Galois covers of arithmetic curves of type $(p, \dots, p)$

Submitted by

Nicholas John Williams,

to the University of Exeter as a thesis for the degree of

Doctor of Philosophy in Mathematics,

March 2014

I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

....

Nicholas John Williams

Abstract

In this thesis we consider the setting where R is a complete discrete valuation ring of mixed characteristic $(0, p)$ where $p > 0$ is prime. Let $(p, ..., p)$ denote either the group $\mathbb{Z}/p\mathbb{Z} \times ... \times \mathbb{Z}/p\mathbb{Z}$ or the product of rank p group schemes.

Given a degree p Galois cover between R-curves, it is understood when the cover has the structure of a torsor under a finite flat group scheme of rank p. We investigate torsors under group schemes of type $(p, ..., p)$ and establish criteria for their existence.

We treat the boundary of the formal fibre and extend our knowledge of the conductor and degree of the different in degree p to the (p, p) setting. We also take the opportunity to explain how this can be naturally extended to the general $(p, ..., p)$ case.

Finally, we generalise a local vanishing cycles formula for curves known in the degree p case to Galois groups of type (p, p), relating the genus of two points in terms of just the cover's ramification data and the conductors acting at the boundaries.

Acknowledgments

I am sincerely grateful to my supervisor, Mohamed Saïdi, for his steadfast guidance, patience and generosity during the last three years. The subject of this thesis was suggested by him and I thank him for sharing his knowledge, expertise and time with me on countless occasions. In short, I could not have hoped for a kinder, more involved mentor.

Rachel Pries at Colorado State University and Robin Chapman at Exeter University kindly agreed to act as my examiners. I am very grateful to them both for their time and also thank them for their helpful comments.

I would like to express my thanks to the Hausdorff Research Institute for hosting me during their Arithmetic and Geometry trimester last year. It was a pleasure working in such a stimulating environment and it was there where much of the writing of this thesis was undertaken. On the subject of institutional acknowledgment, I am of course especially grateful to all the people at Exeter who have assisted me during my time here.

Francisco, Julio, Dave and Daniele punctuated this process with many enjoyable moments of recreation. Francisco also assisted in sourcing reading material and took the time to comment on a draft of this thesis.

My studies were supported by a PhD scholarship and I would like to express my gratitude for the award. I also benefited from a travel grant which gave me the opportunity to attend several conferences and workshops.

On a personal note, let me also take this opportunity to acknowledge the many forms of support afforded me by my family. Finally, I thank Duygu.

Contents

Chapter 1

Introduction

1.1 Notation, background and an overview of related results

Let R be a complete discrete valuation ring or DVR—a principal ideal domain with unique maximal ideal $\mathfrak{m} = (\pi)$ where π is the uniformiser—of mixed or unequal characteristic (i.e. $\mathrm{char}(R) = 0$ but there exists an ideal I such that $\mathrm{char}(R/I) > 0$). Further, suppose R contains a primitive p-th root of unity ζ and let $\lambda = \zeta - 1$ where $p > 0$ is a prime. Let $K = \mathrm{Frac}(R)$ denote the fraction field and $k = R/\mathfrak{m}$ denote the residue field of R. We ask that the residue field has characteristic p and we additionally assume that it is algebraically closed, $k = k^{\mathrm{alg}}$. We denote by v_K the valuation of K which we normalise so that $v_K(\pi) = 1$. The étale cohomology group is denoted by $H^1_{\text{ét}}$, while H^1_{fppf} represents cohomology in the flat topology. If A is a ring (or a scheme) we denote its normalisation in its field of fractions (if A is integral) or in its total ring of fractions (if A is not integral) by $\widetilde{A}$ and if S is an R-scheme or an R-algebra then we denote its π-adic formal completion by $\widehat{S}$. Our notation $(p, ..., p)$ refers to the Galois cover group $\mathbb{Z}/p\mathbb{Z} \times ... \times \mathbb{Z}/p\mathbb{Z} = (\mathbb{Z}/p\mathbb{Z})^n$ or, in the case of torsors under group schemes, the product of degree p group schemes.

We say a scheme $X = (X, \mathcal{O}_X)$ is an R-scheme if there exists a morphism $X \to \mathrm{Spec}(R)$ and we call the preimages of points in $\mathrm{Spec}(R)$ under this morphism the fibres of X. By an R-curve we mean an R-scheme of finite type which is normal, flat and whose fibres have dimension 1. Separately, $X_K := X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$ and $X_k := X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ denote respectively the so-called generic and special fibres of an R-scheme X. Incidentally, these are often abbreviated to $X_K = X \times_R K$ and $X_k = X \times_R k$ and this will be our convention throughout also. The stalk of $\mathcal{O}_X$ at the point $x \in X$ is defined as $\mathcal{O}_{X,x} := \lim_{\rightarrow x \in U} \mathcal{O}_X(U)$, the direct limit of the scheme's structure sheaf over all open neighbourhoods containing x and capturing the local behaviour of $\mathcal{O}_X$ around the point x. By a germ $\mathcal{X}$ of an R-curve X we mean that $\mathcal{X} := \mathrm{Spec}\,(\mathcal{O}_{X,x})$ is the spectrum of the stalk of X at x. To produce the definition of a formal scheme, formal curve, formal germ and so on, replace the normal Zariski spectrum Spec with the formal spectrum Spf and replace the algebras or the schemes in consideration by their π-adic completions.

A cover between schemes X, Y (or, indeed, curves) is a finite and generically separable morphism of schemes $f : Y \to X$. If X, Y are integral, normal schemes and the corresponding extension of function fields $K(X) \to K(Y)$ is a Galois extension with Galois group G such that the quotient $Y/G \simeq X$ then f is called a Galois cover with Galois group G. The degree of the cover, $\deg(f)$, is the cardinality of G. When X, Y are smooth curves, the ramification index $e_y \geq 1$ of a point $y \in Y$ is the ramification index of the extension $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ of DVRs. The point y is a ramification point if $e_y \geq 2$ and is unramified if $e_y = 1$. There are only finitely many ramification points forming the ramification locus. The image of the ramification locus of f is called the branch locus and is denoted by $B \subset X$, the set of points $x \in X$ such that the fibre $f^{-1}(x) \subset Y$ contains a ramification point. Alternatively, over an algebraically closed field, a branch point is simply a point $x \in X$ such that the cardinality of $f^{-1}(x)$ does not have size $\deg(f)$. If y is a ramification point, the decomposition group of the point y is the subgroup of G which fixes y, namely:

$$D_y = \{g \in G : g(y) = y\} = \{\sigma \in \mathrm{Aut}(Y/X) : \sigma(y) = y\}$$

For a point $y \in Y$, set $x = f(y)$. Then, again over an algebraically closed field, the cardinality of $f^{-1}(x) \subset Y$ is equal to the cardinality of G divided by the cardinality of D_y. The inertia group I_y is the subgroup of D_y acting by identity on the residue field at y, namely $\mathcal{O}_{Y,y}/\mathfrak{m}_y$. The cover is said to be wildly ramified at the ramification point y if $\mathrm{char}(k) = p$ divides the cardinality of I_y. Note that when k is algebraically closed, as is the case here, $I_y = D_y$ and so reference to the inertia group is in fact redundant.

In fact, we are particularly interested in the case where G is of the form $(p, ..., p)$ and therefore the cover under consideration along with its generic and special fibres, should be pictured as follows:

$$\begin{tikzcd}
Y_K = Y \times_R K \arrow[r, "G"] \arrow[d] & X_K = X \times_R K \arrow[r] \arrow[d] & K = \mathrm{Frac}(R) \arrow[d] \\
Y \arrow[r, "G=\mathbb{Z}/p\mathbb{Z}\times\ldots\times\mathbb{Z}/p\mathbb{Z}", "\text{Galois cover } f"'] & X \arrow[r] & R \\
Y_k = Y \times_R k \arrow[r] \arrow[u] & X_k = X \times_R k \arrow[r] \arrow[u] & k = R/\mathfrak{m} \arrow[u]
\end{tikzcd}$$

For our ring R, $R[[T]]$ denotes the ring of power series:

$$R[[T]] := \left\{ \sum_{i=0}^{\infty} a_i T^i : a_i \in R \right\}$$

If R is a ring with a non-archimedean absolute value $|\ |$ then $R\{T\}$ is the ring of power series $\sum_{i=0}^{\infty} a_i T^i$ with $\lim_{i\to\infty}|a_i| = 0 \Leftrightarrow \lim_{i\to\infty} v(a_i) = \infty$. Note that $R\{T\}$ is clearly a subring of $R[[T]]$. If F is a field then $F((t))$ denotes the field of power series with variable t and coefficients in F. The scheme $\mathrm{Spec}(R[[T]])$ is called the open unit disc and the scheme $\mathrm{Spec}(R\{T\})$ is called the closed unit disc. The boundary of the open unit disc is the scheme $\mathrm{Spec}(R[[T]]\{T^{-1}\})$ where

$$R[[T]]\{T^{-1}\} := \left\{ \sum_{i\in\mathbb{Z}} a_i T^i : \lim_{i\to-\infty}|a_i| = 0 \right\}$$

is the ring of formal power series with a convergence condition on negative powers of T where the absolute value $|\ |$ is associated to the valuation of K. It is a complete DVR with uniformiser π and residue field $k((t)) = R[[T]]\{T^{-1}\}/(\pi)$

where $t \equiv T \bmod \pi$, the reduction of the parameter T. Corresponding to each of these unit discs are the formal unit discs. The formal open unit disc is given by $\mathrm{Spf}(R[[T]])$ and its boundary is $\mathrm{Spf}(R[[T]]\{T^{-1}\})$. The boundaries of the (formal) open unit disc are isomorphic to (formal) germs of R-curves.

A group scheme G is a scheme over a base scheme R (also known as an R-group scheme) such that $G(S) := \mathrm{Hom}(S, G) = \{\text{morphisms of } R\text{-schemes } S \to G\}$ is a group for every scheme R-scheme S (see [21] for more details). The group scheme G is commutative if $G(S)$ is an abelian group for every scheme S. We will work with three group schemes in particular, $\mathbb{Z}/p\mathbb{Z}$, μ_p and α_p, which we define explicitly in the next chapter. These group schemes are commutative, finite, flat group schemes of rank p and our concern will be with torsors operating under these group schemes. In this context, a torsor $f : Y \to X$ under a group scheme G will mean that Y is equipped with an action of G which preserves the fibres of f and which is simply transitive on those fibres. We will attach a conductor to torsor equations illustrating, as in class field theory, an integer measure of ramification in the extension of the morphism.

A scheme X is reduced (integral or normal, respectively) if $\mathcal{O}_{X,x}$ is a reduced ring (an integral domain or integrally closed, respectively) for every point $x \in X$. The affine picture is worth noting: $X = \mathrm{Spec}(A)$ is called reduced (integral or normal, respectively) if the ring A is reduced (an integral domain or integrally closed, respectively). It is known that if X is an integral, flat R-scheme with reduced special fibre and normal generic fibre then X is a normal scheme.

The Riemann-Hurwitz formula is a classical result from the 1800s which establishes a relationship between the genus of two curves by calling on the cover's ramification data. Understood to have been given by Riemann but proved by Hurwitz, the formula can be expressed in various contexts but for us the most pertinent is the following, given here in its characteristic zero form:

Theorem (Riemann-Hurwitz formula, [7]) *Let $f : Y \to X$ be a Galois cover with Galois group G between smooth, proper curves X, Y over a field of characteristic zero and whose genus we denote respectively by g_X, g_Y. Then* *we have the formula:*

$$2g_Y - 2 = |G|(2g_X - 2) + \sum_{y \in f^{-1}(B)} \left(|D_y| - 1\right)$$

where B is the branch locus of f and D_y is the decomposition group of the point y .

And so it must be observed that the genus g_Y is dependent only on the genus g_X, $|G| = \deg(f)$, and $|D_y|$.

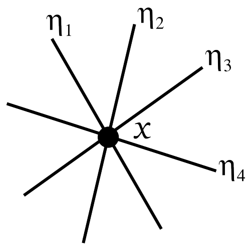
Postponing for the moment its technical definition, the genus of a point x, denoted by g_x, can be thought of as an integer measure of the singularity at the point. So given a Galois cover $Y \to X$ which maps points $y \in Y \to x \in X$, we would like to compute g_y with knowledge of g_x. It makes sense to look at the very objects which capture the local behaviour of the curve at these points, namely the (formal) germs $\mathrm{Spf}\left(\widehat{\mathcal{O}}_{Y,y}\right) \to \mathrm{Spf}\left(\widehat{\mathcal{O}}_{X,x}\right)$ whose map is induced by the cover $Y \to X$. We can read the genus from the point's germ since the germ contains all the necessary data to compute the genus of a point.

$$\begin{tikzcd}
Y \arrow[r] & X \\
\mathrm{Spf}\left(\widehat{\mathcal{O}}_{Y,y}\right) \arrow[u] \arrow[r] & \mathrm{Spf}\left(\widehat{\mathcal{O}}_{X,x}\right) \arrow[u]
\end{tikzcd}$$

For the curve X and the point x belonging to the special fibre X_k, we have the following picture:

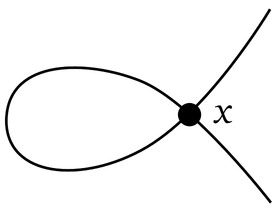
$$\begin{tikzcd}
\mathrm{Spec}\left(\widehat{\mathcal{O}}_{X,x} \otimes K\right) \arrow[r] \arrow[d, dashed] & X_K \arrow[r] \arrow[d, dashed] & \mathrm{Spec}\,(K) \arrow[d, dashed] \\
\mathrm{Spf}\left(\widehat{\mathcal{O}}_{X,x}\right) \arrow[r] & X \arrow[r] & \mathrm{Spf}\,(R) \\
\mathrm{Spec}\left(\widehat{\mathcal{O}}_{X,x} \otimes k\right) \arrow[r] \arrow[u] & X_k \arrow[r] \arrow[u] & \mathrm{Spec}\,(k) \arrow[u]
\end{tikzcd}$$

The formal scheme Spf $\left(\widehat{\mathcal{O}}_{X,x}\right)$ contains the maximal ideal corresponding to the closed point x (since non-zero maximal ideals are prime) and it contains the zero ideal (0) (since it is integral). It is also home to the finitely many prime ideals of height 1 that contain π and which we denote by $(\wp_i)_{i=1}^{n}$. They correspond exactly to the branches $(\eta_i)_{i=1}^{n}$ of the point x.



On the generic fibre Spec $\left(\widehat{\mathcal{O}}_{X,x} \otimes K\right) = \text{Spec}\left(\widehat{\mathcal{O}}_{X,x}[\frac{1}{\pi}]\right)$, the prime ideals $\wp_i$ containing π become units on the generic fibre and effectively disappear which makes sense as they live on the special fibre. The prime ideals of $\widehat{\mathcal{O}}_{X,x}$ which do not contain π correspond to points on the generic fibre Spec $\left(\widehat{\mathcal{O}}_{X,x} \otimes K\right)$.

Note that n is really the number of branches at the local level. Consider the picture below which illustrates where, for example, we can have one irreducible component passing through x but with x having two branches locally.



This is why we take the completion, which will contain two prime ideals, and gives us the local picture at the point rather than just the uncompleted local ring which will contain just one prime ideal corresponding to the single irreducible component.

For the open disc, when x is a smooth point we have that

$$\widehat{\mathcal{O}}_{X,x} \simeq R[[T]].$$

The maximal ideal in this picture is (π, T). There is only one prime ideal which contains π, it is (π) itself, and all the other prime ideals are points on the generic fibre. To obtain the boundary we localise $R[[T]]$ at π and complete to obtain:

$$R[[T]]\widehat{_{(\pi)}} \simeq R[[T]]\{T^{-1}\}$$

because when we localise at π, we invert everything that contains π—in particular T is inverted because the ideal (π, T) contains (π)—and by completing our power series becomes convergent for negative powers of T.

When x is a double point we have that

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{R[[S,T]]}{(ST - \pi^e)}$$

This makes sense since we have two prime ideals containing π, namely (π, T) and (π, S) which are both contained in the maximal ideal (π, S, T). In the same way as before, these two prime ideals yield two boundaries, $\operatorname{Spf}\left(R[[T]]\{T^{-1}\}\right)$ and $\operatorname{Spf}\left(R[[S]]\{S^{-1}\}\right)$ which is what we would expect given that two branches occur at a double point.

For a Galois cover of degree p, Saïdi gives in [15] an explicit local Riemann-Hurwitz type-formula comparing the genus of two points between formal germs of curves over a complete DVR of unequal characteristic:

Theorem (Theorem 3.4 in [15]) *Let $X := \operatorname{Spf}\left(\widehat{\mathcal{O}}_x\right)$ be the formal germ of an R-curve at a closed point $x \in X$ with X_k reduced. Let $f : Y \to X$ be a Galois cover of degree p with Y normal and local. Assume that the special fibre Y_k of Y is reduced. Let X_{b_i} denote the boundaries of X. Then, for each i, the cover f induces a torsor $f_{b_i} : Y_{b_i} \to X_{b_i}$ under a finite and flat R-group scheme G_i of rank p above the boundary X_{b_i} with conductor c_i. If $y \in Y$ is the closed point of Y, then:*

$$2g_y - 2 = p(2g_x - 2) + d_\eta - d_s$$

where g_y (resp. g_x) denotes the genus of the singularity at y (resp. x), d_η is the degree of the divisor of ramification in the morphism $f_K : Y_K \to X_K$ induced by f on the generic fibre, and $d_s = \sum_{i \in I}(c_i - 1)(p - 1)$.

Again, the critical observation must be that g_y depends on g_x, the ramification data on the generic fibre and the conductor associated to the torsor. The proof of this formula was achieved using formal patching, a technique which permits the construction and study of global objects via more local objects using a so-called 'cut-and-paste' approach that build covers locally and then combines them to form a global cover taking care on the 'overlaps' [5]. The technique was developed by Harbater in the 1980s and gained prominence for the proof of Abhyankar's conjecture [1], [12]. As one might guess, 'formal' patching involves Grothendieck's formal completions of schemes where the aforementioned overlaps are the formal boundaries. In fact, this will explain why we tend to work mainly with the formal spectrum rather than the ordinary spectrum, since we depend on results which rely on formal patching. Apart from this, we could equally have worked with Spec as all else is effectively the same.

Comparing the genus of points is in some way the same as comparing the dimensions of the spaces of vanishing cycles and, in this respect, such a formula is regarded as a 'vanishing cycles formula'. Indeed, prior to this result, Kato [6] had given in 1986 a local 'Riemann-Hurwitz type' formula which established a relationship between the dimensions of the spaces of vanishing cycles in a finite morphism between formal germs of curves over a complete DVR. His formula, however, is explicit only in the case where this morphism is generically separable on the level of special fibres. The result by Saïdi includes the case where we have inseparability on the level of special fibres.

The term 'vanishing cycles' is borrowed from topology and its usage in this context is well-established but perhaps we should, by way of a historical note, explain how it came to be inherited. The analogy in geometry of what we study here is to consider a family of analytic manifolds or complex varieties over the open disk $\{Z : |Z| < 1\}$ where the special fibre corresponds to $Z = 0$ and the generic fibre corresponds to $Z \neq 0$. The variety or manifold is defined by some equations (e.g. analytic equations or a power series) whose coefficients involve the parameter Z. Our case of good reduction corresponds to a smooth family.

However, manifolds which degenerate at $Z = 0$ and the cycles or loops which generate their topological fundamental group have been the focus of study. In particular, if we take any fibre close to $Z = 0$, then on this fibre we have some topological shape and can examine what happens when we approach zero. Some cycles disappear and this is how the term vanishing cycles was coined. Deligne generalised this to schemes via cohomology so that the dimension of the space of vanishing cycles, expressed as an H^1 étale cohomological group, corresponds to what we essentially mean by the genus of a singularity at a point. Indeed, if the point is smooth the dimension of the space of vanishing cycles is 0 which means there are no vanishing cycles.

Note that the results of Kato and Saïdi are by no means the only such vanishing cycles formulae. The literature is rich with examples in a variety of different settings but one key formula responsible for reigniting interest is the 1999 vanishing cycles formula given by Raynaud [11]. When p exactly divides the Galois group G of a branched covering of curves, his formula relates certain invariants on the so-called tail covers in the case of stable reduction. All these formulae have many interesting applications. For instance, Saïdi's formula can in fact help determine whether a cover of a curve is semi-stable.

1.2 Thesis outline and our main results

The motivating idea of this thesis is to move beyond degree p Galois covers and attempt to treat covers with group (p, p) and, if possible, the general setting of $(p, ..., p)$. We seek to extend known results and examine how things may differ. Note that we can factorise a Galois cover with group $G = G_1 \times G_2$ in the following way:

$$\begin{tikzcd} & Y \arrow[dl, "G_2"'] \arrow[dr, "G_1"] \arrow[dd] & \\ X_1 \arrow[dr, "G_1"'] & & X_2 \arrow[dl, "G_2"] \\ & X & \end{tikzcd}$$

by taking $X_1 = Y/G_2$, $X_2 = Y/G_1$ and $X = Y/G$ where $G = G_1 \times G_2$. This means that $Y \to X_i \to X$ is composed of two degree p Galois covers each with either Galois group G_1 or G_2 at each step for $i = 1, 2$. When we treat the (p, p) case, we have that $G_1 = G_2 = \mathbb{Z}/p\mathbb{Z}$. The $(p, ..., p)$ situation is illustrated by extending this picture in the natural way.

This thesis is composed as follows. Chapter 2 provides the reader with a more detailed background to group schemes and torsors than this introduction could offer. Chapter 3 through to 5 then house our new work and main results on the (p, p) and $(p, ..., p)$ settings, modulo a terse overview at the start of each chapter of the corresponding and known degree p case as it exists in the literature.

If $f : Y \to X$ is a Galois cover with group $G = \mathbb{Z}/p\mathbb{Z}$ for a smooth, normal, connected and flat R-curve of finite type (where Y is normal with Y_k reduced) then the cover f has the structure of a torsor under a finite and flat R-group scheme of rank p over X. However, this does not automatically apply to all finite groups G, including $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and so in Chapter 3 we investigate under what conditions we have the structure of a torsor for Galois covers of type $(p, ..., p)$.

Theorem 3.2.5 *Let X be smooth, geometrically connected R-scheme of any dimension whose special fibre X_k is reduced. Suppose $X_i \to X$ is a torsor under a R-group scheme G_i of rank p for $i = 1, ..., n$ and let $\widetilde{X} = X_1 \times_{X} \widetilde{...} \times_X X_n$.*

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[d] \arrow[dll, bend right] \arrow[dl, bend right] \arrow[dr, bend left] \arrow[drr, bend left] & & \\
 & & X_1 \times_X X_2 \times_X \ldots \times_X X_n \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
X_1 \arrow[drr, "G_1"'] & X_2 \arrow[dr, "G_2"] & X_3 \arrow[d, "G_3"] & \ldots \arrow[dl] & X_n \arrow[dll, "G_n"] \\
 & & X & &
\end{tikzcd}$$

Then the following statements are equivalent

1. *$\widetilde{X}$ is a torsor over X under some group scheme G (where, in this case, $G = G_1 \times ... \times G_n$ necessarily)*
2. *$\widetilde{X} = X_1 \times_X X_2 \times_X ... \times_X X_n$ (i.e. $X_1 \times_X X_2 \times_X ... \times_X X_n$ is normal)*
3. *$(X_1 \times_X X_2 \times_X ... \times_X X_n)_k$ is reduced*

We also offer an additional criterion in the case of a curve:

> **Theorem 3.2.6** *Let X be a smooth formal affine connected R-curve. Then $\widetilde{X}$ is a torsor over X if and only if at least $n-1$ of the n finite flat R-group schemes G_i of rank p acting on $X_i \to X$ are étale.*

In **Example 3.2.7** we illustrate why this criterion does not hold beyond dimension one.

Likewise, on the boundary of the formal fibre in the degree p case one has the structure of a torsor. We can attach two invariants to the torsor induced by the cover on the boundary—namely, a conductor c and the degree of the different δ of the corresponding extension. In **Theorem 4.2.1** and **Corollary 4.2.2** of Chapter 4, using Kummer and Artin-Schreier theory, we compute the conductor in the (p,p) setting for all possible combinations of acting rank p group schemes. To the best of our knowledge, only the case of two étale group schemes had been previously treated in the literature, [4]. This then enables us to explicitly state in which of the cases we have the structure of a torsor, a result we articulate in **Theorem 4.2.3**. Our results for the degree of the different in the (p,p) setting are housed in **Theorem 4.2.7**. With the (p,p) theory to hand it is easy to extend all these results to $(p,...,p)$ by iteration. We illustrate this in **Example 4.2.8** for the case of (p,p,p). The results of Chapter 4 are significant for our work in Chapter 5 and even underpin key elements in Chapter 3. However, due to the statements being rather involved, we refrain from citing them here and instead refer the reader to the text.

In Chapter 5 we prove that one can generalise Saïdi's vanishing cycles formula—relating the genus of two points in terms of just the cover's ramification data and the conductors acting at the boundaries—to Galois covers of type (p,p). This is achieved by decomposing the cover and applying the corresponding degree p result twice, taking care to count for the distinctions exhibited by such a cover. We hope it will be clear to the reader how such a formula can be extended to curves with Galois groups of type $(p, ..., p)$. Our main result from the chapter is the following:

> **Theorem 5.2.1** (Local Riemann-Hurwitz formula for (p, p)) *Let $X := \mathrm{Spf}\left(\widehat{\mathcal{O}}_x\right)$ be the formal germ of an R-curve at a closed point x with X_k reduced. Let $f : Y \to X$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$—that is, of type (p, p)—where Y is normal and local and the special fibre Y_k of Y is reduced. Let $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ be two degree p Galois covers such that Y is the compositum of Y_1 and Y_2. Let X_{b_i} denote the boundaries of X. The Galois cover f_1 induces a torsor $Y_{1,b_i} \to X_{b_i}$ under a finite and flat R-group scheme of type p with conductor $c_{1,i}$ for each i. Similarly, $c'_{1,i}$ denotes the conductor associated to the torsor of the cover $Y_{b_i} \to Y_{1,b_i}$. We let r_1 (resp. r_2) denote the number of ramified points in $Y_1 \to X$ (resp. $Y \to Y_1$). If $y \in Y$ is the closed point of Y, then:*
>
> $$2g_y - 2 = p^2(2g_x - 2) + d_\eta - d_s$$
>
> *where g_y (resp. g_x) denotes the genus of the singularity at y (resp. x), $d_\eta := (r_1 + r_2)p(p-1)$ is the degree of the divisor of ramification in the morphism $f_K : Y_K \to X_K$ induced by f on the generic fibre and*
>
> $$d_s = \sum_{\substack{\text{boundary unibranched} \\ \text{throughout}}} (c'_{1,i} - 1)(p-1) + (c_{1,i} - 1)p(p-1)$$
> $$+ \sum_{\substack{\text{boundary unibranched} \\ \text{then } p\text{-branched}}} (c_{1,i} - 1)p(p-1) + \sum_{\substack{\text{boundary } p\text{-branched} \\ \text{then unibranched}}} (c'_{1,i} - 1)(p-1)$$

It is appropriate to bring to the reader's attention that in his 2004 paper *Cyclic p-groups and semi-stable reduction of curves in equal characteristic $p > 0$* [14], Saïdi investigated the case where the Galois group is a cyclic degree p^2 cover, namely $G = \mathbb{Z}/p^2\mathbb{Z}$. He demonstrated that his local Riemann Hurwitz result can be formulated for this cyclic group. This thesis, seeking to answer similar questions for the non-cyclic p^2 cover as well as in the general p^n case, will hopefully complement and add to the existing body of literature.

Chapter 2

Review of torsors under group schemes of type p

This chapter begins with an elementary introduction to group schemes, the material of which is either restated or adapted from [21]. The appendix on group schemes in [3] was also helpful and [7] was relied on for certain technical definitions. References [19] and [9] were referred to when writing the next section of this chapter where we define torsors under finite flat group schemes. Our attention then begins to turn to characteristic $p > 0$ and so in the third section we introduce torsors under rank p group schemes, namely $\mathbb{Z}/p\mathbb{Z}$, μ_p and α_p. We explain how to describe torsors locally with explicit equations under these group schemes. In the étale case of $\mathbb{Z}/p\mathbb{Z}$, it is Artin-Schreier theory which provides the necessary tools to produce these equations. Finally, in the fourth section, we discuss degeneration of μ_p torsors. While all definitions and results of this chapter are well known and established in the literature, this background material is essential for our work in the subsequent chapters.

2.1 Group schemes

Let C be a category with finite products and final object S. Let G be an object in $\mathcal{C}$ and $m : G \times G \to G$ a composition morphism on G. For every object T in $\mathcal{C}$, m

induces a law of composition on the set $G(T) := \mathrm{Hom}(T, G)$ as follows:

> Let $g_1, g_2 \in G(T)$. Set $g_1 g_2 = m \circ (g_1, g_2)$ where $(g_1, g_2) : T \to G \times G$ is the unique arrow such that $\mathrm{pr}_1 \circ (g_1, g_2) = g_1$ and $\mathrm{pr}_2 \circ (g_1, g_2) = g_2$ for projections $\mathrm{pr}_1, \mathrm{pr}_2 : G \times G \to G$.
>
> $$\begin{tikzcd} & T \arrow[dl, "g_1"'] \arrow[d, dashed, "(g_1,g_2)"] \arrow[dr, "g_2"] & \\ G & G \times G \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & G \end{tikzcd}$$
>
> So $g_1 g_2$ is an element of $\mathrm{Hom}(T, G) = G(T)$ since the composition $m \circ (g_1, g_2)$ gives $T \to G \times G \to G$.

It is now clear that $G(T)$ is a magma, namely a set with a law of composition.

Let T and T' be objects in $\mathcal{C}$ and suppose we have a morphism $f : T' \to T$. Then f induces a map $f^* : G(T) \to G(T')$ by setting $f^*(g) := g \circ f$ since $g : T \to G$ and $f : T' \to T$ and so $g \circ f : T' \to G \in G(T')$.

It can easily be seen that f^* is multiplicative since:

$$\begin{aligned} f^*(g_1 g_2) &= f^*(m \circ (g_1, g_2)) = (m \circ (g_1, g_2)) \circ f \\ &= m \circ ((g_1, g_2) \circ f) = m \circ (g_1 \circ f, g_2 \circ f) \\ &= (g_1 \circ f)(g_2 \circ f) = f^*(g_1) f^*(g_2) \end{aligned}$$

Proposition 2.1.1. *The association $T \mapsto G(T)$ is a contravariant functor from $\mathcal{C}$ to the category of magmas.*

Proof. Let $\mathcal{D}$ denote the category of magmas. Then the functor $F : \mathcal{C} \to \mathcal{D}$ which maps an object T of $\mathcal{C}$ to an object $F(T) := G(T)$ of $\mathcal{D}$ is clearly a contravariant functor since $f \in \mathrm{Hom}(T', T) \mapsto F(f) := f^* \in \mathrm{Hom}(G(T), G(T')) = \mathrm{Hom}(F(T), F(T'))$. $\square$

We will now say what it means for the $G(T)$ to be commutative, associative and possess inverses and unit elements.

- (Associativity) $G(T)$ is associative for all T if and only if $(\mathrm{pr}_1\mathrm{pr}_2)\mathrm{pr}_3 = \mathrm{pr}_1(\mathrm{pr}_2\mathrm{pr}_3)$ holds in $G(G \times G \times G)$, if and only the following diagram commutes:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times \mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_G \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

- (Identity) $G(T)$ has two-sided identity elements for all T if and only if there is a point $\epsilon \in G(S)$ where S is the final object in $\mathcal{C}$ such that $\pi^*(\epsilon) \cdot \mathrm{id} = \mathrm{id} = \mathrm{id} \cdot \pi^*(\epsilon)$ holds in $G(G)$ where $\pi = \pi_G$ is the unique arrow $G \to S$ if and only if each triangle in the diagram commutes:

$$\begin{tikzcd}
G = S \times G = G \times S \arrow[r, "\mathrm{id} \times \epsilon"] \arrow[d, "\epsilon \times \mathrm{id}"'] \arrow[dr] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

  In this case, $\epsilon_T := \pi_T^*(\epsilon)$ is the identity element in $G(T)$.

- (Inverse) If the magmas $G(T)$ have two-sided identity elements for all T then every element $g \in G(T)$ has a left inverse for every T if and only if there exists an element $\mathrm{inv} \in G(G)$ such that $\mathrm{inv} \cdot \mathrm{id}_G = \epsilon_G$ if and only if the following diagram commutes:

$$\begin{tikzcd}
G \times G \arrow[r, "\mathrm{inv} \times \mathrm{id}"] & G \times G \arrow[d, "m"] \\
G \arrow[u] \arrow[r, "\epsilon \circ \pi"'] & G
\end{tikzcd}$$

  In this case, $(\mathrm{inv} \circ g) \cdot g = \epsilon_T$ for every $g \in G(T)$ and for any T.

- (Commutativity) $G(T)$ is commutative for all T if and only if $\mathrm{pr}_1\mathrm{pr}_2 =$

$\mathrm{pr}_2\mathrm{pr}_1$ holds in $G(G \times G)$ if and only the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[rr, "(\mathrm{pr}_2,\mathrm{pr}_1)"] \arrow[dr, "m"'] & & G \times G \arrow[dl, "m"] \\ & G & \end{tikzcd}$$

where $(\mathrm{pr}_2, \mathrm{pr}_1)$ denotes the interchanging automorphism.

Definition 2.1.2. *A* ***group object*** *is an object G in $\mathcal{C}$ together with a morphism $m : G \times G \to G$ such that the induced law of composition $G(T) \times G(T) \to G(T)$ makes $G(T)$ a group for every object T in $\mathcal{C}$.*

A group object G is commutative if the group $G(T)$ is an abelian group for every T.

Definition 2.1.3. *Let G and G' be group objects in $\mathcal{C}$. A* ***homomorphism of group objects*** *$\phi : G \to G'$ is a morphism such that the induced map $G(T) \to G'(T)$ given by $g \mapsto \phi \circ g$ is a homomorphism of groups for every object T in $\mathcal{C}$.*

If m and m' are the composition morphisms of group objects G and G' respectively. A morphism $\phi : G \to G'$ is a homomorphism of group objects if and only if the equality $\phi^*(\mathrm{pr}_1\mathrm{pr}_2) = \phi^*(\mathrm{pr}_1)\phi^*(\mathrm{pr}_2)$ holds in $G'(G \times G)$ if and only if the following diagram is commutative:

$$\begin{tikzcd} G \times G \arrow[r, "\phi\times\phi"] \arrow[d, "m"'] & G' \times G' \arrow[d, "m'"] \\ G \arrow[r, "\phi"'] & G' \end{tikzcd}$$

Here is another way to construct a group object. Let G be an object in $\mathcal{C}$ and rather than a composition morphism $m : G \times G \to G$, suppose for each object T in $\mathcal{C}$ we are given a group structure on $G(T)$ such that for each $f : T' \to T$ the induced map $f^* : G(T) \to G(T')$ is a group homomorphism. The composition morphism $m : G \times G \to G$ can be recovered from the law of composition on the group $G(G \times G)$. It is in fact the product of the two projections $m = \mathrm{pr}_1\mathrm{pr}_2$ in $G(G \times G)$.

Definition 2.1.4. *Let $\mathcal{C}$ be the category of schemes over a base scheme S. A **group scheme** or an **S-group scheme** is a group object in $\mathcal{C}$.*

Let us decompose this definition somewhat. A scheme G over a base scheme S (or an S-scheme) simply means that G is endowed with a morphism of schemes $G \to S$. In the category of schemes over a base scheme S the 'product' is the 'fibred product', namely:

Definition 2.1.5. *Let S be a scheme, and let X, Y be two S-schemes. We define the **fibred product of schemes** X, Y over S to be an S-scheme $X \times_S Y$, together with two projection morphisms of S-schemes $\mathrm{pr}_1 : X \times_S Y \to X, \mathrm{pr}_2 : X \times_S Y \to Y$, verifying the following universal property:*

> *Let $f : Z \to X, g : Z \to Y$ be two morphisms of S-schemes. Then there exists a unique morphism of S-schemes $(f, g) : Z \to X \times_S Y$ making the following diagram commutative:*

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[d, dashed, "{(f,g)}"'] \arrow[dr, "g"] & \\
X & X \times_S Y \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & Y
\end{tikzcd}$$

In other words, a group scheme G is a scheme over a base scheme S together with three morphisms of schemes over S

> (Multiplication) $m : G \times_S G \to G$
>
> (Identity) $\epsilon : S \to G$
>
> (Inverse) $\mathrm{inv} : G \to G$

such that the following diagrams commute:

- (Associativity)

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "m\times \mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_G \times m"'] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"'] & G \end{tikzcd}$$

- (Identity)

$$\begin{tikzcd} G = G \times_S S \arrow[r] \arrow[dr, "\mathrm{Id}_G"'] & G \times_S G \arrow[d, "m"] \\ & G \end{tikzcd}$$

- (Inverse)

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G \times_S G \arrow[r, "\mathrm{Id}_G \times \mathrm{inv}"] & G \times_S G \arrow[d, "m"] \\ S \arrow[rr, "\epsilon"'] & & G \end{tikzcd}$$

Therefore, a group scheme G is a scheme over a base scheme S such that $G(T) = \mathrm{Hom}(T, G) = \{\text{morphisms of } S\text{−schemes } T \to G\}$ is a group for every scheme T over S.

Definition 2.1.6. *A* ***subgroup scheme*** *of a group scheme G is a (closed) subscheme H of G such that $H(T)$ is a subgroup of $G(T)$ for all S-schemes T.*

The kernel of a group scheme homomorphism is a group scheme. Suppose $f : G \to G'$ is a homomorphism between group schemes G, G'. Then, for all S-schemes T, we have a natural identification $\mathrm{Ker}(f)(T) = \mathrm{Ker}\,(G(T) \to G'(T))$. More accurately, the kernel is the fibred product in the category of S-schemes given by the following Cartesian diagram:

$$\begin{tikzcd} \mathrm{Ker}(f) = G \times_{G'} S \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & S \arrow[d, "\epsilon"] \\ G \arrow[r, "f"'] & G' \end{tikzcd}$$

The projection morphism pr_1 provides the identification between $\mathrm{Ker}(f)(T)$ and the group $\mathrm{Ker}\,(G(T) \to G'(T))$, while the projection pr_2 gives the structure morphism establishing $\mathrm{Ker}(f)$ as an S-scheme in its own right. So the kernel of a group scheme exists so long as we are working in a domain where we can take fibre products and that is certainly true in the category of S-schemes.

Suppose the base scheme S is affine. So we can write $S = \mathrm{Spec}(R)$ for some commutative ring R. With this established, we often replace S by R in the notation and terminology; instead speaking of R-schemes and R-group schemes for example.

There is an arrow-reversing equivalence between the category of commutative R-Hopf algebras and the category of affine R-group schemes:

Let $G = \mathrm{Spec}(A)$ be an affine R-scheme (i.e. an S-scheme where $S = \mathrm{Spec}(R)$). Then to make G into an R-group scheme we need to give three R-algebra homomorphisms:

(Comultiplication) $\widetilde{m} : A \to A \otimes_R A$

(Counit) $\widetilde{\epsilon} : A \to R$

(Antipode) $\widetilde{\mathrm{inv}} : A \to A$

corresponding to the earlier morphisms m, ϵ and inv (by reversing arrows, replacing S by R and G by A) such that the following diagrams are commutative:

- (Associativity)

$$\begin{tikzcd} A \otimes_R A \otimes_R A & A \otimes_R A \arrow[l, "\widetilde{m} \times \mathrm{Id}_A"'] \\ A \otimes_R A \arrow[u, "\mathrm{Id}_A \times \widetilde{m}"] & A \arrow[l, "\widetilde{m}"] \arrow[u, "\widetilde{m}"'] \end{tikzcd}$$

- (Identity)

$$\begin{tikzcd}
A = A \otimes_R R & & A \otimes_R A \arrow[ll] \\
& & A \arrow[u, "\widetilde{m}"] \arrow[ull, "\mathrm{Id}_A"]
\end{tikzcd}$$

- (Inverse)

$$\begin{tikzcd}
A & A \otimes_R A \arrow[l] & A \otimes_R A \arrow[l, "\mathrm{Id}_A \times \widetilde{\mathrm{inv}}"'] \\
R \arrow[u] & & A \arrow[ll, "\widetilde{\epsilon}"'] \arrow[u, "\widetilde{m}"]
\end{tikzcd}$$

A commutative R-algebra A with homomorphisms $\widetilde{m}$, $\widetilde{\epsilon}$ and $\widetilde{\mathrm{inv}}$ as above is called a commutative Hopf algebra. Therefore the category of affine R-group schemes is antiequivalent to the category of commutative Hopf algebras over R. From now on, we refer to the triple $\widetilde{m}$, $\widetilde{\epsilon}$ and $\widetilde{\mathrm{inv}}$ as the Hopf morphisms. With this construction, one can readily translate much of the theory so far established. For instance, the idea of a homomorphism between two S-group schemes $G \to G'$ is, by this equivalence, the same as giving a homomorphism between the corresponding R-algebras $R_{G'} \to R_G$ which commutes with the Hopf morphisms. We now turn to some examples and make a point of defining them using both constructions.

Example 2.1.7. *The* ***additive group scheme*** *is defined as* $\mathbb{G}_a := \mathrm{Spec}(R[u])$ *for an indeterminate* u *(essentially the affine line* $\mathbb{A}^1$*) where the composition law can be determined from the Hopf morphisms:*

$$\begin{aligned}
\widetilde{m} &: R[u] \to R[u] \otimes_R R[u] \\
&\quad u \mapsto u \otimes 1 + 1 \otimes u \\
\widetilde{\epsilon} &: R[u] \to R \\
&\quad u \mapsto 0 \\
\widetilde{\mathrm{inv}} &: R[u] \to R[u] \\
&\quad u \mapsto -u
\end{aligned}$$

To an S-scheme T, $\mathbb{G}_a$ associates the group $\mathbb{G}_a(T) := \Gamma(T, \mathcal{O}_T)$, namely the additive group of global sections.

Example 2.1.8. *The* ***multiplicative group scheme*** *is defined as $\mathbb{G}_m := \mathrm{Spec}(R[u, u^{-1}])$ for an indeterminate u where the composition law can be determined from the Hopf morphisms:*

$$\begin{aligned}
&\widetilde{m} : R[u, u^{-1}] \to R[u, u^{-1}] \otimes_R R[u, u^{-1}] \\
&\quad u \mapsto (u \otimes 1)(1 \otimes u) = u \otimes u \\
&\widetilde{\epsilon} : R[u, u^{-1}] \to R \\
&\quad u \mapsto 1 \\
&\widetilde{\mathrm{inv}} : R[u, u^{-1}] \to R[u, u^{-1}] \\
&\quad u \mapsto u^{-1}
\end{aligned}$$

Another natural expression for $\mathbb{G}_m$ is $\mathrm{Spec}(R[x, y]/(xy - 1))$ for indeterminates x and y. To an S-scheme T, $\mathbb{G}_m$ associates the group $\mathbb{G}_m(T) := \Gamma(T, \mathcal{O}_T)^*$, namely the multiplicative group of invertible elements of $\Gamma(T, \mathcal{O}_T)$.

Example 2.1.9. *The* ***constant group scheme*** *is constructed using a finite group Γ by taking the disjoint union of $|\Gamma|$ copies of our base scheme S. Perhaps confusingly, we denote the constant group scheme associated to the group Γ by Γ. The group Γ doesn't have to be finite but we will take it to be so. To an S-scheme T, the group scheme Γ associates the direct sum group $\Gamma(T) := \Gamma \times ... \times \Gamma$ where the number of copies of Γ is equal to the number of connected components in the scheme T. So that $\Gamma(T) = \Gamma$ only when the scheme T is connected. The rank of the group scheme Γ is equal to the order of the group Γ. If we take $\Gamma = \mathbb{Z}/p\mathbb{Z}$, then the group scheme Γ has rank p. Constant group schemes are étale.*

Definition 2.1.10. *A group scheme G over a base scheme S is* ***flat*** *(respectively* ***finite****) if the morphism $G \to S$ is a flat (respectively finite) morphism.*

The morphism $f : G \to S$ is a flat morphism if for every point $x \in G$, the homomorphism $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{G,x}$ is flat which means that $\mathcal{O}_{G,x}$ is a flat $\mathcal{O}_{S,f(x)}$-module (i.e. tensoring preserves exact sequences). The morphism $f : G \to S$

is a morphism of finite type if f is quasi-compact (i.e. the inverse image of any affine open subset is quasi-compact, by which we mean we can extract a finite subcovering from any open covering of the topological space) and if for every affine open subset V of S, and for every affine open subset U of $f^{-1}(V)$ in G, the canonical homomorphism $\mathcal{O}_S(V) \to \mathcal{O}_G(U)$ makes $\mathcal{O}_G(U)$ into a finitely generated $\mathcal{O}_S(V)$-algebra. So, if $V = \mathrm{Spec}(B)$ and $U = \mathrm{Spec}(A)$ then $A = \mathcal{O}_G(U)$ is a finitely generated $B = \mathcal{O}_S(V)$-module. Needless to say, group schemes that are both finite and flat are what we regard as finite flat group schemes.

Definition 2.1.11. *Suppose G is a finite flat group scheme over a base scheme S. Then the **rank** of G is the rank of the finitely generated modules induced by the definition of a finite group scheme.*

The rank of these finitely generated modules should be the same and we will take it to be so. In fact, the rank is constant when S is a connected scheme which will be the case in the examples we treat. With subgroup schemes defined, it is acceptable to take the quotient of finite group schemes. For instance, if H is subgroup scheme of a group scheme G then we can construct G/H and the rank of G is the product of the ranks of H and G/H. Likewise, if G and G' are two group schemes over the same base scheme S withs ranks n_1 and n_2 respectively, then we can take their product which will have rank $n_1 n_2$.

2.2 Groups schemes in characteristic p

Let $p > 0$ be prime. We will define three new group schemes $\mathbb{Z}/p\mathbb{Z}$, μ_p and α_p in characteristic p. In this section, we work with schemes over $\mathbb{F}_p$.

- **$\mathbb{Z}/p\mathbb{Z}$, étale group scheme:** The constant group scheme $\mathbb{Z}/p\mathbb{Z}$ is known as the étale group scheme. The map $f : \mathbb{G}_a \to \mathbb{G}_a$ defined by $x \mapsto f(x) := x^p - x$, yields the exact sequence in the étale topology:

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{x^p - x} \mathbb{G}_a \longrightarrow 0$$

as $\mathrm{Ker}(f : \mathbb{G}_a \to \mathbb{G}_a) = \{x \in \mathbb{G}_a : f(x) = 0\} = \{x \in \mathbb{G}_a : x^p - x = 0\} = \{x \in \mathbb{G}_a : x^p = x\} = \mathbb{Z}/p\mathbb{Z}$. This sequence is known as the Artin-Schreier sequence.

- μ_p**, p-th roots of unity group scheme:** For each p, the group scheme $\{x \in \mathbb{G}_m : x^p = 1\}$ of p-th roots of unity is called μ_p. To an R-scheme S, μ_p associates the subgroup $\mu_p(S)$ of elements of $\mathbb{G}_m(S)$ whose order divides p. It has the same group structure as in the case of $\mathbb{G}_m$. So μ_p is a closed subgroup subscheme of $\mathbb{G}_m$. Equivalently one can express μ_p as $\mathrm{Spec}(R[u, u^{-1}]/(u^p - 1))$. It is an important observation that μ_p is the kernel of the p-th power map $f : \mathbb{G}_m \to \mathbb{G}_m$ defined by $x \mapsto f(x) := x^p$:

$$1 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{x^p} \mathbb{G}_m \longrightarrow 1$$

  as $\mathrm{Ker}(f : \mathbb{G}_m \to \mathbb{G}_m) = \{x \in \mathbb{G}_m : f(x) = 1\} = \{x \in \mathbb{G}_m : x^p = 1\} = \mu_p$. This sequence is known as the Kummer sequence and is exact in the fppf topology.

- α_p**, p-th roots of zero group scheme:** We write α_p for the group scheme $\{x \in \mathbb{G}_a : x^p = 0\}$. To an R-scheme S, α_p associates the subgroup $\alpha_p(S) := \{x \in \mathbb{G}_a(S) = \Gamma(S, \mathcal{O}_S) : x^p = 0\}$ of $\mathbb{G}_a(S)$. It has the same group structure as in the case of $\mathbb{G}_a$. So α_p is a closed subgroup subscheme of $\mathbb{G}_a$. Equivalently one can express α_p as $\mathrm{Spec}(R[u]/(u^p))$. Much like before, it is an important observation that α_p is the kernel of the p-th power map $f : \mathbb{G}_a \to \mathbb{G}_a$ defined by $x \mapsto f(x) := x^p$:

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{x^p} \mathbb{G}_a \longrightarrow 0$$

  as $\mathrm{Ker}(f : \mathbb{G}_a \to \mathbb{G}_a) = \{x \in \mathbb{G}_a : f(x) = 0\} = \{x \in \mathbb{G}_a : x^p = 0\} = \alpha_p$. This sequence is exact in the fppf topology.

We refer to μ_p and α_p as the radicial group schemes and $\mathbb{Z}/p\mathbb{Z}$ as the étale group scheme. Both μ_p and $\mathbb{Z}/p\mathbb{Z}$ can be defined using the same construction for arbitrary characteristic and in the case of characteristic 0, it turns out that non-canonically we have $\mu_p \simeq \mathbb{Z}/p\mathbb{Z}$.

Theorem 2.2.1 (Theorem 1 in [22]). *Every finite group scheme of rank p is commutative.*

So, $\mathbb{Z}/p\mathbb{Z}$, μ_p and α_p are all commutative and, in fact, these three are the only finite group schemes of rank p that exist in characteristic p.

2.3 Torsors

We begin by briefly reviewing the theory of torsors under group schemes.

Let G be a group with composition $*$ and S be a set. Recall that G is said to act (on the right of) S if there is a function $\circ : S \times G \to S$ such that for all $g_1, g_2 \in G$ and $s \in S$:

1. $s \circ (g_1 * g_2) = (s \circ g_1) \circ g_2$,

2. $s \circ e = s$.

Definition 2.3.1. *Let G be a group and S be a set on which G acts on the right. S is said to be a **torsor** for G if for every $s \in S$ the map from $G \to S$ given by $g \mapsto sg$ is a bijection.*

Definition 2.3.2. *Let S be a scheme and let G be an S-group scheme which is finite and flat. A G-**torsor** or a **torsor under the group scheme** G is defined to be an S-scheme X with a G-action $X \times_S G \to X$ such that:*

- *$X \times_S G \to X \times_S X$ where $(x, g) \mapsto (x, xg)$ is an isomorphism; and*

- *$X \to S$ is faithfully flat.*

A trivial torsor is a torsor X which is isomorphic to G acting on itself on the right by translation.

Proposition 2.3.3 (Proposition 4.1 of Chapter 3 in [9]). *The following two statements are equivalent*

1. *$X \to S$ is faithfully flat and $X \times_S G \to X \times_S X$ where $(x, g) \mapsto (x, xg)$ is an isomorphism.*

2. *There exists a covering $\mathcal{U} = (\mathcal{U}_i \to S)$ in the flat topology such that for each i the pair ($X_{U_i} = X \times_S U_i$, the action of $G_{U_i} = U_i \times_S G$) is isomorphic to the pair (G_{U_i}, the right action of G_{U_i} on itself).*

Another, perhaps more accessible, way of thinking about torsors is to consider a torsor under a finite group G as a surjective morphism $f : Y \to X$ with Y being equipped with an action of G which preserves the fibres of f and which is simply transitive on the geometric fibres.

Note that whenever we have a torsor above an R-scheme where R is of mixed characteristic, the generic fibre is étale automatically since every group scheme in characteristic 0 is étale.

It so happens that when the group scheme G is commutative we can classify torsors as elements of the fppf cohomology group H^1_{fppf}, as explained in Section 2.2 of [19]. In particular:

$$H^1_{fppf}(X, G) \;\; \simeq \;\; \{X\text{-torsors under } G\} \; / \; \sim$$

where $H^1_{fppf}(X, G)$ denotes the first cohomology group for the scheme X with values in the sheaf which is represented by G and the modulo relation is up to isomorphism. Since the group schemes we study are all commutative this descriptions works fine for us.

In fact, we use this description to go further and relate α_p and μ_p torsors to differentials by way of the following result:

Theorem 2.3.4 (Proposition 4.14 of Chapter 3 in [9])**.** *Let X be a smooth scheme over a perfect field k of characteristic $p > 0$. Then:*

$$H^1_{fppf}(X, \alpha_p) \simeq \{\omega \in \Gamma(X, \Omega^1_X) : d\omega = 0, C\omega = 0\}$$

$$H^1_{fppf}(X, \mu_p) \simeq \{\omega \in \Gamma(X, \Omega^1_X) : d\omega = 0, C\omega = \omega\}$$

where Ω^1_X denotes the sheaf of Kahler differentials on X and C is the Cartier operator.

So the differential form associated to a μ_p torsor is fixed under the Cartier operation C while for an α_p torsor the associated differential form is annihilated by the Cartier operation C. See Chapter 3, Section 4 of [9] for more details.

2.4 Torsors under group schemes of rank p in characteristic $p > 0$

Recall that Artin-Schreier theory is concerned with the construction of prime p degree cyclic Galois extensions over fields of characteristic p. We will see that when working in characteristic p, the theory provides explicit local equations for torsors which opens the door for us to perform a host of calculations that would otherwise have been very difficult.

Recall the Artin-Schreier sequence:

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{x^p - x} \mathbb{G}_a \longrightarrow 0$$

From this short exact sequence we can obtain a long exact sequence in terms of cohomology groups:

$$\begin{aligned} 0 &\to \Gamma(X, \mathbb{Z}/p\mathbb{Z}) \to \Gamma(X, \mathbb{G}_a) \xrightarrow{x^p - x} \Gamma(X, \mathbb{G}_a) \\ &\to H^1_{fppf}(X, \mathbb{Z}/p\mathbb{Z}) \to H^1_{fppf}(X, \mathbb{G}_a) \xrightarrow{x^p - x} H^1_{fppf}(X, \mathbb{G}_a) \to ... \end{aligned}$$

where X is a scheme of characteristic p. When G is a smooth commutative group scheme, as in the case of $\mathbb{Z}/p\mathbb{Z}$, then $H^1_{fppf}(X, G) = H^1_{\text{ét}}(X, G)$. Suppose that $X = \text{Spec}(A)$ is affine which means we have $H^1_{fppf}(X, \mathbb{G}_a) = 0$ (by Corollary 4.1.3 in [20]). Then our long exact sequence simplifies to:

$$... \to \Gamma(\text{Spec}(A), \mathbb{G}_a) \xrightarrow{x^p - x} \Gamma(\text{Spec}(A), \mathbb{G}_a) \to H^1_{fppf}(\text{Spec}(A), \mathbb{Z}/p\mathbb{Z}) \to 0$$

Hence, we have an isomorphism $H^1_{fppf}(X, \mathbb{Z}/p\mathbb{Z}) \simeq \Gamma(X, \mathbb{G}_a)/\text{Im}(x^p - x)$ which means that every étale $\mathbb{Z}/p\mathbb{Z}$ torsor $f : Y \to X$ is given by an equation $T^p - T = a$ where T is an indeterminate and $a \in \Gamma(X, \mathcal{O}_X) = \Gamma(X, \mathbb{G}_a)$. The function a is uniquely determined, modulo addition of elements of the form $b^p - b$.

Of course, the assumption that X is affine is very restrictive. In general we cannot assume $H^1_{fppf}(X, \mathbb{G}_a) = 0$ and so we have that the method only provides equations for the torsor locally. We now run through the same process to generate local equations for μ_p and α_p torsors in the same way.

To generate a local equation for a μ_p torsor, consider the short exact Kummer sequence:

$$1 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{x^p} \mathbb{G}_m \longrightarrow 1$$

It induces this long exact sequence in terms of cohomology groups:

$$\begin{aligned} 0 \to \Gamma(X, \mu_p) \to \Gamma(X, \mathbb{G}_m) &\xrightarrow{x^p} \Gamma(X, \mathbb{G}_m) \\ \to H^1_{fppf}(X, \mu_p) \to H^1_{fppf}(X, \mathbb{G}_m) &\xrightarrow{x^p} H^1_{fppf}(X, \mathbb{G}_m) \to ... \end{aligned}$$

where, as before, X is a scheme of characteristic p. In characteristic p, recall that μ_p (just like α_p) is not a smooth group scheme so $H^1_{fppf}(X, \mu_p)$ does not coincide with its étale cohomology group. $\mathbb{G}_m$ is smooth, however, and so the cohomology groups are the same

$$H^1_{fppf}(X, \mathbb{G}_m) = H^1_{\text{ét}}(X, \mathbb{G}_m)$$

and by Hilbert's 90 Theorem (Proposition 4.9 of Chapter 3 in [9]) we have that

$$H^1_{fppf}(X, \mathbb{G}_m) = H^1_{\text{ét}}(X, \mathbb{G}_m) \simeq \text{Pic}(X).$$

As before, let us assume $X = \text{Spec}(A)$ is affine and also that the Picard group $\text{Pic}(X)$ is trivial then $H^1(X, \mathbb{G}_m) = 0$ which means we are in the following situation:

$$... \to \Gamma(\text{Spec}(A), \mathbb{G}_m) \xrightarrow{x^p} \Gamma(\text{Spec}(A), \mathbb{G}_m) \to H^1_{fppf}(\text{Spec}(A), \mu_p) \to 0$$

Therefore, $H^1_{fppf}(X, \mu_p) \simeq \Gamma(\text{Spec}(A), \mathbb{G}_m)/\text{Im}(x^p)$ which means that every μ_p torsor $f : Y \to X = \text{Spec}(A)$ corresponds to an element $u \in \Gamma(X, \mathcal{O}_X)^* = \Gamma(\text{Spec}(A), \mathbb{G}_m)$ which is uniquely determined, modulo multiplication of elements of the form b^p, namely p-powers, so that equations of the form

$$T^p = u$$

where T is an indeterminate and $u \in \Gamma(X, \mathcal{O}_X)^*$, provide the explicit equations for the torsor f. The invertible function u is uniquely determined up to multiplication by p-powers.

For an α_p torsor, the short exact sequence

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{x^p} \mathbb{G}_a \longrightarrow 0$$

induces the long exact sequence

$$\begin{aligned}0 &\to \Gamma(X, \alpha_p) \to \Gamma(X, \mathbb{G}_a) \xrightarrow{x^p} \Gamma(X, \mathbb{G}_a) \\ &\to H^1_{fppf}(X, \alpha_p) \to H^1_{fppf}(X, \mathbb{G}_a) \xrightarrow{x^p} H^1_{fppf}(X, \mathbb{G}_a) \to \ldots\end{aligned}$$

As before, let us assume $X = \text{Spec}(A)$ is affine so that $H^1(X, \mathbb{G}_a) = 0$. We then have the exact sequence:

$$\ldots \to \Gamma(\text{Spec}(A), \mathbb{G}_a) \xrightarrow{x^p} \Gamma(\text{Spec}(A), \mathbb{G}_a) \to H^1_{fppf}(\text{Spec}(A), \alpha_p) \to 0$$

This means $H^1_{fppf}(X, \alpha_p) \simeq \Gamma(\text{Spec}(A), \mathbb{G}_a)/\text{Im}(x^p)$ which means that every α_p torsor $f : Y \to X = \text{Spec}(A)$ corresponds to an element $a \in \Gamma(X, \mathcal{O}_X) = \Gamma(\text{Spec}(A), \mathbb{G}_a)$ which is uniquely determined, modulo addition of elements of the form b^p, namely p-powers, so that equations of the form

$$T^p = a$$

where T is an indeterminate and $a \in \Gamma(X, \mathcal{O}_X)$, are the explicit equations for the torsor f.

Therefore, with X a scheme of characteristic p, we can give explicitly local equations for the $\mathbb{Z}/p\mathbb{Z}$, μ_p and α_p torsors. Note that when $X = \text{Spec}(A)$, we have that $\Gamma(X, \mathcal{O}_X) = A$ (by Proposition 3.1 of Chapter 2 in [7]) and, likewise, $\Gamma(X, \mathcal{O}_X)^\times = A^\times$.

Table 2.1: Local torsor equations for $\mathbb{Z}/p\mathbb{Z}$, μ_p and α_p

Group scheme	Local torsor equation	Modulo
$\mathbb{Z}/p\mathbb{Z}$	$T^p - T = a, \ \ a \in \Gamma(X, \mathcal{O}_X)$	$+ \, b^p - b$
μ_p	$T^p = u, \ \ u \in \Gamma(X, \mathcal{O}_X)^\times$	$\times \, b^p$
α_p	$T^p = a, \ \ a \in \Gamma(X, \mathcal{O}_X)$	$+ \, b^p$

Keeping notation consistent with [13], we define the conductor of a torsor.

Definition 2.4.1. *Let $f : V \to U$ be a G-torsor where U is a smooth, geometrically-connected curve over the field k with characteristic p. Let X and Y denote the smooth compactification of U and V respectively and $f' : Y \to X$ the induced morphism. By* *smooth compactification of U we mean the unique complete, smooth curve X containing U.*

1. $G = \mathbb{Z}/p\mathbb{Z}$

   There exists an open covering $\mathcal{U} := (U_s)_s$ of U and regular functions $a_s \in \Gamma(U_s, \mathcal{O}_X)$ such that above U_s (i.e. locally) the torsor f is given by an equation $T_s^p - T_s = a_s$, where a_s is defined up to addition of elements of the form $b_s^p - b_s$. Above a point $x \in X \backslash U$ and after an étale localisation, the morphism f' is given by an equation $T^p - T = t^m$ where m is an integer coprime to p and t is a uniformising parameter at x. Then the ***conductor*** *c of the torsor f at the point x is defined as*

$$c = \mathrm{cond}_x(f) = \begin{cases} 0, & m > 0 \text{ (f' is étale above x)} \\ -m, & \text{otherwise (f' ramifies above x)} \end{cases}$$

2. $G = \mu_p$

   There exists an open covering $\mathcal{U} := (U_s)_s$ of U and invertible functions $u_s \in \Gamma(U_s, \mathcal{O}_X)^*$ such that above U_s (i.e. locally) the torsor f is given by an equation $T_s^p = u_s$, where u_s is defined up to multiplication by a p-power b_s^p. The ***differential form*** *associated to f is $\omega := du_s/u_s$. For a point $x \in X$ we define the* ***conductor*** *of the torsor f at the point x as:*

$$c = \mathrm{cond}_x(f) = -(\mathrm{ord}_x(\omega) + 1)$$

3. $G = \alpha_p$

   There exists an open covering $\mathcal{U} := (U_s)_s$ of U and regular functions $a_s \in \Gamma(U_s, \mathcal{O}_X)$ such that above U_s (i.e. locally) the torsor f is given by an equation $T_s^p = a_s$, where a_s is defined up to addition by a p-power b_s^p. The ***differential form*** *associated to f is $\omega := da_s$. For a point $x \in X$ we define the* ***conductor*** *of the torsor f at the point x as:*

$$c = \mathrm{cond}_x(f) = -(\mathrm{ord}_x(\omega) + 1)$$

*For any of these three cases we refer to m as the **conductor variable** on which c depends.*

When the group scheme is radicial, the conductor doesn't depend on the choice of the differential form ω.

2.5 Degeneration of μ_p-torsors

Recall from the introduction that R is a complete DVR of mixed characteristic with uniformiser π, ζ is a primitive p-th root of unity contained in R and we set $\lambda = \zeta - 1$ where the prime $p > 0$ is the residue characteristic of R. Here K denotes the fraction field of R and v_K the π-adic valuation of K while k denotes the residue field of R, which has characteristic p and we assume to be algebraically closed. We will now detail the degeneration of μ_p torsors from zero characteristic to positive characteristic using the construction provided in [16].

Definition 2.5.1. *For a positive integer n, we define the commutative R-group scheme $\mathcal{G}_R^n$ by*

$$\mathcal{G}_R^n = \mathrm{Spec}(R[x, 1/(\pi^n x + 1)])$$

Its Hopf morphisms are given by

$$\widetilde{m} : R[x, 1/(\pi^n x + 1)]) \to R[x, 1/(\pi^n x + 1)]) \otimes_R R[x, 1/(\pi^n x + 1)])$$
$$x \mapsto x \otimes 1 + 1 \otimes x + \pi^n x \otimes x$$
$$\widetilde{\epsilon} : R[x, 1/(\pi^n x + 1)]) \to R$$
$$x \mapsto 0$$
$$\widetilde{\mathrm{inv}} : R[x, 1/(\pi^n x + 1)]) \to R[x, 1/(\pi^n x + 1)])$$
$$x \mapsto -x/(\pi^n x + 1)$$

Note the following interesting property of the generic and special fibres of $\mathcal{G}_R^n$.

Proposition 2.5.2 (1.2.2 and 1.2.3 in [16]). *We have the following equality of group schemes: $(\mathcal{G}_R^n)_K = \mathbb{G}_{m,K}$ and $(\mathcal{G}_R^n)_k = \mathbb{G}_{a,k}$.*

Proof.

$$\begin{aligned}
(\mathcal{G}_R^n)_K &:= \mathcal{G}_R^n \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K) \\
&= \mathrm{Spec}(R[x, 1/(\pi^n x + 1)]) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K) \\
&= \mathrm{Spec}(R[x, 1/(\pi^n x + 1)] \otimes_R K) \\
&= \mathrm{Spec}(K[x, 1/(\pi^n x + 1)]) \\
&\simeq \mathrm{Spec}(K[y, y^{-1}]) \\
&= \mathbb{G}_{m,K}
\end{aligned}$$

since tensoring with K effectively means we make π invertible. The change of variables $\pi^n x + 1 \mapsto y$ (so that $x \mapsto \frac{y-1}{\pi^n}$) is an isomorphism over K.

$$\begin{aligned}
(\mathcal{G}_R^n)_k &:= \mathcal{G}_R^n \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k) \\
&= \mathrm{Spec}(R[x, 1/(\pi^n x + 1)]) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k) \\
&= \mathrm{Spec}(R[x, 1/(\pi^n x + 1)]) \otimes_R k) \\
&= \mathrm{Spec}(k[x, 1/(0^n x + 1)])) \\
&= \mathrm{Spec}(k[x]) \\
&= \mathbb{G}_{a,k}
\end{aligned}$$

since here tensoring with k means that π becomes 0. $\square$

Consider the map $f : \mathcal{G}_R^n \to \mathcal{G}_R^{pn} = \mathrm{Spec}(R[x, 1/(\pi^{pn} x + 1)])$ defined in terms of its Hopf algebras given by:

$$R\left[x, \frac{1}{\pi^{pn} x + 1}\right] \longrightarrow R\left[x, \frac{1}{\pi^n x + 1}\right]$$

$$x \mapsto \frac{(\pi^n x + 1)^p - 1}{\pi^{pn}}$$

This surjective maps yields the exact sequence

$$0 \longrightarrow \mathcal{H}_{n,R} \longrightarrow \mathcal{G}_R^n \xrightarrow{\ f\ } \mathcal{G}_R^{pn} \longrightarrow 0$$

where $\mathcal{H}_{n,R} := \mathrm{Ker}(f : \mathcal{G}_R^n \to \mathcal{G}_R^{pn})$. Thus, $\mathcal{H}_{n,R}$ is a finite, flat R-group scheme of rank p due to f being an isogeny (i.e. surjective and has finite kernel). We have the commutative diagram of exact sequences in the fppf topology:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}_{n,R} \arrow[r] \arrow[d] & \mathcal{G}_R^n \arrow[r, "f"] \arrow[d] & \mathcal{G}_R^{pn} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mu_{p,R} \arrow[r] & \mathbb{G}_{m,R} \arrow[r, "x^p"] & \mathbb{G}_{m,R} \arrow[r] & 0 \end{tikzcd}$$

where $\mathcal{G}_R^n \to \mathbb{G}_{m,R}$ (and, likewise, for $\mathcal{G}_R^{pn} \to \mathbb{G}_{m,R}$) is given in terms of Hopf algebras by

$$R[u, u^{-1}] \to R\left[x, \frac{1}{\pi^n x + 1}\right]$$
$$u \to \pi^n x + 1$$

In fact, it is this morphism which gives the group law for the group scheme $\mathcal{G}_R^n$, in particular when this morphism is a group scheme homomorphism. Now, the upper exact sequence of this commutative diagram has the Kummer sequence as its generic fibre:

$$0 \longrightarrow \mu_{p,K} \longrightarrow \mathbb{G}_{m,K} \xrightarrow{x^p} \mathbb{G}_{m,K} \longrightarrow 0,$$

and the following exact sequence as its special fibre if $0 < n < v_K(\lambda)$:

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_{a,k} \xrightarrow{x^p} \mathbb{G}_{a,k} \longrightarrow 0,$$

or the Artin-Schreier sequence as its special fibre if $n = v_K(\lambda)$:

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_{a,k} \xrightarrow{x^p - x} \mathbb{G}_{a,k} \longrightarrow 0.$$

Note that K is a field of characteristic 0 and so here $\mu_p \simeq \mathbb{Z}/p\mathbb{Z}$ (non-canonically) over K.

The upper exact sequence of the commutative diagram gives us local equations for torsors $f : Y \to X$ under the group scheme $\mathcal{H}_{n,R}$ in the same way as we obtained them before for μ_p, α_p and $\mathbb{Z}/p\mathbb{Z}$. In particular, we consider the corresponding long exact sequence:

$$\begin{aligned} 0 &\to \Gamma(X, \mathcal{H}_{n,R}) \to \Gamma(X, \mathcal{G}_R^n) \xrightarrow{f} \Gamma(X, \mathcal{G}_R^{pn}) \\ &\to H^1_{fppf}(X, \mathcal{H}_{n,R}) \to H^1_{fppf}(X, \mathcal{G}_R^n) \xrightarrow{f} H^1_{fppf}(X, \mathcal{G}_R^{pn}) \to \ldots \end{aligned}$$

By Theorem 1.2 of [17] we can consider $H^1_{fppf}(X, \mathcal{G}_R^n) = 0$ when X satisfies some local conditions and so locally we have

$$... \to \Gamma(X, \mathcal{G}_R^n) \xrightarrow{f} \Gamma(X, \mathcal{G}_R^{pn}) \to H^1_{fppf}(X, \mathcal{H}_{n,R}) \to 0$$

Therefore,

$$H^1_{fppf}(X, \mathcal{H}_{n,R}) = \frac{\Gamma(X, \mathcal{G}_R^{pn})}{\mathrm{Im}(f)}$$

To every $\mathcal{H}_{n,R}$-torsor is associated an element of $\Gamma(X, \mathcal{G}_R^{pn})$ modulo elements of the form $\frac{(\pi^n x+1)^p-1}{\pi^{pn}}$ so that the equation

$$\frac{(\pi^n x+1)^p - 1}{\pi^{pn}} = a \Leftrightarrow (\pi^n T + 1)^p = \pi^{pn} a + 1$$

where T is an indeterminate and $a \in \Gamma(X, \mathcal{G}_R^{pn})$, are the explicit local equations for the torsor f. As $X = \mathrm{Spec}(A)$ is affine, the global sections $\Gamma(X, \mathcal{G}_R^{pn})$ is the set of all elements $a \in A$ such that $\pi^{pn}a + 1$ is a unit in A. This is because we view $\mathcal{G}_R^{pn}$ as a sheaf and the map $X \to \mathcal{G}_R^{pn}$ on the level of algebras is given by

$$R\left[x, \frac{1}{\pi^{pn}x+1}\right] \longrightarrow A$$

$$x \mapsto a$$

which means

$$\frac{1}{\pi^{pn}x+1} \mapsto \frac{1}{\pi^{pn}a+1}$$

and so we require $\pi^{pn}a + 1$ to be invertible.

When $n = v_K(\lambda)$, it can be shown that $(\pi^n T + 1)^p = \pi^{pn}a + 1$ reduces modulo π to $t^p - t = \bar{a}$ where $t = T \bmod \pi$ is an indeterminate and $\bar{a} = a \bmod \pi$. When $n < v_K(\lambda)$—that is, where $v_K(\lambda) = \frac{v_K(p)}{p-1}$ so that $v_K(p) - n(p-1) > 0$—it can also be shown that $(\pi^n T + 1)^p = \pi^{pn}a + 1$ reduces to $t^p = \bar{a}$. These are exactly the torsor equations we saw in characteristic p which makes sense since reducing modulo π means we are operating on the special fibre over k, a field of characteristic p.

We house this information in the following proposition:

Proposition 2.5.3. *The group scheme $\mathcal{H}_n$, defined to be the kernel of the group scheme homomorphism $\mathcal{G}^n \to \mathcal{G}^{pn}$, is a finite flat R-group scheme of rank p and is locally given by a torsor equation of the form*

$$(1+\pi^n T)^p = 1+\pi^{pn}a$$

where $0 < n \leq v_K(\lambda)$ and $a \in \Gamma(X, \mathcal{G}^{pn})$. When $n = v_K(\lambda)$, we have that the torsor equation for $\mathcal{H}_{v_K(\lambda)}$ is given locally by:

$$(1+\lambda T)^p = 1+\lambda^p a$$

The k-group scheme on the special fibre of $\mathcal{H}_n$ corresponds to α_p for $0 < n < v_K(\lambda)$ and to $\mathbb{Z}/p\mathbb{Z}$ for $n = v_K(\lambda)$.

We have demonstrated how the degeneration of μ_p-torsors from zero characteristic in K to positive characteristic p in k results in the natural appearance of α_p, μ_p, and $\mathbb{Z}/p\mathbb{Z}$ torsors on the special fibre. In particular, if $Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$, we have the following three cases.

When $n = v_K(\lambda)$:

$$\begin{tikzcd} B_K & B_R & B_k \\ A_K \arrow[u, "G_K=\mathbb{Z}/p\mathbb{Z}\simeq\mu_p"'] & A_R \arrow[u, "G_R=\mathcal{H}_{v_K(\lambda)}"'] & A_k \arrow[u, "G_k=\mathbb{Z}/p\mathbb{Z}"'] \end{tikzcd}$$

When $0 < n < v_K(\lambda)$:

$$\begin{tikzcd} B_K & B_R & B_k \\ A_K \arrow[u, "G_K=\mathbb{Z}/p\mathbb{Z}\simeq\mu_p"'] & A_R \arrow[u, "G_R=\mathcal{H}_n"'] & A_k \arrow[u, "G_k=\alpha_p"'] \end{tikzcd}$$

In our definition for $\mathcal{G}_R^n$ we specified that n must be positive. However, one can think of the last case as corresponding to when $n = 0$:

$$\begin{tikzcd}
B_K & B_R & B_k \\
A_K \arrow[u, "G_K=\mathbb{Z}/p\mathbb{Z}\simeq\mu_p"'] & A_R \arrow[u, "G_R=\mu_p"'] & A_k \arrow[u, "G_k=\mu_p"']
\end{tikzcd}$$

Chapter 3

Classification of torsors under group schemes of type $(p, ..., p)$

3.1 The existence of torsors under degree p group schemes

Consider the following result which establishes when we have a torsor structure and under what group scheme the torsor acts.

Theorem 3.1.1 (Proposition 2.4 in [13]; Theorem 5.0.1 in [23]). *Let X be a formal R-scheme of finite type which is normal, connected and flat over R. Suppose X is smooth. Let η be the generic point of the special fibre X_k and let $\mathcal{O}_\eta$ be the local ring of X at η, which is a discrete valuation ring with fraction field $K(X)$, the function field of X.*

Let $f_K : Y_K \to X_K$ be a μ_p-torsor and let $K(X) \to L$ be the corresponding extension of function fields such that the ramification index above $\mathcal{O}_\eta$ in the extension equals 1. Let δ denote the degree of the different above η in this extension.

Then we have that the torsor $f_K : Y_K \to X_K$ extends to a torsor $f : Y \to X$ under a finite and flat R-group scheme of rank p, with Y normal. In particular, the following cases occur:

1. *$\delta = 0$ in which case f is a torsor under the group scheme $\mathcal{H}_{v_K(\lambda),R}$ and f_k is an étale torsor under $\mathbb{Z}/p\mathbb{Z}$.*

2. *$0 < \delta < v_K(\lambda)$ in which case $\delta = v_K(\lambda) - n(p-1)$ for a certain integer $n \geq 1$,*

f is a torsor under $\mathcal{H}_{n,R}$ and f_k is a radicial torsor under α_p.

3. *$\delta = v_K(\lambda)$, f is a torsor under μ_p and f_k is a radicial torsor under μ_p.*

Note that we do not require X to be proper. The condition that the ramification index equals 1 is always satisfied after a finite extension of R due to the result in [2]. We will see in the next chapter that Theorem 3.1.1 also holds for the boundary $\mathrm{Spf}(R[[T]]\{T^{-1}\})$ of a formal germ of an R-curve.

3.2 The existence of torsors under $(p, ..., p)$ group schemes

In this section we prove how the results of degeneration from the degree p case extend to $(p, ..., p)$. We begin with two elementary results whose proofs follow directly from the relevant definitions. Both statements are certainly known but due to the nature of this chapter, it is perhaps good form to include them for completeness.

Lemma 3.2.1. *Let G_i be an S-group scheme for $i = 1, ..., n$. Then the product $G_1 \times G_2 \times ... \times G_n$ is an S-group scheme.*

Proof. For each i, G_i associates to every S-scheme T, the group $G_i(T)$. Let $G = G_1 \times G_2 \times ... \times G_n$. By setting $G(T) := G_1(T) \times G_2(T) \times ... \times G_n(T)$, we have that G inherits a group scheme structure. □

Lemma 3.2.2. *Let $X_i \to X$ be a torsor under the group scheme G_i for $i = 1, ..., n$. Then $X_1 \times_X X_2 \times_X ... \times_X X_n$ is a torsor under the group scheme $G_1 \times G_2 \times ... \times G_n$*

Proof. It suffices to prove this in the case of $n = 2$.

Let $X_i \xrightarrow{G_i} X$ be a torsor for $i = 1, 2$. Then, by definition, we have isomorphisms $X_i \times_X G_i \xrightarrow{\sim} X_i \times_X X_i$ defined by the map $(x_i, g_i) \mapsto (x_i, x_i g_i)$ and the maps $X_i \to X$ are faithfully flat.

It follows immediately that the map

$$(X_1 \times_X X_2) \times_X (G_1 \times G_2) \to (X_1 \times_X X_2) \times_X (X_1 \times_X X_2)$$

is also an isomorphism by taking $(x_1, x_2, g_1, g_2) \mapsto (x_1, x_2, x_1g_1, x_2g_2)$.

The property of being faithfully flat is preserved by base change so that the fibred product $X_1 \times_X X_2 \to X$ is also faithfully flat. See Chapter 4 of [7].

With these two conditions satisfied we have that $X_1 \times_X X_2 \xrightarrow{G_1 \times G_2} X$ is a torsor.

$\square$

The following is a result from [7] and can also be found, albeit in a slightly different form, in [8]. We take the opportunity to write out its proof in a little more detail than can be found in the original sources.

Theorem 3.2.3 (Lemma 1.18 of Section 4.1.1 in [7]; Lemma 1.1 in [8]). *Let X be an integral, flat R-scheme such that:*

- *its special fibre X_k is reduced*
- *its generic fibre X_K is normal*

Then X is normal.

Proof. We take X to be affine since normality is a local property by the fact that X being normal is equivalent to $\mathcal{O}_X(U)$ being a normal integral domain for every open $U \subset X$. So let $X = \mathrm{Spec}(A)$ for some commutative ring A. We need to show A is a normal integral domain. Now, A is of course an integral domain since X is affine and integral (note that a scheme X is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for every U). It remains to prove A is normal: we assume $\alpha \in \mathrm{Frac}(A)$ is integral over A and seek to show that $\alpha \in A$.

Note that X is flat over R which ensures that $R \hookrightarrow A$ is injective which means that elements of R, including its uniformiser π, can be viewed as elements of A.

The special fibre X_k is reduced:

$$\begin{aligned} X_k &= X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k) \\ &= \mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/(\pi)) \\ &= \mathrm{Spec}\left(A \otimes_R R/(\pi)\right) \\ &= \mathrm{Spec}\left(A/(\pi)\right) \end{aligned}$$

This implies that $A/(\pi)$ is a reduced ring (i.e. without non-zero nilpotents) by the fact X_k is affine.

The generic fibre X_K is normal:

$$\begin{aligned} X_K &= X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K) \\ &= \mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R[1/\pi]) \\ &= \mathrm{Spec}\left(A \otimes_R R[1/\pi]\right) \\ &= \mathrm{Spec}\left(A[1/\pi]\right) \end{aligned}$$

This implies that $A[1/\pi]$ is integrally closed by the fact X_K is affine.

We have the following integral relation for α:

$$\alpha^n + a_{n-1}\alpha^{n-1} + ... + a_1\alpha + a_0 = 0$$

where $a_i \in A$. However, $\alpha \in A[1/\pi]$ since $A[1/\pi]$ is integrally closed. As an element of $\mathrm{Frac}(A)$, we can write $\alpha = a/b$ where $a, b \in A$ but, as $\alpha \in A[1/\pi]$, this means we can go further and write $\alpha = a/\pi^r$ for some $a \in A$ and $r \in \mathbb{Z}$. We can additionally suppose that $a \notin (\pi)$ by letting the denominator absorb any instances of π that might have occurred in a. Our integral relation then becomes:

$$(a/\pi^r)^n + a_{n-1}(a/\pi^r)^{n-1} + ... + a_1(a/\pi^r) + a_0 = 0$$

Multiplying through by π^{rn} we have that:

$$a^n + a_{n-1}a^{n-1}\pi^r + ... + a_0\pi^{rn} = 0$$

Suppose $r > 0$. Then, modulo (π), we have that $a^n = 0 \bmod (\pi)$ so $a^n = 0$ in the special fibre but as the special fibre X_k is reduced, $a = 0$ in X_k. Therefore, $a \in (\pi)$ which is a contradiction. So we must have that $r \leq 0$ which means we can express $\alpha = a\pi^j$ where $j \geq 0$ and so $\alpha \in A$, as required, by the fact that both $a, \pi \in A$. □

With these results to hand, we now describe the precise setting we will treat. Let X be a formal R-scheme of finite type which is normal, geometrically connected and flat over R. Suppose X is smooth. Recall R is a complete discrete valuation ring with uniformiser π. Let $K = \mathrm{Frac}(R)$ denote the fraction field of R and $k = R/\pi$ the residue field of R which we assume to be algebraically closed.

Let X_K denote the generic fibre $X \times_R K$ and suppose $X_{i,K} \to X_K$ are μ_p torsors which are generically disjoint for $i = 1, ..., n$. Then, the picture on the generic fibre is as follows:

$$
\begin{tikzcd}
 & & \widetilde{X_K} := X_{1,K} \times ... \times X_{n,K} \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
X_{1,K} \arrow[drr, "G_{1,K}"'] & X_{2,K} \arrow[dr, "G_{2,K}"] & X_{3,K} \arrow[d] & ... \arrow[dl] & X_{n,K} \arrow[dll, "G_{n,K}"] \\
 & & X_K & &
\end{tikzcd}
$$

where $G_{i,K} = \mu_p$ are étale group schemes and $\widetilde{X_K}$ is the fibre product of $X_{i,K}$ for $i = 1, ..., n$.

Now we consider the normalisation of this system of extensions:

$$
\begin{tikzcd}
 & & \widetilde{X} \arrow[d] \arrow[ddll, bend right] \arrow[ddl, bend right] \arrow[ddr, bend left] \arrow[ddrr, bend left] & & \\
 & & X_1 \times_X X_2 \times_X ... \times_X X_n \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
X_1 \arrow[drr, "G_1"'] & X_2 \arrow[dr, "G_2"] & X_3 \arrow[d, "G_3"] & ... \arrow[dl] & X_n \arrow[dll, "G_n"] \\
 & & X & &
\end{tikzcd}
$$

where:

- $\widetilde{X}$ is the normalisation of X in $\widetilde{X_K}$ and X_i is the normalisation of X in $X_{i,K}$
- $X_i \to X$ is a torsor under G_i by Theorem 3.1.1
- $X_1 \times_X X_2 \times_X ... \times_X X_n$ is the fibre product of the X_i over X and, by Lemma 3.2.2, $X_1 \times_X X_2 \times_X ... \times_X X_n \to X$ is a torsor under the group scheme $G_1 \times G_2 \times ... \times G_n$.

Note that opposite group schemes are the same by base change

$$\begin{tikzcd} & \widetilde{X} \arrow[d] & \\ & X_1 \times_X X_2 \arrow[dl, "G_2"'] \arrow[dr, "G_1"] & \\ X_1 \arrow[dr, "G_1"'] & & X_2 \arrow[dl, "G_2"] \\ & X & \end{tikzcd}$$

This is because by base change we have the fibred product diagram:

$$\begin{tikzcd} X_1 \times_X X_2 \arrow[r, "G_1"] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r, "G_1"'] & X \end{tikzcd} \qquad \begin{tikzcd} X_1 \times_X X_2 \arrow[r, "G_2"] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r, "G_2"'] & X \end{tikzcd}$$

The key question we are interested in understanding in this chapter is when $\widetilde{X}$ has the structure of a torsor over X.

Proposition 3.2.4. *Let G be a finite, flat commutative group scheme over R whose generic fibre is a product of group schemes of the form*

$$G_K = G'_1 \times ... \times G'_n$$

where G'_i are finite commutative K-group schemes of rank p. Then G is a product of finite commutative R-group schemes of the form

$$G = G_1 \times ... \times G_n$$

where $G_{i,K} = G_i \times_R K = G'_i$.

Proof. It suffices to treat the case of $n = 2$. We have $G_K = G'_1 \times G'_2$ and need to show $G \simeq G_1 \times G_2$ where $G_{i,K} = G'_i$ for $i = 1, 2$. Let $G_i :=$ schematic closure of G'_i in G where the schematic closure is the smallest closed subscheme of G with G'_i as its generic fibre. Therefore, G_1, G_2 are contained in G. All these group schemes are commutative finite group schemes over a discrete valuation ring R which means we can treat them as abstract groups in so far as being able to take quotients; something which is not true in general. See [10] for more details.

We have a short exact sequence:

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G/G_1 \longrightarrow 1$$

and, of course, likewise:

$$1 \longrightarrow G_2 \longrightarrow G \longrightarrow G/G_2 \longrightarrow 1$$

Now, we can build this sequence to include $G_2 \hookrightarrow G$:

$$\begin{tikzcd} & & G_2 \arrow[d] \arrow[dr, dashed] & & \\ 1 \arrow[r] & G_1 \arrow[r] & G \arrow[r] & G/G_1 \arrow[r] & 1 \end{tikzcd}$$

It remains to show $G_2 \to G/G_1$ is an isomorphism.

The map $G \to G/G_1$ is a finite map because it is a quotient by a finite group. It is also a proper morphism. $G_2 \to G$ is a closed immersion so is proper since closed immersions are always proper morphisms. As the maps $G_2 \to G$ and $G \to G/G_1$ are both proper, their composite $G_2 \to G/G_1$ must be proper.

The map $G_2 \to G/G_1$ is also quasi-finite because the fibres are finite because the fibres of $G_2 \to G/G_1$ are contained in the fibres of $G \to G/G_1$ by the fact G_2 is contained in G. A morphism is finite if and only if it is proper and quasi-finite. Therefore, $G_2 \to G/G_1$ is finite. A finite, flat morphism of degree 1 is an isomorphism.

The map $G_2 \to G/G_1$ is injective because its kernel is trivial due to the fact that on the generic fibre the kernel is trivial: $G_1 \cap G_2 = \{1\}$ since $(G_1 \cap G_2)_K = G_1' \cap G_2' = \{1\}$ where $G_1 \cap G_2$ is a finite subgroup scheme of G. The map is also surjective and, as such, an isomorphism.

Similarly, $G_1 \to G/G_2$ is an isomorphism in the following diagram

$$\begin{tikzcd} & & G_1 \arrow[d] \arrow[dr, dashed] & & \\ 1 \arrow[r] & G_2 \arrow[r] & G \arrow[r] & G/G_2 \arrow[r] & 1 \end{tikzcd}$$

Therefore, $G \simeq G_1 \times G_2$, as required.

□

The following important theorem establishes conditions for the existence of the structure of a torsor.

Theorem 3.2.5. *Let X be smooth, geometrically connected R-scheme of any dimension. Assume the special fibre $\widetilde{X}_k$ of $\widetilde{X}$ is reduced. Then the following statements are equivalent*

1. *$\widetilde{X}$ is a torsor over X under some group scheme G (necessarily $G = G_1 \times ... \times G_n$),*
2. *$\widetilde{X} = X_1 \times_X X_2 \times_X ... \times_X X_n$ (i.e. $X_1 \times_X X_2 \times_X ... \times_X X_n$ is normal),*
3. *$(X_1 \times_X X_2 \times_X ... \times_X X_n)_k$ is reduced.*

Proof. $(1 \Rightarrow 2)$

Assume $\widetilde{X}$ is a torsor over X under some group scheme G. It suffices to treat the case $n = 2$.

$$\begin{tikzcd}
 & \widetilde{X} \arrow[d] \arrow[ddl, bend right, "G_2'"'] \arrow[ddr, bend left, "G_1'"] \arrow[ddd, bend right=40, "G"] & \\
 & X_1 \times_X X_2 \arrow[dl, "G_2"'] \arrow[dr, "G_1"] & \\
X_1 \arrow[dr, "G_1"'] & & X_2 \arrow[dl, "G_2"] \\
 & X &
\end{tikzcd}$$

By Proposition 3.2.4 we have the following picture on the generic fibre

$$\begin{tikzcd}
 & \widetilde{X}_K = X_{1,K} \times_X X_{2,K} \arrow[dl, "G_{2,K}"'] \arrow[dr, "G_{1,K}"] \arrow[dd, "G_K"] & \\
X_{1,K} \arrow[dr, "G_{1,K}"'] & & X_{2,K} \arrow[dl, "G_{2,K}"] \\
 & X_K &
\end{tikzcd}$$

where $G_1' = (G_{1,K})^{\text{schematic closure}}$ and $G_2' = (G_{2,K})^{\text{schematic closure}}$ so that $G = G_1' \times G_2'$.

We want to show $\widetilde{X} = X_1 \times_X X_2$ and that reduces to showing $G = G_1 \times_R G_2$ or indeed just $G_1 = G_1'$ and $G_2 = G_2'$. In fact, we only need to demonstrate that one of these two equalities holds since if $G_2 = G_2'$, we have the following picture:

$$\begin{tikzcd} & \widetilde{X} \arrow[d] \arrow[ddl, bend right, "G_2"'] \\ & X_1 \times_X X_2 \arrow[dl, "G_2"'] \\ X_1 & \end{tikzcd}$$

When this is the case, $\widetilde{X} \simeq X_1 \times_X X_2$ because if one has two torsors sitting above the same object, X_1, under the same group scheme, G_2, and we have a map $\widetilde{X} \to X_1 \times_X X_2$ between them which is compatible with the structure of torsor, it must imply they are isomorphic (a consequence of Lemma 4.1.2 of [23]). Otherwise the acting group schemes must be different in order to have two such different torsors but that is not the case here. If $G_2 = G_2'$ is established, the other equality, in this case $G_1 = G_1'$, will follow automatically.

We have two short exact sequences. The first by taking the quotient by G_2'

$$1 \longrightarrow G_2' \longrightarrow G \longrightarrow G_1 \simeq G/G_2' \longrightarrow 1$$

and the second by taking the quotient by G_1'

$$1 \longrightarrow G_1' \longrightarrow G \longrightarrow G_2 \simeq G/G_1' \longrightarrow 1$$

The reason why $G_1 \simeq G/G_2'$ and $G_2 \simeq G/G_1'$ stems from the fact that $\widetilde{X}$ has a torsor structure for everything sitting below it. This means that every intermediate group scheme must be a quotient by some subgroup scheme. The only choice for G/G_2' where $X_1 = \widetilde{X}/G_2'$ must be G_1 and, likewise, the only choice for G/G_1' where $X_2 = \widetilde{X}/G_1'$ is G_2.

The maps from $G \to G_1$, $G \to G_2$ are finite maps. We can naturally define another finite map ϕ induced by these two maps from G to the fibred product over R of G_1 and G_2.

$$1 \longrightarrow \mathrm{Ker}(\phi) \longrightarrow G \xrightarrow{\phi} G_1 \times_R G_2 \longrightarrow 1$$

We want to show the map $G \to G_1 \times_R G_2$ is an isomorphism. We have $\mathrm{Ker}(\phi) = G_1' \cap G_2'$ since the kernel of this map are the elements which map trivially to G_1 and the elements which map trivially to G_2. However, $G_1' \cap G_2' = \{1\}$ since $G = G_1' \times G_2'$ by Proposition 3.2.4 and, therefore, $\mathrm{Ker}(\phi) = \{1\}$ which means $G \to G_1 \times_R G_2$ is a closed immersion. Finally, note that G and $G_1 \times_R G_2$ have the same rank as group schemes which means $G \simeq G_1 \times_R G_2$, as required.

($2 \Rightarrow 3$) This is almost an empty statement as 2 plays no role and the reason $(X_1 \times_X X_2 \times_X ... \times_X X_n)_k$ is reduced is simply because we impose that $\widetilde{X}_k$ is reduced and $\widetilde{X}$ equals $X_1 \times_X X_2 \times_X ... \times_X X_n$.

($3 \Rightarrow 1$) $(X_1 \times_X X_2 \times_X ... \times_X X_n)_k$ is reduced and $(X_1 \times_X X_2 \times_X ... \times_X X_n)_K$ is normal. So by Theorem 3.2.3, $X_1 \times_X X_2 \times_X ... \times_X X_n$ is normal and we can write $\widetilde{X} = X_1 \times_X X_2 \times_X ... \times_X X_n$. We know $X_1 \times_X X_2 \times_X ... \times_X X_n$ is a torsor under the group scheme $G_1 \times G_2 \times .... \times G_n$ by Lemma 3.2.2. So now $\widetilde{X}$ is a torsor under the group scheme $G_1 \times G_2 \times .... \times G_n$.

□

Note that this proof is also valid for equal characteristic and the statement is true for commutative finite flat group schemes of any rank.

Now we can say something more precise in the case of dimension 1:

Theorem 3.2.6. *Let X be a smooth formal affine connected R-**curve**. Then $\widetilde{X}$ is a torsor over X if and only if at least $n-1$ of the finite, flat R-group schemes G_i of rank p acting on $X_i \to X$ are étale.*

Proof. This will be a proof by induction on n.

($\Rightarrow$) Suppose $\widetilde{X}$ is a torsor over X; in which case, $\widetilde{X} = X_1 \times_X X_2 \times_X ... \times_X X_n$.

Base case: The base case pertains to $n = 2$. We assume $\widetilde{X} = X_1 \times_X X_2$ and prove that at least one of the two group schemes are étale.

$$\begin{tikzcd} & \widetilde{X} = X_1 \times_X X_2 \arrow[dl, "G_2"'] \arrow[dr, "G_1"] & \\ X_1 \arrow[dr, "G_1"'] & & X_2 \arrow[dl, "G_2"] \\ & X & \end{tikzcd}$$

Suppose, for contradiction, that G_1 and G_2 are both non-étale group schemes. We have that X is an affine curve so is of the form $X = \mathrm{Spf}\,(A)$. Take a closed point x of X, localise and complete at x and then from its germ at x, which is an open disc, we obtain the corresponding boundary of the disc. We base change this situation from the boundary back to the open disc and then again base change back to our curve X.

$$\begin{tikzcd} X = \mathrm{Spf}\,(A) & \mathrm{Spf}\left(\widehat{\mathcal{O}}_{X,x}\right) = \mathrm{Spf}\,(R[[T]]) \arrow[l] & \mathrm{Spf}\left(R[[T]]\{T^{-1}\}\right) \arrow[l] \arrow[ll, bend right] \end{tikzcd}$$

following from

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left] & R[[T]] = \widehat{\mathcal{O}}_{X,x} \arrow[r] & R[[T]]\widehat{}_{(\pi)} = R[[T]]\{T^{-1}\} \end{tikzcd}$$

If we do not have the structure of torsor above this boundary then this means that globally on the curve we do not have it either, for if we had a torsor globally then on restriction to the boundary we would also have a torsor. It is shown in Theorem 4.2.3 in the next chapter that above the boundary $\mathrm{Spf}\left(R[[T]]\{T^{-1}\}\right)$ we only have a structure of a torsor when at least one of the two group schemes are étale. This resolves the matter but the underpinning idea is as follows. As they are not étale group schemes, the special fibres of G_1 and G_2 are either μ_p or α_p group schemes. So, for example, they are given by equations of the form $Z_1 = 1 + \pi^{(\dots)} f(T)$ and $Z_2^p = T^m$. However, when we base change the equation $Z_1 = 1 + \pi^{(\dots)} f(T)$, the torsor above X_2 is given by the same equation but the special fibre is not reduced because we extract a p-th root of t residually and so after base change $f(T)$, which was not a p-power modulo π originally, becomes a p-power modulo π because t becomes a p-power modulo π. This means that the torsor is under the same group scheme and given by the same equation but the special fibre of $X_1 \times_X X_2$ is not reduced, in so far as the equation is not the equation corresponding to the normalisation. And so we obtain the contradiction $\widetilde{X} \neq X_1 \times_X X_2$. Note that this is unlike with an étale extension, where nothing becomes a p-power modulo π, so that we can base change the other equation and even if it is radicial we obtain the equation of the normalisation.

Inductive hypothesis: For $\widetilde{X}_1 = X_1 \times_X X_2 \times_X \dots \times_X X_{n-1}$, at least $n-2$ of the corresponding G_i are étale. In particular and without loss of generality, we assume $G_i = \mathbb{Z}/p\mathbb{Z}$ for $1 \leq i \leq n-2$.

Inductive step: We have the following picture for our inductive step (the case for n):

$$\begin{tikzcd}
& & & \widetilde{X} \arrow[dl] \arrow[dr] & & \\
& & \widetilde{X}_1 \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \widetilde{X}_2 \arrow[d] \arrow[dr] & \\
X_1 \arrow[drrr, "\mathbb{Z}/p\mathbb{Z}"'] & X_2 \arrow[drr, "\mathbb{Z}/p\mathbb{Z}"] & \dots & X_{n-2} \arrow[d, "\mathbb{Z}/p\mathbb{Z}"'] & X_{n-1} \arrow[dl, "G_{n-1}"'] & X_n \arrow[dll, "G_n"] \\
& & & X & &
\end{tikzcd}$$

Suppose, for contradiction, that neither G_{n-1} nor G_n is étale. This would mean that $\widetilde{X}_2 = X_{n-1} \times_X X_n$ does not have the structure of a torsor (see above discussion for $n = 2$) which in turn suggests that $\widetilde{X}$ does not have the structure of a torsor since $\widetilde{X}_2$ sits below $\widetilde{X}$. Of course, $\widetilde{X}$ is a torsor and so this is a contradiction. Therefore, at least one of G_{n-1} and G_n is étale, as required.

($\Leftarrow$) Suppose at least $n-1$ of the G_i are étale.

Base case: The base case pertains to $n = 2$. We assume at least one of the two group schemes are étale and prove that $\widetilde{X} = X_1 \times_X X_2$

$$
\begin{tikzcd}
& \widetilde{X} \arrow[d] & \\
& X_1 \times_X X_2 \arrow[dl, "G_i"'] \arrow[dr, "\text{étale}"] & \\
X_1 \arrow[dr, "\text{étale}"'] & & X_2 \arrow[dl, "G_i"] \\
& X &
\end{tikzcd}
$$

X_1 is normal because X is normal and $X_1 \to X$ is étale (since an étale cover above something that is normal is normal itself). Likewise, $X_1 \times_X X_2$ is normal because the special fibre of $X_1 \times_X X_2$ is reduced and the map $X_1 \times_X X_2 \to X_2$ is étale which means the generic fibre of $X_1 \times_X X_2$ is normal. This implies $\widetilde{X} = X_1 \times_X X_2$ and $\widetilde{X}$ has torsor structure.

Inductive hypothesis: Suppose at least $n-1$ of the G_i are étale.

Inductive step: Without loss of generality, set $G_i = \mathbb{Z}/p\mathbb{Z}$ for $1 \leq i \leq n-1$ which means we have an étale $\mathbb{Z}/p\mathbb{Z} \times ... \times \mathbb{Z}/p\mathbb{Z}$ torsor between $X_1 \times_X ... \times_X X_{n-1}$ and X. We will show that $\widetilde{X} = \widetilde{X}_1 \times_X X_n = X_1 \times_X X_2 \times_X ... \times_X X_n$. The picture is as follows with :

$$
\begin{tikzcd}
& & \widetilde{X} \arrow[d] & \\
& & X_1 \times_X ... \times_X X_n \arrow[dl] \arrow[ddr, no head] & \\
& \widetilde{X}_1 = X_1 \times_X ... \times_X X_{n-1} \arrow[dl] \arrow[dd, bend left] \arrow[dr, no head] & & \\
X_1 \arrow[dr, "\mathbb{Z}/p\mathbb{Z}"'] & ... & X_{n-1} \arrow[dl, "\mathbb{Z}/p\mathbb{Z}"'] & X_n \arrow[dll, "G_n"] \\
& X & &
\end{tikzcd}
$$

By base change we have:

$$\begin{tikzcd}
 & \widetilde{X} \arrow[d] & \\
 & \widetilde{X}_1 \times X_n \arrow[dl, "G_n"'] \arrow[dr, "\text{étale}"] & \\
\widetilde{X}_1 \arrow[dr, "\text{étale}"'] & & X_n \arrow[dl, "G_n"] \\
 & X &
\end{tikzcd}$$

where $\widetilde{X}_1 = X_1 \times_X ... \times_X X_{n-1}$ and the opposite group schemes are equal by base change. Also $\widetilde{X}_1 \times X_n$ is in fact the same as $\widetilde{X}$ by the same reasoning as explained in the base case and so

$$\widetilde{X} = \widetilde{X}_1 \times_X X_n = X_1 \times_X X_2 \times_X ... \times_X X_n,$$

as required. $\square$

Example 3.2.7. *This last theorem is not valid for R-schemes of relative dimension* $\dim \geq 2$. *Take, for example,* $X = \mathrm{Spf}(A)$ *where* $A = R < T_1, T_2 >$ *and take* $G_1 = G_2 = \mu_p$*, neither being étale, where one equation is given by* $Z_1^p = T_1$ *and the other is given, in a different variable, by* $Z_2^p = T_2$.

$$\begin{tikzcd}
 & \widetilde{X} = X_1 \times X_2 \arrow[dl, "\mu_p"', "Z_2^p = T_2"] \arrow[dr, "\mu_p", "Z_1^p = T_1"'] & \\
X_1 \arrow[dr, "Z_1^p = T_1", "\mu_p"'] & & X_2 \arrow[dl, "Z_2^p = T_2"', "\mu_p"] \\
 & X = \mathrm{Spf}\,(R < T_1, T_2 >) &
\end{tikzcd}$$

After base change they remain μ_p *torsors because the variables* T_1 *and* T_2 *do not become p-powers after base change so we obtain the equation of the normalisation and have that* $X_1 \times X_2 = \widetilde{X}$ *and* $\widetilde{X}$ *is a torsor over X with group* $\mu_p \times \mu_p$.

Note that cases where $\widetilde{X} \neq X_1 \times_X X_2 \times_X ... \times_X X_n$ *do exist, as we will see in the very next chapter where* $A = R[[T]]\{T^{-1}\}$.

As a group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ has $p+1$ subgroups of degree p and, accordingly, a Galois cover of curves with such a Galois group has $p+1$ quotient covers of degree p. In our situation the same principle extends to torsors and so in the case of $n = 2$ we have a natural corollary to Theorem 3.2.6 which tells us how many of these torsor quotients must be étale if we have torsor structure overall:

Corollary 3.2.8. *Let $\widetilde{X} \to X$ be a torsor of type (p, p) where X satisfies the conditions of Theorem 3.2.6 and $\widetilde{X}$ is normal. Let $X_i \to X$ denote its rank p quotient torsors where $i = 1, ..., p+1$. Then at least p of the quotient torsors are étale.*

Proof. As a group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ contains $p^2 - 1$ elements of order p, each of which can generate a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z}$. However, each of the $p-1$ order p elements in these degree p subgroups generate the same group. This implies there are total of $p^2 - 1/p - 1 = p + 1$ distinct quotient subgroups.

As previously mentioned, we work with commutative finite group schemes over a discrete valuation ring which means we can treat them as abstract groups in so far as being able to take quotients. Therefore, if $\widetilde{X} \to X$ is a torsor under a finite group scheme G of type (p, p), we can consider the rank p torsors under the group schemes obtained by taking quotients of the $p+1$ subgroups of the group scheme G. Note, $X_i \to X$ must be a torsor since it sits inside the torsor $\widetilde{X} \to X$.

Now, by contrapositive, suppose at least two of the torsors had group schemes which were not étale. Then, by Theorem 3.2.6, we have a contradiction to $\widetilde{X} \to X$ being a torsor. □

Chapter 4

The conductor on the boundary of the formal fibre

4.1 The degree p case

Let A be the ring of the restricted Laurent series $R[[T]]\{T^{-1}\}$ over R and let $f : \mathrm{Spf}\,(B) \to \mathrm{Spf}\,(A)$ be a non-trivial torsor under a finite flat R-group scheme of rank p. The ramification index of this extension can be assumed to equal 1, a result due to [2] in which Epp proves that after a finite extension this is true for DVRs. Then Proposition 2.3 of [15] gives us the structure of a torsor on the boundary of the formal fibre $\mathrm{Spf}\,(A)$ under the three group schemes, μ_p, $\mathcal{H}_n$ where $0 < n < v_K(\lambda)$, and $\mathcal{H}_{v_K(\lambda)}$.

We summarise below the pertinent information concerning these torsors; in particular, their presentation over R including their explicit local torsor equations and corresponding conductor. Recall that c denotes the **conductor** while m denotes the **conductor variable** on which c depends, owing to the relationship

$$c = -m.$$

It is in particular the proof of Proposition 2.3 of [15] that provides the following details in the three occurring cases:

(a) For the **group scheme $\boldsymbol{\mu_p}$** where $\delta = v_K(p)$, the torsor equation is of the form

$$Z^p = u$$

where $u = \sum_{i\in\mathbb{Z}} a_i T^i$ is a unit element in $A = R[[T]]\{T^{-1}\}$ such that modulo π it is not a p-power.

Reducing modulo π the equation is given on the special fibre by $z^p = \overline{u}$ where $\overline{u} = \sum_{i\geq l} \overline{a}_i t^i \in k((t))$ for some integer l with $\overline{a}_l \neq 0$ and which is defined up to multiplication by p-power. There are two cases to consider:

(a1) $\gcd(l, p) = 1$, in which case the conductor variable $m = 0$.

(a2) $\gcd(l, p) > 1$, in which case l is divisible by p so that $l = \alpha p^n$ for some $\alpha, n \in \mathbb{Z}$ and such that $\gcd(\alpha, p) = 1$ and $n \neq 0$. In this case, after a possible finite extension of R, the torsor equation can be written in the form:

$$Z^p = 1 + a_m T^m + \sum_{\substack{v_K(a_i)=0\\ i>m}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i$$

where $m := \min\{i | v_K(a_i) = 0, \gcd(i - l, p) = 1\} > 0$

Simplified form: After a change of the parameter T and a possible finite extension of R, the torsor equation $Z^p = u$ can be reduced to either the form

$$\textbf{(a1)} \quad Z^p = T^h$$

where $h \in \mathbb{F}_p^\times$ or of the form

$$\textbf{(a2)} \quad Z^p = 1 + T^m$$

where m is as defined above for these two cases. The conductor is given by

$$(a1) \;\; c = 0 \qquad\qquad (a2) \;\; c = -m$$

(b) For the **group scheme $\mathcal{H}_n$** (as given in Proposition 2.5.3) where $0 < n < v_K(\lambda)$ and $\delta = v_K(p) - n(p-1)$, the torsor equation is of the form

$$(1 + \pi^n Z)^p = 1 + \pi^{np} u$$

where $u = \sum_{i\in\mathbb{Z}} a_i T^i$ is a unit element in $A = R[[T]]\{T^{-1}\}$ such that modulo π it is not a p-power.

Reducing modulo π, on the special fibre the group scheme is α_p and the torsor is given by an equation $z^p = \overline{u}$ where $\overline{u} = \sum_{i\geq l} \overline{a}_i t^i \in k((t))$ for some integer l with $\overline{a}_l \neq 0$ and which is defined up to addition by p-power. We define

$$m := \min\{i | v_K(a_i) = 0, \gcd(i,p) = 1\} \in \mathbb{Z}$$

and then one can take

$$u = \sum_{\substack{v_K(a_i)=0 \\ i\geq m}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i$$

Simplified form: After a change of the parameter T and a possible finite extension of R, the torsor equation can be reduced to the form

$$Z^p = 1 + \pi^{np} T^m$$

where m is as defined above. The conductor is given by

$$c = -m$$

(c) For the **group scheme** $\boldsymbol{\mathcal{H}}_{v_K(\lambda)}$ (as given in Proposition 2.5.3) where $\delta = 0$, the torsor equation is of the form

$$(1 + \lambda Z)^p = 1 + \lambda^p u$$

where $u = \sum_{i\in\mathbb{Z}} a_i T^i$ is a unit element in $A = R[[T]]\{T^{-1}\}$.

On the special fibre the group scheme is $\mathbb{Z}/p\mathbb{Z}$ and the torsor is given by an equation $z^p - z = \overline{u}$ where $\overline{u} = \sum_{i\geq l} \overline{a}_i t^i$ for some integer l with $\overline{a}_l \neq 0$ and which is defined up to addition by an Artin-Schreier element of the form $b^p - b$. In fact, after removing sufficient elements of this form, $\overline{u}$ can be represented as:

$$\overline{u} = \overline{a}_m t^m + \overline{a}_{m+1} t^{m+1} + ... + \overline{a}_{-1} t^{-1} = \sum_{i=m}^{-1} \overline{a}_i t^i$$

where $\overline{a}_m \neq 0$ and $m < 0$ is the conductor variable such that $\gcd(m, p) = 1$ and $v_K(a_m) = 0$.

Simplified form: After a change of the parameter T and a possible finite extension of R, the torsor equation over R can be simplified to the form

$$Z^p = 1 + \lambda^p T^m$$

where m is as defined above. The conductor is given by

$$c = -m$$

The condition $\gcd(m, p) = 1$ implies that the conductor variable m is never a multiple of p. The condition that $v_K(a_i) = 0$ means $\overline{a}_i \neq 0$ in the residue field k. Finally, note that the description given for an $\mathcal{H}_n$ torsor $(1 + \pi^n Z)^p = 1 + \pi^{np} T^m$ and its degree of different $\delta = v_K(p) - n(p-1)$ actually hold in the other two cases by substituting their corresponding n value. For $\mathcal{H}_{v_K(\lambda)}$, we have that $n = v_K(\lambda)$ so that

$$(1 + \pi^{v_K(\lambda)} Z)^p = 1 + \pi^{p v_K(\lambda)} T^m \Leftrightarrow (1 + \lambda Z)^p = 1 + \lambda^p T^m,$$

where strictly speaking $\pi^{v(\lambda^p)}$ equals λ^p multiplied by a unit, and

$$\delta = v_K(p) - n(p-1) = v_K(p) - v_K(\lambda)(p-1) = 0.$$

Likewise for μ_p, we take $n = 0$ so that

$$(1 + \pi^0 Z)^p = 1 + \pi^0 T^m \Leftrightarrow (1 + Z)^p = 1 + T^m,$$

which we recognise, after a change of variables, as the μ_p (a2) simplified form torsor equation, and

$$\delta = v_K(p) - n(p-1) = v_K(p).$$

This observation will play a role in Theorem 4.2.7.

4.2 The degree (p, p) case

In this section A continues to denote the ring of the restricted Laurent series $R[[T]]\{T^{-1}\}$ and we take $X_b = \mathrm{Spf}\,(A)$ to be the formal boundary. We take the opportunity to remind the reader that we use the notation X_b as, for the purposes of context, this is the boundary of $X = \mathrm{Spf}\,(R[[T]])$ or, for that matter, any germ of an R-curve. Suppose that we have two disjoint non-trivial degree p Galois covers $f_{i,K} : (X_{i,b})_K \to (X_b)_K$ where $(X_b)_K$ denotes the generic fibre of X_b for $i = 1, 2$. By taking $(Y_b)_K$ to be the compositum of these covers, we have the following picture:

$$\begin{tikzcd}
 & (Y_b)_K = (X_{1,b})_K \times (X_{2,b})_K \arrow[dl, "G'_{1,K}"'] \arrow[dd, "G_{1,K}\times G_{2,K}"] \arrow[dr, "G'_{2,K}"] & \\
(X_{1,b})_K \arrow[dr, "G_{1,K}"'] & & (X_{2,b})_K \arrow[dl, "G_{2,K}"] \\
 & (X_b)_K &
\end{tikzcd}$$

where $G_{i,K}$ and $G'_{i,K}$ are the acting group schemes and as $\mathrm{char}(K) = 0$, we have $G_{i,K} = G'_{i,K} = \mathbb{Z}/p\mathbb{Z} \simeq \mu_p$ for all i.

For $i = 1, 2$, let $f_i : X_{i,b} \to X_b$ be the Galois covers of degree p on the formal boundary obtained by taking $X_{i,b}$ to be the normalisation of X_b in $(X_{i,b})_K$. Note that for a suitable choice of parameter T_i, the boundary $X_{i,b}$ is also of the form $\mathrm{Spf}\left(R[[T_i]]\{T_i^{-1}\}\right)$ for $i = 1, 2$. Similarly, Y_b is the normalisation of X_b in $(Y_b)_K$ so that $f : Y_b \to X_b$ is a non-trivial Galois cover of type (p, p) on the formal boundary and we have the following picture:

$$
\begin{tikzcd}
 & Y_b = X_{1,b} \widetilde{\times_{X_b}} X_{2,b} \arrow[ddl, "G_1'", "c_1'"'] \arrow[d] \arrow[ddr, "G_2'"', "c_2'"] & \\
 & X_{1,b} \times_{X_b} X_{2,b} \arrow[dl, "G_2"'] \arrow[dr, "G_1"] \arrow[dd] & \\
X_{1,b} = \mathrm{Spf}(A_1) \arrow[dr, "c_1", "G_1"'] & & X_{2,b} = \mathrm{Spf}(A_2) \arrow[dl, "c_2"', "G_2"] \\
 & X_b = \mathrm{Spf}(A) &
\end{tikzcd}
$$

where:

- Y_b and $X_{i,b}$ are normal for $i = 1, 2$.

- c_i (respectively c_i') denotes the conductor of the torsor $X_{i,b} \to X_b$ (respectively $Y_b \to X_{i,b}$). The conductor c_i (respectively c_i') is dependent on the conductor variable m_i (respectively m_i').

- G_i (respectively G_i') denotes the finite flat group scheme over R of the torsor $X_{i,b} \to X_b$ (respectively $Y_b \to X_{i,b}$). We know G_i, G_i' are one of the R-group schemes $\mathcal{H}_{v_K(\lambda)}$, μ_p, or $\mathcal{H}_n$ for $0 < n < v_K(\lambda)$.

On the level of the special fibre over k we have:

$$\begin{tikzcd}
 & (Y_b)_k \arrow[ddl, "G'_{1,k}"'] \arrow[d] \arrow[ddr, "G'_{2,k}"] & \\
 & (X_{1,b})_k \times_{(X_b)_k} (X_{2,b})_k \arrow[dl, "G_{2,k}"'] \arrow[dr, "G_{1,k}"] \arrow[dd] & \\
(X_{1,b})_k = \mathrm{Spec}\, k((t_1)) \arrow[dr, "G_{1,k}"'] & & (X_{2,b})_k = \mathrm{Spec}\, k((t_2)) \arrow[dl, "G_{2,k}"] \\
 & (X_b)_k = \mathrm{Spec}\, k((t)) &
\end{tikzcd}$$

where:

- $(Y_b)_k$ and $(X_{i,b})_k$ are reduced for $i = 1, 2$.
- $k((t)) = A/(\pi)$ where t (respectively t_1 and t_2) represents the reduction modulo π of T (respectively T_1 and T_2 where T_i is some suitable parameter of $X_{i,b}$ for $i = 1, 2$).
- $G_{i,k} = G_i \times_R k$ and $G'_{i,k} = G'_i \times_R k$ are the acting group schemes over k, a field with characteristic p, so that these group schemes are necessarily $\mathbb{Z}/p\mathbb{Z}$, μ_p or α_p.

We aim to express the **conductor** c'_1 in terms of c_1 and c_2 for the various torsor combinations and likewise for c'_2. To achieve this, we express the **conductor variables** m'_1 and m'_2 in terms of m_1 and m_2. We have six cases to consider by taking all possible pairs of the group schemes $\mathcal{H}_{v_K(\lambda)}$, μ_p and $\mathcal{H}_n$ over R.

This leads us to one of the most important theorems in this chapter and, indeed, of this thesis. Several key results will follow from this theorem.

Theorem 4.2.1. *Let $X_b = \mathrm{Spf}\,(A)$ and suppose we have two disjoint non-trivial degree p Galois covers $f_{i,K} : (X_{i,b})_K \to (X_b)_K$, where $(X_b)_K$ denotes the generic fibre of X_b, for $i = 1, 2$. We take $(Y_b)_K$ to be the compositum of these covers.*

For $i = 1, 2$, let $f_i : X_{i,b} \to X_b$ be the Galois covers of degree p on the formal boundary obtained by taking $X_{i,b}$ to be the normalisation of X_b in $(X_{i,b})_K$. These are non-trivial torsors under finite flat R-group schemes G_i of rank p on the formal boundary which are generically disjoint with conductor variable m_i for $i = 1, 2$. Set Y_b as the normalisation of X_b in $(Y_b)_K$ so that the normalisation of the compositum of the two covers f_i, namely $f : Y_b \to X_b$, is a non-trivial Galois cover of type (p, p) on the formal boundary. We assume that the ramification index of this extension equals 1 and that the special fibre of Y_b is reduced.

Let m'_i denote the conductor variable of the torsor $Y_b \to X_{i,b}$. Then, for all possible pairs of G_1 and G_2, we can express the m'_i in terms of the m_i conductor variables for $i = 1, 2$ as follows:

$$
\begin{tikzcd}
 & Y_b \arrow[dl, "m'_1"] \arrow[dr, "m'_2"'] & \\
X_{1,b} \arrow[dr, "m_1"'] & & X_{2,b} \arrow[dl, "m_2"] \\
 & X_b &
\end{tikzcd}
$$

1. *For $G_1 = G_2 = \mathcal{H}_{v_K(\lambda)}$ we have that $m'_1 = m_2$ and $m'_2 = m_1 p - m_2(p-1)$ when $m_1 \leq m_2$, and $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$ when $m_1 > m_2$.*

2. *For $G_1 = \mathcal{H}_{v_K(\lambda)}$ and $G_2 = \mu_p$ we have that $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$.*

3. *For $G_1 = \mathcal{H}_{v_K(\lambda)}$ and $G_2 = \mathcal{H}_n$ we have that $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$.*

4. *For $G_1 = \mathcal{H}_n$ and $G_2 = \mu_p$ we have that $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$.*

5. *For $G_1 = G_2 = \mu_p$ we have that $m'_1 = m_2$ and $m'_2 = m_1 p - m_2(p-1)$ when $m_1 \leq m_2$, and $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$ when $m_1 > m_2$. Note that*

in this case these results are only valid when at least one of m_1 and m_2 is non-zero.

6. *For $G_1 = \mathcal{H}_{n_1}$ and $G_2 = \mathcal{H}_{n_2}$ we have that $m_1' = m_2$ and $m_2' = m_1 p - m_2(p-1)$ when $n_1 < n_2$, that $m_1' = m_2 p - m_1(p-1)$ and $m_2' = m_1$ when $n_1 > n_2$, that $m_1' = m_2$ and $m_2' = m_1 p - m_2(p-1)$ when both $n_1 = n_2$ and $m_1 < m_2$, and $m_1' = m_2 p - m_1(p-1)$ and $m_2' = m_1$ when both $n_1 = n_2$ and $m_1 \geq m_2$.*

where $0 < n, n_1, n_2 < v_K(\lambda)$. We summarise these results in the following table:

Table 4.1: Conductor variables m_1', m_2' in terms of m_1, m_2

(G_1, G_2)	m_1'	m_2'
$\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_{v_K(\lambda)}$	*If $m_1 \leq m_2$ then* $m_1' = m_2$ *If $m_1 > m_2$ then* $m_1' = m_2 p - m_1(p-1)$	*If $m_1 \leq m_2$ then* $m_2' = m_1 p - m_2(p-1)$ *If $m_1 > m_2$ then* $m_2' = m_1$
$\mathcal{H}_{v_K(\lambda)}, \mu_p$	$m_1' = m_2 p - m_1(p-1)$	$m_2' = m_1$
$\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_n$	$m_1' = m_2 p - m_1(p-1)$	$m_2' = m_1$
$\mathcal{H}_n, \mu_p$	$m_1' = m_2 p - m_1(p-1)$	$m_2' = m_1$

Continued on next page

<table>
<tr><th>(G_1, G_2)</th><th>m'_1</th><th>m'_2</th></tr>
<tr><td>μ_p, μ_p

Assuming m_1 or m_2 is non-zero.</td><td>*If $m_1 \leq m_2$ then*
$$m'_1 = m_2$$
If $m_1 > m_2$ then
$$m'_1 = m_2 p - m_1(p-1)$$</td><td>*If $m_1 \leq m_2$ then*
$$m'_2 = m_1 p - m_2(p-1)$$
If $m_1 > m_2$ then
$$m'_2 = m_1$$</td></tr>
<tr><td>$\mathcal{H}_{n_1}, \mathcal{H}_{n_2}$</td><td>*If $n_1 < n_2$ then*
$$m'_1 = m_2$$
If $n_1 > n_2$ then
$$m'_1 = m_2 p - m_1(p-1)$$
If $n_1 = n_2$ and $m_1 < m_2$ then
$$m'_1 = m_2$$
If $n_1 = n_2$ and $m_1 \geq m_2$ then
$$m'_1 = m_2 p - m_1(p-1)$$</td><td>*If $n_1 < n_2$ then*
$$m'_2 = m_1 p - m_2(p-1)$$
If $n_1 > n_2$ then
$$m'_2 = m_1$$
If $n_1 = n_2$ and $m_1 < m_2$ then
$$m'_2 = m_1 p - m_2(p-1)$$
If $n_1 = n_2$ and $m_1 \geq m_2$ then
$$m'_2 = m_1$$</td></tr>
</table>

Proof. We treat the six occurring cases individually. However, there is an important distinction between the first three cases and the last three cases.

In the first three cases, that is when at least one of the acting group schemes is the étale group scheme $\mathcal{H}_{v_K(\lambda)}$, one can work modulo π on the special fibre for, in this case, $Y_b = X_{1,b} \times_{X_b} X_{2,b}$ which of course means that on the special fibre $(Y_b)_k = (X_{1,b})_k \times_{(X_b)_k} (X_{2,b})_k$. This result is due to the following: suppose $G_1 = \mathcal{H}_{v_K(\lambda)}$ so that the torsor $X_{1,b} \to X_b$ is étale. Then, by base change, the torsor $X_{1,b} \times X_{2,b} \to X_{2,b}$ is automatically étale. The special fibre of $X_{2,b}$ is reduced (because it is dominated by Y_b which is reduced) but as $X_{1,b} \times X_{2,b} \to X_{2,b}$ is étale, this means the special fibre of $X_{1,b} \times X_{2,b}$ is also reduced. Then, by Theorem 3.2.3, $X_{1,b} \times_{X_b} X_{2,b}$ is normal and equal to Y_b, as required.

In the last three cases, we do not have this privilege, which means one must work above X_b over R without being permitted to reduce to the special fibre. However, we still proceed in a similar fashion, even if the computations are more involved, due to being unable to eliminate π. In particular, we base change and make appropriate transformations in order to find the torsor equations $Y_b \to X_{i,b}$ and read off the conductors m'_i for $i = 1, 2$. It is in fact by applying a binomial/multinomial identity that we get the torsor equation of the normalisation. This process could be interpreted geometrically as blowing-up.

Note that in each case we can perform a change of the parameter T in $A = R[[T]]\{T^{-1}\}$ so that one of the two torsor equations above $X_b = \mathrm{Spf}\,(A)$ is in its simplified form (as stipulated in section 4.1) but we must assume the other equation remains in its original full power series form.

1. $(\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_{v_K(\lambda)})$

Note that this case was previously treated in [4].

Here $m_1, m_2 < 0$. The first $\mathcal{H}_{v_K(\lambda)}$ torsor equation is given by $(1 + \lambda Z_1)^p = 1 + \lambda^p u_1$ and the second is given by $(1 + \lambda Z_2)^p = 1 + \lambda^p u_2$ where $u_1, u_2 \in A^\times$. Modulo π, these torsor equations reduce to $z_1^p - z_1 = \overline{u}_1$ and $z_2^p - z_2 = \overline{u}_2$ respectively on the special fibre. On the special fibre the acting group schemes are $\mathbb{Z}/p\mathbb{Z}$, the reduction of $\mathcal{H}_{v_K(\lambda)}$.

We start by computing m'_1. We can choose the parameter T so that $u_1 = T^{m_1}$ but $u_2 = \sum_{i\in\mathbb{Z}} a_i T^i$ remains as a power series and therefore, accordingly, $\overline{u}_1 = t^{m_1}$ and $\overline{u}_2 = \sum_{i=m_2}^{-1} \overline{a}_i t^i$ where $\overline{a}_{m_2} \neq 0$. We can write t in terms of z_1 of $(X_{1,b})_k$:

$$\begin{aligned} z_1^p - z_1 = t^{m_1} &\Leftrightarrow z_1^p\left(1 - z_1^{1-p}\right) = t^{m_1} \\ &\Leftrightarrow t = \left(z_1^{1/m_1}\right)^p \left(1 - z_1^{1-p}\right)^{1/m_1} \end{aligned}$$

By considering the special fibre, residually we have ramified degree p extensions of complete discrete valuation fields above $k((t))$, where $(X_b)_k = \mathrm{Spec}\, k((t))$. For $(X_b)_k$ the parameter is t which means that for any degree p extension above $(X_b)_k$ a uniformising parameter will be anything that as a p-power multiplied by a unit equals t. Note that anything that is a parameter residually—that is, modulo π on the special fibre—is a parameter over R. This suggests that in this case the parameter of $(X_{1,b})_k$ is z_1^{1/m_1} and so by taking $z := z_1^{1/m_1}$ we can write

$$t = z^p \left(1 - z^{-m_1(p-1)}\right)^{1/m_1}.$$

We can now proceed to base change the torsor equation of $(X_{2,b})_k \to (X_b)_k$ to obtain the torsor equation for $(Y_b)_k \to (X_{1,b})_k$:

$$\begin{aligned} z_2^p - z_2 &= \sum_{i=m_2}^{-1} \overline{a}_i t^i = \sum_{i=m_2}^{-1} \overline{a}_i z^{ip} \left(1 - z^{-m_1(p-1)}\right)^{i/m_1} \\ &= \sum_{i=m_2}^{-1} \overline{a}_i z^{ip} \left(1 - \frac{i}{m_1} z^{-m_1(p-1)} + ...\right) \\ &= \overline{a}_{m_2} z^{m_2 p} \left(1 - \frac{m_2}{m_1} z^{-m_1(p-1)} + ...\right) + \text{higher order terms} \\ &= \overline{a}_{m_2} z^{m_2 p} - \frac{m_2 \overline{a}_{m_2}}{m_1} z^{m_2 p - m_1(p-1)} + \text{higher order terms} \end{aligned}$$

Note that leading term is a multiple of p but removing from it an Artin-Schreier element—in particular, expressing $z^{m_2 p}$ as $z^{m_2 p} - z^{m_2} + z^{m_2} \simeq z^{m_2}$—gives rise to an equation of the form:

$$z_2^p - z_2 = \overline{a}_{m_2} z^{m_2} - \frac{m_2 \overline{a}_{m_2}}{m_1} z^{m_2 p - m_1(p-1)} + \text{higher order terms}$$

The conductor variable m_1' is the smallest power of z which is coprime to p. The expression above indicates there are two candidates, namely m_2 and $m_2 p - m_1(p-1)$. Note that $m_2 p - m_1(p-1) \leq m_2$ is equivalent to $m_1 \geq m_2$. Therefore, when $m_1 \geq m_2$ we have that $m_1' = m_2 p - m_1(p-1)$ and when $m_1 < m_2$ we have that $m_1' = m_2$.

We now determine m_2'. However, this is effectively entirely the same procedure. This time we choose T so that $u_1 = \sum_{i \in \mathbb{Z}} a_i T^i$ and $u_2 = T^{m_2}$ and, after reducing modulo π, we have $z_1^p - z_1 = \sum_{i=m_1}^{-1} \overline{a}_i t^i$ and $z_2^p - z_2 = t^{m_2}$ on the special fibre. As a result, the parameter of $(X_{2,b})_k$ is given by $z := z_2^{1/m_2}$ and we have $t = z^p \left(1 - z^{-m_2(p-1)}\right)^{1/m_2}$. By base changing to obtain the torsor equation for $(Y_b)_k \to (X_{2,b})_k$, we find the same result but of course with m_1 and m_2 interchanged. Therefore, when $m_1 \geq m_2$ we have that $m_2' = m_1$ and when $m_1 < m_2$ we have that $m_2' = m_1 p - m_2(p-1)$.

---

2. $(\mathcal{H}_{v_K(\lambda)}, \mu_p)$

Here $m_1 < 0$ while $m_2 \geq 0$ which naturally gives us that $m_1 \leq m_2$. The torsor equation for $\mathcal{H}_{v_K(\lambda)}$ is given by $(1+\lambda Z_1)^p = 1 + \lambda^p u_1$ and for μ_p by $Z_2^p = u_2$ where $u_1, u_2 \in A^\times$. Modulo π, these torsor equations reduce to $z_1^p - z_1 = \overline{u}_1$ and $z_2^p = \overline{u}_2$ with acting group schemes $\mathbb{Z}/p\mathbb{Z}$ and μ_p respectively on the special fibre.

We start by computing m_1'. We can choose the parameter T so that $u_1 = T^{m_1}$ but $u_2 = \sum_{i\in\mathbb{Z}} a_i T^i$ remains as a power series and therefore, accordingly, $\overline{u}_1 = t^{m_1}$ and $\overline{u}_2 = \sum_{i\geq l} \overline{a}_i t^i$ for some integer l where $\overline{a}_l \neq 0$. As in case 1 of this proof, we have that the parameter of $(X_{1,b})_k$ is $z := z_1^{1/m_1}$ and we can write $t = z^p\left(1 - z^{-m_1(p-1)}\right)^{1/m_1}$. We now have two cases to treat, namely $(a1)$ and $(a2)$, depending on whether or not l is coprime to p.

(a1) In this case, $\gcd(l,p) = 1 \Rightarrow m_2 = 0$. We base change the torsor equation of $(X_{2,b})_k \to (X_b)_k$ to obtain the torsor equation for $(Y_b)_k \to (X_{1,b})_k$:

$$\begin{aligned} z_2^p &= \sum_{i\geq l} \overline{a}_i t^i = \sum_{i\geq l} \overline{a}_i z^{ip} (1 - z^{-m_1(p-1)})^{i/m_1} \\ &= \sum_{i\geq l} \overline{a}_i z^{ip} (1 - \frac{i}{m_1} z^{-m_1(p-1)} + ...) \\ &= \overline{a}_l z^{lp}\left(1 - \frac{m_2}{m_1} z^{-m_1(p-1)}\right) + \text{higher order terms} \end{aligned}$$

As this is a μ_p torsor equation, the multiplicative factor $\overline{a}_l z^{lp}$ can be eliminated by multiplication of a p-power. Hence, we now have:

$$z_2^p = 1 - \frac{m_2}{m_1} z^{-m_1(p-1)} + \text{higher order terms}$$

So the conductor variable is evidently $m_1' = -m_1(p-1)$, as this is the smallest power of z and satisfies the condition that it must be coprime to p.

(a2) In this case $\gcd(l,p) \neq 1 \Rightarrow l = p^n\alpha$ for some $\alpha, n \in \mathbb{Z}$ such that $\gcd(\alpha, p) = 1$. By section 4.1, we know that the $(X_{2,b})_k \to (X_b)_k$ torsor equation can be expressed as follows:

$$z_2^p = 1 + \overline{a}_{m_2} t^{m_2} + \sum_{i>m_2} \overline{a}_i t^i = 1 + \sum_{i\geq m_2} \overline{a}_i t^i$$

By a change of variables, we can express this μ_p torsor as:

$$z_2^p = 1 + \sum_{i \geq m_2} \overline{a}_i t^i \Leftrightarrow (z_2 - 1)^p = \sum_{i \geq m_2} \overline{a}_i t^i \Rightarrow z_2^p = \sum_{i \geq m_2} \overline{a}_i t^i$$

We can now proceed to base change the torsor equation of $(X_{2,b})_k \to (X_b)_k$ to obtain the torsor equation for $(Y_b)_k \to (X_{1,b})_k$:

$$\begin{aligned} z_2^p &= \sum_{i \geq m_2} \overline{a}_i z^{pi} \left(1 - z^{-m_1(p-1)}\right)^{i/m_1} \\ &= \sum_{i \geq m_2} \overline{a}_i z^{ip} \left(1 - \frac{i}{m_1} z^{-m_1(p-1)} + \ldots\right) \\ &= \overline{a}_{m_2} z^{m_2 p} \left(1 - \frac{m_2}{m_1} z^{-m_1(p-1)} + \ldots\right) + \text{higher order terms} \\ &= \overline{a}_{m_2} z^{m_2 p} - \frac{m_2 \overline{a}_{m_2}}{m_1} z^{m_2 p - m_1(p-1)} + \text{higher order terms} \end{aligned}$$

We can't eliminate the leading term as it is not a multiplicative factor of every term in the expression. So $m_1' = m_2 p - m_1(p-1)$ as this is the smallest power of z which is not divisible by p. Note that the expression for m_1' in this case is consistent with its (a1) counterpart, because by substituting $m_2 = 0$ one obtains $m_1' = -m_1(p-1)$.

We now determine m_2'. We choose T so that $u_1 = \sum_{i \in \mathbb{Z}} a_i T^i$ and u_2 is given by T^h in the case (a1) and by $1 + T^{m_2}$ in the case (a2). After reducing these equations modulo π, we have $z_1^p - z_1 = \sum_{i=m_1}^{-1} \overline{a}_i t^i$ and (a1) $z_2^p = t^h$ or (a2) $z_2^p = 1 + t^{m_2}$ on the special fibre.

(a1) We can write t in terms of z_2 of $(X_{2,b})_k$ since $z_2^p = t^h \Leftrightarrow t = \left(z_2^{1/h}\right)^p$. This implies that $z := z_2^{1/h}$ is the parameter of $X_{2,b}$ and we have that $t = z^p$. We base change to obtain the torsor equation for $(Y_b)_k \to (X_{2,b})_k$:

$$\begin{aligned} z_1^p - z_1 = \sum_{i=m_1}^{-1} \overline{a}_i t^i &= \sum_{i=m_1}^{-1} \overline{a}_i z^{ip} \\ &= \overline{a}_{m_1} z^{m_1 p} + \text{higher order terms} \end{aligned}$$

The leading term $z^{m_1 p}$ is a multiple of p but, as in Case 1 of this proof, modulo an Artin-Schreier element we have:

$$z_1^p - z_1 = \overline{a}_{m_1} z^{m_1} + \text{higher order terms}$$

Therefore, the conductor variable $m_2' = m_1$.

(a2) As before, we write t in terms of z_2 of $(X_{2,b})_k$:

$$\begin{aligned} z_2^p = 1 + t^{m_2} &\Leftrightarrow z_2^p - 1 = t^{m_2} \\ &\Leftrightarrow (z_2 - 1)^p = t^{m_2} \\ &\Leftrightarrow t = \left((z_2 - 1)^{1/m_2}\right)^p \end{aligned}$$

This means that the parameter of $(X_{2,b})_k$ is $z := (z_2 - 1)^{1/m_2}$ and so, from the above, we obtain $t = z^p$. Now, we base change to obtain the torsor equation for $(Y_b)_k \to (X_{2,b})_k$:

$$\begin{aligned} z_1^p - z_1 = \sum_{i=m_1}^{-1} \overline{a}_i t^i &= \sum_{i=m_1}^{-1} \overline{a}_i z^{ip} \\ &= \overline{a}_{m_1} z^{m_1 p} + \text{higher order terms} \end{aligned}$$

Modulo an Artin-Schreier element applied to the leading term we have a torsor equation of the form:

$$z_1^p - z_1 = \overline{a}_{m_1} z^{m_1} + \text{higher order terms}$$

Therefore, as in the (a1) case, the conductor variable $m_2' = m_1$.

---

3. $(\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_n)$

Here $m_1 < 0$ while $m_2 \in \mathbb{Z}$. The torsor equation for $\mathcal{H}_{v_K(\lambda)}$ is given by $(1 + \lambda Z_1)^p = 1 + \lambda^p u_1$ and for $\mathcal{H}_n$ by $(1 + \pi^n Z_2)^p = 1 + \pi^{np} u_2$ where $u_1, u_2 \in A^\times$. Modulo π, these torsor equations reduce to $z_1^p - z_1 = \overline{u}_1$ and $z_2^p = \overline{u}_2$ with acting group schemes $\mathbb{Z}/p\mathbb{Z}$ and α_p respectively on the special fibre.

We start by computing m_1'. We can choose the parameter T so that $u_1 = T^{m_1}$ but $u_2 = \sum_{i \in \mathbb{Z}} a_i T^i$ remains as a power series and therefore, accordingly, $\overline{u}_1 = t^{m_1}$ and $\overline{u}_2 = \sum_{i \geq l} \overline{a}_i t^i$ for some integer l where $\overline{a}_l \neq 0$. As in Case 1 of this proof, we have that the parameter of $(X_{1,b})_k$ is $z := z_1^{1/m_1}$ and we can write $t = z^p \left(1 - z^{-m_1(p-1)}\right)^{1/m_1}$. We base change the torsor equation of $(X_{2,b})_k \to (X_b)_k$ to obtain the torsor equation for $(Y_b)_k \to (X_{1,b})_k$:

$$\begin{aligned} z_2^p &= \sum_{i \geq l} \overline{a}_i t^i = \sum_{i \geq l} \overline{a}_i z^{pi} \left(1 - z^{-m_1(p-1)}\right)^{i/m_1} \\ &= \sum_{i \geq l} \overline{a}_i z^{pi} \left(1 - \frac{i}{m_1} z^{-m_1(p-1)} + ...\right) \\ &= \overline{a}_l z^{pl} \left(1 - \frac{l}{m_1} z^{-m_1(p-1)} + ...\right) + \text{higher order terms} \\ &= \overline{a}_l z^{pl} - \frac{l\overline{a}_l}{m_1} z^{lp - m_1(p-1)} + \text{higher order terms} \end{aligned}$$

This is an α_p torsor equation which we remind the reader operates modulo addition of p-powers and, as such, the leading term z^{pl} can be removed. So the conductor variable $m_1' = m_2 p - m_1(p-1)$, as this is the first power of z which is not divisible by p.

It remains to compute m_2' in this case. This time we choose the parameter T so that $u_1 = \sum_{i \in \mathbb{Z}} a_i T^i$ is the power series and $u_2 = T^{m_2}$. After reducing modulo π, we have $\overline{u}_1 = \sum_{i=m_1}^{-1} \overline{a}_i t^i$ and $\overline{u}_2 = t^{m_2}$ on the special fibre. We can write t in terms of z_2 of $(X_{2,b})_k$ since $z_2^p = t^{m_2} \Leftrightarrow t = \left(z_2^{1/m_2}\right)^p$. This implies that $z := z_2^{1/m_2}$ is the parameter of $X_{2,b}$ and we have that $t = z^p$. We base change to obtain the torsor equation for $(Y_b)_k \to (X_{2,b})_k$:

$$\begin{aligned} z_1^p - z_1 &= \sum_{i=m_1}^{-1} \overline{a}_i t^i = \sum_{i=m_1}^{-1} \overline{a}_i z^{ip} \\ &= \overline{a}_{m_1} z^{m_1 p} + \text{higher order terms} \end{aligned}$$

The leading term $z^{m_1 p}$ is a multiple of p but, just as in Case 1 of this proof, modulo an Artin-Schreier element we have:

$$z_1^p - z_1 = \overline{a}_{m_1} z^{m_1} + \text{higher order terms}$$

Therefore, the conductor variable $m_2' = m_1$.

We remind the reader that in the remaining three cases, we cannot reduce modulo π and work on the special fibre. Thus, the computations here are slightly more involved. It will be useful to recall in advance here the following equality given by the binomial theorem

$$1 + (\pi^n bZ)^p = (1 + \pi^n bZ)^p - \sum_{k=1}^{p-1} \binom{p}{k} (\pi^n bZ)^k,$$

which can be extended by the multinomial theorem to

$$1 + \sum_i \left(\pi^n b_i Z^i\right)^p = \left(1 + \sum_i \pi^n b_i Z^i\right)^p - p \sum_i \pi^n b_i Z^i + \text{higher order terms.}$$

We also mention here that to circumvent our inability to take, say, p-th roots of coefficients belonging to the ring R (which, unlike k, is not an algebraically closed field), we can adopt the following technique for a given element a_i where $v_K(a_i) = 0$; namely, it can be expressed as $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ such that $v_K(b_i) = 0$ but $v_K(c_i) > 0$.

Finally, note that the expression 'higher order terms' appearing in the computations will broaden to also include terms of higher valuation (i.e. higher powers or 'orders' of π).

4. $(\mathcal{H}_n, \mu_p)$

Here $m_1 \in \mathbb{Z}$ while $m_2 \geq 0$. The torsor equation for $\mathcal{H}_n$ is given by $(1 + \pi^n Z_1)^p = 1 + \pi^{np} u_1$ and for μ_p by $Z_2^p = u_2$ where $u_1, u_2 \in A^\times$.

We start by computing m_1'. We can choose the parameter T so that $u_1 = T^{m_1}$ but $u_2 = \sum_{i \in \mathbb{Z}} a_i T^i$ remains as a power series. We can express T in terms of Z_1 in order to read off the parameter for $X_{1,b}$:

$$\begin{aligned}
(1 + \pi^n Z_1)^p = 1 + \pi^{np} T^{m_1} &\Leftrightarrow \pi^{np} Z_1^p + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{nk} Z_1^k + 1 = 1 + \pi^{np} T^{m_1} \\
&\Leftrightarrow \pi^{np} Z_1^p + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{nk} Z_1^k = \pi^{np} T^{m_1} \\
&\Leftrightarrow Z_1^p \left(\pi^{np} + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{nk} Z_1^{k-p}\right) = \pi^{np} T^{m_1} \\
&\Leftrightarrow Z_1^p \left(1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{n(k-p)} Z_1^{k-p}\right) = T^{m_1} \\
&\Leftrightarrow \left(Z_1^{1/m_1}\right)^p \left(1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{n(k-p)} Z_1^{k-p}\right)^{1/m_1} = T
\end{aligned}$$

We know from case 3 that $Z := Z_1^{1/m_1}$ is the parameter of $X_{1,b}$ modulo π and so we can write:

$$T = Z^p \left(1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{-n(p-k)} Z^{-m_1(p-k)}\right)^{1/m_1}.$$

For convenience, set $B = \sum_{k=1}^{p-1} \binom{p}{k} \pi^{-n(p-k)} Z^{-m_1(p-k)}$ so that $T = Z^p (1+B)^{1/m_1}$. We now have two cases to treat, namely $(a1)$ and $(a2)$, depending on whether or not $l = \min\{i | v_K(a_i) = 0\}$ is coprime to p.

(a1) In this case, $\gcd(l, p) = 1 \Rightarrow m_2 = 0$. We base change the torsor equation of $X_{2,b} \to X_b$ to obtain:

$$\begin{aligned} Z_2^p &= \sum_{i\in\mathbb{Z}} a_i T^i = \sum_{i\in\mathbb{Z}} a_i Z^{ip} (1+B)^{i/m_1} \\ &= \sum_{i\in\mathbb{Z}} a_i Z^{ip} \left(1 + \frac{i}{m_1} B + ...\right) \\ &= \sum_{i\in\mathbb{Z}} a_i Z^{ip} + \sum_{i\in\mathbb{Z}} \frac{ia_i}{m_1} Z^{ip} B + ... \end{aligned}$$

We can partition any summation over the index i into the terms where $v_K(a_i) = 0$ and the terms where $v_K(a_i) > 0$. For now we do this to only the first summation:

$$Z_2^p = \sum_{v_K(a_i)=0} a_i Z^{ip} + \sum_{v_K(a_i)>0} a_i Z^{ip} + \sum_{i\in\mathbb{Z}} \frac{ia_i}{m_1} Z^{ip} B + ...$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ but $v_K(c_i) > 0$.

$$\begin{aligned} Z_2^p &= \sum_{v_K(b_i)=0} b_i^p Z^{ip} + \sum_{v_K(c_i)>0} c_i Z^{ip} + \sum_{v_K(a_i)>0} a_i Z^{ip} + \sum_{i\in\mathbb{Z}} \frac{ia_i}{m_1} Z^{ip} B + ... \\ &= \sum_{v_K(b_i)=0} \left(b_i Z^i\right)^p + \sum_{v_K(d_i)>0} d_i Z^{ip} + \sum_{i\in\mathbb{Z}} \frac{ia_i}{m_1} Z^{ip} B + ... \end{aligned}$$

where, for purposes of simplicity, we set $d_i = c_i + a_i$ so that we can unify both positive valuation summations. Now, as this is a μ_p torsor we can take the p-power term $\left(b_l Z^l\right)^p$ into factor, so that we have:

$$Z_2^p = 1 + \sum_{\substack{v_K(b_i)=0 \\ i\neq l}} \left(b_l^{-1} b_i Z^{i-l}\right)^p + \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} + \sum_{i\in\mathbb{Z}} \frac{ib_l^{-p} a_i}{m_1} Z^{p(i-l)} B + ...$$

The summation $\sum_{v_K(d_i)>0} d_i b_l^{-p} Z^{p(i-l)}$ does not contribute to the conductor variable since it has positive valuation so we can safely exclude it. Even if this were not the case, its π valuation would rule it out. For the rest of this proof we automatically operate in this way. Now, by the multinomial identify stated above the first two terms $1 + \sum \left(b_l^{-1} b_i Z^{i-l}\right)^p$ can be expressed in the form

$$\left(1 + \sum_{\substack{v_K(b_i)=0 \\ i \neq l}} b_l^{-1} b_i Z^{i-l}\right)^p - p \sum_{\substack{v_K(b_i)=0 \\ i \neq l}} b_l^{-1} b_i Z^{i-l} + \text{higher order terms}$$

so that we can write the torsor equation as:

$$\begin{aligned} Z_2^p = &\left(1 + \sum_{\substack{v_K(b_i)=0 \\ i \neq l}} b_l^{-1} b_i Z^{i-l}\right)^p - p \sum_{\substack{v_K(b_i)=0 \\ i \neq l}} b_l^{-1} b_i Z^{i-l} \\ &+ \sum_{i \in \mathbb{Z}} \frac{i b_l^{-p} a_i}{m_1} Z^{p(i-l)} B + \text{higher order terms} \end{aligned}$$

Then, by multiplying our torsor equation by the inverse p-power

$$\left(1 + \sum_{\substack{v_k(b_i)=0 \\ i \neq l}} b_l^{-1} b_i Z^{i-l}\right)^{-p} = 1 - p \sum_{\substack{v_k(b_i)=0 \\ i \neq l}} b_l^{-1} b_i Z^{i-l} + \text{higher order terms}$$

gives us an equation of the form:

$$Z_2^p = 1 - p \sum_{\substack{v_K(b_i)=0 \\ i \neq l}} b_l^{-1} b_i Z^{i-l} + \sum_{i \in \mathbb{Z}} \frac{i b_l^{-p} a_i}{m_1} Z^{p(i-l)} B + \text{higher order terms}$$

Then, up to units we have:

$$\begin{aligned} Z_2^p = 1 - \pi^{v_K(p)} &\sum_{\substack{v_K(b_i)=0 \\ i \neq l}} b_l^{-1} b_i Z^{i-l} \\ &+ \pi^{v_K(p) - n(p-1)} \sum_{i \in \mathbb{Z}} \frac{i b_l^{-p} a_i}{m_1} Z^{p(i-l) - m_1(p-1)} + \text{higher order terms} \end{aligned}$$

Clearly the smallest power of π is $v_K(p) - n(p-1)$ and so we look to that summation for the conductor variable. For zero valuation coefficients, the index of the two summation will start at the integer l, the index corresponding to lowest zero valuation coefficient. As m_1' is the smallest exponent appearing in this leading summation which is coprime to p, we have that $m_1' = -m_1(p-1)$.

(a2) In this case $\gcd(l,p) \neq 1 \Rightarrow l = p^n \alpha$ for some $\alpha, n \in \mathbb{Z}$ such that $\gcd(\alpha, p) = 1$. Again, we take $T = Z^p (1+B)^{1/m_1}$ where B is as defined previously. We then base change the μ_p torsor equation $X_{2,b} \to X_b$ to obtain:

$$\begin{aligned} Z_2^p &= \sum_{i \in \mathbb{Z}} a_i T^i = 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i \\ &= 1 + \sum_{v_K(a_i)=0} a_i Z^{ip} (1+B)^{i/m_1} + \sum_{v_K(a_i)>0} a_i Z^{ip} (1+B)^{i/m_1} \end{aligned}$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ but $v_K(c_i) > 0$ so that:

$$\begin{aligned} Z_2^p = 1 &+ \sum_{v_K(b_i)=0} b_i^p Z^{ip} (1+B)^{i/m_1} + \sum_{v_K(c_i)>0} c_i Z^{ip} (1+B)^{i/m_1} \\ &+ \sum_{v_K(a_i)>0} a_i Z^{ip} (1+B)^{i/m_1} \\ = 1 &+ \sum_{v_K(b_i)=0} b_i^p Z^{ip} (1+B)^{i/m_1} + \sum_{v_K(d_i)>0} d_i Z^{ip} (1+B)^{i/m_1} \end{aligned}$$

where, for purposes of simplicity, we again set $d_i = c_i + a_i$ so that we can unify both positive valuation summations. Now we continue by expansion of the binomial terms:

$$\begin{aligned} Z_2^p &= 1 + \sum_{v_K(b_i)=0} b_i^p Z^{ip} (1+B)^{i/m_1} + \sum_{v_K(d_i)>0} d_i Z^{ip} (1+B)^{i/m_1} \\ &= 1 + \sum_{v_K(b_i)=0} b_i^p Z^{ip} \left(1 + \frac{i}{m_1} B + \ldots\right) + \sum_{v_K(d_i)>0} d_i Z^{ip} \left(1 + \frac{i}{m_1} B + \ldots\right) \\ &= 1 + \sum_{v_K(b_i)=0} \left(b_i Z^i\right)^p + \sum_{v_K(b_i)=0} \frac{i}{m_1} b_i^p Z^{ip} B \\ &\quad + \sum_{v_K(d_i)>0} d_i Z^{ip} + \sum_{v_K(d_i)>0} \frac{i}{m_1} d_i Z^{ip} B + \text{higher order terms} \end{aligned}$$

By the multinomial identify the first two terms $1 + \sum \left(b_i Z^i\right)^p$ can be expressed in the form

$$\left(1 + \sum b_i Z^i\right)^p - p \sum b_i Z^i + \text{higher order terms}$$

so that we can write:

$$Z_2^p = \left(1 + \sum_{v_K(b_i)=0} b_i Z^i\right)^p - p \sum_{v_K(b_i)=0} b_i Z^i$$
$$+ \sum_{v_K(b_i)=0} \frac{i}{m_1} b_i^p Z^{ip} B + \text{higher order terms}$$

Then by multiplying our torsor equation by the inverse p-power

$$\left(1 + \sum_{v_K(b_i)=0} b_i Z^i\right)^{-p} = 1 - p \sum_{v_K(b_i)=0} b_i Z^i + \text{higher order terms}$$

gives us an equation of the form:

$$\begin{aligned}
Z_2^p &= 1 - p \sum_{v_K(b_i)=0} b_i Z^i + \sum_{v_K(b_i)=0} \frac{i}{m_1} b_i^p Z^{ip} B + \text{higher order terms} \\
&= 1 - p \sum_{i \geq m_2} b_i Z^i + \sum_{i \geq m_2} \frac{i}{m_1} b_i^p Z^{ip} B + \text{higher order terms} \\
&= 1 - p \sum_{i \geq m_2} b_i Z^i + \sum_{i \geq m_2} \frac{i}{m_1} b_i^p Z^{ip} \left(\sum_{k=1}^{p-1} \binom{p}{k} \pi^{-n(p-k)} Z^{-m_1(p-k)} \right) \\
&\qquad + \text{higher order terms} \\
&= 1 - \pi^{v_K(p)} \sum_{i \geq m_2} b_i Z^i + \pi^{v_K(p)-n(p-1)} \sum_{i \geq m_2} \frac{i b_i^p}{m_1} Z^{ip - m_1(p-1)} \\
&\qquad + \text{higher order terms}
\end{aligned}$$

up to units. The second summation has the smallest π valuation and so the conductor variable is $m_1' = m_2 p - m_1(p-1)$. Like in the previous case, the expression for m_1' here is also consistent with the case $(a1)$ since there we have that $m_2 = 0$ and one obtains $m_1' = -m_1(p-1)$ from $m_1' = m_2 p - m_1(p-1)$.

We now determine m_2'. We choose the parameter T so that $u_1 = \sum_{i \in \mathbb{Z}} a_i T^i$ and $u_2 = T^h$ in the case (a1) while $u_2 = 1 + T^{m_2}$ in the case (a2).

(a1) In this case, $m_2 = 0$. The parameter of $X_{2,b}$ is $Z := Z_2^{1/h}$ where $T = Z^p$ is obtained from the torsor equation $Z_2^p = T^h \Leftrightarrow \left(Z_2^{1/h}\right)^p = T$. We base change the torsor equation $X_{1,b} \to X_b$ to obtain:

$$\begin{aligned}
(1 + \pi^n Z_1)^p = 1 + \pi^{np} \sum_{i \in \mathbb{Z}} a_i T^i &= 1 + \pi^{np} \sum_{i \in \mathbb{Z}} a_i Z^{pi} \\
&= 1 + \pi^{np} \sum_{v_K(a_i)=0} a_i Z^{pi} + \pi^{np} \sum_{v_K(a_i)>0} a_i Z^{pi}
\end{aligned}$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ but $v_K(c_i) > 0$.

$$\begin{aligned}(1+\pi^n Z_1)^p &= 1 + \pi^{np} \sum_{v_K(b_i)=0} b_i^p Z^{pi} + \pi^{np} \sum_{v_K(c_i)>0} c_i Z^{pi} + \pi^{np} \sum_{v_K(a_i)>0} a_i Z^{pi} \\ &= 1 + \sum_{v_K(b_i)=0} \left(\pi^n b_i Z^i\right)^p + \sum_{v_K(c_i)>0} \pi^{np} c_i Z^{pi} + \sum_{v_K(a_i)>0} \pi^{np} a_i Z^{pi} \\ &= 1 + \sum_{v_K(b_i)=0} \left(\pi^n b_i Z^i\right)^p + \sum_{v_K(d_i)>0} \pi^{np} d_i Z^{pi}\end{aligned}$$

where, for purposes of simplicity, we again set $d_i = c_i + a_i$ so that we can unify both positive valuation summations. Note that the first two terms $1 + \sum \left(\pi^n b_i Z^i\right)^p$ can be expressed using the multinomial identity as:

$$1 + \sum \left(\pi^n b_i Z^i\right)^p = \left(1 + \sum \pi^n b_i Z^i\right)^p - p \sum \pi^n b_i Z^i + \text{higher order terms}$$

and then by multiplying this μ_p-torsor equation by the p-power $\left(1 + \sum \pi^n b_i Z^i\right)^{-p}$ yields:

$$\begin{aligned}(1+\pi^n Z_1)^p &= 1 - p \sum_{v_K(b_i)=0} \pi^n b_i Z^i + \sum_{v_K(d_i)>0} \pi^{np} d_i Z^{pi} + \text{higher order terms} \\ &= 1 - \pi^{v_K(p)+n} \sum_{i \geq m_1} b_i Z^i + \text{higher order terms}\end{aligned}$$

up to units. Now, m_2' is the smallest exponent appearing in this leading summation which is coprime to p and so $m_2' = m_1$.

(a2) In this case $\gcd(l, p) \neq 1 \Rightarrow l = p^n \alpha$ for some $\alpha, n \in \mathbb{Z}$ such that $\gcd(\alpha, p) = 1$. The torsor equation $Z_2^p = 1 + T^{m_2}$ can be written with T as the subject:

$$\begin{aligned}Z_2^p = 1 + T^{m_2} \Leftrightarrow Z_2^p - 1 = T^{m_2}& \\ \Leftrightarrow (Z_2 - 1)^p - \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k Z_2^k &= T^{m_2} \\ \Leftrightarrow (Z_2 - 1)^p \left(1 - (Z_2 - 1)^{-p} \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k Z_2^k\right) &= T^{m_2} \\ \Leftrightarrow \left((Z_2 - 1)^{\frac{1}{m_2}}\right)^p \left(1 - (Z_2 - 1)^{-p} \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k Z_2^k\right)^{\frac{1}{m_2}} &= T\end{aligned}$$

and so we take $Z := (Z_2 - 1)^{1/m_2}$, which we already know to be the parameter of $X_{2,b}$ by Case 2 of this proof, in order to write:

$$T = Z^p \left(1 - Z^{-m_2 p} \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (1 + Z^{m_2})^k \right)^{\frac{1}{m_2}}$$

For simplicity, let us denote $\sum_{k=1}^{p-1} \binom{p}{k}(-1)^k(1+Z^{m_2})^k$ by B so that we can write $T = Z^p \left(1 - Z^{-m_2 p} B\right)^{\frac{1}{m_2}}$. We can now base change the torsor equation of $X_{1,b} \to X_b$ to obtain:

$$\begin{aligned}
(1 + \pi^n Z_1)^p &= 1 + \pi^{np} \sum_{i \in \mathbb{Z}} a_i T^i \\
&= 1 + \pi^{np} \sum_{i \in \mathbb{Z}} a_i Z^{pi} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} \\
&= 1 + \pi^{np} \sum_{i \in \mathbb{Z}} a_i Z^{pi} \left(1 - \frac{i}{m_2} Z^{-m_2 p} B + ...\right) \\
&= 1 + \pi^{np} \sum_{i \in \mathbb{Z}} a_i Z^{pi} - \pi^{np} \sum_{i \in \mathbb{Z}} \frac{i a_i}{m_2} Z^{p(i - m_2)} B + ...
\end{aligned}$$

Partitioning any summation over the index i into the terms where $v_K(a_i) = 0$ and the terms where $v_K(a_i) > 0$ gives:

$$\begin{aligned}
(1 + \pi^n Z_1)^p = 1 + \pi^{np} &\sum_{v_K(a_i)=0} a_i Z^{pi} - \pi^{np} \sum_{v_K(a_i)=0} \frac{i a_i}{m_2} Z^{p(i-m_2)} B \\
&+ \pi^{np} \sum_{v_K(a_i)>0} a_i Z^{pi} - \pi^{np} \sum_{v_K(a_i)>0} \frac{i a_i}{m_2} Z^{p(i-m_2)} B + ...
\end{aligned}$$

Again, for the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ but $v_K(c_i) > 0$ and so we have:

$$\begin{aligned}
(1 + \pi^n Z_1)^p &= 1 + \pi^{np} \sum_{v_K(b_i)=0} b_i^p Z^{pi} + \pi^{np} \sum_{v_K(c_i)>0} c_i Z^{pi} \\
&\quad - \pi^{np} \sum_{v_K(b_i)=0} \frac{i b_i}{m_2} Z^{p(i-m_2)} B - \pi^{np} \sum_{v_K(c_i)>0} \frac{i c_i}{m_2} Z^{p(i-m_2)} B \\
&\quad + \pi^{np} \sum_{v_K(a_i)>0} a_i Z^{pi} - \pi^{np} \sum_{v_K(a_i)>0} \frac{i a_i}{m_2} Z^{p(i-m_2)} B \\
&\quad + \text{higher order terms} \\
&= 1 + \sum_{v_K(b_i)=0} \left(\pi^n b_i Z^i\right)^p - \pi^{np} \sum_{v_K(b_i)=0} \frac{i b_i}{m_2} Z^{p(i-m_2)} B \\
&\quad + \text{higher order terms}
\end{aligned}$$

excluding positive valuation terms. By the multinomial identify the first two terms can be expressed in the form

$$\left(1+\sum_{v_K(b_i)=0}\pi^n b_i Z^i\right)^p - p\sum_{v_K(b_i)=0}\pi^n b_i Z^i + \text{higher order terms}$$

and so we can write:

$$\begin{aligned}(1+\pi^n Z_1)^p = &\left(1+\sum_{v_K(b_i)=0}\pi^n b_i Z^i\right)^p - p\sum_{v_K(b_i)=0}\pi^n b_i Z^i\\ &- \pi^{np}\sum_{v_K(b_i)=0}\frac{ib_i}{m_2}Z^{p(i-m_2)}B + \text{higher order terms}\end{aligned}$$

Then by multiplying this μ_p-torsor equation by the p-power

$$\left(1+\sum\pi^n b_i Z^i\right)^{-p} = 1 - p\sum\pi^n b_i Z^i + \text{higher order terms}$$

yields:

$$\begin{aligned}(1+\pi^n Z_1)^p &= 1 - p\sum_{v_K(b_i)=0}\pi^n b_i Z^i - \pi^{np}\sum_{v_K(b_i)=0}\frac{ib_i}{m_2}Z^{p(i-m_2)}B\\ &\quad + \text{higher order terms}\\ &= 1 - \pi^{v_K(p)+n}\sum_{i\geq l} b_i Z^i\\ &\quad - \pi^{np}\sum_{v_K(b_i)=0}\frac{ib_i}{m_2}Z^{p(i-m_2)}\left(\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k(1+Z^{m_2})^k\right)\\ &\quad + \text{higher order terms}\\ &= 1 - \pi^{v_K(p)+n}\sum_{i\geq l} b_i Z^i\\ &\quad + \pi^{v_K(p)+np}\sum_{v_K(b_i)=0}\frac{ib_i}{m_2}Z^{p(i-m_2)}(1+Z^{m_2})\\ &\quad + \text{higher order terms}\end{aligned}$$

up to units. Since $v_K(p)+n$ is the smallest exponent of π and $l=\min\{i|v_K(a_i)=0, \gcd(i,p)=1\}=m_1$ is the starting index, we have that $m_2'=m_1$ is the conductor variable.

---

5. (μ_p, μ_p)

Here $m_1, m_2 \geq 0$. The first μ_p torsor is given by $Z_1^p = u_1$ and the second μ_p torsor by $Z_2^p = u_2$ where $u_1, u_2 \in A^\times$. Modulo π, these torsor equations reduce to $z_1^p = \overline{u}_1$ and $z_2^p = \overline{u}_2$ with acting group schemes μ_p on the special fibre. On the special fibre the reduced power series are of the form $\overline{u} = \sum_{i \geq l} \overline{a}_i t^i$ for some integer l. Depending on whether or not these l are coprime to p or not, there are three cases to consider. In particular, the pairs $(a1, a1)$, $(a1, a2)$ and $(a2, a2)$.

(a1, a1) Here $m_1 = m_2 = 0$. In this case one can show that the group schemes acting under the torsors $Y_b \to X_{i,b}$ with conductor variables m_i' are $\mathcal{H}_{n_i'}$ with $0 < n_i' < v_K(\lambda)$ for $i = 1, 2$. Moreover, one can show $n_1' = n_2'$ and $m_1' = m_2'$. Suppose $u_1 = T^h$ and $u_2 = T^l\,(v(T))$ where $v(T) = \sum_{i \geq 0} a_i T^i$ such that $a_0 \neq 0$. In other words, $v(T)$ is a unit and when restricted modulo π it has no zeros at T, hence the $a_0 \neq 0$ condition so that it doesn't have a factor of T. The conductor variables m_1', m_2' are in fact encoded in $v(T)$. The proof is complicated to present in general and, moreover, the result is not dependent on m_1 and m_2. Instead, we treat an instructive example to illustrate the computations involved:

Suppose $p \neq 2$ and $u_1 = T$ and $u_2 = T + T^3 = T(1 + T^2)$. Then the coverings $Z_1^p = T$ and $Z_2^p = T(1 + T^2)$ are linearly disjoint because they are not related by a p-power factor.

We begin by computing m_1'. We can write $T = Z^p$ where $Z = Z_1$ is the parameter of $X_{1,b}$. Then we can base change the torsor equation $X_{2,b} \to X_b$:

$$Z_2^p = T(1 + T^2) = Z^p(1 + Z^{2p})$$

Removing the multiplicative factor Z^p gives rise to an equation of the form

$$Z_2^p = 1 + Z^{2p} = \left(1 + Z^2\right)^p - \sum_{k=1}^{p-1} \binom{p}{k} Z^{2k}$$

Multiplying this equation by the p-power $\left(1 + Z^2\right)^{-p} = 1 - pZ^2 + \ldots$, results in an equation of the form

$$Z_2^p = 1 - \sum_{k=1}^{p-1} \binom{p}{k} Z^{2k} + \text{higher order terms}$$

The smallest power of Z which is coprime to p is obtained when $k = 1$. Therefore, $m_1' = 2$.

Now we want to determine m_2'. We have that $Z_2^p = T(1 + T^2)$ but suppose $Z_2^p = T' \Leftrightarrow Z^p = T'$ where the parameter of $X_{2,b}$ is $Z = Z_2$. The relation $T' = T(1 + T^2)$ implies:

$$\begin{aligned} T &= T'\left(1 + T^2\right)^{-1} = T'\left(1 - T^2 + T^4 - T^6 + ...\right) \\ &= T' + \left(-T'T^2 + T'T^4 - T'T^6 + ...\right) \end{aligned}$$

From this we deduce that T can be interpreted as $T'+$ higher powers of T'. In particular, $T = T' - T'^3 + T'^5 - T'^7 + ...$ We can now proceed to base change the torsor equation $X_{1,b} \to X_b$:

$$\begin{aligned} Z_1^p &= T = T' - T'^3 + T'^5 - T'^7 + \text{higher order terms} \\ &= Z^p - Z^{3p} + Z^{5p} - Z^{7p} + \text{higher order terms} \\ &= Z^p\left(1 - Z^{2p} + Z^{4p} - Z^{6p} + \text{higher order terms}\right) \end{aligned}$$

Removing the multiplicative factor Z^p, gives rise to an equation of the form:

$$\begin{aligned} Z_1^p &= 1 - Z^{2p} + Z^{4p} - Z^{6p} + \text{higher order terms} \\ &= \left(1 - Z^2 + Z^4 ...\right)^p - p\sum(Z^2 + Z^4 + ...) + \text{higher order terms} \end{aligned}$$

by the multinomial theorem. Multiplying this equation by the inverse p-power $\left(1 - Z^2 + Z^4 ...\right)^{-p}$, results in an equation of the form

$$\begin{aligned} Z_1^p &= 1 - p\sum(Z^2 + Z^4 + ...) + \text{higher order terms} \\ &= 1 - \pi^{v_K(p)}\sum(Z^2 + Z^4 + ...) + \text{higher order terms} \end{aligned}$$

up to units. Therefore, the conductor variable is $m_2' = 2 = m_1'$ and $n_2' = \frac{v_K(p)}{p} = n_1'$.

(a1, a2) Here $m_1 = 0$ and $m_2 > 0$. We start by computing m_1'. We can choose the parameter T so that $u_1 = T^h$ but $u_2 = 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i$ remains as a power series. By the torsor equation $Z_1^p = T^h$, we can write $T = Z^p$ where $Z = Z_1^{1/h}$ is the parameter of $X_{1,b}$. Then we can base change the torsor equation $X_{2,b} \to X_b$:

$$\begin{aligned} Z_2^p &= 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i \\ &= 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi} \end{aligned}$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ but $v_K(c_i) > 0$ and so we have:

$$\begin{aligned} Z_2^p &= 1 + \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{pi} + \sum_{\substack{v_K(c_i)>0 \\ i \geq m_2}} c_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi} \\ &= 1 + \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} \left(b_i Z^i\right)^p + \sum_{\substack{v_K(c_i)>0 \\ i \geq m_2}} c_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi} \end{aligned}$$

Then, by the multinomial identity, we can rewrite the first two terms and obtain an equation of the form:

$$\begin{aligned} Z_2^p = &\left(1 + \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i\right)^p - p \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i \\ &+ \sum_{\substack{v_K(c_i)>0 \\ i \geq m_2}} c_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi} + \text{higher order terms} \end{aligned}$$

Then by multiplying this μ_p torsor equation by the p-power $\left(1 + \sum b_i Z^i\right)^{-p}$ yields:

$$\begin{aligned} Z_2^p &= 1 - p \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i + \sum_{\substack{v_K(c_i)>0 \\ i \geq m_2}} c_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi} + \text{higher order terms} \\ &= 1 - \pi^{v_K(p)} \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i + \sum_{\substack{v_K(c_i)>0 \\ i \geq m_2}} c_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi} + \text{higher order terms} \end{aligned}$$

Therefore, $m_1' = m_2$.

Now we want to determine m_2'. We can choose the parameter T so that $u_1 = \sum_{i \in \mathbb{Z}} a_i T^i$ is a power series and $u_2 = 1 + T_2^m$ is in simplified form. We know from Case 4 that the torsor equation $Z_2^p = 1 + T_2^m$ gives rise to:

$$T = Z^p \left(1 - Z^{-m_2 p} \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (1 + Z^{m_2})^k\right)^{\frac{1}{m_2}}$$

where $Z := (Z_2 - 1)^{1/m_2}$ is the parameter of $X_{2,b}$. We have that

$$Z_1^p = \sum_{i \in \mathbb{Z}} a_i T^i = \sum_{\substack{v_K(a_i)=0 \\ i \geq l}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i$$

where l is such that $\gcd(l, p) = 1$. We can write $T = Z^p \left(1 - Z^{-m_2 p} B\right)^{\frac{1}{m_2}}$ if we take $B = \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (1 + Z^{m_2})^k$. Then we can base change the torsor equation $X_{1,b} \to X_b$:

$$Z_1^p = \sum_{\substack{v_K(a_i)=0 \\ i \geq l}} a_i Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} + \sum_{v_K(a_i)>0} a_i Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}}$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ but $v_K(c_i) > 0$ and so we have:

$$\begin{aligned} Z_1^p &= \sum_{\substack{v_K(b_i)=0 \\ i \geq l}} b_i^p Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} + \sum_{\substack{v_K(c_i)>0 \\ i \geq l}} c_i Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} \\ &\quad + \sum_{v_K(a_i)>0} a_i Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} \\ &= \sum_{\substack{v_K(b_i)=0 \\ i \geq l}} b_i^p Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} + \sum_{v_K(d_i)>0} d_i Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} \end{aligned}$$

where $d_i = c_i + a_i$ is taken to unify the positive valuation summations. We continue:

$$\begin{aligned} Z_1^p &= \sum_{\substack{v_K(b_i)=0 \\ i \geq l}} b_i^p Z^{ip} \left(1 - \frac{i}{m_2} Z^{-m_2 p} B + ...\right) + \sum_{v_K(d_i)>0} d_i Z^{ip} \left(1 - \frac{i}{m_2} Z^{-m_2 p} B + ...\right) \\ &= \sum_{\substack{v_K(b_i)=0 \\ i \geq l}} b_i^p Z^{ip} - \sum_{\substack{v_K(b_i)=0 \\ i \geq l}} b_i^p Z^{ip} \frac{i}{m_2} Z^{-m_2 p} B \\ &\quad + \sum_{v_K(d_i)>0} d_i Z^{ip} - \sum_{v_K(d_i)>0} d_i Z^{ip} \frac{i}{m_2} Z^{-m_2 p} B + \text{higher order terms} \end{aligned}$$

Take into factor the p-power $b_l^p Z^{lp} = \left(b_l Z^l\right)^p$ so that we have:

$$\begin{aligned}
Z_1^p &= 1 + \sum_{\substack{v_K(b_i)=0\\ i\geq l}} b_l^{-p} b_i^p Z^{p(i-l)} - \sum_{\substack{v_K(b_i)=0\\ i\geq l}} b_l^{-p} b_i^p Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B \\
&\quad + \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} - \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B \\
&\quad + \text{higher order terms} \\
&= 1 + \sum_{\substack{v_K(b_i)=0\\ i\geq l}} \left(b_l^{-1} b_i Z^{i-l}\right)^p - \sum_{\substack{v_K(b_i)=0\\ i\geq l}} b_l^{-p} b_i^p Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B \\
&\quad + \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} - \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B \\
&\quad + \text{higher order terms}
\end{aligned}$$

The first two terms $1 + \sum \left(b_l^{-1} b_i Z^{i-l}\right)^p$ can be expressed by the multinomial identity as

$$\left(1 + \sum_{\substack{v_K(b_i)=0\\ i>l}} b_l^{-1} b_i Z^{i-l}\right)^p - p \sum_{v_K(b_i)=0, i>l} b_l^{-1} b_i Z^{i-l} + \text{higher order terms}$$

which means we can proceed to write:

$$\begin{aligned}
Z_1^p = &\left(1 + \sum_{\substack{v_K(b_i)=0\\ i>l}} b_l^{-1} b_i Z^{i-l}\right)^p - p \sum_{\substack{v_K(b_i)=0\\ i>l}} b_l^{-1} b_i Z^{i-l} \\
&- \sum_{v_K(b_i)=0, i\geq l} b_l^{-p} b_i^p Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B + \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \\
&- \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B + \text{higher order terms}
\end{aligned}$$

Multiplying by the inverse p-power

$$\left(1 + \sum_{\substack{v_K(b_i)=0\\ i>l}} b_l^{-1} b_i Z^{i-l}\right)^{-p} = 1 - p \sum_{\substack{v_K(b_i)=0\\ i>l}} b_l^{-1} b_i Z^{i-l} + \text{higher order terms}$$

gives us an equation of the form

$$\begin{aligned}
Z_1^p &= 1 - p \sum_{\substack{v_K(b_i)=0 \\ i>l}} b_l^{-1} b_i Z^{i-l} - \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_l^{-p} b_i^p Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B \\
&\quad + \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} - \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B \\
&\quad + \text{higher order terms} \\
&= 1 - p \sum_{\substack{v_K(b_i)=0 \\ i>l}} b_l^{-1} b_i Z^{i-l} \\
&\quad - \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_l^{-p} b_i^p Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} \left(\sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (1+Z^{m_2})^k \right) \\
&\quad + \text{higher order terms} \\
&= 1 - \pi^{v_k(p)} \sum_{\substack{v_K(b_i)=0 \\ i>l}} b_l^{-1} b_i Z^{i-l} \\
&\quad + \pi^{v_k(p)} \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_l^{-p} b_i^p \frac{i}{m_2} Z^{p(i-l)-m_2 p} (1+Z^{m_2}) + \text{higher order terms}
\end{aligned}$$

up to units. Then, clearly, $m_2' = -m_2(p-1)$ is the conductor variable.

(a2, a2) Here both $m_1, m_2 > 0$ and, as usual, we start by computing m_1'. The parameter T can be chosen so that $u_1 = 1 + T^{m_1}$ but $u_2 = \sum_{i \in \mathbb{Z}} a_i T^i$ must remain as a power series. By the torsor equation $Z_1^p = 1 + T_1^m$, we know from Case 4 that we can write T as:

$$T = Z^p \left(1 - Z^{-m_1 p} \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (1+Z^{m_1})^k \right)^{\frac{1}{m_1}}$$

where $Z := (Z_2 - 1)^{1/m_1}$ is the parameter of $X_{2,b}$. Again, we know this is the correct parameter because this is what we obtained on the special fibre. As before, for purposes of convenience, we simplify this expression for T to $Z^p \left(1 - Z^{-m_1 p} B\right)^{\frac{1}{m_1}}$ where $B = \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (1+Z^{m_1})^k$. Then we can base change the torsor equation $X_{2,b} \to X_b$:

$$\begin{aligned}
Z_2^p &= 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i \\
&= 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} \left(1 - Z^{-m_1 p} B\right)^{\frac{i}{m_1}} + \sum_{v_K(a_i)>0} a_i Z^{ip} \left(1 - Z^{-m_1 p} B\right)^{\frac{i}{m_1}} \\
&= 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} \left(1 - \frac{i}{m_1} Z^{-m_1 p} B + \ldots\right) \\
&\quad + \sum_{v_K(a_i)>0} a_i Z^{ip} \left(1 - \frac{i}{m_1} Z^{-m_1 p} B + \ldots\right) \\
&= 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} - \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} \frac{i}{m_1} Z^{-m_1 p} B \\
&\quad + \sum_{v_K(a_i)>0} a_i Z^{ip} - \sum_{v_K(a_i)>0} a_i Z^{ip} \frac{i}{m_1} Z^{-m_1 p} B + \text{higher order terms}
\end{aligned}$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ but $v_K(c_i) > 0$ and so we have:

$$\begin{aligned}
Z_2^p &= 1 + \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{ip} - \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} Z^{-m_1 p} B \\
&\quad + \sum_{v_K(c_i)>0} c_i Z^{ip} - \sum_{v_K(c_i)>0} c_i Z^{ip} \frac{i}{m_1} Z^{-m_1 p} B \\
&\quad + \sum_{v_K(a_i)>0} a_i Z^{ip} - \sum_{v_K(a_i)>0} a_i Z^{ip} \frac{i}{m_1} Z^{-m_1 p} B + \text{higher order terms} \\
&= 1 + \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} \left(b_i Z^i\right)^p - \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} Z^{-m_1 p} B \\
&\quad + \sum_{v_K(d_i)>0} d_i Z^{ip} - \sum_{v_K(d_i)>0} d_i Z^{ip} \frac{i}{m_1} Z^{-m_1 p} B + \text{higher order terms}
\end{aligned}$$

where we take $d_i = c_i + a_i$ to unify the positive valuation summations. Now, by the multinomial theorem, we can write the first two terms $1 + \sum \left(b_i Z^i\right)^p$ as:

$$\left(1 + \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i\right)^p - p \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i + \text{higher order terms}$$

so that the torsor equation can now be expressed as

$$\begin{aligned}Z_2^p = &\left(1 + \sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i Z^i\right)^p - p\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i Z^i - \sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i^p Z^{ip}\frac{i}{m_1}Z^{-m_1 p}B\\ &+ \sum_{v_K(d_i)>0} d_i Z^{ip} - \sum_{v_K(d_i)>0} d_i Z^{ip}\frac{i}{m_1}Z^{-m_1 p}B + \text{higher order terms}\end{aligned}$$

Then, after multiplying by the inverse p-power,

$$\left(1 + \sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i Z^i\right)^{-p} = 1 - p\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i Z^i + \text{higher order terms}$$

we have:

$$\begin{aligned}Z_2^p &= 1 - p\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i Z^i - \sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i^p Z^{ip}\frac{i}{m_1}Z^{-m_1 p}B + \text{higher order terms}\\ &= 1 - p\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i Z^i - \sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i^p Z^{ip}\frac{i}{m_1}Z^{-m_1 p}\left(\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k(1+Z^{m_1})^k\right)\\ &\qquad + \text{higher order terms}\end{aligned}$$

Then up to units we have:

$$\begin{aligned}Z_2^p = 1 - \pi^{v_K(p)}\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i Z^i + \pi^{v_K(p)}\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i^p\frac{i}{m_1}Z^{(i-m_1)p}(1+Z^{m_1})\\ + \text{higher order terms}\end{aligned}$$

In order to determine which is the smallest power of Z and hence the conductor variable m_1', we need to compare m_2 with $m_2 p - m_1(p-1)$ since both summations have the same π valuation. Note that $m_2 \leq m_2 p - m_1(p-1)$ is equivalent to $m_1 \leq m_2$. Taking $m_1 \leq m_2$, we have $m_1' = m_2$ and, by symmetry, $m_2' = m_1 p - m_2(p-1)$

---

6. $(\mathcal{H}_{n_1}, \mathcal{H}_{n_2})$

Here $m_1, m_2 \in \mathbb{Z}$ and both are coprime to p. The torsor equation for $\mathcal{H}_{n_1}$ is given by $(\pi^{n_1}Z_1 + 1)^p = 1 + \pi^{pn_1}u_1$ and the second $\mathcal{H}_{n_2}$ torsor by $(\pi^{n_2}Z_2 +$

$1)^p = 1 + \pi^{pn_2} u_2$ where $u_1, u_2 \in A^\times$. The two torsors have associated conductor variables m_1, m_2 respectively.

We begin by computing m_1'. We can choose the parameter T so that $u_1 = T^{m_1}$ but $u_2 = \sum_{i \in \mathbb{Z}} a_i T^i$ remains as a power series. By Case 4, we can express T in terms of $Z := Z_1^{1/m_1}$, the parameter for $X_{1,b}$, as:

$$T = Z^p \left(1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{-n_1(p-k)} Z^{-m_1(p-k)} \right)^{\frac{1}{m_1}}$$

and, for convenience, we can set $B = \sum_{k=1}^{p-1} \binom{p}{k} \pi^{-n_1(p-k)} Z^{-m_1(p-k)}$ so that we can write $T = Z^p (1+B)^{\frac{1}{m_1}}$. Then we can base change the torsor equation $X_{2,b} \to X_b$:

$$\begin{aligned}
(\pi^{n_2} Z_2 + 1)^p &= 1 + \pi^{pn_2} \sum_{i \in \mathbb{Z}} a_i T^i \\
&= 1 + \pi^{pn_2} \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i T^i + \pi^{pn_2} \sum_{v_K(a_i)>0} a_i T^i \\
&= 1 + \pi^{pn_2} \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} (1+B)^{\frac{i}{m_1}} + \pi^{pn_2} \sum_{v_K(a_i)>0} a_i Z^{ip} (1+B)^{\frac{i}{m_1}} \\
&= 1 + \pi^{pn_2} \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} \left(1 + \frac{i}{m_1} B + \ldots \right) \\
&\quad + \pi^{pn_2} \sum_{v_K(a_i)>0} a_i Z^{ip} \left(1 + \frac{i}{m_1} B + \ldots \right) \\
&= 1 + \pi^{pn_2} \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} + \pi^{pn_2} \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} \frac{i}{m_1} B \\
&\quad + \pi^{pn_2} \sum_{v_K(a_i)>0} a_i Z^{ip} + \pi^{pn_2} \sum_{v_K(a_i)>0} a_i Z^{ip} \frac{i}{m_1} B \\
&\quad + \text{higher order terms}
\end{aligned}$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$

with $v_K(b_i) = 0$ but $v_K(c_i) > 0$ and so we have:

$$\begin{aligned}
(\pi^{n_2} Z_2 + 1)^p = 1 &+ \pi^{pn_2} \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{ip} + \pi^{pn_2} \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} B \\
&+ \pi^{pn_2} \sum_{v_K(c_i)>0} c_i Z^{ip} + \pi^{pn_2} \sum_{v_K(c_i)>0} c_i Z^{ip} \frac{i}{m_1} B \\
&+ \pi^{pn_2} \sum_{v_K(a_i)>0} a_i Z^{ip} + \pi^{pn_2} \sum_{v_K(a_i)>0} a_i Z^{ip} \frac{i}{m_1} B \\
&+ \text{higher order terms} \\
= 1 &+ \sum_{v_K(b_i)=0, i \geq m_2} \left(\pi^{n_2} b_i Z^i\right)^p + \pi^{pn_2} \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} B \\
&+ \pi^{pn_2} \sum_{v_K(d_i)>0} d_i Z^{ip} + \pi^{pn_2} \sum_{v_K(d_i)>0} d_i Z^{ip} \frac{i}{m_1} B \\
&+ \text{higher order terms}
\end{aligned}$$

where $d_i = c_i + a_i$. By the multinomial identity, we can express the two leading terms of this equation $1 + \sum \left(\pi^{n_2} b_i Z^i\right)^p$ as follows

$$\left(1 + \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} \pi^{n_2} b_i Z^i\right)^p - p \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} \pi^{n_2} b_i Z^i + \text{higher order terms}$$

and then by substitution we obtain an equation of the form:

$$\begin{aligned}
(\pi^{n_2} Z_2 + 1)^p = &\left(1 + \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} \pi^{n_2} b_i Z^i\right)^p - p \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} \pi^{n_2} b_i Z^i \\
&+ \pi^{pn_2} \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} B + \pi^{pn_2} \sum_{v_K(d_i)>0} d_i Z^{ip} \\
&+ \pi^{pn_2} \sum_{v_K(d_i)>0} d_i Z^{ip} \frac{i}{m_1} B + \text{higher order terms}
\end{aligned}$$

Multiplying by the inverse p-power of the leading term, $\left(1 + \sum \pi^{n_2} b_i Z^i\right)^{-p}$, gives rise to a torsor equation:

$$\begin{aligned}(\pi^{n_2}Z_2+1)^p &= 1 - p\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} \pi^{n_2}b_iZ^i \\ &\quad + \pi^{pn_2}\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i^pZ^{ip}\frac{i}{m_1}B + \text{higher order terms} \\ &= 1 - p\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} \pi^{n_2}b_iZ^i \\ &\quad + \pi^{pn_2}\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i^pZ^{ip}\frac{i}{m_1}\left(\sum_{k=1}^{p-1}\binom{p}{k}\pi^{n_1(k-p)}Z^{-m_1(p-k)}\right) \\ &\quad + \text{higher order terms}\end{aligned}$$

Then up to units we have:

$$\begin{aligned}(\pi^{n_2}Z_2+1)^p &= 1 - \pi^{v_K(p)+n_2}\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_iZ^i \\ &\quad + \pi^{v_K(p)+pn_2-n_1(p-1)}\sum_{\substack{v_K(b_i)=0\\ i\geq m_2}} b_i^pZ^{ip}\frac{i}{m_1}Z^{-m_1(p-1)} \\ &\quad + \text{higher order terms}\end{aligned}$$

We have to compare $v_K(p)+n_2$ with $v_K(p)+pn_2-n_1(p-1)$ in order to determine the smallest power of π so that we know where to find the conductor variable. It turns out that $v_K(p)+n_2 < v_K(p)+pn_2-n_1(p-1)$ is equivalent to $n_1 < n_2$ and so when this happens, $m_1' = m_2$ and when $n_1 > n_2$, we have $m_1' = m_2p - m_1(p-1)$. We also need to consider the case where $n_1 = n_2$. This results in comparing m_2 with $m_2p - m_1(p-1)$ and from earlier we have that:

$$m_1' = \min\{m_2, m_2p - m_1(p-1)\} = \begin{cases} m_2 & \text{if } m_1 < m_2 \\ m_2p - m_1(p-1) & \text{if } m_1 \geq m_2 \end{cases}$$

Now we want to determine m_2' but by symmetry of this case we already have the result. In particular, if $n_1 < n_2$ then $m_2' = m_1p - m_2(p-1)$, if $n_1 > n_2$ then

$m_2' = m_1$ and, finally, if $n_1 = n_2$ then:

$$m_2' = \min\{m_2, m_2 p - m_1(p-1)\} = \begin{cases} m_1 p - m_2(p-1) & \text{if } m_1 < m_2 \\ m_1 & \text{if } m_1 \geq m_2 \end{cases}$$

All six possible cases have now been treated. □

The following is an immediate corollary to Theorem 4.2.1. In particular, we move from the **conductor variable** m and express the same result in terms of the **conductor** c.

Corollary 4.2.2. *With the situation and notations described in Theorem 4.2.1, let c_i' denote the conductor of the torsor $Y_b \to X_{i,b}$. Then, for all possible pairs of G_1 and G_2, we can express the c_i' in terms of the c_i conductors for $i = 1, 2$ as follows:*

Table 4.2: Conductors c_1', c_2' in terms of conductors c_1, c_2

(G_1, G_2)	c_1'	c_2'
$\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_{v_K(\lambda)}$	*If $c_1 \geq c_2$ then* $c_1' = c_2$ *If $c_1 < c_2$ then* $c_1' = c_2 p - c_1(p-1)$	*If $c_1 \geq c_2$ then* $c_2' = c_1 p - c_2(p-1)$ *If $c_1 < c_2$ then* $c_2' = c_1$
$\mathcal{H}_{v_K(\lambda)}, \mu_p$	$c_1' = c_2 p - c_1(p-1)$	$c_2' = c_1$
$\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_n$	$c_1' = c_2 p - c_1(p-1)$	$c_2' = c_1$

Continued on next page

(G_1, G_2)	c_1'	c_2'
$\mathcal{H}_n, \mu_p$	$c_1' = c_2 p - c_1(p-1)$	$c_2' = c_1$
μ_p, μ_p *Assuming c_1 or c_2 is non-zero.*	*If $c_1 \geq c_2$ then* $c_1' = c_2$ *If $c_1 < c_2$ then* $c_1' = c_2 p - c_1(p-1)$	*If $c_1 \geq c_2$ then* $c_2' = c_1 p - c_2(p-1)$ *If $c_1 < c_2$ then* $c_2' = c_1$
$\mathcal{H}_{n_1}, \mathcal{H}_{n_2}$	*If $n_1 < n_2$ then* $c_1' = c_2$ *If $n_1 > n_2$ then* $c_1' = c_2 p - c_1(p-1)$ *If $n_1 = n_2$ and $c_1 > c_2$ then* $c_1' = c_2$ *If $n_1 = n_2$ and $c_1 \leq c_2$ then* $c_1' = c_2 p - c_1(p-1)$	*If $n_1 < n_2$ then* $c_2' = c_1 p - c_2(p-1)$ *If $n_1 > n_2$ then* $c_2' = c_1$ *If $n_1 = n_2$ and $c_1 > c_2$ then* $c_2' = c_1 p - c_2(p-1)$ *If $n_1 = n_2$ and $c_1 \leq c_2$ then* $c_2' = c_1$

Proof. Substituting $c_1 = -m_1$, $c_2 = -m_2$, $c_1' = -m_1'$ and $c_2' = -m_2'$, as per the conductor definition, into Theorem 4.2.1 gives this result. □

We are also able to state when we have the structure of a torsor by taking into account when base changing in the above proof without additional modification resulted in the equation of the normalisation:

Theorem 4.2.3. *Let $f_i : X_{i,b} \to X_b$ be non-trivial Galois covers of degree p on the formal boundary which are generically disjoint for $i = 1, 2$. Let G_i denote the group schemes corresponding to the torsor acting on each of these covers for $i = 1, 2$ and let Y_b be as defined in Theorem 4.2.1. We have that $Y_b = X_{1,b} \times_{X_b} X_{2,b}$, in which case $Y_b \to X_b$ is a torsor under $G_1 \times G_2$, if and only if at least one of the two group schemes G_i is the étale group scheme $\mathcal{H}_{v_K(\lambda)}$.*

Proof. Note that Theorem 3.2.5 gives us a torsor structure under $G_1 \times G_2$ on $Y_b \to X_b$ if and only if $Y_b = X_{1,b} \times_{X_b} X_{2,b}$. The remaining argument here relates to the proof of Theorem 4.2.1 where the six cases are treated individually.

($\Leftarrow$) Demonstrated at the start of the proof of Theorem 4.2.1 when justifying working on the special fibre when at least one of the group schemes is étale.

($\Rightarrow$) By contrapositive, suppose that neither of the group schemes are étale; namely cases 4-6 in Theorem 4.2.1. The proof of that theorem revealed that after base change in each of these cases one obtained an equation which when reduced modulo π would be a p-power. In particular, on the special fibre one has a torsor equation of the form $z^p = \overline{f} = \overline{g}^p \Leftrightarrow (z - \overline{g})^p = 0$ with $z - \overline{g} \neq 0$. This is a contradiction to the special fibre being reduced as $z - \overline{g}$ would be a nilpotent element. $\square$

Definition 4.2.4. *For the extension B/A where $X_b = \mathrm{Spf}\,(A)$ and $Y_b = \mathrm{Spf}\,(B)$ as in Theorem 4.2.1, we define the **special different** by*

$$d_{s_1} = (c_1 - 1)p(p-1) + (c_1' - 1)(p-1)$$

$$d_{s_2} = (c_2 - 1)p(p-1) + (c_2' - 1)(p-1)$$

This d_s is really the same as the $\varphi(s)$ which appears in Kato's vanishing cycles formula (Theorem 6.7 in [6]). We will also see this variable makes an appearance in our genus formula in Theorem 5.2.1 in the next chapter.

Lemma 4.2.5. *The two special differents are in fact equal: $d_{s_1} = d_{s_2}$*

Proof. This result is immediately given by substituting the possible values for c_1' and c_2' under each of the six cases given in Corollary 4.2.2. $\square$

Corollary 4.2.6. *We have the following relationship between the acting conductors:*

$$c_2' - c_1' = (c_1 - c_2)p$$

Proof. This result follows from rearranging the relationship between conductors given by $d_{s_1} = d_{s_2}$. □

Had this corollary been given, we would have only needed to perform half of the computations in the proof of Theorem 4.2.1. In particular, with m_1' (or equivalently c_1') obtained, m_2' (or equivalently c_2') would be determined by this relationship. In fact, this formula could have been derived independently using the theory of higher ramification groups as per [18] but only for $\left(\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_{v_K(\lambda)}\right)$, the first of the six cases in Theorem 4.2.1. This is because this ramification theory only holds when the residue field extension is separable and so the group scheme must be étale.

From the calculations in the proof of Theorem 4.2.1 it is also possible to compute the degree of the different δ; namely, the degree of the different ideal $\mathfrak{D}$ in the extension $Y_b \to X_b$. In a DVR every ideal I is of the form $I = (\pi^n)$ where π is a uniformiser and n is an integer. When $I = \mathfrak{D}$ we call n the degree of the different δ. Since the ramification index $e = 1$ by Epp's result, we have that $\mathfrak{D}_{B/A} = \mathfrak{D}_{B/A_1}\mathfrak{D}_{A_1/A}$ is multiplicative over extensions and $\delta = e\delta_1 + \delta_1' = \delta_1 + \delta_1'$. Climbing up the right-hand side of the diagram instead gives us $\delta = \delta_2 + \delta_2'$ similarly.

$$\begin{tikzcd}
 & Y_b \arrow[dl, "\delta_1'"'] \arrow[dd, "\delta"] & \\
X_{1,b} \arrow[dr, "\delta_1"'] & & X_{2,b} \arrow[ul, "\delta_2'"'] \arrow[dl, "\delta_2"] \\
 & X_b &
\end{tikzcd}$$

Theorem 4.2.7. *With the situation described in Theorem 4.2.1, let δ_i' denote the degree of the different corresponding to the extension $Y_b \to X_{i,b}$. Then, for all possible pairs of G_1 and G_2, we can explicitly state the values for δ_i' as follows:*

Table 4.3: Degree of the differents δ_1', δ_2'

(G_1, G_2)	δ_1'	δ_2'
$\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_{v_K(\lambda)}$	0	0
$\mathcal{H}_{v_K(\lambda)}, \mu_p$	$v_K(p)$	0
$\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_n$	$v_K(p) - n(p-1)$	0
$\mathcal{H}_n, \mu_p$	$v_K(p) - \frac{v_K(p)-n(p-1)}{p}(p-1)$	$v_K(p) - \frac{v_K(p)+n}{p}(p-1)$
μ_p, μ_p	$v_K(p) - \frac{v_K(p)}{p}(p-1)$	$v_K(p) - \frac{v_K(p)}{p}(p-1)$
$\mathcal{H}_{n_1}, \mathcal{H}_{n_2}$	*If* $n_1 \leq n_2$ *then* δ_1' *equals* $v_K(p) - \frac{v_K(p)+n_2}{p}(p-1)$ *If* $n_1 > n_2$ *then* δ_2' *equals* $v_K(p) - \frac{v_K(p)+n_2p-n_1(p-1)}{p}(p-1)$	*If* $n_1 \leq n_2$ *then* δ_2' *equals* $v_K(p) - \frac{v_K(p)+n_1p-n_2(p-1)}{p}(p-1)$ *If* $n_1 > n_2$ *then* δ_2' *equals* $v_K(p) - \frac{v_K(p)+n_1}{p}(p-1)$

Proof. For an arbitrary torsor with conductor variable m and general torsor equation $Z^p = 1 + \pi^{np}T^m +$ higher terms, the degree of the different is given by $\delta = v_K(p) - n(p-1)$ where $0 \leq n \leq v_K(\lambda)$. This means that computing the degree of the different δ reduces to obtaining n from the exponent of π in the term corresponding to the conductor variable m. From our calculations obtained in the proof of Theorem 4.2.1, we can simply read off the n value in each of the cases and substitute into the formula $v_K(p) - n(p-1)$ to obtain the degree of the different at that particular stage. Strictly speaking, we can only rely on this approach for the last three cases because in the first three cases we worked modulo π on the special fibre. However, by the equality present for the étale cases as per Theorem 4.2.3, we know the degree of the different is preserved and this can be verified, if necessary, by working above X_b over R. $\square$

Example 4.2.8. *With the (p,p) case established it is possible to manually perform the same calculations in the $(p,...,p)$ setting. We illustrate this with a (p,p,p) example, using the (p,p) theorem iteratively at each stage to determine the conductors in terms of the base level conductors.*

$$\begin{tikzcd}
 & & Y \arrow[dl, "c_1''"'] \arrow[dr, "c_2''"] & & \\
 & \widetilde{X_1 \times X_2} \arrow[dl, "c_1'"] \arrow[dr, "c_2'"'] & & \widetilde{X_2 \times X_3} \arrow[dl, "c_3'"] \arrow[dr, "c_4'"'] & \\
X_1 \arrow[drr, "c_1"'] & & X_2 \arrow[d, "c_2"'] & & X_3 \arrow[dll, "c_3"] \\
 & & X & &
\end{tikzcd}$$

Suppose $X_i \to X$ are torsors under the group scheme G_i for $i = 1,2,3$ which are pairwise disjoint. For the purposes of an example, let $G_i = \mathcal{H}_{v_K(\lambda)}$ for all i and assume $c_1 \leq c_2 \leq c_3$ and $c_2' \leq c_3'$. Then, by applying the type (p,p) formula iteratively we have:

$$\begin{aligned}
c_1'' &= c_3'p - c_2'(p-1) \\
&= (c_3p - c_2(p-1))\,p - (c_1)\,(p-1) \\
&= c_3p^2 - c_2p(p-1) - c_1(p-1)
\end{aligned}$$

Similarly for c_2''.

Chapter 5

Vanishing cycles in a Galois cover of type (p, p)

5.1 Computation of vanishing cycles in degree p

Postponed from the introduction, we now formally define the genus of a point.

Definition 5.1.1. *For an R-curve X and a closed point x belonging to the curve's special fibre X_k, we let $\mathcal{X} := \operatorname{Spf}\left(\widehat{\mathcal{O}}_{X,x}\right)$ denote the formal spectrum of the completion of the local ring of X at x. Then the* ***genus of the point*** *x is given by:*

$$g_x := \delta_x - r_x + 1$$

where

- $\delta_x = \dim_k \left(\widetilde{\mathcal{O}_x} / \mathcal{O}_x\right)$
- *r_x is the number of maximal ideals in $\widetilde{\mathcal{O}_x}$.*

where $\mathcal{O}_x$ denotes the stalk of the special fibre $\mathcal{X}_k$ at x and $\widetilde{\mathcal{O}_x}$ denotes its normalisation in its total ring of fractions.

Perhaps not unlike the genus of a curve which is a topological invariant and offers a measure for the geometric complexity of the curve, the genus of a point can be regarded as a measure of the singularity at the point. If $g_x = 0$, the point

x is either a smooth or ordinary multiple point (where $\delta_x = r_x - 1$). An ordinary double point is a particular case of an ordinary multiple point where $r_x = 2$ and $\delta_x = 1$. For $g_x \geq 1$ the point is not smooth and has some singularity.

Here is a result which provides an explicit formula—a local Riemann-Hurwitz formula—comparing the genus in a Galois cover of degree p:

Theorem 5.1.2 (Theorem 3.4 in [15]). *Let $X := \mathrm{Spf}\left(\widehat{\mathcal{O}}_x\right)$ be the formal germ of an R-curve at a closed point x with X_k reduced. Let $f : Y \to X$ be a Galois cover of degree p with Y normal and local. Assume that the special fibre Y_k of Y is reduced. Let $\{\wp_i\}_{i \in I}$ be the minimal prime ideals of $\widehat{\mathcal{O}}_x$ which contain π, and let $X_{b_i} := \mathrm{Spf}\left(\widehat{\mathcal{O}}_{\wp_i}\right)$ be the formal completion of the localisation of X at $\wp_i$. For each $i \in I$ the above cover f induces a torsor $f_{b_i} : Y_{b_i} \to X_{b_i}$ under a finite and flat R-group scheme G_i of rank p above the boundary X_{b_i} with conductor c_i, as occurring in Definition 2.4.1. If $y \in Y$ is the closed point of Y, then:*

$$2g_y - 2 = p(2g_x - 2) + d_\eta - d_s$$

where g_y (resp. g_x) denotes the genus of the singularity at y (resp. x), d_η is the degree of the divisor of ramification in the morphism $f_K : Y_K \to X_K$ induced by f on the generic fibre, and $d_s = \sum_{i \in I}(c_i - 1)(p - 1)$.

We will refer to this formula simply as the genus formula.

Proof. See [15] for a detailed proof of this result. In short, a compactification of the Galois cover is constructed using formal patching and the result is deduced from comparing the genus of the special and generic fibres in the compactification. □

The following two corollaries are given in [15] where their proofs are omitted due to their elementary nature. For completeness, we give their proofs.

Corollary 5.1.3 (Proposition 4.1.1 in [15]). *Let $X = \mathrm{Spf}\left(R[[T]]\right)$ be the formal germ of an R-curve at a smooth point x and let $Y \to X$ be a degree p Galois cover with Y local and Y_k reduced. Let $Y_b \to X_b = \mathrm{Spf}\left(R[[T]]\{T^{-1}\}\right)$ denote the corresponding degree p Galois cover on the boundary of this germ. Let y be the unique closed point of Y_k. Let $d_\eta = r(p-1)$, where r is the number of points in X_K which ramify, denote the degree of* *the divisor of ramification in the morphism* $f_K : Y_K \to X_K$*, while* c *is the conductor on the boundary. If* Y_k *is unibranched at* y *then*

$$g_y = \frac{(r-c-1)(p-1)}{2}$$

and if Y_k *is* p*-branched at* y *then*

$$g_y = \frac{(r-2)(p-1)}{2}$$

Proof. We use the genus formula and, of course, $g_x = 0$ since x is a smooth point. Recall that d_η is the number of ramified points multiplied by the ramification index minus 1 giving $d_\eta = r(p-1)$ in both cases.

1. Here $d_s = (c-1)(p-1)$ as in general.

$$\begin{aligned} 2g_y - 2 &= p(2g_x - 2) + d_\eta - d_s \\ &= p(0-2) + r(p-1) - (c-1)(p-1) \\ &= -2p + (r-c+1)(p-1) \end{aligned}$$

   and so $2g_y = -2(p-1) + (r-c+1)(p-1)$ which gives the result.

2. Here $d_s = 0$ as the boundary decomposes and so there is no conductor.

$$\begin{aligned} 2g_y - 2 &= p(2g_x - 2) + d_\eta - d_s \\ &= p(0-2) + r(p-1) - 0 \\ &= -2p + r(p-1) \end{aligned}$$

   and so $2g_y = -2(p-1) + r(p-1)$ which gives the result.

□

Corollary 5.1.4 (Proposition 4.2.1 in [15]). *Let* $X := \mathrm{Spf}\left(\frac{R[[S,T]]}{(ST-\pi^e)}\right)$ *be the formal germ of an* R*-curve at an ordinary double point* x *of thickness* e*. We denote by* $X_{b_1} :=$ $\mathrm{Spf}(R[[S]]\{S^{-1}\})$ *and* $X_{b_2} := \mathrm{Spf}(R[[T]]\{T^{-1}\})$ *the two boundaries of* X*. Let* $f : Y \to X$ *be a Galois cover with group* $\mathbb{Z}/p\mathbb{Z}$*. Assume* Y *is normal and local and that the special fibre* Y_k *of* Y *is reduced. We assume that* Y_k *has two branches at the point* y*. Let*

$d_\eta := r(p-1)$ be the degree of the divisor of ramification in the morphism $f_K : Y_K \to X_K$ and let c_i denote the conductors on the boundaries X_{b_i} of X for $i = 1, 2$. Then:

$$g_y = \frac{(r - c_1 - c_2)(p-1)}{2}$$

and, in particular, if y is a double point and $r = 0$, we have that $c_1 = -c_2$.

Note that in this situation y cannot be smooth because its image is a double point so it is at least a double point.

Proof. We have that $g_x = 0$ as x is an ordinary double point, $d_\eta := r(p-1)$ and $d_s = (c_1 - 1)(p-1) + (c_2 - 1)(p-1)$ as there are two boundaries. Then, by the vanishing cycles formula,

$$\begin{aligned} 2g_y - 2 &= p(2g_x - 2) + d_\eta - d_s \\ &= p(0-2) + r(p-1) - (c_1 - 1)(p-1) - (c_2 - 1)(p-1) \end{aligned}$$

and so

$$\begin{aligned} 2g_y &= 2 - 2p + r(p-1) - (c_1 - 1)(p-1) - (c_2 - 1)(p-1) \\ &= -2(p-1) + r(p-1) - (c_1 - 1)(p-1) - (c_2 - 1)(p-1) \\ &= (p-1)\,(-2 + r - c_1 + 1 - c_2 + 1) \\ &= (p-1)\,(r - c_1 - c_2) \end{aligned}$$

If y is a smooth or double point, then $g_y = 0$ and so when $r = 0$ we have by the double-point formula:

$$(-c_1 - c_2)(p-1) = 0 \Rightarrow -c_1 - c_2 = 0 \Rightarrow c_1 = -c_2$$

□

5.2 Computation of vanishing cycles in degree (p, p)

In this section we will prove that the degree p genus formula can be extended to the case of (p, p). In short, our mode of attack will see us apply the degree p formula twice iteratively but to do this we need to ensure we have all the relevant data at each of the two steps. In particular, we require knowledge of the ramification at each step as well as information on the group schemes and conductors acting on the boundaries. We will then be in a position to apply the formula once on the first step to find g_{y_1} (dependent on g_x) and then again on the second step to find g_{y_2} (dependent on g_{y_1}) and substitute one into the other giving a genus formula for g_{y_2} in terms of g_x across the Galois cover with group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ (see the picture below).

Suppose $f : Y \to X$ is a Galois cover of type (p, p) where X is a formal germ of an R-curve and Y is local and normal and whose special fibre Y_k is reduced. We can express it as the compositum of two degree p Galois covers $Y_i \to X$ where Y_i is normal for $i = 1, 2$ as follows:

$$\begin{tikzcd}
y_2 \arrow[d] \\
y_1 \arrow[d] \arrow[u, bend left, "g_{y_2}"] \\
x \arrow[u, bend left, "g_{y_1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr] \arrow[dd] & \\
Y_1 \arrow[dr, "\mathbb{Z}/p\mathbb{Z}"'] & & Y_2 \arrow[dl, "\mathbb{Z}/p\mathbb{Z}"] \\
 & X &
\end{tikzcd}$$

Let $\{x_i\}_i \subset X_K$ be the set of the (finitely many) branch points of the cover. Suppose there are r ramification points of the cover and these are denoted by $\{y_{ij}\}_{i,j}$, the set of all ramified points in Y, so that $r = \text{Card}\{\{y_{ij}\}_{i,j}\}$. We can view these in two steps where we assume r_1 points ramify at the first step—that is, between X and Y_1— and r_2 points ramify at the second—between, Y_1 and Y. Because at each individual stage the cover has group $G = \mathbb{Z}/p\mathbb{Z}$ consisting of p elements, any decomposition subgroup of G will have cardinality 1 or p and this in turn means that the inverse image of a branch point has cardinality p or 1. However, at each individual stage the degree of the cover is p and so a point is ramified if its inverse image contains just one point. At each of these two stages, either a single point or p points can sit above a branch point which results in four cases. The case of a branch point sitting below p points, each of which sit below p

points contradicts the definition of a branch point since over both steps its inverse image will contain p^2 points and this is exactly the degree of the (p, p) cover. The case of a branch point sitting below one point and which itself sits below one point is also not possible. If it were to happen, the decomposition group would be full, namely $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, which is not cyclic but the decomposition group has to be cyclic in characteristic 0. Therefore, there remain just two cases: If at the first stage we have one point above a branch point then at the second stage there must be p points which sit above it *or* if at the first stage we have p points above a branch point then at the second stage there must be just 1 point above each of these p points.

For a branch point x, we have one of the two situations occurring:

$$
\begin{tikzcd}
\widetilde{y}_1 \arrow[drr] & \widetilde{y}_2 \arrow[dr] & & \cdots & \widetilde{y}_p \arrow[dll] \\
 & & y \arrow[d] & & \\
 & & x & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{y}_1 \arrow[d] & \widetilde{y}_2 \arrow[d] & & \cdots & \widetilde{y}_p \arrow[d] \\
y_1 \arrow[drr] & y_2 \arrow[dr] & & \cdots & y_p \arrow[dll] \\
 & & x & &
\end{tikzcd}
$$

The diagram on the left depicts ramification occurring at the first stage while the diagram on the right depicts ramification occurring at the second stage. Since the cases are disjoint, this gives us that $r = r_1 p + r_2$ where:

- $r_1 = \text{Card}\{$ ramified points in $Y_1 \to X$ $\}$
- $r_2 = \text{Card}\{$ ramified points in $Y \to Y_1$ $\}$

For the branch points $\{x_i\} \subset X_K$ in the cover $f : Y \to X$, we can visualise the general picture, including decomposition groups, as follows:

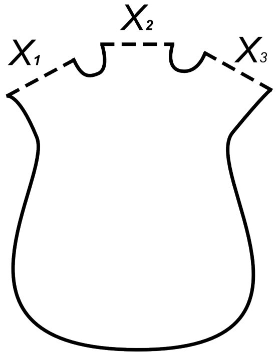
$$\begin{tikzcd}
\{\widetilde{y}_{i,j}\} \subset Y_K \arrow[d, bend left, "D_{\widetilde{y}_{i,j}}"] \arrow[dd, bend right, "f"'] \\
\{y_{i,j}\} \subset Y_{1.K} \arrow[d, bend left, "D_{y_{i,j}}"] \\
\{x_i\} \subset X_K
\end{tikzcd}$$

where $D_{y_{i,j}} \leq \mathbb{Z}/p\mathbb{Z}$ and $D_{\widetilde{y}_{i,j}} \leq \mathbb{Z}/p\mathbb{Z}$ denote the decomposition groups of the point $y_{i,j}$ at the first stage and the point $\widetilde{y}_{i,j}$ at the second stage respectively. If only one points sits immediately above x_i in Y_1 then the order of the decomposition group $D_{y_{i,j}}$ will equal p and, otherwise, the opposite is true. This means we have a natural test for ramification in the first and second step.

$$p = |D_{y_{i,j}}| \Leftrightarrow p \neq |D_{\widetilde{y}_{i,j}}| = 1 \Leftrightarrow x_i \text{ ramifies at 1st stage}$$

$$p \neq |D_{y_{i,j}}| = 1 \Leftrightarrow p = |D_{\widetilde{y}_{i,j}}| \Leftrightarrow x_i \text{ ramifies at 2nd stage}$$

Now we turn to address the question of the boundaries. Let $\{X_{b_i}\}_i$ denote the boundaries of X. If $X := \operatorname{Spf}\left(\widehat{\mathcal{O}}_x\right)$ is the formal germ of an R-curve at a closed point x and $\{\wp_i\}_{i\in I}$ are the minimal prime ideals of $\widehat{\mathcal{O}}_x$ which contain π, then the boundaries are given by $X_{b_i} := \operatorname{Spf}\left(\widehat{\mathcal{O}}_{\wp_i}\right)$, the formal completion of the localisation of X at $\wp_i$.



For each i, the Galois cover $f : Y \to X$ induces a Galois cover $f_i : Y_{b_{i,j}} \to X_{b_i}$ on the boundaries where $\{Y_{b_{i,j}}\}_j$ are the set of boundaries above X_{b_i}. Unlike the degree p case, this cover is not a (p, p) torsor under $G_{1,i} \times G_{2,i}$ unless at least one of the group schemes $G_{1,i}$ and $G_{2,i}$ is étale, for the reasons discussed in the two previous chapters. However, at each degree p stage, the cover is indeed a torsor where $c_{1,i}$ and $c_{2,i}$ (respectively $c'_{1,i}$ and $c'_{2,i}$) are the conductors associated to the torsor under the finite flat R-group schemes $G_{1,i}$ and $G_{2,i}$ respectively (respectively $G'_{1,i}$ and $G'_{2,i}$).

Here is the picture when there is a single boundary sitting above X_{b_i} at each stage; the case we refer to as being unibranched throughout.

$$\begin{tikzcd}
 & Y_{b_i} \arrow[dl, "{(G'_{1,i},c'_{1,i})}"'] \arrow[dr, "{(G'_{2,i},c'_{2,i})}"] & \\
Y_{1,b_i} \arrow[dr, "{(G_{1,i},c_{1,i})}"'] & & Y_{2,b_i} \arrow[dl, "{(G_{2,i},c_{2,i})}"] \\
 & X_{b_i} &
\end{tikzcd}$$

Our main theorem of this chapter compares the genus in a Galois cover of type (p, p):

Theorem 5.2.1. *Let $X := \mathrm{Spf}\left(\widehat{\mathcal{O}}_x\right)$ be the formal germ of an R-curve at a closed point x with X_k reduced. Let $f : Y \to X$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$—that is, of type (p, p)—where Y is normal and local and the special fibre Y_k of Y is reduced.*

Let $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ be two degree p Galois covers such that Y is the compositum of Y_1 and Y_2. Let X_{b_i} denote the boundaries of X. The Galois cover f_1 induces a torsor $Y_{1,b_i} \to X_{b_i}$ under a finite and flat R-group scheme of type p with conductor $c_{1,i}$ for each i. Similarly, $c'_{1,i}$ denotes the conductor associated to the torsor of the cover $Y_{b_i} \to Y_{1,b_i}$. We let r_1 (resp. r_2) denote the number of ramified points in $Y_1 \to X$ (resp. $Y \to Y_1$).

If $y \in Y$ is the closed point of Y, then:

$$2g_y - 2 = p^2(2g_x - 2) + d_\eta - d_s$$

where g_y (resp. g_x) denotes the genus of the singularity at y (resp. x), $d_\eta := (r_1 + r_2)p(p-1)$ is the degree of the divisor of ramification in the morphism $f_K : Y_K \to X_K$ induced by f on the generic fibre and

$$\begin{aligned} d_s = & \sum_{\substack{\text{boundary unibranched} \\ \text{throughout}}} (c'_{1,i} - 1)(p-1) + (c_{1,i} - 1)p(p-1) \\ & + \sum_{\substack{\text{boundary unibranched} \\ \text{then } p\text{-branched}}} (c_{1,i} - 1)p(p-1) + \sum_{\substack{\text{boundary } p\text{-branched} \\ \text{then unibranched}}} (c'_{1,i} - 1)(p-1) \end{aligned}$$

Proof. By Theorem 5.1.2 we have the following genus formula for the degree p Galois cover $Y_1 \to X$ expressing g_{y_1} in terms of g_x:

$$2g_{y_1} - 2 = p(2g_x - 2) + r_1(p-1) - \sum_{i \in I} (c_{1,i} - 1)(p-1)$$

Any boundary can find itself unibranched or p-branched and so we can break up the d_s summation as follows:

$$\begin{aligned} &= p(2g_x - 2) + r_1(p-1) \\ &- \sum_{\substack{i \in I \\ X_{b_i} \text{ unibranched}}} (c_{1,i} - 1)(p-1) - \sum_{\substack{i \in I \\ X_{b_i}\ p\text{-branched}}} (c_{1,i} - 1)(p-1) \end{aligned}$$

Also by Theorem 5.1.2 we have the following genus formula for the degree p Galois cover $Y \to Y_1$ expressing g_{y_2} in terms of g_{y_1}:

$$2g_{y_2} - 2 = p\left(2g_{y_1} - 2\right) + r_2 p(p-1) - \sum_{i \in I} (c'_{1,i} - 1)(p-1)$$

Again, we rewrite the d_s summation into unibranched or p-branched cases:

$$\begin{aligned} &= p(2g_{y_1} - 2) + r_2 p(p-1) \\ &- \sum_{\substack{i \in I \\ Y_{1,b_i} \text{ unibranched}}} (c'_{1,i} - 1)(p-1) - \sum_{\substack{i \in I \\ Y_{1,b_i}\ p\text{-branched}}} (c'_{1,i} - 1)(p-1) \end{aligned}$$

Tracing a boundary X_i through the entire type (p, p) Galois cover $f : Y \to X$, keeping in mind that under the first cover $Y_1 \to X$ the boundary can be p-branched or unibranched and, likewise, under the second cover $Y \to Y_1$, we have four possible cases which can arise. In particular, the boundary is unibranched throughout, unibranched and then p-branched, p-branched and then unibranced or, finally, p-branched throughout. Now, substituting, our first genus formula expressing g_{y_1} in terms of g_x into the second genus formula g_{y_2} in terms of g_{y_1}, will give us a genus formula expressing g_{y_2} in terms of g_x, as required.

$$\begin{aligned}
2g_{y_2} - 2 &= p(2g_{y_1} - 2) + r_2 p(p-1) - \sum_{\substack{i \in I \\ Y_{1,b_i} \text{ uni.}}} (c'_{1,i} - 1)(p-1) - \sum_{\substack{i \in I \\ Y_{1,b_i}\ p\text{-b.}}} (c'_{1,i} - 1)(p-1) \\
&= p \left(p(2g_x - 2) + r_1(p-1) - \sum_{\substack{i \in I \\ X_{b_i} \text{ uni.}}} (c_{1,i} - 1)(p-1) - \sum_{\substack{i \in I \\ X_{b_i}\ p\text{-b.}}} (c_{1,i} - 1)(p-1) \right) \\
&+ r_2 p(p-1) - \sum_{\substack{i \in I \\ Y_{1,b_i} \text{ uni.}}} (c'_{1,i} - 1)(p-1) - \sum_{\substack{i \in I \\ Y_{1,b_i}\ p\text{-b.}}} (c'_{1,i} - 1)(p-1) \\
&= p^2(2g_x - 2) + (r_1 + r_2)\, p(p-1) - \sum_{\substack{i \in I \\ \text{uni., uni.}}} (c_{1,i} - 1)p(p-1) + (c'_{1,i} - 1)(p-1) \\
&- \sum_{\substack{i \in I \\ \text{uni., } p\text{-b.}}} (c_{1,i} - 1)p(p-1) - \sum_{\substack{i \in I \\ p\text{-b., uni.}}} (c'_{1,i} - 1)(p-1) - \sum_{\substack{i \in I \\ p\text{-b., } p\text{-b.}}} 0
\end{aligned}$$

Recall that if the boundary decomposes then $(c-1)(p-1) = 0$. So, we obtain a genus formula in the form

$$2g_y - 2 = p^2(2g_x - 2) + d_\eta - d_s$$

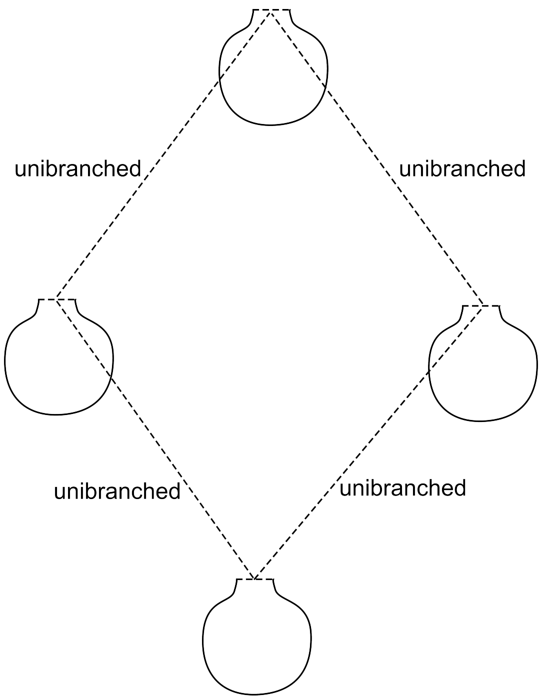
where d_η and d_s are as expressed in the theorem. □

Note that we have produced a formula which depends on ramification data and, by Corollary 4.2.2, solely on base level conductors. Let us now make some remarks on some of the assumptions on which the theorem operates. First, we require that each individual stage is a torsor so that we can apply the vanishing cycles formula at each stage. In particular, Y/Y_i and Y_i/Y must both be torsors and so we require that the special fibres Y_k and $Y_{i,k}$ of Y and Y_i respectively be reduced schemes. However, if Y_k is reduced then $Y_{i,k}$ are automatically reduced. For, to be reduced, it means that when localised at the generic point of the special fibre, we obtain a DVR with uniformiser π but as everything dominates R, π

must also be the uniformiser at the corresponding points in Y_i. Secondly, the assumption Y is local means Y is the spectrum of a local ring (a ring with a unique maximal ideal). This ensures there is one point above x, namely y. If there were several points, albeit finitely many, sitting above x then Y would be a semi local ring (a ring with finitely many maximal ideals).

For illustration purposes, we explain this picture on the boundary with an open disc so that for purposes of simplicity we can assume $g_x = 0$ and we are working with one boundary.

Case 1: unibranched throughout



By Corollary 5.1.3 we have that

$$g_{y_1} = \frac{(r_1 - c_1 - 1)(p-1)}{2}$$

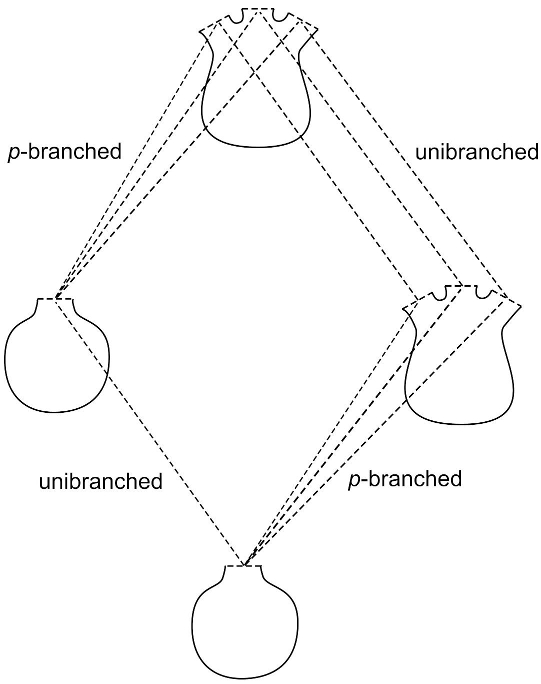
and by the genus formula given in Theorem 5.1.2 we can write

$$2g_{y_2} - 2 = p(2g_{y_1} - 2) + r_2 p(p-1) - (c_1' - 1)(p-1)$$

and so substituting the first equation into the first results in

$$
\begin{aligned}
2g_y - 2 &= p(2g_{y_1} - 2) + r_2 p(p-1) - (c_1' - 1)(p-1) \\
&= p\left((r_1 - m_1 - 1)(p-1) - 2\right) + r_2 p(p-1) - (c_1' - 1)(p-1) \\
&= -2p^2 + 2p^2 - 2p + (r_1 + r_2)p(p-1) - (p-1)(c_1' - 1) + p(c_1 + 1) \\
&= \underbrace{-2p^2 + 2p^2}_{0} - 2p + (r_1 + r_2)p(p-1) - (p-1)\left(c_1' - 1 + p(c_1 + 1)\right) \\
&= p^2(0-2) + (r_1 + r_2)p(p-1) - (p-1)\left(c_1' - 1 + p(c_1 + 1) - 2p\right) \\
&= p^2(0-2) + \underbrace{(r_1 + r_2)p(p-1)}_{d_\eta} - \underbrace{(p-1)\left((c_1' - 1) + p(c_1 - 1)\right)}_{d_s}
\end{aligned}
$$

Case 2: unibranched then p-branched



The conductor c operating in the bottom left and the top right of this diagram remains the same by base change.

By Corollary 5.1.3 we have that

$$g_{y_1} = \frac{(r_1 - c - 1)(p-1)}{2}$$

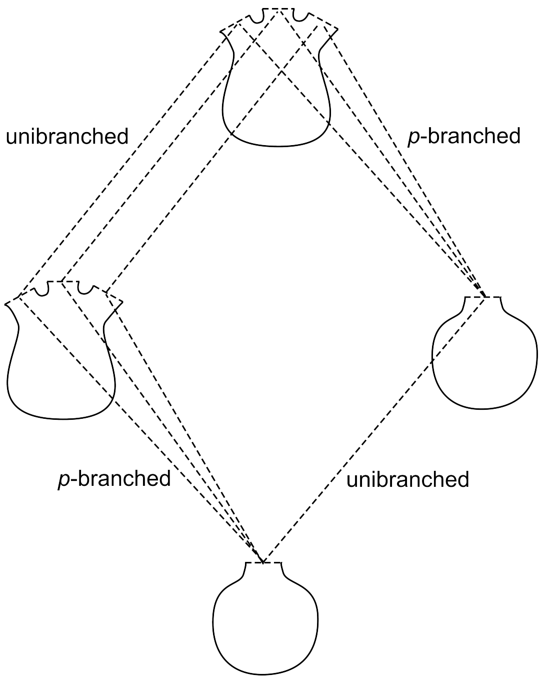
and by the genus formula given in Theorem 5.1.2 we can write

$$2g_{y_2} - 2 = p(2g_{y_1} - 2) + r_2 p(p-1) - 0$$

and so substituting the first equation into the first results in

$$\begin{aligned}
2g_y - 2 &= p(2g_{y_1} - 2) + r_2 p(p-1) \\
&= p\left((r_1 - m - 1)(p-1) - 2\right) + r_2 p(p-1) \\
&= -2p + (r_1 + r_2)p(p-1) - p(p-1)(c+1) \\
&= \underbrace{-2p^2 + 2p^2}_{0} - 2p + (r_1 + r_2)p(p-1) - p(p-1)(c+1) \\
&= p^2(0-2) + (r_1 + r_2)p(p-1) - (p-1)(p(c+1) - 2p) \\
&= p^2(0-2) + \underbrace{(r_1 + r_2)p(p-1)}_{d_\eta} - \underbrace{p(p-1)(c-1)}_{d_s}
\end{aligned}$$

Case 3: p-branched then unibranched



This is similar to the second case due to presence of exactly one conductor. By Corollary 5.1.3 we have that

$$g_{y_1} = \frac{(r_1 - 2)(p-1)}{2}$$

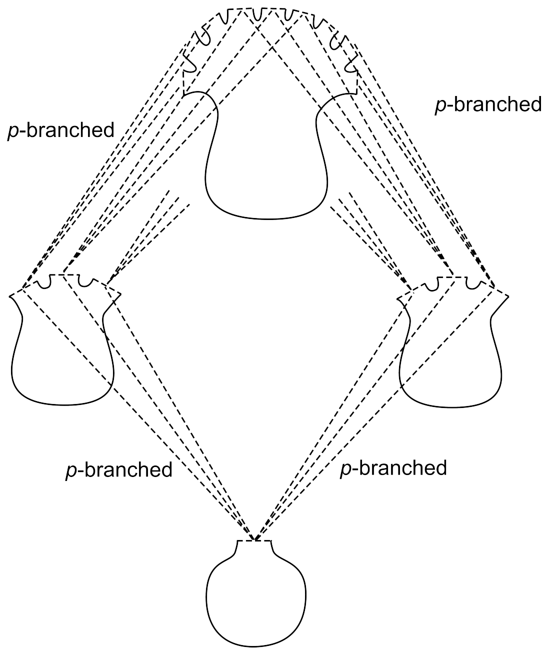
and by the genus formula given in Theorem 5.1.2 we can write

$$2g_{y_2} - 2 = p(2g_{y_1} - 2) + r_2 p(p-1) - (c-1)(p-1)$$

and so substituting the first equation into the first results in

$$\begin{aligned}
2g_y - 2 &= p(2g_{y_1} - 2) + r_2 p(p-1) - (c-1)(p-1) \\
&= p\left((r_1 - 2)(p-1) - 2\right) + r_2 p(p-1) - (c-1)(p-1) \\
&= -2p^2 + 2p - 2p + (r_1 + r_2)p(p-1) - (c-1)(p-1) \\
&= p^2(0-2) + \underbrace{(r_1 + r_2)p(p-1)}_{d_\eta} - \underbrace{(p-1)(c-1)}_{d_s}
\end{aligned}$$

Case 4: p-branched throughout



By Corollary 5.1.3 we have that

$$g_{y_1} = \frac{(r_1 - 2)(p-1)}{2}$$

and by the genus formula given in Theorem 5.1.2 we can write

$$2g_{y_2} - 2 = p(2g_{y_1} - 2) + r_2 p(p-1) - 0$$

and so substituting the first equation into the first results in

$$
\begin{aligned}
2g_y - 2 &= p(2g_{y_1} - 2) + r_2 p(p-1) \\
&= p\left((r_1 - 2)(p-1) - 2\right) + r_2 p(p-1) \\
&= \underbrace{-2p^2 + 2p^2}_{0} - 2p + (r_1 + r_2)p(p-1) - 2p(p-1) \\
&= p^2(0-2) + \underbrace{(r_1 + r_2)p(p-1)}_{d_\eta} - \underbrace{0}_{d_s}
\end{aligned}
$$

We can summarise our results in the following table:

Table 5.1: Values for d_η and d_s in the (p,p) setting above one boundary

1st step	2nd step	d_η	d_s
uni	uni	$(r_1+r_2)p(p-1)$	$(c_1'-1)(p-1)+(c_1-1)p(p-1)$
uni	p	$(r_1+r_2)p(p-1)$	$p(p-1)(c-1)$
p	uni	$(r_1+r_2)p(p-1)$	$(p-1)(c-1)$
p	p	$(r_1+r_2)p(p-1)$	0

so that the general form for the genus formula for (p,p) above one boundary is given by

$$2g_y - 2 = p^2(2g_x - 2) + d_\eta - d_s$$

where $d_\eta = (r_1 + r_2)p(p-1)$ and where

$$
d_s = \begin{cases}
(c_1' - 1)(p-1) + (c_1 - 1)p(p-1) & \text{boundary unibranched throughout} \\
(c-1)p(p-1) & \text{boundary unibranched, then } p\text{-branched} \\
(c-1)(p-1) & \text{boundary } p\text{-branched, then unibranched} \\
0 & \text{boundary } p\text{-branched throughout}
\end{cases}
$$

As with the degree p case, we can derive from the formula some helpful results:

Proposition 5.2.2. *Let $X = \operatorname{Spf}(R[[T]])$ be the formal germ of an R-curve at a smooth point x and let $f : Y \to X$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Assume Y is normal and local and that the special fibre Y_k of Y is reduced. Let $X_b =$*

$\mathrm{Spf}(R[[T]]\{T^{-1}\})$ *be the boundary of the germ and let* $Y_b \to X_b$ *be the corresponding Galois cover on the boundary. Let* y *be the unique closed point of* Y_k *and* $d_\eta := (r_1 + r_2)p(p-1)$ *be the degree of the divisor of ramification in the morphism* $f_K : Y_K \to X_K$ *induced by* f *on the generic fibre and* c_1 *and* c_1' *are as in Theorem 5.2.1 and where* c *is the only acting conductor at the relevant unibranched stage. Then:*

1. *If* Y_k *is unibranched at* y *then* $g_y = \frac{(p(r_1+r_2-c_1-1)-c_1'-1)(p-1)}{2}$
2. *If* Y_k *is unibranched and then* p*-branched at* y *then* $g_y = \frac{(p(r_1+r_2-c-1)-2)(p-1)}{2}$
3. *If* Y_k *is* p*-branched and then unibranched at* y *then* $g_y = \frac{(p(r_1+r_2-2)-c-1)(p-1)}{2}$
4. *If* Y_k *is* p^2*-branched at* y *then* $g_y = \frac{(p(r_1+r_2-2)-2)(p-1)}{2}$

Proof. Follows directly from rearranging the (p, p) vanishing cycles formula with $g_x = 0$. □

In this situation, we have the following test for whether y is a smooth point or not.

Corollary 5.2.3. *In the same situation as the above proposition,* y *is a smooth point if and only if* $g_y = 0$ *if and only if* $p(r_1 + r_2 - 1) = 1 + c_1' + c_1 p$ *(i.e. case 1 in the above corollary).*

Proof. ($\Rightarrow$) Suppose y is a smooth point. Then $\delta_y = \dim_k(\widetilde{\mathcal{O}_y}/\mathcal{O}_y) = 0$ and $r_y = 1$ since there is one branch and so $g_y = \delta_y - r_y + 1 = 0 - 1 + 1 = 0$. If $g_y = 0$ in the unibranched case then, by the previous proposition, $p(r_1 + r_2 - c_1 - 1) - c_1' - 1 = 0$ which rearranges to $p(r_1 + r_2 - 1) = 1 + c_1' + c_1 p$.

($\Leftarrow$) Suppose $p(r_1 + r_2 - 1) = 1 + c_1' + c_1 p$, then we are in the unibranched case and $g_y = 0$. As there is one branch $r_y = 1$ and so we have that $\delta_y = g_y + r_y - 1 = 0 + 1 - 1$ which in turn implies y is a smooth point. □

Bibliography

[1] Antoine Chambert-Loir, *La conjecture d'Abhyankar I: construction de revêtements en géométrie rigide*, Courbes Semi-stables et Groupe Fondamental en Géométrie Algébrique (Luminy, Décembre 1998) (Jean-Benoît Bost Michel Raynaud, François Loeser, ed.), (Progress in Mathematics, 187) Birkhauser, 2000, pp. 233–248.

[2] Helmut P Epp, *Eliminating wild ramification*, Inventiones Mathematicae **19** (1973), 235–249.

[3] Eyal Z. Goren, *Lectures on Hilbert Modular Varieties and Modular Forms*, American Mathematical Society, 2002.

[4] Barry Green and Michel Matignon, *Liftings of Galois Covers of Smooth Curves*, Compositio Mathematica **113 (3)** (1998), 237–272.

[5] David Harbater, *Patching and Galois theory*, Galois Groups and Fundamental Groups (L. Schneps, ed.), (MSRI Publications series, 41) Cambridge University Press, 2003, pp. 313–424.

[6] Kazuya Kato, *Vanishing cycles, ramification of valuations, and class field theory*, Duke Mathematical Journal **55 (3)** (1987), 629–659.

[7] Qing Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford University Press, 2009.

[8] Qing Liu and Dino Lorenzini, *Models of curves and finite covers*, Compositio Mathematica **118** (1999), 61–102.

[9] James S Milne, *Etale cohomology*, Princeton University Press, 1980.

[10] Michel Raynaud, *Schémas en groupes de type* $(p, ..., p)$, Bulletin de la Société Mathématique de France **102** (1974), 241–280.

[11] ______, *Spécialisation des revêtements en caractéristique* $p > 0$, Annales scientifiques de l'É.N.S. série 4 **32 (1)** (1999), 87–126.

[12] Mohamed Saïdi, *Abhyankar's Conjecture II: the use of semi-stable curves*, Courbes Semi-stables et Groupe Fondamental en Géométrie Algébrique (Luminy, Décembre 1998) (Jean-Benoît Bost Michel Raynaud, François Loeser, ed.), (Progress in Mathematics, 187) Birkhauser, 2000, pp. 249–265.

[13] ______, *Torsors under finite and flat group schemes of rank p with Galois action*, Mathematische Zeitschrift **245** (2003), 695–710.

[14] ______, *Cyclic p-groups and semi-stable reduction of curves in equal characteristic* $p > 0$, arXiv:math/0405529v1 (2004), 1–54.

[15] ______, *Wild ramification and a vanishing cycles formula*, Journal of Algebra **273** (2004), 108–128.

[16] Tsutomu Sekiguchi, Frans Oort, and Noriyuki Suwa, *On the deformation of Artin-Schreier to Kummer*, Annales scientifiques de l'É.N.S. série 4 **22 (3)** (1989), 345–375.

[17] Tsutomu Sekiguchi and Noriyuki Suwa, *Théories de Kummer-Artin-Schreier-Witt*, Comptes rendus de l'Académie des sciences. Série 1, Mathématique **319** (1994), 105–110.

[18] Jean-Pierre Serre, *Local fields*, Springer Verlag, 1979.

[19] Alexei Skorobogatov, *Torsors and Rational Points*, Cambridge University Press, 2001.

[20] Günter Tamme, *Introduction to Étale Cohomology*, Springer-Verlag, 1994.

[21] John Tate, *Finite flat group schemes*, Modular Forms and Fermat's Last Theorem (Glenn Stevens Gary Cornell, Joseph H. Silverman, ed.), Springer, 1997, pp. 121–154.

[22] John Tate and Frans Oort, *Group schemes of prime order*, Annales scientifiques de l'É.N.S. série 4 **3 (1)** (1970), 1–21.

[23] Dajano Tossici, *Effective models and extension of torsors over a discrete valuation ring of unequal characteristic*, International Mathematics Research Notices (2008), 1–68.