiewicz, 1977), the large matrix $A$ is assembled. But by EBE, all element matrices $A^e$ are stored and distributed in different processors. All algebraic operations such as vector addition, inner product and matrix-vector multiplication, are EBE parallelized.

The parallelization of vector addition is natural, as follows

$$x + y \Leftrightarrow x_{pe} + y_{pe}, \quad pe = 1, 2, ..., NP.$$  

All additions $x_{pe} + y_{pe}$ deal with components of vectors which are locally stored, thus no inter-processor communication is required.

To parallelize the inner product $d = x \cdot y$, first we compute the value of local inner product $d_{pe} = x_{pe} \cdot y_{pe}$ on each processor $pe$, and then reduce (using inter-processor communications) $d$ from $d_{pe}$’s on all processors.

For the parallelization of the most time-consuming operation, the matrix-vector product $v = Ax$, we first compute the product in an EBE manner as

$$Ax = \left( \sum_{k=1}^{Nel} A^e_k \right)x = \sum_{k=1}^{Nel} A^e_k x^e_k = \sum_{k=1}^{Nel} v^e_k = v.$$  

where $x^e_k$ and $v^e_k$ are the parts of $x$ and $v$ associated with $k$’th element. Then the EBE sum is decomposed into different parts on different processors

$$\sum_{k=1}^{Nel} A^e_k x^e_k = \sum_{pe=1}^{NP} \sum_{k=1}^{Nel_{pe}} A^e_k x^e_k = \sum_{pe=1}^{NP} \sum_{k=1}^{Nel_{pe}} v^e_k.$$  

Note that the equations are solved under the boundary conditions

$$\mathbf{u} = 0 \quad \text{on} \quad \partial \Omega.$$  

To meet this requirement, we can just omit those surface unknowns (leaving them zero). Further, because the pressure $P$ is variable by a constant, to fix it we let $P = 0$ at a certain node on the surface.

Also we use a Grad-Div stabilizer to make the solving procedure stable. See Olshanskii (2004).

After solving the linear equations, we get the 34 values

$$U_1, ..., U_{10}, V_1, ..., V_{10}, W_1, ..., W_{10}, P_1, ..., P_4$$  

for every element in the mesh. But this finite element mesh is quite irregular. In order to draw figures giving useful information of the velocity and pressure field, we first need to get the values of the variables on a regular mesh. This is done using
interpolation. For example, suppose \( Q \) is a point in the regular mesh. We can locate it in the finite element mesh to a certain tetrahedron element, say \( ABCD \). Then we can compute \( Q \)'s volume coordinates in \( ABCD \); suppose they are \( L_i, i = 1, ..., 4 \). Because the shape functions (4.11) are a basis of the velocity function space, we can get a velocity field through the following linear sum

\[
U = \sum_{i=1}^{10} U_i N_i.
\]

Then for the point \( Q \) we have

\[
U(Q) = \sum_{i=1}^{10} U_i N_i(L_1, ..., L_4).
\]

For \( V, W, P \) at \( Q \) we have similar formulas.

Finally, most figures in this chapter are drawn with MATLAB.

4.4 An analytical solution for the spherical-like cylinder

Since the Poincaré force resonates directly with a single inertial mode \( u_{111} \) in the spherical-like cylinder with \( \Gamma = 0.502559 \) (for example, Gans,1970; Liao and Zhang,2012), other inertial modes may be omitted to leading-order approximation in deriving an asymptotic solution of the precessing flow. This dramatically simplifies the mathematical analysis and, more significantly, an analytical solution in closed form for \( 0 < Ek \ll 1 \) becomes available in the weakly precessing cylinder.

The asymptotic solution describing the weakly precessing flow for \( \Gamma = 0.502559 \) at \( Ek \ll 1 \) can be expanded in the form

\[
\begin{align*}
\mathbf{u}(s, \phi, z, t) & = [A_{111} u_{111}(s, \phi, z) + \hat{\mathbf{u}} + \tilde{\mathbf{u}}] e^{\text{i} t}, \\
p(s, \phi, z, t) & = [A_{111} p_{111}(s, \phi, z) + \hat{p} + \tilde{p}] e^{\text{i} t},
\end{align*}
\]

(4.14) (4.15)

where \( \hat{\mathbf{u}} \) and \( \hat{p} \) represent small interior perturbations, caused by the viscous effect, to the inertial wave mode \( (u_{111}, p_{111}) \), \( A_{111} \) is the amplitude of the flow to be determined, \( \tilde{\mathbf{u}} \) and \( \tilde{p} \) denote the flow and the pressure in the viscous boundary layer. For
the spherical-like cylinder, the un-normalized inertial mode \((u_{111}, p_{111})\) is given by

\[
\hat{s} \cdot u_{111}(s, \phi, z) = -\frac{i}{3} \left[ \sqrt{3} \pi J_0 \left( \frac{\xi_{111} s}{\Gamma} \right) + \frac{1}{s} J_1 \left( \frac{\xi_{111} s}{\Gamma} \right) \right] \cos (\pi z) e^{i\phi},
\]

\[
\hat{\phi} \cdot u_{111}(s, \phi, z) = \frac{2}{3} \left[ \sqrt{3} \pi J_0 \left( \frac{\xi_{111} s}{\Gamma} \right) - \frac{1}{2s} J_1 \left( \frac{\xi_{111} s}{\Gamma} \right) \right] \cos (\pi z) e^{i\phi},
\]

\[
\hat{z} \cdot u_{111}(s, \phi, z) = -i\pi J_1 \left( \frac{\xi_{111} s}{\Gamma} \right) \sin (\pi z) e^{i\phi}.
\]

\[
p_{111} = J_1 \left( \frac{\xi_{111} s}{\Gamma} \right) \cos (\pi z) e^{im\phi},
\]

where \(\Gamma = 0.502559\) and \(\xi_{111} = 2.73462\). For a weakly \((0 < Po \ll 1)\) precessing cylinder, we have the property

\[
|\hat{u}| \ll |A_{111} u_{111}| \quad \text{and} \quad |\tilde{u}| = O |A_{111} u_{111}|
\]

which will be used in the asymptotic analysis. On substituting the expansions into (4.5)–(4.6), we obtain an inhomogeneous system for the small perturbation \(\hat{u}\) and \(\hat{p}\)

\[
i\hat{u} + 2\hat{z} \times \hat{u} + \nabla \hat{p} = Ek \nabla^2 (A_{111} u_{111}) - 2\hat{z} s Po \sin \alpha_p e^{i\phi},
\]

\[
\nabla \cdot \hat{u} = 0.
\]

Next we will try to solve these equations. Multiplying the complex conjugate \(u_{111}^*\) in both sides of (4.16) and integrating over the cylinder, we get

\[
4\pi^2 Ek A_{111} \int_0^1 \int_0^\Gamma \int_0^{2\pi} |u_{111}|^2 s \, d\phi \, ds \, dz 
+ \int_S p_{111}^* (\hat{n} \cdot \hat{u}) \, dS = -2 Po \sin \alpha_p \int_0^1 \int_0^\Gamma \int_0^{2\pi} (\hat{z} \cdot u_{111}^*) e^{i\phi} \, s^2 \, d\phi \, ds \, dz = -i8\pi Po \sin \alpha_p \int_0^\Gamma J_1 \left( \frac{\xi_{111} s}{\Gamma} \right) s^2 ds.
\]

We know the exact form of \(u_{111}^*\), so the volume integration could be calculated:

\[
-2 Po \sin \alpha_p \int_0^1 \int_0^\Gamma \int_0^{2\pi} (\hat{z} \cdot u_{111}^*) e^{i\phi} \, s^2 \, d\phi \, ds \, dz = -i8\pi Po \sin \alpha_p \int_0^\Gamma J_1 \left( \frac{\xi_{111} s}{\Gamma} \right) s^2 ds
= i \left( \frac{8\Gamma Po \sin \alpha_p}{\pi} \right) J_1 (\sqrt{3} \Gamma \pi).
\]

The surface integral in (4.18), \(\int_S p_{111}^* (\hat{n} \cdot \hat{u}) \, dS\), representing the influx from the viscous boundary layers at the bounding surface of the cylinder, can be expressible
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\[
\int_\mathcal{S} p_{111}^* (\hat{n} \cdot \hat{u}) \, d\mathcal{S} = -\sqrt{E_k} \int_0^1 \int_0^{2\pi} \left[ (u_{111}^* - 2\hat{z} \times u_{111}^*)_{z=\Gamma_k} \cdot \left( \int_0^\infty \tilde{u}_{\text{sidewall}} \, d\eta \right) \right] \Gamma \, d\phi \, dz

- \sqrt{E_k} \int_0^{\Gamma_k} \int_0^{2\pi} \left[ (u_{111}^* - 2\hat{z} \times u_{111}^*)_{z=0} \cdot \left( \int_0^\infty \tilde{u}_{\text{bottom}} \, d\eta \right) \right] s \, d\phi \, ds

- \sqrt{E_k} \int_0^{\Gamma_k} \int_0^{2\pi} \left[ (u_{111}^* - 2\hat{z} \times u_{111}^*)_{z=1} \cdot \left( \int_0^\infty \tilde{u}_{\text{top}} \, d\eta \right) \right] s \, d\phi \, ds,
\]

where \( \eta \) is the stretched boundary-layer variable and \( \tilde{u}_{\text{sidewall}}, \tilde{u}_{\text{bottom}} \) and \( \tilde{u}_{\text{top}} \) denote the viscous boundary layers at the sidewall, bottom and top of the spherical-like cylinder which can be readily derived. First, the boundary flow \( \tilde{u}_{\text{bottom}} \) satisfies the fourth-order differential equation

\[
\left( \frac{\partial^2}{\eta_0^2} - i \right)^2 \tilde{u}_{\text{bottom}} + 4\tilde{u}_{\text{bottom}} = 0,
\]

where \( \eta_0 = z/\sqrt{E_k} \), and the four boundary conditions

\[
\left( \tilde{u}_{\text{bottom}} \right)_{\eta=0} = -A_{111} (u_{111})_{z=0},

\left( \frac{\partial^2 \tilde{u}_{\text{bottom}}}{\partial \eta_0^2} \right)_{\eta=0} = -A_{111} (u_{111} + 2\hat{z} \times u_{111})_{z=0},

\left( \tilde{u}_{\text{bottom}} \right)_{\eta=\infty} = 0,

\left( \frac{\partial^2 \tilde{u}_{\text{bottom}}}{\partial \eta_0^2} \right)_{\eta=\infty} = 0.
\]

Thus \( \tilde{u}_{\text{bottom}} \) could be solved as

\[
\tilde{u}_{\text{bottom}} = -\frac{1}{2} A_{111} \{ (u_{111} - i\hat{z} \times u_{111})_{z=0} e^{-\sqrt{6}(1+i)\eta_0/2} - (u_{111} + i\hat{z} \times u_{111})_{z=0} e^{-\sqrt{2}(1-i)\eta_0/2} \}.
\]

By vertical symmetry we also have

\[
\tilde{u}_{\text{top}} = -\frac{1}{2} A_{111} \{ (u_{111} - i\hat{z} \times u_{111})_{z=0} e^{-\sqrt{6}(1+i)\eta_1/2} - (u_{111} + i\hat{z} \times u_{111})_{z=0} e^{-\sqrt{2}(1-i)\eta_1/2} \},
\]

where \( \eta_1 = (1 - z)/\sqrt{E_k} \). For the sidewall boundary flow \( \tilde{u}_{\text{sidewall}} \), the boundary conditions are

\[
\left( \tilde{u}_{\text{sidewall}} \right)_{\eta=0} = -A_{111} (u_{111})_{s=\Gamma},

\left( \tilde{u}_{\text{sidewall}} \right)_{\eta=\infty} = 0.
\]
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where \( \eta_s = (\Gamma - s)/\sqrt{E_k} \). So we have

\[
\tilde{u}_{\text{sidewall}} = -A_{111} (u_{111})_s \Gamma e^{-\sqrt{2}(\Gamma - s)(1+i)/2\sqrt{E_k}}.
\]

Putting all of these into (4.18) leads to the determination of the coefficient \( A_{111} \). Inserting the coefficient \( A_{111} \) and the boundary layers \( \tilde{u} = \tilde{u}_{\text{sidewall}} + \tilde{u}_{\text{bottom}} + \tilde{u}_{\text{top}} \) into the expansion (4.14), we obtain an analytical solution in closed form describing weakly precessing flow in the rotating frame for the spherical-like cylinder

\[
\begin{align*}
\mathbf{u}(s, \phi, z, t) &= \frac{1}{2} \left( \frac{P \sin \alpha_p}{\sqrt{E_k}} \right) \begin{bmatrix}
-5.6729i \\
945.18\sqrt{E_k} + 69.112
\end{bmatrix} - i2.8276 \\
&- (u_{111} - i\hat{z} \times u_{111})_{z=0} e^{-\sqrt{2}(1+i)\eta_0/2} - (u_{111} + i\hat{z} \times u_{111})_{z=0} e^{-\sqrt{2}(1-i)\eta_0/2} \\
&- (u_{111} - i\hat{z} \times u_{11k})_{z=1} e^{-\sqrt{2}(1+i)m/2} - (u_{111} + i\hat{z} \times u_{111})_{z=1} e^{-\sqrt{2}(1-i)m/2} \\
&+ 2u_{111} - (2u_{111})_{s=0.502559} e^{-\sqrt{2}\eta_0/(1+i)/2} \right) e^{it},
\end{align*}
\]

(4.19)

whose real part will be taken as the physical solution. The corresponding kinetic energy density \( E_{\text{kin}} \) of the precessing flow is

\[
E_{\text{kin}} = \left[ \frac{2(0.50256\pi)^2 + 1}{6(0.50256)^2 E_k} \right] \left[ \frac{-P \sin \alpha_p 5.6729i}{945.18\sqrt{E_k} + 69.112} - i2.8276 \right]^2 J_1^2(2.7346)
\]

(4.20)

The analytical formulas (4.19) and (4.20) in close form are valid in the mantle frame of reference for the spherical-like cylinder with \( \Gamma = 0.502559 \) with \( 0 < E_k \ll 1 \) and \( 0 < P / \sqrt{E_k} \ll 1 \), providing an effective way of checking the finite element code of direct numerical simulation.

4.5 Nonlinear precessing flow in the spherical-like cylinder

4.5.1 Decomposition of a nonlinear flow

On the basis of the two assumptions that the inertial modes \((u_{mnk}, p_{mnk}, \sigma_{mnk})\) are mathematically complete and that the precessing flow \( \mathbf{u} \) is piecewise continuous and
differentiable, we can always expand the velocity $u$ and its pressure $p$ in the form

$$
u = \tilde{u} + \sum_{k=1}^{K} A_{00k}(t) u_{00k}(s) + \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{1}{2} \left[ A_{m0k}(t) u_{m0k}(s, \phi) + c.c. \right]$$

$$+ \sum_{n=1}^{N} \sum_{k=1}^{K} \frac{1}{2} \left[ A_{0nk}(t) u_{0nk}(s, z) + c.c. \right]$$

$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{k=1}^{2K} \frac{1}{2} \left[ A_{mnk}(t) u_{mnk}(s, z, \phi) + c.c. \right],$$

(4.21)

$$p = \tilde{p} + \sum_{k=1}^{K} A_{00k}(t) p_{00k}(s) + \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{1}{2} \left[ A_{m0k}(t) p_{m0k}(s, \phi) + c.c. \right]$$

$$+ \sum_{n=1}^{N} \sum_{k=1}^{K} \frac{1}{2} \left[ A_{0nk}(t) p_{0nk}(s, z) + c.c. \right]$$

$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{k=1}^{2K} \frac{1}{2} \left[ A_{mnk}(t) p_{mnk}(s, z, \phi) + c.c. \right],$$

(4.22)

where $c.c.$ denotes the complex conjugate of the preceding term, $(\tilde{u}, \tilde{p})$ represents the viscous boundary layer, $(u_{00k}, p_{00k})$ and $(u_{m0k}, p_{m0k})$ represent the axisymmetric and non-axisymmetric geostrophic components respectively, $(u_{0nk}, p_{0nk})$ is the axisymmetric oscillatory inertial mode and $(u_{mnk}, p_{mnk})$ with $m \geq 1$ and $n \geq 1$ represents the non-axisymmetric inertial wave mode. As the truncation parameters $M, N$ and $K$ become larger, the partial sum would get nearer and nearer to any precessing flow $u$. The inertial-mode spectral analysis is pivotal in unveiling the structure and transition of a nonlinear precessing flow that is dynamically controlled by the effect of rotation.

In the inertial-mode spectral analysis, it is mathematically convenient to employ normalized inertial modes instead of the un-normalized modes previously discussed. This is because the size of the coefficients $A_{mnk}$ has no physical significance unless $u_{mnk}$ is properly normalized. The normalized axisymmetric geostrophic component is expressible as

$$u_{00k}(s) = \left[ \frac{1}{N_{00k}} \right] J_1 \left( \frac{\xi_{00k} s}{\Gamma} \right) \hat{\phi},$$

$$p_{00k}(s) = -\left[ \frac{2 \Gamma}{N_{00k} \xi_{00k}} \right] J_0 \left( \frac{\xi_{00k} s}{\Gamma} \right),$$

where $u_{00k}$ and $p_{00k}$ are real,

$$N_{00k} = \sqrt{\pi} \Gamma |J_0(\xi_{00k})|,$$
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and the values of $\xi_{00k}$ are solutions of the equation

$$J_1 (\xi_{00k}) = 0$$

with $0 < \xi_{001} < \xi_{002} < \ldots < \xi_{0nk} < \ldots$. For a given nonlinear precessing flow $\mathbf{u}$ from a numerical experiment, the coefficient $A_{00k}$ at any instant $t$, because of the orthogonal and normal (orthonormal) property, is given by

$$A_{00k}(t) = \int_0^1 \int_0^\Gamma \int_0^{2\pi} (\mathbf{u}_{00k} \cdot \mathbf{u}) \, s \, d\phi \, ds \, dz$$

where a small contribution of $O(Ek^{1/2})$ from the top and bottom viscous boundary layers is neglected. It follows that the axisymmetric geostrophic component of the flow $\mathbf{u}$ is expressible as

$$U_G(s) \hat{\phi} = \sum_{k=1}^K \left[ \int_0^1 \int_0^\Gamma \int_0^{2\pi} (\mathbf{u}_{00k} \cdot \mathbf{u}) \, s \, d\phi \, ds \, dz \right] \mathbf{u}_{00k}(s)$$

for a sufficiently large $K$.

The normalized non-axisymmetric geostrophic component is given by

$$\mathbf{\hat{s}} \cdot \mathbf{u}_{m0k} = -\frac{im}{2sN_{m0k}} J_m \left( \frac{\xi_{m0k}s}{\Gamma} \right) e^{im\phi},$$

$$\mathbf{\hat{\phi}} \cdot \mathbf{u}_{m0k} = \frac{1}{2N_{m0k}} \left[ \frac{\xi_{m0k}}{\Gamma} J_{m-1} \left( \frac{\xi_{m0k}s}{\Gamma} \right) - \frac{m}{s} J_m \left( \frac{\xi_{m0k}s}{\Gamma} \right) \right] e^{im\phi},$$

$$\mathbf{\hat{z}} \cdot \mathbf{u}_{m0k} = 0,$$

$$p_{m0k} = \frac{1}{N_{m0k}} J_m \left( \frac{\xi_{m0k}s}{\Gamma} \right) e^{im\phi},$$

where $m \geq 1$ and $\xi_{m0k}$ are solutions of the transcendental equation

$$J_m (\xi_{m0k}) = 0$$

which is arranged using the subscript notation

$$0 < \xi_{m01} < \xi_{m02} < \xi_{m03}, \ldots < \xi_{m0k} < \ldots,$$

with $\xi_{m0k}$ standing for the $k$th smallest positive root. The normalization factor $N_{0nk}$ is

$$N_{m0k} = \sqrt{\pi} \frac{\sqrt{2}}{2} \left\{ \xi_{m0k}^2 J_{m-1} (\xi_{m0k}) + \left[ \xi_{m0k}^2 - (m + 1)^2 \right] J_{m+1}^2 (\xi_{m0k}) \right. + \left[ (m + 1) J_{m+1} (\xi_{m0k}) - \xi_{m0k} J_{m+2} (\xi_{m0k}) \right]^2 \right\}^{1/2}.$$
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can be calculated from the integral

$$A_{m0k}(t) = 2 \int_0^1 \int_0^\Gamma \int_0^{2\pi} (u_{m0k}^* \cdot u) \, s \, d\phi \, ds \, dz$$

for $m \geq 1$ and $k \geq 1$, where a small contribution of $O(Ek^{1/2})$ from the thin viscous boundary layer is neglected.

The normalized axisymmetric oscillatory modes $(u_{0nk}, p_{0nk})$ are

\[
\hat{s} \cdot u_{0nk} = \left[ \frac{i \sigma_{0nk} \xi_{0nk}}{2N_{0nk} \pi (1 - \sigma_{0nk}^2)} \right] J_1 \left( \frac{\xi_{0nk} s}{\pi} \right) \cos n\pi z,
\]

\[
\hat{\phi} \cdot u_{0nk} = - \left[ \frac{\xi_{0nk}}{2N_{0nk} \pi (1 - \sigma_{0nk}^2)} \right] J_1 \left( \frac{\xi_{0nk} s}{\pi} \right) \cos (n\pi z),
\]

\[
\hat{z} \cdot u_{0nk} = - \left( \frac{\ln \pi}{2N_{0nk} \pi} \right) J_0 \left( \frac{\xi_{0nk} s}{\pi} \right) \sin (n\pi z),
\]

\[
p_{0nk} = \frac{1}{2N_{0nk}} J_0 \left( \frac{\xi_{0nk} s}{\pi} \right) \cos (n\pi z),
\]

where $u_{0nk}$ and $p_{0nk}$ are complex, $n \geq 1$ and $k \geq 1$, $\sigma_{0nk} = \pm [1 + (\xi_{0nk}/n\pi)^2]^{-1/2}$, $\xi_{0nk}$ is solutions of the transcendental equation

$$J_1 (\xi_{0nk}) = 0,$$

and the normalization factor $N_{0nk}$ is

$$N_{0nk} = \left[ \frac{\pi (n\pi \Gamma) \sqrt{\pi}}{2 |\sigma_{0nk}| \sqrt{1 - \sigma_{0nk}^2}} \right] |J_0(\xi_{0nk})|.$$

By virtue of the orthonormal property, we can obtain the coefficient $A_{0nk}$ by performing the integration

$$A_{0nk}(t) = 2 \int_0^1 \int_0^\Gamma \int_0^{2\pi} (u_{0nk}^* \cdot u) \, s \, d\phi \, ds \, dz$$

for $n \geq 1$ and $k \geq 1$, where a small contribution of $O(Ek^{1/2})$ from the thin viscous
boundary layer is again omitted. The normalized inertial wave modes \( u_{mnk} \) read

\[
\hat{s} \cdot u_{mnk} = \left[ \frac{-i \xi_{mnk}}{4\Gamma(1 - \sigma_{mnk}^2)} \right] \left[ (1 + \sigma_{mnk}) J_{m-1} \left( \frac{\xi_{mnk}s}{\Gamma} \right) \right] (1 - \sigma_{mnk}^2) J_m \left( \frac{\xi_{mnk}s}{\Gamma} \right) \frac{\cos (n\pi z)}{N_{mnk}} e^{im\phi},
\]

\[
\hat{\phi} \cdot u_{mnk} = \left[ \frac{\xi_{mnk}}{4\Gamma(1 - \sigma_{mnk}^2)} \right] \left[ (1 + \sigma_{mnk}) J_{m-1} \left( \frac{\xi_{mnk}s}{\Gamma} \right) \right] (1 + \sigma_{mnk}) J_{m+1} \left( \frac{\xi_{mnk}s}{\Gamma} \right) \frac{\cos (n\pi z)}{N_{mnk}} e^{im\phi},
\]

\[
\hat{z} \cdot u_{mnk} = \frac{-in\pi}{2\sigma_{mnk}} J_m \left( \frac{\xi_{mnk}s}{\Gamma} \right) \frac{\sin (n\pi z)}{N_{mnk}} e^{im\phi},
\]

\[
p_{mnk} = J_m \left( \frac{\xi_{mnk}s}{\Gamma} \right) \frac{\cos (n\pi z)}{N_{mnk}} e^{im\phi},
\]

for positive integers \( m \geq 1, n \geq 1 \) and \( k \geq 1 \). Here \( u_{mnk} \) and \( p_{mnk} \) are complex, \( \sigma_{mnk} = \pm [1 + (\frac{\xi_{mnk}}{n\pi})^2]^{-1/2} \), \( \xi_{mnk} \) are solutions of the transcendental equation

\[
\xi_{mnk} J_{m-1}(\xi_{mnk}) + m J_m(\xi_{mnk}) \left\{ \frac{\sigma_{mnk}}{\xi_{mnk}} \left[ 1 + \left( \frac{\xi_{mnk}}{\Gamma n\pi} \right)^2 \right]^{1/2} - 1 \right\} = 0 \quad (4.23)
\]

and the normalization factor \( N_{mnk} \) is

\[
N_{mnk} = \frac{\sqrt{\pi}}{2|\sigma_{mnk}|} \left[ \frac{(n\pi \Gamma)^2 + m(m - \sigma_{mnk})}{(1 - \sigma_{mnk}^2)} \right]^{1/2} |J_m(\xi_{mnk})|.
\]

Similarly, the coefficient \( A_{mnk} \) in the expansion (4.21) at any instant \( t \) is derivable from the flow \( u \),

\[
A_{mnk}(t) = 2 \int_0^1 \int_0^\Gamma \int_0^{2\pi} (u_{mnk}^* \cdot u) s \, d\phi \, ds \, dz,
\]

for \( n \geq 1, m \geq 1 \) and \( k \geq 1 \).

With all the coefficients \( A_{mnk} \), the total kinetic energy density \( E_{\text{kin}} \) of a precessing flow at any instant \( t \) can be expressed as

\[
E_{\text{kin}}(t) &= \frac{1}{2\pi} \int_0^1 \int_0^\Gamma \int_0^{2\pi} |u|^2 s \, d\phi \, ds \, dz
\]

\[
= \frac{1}{2\pi \Gamma^2} \left\{ \left[ \frac{K}{k=1} \sum |A_{00k}(t)|^2 + \frac{1}{2} \sum_{m=1}^M \sum_{k=1}^K |A_{m0k}(t)|^2 + \frac{1}{2} \sum_{n=1}^N \sum_{k=1}^K |A_{0nk}(t)|^2 \right] + O(\sqrt{E_k}) \right\},
\]

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4. Precessing circular cylinder

Figure 4.2: Four different kinetic energy densities are shown as a function of $Po$ for $\alpha_p = \pi/4$ and $Ek = 10^{-4}$ in the spherical-like cylinder $\Gamma = 0.502559$. The dashed line is computed from the analytical expression (4.20).

where we have used the orthonormal property

$$\int_0^1 \int_0^\Gamma \int_0^{2\pi} (u^*_{mnk} \cdot u_{mnk}) s \, d\phi \, ds \, dz = 1$$

for all possible $m, n$ and $k$.

A useful check can be provided by comparing $E_{\text{kin}}$ computed from the inertial-mode decomposition to that calculated directly from the finite element solution. It is found that the difference is small, arising primarily from the contribution of the viscous boundary layer $O(Ek^{1/2})$. The variation and size of $A_{mnk}(t)$ as a function of $m, n, k$ and $t$ in the spectrum offers valuable insight into the structure and possible instabilities of a nonlinear precessing flow.

### 4.5.2 Structure of nonlinear precessing flow

To facilitate the characterization of a nonlinear precessing flow, we introduce, in addition to the total kinetic energy density $E_{\text{kin}}$, the kinetic energy density of the axisymmetric geostrophic component defined as

$$E_{\text{geo}} = \frac{1}{2V} \int_0^1 \int_0^\Gamma \int_0^{2\pi} |U_G \hat{\phi}|^2 s \, d\phi \, ds \, dz = \frac{1}{2\pi \Gamma^2} \sum_{k=1}^K |A_{00k}(t)|^2,$$

and the kinetic energy density of the forced inertial mode

$$E_{\text{for}} = \frac{1}{2V} \int_0^1 \int_0^\Gamma \int_0^{2\pi} \frac{1}{2} |A_{111} u_{111}|^2 s \, d\phi \, ds \, dz = \frac{1}{2\pi \Gamma^2} \left( \frac{|A_{111}(t)|^2}{2} \right).$$
4. Precessing circular cylinder

Figure 4.3: (a) The total kinetic energy density, \( E_{\text{kin}} \), of the nonlinear precessing flow as a function of time for \( \alpha_p = \pi/4 \) and at \( Ek = 10^{-4} \) in the spherical-like cylinder \( \Gamma = 0.502559 \): (a) for \( Po = 0.075 \), (b) for \( Po = 0.25 \) and (c) for \( Po = 0.5 \).

where \( u_{111} \) denotes the normalized inertial mode. It should be noted that we are only interested in the fluid motion in the nonlinear equilibrium controlled by the effect of the viscous boundary layer after sufficiently long simulation with \( t \geq O(1/\sqrt{Ek}) \) at \( \sqrt{Ek} \ll 1 \). For example, our experience indicates that the required length of a nonlinear simulation at \( Ek = 10^{-4} \) with \( 1/\sqrt{Ek} = 100 \) would be typically \( t \geq 700 \), i.e., more than a hundred of rotation periods.

Nonlinear results obtained from direct numerical simulation are summarized in Figure 4.2, Figure 4.3 and Figure 4.4 for \( Ek = 10^{-4} \) and \( \alpha_p = \pi/4 \), along with the result of the linear asymptotic solution for comparison. Several interesting features emerge from the fully nonlinear simulation. First, the linear approximation, given by the analytical expression in closed form (4.19) and (4.20), appears to be adequate when \( Po/\sqrt{Ek} < O(1) \). This is clearly suggested by Figure 4.2, showing a good agreement between the asymptotic solution \( E_{\text{lin}} \) (the dashed line given by (4.20)) and the total kinetic energy \( E_{\text{kin}} \) when \( 0 < Po \leq 0.01 \) at \( Ek = 10^{-4} \). There exist no noticeable differences between the precessing flows obtained from the linear and weakly nonlinear solutions when \( Po/\sqrt{Ek} < O(1) \). Contours of \( u_z \) of the precessing flow from the nonlinear simulation even for \( Po = 0.025 \) at \( Ek = 10^{-4} \), which is displayed in Figures 4.5(a,b), largely resemble the linear asymptotic solution (4.19). Second, the essential characters of nonlinear precessing flow – the total kinetic energy density \( E_{\text{kin}} \), the structure of the flow, the amplitude of the forced mode \( u_{111} \) and the amplitude/profile of the geostrophic flow \( U_G(s) \) – change only slightly as a function of time after our numerical simulation reaches the nonlinear equilibrium with \( t \geq O(1/\sqrt{Ek}) \). Apart from the expected retrograde propagation of non-axisymmetric components, the spatial structure of even strongly precessing flow
4. Precessing circular cylinder

Figure 4.4: (a) The scaled geostrophic flow $U_G(s)$ as a function of $s$ for different values of $Po$ for $\alpha_p = \pi/4$ and $Ek = 10^{-4}$ at $\Gamma = 0.502559$. (b) The geostrophic flow $U_G(s)$ as a function of $s$ at $Po = 0.25$ and $Po = 0.5$ for $\alpha_p = \pi/4$ and $Ek = 10^{-4}$ at $\Gamma = 0.502559$. 
4. Precessing circular cylinder

shows little variation at different instants, which are illustrated in Figures 4.5(c,d) for \( Po = 0.075 \) and \( Ek = 10^{-4} \) at two different instants. This quasi-steady feature is also reflected in Figures 4.3 where the total kinetic energy varies irregularly but only within a few percent even for \( Po/\sqrt{Ek} = 10 \). Third, the disordered flow takes place in the spherical-like cylinder with \( \Gamma = 0.502559 \) when \( Po/\sqrt{Ek} > O(1) \), which is illustrated in Figure 4.5(c-f) at \( Ek = 10^{-4} \). The most remarkable feature is the formation and growth of rigid-body rotation in the central region. When \( Po \) is moderate, as depicted in Figure 4.4(a), the geostrophic flow \( U_G(s) \) arises in the central region in the form of nearly rigid-body rotation. At \( Po = 10^{-2} \), for instance, the rigid-body rotation is confined in the central region \( 0 \leq s < s_{rigid} \approx 0.13 \). As \( Po \) gradually increases, however, the central rigid-body rotation grows in both size and strength which is clearly illustrated in Figure 4.4(a,b). Irregular pattern of the non-axisymmetric flow in Figures 4.5(c,d) for \( Po = 0.075 \) highlights the resulting distortion caused by strong geostrophic flow \( U_G(s) \). When \( Po = 0.5 \), the non-axisymmetric component is largely compelled to be sidewall-localized by the dominant central rigid-body rotation depicted in Figure 4.5(e,f). It follows that the disordered strongly precessing flow can be decomposed into the three major components: a weak non-axisymmetric component in the form of retrogradely propagating wave, displayed in Figure 4.5(e,f), that is largely sidewall-localized; an axisymmetric wall-localized shear in the vicinity of the sidewall shown in Figure 4.4(b); and a predominant rigid-body rotation in the interior which has the opposite direction of the basic rotation \( \Omega_0 \) and is approximately given by the formula

\[
U_G(s) \hat{\phi} = \Omega_G \hat{z} \times \hat{r} \quad \text{in} \quad 0 < s \leq s_{rigid} \approx (\Gamma - \delta) = 0.43
\]

with \( \Omega_G \approx -0.8 \) and the sidewall-localized shear in the Stewartson-type layer (for example, Hollerbach, 2003) of the thickness \( \delta \approx 0.07 \).

4.5.3 The effect of precession angle \( \alpha_p \)

In experimental studies, it is perhaps mechanically convenient to fix the precession angle \( \alpha_p = 90^\circ \) (for example, Mouhali, 2012). As indicated by (4.1)-(4.2), larger precession angles generally exert stronger precessional forcing on the fluid system. In order to understand the effect of varying precession angle on the flow, we have also computed several nonlinear solutions in in the precessing spherical-like cylinder with \( \alpha_p = 90^\circ \).

The results of nonlinear simulation in the spherical-like cylinder with \( \alpha_p = 90^\circ \) are summarized in Figure 4.6, Figure 4.7 and Figure 4.8 at \( Ek = 10^{-4} \). Despite its irregular variations with time, as shown in Figure 4.6 for \( Po = 0.075 \) and \( Po = 0.25 \), both the amplitude and the profile of the strongly precessing flow change only slightly as a function of time. This quasi-steady feature is also reflected in Figures 4.8
Figure 4.5: Contours of $u_z$ of a precessing flow in the mantle frame at the $z = 1/2$ plane for $\alpha_p = \pi/4$ and $Ek = 10^{-4}$ in the spherical-like cylinder $\Gamma = 0.502559$: (a-b) for $Po = 0.025$ at two different instants; (c-d) for $Po = 0.075$ at two different instants; and (e-f) for $Po = 0.5$ at two different instants.
4. Precessing circular cylinder

Figure 4.6: (a) The total kinetic energy density, $E_{\text{kin}}$, of the nonlinear precessing flow as a function of time for $\alpha_p = \pi/2$ and at $Ek = 10^{-4}$ in the spherical-like cylinder $\Gamma = 0.502559$: (a) for $Po = 0.075$, (b) for $Po = 0.25$.

Figure 4.7: The geostrophic flow $U_G(s)$ as a function of $s$ at $Po = 0.25$ and $Po = 0.5$ for $\alpha_p = \pi/2$ and $Ek = 10^{-4}$ at $\Gamma = 0.502559$.

<table>
<thead>
<tr>
<th>$(m, n, k)$</th>
<th>$A_{mnk}$</th>
<th>$\xi_{mnk}$</th>
<th>$\sigma_{mnk}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1, 1)</td>
<td>$1.6230 \times 10^{-1}$</td>
<td>2.7346</td>
<td>0.5000</td>
</tr>
<tr>
<td>(0, 0, 1)</td>
<td>$9.8678 \times 10^{-2}$</td>
<td>3.8317</td>
<td>0.0000</td>
</tr>
<tr>
<td>(1, 1, 2)</td>
<td>$5.0699 \times 10^{-2}$</td>
<td>5.9578</td>
<td>0.2562</td>
</tr>
<tr>
<td>(1, 1, 1)$^-$</td>
<td>$3.7665 \times 10^{-2}$</td>
<td>4.7528</td>
<td>-0.3153</td>
</tr>
<tr>
<td>(0, 0, 2)</td>
<td>$3.2185 \times 10^{-2}$</td>
<td>7.0156</td>
<td>0.0000</td>
</tr>
<tr>
<td>(1, 1, 2)$^-$</td>
<td>$1.9910 \times 10^{-2}$</td>
<td>7.9731</td>
<td>-0.1942</td>
</tr>
<tr>
<td>(1, 1, 3)</td>
<td>$1.8176 \times 10^{-2}$</td>
<td>9.1316</td>
<td>0.1704</td>
</tr>
<tr>
<td>(1, 3, 1)</td>
<td>$1.8080 \times 10^{-2}$</td>
<td>2.4557</td>
<td>0.8878</td>
</tr>
<tr>
<td>(0, 0, 3)</td>
<td>$1.6921 \times 10^{-2}$</td>
<td>10.1734</td>
<td>0.0000</td>
</tr>
<tr>
<td>(2, 2, 1)</td>
<td>$1.6314 \times 10^{-2}$</td>
<td>4.1050</td>
<td>0.6097</td>
</tr>
</tbody>
</table>

Table 4.1: Largest ten coefficients $A_{mnk}$ of the nonlinear resonant precessing flow in the spherical-like cylinder with $\alpha_p = \pi/2$, $\Gamma = 0.502559$ and $Po = 0.075$ at $Ek = 10^{-4}$.
Figure 4.8: Contours of $u_z$ of a precessing flow in the mantle frame at the $z = 1/2$ plane for $\alpha_p = \pi/2$ and $Ek = 10^{-4}$ in the spherical-like cylinder $\Gamma = 0.502559$: (a-b) for $Po = 0.025$ at two different instants; (c-d) for $Po = 0.075$ at two different instants.

In which the flow profiles at two different instants are presented. Similar to the solutions obtained at $\alpha_p = 45^\circ$, the geostrophic flow $U_G(s)$ arises in the central region in the form of nearly rigid-body rotation whose size and amplitude grows as $Po$ gradually increases, which is illustrated in Figure 4.8. When $Po = 0.25$, the non-axisymmetric component is largely compelled to be sidewall-localized by the dominant central rigid-body rotation depicted in Figure 4.7. The strongly disordered flow at $Po = 0.25$ consists of a wall-localized traveling wave, an axisymmetric wall-localized shear and a predominant rigid-body rotation approximately given by the formula

$$U_G(s)\hat{\phi} = \Omega_G \hat{z} \times \hat{r} \text{ in } 0 < s \leq s_{\text{rigid}} \approx (\Gamma - \delta) = 0.43$$

with $\Omega_G \approx -1.0$ and $\delta \approx 0.07$. In short, the precession angle $\alpha_p = 90^\circ$ does not seem to play a key role in the dynamics of precessing flow. Table 4.1 shows largest ten coefficients $|A_{mnk}|$ of the nonlinear precessing flow for $Po = 0.075$ and $\alpha_p = \pi/2$ at $Ek = 10^{-4}$.

It is noteworthy that our numerical solutions in the nonlinear equilibrium are stable without showing any indication of instabilities. To fully capture the effect of the viscous boundary layer, we have run our simulation sufficiently long with $t \geq O(1/\sqrt{Ek})$ at $Ek = 10^{-4}$. For example, Figure 4.6 shows that the length of
our numerical simulation is about \( t = 1500 \) which is much larger than \( O(1/\sqrt{Ek}) = O(100) \), representing more than 200 rotation periods.

### 4.6 Summary and some remarks

We have studied, through both asymptotic analysis and direct numerical simulation, precessionally driven flow confined in fluid-filled circular cylinders. Our emphasis is placed on the spherical-like cylinder with the radius-height ratio \( \Gamma = 0.502559 \) at which the strongest resonance can take place. We have derived an asymptotic analytical solution in closed form in the mantle frame of reference for describing weakly precessing flow in the spherical-like cylinder at asymptotically small Ekman numbers. A three-dimensional finite element code is developed to elucidate the structure of the nonlinear flow. Moreover, the complete spectrum of an inertial-mode decomposition is employed. It is slightly surprising that the structure of very strongly nonlinear flow, even with \( Po/\sqrt{Ek} = O(10) \), in the spherical-like precessing cylinder is temporally quasi-steady and spatially quite simple, consisting of a sidewall-localized non-axisymmetric traveling wave and a wall-localized axisymmetric shear along with a dominant interior rigid-body rotation.

Our current analysis has concentrated on the spherical-like cylinder with \( \Gamma = 0.502559 \) in which the precessional forcing resonates directly with the spatially simplest inertial mode. It would be, of course, interesting to find out whether the unusual nonlinear behavior like the growing rigid-body rotation and the wall-localized mode also occur in other precessing cylinders such as \( \Gamma = 1.045945 \) at which the precessional forcing resonates directly with the second inertial mode \( \mathbf{u}_{112} \). It is also interesting to investigate other aspect ratios at which there are no direct resonance between the precessional forcing and spatially simple inertial modes. The finite element nonlinear computations in larger aspect ratios are numerically much more expensive, which will be carried out in the future.
Chapter 5

Conclusions

In the study of bacteria swimming, we made an emphasis on the influence of the shape of the bacterium on its swimming motion. It turned out that the trajectory of the bacterium is highly sensitive to its eccentricity. Future work could be done to study more details of the impact of the shape on the dynamics. A basic calculation shows that the current model does not work very well when the eccentricity of the bacterium is very high. This means that there are fundamental differences in swimming mechanism for bacteria of different shapes.

Swimming microorganisms exist everywhere around our daily life, not only magnetotactic bacteria. The subject of microorganism swimming has attracted lots of study (Lauga, 2009). Usually, microorganisms can detect gradients of nutrients and oxygen concentration and swim to regions where it is higher. The physics at the micron scale is different from that at macroscopic scale. The former is a world of low Reynolds number, where inertia is not important, and viscosity plays a dominant role. Microorganisms usually swim with the driving force of flagellums. Nowadays atomic force microscopy allows direct measurement of the force exerted by cilia (Teff et al, 2007), thus detailed study is possible. Microorganisms swimming in viscous fluids could form a dense population, where a single organism may cause a flow which will affect cells nearby, and possibly affect the dynamics of the entire population. For studies of interaction see Dombrowski et al (2004); Cisneros et al (2007); Wolgemuth (2008); Ishikawa et al (2006). If there is a boundary in the fluid, then the motion of organisms will also be changed. The velocity may be lower because of increasing viscous drag near the boundary (Katz, 1974). Also the trajectories of the organisms may also change near the boundary, for example, E.coli (a kind of bacteria with helical flagella) change their trajectory from straight to circular near a surface. Usually, a bacteria have many flagella, not only one. Interactions between flagella also play a key role for bacteria motion. In this subject an interesting phenomenon is flagellar bundling (Macnab, 1977).

There is an open problem whether the magnetic behavior of magnetotactic bacteria is passive or active. By the word “passive” we mean that, the bacteria itself does
Conclusions

not “know” about the external magnetic field, so that the magnetotactical behavior of these bacteria is just a consequence of the equipped magnetic moment. By the word “active”, in contrast, we mean that the bacteria can detect the external magnetic field, and then change their biological-physiological process, such that their motion become magnetotactic. Our study support the passive assumption. Because the magnetic moment is assumed to be only equipped to the bacterium body, and all magnetic behaviors of the bacterium are caused by this moment. But taking into consideration the accuracy of the experiments, this problem still needs more study.

In chapter 4 we find that inertial modes are important in asymptotic analysis of rotating flows. Some other study has been carried out for similar or different problems (for example Zhang, 2010; Liao, 2010). The validation of this method depends on the completeness of the inertial modes as an orthogonal basis. Most known examples are involved with self-adjoint compact operator. While in our case the operator is not self-adjoint. This is why the problem is unusual and difficult. Our proof of the completeness of inertial modes in chapter 3 is only concerned with the most basic geometry, that is an annular channel. The completeness in other geometries still need study. For the channel geometry, the completeness of inertial waves has a basic connection with that of trigonometric functions. But in other geometry, such as a cylinder, the functions are related to Bessel functions, whose completeness itself is a difficult problem. For spherical case it is more complicated.

The study of precessional fluids confined in a cylinder container has a background from astrophysics. For example our Earth is precessing slowly under the torque imposed by the Sun and the Moon. The inner fluid core of the Earth could be modeled as fluids confined in a narrow annular channel (Zhang, 2010) or a cylinder container rotating and precessing. These motion in fluids could be the cause of the magnetic fields through the dynamo mechanism. In our study, we have derived an asymptotic analytical solution in closed form in the mantle frame of reference to describe weakly precessing flow in the spherical-like cylinder at asymptotically small Ekman numbers. In the mean time a three-dimensional finite element code is developed to elucidate the structure of the nonlinear flow. Future work could be done for other aspect ratio $\Gamma$, say $\Gamma = 1.045945$. Numerical simulation for larger $\Gamma$ is more expensive. In our study we find an interesting phenomenon. When the Poincaré number $Po$ is large, the rotating flow transfers into a quite irregular style. The precise mechanism of this transition from the laminar to disordered flow is not fully understood. There exist several possible scenarios that have been proposed to explain the breakdown of the precessionally forced inertial mode at resonance. A popular scenario is that the breakdown of the laminar precessing flow at resonance is caused by the mechanism of triadic resonance involving the nonlinear interaction of three inertial modes in a rotating cylinder (see, for example, Kobine, 1996; Lagrange et al, 2008). However, we have failed to see any evidence of triadic resonance in our
Conclusions

Finally, the aspect ratio $\Gamma$ needs more discussions. Given the governing equations under Poincaré forcing

$$\frac{\partial \mathbf{u}}{\partial t} + 2\hat{z} \times \mathbf{u} + \nabla p = Ek \nabla^2 \mathbf{u} - 2\hat{z} s Po \sin \alpha p e^{i(\phi + t)}, \quad (5.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (5.2)$$

we can derive that, in order for the solution to have resonance with some special inertial modes, these modes should contain some characters of the external forcing. That is, we require the following integration not equal zero:

$$\int_0^\Gamma \int_0^1 \int_0^{2\pi} \mathbf{u}_{\text{res}}^* \cdot [2\hat{z} s Po \sin \alpha p e^{i\phi}] s d\phi ds dz \neq 0. \quad (5.3)$$

As a result, we must have $m = 1, n = 1, 3, 5, 7, \ldots$. And the frequency $\sigma = \frac{1}{2}$, which leads to the following two equations, called Kelvin dispersion relation:

$$0 = \xi_{1n} J_0(\xi_{1n}) + J_1(\xi_{1n}), \quad (5.4)$$

$$\Gamma_{\text{resonant}} = \Gamma_{nk} = \frac{\xi_{1n}}{\sqrt{3n\pi}}. \quad (5.5)$$

Here the radial wavenumber index $k$ is labeled in an ascending order

$$0 < \xi_{1n1} < \xi_{1n2} < \xi_{1n3} \cdots < \xi_{1nk} \cdots \quad (5.6)$$

Now from (5.5) we know that, possibly for any aspect ratio $\Gamma$ there exists $n$ and $k$ that get arbitrarily close to resonance. That is, for any $\Gamma$ of the cylinder, there would be certain inertial modes which would almost be triggered.
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I should thank a lot of Dr. Dali Kong from Institute of Mathematics, University of Exeter. When I met with problems in coding, I always go to him for help. Also in daily life he helped me to solve many difficult issues.

Finally, thank my parents for the care and support.
Appendix A

The subroutine “DragTorqCalc”

function [F,N,U,p]=DragTorqCalc(V,W,d,r,A,a)

(1) DragTorqCalc computes the interaction between the moving prolate spheroid and the fluid into which it is moving.

(2) Input Arguments:

V(3): Cartesian components of translation velocity of the spheroid in the laboratory frame

W(3): Cartesian components of rigid body rotation angular velocity of the spheroid in the laboratory frame

d(3): Orientation of the spheroid defined by the components of the unit vector, aligned in the direction of its axisymmetric axis, in the laboratory frame

r(3): The coordinates of a spatial point located outside the spheroid, Components w.r.t a body fixed inertial frame (denoted by C)

A : The semimajor axis of the spheroid

a : The semiminor axis of the spheroid

(3) Output Arguments:

F(3): Total force imposed by the fluid on the spheroid, w.r.t. the laboratory frame

N(3): Total couple imposed by the fluid on the spheroid, w.r.t. the laboratory frame

U(3): Flow velocity at the exterior point specified by
Appendix

```matlab
if (size(V,1)*size(V,2)~=3) %< Dimension Check for V >
    error('Main Input: V should be a vector of dimension 3!!!
end
if (size(W,1)*size(W,2)~=3) %< Dimension Check for W >
    error('Main Input: W should be a vector of dimension 3!!!
end
if (size(d,1)*size(d,2)~=3) %< Dimension Check for d >
    error('Main Input: d should be a vector of dimension 3!!!
end
if (size(r,1)*size(r,2)~=3) %< Dimension Check for r >
```
Appendix

```matlab
error('Main Input: r should be a vector of dimension 3 !!!'); end
if (A<=0 || a<=0) %< Positive Semimajor and Semiminor Axes >
    error('Main Input: Semimajor and Semiminor Axes should be of positive length !!!'); end
if (A<=a) %< a Right Comparison between Semimajor and Semiminor Axes >
    error('Main Input: The semimajor must be longer than the semiminor axis !!!'); end
if (abs(norm(d)-1)>1e-4) %< Confirm d is a unit vector >
    error('Main Input: Vector d should be normalized !!!'); end

VTransL=[V(1) V(2) V(3)];
OmegaL=[W(1) W(2) W(3)];
OrientL=[d(1) d(2) d(3)];
LocExtC=[r(1) r(2) r(3)];
%< All input vector converted to line vector >
SemiMajr=A;
SemiMinr=a;
%------------------- Main Computation Starts -------------------
c=sqrt(SemiMajr^2-SemiMinr^2); %< Focal Length of the shape >
e=c/SemiMajr; %< Eccentricity >
ForceL=GetDrag(VTransL, OrientL, e, c);
TorqL=GetTorq(OmegaL, OrientL, e, c);
UExtC=GetExtFlow(LocExtC);
%< Important Release Note: Not implemented in >
%< this version >
ShrTnsrC=GetExtStress(LocExtC);
%< Important Release Note: Not implemented in this version >
%-------------------
F=ForceL;N=TorqL;U=UExtC;p=ShrTnsrC;
return;
end

function ExtFlow=GetExtFlow(LocExtC)
```

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% Dummy function
ExtFlow=zeros(size(LocExtC,1),size(LocExtC,2));
end

function ExtStress=GetExtStress(LocExtC)
% Dummy function
ExtStress=zeros(size(LocExtC,1),size(LocExtC,2));
end

function Torq=GetTorq(OmegaL, OrientL, e, c)
% GetTorq computes the components of torque in
% the laboratory frame
% Input:
% OmegaL :
% Angular velocity components in the laboratory frame
% OrientL: The unit vector of the orientation of
%  the semimajor axis in the laboratory frame
% e: Shape eccentricity of the spheroid
% c: Shape focal length
% Output:
% Torq: Total torque imposed on the spheroid arising from the
%  rotation
% Licensing:
% + This code is distributed under the GNU GPL license.
% + Last modified on 26 July 2011
% + Author: Dali Kong
% + email: kongdljnju@gmail.com

if(size(OmegaL,1)*size(OmegaL,2)^2=3)
  error('GetTorq:Input:Angular velocity should be
defined by a 3-D vector!!!')
end
if(size(OrientL,1)*size(OrientL,2)^2=3)
  error('GetTorq:Input:Orientation should be defined
by a 3-D vector!!!')
end
if(abs(norm(OrientL)-1)>1e-4)
  error('GetTorq:Input:Orientation vector must be
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Appendix

if (e >= 1 || e <= 0) %< Valid range of eccentricity >
error ('GetTorq. Input: Invalid value of eccentricity !!! ')
end
if (c <= 0) %< Valid range of focal length >
error ('GetTorq. Input: Invalid value of focal length !!! ')
end
if (norm(OmegaL) < 1e-8)
Torq = zeros(1,3);
return;
%< If angular speed is null, torque vanishes. >
end
xio = 1/e;
Omega0 = norm(OmegaL); %< Angular speed >
[theta, phi, psi] = GetEulerAng(OrientL);
OmegaC = Lab2Bdy(OmegaL, theta, phi, psi);
if (abs(OmegaC) > 1e-4)
    disp(['Angular speed computed by L-frame Omega: ',
        num2str(Omega0)])
    disp(['Angular speed computed by C-frame Omega: ',
        num2str(norm(OmegaC))])
    error('GetTorq. Run: Ambiguous angular speed !!! ')
end
OmegaCDirect = OmegaC/Omega0;
alpha = acos(OmegaCDirect(3));
if (norm(OmegaCDirect(1:2)) < 1e-10)
    beta = 0;
else
    sinbeta = OmegaCDirect(2)/norm(OmegaCDirect(1:2));
    cosbeta = OmegaCDirect(1)/norm(OmegaCDirect(1:2));
    if (sinbeta >= 0 && cosbeta >= 0)
        beta = asin(sinbeta);
    elseif (sinbeta >= 0 && cosbeta <= 0)
        beta = pi-asin(sinbeta);
    elseif (sinbeta <= 0 && cosbeta <= 0)
        beta = pi-asin(sinbeta);
    elseif (sinbeta <= 0 && cosbeta > 0)
        beta = 2*pi+asin(sinbeta);
    else
        %< Handling other cases >
    end
end
Appended

```matlab
error('GetTorq_Run: Strange Exceptional of beta angle !!!')
end
TorqC_sigma=zeros(size(OmegaL,1),size(OmegaL,2));
TorqC_p=zeros(size(OmegaL,1),size(OmegaL,2));
TorqC_sigma(1)=Omega0*c^3*sin(alpha)*cos(beta)*(-8*xio^2)... 
/(2*xio+(xio^2+1)*log((xio+1)/(xio-1)))*... 
(2*pi*(xio^2-1)*atanh(1/xio)/xio+pi/3);
TorqC_sigma(2)=Omega0*c^3*sin(alpha)*sin(beta)*(-8*xio^2)... 
/(2*xio+(xio^2+1)*log((xio+1)/(xio-1)))*... 
(2*pi*(xio^2-1)*atanh(1/xio)/xio+pi/3);
TorqC_sigma(3)=Omega0*c^3*cos(alpha)*8*pi/3*(4*(xio^2-1))... 
/(2*xio+(xio^2-1)*log((xio+1)/(xio-1)));
TorqC_p(1)=-2*pi/3*Omega0*c^3*sin(alpha)*cos(beta)*(xio^2-1)... 
*(-4+6*xio^2-3*xio*(xio^2-1)*log((xio+1)/(xio-1)))*... 
/(-xio/2+(xio^2+1)/4*log((xio+1)/(xio-1)));
TorqC_p(2)=-2*pi/3*Omega0*c^3*sin(alpha)*sin(beta)*(xio^2-1)... 
*(-4+6*xio^2-3*xio*(xio^2-1)*log((xio+1)/(xio-1)))*... 
/(-xio/2+(xio^2+1)/4*log((xio+1)/(xio-1)));
TorqC_p(3)=0;
Torq=TorqC_sigma+TorqC_p;
Torq=Bdy2Lab(TorqC,theta,phi,psi);
end

function Drag=GetDrag(VTransL,OrientL,e,c)
% GetDrag computes the components of drag force
% in the laboratory frame
% Input:
% VTransL: Translation motion velocity in the laboratory frame
% OrientL: The unit vector of the orientation of the
% semimajor axis in the laboratory frame
% e: Shape eccentricity of the spheroid
% c: Shape focal length
% Output:
% Drag: Total drag force arising from the translation motion
% of the spheroid
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% + Last modified on 25 July 2011
% + Author: Dali Kong
```

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if(size(VTransL,1)*size(VTransL,2)~=3)
  error('GetDrag Input: Velocity should be defined by a 3-D vector!!! ')
end
if(size(OrientL,1)*size(OrientL,2)~=3)
  error('GetDrag Input: Orientation should be defined by a 3-D vector!!! ')
end
if(abs(norm(OrientL)-1)>1e-4)
  error('GetDrag Input: Orientation vector must be normalized!!! ')
end
if(e>=1 || e<=0) %< Valid range of eccentricity >
  error('GetDrag Input: Invalid value of eccentricity !!! ')
end
if(c<=0) %< Valid range of focal length >
  error('GetDrag Input: Invalid value of focal length !!! ')
end
if(norm(VTransL)<1e-8)
  Drag=zeros(1,3);
  return;%< If the velocity is zero, so is the drag force. >
end
xio=1/e;
gamma1=acos(dot(VTransL,OrientL)/norm(VTransL));
[theta,phi,psi]=GetEulerAng(OrientL);
VTransC=Lab2Bdy(VTransL,theta,phi,psi);
gamma2=acos(VTransC(3)/norm(VTransC));
if(abs(gamma1-gamma2)>1e-4)
  disp(['gamma computed from L-frame vectors: ' num2str(gamma1)]);
  disp(['gamma computed from C-frame vectors: ' num2str(gamma2)]);
  error('GetDrag Run: Ambiguous AOA!!! ')
end
gamma=gamma2;
VDirectC=VTransC/norm(VTransC);
if(gamma<1e-8)
  shift=0;
else

    sin shift = VDirectC(2)/norm(VDirectC(1:2));
    cos shift = VDirectC(1)/norm(VDirectC(1:2));

    if (sin shift >= 0 & & cos shift >= 0)
        shift = asin(sin shift);
    elseif (sin shift >= 0 & & cos shift <= 0)
        shift = pi - asin(sin shift);
    elseif (sin shift <= 0 & & cos shift <= 0)
        shift = pi - asin(sin shift);
    elseif (sin shift <= 0 & & cos shift > 0)
        shift = 2*pi + asin(sin shift);
    else
        disp(['gamma=', num2str(gamma)]);
        disp(['VDirectC=', num2str(VDirectC)]);
        disp(['VTransL=', num2str(VTransL)]);
        disp(['VTransC=', num2str(VTransC)]);
        error('GetDragRun: Strange Exceptional of shift angle !!! ')
    end
end
U0 = norm(VTransC);

Drag_temp(1) = ...

-4*pi*c*U0*(-4+2*(xio^2-1)*(-2+xio*log((xio+1)/(xio-1))))... /
(2*xio - (xio^2-3)*log((xio+1)/(xio-1)))*sin(gamma)...
-2*pi*c*U0*(xio^2*(2-(xio^2-1))/xio*log((xio+1)/(xio-1))))... /
(xio/2-(xio^2-3)/4*log((xio+1)/(xio-1)))*sin(gamma);

Drag_temp(2) = 0;

Drag_temp(3) = ...

-4*pi*c*U0*(2*xio^2*(xio^2-1)*(2-(xio^2-1))/xio*log((xio+1)... /
(xio-1))))/(2*xio - 2*xio^3+(xio^4-1)*log((xio+1)/(xio-1)))... *
cos(gamma)-2*pi*c*U0*((xio^2-1)*(2-xio*log((xio+1)/(xio-1))))... /
(xio/2-(xio^2+1)/4*log((xio+1)/(xio-1)))*cos(gamma);

DragC = Rz(-shift, Drag_temp);

% Drag_temp is the drag force in the frame >
% in which the velocity does not have y >
% component. DragC then consists of components >
% in the C frame. >

Drag = Bdy2Lab(DragC, theta, phi, psi);

% Drag is now finally converted to >
% the laboratory frame. >
end
function xC=Lab2Bdy(xL,theta,phi,psi)
% Lab2Bdy implements the coordinates transformation
% from the laboratory frame to the body fixed frame.
% Input:
% xL: The vector coordinates in the laboratory frame
% theta: Euler angle (Nutation)
% phi: Euler angle (Precession)
% psi: Euler angle (Spin)
% Output:
% xC: The vector coordinates in the body fixed frame

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% + Author: Dali Kong
% + email: kongdlnju@gmail.com
if(size(xL,1)*)size(xL,2)^=3) %< Double check the dimension >
    error('Bdy2Lab: Input: 3-D vector required !!!')
end
if(theta<0 || theta>pi) %< Nutation range check >
    error('Bdy2Lab: Input: Nutation angle should be within 0 to pi !!!')
end
xL_temp=xL;
[xL_temp,Rz(phi,xL_temp)];
[xL_temp,Ry(theta,xL_temp)];
[xL_temp,Rz(psi,xL_temp)];
xC=xL_temp;
end

function xL=Bdy2Lab(xC,theta,phi,psi)
% Bdy2Lab implements the coordinates transformation
% from the body fixed frame to the laboratory frame.
% Input:
% xC: The vector coordinates in the body frame
% theta: Euler angle (Nutation)
% phi: Euler angle (Precession)
% psi: Euler angle (Spin)
% Output:
% xL: The vector coordinates in the laboratory frame
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if (size(xC,1)*size(xC,2)~=3) %< Double check the dimension >
error( 'Bdy2Lab:Input:3-D vector required !!! ' )
end
if (theta<0 || theta>pi) %< Nutation range check >
error( 'Bdy2Lab:Input:Nutation angle should be within 0 to pi !!! ' )
end
xC_temp=xC;
xC_temp=Rz(-psi , xC_temp);
xC_temp=Ry(-theta , xC_temp);
xC_temp=Rz(-phi , xC_temp);
xL=xC_temp;
end

function [theta , phi , psi]=GetEulerAng ( Orient )
% GetEulerAng obtains the three Euler angle out of the
given orientation of the spheroid in the laboratory frame.
% Input:
% Orient: Unit vector defining the orientation of the
% semimajor axis in the laboratory frame.
% Output:
% theta: Nutation angle (uniquely determined)
% phi: Precession angle (uniquely determined)
% psi: Spin angle (set to zero because of the
% omni-directional freedom)
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if (size(Orient ,1)*size(Orient ,2)~=3) %< Double check the dimension >
error( 'GetEulerAng:Input:Orientation should be defined by
3-D vector !!! ' )
end
if (abs(norm(Orient)-1)>1e-4) %< Double check the normalization >
error('GetEulerAng\nInput: Orientation vector must be normalized!!!')
end
theta=acos(Orient(3));
if(norm(Orient(1:2))<1e-10)
    phi=0;
else
    sinphi=Orient(2)/norm(Orient(1:2));
    cosphi=Orient(1)/norm(Orient(1:2));
    if(sinphi>=0 && cosphi>=0)
        phi=asin(sinphi);
    elseif(sinphi>=0 && cosphi<=0)
        phi=pi-asin(sinphi);
    elseif(sinphi<=0 && cosphi<=0)
        phi=pi-asin(sinphi);
    elseif(sinphi<=0 && cosphi>=0)
        phi=2*pi+asin(sinphi);
    else
        disp(['GetEulerAng\nDebug Info: theta=', num2str(theta)]);
        disp(['GetEulerAng\nDebug Info: sin(phi)=', num2str(sinphi)]);
        disp(['GetEulerAng\nDebug Info: cos(phi)=', num2str(cosphi)]);
        error('GetEulerAng\nRun: Strange Exceptional')
    end
end
psi=0;
%< The absolute value of the spin angle does not matter in the computation of torque. >
end

function x2=Ry(ang,x1)
% Ry does the coordinates transformation induced by reference frame rotating about the y-axis.
% Input:
% ang: rotation angle
% x1: The coordinates in the pre-rotation frame
% Output:
% x2: The coordinates in the post-rotation frame
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if (size(x1,1)*size(x1,2)==3) %< Double check the dimension >
    error( 'Rz Input: 3-D vector required !!!' )
end
x2=zeros(size(x1,1),size(x1,2));
x2(1)=cos(ang)*x1(1)-sin(ang)*x1(3);
x2(2)=x1(2);
x2(3)=sin(ang)*x1(1)+cos(ang)*x1(3);
end

function x2=Rz(ang,x1)
% Rz does the coordinates tranformation induced by reference
% frame rotating about the z-axis.
% Input:
% ang: rotation angle
% x1: The coordinates in the pre-rotation frame
% Output:
% x2:  The coordinates in the post-rotation frame
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% + Last modified on 24 July 2011
% + Author: Dali Kong
% + email: kongdlnju@gmail.com
if (size(x1,1)*size(x1,2)==3) %< Double check the dimension >
    error( 'Rz Input: 3-D vector required !!!' )
end
x2=zeros(size(x1,1),size(x1,2));
x2(1)=cos(ang)*x1(1)-sin(ang)*x1(3);
x2(2)=-sin(ang)*x1(1)+cos(ang)*x1(2);
x2(3)=x1(3);
end
References


