OPTIMISM AND PESSIMISM IN GAMES[*]

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12th February 2013

Abstract

This paper considers the impact of ambiguity in strategic situations. It extends the existing literature on games with ambiguity-averse players by allowing for optimistic responses to ambiguity. We use the CEU model of ambiguity with a class of capacities introduced by Jaffray and Philippe (1997), which allows us to distinguish ambiguity from ambiguity-attitude, and propose a new solution concept, Equilibrium under Ambiguity (EUA), for players who may be characterized by ambiguity-preference. Applying EUA, we study comparative statics of changes in ambiguity-attitude in games with strategic complements. This extends work in Eichberger and Kelsey (2002) on the effects of increasing ambiguity if players are ambiguity averse.

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Keywords   Ambiguity in games, support, strategic complementarity, optimism, multiple equilibria.

JEL Classification   C72, D81.

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1 INTRODUCTION

Beginning with the seminal work of von-Neumann and Morgenstern (1944), expected utility theory (EUT) has been closely related to the analysis of strategic decision-making. Equilibrium concepts for games combine consistency properties for the beliefs of players with assumptions about decision-making in the light of these beliefs. Many researchers (e.g. Allais (1953), Ellsberg (1961) and Kahneman and Tversky (1979)) questioned the descriptive validity of EUT. Despite this, progress in applying the ensuing theoretical developments of Schmeidler (1989), Gilboa and Schmeidler (1989), and Sarin and Wakker (1992) to game theory has been slow.

There is a small literature about strategic behaviour in games, reviewed below, which deals with the three main issues arising in games by departing from expected utility theory:

1. How much consistency in beliefs does one want to impose in equilibrium?

2. To what extent do beliefs about the opponents' behaviour have to be independent?

3. How do attitudes towards ambiguity affect behaviour in games?

In this paper, our research will focus on the third question. The literature which followed Ellsberg (1961) has focused on ambiguity-aversion. As we will show, extending the analysis to include ambiguity-seeking requires us to reconsider the question of consistency in beliefs.

Considering optimistic attitudes towards ambiguity introduces non-convexities in preferences. Thus, it is not possible to prove existence of equilibrium with standard techniques based on fixed-point theorems. Instead we use lattice theory to demonstrate existence of equilibrium in games with increasing differences.[1] We view it as an advantage of this approach that it enables us to study existence and comparative statics in a common framework. Moreover, games with increasing differences have many economic applications, for instance in industrial organization, macroeconomics, and public good provision.

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[1] Jungbauer and Ritzberger (2011) propose an alternative solution to this problem using a set-based solution concept.

1.1 Ambiguity and Ambiguity-Attitudes

Following Ellsberg (1961)'s criticism of the subjective expected utility (SEU) approach of Savage (1954), both experimental and theoretical research has been directed towards a better understanding of ambiguity and modelling it in individual decision making. Ambiguity describes situations where individuals cannot or do not assign subjective probabilities to uncertain events. By now, there is also a substantial body of experimental evidence which shows that people behave differently in the presence of ambiguity (see for example Camerer and Weber (1992)).

Most of the literature following Ellsberg (1961) has assumed that individuals are *ambiguity-averse*, i.e. they would pay some positive amount of money to avoid a situation where probabilities are poorly defined. Experimental evidence, however, does not uniformly confirm this view (see for instance Abdellaoui, Vossmann, and Weber (2005), Gonzalez and Wu (1999) and Kilka and Weber (2001)). Though many individuals behave cautiously when there is ambiguity, a significant group of individuals behave in the opposite way, a behaviour which we shall call *ambiguity-loving* or *ambiguity-seeking.* Moreover, the same individual may express both ambiguity-seeking and ambiguity-aversion in different contexts. In an experimental study of ambiguity in games, Ivanov (2011) finds that more subjects are ambiguity-loving than ambiguity-averse.

In this paper we study the influence of ambiguity and ambiguity-attitude on behaviour in games. Informally, ambiguity refers to how much uncertainty there is about the probabilities and ambiguity-attitude measures how the individual reacts to unknown probabilities. We believe that ambiguity-attitude is a personal characteristic of an individual player and thus should be taken as exogenous.

We shall use Choquet Expected Utility (CEU), in which individuals have beliefs represented by capacities, (non-additive subjective probabilities). CEU represents preferences as maximizing the expected value of a utility function, where the expectation is represented as a Choquet integral, Choquet (1953-4). These preferences have been axiomatized by Schmeidler (1989) and Sarin and Wakker (1992). Jaffray and Philippe (1997) propose a class of capacities (henceforth JP-capacities) that depend on a parameter which measures ambiguity-attitude. We shall use their approach to represent preferences of players by the Choquet integral with respect to a

JP-capacity[2].

There is a small literature on strategic ambiguity[3]. Dow and Werlang (1994) use CEU preferences to represent players' ambiguity about their opponents' strategy choice. Best responses of a player depend on the capacity (i.e. their beliefs) and the degree of ambiguity embodied in it. In equilibrium, consistency requirements relating beliefs about opponents to their strategy choices have to be satisfied. Dow and Werlang (1994) represented the strategies an individual believes the opponents would play by the *support* of the capacity[4]. However there is more than one way to define the support of a capacity. Equilibrium concepts in the literature differ mainly in the notion of support used to define an equilibrium. One of the surprising facts is that the support notions for capacities used in context with ambiguity-averse preferences are no longer suitable if players may be ambiguity-loving. Hence, we propose a new definition of support, which we believe is more appropriate, and use it as the basis of an equilibrium concept for games with ambiguity.

1.2 Ambiguity in Games with Positive Externalities

Many games in economics have prices or quantities as strategies which can be ordered in a natural way. Moreover, many of these applications are games of positive aggregate externalities with increasing differences. For such games, we show that an increase in ambiguity-seeking increases equilibrium strategies. Intuitively, as a player becomes more ambiguity-loving, he will place more weight on outcomes which are perceived as good. If there are positive externalities, good outcomes are associated with opponents' playing high strategies. Increasing differences imply that raising the decision-weight on high strategies of the opponents will increase incentives to play a high strategy. Hence, the best response function of the player will shift up, which increases the equilibrium strategies of all players.

Strategic complementarity can lead to multiple equilibria. In this case, we show that for

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[2]We explain the reasons for this modelling choice in more detail in Section 2.3 below.

[3]Ambiguity in games may also concern the type spaces of players in games with incomplete information. We do not consider these models in this paper, since the equilibrium concept remains a type-contingent strategy Nash equilibrium as introduced by Harsanyi (1967-68). This is not to say that there are no interesting economic applications for such games. For some recent contributions to this literature, compare Azrieli and Teper (2011) and Ui (2009).

[4]The support of a capacity is the analogue of the usual support of a probability distribution. It is explained in more detail in Definitions 3.1 and 3.3 and the related discussion.

sufficient ambiguity, equilibrium will be unique. If players are sufficiently optimistic (resp. pessimistic) the equilibrium strategies will be higher (resp. lower) than in the highest (resp. lowest) equilibrium without ambiguity. Ambiguity and ambiguity-attitude have distinct effects. Ambiguity causes the set of equilibria to collapse to a single equilibrium, while an increase (decrease) in optimism causes the set of equilibria to move up (down).

Organization of the Paper   In section 2 we present our framework and definitions. Section 3 introduces the equilibrium concept for games with players of differing ambiguity-attitudes. For the case of games with strategic complements, we prove existence of equilibrium and derive the comparative statics results in section 4. Related literature is discussed in Section 5 and Section 6 contains our conclusions. Appendix A relates a number of alternative notions of the support of a capacity and contains the proofs for results on capacities. All other proofs are gathered in Appendix B. Our results are illustrated by an application to the centipede game, which can be found in Section B.3 of the appendix.

2   MODELLING OPTIMISM IN GAMES

We consider a game $\Gamma = \langle N; (S_i), (u_i) : 1 \leqslant i \leqslant n \rangle$ with finite pure strategy sets S_i for each player and payoff functions u_i. The notation, s_{-i}, indicates a strategy combination for all players except i. The space of all strategy profiles for i's opponents is denoted by S_{-i}. The space of all strategy profiles is denoted by S. Player i has utility function $u_i : S \to \mathbb{R}$, for $i = 1, ..., n$. When convenient we shall write $u_i(s) = u_i(s_i, s_{-i})$.

2.1   Non-Additive Beliefs and Choquet Integrals

In CEU, beliefs, ambiguity and ambiguity-attitude are represented as capacities, which assign non-additive values to subsets of S_{-i}. Formally, capacities are defined as follows.

Definition 2.1 *A* capacity *on S_{-i} is a real-valued function ν on the subsets of S_{-i} such that $A \subseteq B \Rightarrow \nu(A) \leqslant \nu(B)$ and $\nu(\varnothing) = 0$, $\nu(S_{-i}) = 1$. The* dual capacity *$\bar{\nu}$ on S_{-i} is defined by $\bar{\nu}(A) = 1 - \nu(A^c)$, where $A^c := S_{-i} \backslash A$.*

The expected utility of the payoff obtained from a given act, with respect to a non-additive belief, ν, can be found using the Choquet integral, defined below.

Definition 2.2 *The* Choquet integral *of* $u_i\left(s_i, s_{-i}\right)$ *with respect to capacity* ν *on* S_{-i} *is:*

$$V_i\left(s_i\right) = \int u_i\left(s_i, s_{-i}\right) d\nu = u_i\left(s_i, s_{-i}^1\right)\nu\left(s_{-i}^1\right) + \sum_{r=2}^R u_i\left(s_i, s_{-i}^r\right)\left[\nu\left(s_{-i}^1, ..., s_{-i}^r\right) - \nu\left(s_{-i}^1, ..., s_{-i}^{r-1}\right)\right],$$

where the strategy profiles in S_{-i} *are numbered so that* $u_i\left(s_i, s_{-i}^1\right) \geqslant u_i\left(s_i, s_{-i}^2\right) \geqslant ... \geqslant u_i\left(s_i, s_{-i}^R\right).$

A simple, though extreme, example of a capacity is the complete uncertainty capacity. Intuitively it describes a situation where the decision maker knows which strategy profiles are possible but has no further information about their likelihood.

Example 2.1 *The* complete uncertainty capacity, ν_0 *on* S_{-i} *is defined by* $\nu_0\left(S_{-i}\right) = 1,$ $\nu_0\left(A\right) = 0$ *for all* $A \subsetneqq S_{-i}.$

Definition 2.3 *A capacity,* ν, *is said to be* convex *if for all* $A, B \subseteq S$, $\nu\left(A \cup B\right) \geqslant \nu\left(A\right) + \nu\left(B\right) - \nu\left(A \cap B\right).$

Convex capacities can be associated in a natural way with a set of probability distributions called *core* of the capacity.

Definition 2.4 *Let* ν *be a capacity on* S_{-i}. *The* core, $\mathcal{C}\left(\nu\right)$, *is defined by,*

$$\mathcal{C}\left(\nu\right) = \left\{p \in \Delta\left(S_{-i}\right); \forall A \subseteq S_{-i}, p\left(A\right) \geqslant \nu\left(A\right)\right\},$$

where $p(A) := \sum_{s_{-i} \in A} p\left(s_{-i}\right).$

Since a capacity and its dual represent upper and lower bounds for the probability distributions in the core, it is natural to define the degree of ambiguity of a player as follows.

Definition 2.5 *Let* ν *be a convex capacity on* S_{-i}. *Define the maximal degree of ambiguity of* ν *by:* $\lambda\left(\nu\right) = \max\left\{\bar{\nu}\left(A\right) - \nu\left(A\right) : \varnothing \subsetneqq A \subsetneqq S_{-i}\right\}$ *and the minimal degree of ambiguity by* $\gamma\left(\nu\right) = \min\left\{\bar{\nu}\left(A\right) - \nu\left(A\right) : \varnothing \subsetneqq A \subsetneqq S_{-i}\right\}.$

These definitions are adapted from Dow and Werlang (1992). They provide upper and lower bounds on the amount of ambiguity which the individual perceives.

Schmeidler (1989) shows that for a convex capacity, ν, the Choquet integral of a pay-off function u_i is equal to the minimum over the core of the expected value of u_i, i.e. $\int u_i\left(s_i, s_{-i}\right) d\nu = \min_{p \in \mathcal{C}(\nu)} \mathbf{E}_p u_i\left(s_i, s_{-i}\right)$, where $\mathbf{E}$ denotes the expected value of u_i with respect to the probability distribution p on S_{-i}. Indeed, Schmeidler (1989) argues that convex capacities represent ambiguity-aversion. More recently Wakker (2001) has shown that convexity is implied by a generalized version of the Allais paradox. This provides another reason to take convex capacities as a representation of ambiguity and the Choquet expected utility as the pessimistic evaluation of acts given this ambiguity.

2.2 Optimism, Pessimism, and JP-Capacities

The present paper is concerned with modelling both ambiguity-averse and ambiguity-seeking behaviour. We achieve this by focusing on the class of JP-capacities introduced by Jaffray and Philippe (1997).

Definition 2.6 *A capacity ν on S_{-i} is a* JP-capacity *if there exists a convex capacity μ and $\alpha \in [0, 1]$, such that $\nu = \alpha\mu + (1 - \alpha)\bar{\mu}$. (Recall that $\bar{\mu}$ denotes the dual capacity of μ.)*

Ambiguity is represented by a convex capacity μ and its core, $\mathcal{C}(\mu)$. A JP-capacity is a convex combination of the capacity μ and its dual. As the following proposition shows, the CEU of a JP-capacity is a weighted average of the minimum and the maximum expected utility over the set of probabilities in $\mathcal{C}(\mu)$.

Proposition 2.1 *(Jaffray and Philippe (1997)) The CEU of a utility function u_i with respect to a* JP-capacity $\nu = \alpha\mu + (1 - \alpha)\bar{\mu}$ *on S_{-i} is:*

$$\int u_i\left(s_i, s_{-i}\right) d\nu\left(s_{-i}\right) = \alpha \min_{p \in \mathcal{C}(\mu)} \mathbf{E}_p u_i\left(s_i, s_{-i}\right) + (1 - \alpha) \max_{p \in \mathcal{C}(\mu)} \mathbf{E}_p u_i\left(s_i, s_{-i}\right).$$

These preferences lie in the intersection of the CEU and α-MEU models. When beliefs are represented by JP-capacities, perceived ambiguity is represented by the capacity μ, while

ambiguity-attitude is represented by α, with higher values of α corresponding to more ambiguity-aversion. If $\alpha = 1$, then we obtain the MEU model axiomatized by Gilboa and Schmeidler (1989). For $\alpha = 0$, we deal with a pure optimist, while in general, for $\alpha \in (0, 1)$, the player's preferences have both optimistic and pessimistic features. Hence, JP-capacities allow a distinction between ambiguity and ambiguity-attitude, which is formalized in the following definition.[5]

Definition 2.7 *Let ν and ν' be two capacities on S. We say that ν is more pessimistic than ν' if for all $A \subseteq S, \nu(A) \leqslant \nu'(A)$.*

It follows that if $\hat{\alpha} \geqslant \tilde{\alpha}$ and μ is convex then $\hat{\nu} = \hat{\alpha}\mu + (1 - \hat{\alpha})\bar{\mu}$ is more pessimistic than $\tilde{\nu} = \tilde{\alpha}\mu + (1 - \tilde{\alpha})\bar{\mu}$. A useful special case of JP-capacities is the neo-additive capacity.

Example 2.2 *A* neo-additive-*capacity ν on S_{-i} is a JP-capacity with convex part $\mu(A) = (1 - \delta)\pi(A)$, for $\emptyset \subsetneqq A \subsetneqq S_{-i}$, where $0 \leqslant \delta < 1$ and π is an additive probability distribution on S_{-i}.[6] The associated JP-capacity is $\nu(A) = \delta(1 - \alpha) + (1 - \delta)\pi(A)$, for $\emptyset \subsetneqq A \subsetneqq S_{-i}$.*

A neo-additive capacity describes a situation where the individual's 'beliefs' are represented by a probability distribution π. However (s)he has some doubts about these beliefs. This ambiguity about the true probability distribution is reflected by the parameter δ. The highest possible level of ambiguity corresponds to $\delta = 1$, while $\delta = 0$ corresponds to no ambiguity. The Choquet expected value of a pay-off function $u_i(s_i, \cdot)$ with respect to the neo-additive-capacity ν is given by:

$$\int u_i(s_i, s_{-i})\, d\nu(s_{-i}) = \delta\alpha \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + \delta(1 - \alpha) \max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + (1 - \delta) \cdot \mathbf{E}_\pi u_i(s_i, s_{-i}).$$

This expression is a weighted averaged of the highest payoff, the lowest payoff and an average payoff. The response to ambiguity is partly optimistic represented by the weight given to the best outcome and partly pessimistic.

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[5] Wakker (2011) presents a related theory of ambiguity and ambiguity-attitude.

[6] Neo-additive capacities are axiomatized by Chateauneuf, Eichberger, and Grant (2007) who write the neo-additive capacity in the form $\nu(A) = \delta\alpha + (1 - \delta)\pi(A)$. In the main text we have modified the definition of a neo-additive capacity to be consistent with the definition of a JP-capacity.

2.3 Modelling Players' Preferences

In order to study the impact of ambiguity, especially ambiguity-loving behaviour, in games, it is necessary to make a clear distinction between ambiguity and ambiguity-attitude. There are, however, only a small number of models which allow one to do this.

1. The α-MEU model, Marinacci (2002), which represents ambiguity by a set of probability distributions and ambiguity-attitude by the parameter α expressing the weight given to the worst possible expected utility.

2. Choquet Expected Utility (CEU), Schmeidler (1989) and Sarin and Wakker (1992) in combination with JP-capacities. Preferences are represented by a Choquet integral with respect to a non-additive belief, $\nu = \alpha\mu + (1-\alpha)\overline{\mu}$, where μ is a convex capacity. The parameter α can similarly be interpreted as a measure of ambiguity-attitude.

3. The smooth model or KMM-model, Klibanoff, Marinacci, and Mukerji (2005).

With the exception of the CEU model for the special case of a neo-additive capacity, none of these has an axiomatization in terms of preferences over Savage acts, which would allow one to distinguish ambiguity from the agent's ambiguity-attitudes[7]. In the case of CEU, preferences over acts determine the capacity ν uniquely but not necessarily the JP-form $\nu = \alpha\mu + (1-\alpha)\overline{\mu}$, which would achieve such a separation. For the α-MEU model, there is no axiomatization so far in the standard single-time period Savage framework[8]. In α-MEU the set of priors is not well defined. Siniscalchi (2006) shows that there may be more than one set of priors and more than one α, which represent the same preferences[9]. In the axiomatization of the smooth model, the attitude towards ambiguity reflected in ϕ is determined by a second preference order over second-order acts, hence, not derived from preferences over Savage acts alone. We consider the problem of the axiomatic separation of ambiguity-attitude and ambiguity as a challenge for future research.

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[7]See Chateauneuf, Eichberger, and Grant (2007).

[8]Ghirardato, Maccheroni, and Marinacci (2004) have proposed a way to define a unique set of priors for the α-MEU model. However as we argue in Eichberger, Grant, Kelsey, and Koshevoy (2011) there are some problems with this approach when the state space is finite. Klibanoff, Mukerji, and Seo (2011) uses an infinite time period model to axiomatize a version of α-MEU, which satisfies a symmetry assumption.

[9]See especially the on-line appendix to Siniscalchi (2006).

Given the unresolved issues surrounding the question of how to distinguish ambiguity of beliefs from ambiguity-attitudes axiomatically[10], we restrict attention to JP-capacities where this distinction between ambiguity-attitude, as reflected by the parameter α, and ambiguous beliefs, as represented by the convex part of the capacity μ, appears natural.

3 EQUILIBRIUM WITH OPTIMISM AND PESSIMISM

3.1 The Support of a Capacity

In the previous literature on games with ambiguity, the support of a player's beliefs represents the profile of strategies that he believes his opponents will play. An equilibrium is defined to occur when all profiles in the support consist only of best responses. In general, capacities reflect both ambiguity and ambiguity-attitudes. It is therefore necessary to separate ambiguity-attitudes from the ambiguous beliefs in order to find an appropriate support notion.

It is not possible for us to use existing definitions of the support unmodified since many of them have implicitly assumed ambiguity-aversion. Two definitions have been used for ambiguity-averse players with convex capacities: the *Dow-Werlang (DW) support* (Dow and Werlang (1994)), and the *Marinacci (M) support* (Marinacci (2000)). The *DW-support* of the capacity ν, $\mathrm{supp}_{DW}\, \nu$ is a set $E \subseteq S_{-i}$, such that $\nu\,(S_{-i}\backslash E) = 0$ and $\nu\,(F) > 0$, for all F such that $S_{-i}\backslash E \subsetneqq F$. It always exists, however it may not be unique. An example is the capacity of complete uncertainty in Example 2.1, where $\{s_{-i}\}$ is a support for any $s_{-i} \in S_{-i}$. Marinacci (2000) defines the support of a capacity ν to be the set of states with positive capacity, $\mathrm{supp}_M\, \nu = \{s_{-i} \in S_{-i} : \nu\,(s_{-i}) > 0\}$. Provided it exists, $\mathrm{supp}_M\, \nu$ is always unique. However, this support can be empty.

Ryan (2002) studies notions of support for sets of probability distributions. Based on his work, we suggest the following definition of support for a convex capacity.

Definition 3.1 *If μ is a* convex *capacity on S_{-i}, we define the support of μ, $\mathrm{supp}\,\mu$, by*

$$\mathrm{supp}\,\mu = \bigcap\nolimits_{p \in \mathcal{C}(\mu)} \mathrm{supp}\, p. \tag{1}$$

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[10] For more discussion of this issue compare Epstein (1999) and Ghirardato and Marinacci (2002).

In Appendix A we show that, for convex capacities, the DW-support and the M-support coincide with the above notion if the DW-support is unique. Moreover, they also agree with the weak support for MEU defined by Ryan (2002). We believe that this consistency of the support notions for MEU and CEU, is a strong argument for using this definition. In the context of games, states correspond to strategy profiles of the opponents. This support notion appears as the natural choice because it does not require best-reply behaviour against any strategy opponents may possibly play but only against those which are unquestionably played.

A natural alternative definition is $\bigcup_{p \in \mathcal{C}(\mu)} \operatorname{supp} p$, which has been studied by Dow and Werlang (1991) and Lo (1996). They show that solution concepts for games based on this definition of support do not differ significantly from Nash equilibrium (henceforth NE). The alternative notion is therefore, not compatible with modelling deviations from NE due to ambiguity[11].

It is not obvious how to extend the definition of support to non-convex capacities. Consider the neo-additive capacity $\nu = \delta (1 - \alpha) + (1 - \delta) \pi$ from Example 2.2. This capacity assigns positive values to all strategy profiles $s_{-i} \in S_{-i}$, provided $\alpha < 1, \delta > 0$. Thus $\operatorname{supp}_M \nu = \operatorname{supp} \nu = S_{-i}$ and $\operatorname{supp}_{DW} \nu = S_{-i}$ provided $\mathcal{C}(\mu) \neq \varnothing$. Hence, none of these concepts are suitable as a support notion for this capacity as they do not allow us to make a distinction between those strategies which a given player believes are possible for his opponents and others.

The problem is that, whenever there is even a small amount of optimism, neo-additive capacities assign positive capacity to all profiles. However, this does not mean the player "believes" in these profiles. Looking at profiles with positive capacity confounds belief and ambiguity-attitude, since optimism increases the capacity values assigned to all profiles.

Sarin and Wakker (1998) argue that the beliefs of a decision-maker can be deduced from the weights used in the Choquet integral. Based on this criterion, one can show that a profile, s_{-i}, always gets positive weight in the Choquet integral of a neo-additive capacity if and only if $\pi(s_{-i}) > 0$. For this reason, it seems desirable that $\operatorname{supp} \nu = \operatorname{supp} \pi$ in this case. This also makes sense in terms of our intuition. A neo-additive capacity describes a situation where the individual's beliefs, although expressed with some doubts, are represented by the probability distribution π. This intuition suggests that the set of profiles in which the player believes is

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[11] With an equilibrium notion, which is similar to the one used by Lo (1996), Bade (2011) also obtains that for two-player games her ambiguous-act equilibria are observationally equivalent to NE.

given by $\operatorname{supp} \pi$.

Thus, we argue that a necessary condition for a player to believe in a profile is that it should always get positive weight in the Choquet integral. This avoids the problem of confounding between ambiguity-attitude and belief described above. The set of all profiles with this property can be defined as follows.

Definition 3.2 *Define the set of* decision-weight increasing profiles *of a capacity v,*

$$\mathcal{B}(\nu) = \{s \in S_{-i} : \forall A \subsetneqq S_{-i}, s \notin A; \nu(A \cup s) > \nu(A)\}.$$

A non-convex capacity has a set of probabilities associated with it. These are the set of decision-weights used in evaluating the Choquet integral. Essentially $\mathcal{B}(\nu)$ is the intersection of the supports of these weights. This is in the spirit of Sarin and Wakker (1998) who argue that an individual's beliefs can be deduced from these decision-weights. For a convex capacity the decision weights are the extreme points of the core. Thus $\mathcal{B}(\nu)$ may be seen as a generalization of our definition of the support for a convex capacity.

One could use $\mathcal{B}(\nu)$ as the definition of the support of a capacity. However, $\mathcal{B}(\nu)$ depends on α. Since α is a measure of ambiguity-attitude, one would prefer the set of strategy profiles in which a player believes to be independent of it. These considerations lead us to propose a closely related set, instead. Hence, for JP-capacities, we propose a support notion which relates only to the convex part μ.

Definition 3.3 *If $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ is a JP-capacity on S_{-i}, we define the support of $\nu, \operatorname{supp}_{JP}\nu$, by $\operatorname{supp}_{JP}\nu = \operatorname{supp}\mu$.*

Our reason for defining the support of a JP-capacity in terms of its convex part is that a capacity and its dual are simply two ways of representing the same information. Since the JP-capacity ν is a weighted average of μ and its dual, μ does contain as much information about the player's beliefs as ν. The following result shows that all elements of $\operatorname{supp}_{JP}\nu$ always receive positive weight in the Choquet integral. Thus they meet the necessary condition described above.

Proposition 3.1 *Let $\nu = \alpha\mu + (1 - \alpha)\bar{\mu}$ be a JP-capacity on S_{-i}, then $\operatorname{supp}\nu \subseteq \mathcal{B}(\nu)$*[12].

If ν is a JP-capacity, $\mathcal{B}(\nu)$ does not depend on the JP-representation. Thus, to a great extent, our support notion is independent of the parameters of the JP-representation[13].

We previously argued that for a neo-additive capacity, $\nu = \delta(1 - \alpha) + (1 - \delta)\pi$, our intuition required that the support of ν be $\operatorname{supp}\pi$. The following result shows that our definition of support has this property.

Proposition 3.2 *Let $\nu = \delta(1 - \alpha) + (1 - \delta)\pi$ be a neo-additive capacity on S_{-i}, where $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \delta < 1$. Then $\operatorname{supp}_{JP}\nu = \operatorname{supp}\pi$.*

3.2 Independent Beliefs and Equilibrium under Ambiguity

In analogy to NE, we define *Equilibrium in Beliefs Under Ambiguity (henceforth EUA)* to be a situation where each player maximizes his (Choquet) expected utility given his ambiguous beliefs about the behaviour of his opponents. These beliefs have to be reasonable in the sense that each player believes that his opponents play best responses. We interpret this as implying that the support of any given player's beliefs should be non-empty and consist only of best responses of his opponents. Let $R_i(\nu_i) = \operatorname{argmax}_{s_i \in S_i}\int u_i(s_i, s_{-i})\,d\nu_i(s_{-i})$ denote the best response correspondence of player i given beliefs ν_i.

Definition 3.4 *An n-tuple of capacities $\hat{\nu} = \langle \hat{\nu}_1, ..., \hat{\nu}_n \rangle$ is an* Equilibrium in Beliefs Under Ambiguity *if for all players, $i \in I$,*

$$\varnothing \neq \operatorname{supp}\hat{\nu}_i \subseteq \underset{j \neq i}{\times} R_j(\hat{\nu}_j).$$

If there is a strategy profile $\hat{s} = \langle \hat{s}_1, ...\hat{s}_n \rangle$ such that for each player $\hat{s}_{-i} \in \operatorname{supp}\hat{\nu}_i$, we say that $\hat{s}$ is an equilibrium strategy profile*. Moreover, if for each player $\operatorname{supp}\hat{\nu}_i$ contains a single strategy profile $\hat{s}_{-i}$ we say that $\hat{s}$ is a* singleton equilibrium*.*

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[12]The converse of this result is not true. There is a counter example available from the authors on request. The counter example is non-generic, thus it is 'almost' the case that $\operatorname{supp}\nu = \mathcal{B}(\nu)$.

[13]In an earlier draft we used $\mathcal{B}(\nu)$ as our support notion. We obtained similar results to those reported in the present version of the paper. This suggests that our results are reasonably robust. It also provides a way to generalize our results to a larger class of capacities.

In equilibrium, a player's evaluation of a particular strategy may, in part, depend on strategies of the opponents which do not lie in the support. We interpret these as events a player views as unlikely but which cannot be ruled out. This may reflect some doubts the player may have about the rationality of the opponents or whether he correctly understands the structure of the game.

In an EUA players choose pure strategies and do not randomize. Non-singleton equilibria cannot be interpreted as randomizations. In such an equilibrium some player i will have two or more best responses. The support of other players' beliefs about i's play, will contain some or all of them. Thus an equilibrium, where the support contains multiple strategy profiles, is an equilibrium in beliefs rather than in randomizations. If there are only two players and the beliefs are additive, then an EUA is a NE.

The model can accommodate observed behaviour which is ruled out in NE. Players may be better off (in an ex-post sense) in an EUA than in the unique NE of a game. An example of this is the centipede game discussed in Appendix B.3.

For games with more than two players, however, an EUA with additive beliefs may not be a NE because players may not believe that their opponents act independently. In addition, it is possible for any two players to have different beliefs about the behaviour of a third player. For NE independence of beliefs follows immediately from the requirement that beliefs coincide with the (mixed) strategies actually played by the opponents. The independent choices of mixed strategies define a unique probability distribution on the product space of strategy sets. Both conditions fail for EUA beliefs.

It is well-known (Denneberg (2000) p. 53-56) that there are several ways of extending the product of capacities from the Cartesian products of the strategy sets to general subsets of the product space. One possibility to obtain a notion of independent beliefs would be to apply the Möbius product, Ghirardato (1997), Hendon, Jacobsen, Sloth, and Tranaes (1996), and use the JP-capacity of the Möbius product of μ as the relevant product capacity[14].

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[14]Technically we need to assume that the convex part of a JP-capacity μ is a Möbius independent product of belief functions defined on the marginals. For a definition of the Möbius independent product and further discussion see Ghirardato (1997).

4 EXISTENCE AND COMPARATIVE STATICS

In this section we prove existence of equilibrium and study the comparative statics of changes in ambiguity and ambiguity-attitude on equilibrium.

4.1 Existence of Equilibrium

In many economic applications strategies are real numbers, such as prices or quantities, which have a natural order. Since strategy sets are finite, we identify strategy sets with an interval of the integers, $S_i = \{\underline{s}_i, \underline{s}_i + 1, ..., \bar{s}_i\}$, for $i = 1, ..., n$[15]. The payoff function $u_i(s_i, s_{-i})$ satisfies *increasing (resp. decreasing) differences* in $\langle s_i, s_{-i} \rangle$ if $\hat{s}_i > \tilde{s}_i$, implies $u_i(\hat{s}_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i})$ is increasing (resp. decreasing) in s_{-i}. If $u_i(s_i, s_{-i})$ satisfies increasing differences in $\langle s_i, s_{-i} \rangle$ then it also has increasing differences in $\langle s_{-i}, s_i \rangle$[16]. Increasing differences implies that a given player, who perceives his opponents increase their strategy, has an incentive to increase his own strategy as well. Bertrand oligopoly with linear demand and constant marginal cost provides an example of a game with increasing differences.

Definition 4.1 *A game,* $\Gamma = \langle N; (S_i), (u_i) : 1 \leqslant i \leqslant n \rangle$, *has positive externalities and increasing differences if* $u_i(s_i, s_{-i})$ *is increasing in* s_{-i} *and has increasing differences in* $\langle s_i, s_{-i} \rangle$ *for* $1 \leqslant i \leqslant n$.

Positive externalities and increasing differences will be a maintained hypothesis throughout the rest of the paper. Negative externalities may be defined in an analogous way.

The following existence result is proved in Appendix B. Fix a vector of ambiguity-attitude parameters $\alpha = (\alpha_1, ..., \alpha_n) \in [0, 1]^n$ and maximal and minimal degrees of ambiguity $(\lambda, \gamma) = ((\lambda_1, \gamma_1), ...(\lambda_n, \gamma_n))$, $0 \leqslant \gamma_i \leqslant \lambda_i \leqslant 1$, then there exists an EUA where players' beliefs are represented by JP-capacities with parameters α, satisfying $\lambda_i(\mu_i) \leqslant \lambda_i$ and $\gamma_i(\mu_i) \geqslant \gamma_i$ for all $i \in I$.

Theorem 4.1 *Let* Γ *be a game of positive externalities and increasing differences. Then for any exogenously given n-tuples of ambiguity-attitudes* α*, maximal degrees of ambiguity* λ*, and*

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[15] The crucial part of this assumption is the restriction to a finite strategy set. It would be straightforward to extend the results to a multi-dimensional strategy space.

[16] See Topkis (1998), p. 42.

minimal degrees of ambiguity $\gamma, (\gamma \leqslant \lambda)$ *the game* Γ *has a singleton Equilibrium in Beliefs under Ambiguity in JP-capacities* $\nu = \langle \nu_1, ..., \nu_n \rangle$, *where* $\nu_i = \alpha_i \mu_i + (1 - \alpha_i) \mu_i$ *for* $1 \leqslant i \leqslant n$. *The convex capacity* μ_i *has maximal degree of ambiguity at most* λ_i *and minimal degree of ambiguity at least* γ_i.

In general, EUA will not be unique. In part this arises because we have allowed a range for the degrees of ambiguity, i.e. $\gamma_i < \lambda_i$. If this range were reduced then EUA would have the same uniqueness properties as NE.

4.2 Comparative statics

Comparative statics exercises are difficult because the capacity represents three distinct concepts: the perceived ambiguity, the attitude to that ambiguity, and beliefs about the opponents' strategies. Moreover, these concepts are interrelated. For instance, if a player's ambiguity-attitude changes this may cause him to play a different strategy. The opponents are likely to change their strategies in response, which would require the first player to revise his beliefs as well so as to maintain consistency.

In order to investigate the comparative statics of ambiguity-attitude, we need to vary ambiguity-attitude while holding perceived ambiguity constant. We do this by placing exogenous bounds on the maximal and minimal degrees of ambiguity. The comparative static results do not depend on the values of these bounds despite the fact they are exogenous. For the comparative static analysis we strengthen positive externalities to the following assumption.

Definition 4.2 *A game,* Γ, *has* positive aggregate externalities *if* $u_i(s_i, s_{-i}) = u_i(s_i, f_i(s_{-i}))$, *for* $1 \leqslant i \leqslant n$, *where* u_i *is increasing in* f_i *and* $f_i : S_{-i} \to \mathbb{R}$ *is increasing in all arguments.*

This is a separability assumption. It says that a player only cares about a one-dimensional aggregate of his opponents' strategies. An example would be a situation of team production, in which the utility of a given team member depends on his own labour input and the total input supplied by all other members of the team. Negative aggregate externalities may be defined in an analogous way.

The following comparative static result shows that an increase in pessimism will reduce the equilibrium strategies in games with positive aggregate externalities and increasing differences. If there are multiple equilibria, the strategies played in the highest and lowest equilibrium will decrease. We assume that when the ambiguity-attitude of one player changes, the ambiguity-attitudes of other players and the perceived ambiguity are held constant.

Theorem 4.2 *Let Γ be a game of* positive aggregate externalities *with increasing differences. Assume that beliefs are represented by JP-capacities and let $\alpha = (\alpha_i, \alpha_{-i})$ denote the vector of ambiguity-attitudes. Let $\underline{s}(\alpha)$ (resp. $\bar{s}(\alpha)$) denote the lowest (resp. highest) equilibrium strategy profile when the minimal (resp. maximal) degree of ambiguity is γ (resp. λ). Then $\underline{s}(\alpha)$ and $\bar{s}(\alpha)$ are decreasing functions of α_i.*

4.3 Multiple Equilibria

Strategic complementarity can give rise to multiple Nash equilibria. Under some assumptions, we can show if there are multiple equilibria without ambiguity and there is sufficient optimism (resp. pessimism), equilibrium will be unique and will correspond to the highest (resp. lowest) equilibrium without ambiguity. To prove this we need the following assumption.

Assumption 4.1 *For $1 \leqslant i \leqslant n$, let $u_i(s_i, \bar{s}_{-i})$ and $u_i(s_i, \underline{s}_{-i})$ have a unique maximizer, i.e. $\left|\operatorname{argmax}_{s_i \in S_i} u_i(s_i, \bar{s}_{-i})\right| = 1$ and $\left|\operatorname{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i})\right| = 1$.*

This assumption is required for technical reasons. If the strategy space were continuous and utility were concave in the player's own strategy, it would be implied by our other assumptions. It says that the gaps in the discrete strategy space do not fall in the "wrong place".

Proposition 4.1 *Consider a game, Γ, of* positive *externalities with increasing differences which satisfies Assumption 4.1. There exist $\bar{\alpha}$ (resp. $\underline{\alpha}$), $0 < \underline{\alpha} \leqslant \bar{\alpha} < 1$, and $\bar{\gamma}$ such that if the minimal degree of ambiguity is $\gamma(\mu_i) \geqslant \bar{\gamma}$ and $\alpha_i \geqslant \bar{\alpha}$, (resp. $\leqslant \underline{\alpha}$) for $1 \leqslant i \leqslant n$, then there is a unique singleton equilibrium with an equilibrium strategy profile that is smaller (resp. larger) than the smallest (resp. largest) equilibrium strategy profile without ambiguity.*

Intuitively as ambiguity increases a player will become less confident in the behaviour of his opponents. Consequently he will respond less to perceived changes in their behaviour.

Thus, increasing ambiguity reduces the slope of the best response functions. When they are sufficiently flat it is only possible for them to intersect once. Hence, the equilibrium strategy profile is unique. Even when Assumption 4.1 is not satisfied, Lemma B.9 shows that as ambiguity increases the Choquet expected pay-offs tend to $\max_{s_i \in S_i} \left\{ \alpha_i u_i \left(s_i, \underline{s}_{-i} \right) + (1 - \alpha_i) u_i \left(s_i, \bar{s}_{-i} \right) \right\}$. Thus, the equilibrium pay-offs will be unique even when the equilibrium strategies are not.

In a game with increasing differences and multiple Nash equilibria, increasing ambiguity causes the multiplicity of equilibria to disappear, while increasing ambiguity-aversion causes the equilibrium strategies to decrease. Hence, ambiguity and ambiguity-attitude have distinct effects.

5 LITERATURE REVIEW

In a broad view, the literature on games with ambiguity can be organized into two strands. One way to approach ambiguity in games is to interpret Nash equilibria as equilibria in beliefs. In this perspective, ambiguity concerns the behaviour of the opponents. Players choose pure strategies but have ambiguous beliefs about the opponents' behaviour. Equilibrium means that there is some degree of consistency between ambiguous beliefs and strategies played. The papers by Dow and Werlang (1994), Marinacci (2000), Eichberger and Kelsey (2000), and the present paper belong to this group. Haller (2000) noted the focus on a support notion for capacities which is a common feature of these equilibrium concepts

The second way to approach equilibrium under ambiguity assumes that players choose mixed strategies. Hence, ambiguity concerns the mixed strategies of the opponents. Equilibrium, once again, requires some degree of consistency with the mixed strategies actually played. Klibanoff (1996), Lo (1996), and more recently Bade (2011), and Lehrer (2011) choose this approach. A characteristic feature of these papers is the "observational equivalence", as Bade (2011) calls it, between Nash equilibrium and these equilibrium notions for two-player games. Section 11 of Bade (2011) provides an excellent and extensive discussion of these equilibrium notions and illustrates their similarities and differences by examples. In particular, she highlights the fact that all these approaches focus on the case of ambiguity aversion. For the remainder of this review, we will concentrate on the more recent literature.

Lehrer (2011) considers decision-making when individuals have incomplete information about the probabilities. In the simplest version, they may have a probability defined on a sub-algebra of events rather than on all measurable events. In more complicated versions, the individuals may only know the expectations of certain random variables. The model is a special case of CEU with a convex capacity. It is less general than CEU since it is capable of modelling ambiguity-aversion but not ambiguity-seeking.

Lehrer proceeds to apply this model to games with a fixed partition of the strategy spaces. Players only know the probabilities of elements of the partition, but not those of individual strategies. The key difference to the present paper is that players regard their own mixed strategies as being as ambiguous. In contrast we assume that a given player views his opponents' behaviour as potentially ambiguous but perceives no ambiguity about his own behaviour.

Bade (2011) investigates whether a player can gain a strategic advantage by deliberately creating ambiguity? Given her assumptions, she finds the answer is no. Since her model uses general preferences over strategies, her results are applicable to a wide range of models. However, her results do not apply to our model, because she makes two assumptions which our model does not satisfy. Firstly, her monotonicity assumption rules out the possibility that a state may have positive weight if associated with a bad outcome but zero weight if associated with a good outcome. CEU preferences do not, in general, satisfy this assumption. Secondly, like Lehrer, she assumes that players agree on the ambiguity of strategies. Hence, a player views his own action as being as ambiguous as the opponent does.

In a recent paper, Riedel and Sass (2011) study whether there is a strategic advantage to creating ambiguity. In their paper players can commit to using an ambiguous randomizing device, like an urn with partial information over the characteristics of the balls in it. As equilibrium concept, they consider a Nash equilibrium in these "ambiguous strategies". Hence, players have common knowledge about "ambiguous strategies". Riedel and Sass (2011) show by example that "deliberate strategic ambiguity" may model behaviour which can be observed in laboratory experiments.

In a dynamic repeated games context, Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2012) use the smooth model of ambiguity. Individuals play the same game an infinite number of

times in a strangers setting. Hence, repeated game effects do not arise. On the equilibrium path players choose best responses to the actual population-wide distribution of strategies played by opponents. Off the equilibrium path, however, non-Nash behaviour can be sustained due to ambiguity about behaviour there. Thus the set of equilibria is larger than the set of Nash equilibria. The dynamic equilibrium notions studied cannot be applied to games in strategic form directly. For the case of ambiguity aversion, Eichberger and Kelsey (2004) provides a first extension of the EUA concept to dynamic games which confirms the analysis of Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2012) that ambiguous out-of-equilibrium beliefs may expand the set of Nash equilibria.

6 CONCLUSION

In this paper, we have studied the impact of ambiguity in games with players who are not necessarily ambiguity-averse. We have extended previous work by proposing new definitions of support and equilibrium which allow for an ambiguity-loving (optimistic) attitude towards uncertainty. The new notion of equilibrium has already been successfully applied to the theoretical analysis of games in Eichberger, Kelsey, and Schipper (2009), and to experimental studies in Eichberger, Kelsey, and Schipper (2008) and **?**.

Modelling optimistic responses to ambiguity introduces non-convexities in preferences. Hence, existence proofs and comparative static results cannot be obtained with the same generality as in the convex case which has been treated in the literature. In this context, the paper makes a number of innovations, for instance developing techniques for analysing the distinct effects of ambiguity and ambiguity-attitude. Compared to Eichberger and Kelsey (2002), we study a significantly broader class of games. In particular, we do not assume symmetry nor concavity in a player's own strategy. Aggregate externalities are only assumed for the comparative statics section.

A couple of issues remain for future research. Extending the analysis to games with more than two players requires one to confront the problem of correlated beliefs. Lo (2009) argues for accepting correlations among beliefs about the opponents' behaviour as the more adequate way of modelling a Nash equilibrium. Bade (2011) suggests notions of independence in beliefs which

need to be related to and compared with notions of independence for capacities and multiple priors. This is a wide field for future research both theoretically and experimentally.

A second open question concerns the treatment of mixed strategies. Given preferences over strategies yielding ambiguous pay-offs, mixed strategies can no longer be a straight-forward extension of pure strategies. Whether mixed strategies should be treated as an ambiguity reducing device (Raiffa (1961)) or as a strategic element in its own right remains an unresolved issue. Earlier work on preferences for randomization under ambiguity by Eichberger and Kelsey (1996) and Klibanoff (2001) has found experimental support in Dominiak and Schnedler (2011) and a new theoretical perspective in Saito (2012). This work will lead to a reconsideration of the role of mixed strategies in games under ambiguity.

A   ALTERNATIVE NOTIONS OF SUPPORT

In Section 3.1, we introduced a support notion for convex capacities, $\operatorname{supp} \mu$, and argued that it is suitable because it coincides with all common definitions of support for convex capacities and a leading support notion for MEU. In this appendix, we will substantiate these claims with some formal results.

Ryan (2002) discusses several notions of a support for MEU preferences[17]. For a set $P \subseteq \Delta(S_{-i})$ of multiple priors, Ryan (2002) (page 56) defines a *strong (resp. weak) support of P* as $\bigcup_{p \in P} \operatorname{supp} p$ (resp. $\bigcap_{p \in P} \operatorname{supp} p$), where $\operatorname{supp} p$ denotes the usual support of a probability distribution. The strong (resp. weak) support comprises the strategy combinations of the opponents which have a positive probability under *some (resp. all)* probability distributions from P. For convex capacities, preferences have both CEU and MEU representations. Hence, these support notions can be applied to a convex capacity, μ, in which case the weak support coincides with $\operatorname{supp} \mu$.

The following Proposition shows that $\operatorname{supp} \mu$ coincides with the M-support and the set of states which always receive positive weight in the Choquet integral, $\mathcal{B}(\nu)$. Even the DW-support, which is in general non-unique, is closely related to the support of Definition 3.1 as Proposition A.1 demonstrates.

Proposition A.1   *For a* convex *capacity μ :*

1. *if supp_{DW} is unique, then $\operatorname{supp} \mu = \mathcal{B}(\mu) = \operatorname{supp}_M \mu = \operatorname{supp}_{DW} \mu$,*

2. *otherwise, $\operatorname{supp} \mu = \mathcal{B}(\mu) = \operatorname{supp}_M \mu \subseteq \operatorname{supp}_{DW} \mu$.*

Proof.   The proof of Proposition A.1 consists of three lemmata (below). Part 1 follows from Lemma A.3 and Lemma A.2. Part 2 follows from Lemma A.3 and Lemma A.1.   ∎

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[17]Ryan (2002) discusses these notions in a model where decision makers have lexicographically ordered beliefs. In this context, Ryan (2002) introduces the concept of *firm beliefs* which coincides with our support notion for the non-lexicographic versions of the CEU and MEU models. An earlier unpublished paper, Ryan (1997), contains a similar discussion in the more familiar context of CEU and MEU.

Lemma A.1 *If ν is a capacity, then $\operatorname{supp}_M \nu \subseteq \operatorname{supp}_{DW} \nu$ for any $\operatorname{supp}_{DW} \nu \in \mathcal{D}(\nu)$, where $\mathcal{D}(\nu)$ denotes the set of all DW-supports of the capacity ν.*

Proof. If $\operatorname{supp}_M \nu = \varnothing$, then the result is trivial. Otherwise, there is $\tilde{s} \in \operatorname{supp}_M \nu$. Suppose, if possible, $\tilde{s} \notin \operatorname{supp}_{DW} \nu$. Then $0 = \nu(S \backslash \operatorname{supp}_{DW} \nu) \geqslant \nu(\tilde{s}) > 0$, which is a contradiction. ∎

Lemma A.2 *Let ν be a capacity on S_{-i} then $\operatorname{supp}_{DW} \nu$ is unique iff $\operatorname{supp}_M \nu$ is a DW-support.*

Proof. Assume that $\operatorname{supp}_{DW} \nu$ is unique. Let E be the DW-support. By Lemma A.1, $\operatorname{supp}_M \nu \subseteq E$. Suppose, if possible, there exists $\hat{s} \in E \backslash \operatorname{supp}_M \nu$, then $\nu(\hat{s}) = 0$. Hence, $F := S_{-i} \backslash \{\hat{s}\}$ satisfies $\nu(S_{-i} \backslash F) = 0$. Let G be a minimal set such that $G \subseteq F$ and $\nu(S_{-i} \backslash G) = 0$. Take $\bar{s} \in G$ and let $G' = G \backslash \{\bar{s}\}$. Then by minimality $\nu(S_{-i} \backslash G') > 0$, which establishes that G is a DW-support different from E. However this contradicts uniqueness. Hence $\operatorname{supp}_M \nu = \operatorname{supp}_{DW} \nu$.

Now assume $M = \operatorname{supp}_M \nu$ is a DW-support. Let F be an arbitrary DW-support. By Lemma A.1, $\operatorname{supp}_M \nu \subseteq F$. Suppose if possible that there exists $\tilde{s} \in F \backslash M$. Let $F' = F \backslash \{\tilde{s}\}$. Then since $F' \subsetneqq F$ and F is a Dow-Werlang support $\nu(S_{-i} \backslash F') > 0$. However since M is a Dow-Werlang support and $S_{-i} \backslash F' \subseteq S_{-i} \backslash M$ we must have $\nu(S_{-i} \backslash F') = \nu(S_{-i} \backslash M) = 0$, which is a contradiction. The result follows. ∎

Lemma A.3 *If μ is an convex capacity with support $\operatorname{supp} \mu$, then $\operatorname{supp} \mu = \operatorname{supp}_M \mu = \mathcal{B}(\mu)$.*

Proof. Let $\tilde{s} \in \operatorname{supp}_M \mu$ and let $\pi \in \mathcal{C}(\mu)$.[18] Then, by definition, $\pi(\tilde{s}) \geqslant \mu(\{\tilde{s}\}) > 0$. Hence, $\operatorname{supp}_M \mu \subseteq \bigcap_{\pi \in \mathcal{C}(\mu)} \operatorname{supp} \pi = \operatorname{supp} \mu$. On the other hand, suppose $s \in \bigcap_{\pi \in \mathcal{C}(\mu)} \operatorname{supp} \pi$. Since μ is convex, $\mu(s) = \min_{\pi \in \mathcal{C}(\mu)} \pi(s) > 0$.[19] Hence $\bigcap_{\pi \in \mathcal{C}(\mu)} \operatorname{supp} \pi \subseteq \operatorname{supp}_M \mu$.

Suppose $s \in \operatorname{supp}_M \mu$. Then $\mu(s) > 0$. For any $A \subseteq S_{-i}$, $s \notin A$, by convexity of μ, $\mu(A \cup s) \geqslant \mu(A) + \mu(s) > \mu(A)$. Hence, $s \in \mathcal{B}(\mu)$. Conversely suppose $s \in \mathcal{B}(\mu)$, then $\mu(s) = \mu(\varnothing \cup s) > \mu(\varnothing) = 0$. Hence, $s \in \operatorname{supp}_M \mu$. Thus $\operatorname{supp}_M \mu = \mathcal{B}(\mu)$. The result follows. ∎

Proof of Proposition 3.1: We shall prove this by demonstrating that for all $A \subsetneqq S_{-i}, \hat{s} \notin A, \nu(A \cup \hat{s}) > \nu(A)$. Take $\hat{s} \in \operatorname{supp} \nu$ and $A \subsetneqq S_{-i}$, $\hat{s} \notin A$. Let $\hat{p} = \operatorname{argmin}_{p \in \mathcal{C}(\mu)} p(A \cup \hat{s})$.

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[18]Recall $\mathcal{C}(\mu)$ denotes the core of the capacity μ, see Definition 2.4.

[19]Although $\mathcal{C}(\mu)$ is an infinite set, the minimum must occur at one of the extremal points. The set of extremal points of a core is finite. Thus the minimum must be positive.

Because μ is convex $\mu(A \cup \hat{s}) - \mu(A) = \hat{p}(A \cup \hat{s}) - \mu(A) = \hat{p}(A) - \mu(A) + \hat{p}(\hat{s}) > 0$, since $\hat{p}(A) \geqslant \mu(A)$ and $\hat{p}(\hat{s}) > 0$ by Lemma A.3. Similarly we may show $\bar{\mu}(A \cup \hat{s}) - \bar{\mu}(A) > 0$. Hence for $A \subsetneqq S_{-i}, s \notin A, \nu(A \cup \tilde{s}) > \nu(A)$. $\blacksquare$

If ν is a JP-capacity, $\mathcal{B}(\nu)$ does not depend on the JP-representation. Thus, to a great extent, our support notion is independent of the parameters of the JP-representation.

We conclude this appendix with the proof of Proposition 3.2, which finds the support of a neo-additive capacity.

Lemma A.4 *Let $\nu = \delta(1-\alpha) + (1-\delta)\pi$ be a neo-additive capacity on S_{-i}, where $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \delta < 1$. Then:*

1. ν may be written in the form $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, where $\mu = (1-\delta)\pi(A) + \delta\nu_0(A)$; [20]

2. the maximal and minimal degrees of ambiguity of μ are $\lambda(\mu) = \gamma(\mu) = \delta$ respectively.

Proof. Clearly $\alpha\mu(\varnothing) + (1-\alpha)\bar{\mu}(\varnothing) = 0 = \nu(\varnothing)$ and $\alpha\mu(S_{-i}) + (1-\alpha)\bar{\mu}(S_{-i}) = 1 = \nu(S_{-i})$. If $\varnothing \subsetneqq A \subsetneqq S_{-i}$ then

$\alpha\mu(A) + (1-\alpha)\bar{\mu}(A) = \alpha(1-\delta)\pi(A) + (1-\alpha)(1-\delta)\pi(A) + (1-\alpha)\delta \cdot 1$

$= \delta(1-\alpha) + (1-\delta)\pi(A) = \nu(A)$.

If $\varnothing \subsetneqq A \subsetneqq S_{-i}$ then $\bar{\mu}(A) - \mu(A) = [1 - (1-\delta)\pi(A^c)] - (1-\delta)\pi(A) = \delta$,

since $\pi(A) + \pi(A^c) = 1$. $\blacksquare$

Proof of Proposition 3.2: By Lemma A.4, $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, where $\mu = (1-\delta)\pi(A) + \delta\nu_0(A)$. By definition, $\text{supp}\,\nu_{JP} = \text{supp}\,\mu$. Suppose that $\hat{s} \in \text{supp}\,\pi$. If $p \in \mathcal{C}(\mu)$ then $p(\hat{s}) \geqslant \mu(\hat{s}) = (1-\delta)\pi(\hat{s}) > 0$. Thus $\forall p \in \mathcal{C}(\mu), p(\hat{s}) > 0$, which implies $\hat{s} \in \text{supp}\,\mu$, hence $\text{supp}\,\pi \subseteq \text{supp}\,\mu$.

To show $\text{supp}\,\mu \subseteq \text{supp}\,\pi$, suppose if possible, there exists $\tilde{s} \in \text{supp}\,\mu \setminus \text{supp}\,\pi$. Then $\mu(\tilde{s}) = (1-\delta)\pi(\tilde{s}) = 0$. If $q = \text{argmin}_{p \in \mathcal{C}(\mu)} p(\tilde{s})$, then $q(\tilde{s}) = 0$. Thus, $\tilde{s} \notin \text{supp}\,q$ and consequentially $\tilde{s} \notin \text{supp}\,\mu = \bigcap_{p \in \mathcal{C}(\mu)} \text{supp}\,p$. However, this is a contradiction, which establishes that $\text{supp}\,\mu \subseteq \text{supp}\,\pi$. The result follows. $\blacksquare$

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[20] Recall ν_0 denotes the complete uncertainty capacity in Example 2.1.

B GAMES WITH AMBIGUITY

This appendix contains proofs of our existence and comparative statics results and some supplementary results. It uses techniques from Topkis (1998).

B.1 Existence

We start with a preliminary definition and Lemma.

Definition B.1 *Suppose that B is a correspondence from a partially ordered set S to a lattice T such that for all $s \in S, B(s)$ is a sub-lattice of T, then we say that B is increasing if when $\hat{s} > \tilde{s}$, and $\hat{t} \in B(\hat{s}), \tilde{t} \in B(\tilde{s})$ then $\min\{\hat{t}, \tilde{t}\} \in B(\tilde{s})$ and $\max\{\hat{t}, \tilde{t}\} \in B(\hat{s})$.*

Lemma B.1 *Let S be a lattice and let $\beta : S \to S$ be an increasing correspondence. Then*

1. *β has a fixed point;*

2. *$\sup\{s : \beta(s) \geqslant s\}$ is the greatest fixed point of β.*

Proof. Let $\mathcal{T} = \{s : \beta(s) \geqslant s\}$. Note that $\mathcal{T}$ is non-empty since $\underline{s} \in \mathcal{T}$, (where $\underline{s} = \min S$). Let $s' = \sup \mathcal{T}$. By definition, if $s'' > s'$ then

$$\beta(s'') < s''. \tag{2}$$

Suppose $\tilde{s} \in \mathcal{T}$, then $\beta(\tilde{s}) \geqslant \tilde{s}$. Since β is increasing, $\beta(s') \geqslant \beta(\tilde{s})$ and $\beta(s') \geqslant s' \geqslant \tilde{s}$. Thus, $\beta(\beta(s')) \geqslant \beta(s') \geqslant s'$, which implies $\beta(s') \in \mathcal{T}$ and hence $s' \geqslant \beta(s') \geqslant s'$.[21] The last equation implies that s' is a fixed point of β. Equation (2) implies that there is no greater fixed point. ∎

Proof of Theorem 4.1 Choose an n-tuple of parameters $\delta = \langle \delta_1, ..., \delta_n \rangle$ such that $\lambda \geqslant \delta \geqslant \gamma$. Let $\nu_i^{\hat{s}_{-i}}$ denote the neo-additive capacity on S_{-i} defined by $\nu_i^{\hat{s}_{-i}}(S_{-i}) = 1, \nu_i^{\hat{s}_{-i}}(A) = \delta_i(1 - \alpha_i)$, if $\hat{s}_{-i} \notin A, \nu_i^{\hat{s}_{-i}}(A) = \delta_i(1 - \alpha_i) + 1 - \delta_i$, otherwise.[22] Define $V_i(s_i, \hat{s}_{-i})$ to be player i's (Choquet) expected utility from playing s_i when his beliefs are represented by $\nu_i^{\hat{s}_{-i}}$ i.e.

$$V_i(s_i, \hat{s}_{-i}) = \int u_i(s_i, s_{-i}) d\nu_i^{\hat{s}_{-i}}(s_{-i}) = \delta_i(1 - \alpha_i) u_i(s_i, \bar{s}_{-i}) + \delta_i \alpha_i u_i(s_i, \underline{s}_{-i}) + (1 - \delta_i) u_i(s_i, \hat{s}_{-i}).$$

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[21] $\beta(\beta(s')) = \{s : \exists \tilde{s} \in \beta(s'), s = \beta(\tilde{s})\}$.

[22] Informally $\nu_i^{\hat{s}_{-i}}$ represents a situation where i has an ambiguous belief that his/her opponents will play $\hat{s}_{-i}$.

Define $\beta_i(\hat{s}_{-i}) = \text{argmax}_{s_i \in S_i} V_i(s_i, \hat{s}_{-i})$ and $\beta(\hat{s}) = \langle \beta_1(\hat{s}_{-1}), ..., \beta_n(\hat{s}_{-n}) \rangle$. Thus $\beta_i(\hat{s}_{-i})$ is the best response of player i, if his beliefs are a neo-additive capacity which represents an ambiguous belief that his opponents will play $\hat{s}_{-i}$. Lemma B.6 implies that $V_i(s_i, \hat{s}_{-i})$ has increasing differences in $\langle s_i, \hat{s}_{-i} \rangle$, hence β_i is an increasing correspondence. (The proof is similar to that of Lemma B.8.) Thus by Lemma B.1, β has a fixed point s^*. This implies $\nu^{s^*} = \left\langle \nu_1^{s^*_{-1}}, ..., \nu_n^{s^*_{-n}} \right\rangle$ is a singleton equilibrium. By Proposition 3.2, $\nu_i^{s^*_{-i}}$ may be written in the form $\nu_i^{s^*_{-i}} = \alpha_i \mu_i + (1 - \alpha_i) \bar{\mu}_i$, where μ_i is convex and $\lambda(\mu_i) = \gamma(\mu_i) = \delta_i$. ∎

B.2   Comparative Statics Proofs

B.2.1   Correspondences on Partially Ordered Sets

This section contains some results about increasing correspondences and selections from them.

Lemma B.2 *Suppose that B_λ is an increasing correspondence from a partially ordered set S to a totally ordered set T for all λ in an index set Λ, then $\bar{B}(s) = \max_{\lambda \in \Lambda} B_\lambda(s)$ and $\underline{B}(s) = \min_{\lambda \in \Lambda} B_\lambda(s)$ are increasing functions from S to T.*

Proof. Suppose that $\hat{s} > \tilde{s}$. Then there exists $\tilde{\lambda} \in \Lambda$ such that $\bar{B}(\tilde{s}) = B_{\tilde{\lambda}}(\tilde{s})$. Since $B_{\tilde{\lambda}}$ is increasing, $\bar{B}(\hat{s}) \succcurlyeq B_{\tilde{\lambda}}(\hat{s}) \succcurlyeq B_{\tilde{\lambda}}(\tilde{s}) = \bar{B}(\tilde{s})$, which demonstrates that $\bar{B}$ is increasing.

There exists $\hat{\lambda} \in \Lambda$ such that $\underline{B}(\hat{s}) = B_{\hat{\lambda}}(\hat{s}) = \min B_{\hat{\lambda}}(\hat{s})$. Since $B_{\hat{\lambda}}$ is increasing, $B_{\hat{\lambda}}(\hat{s}) \succcurlyeq B_{\hat{\lambda}}(\tilde{s})$. Finally $B_{\hat{\lambda}}(\tilde{s}) \succcurlyeq \underline{B}(\tilde{s})$, which establishes that $\underline{B}(\hat{s}) \succcurlyeq \underline{B}(\tilde{s})$. ∎

The following lemma describes some properties of fixed points of functions on partially ordered sets.

Lemma B.3 *Let S and A be partially ordered sets and let $f : S \times A \to S$ be a function which is increasing in s and α. Then the greatest fixed point of $f(\cdot, \alpha)$ is an increasing function of α.*

Proof. Let $s(\alpha)$ denote the greatest fixed point of $f(\cdot, \alpha)$. Since f is increasing in α, if $\hat{\alpha} > \tilde{\alpha}, \{s : f(s, \tilde{\alpha}) \geqslant s\} \subseteq \{s : f(s, \hat{\alpha}) \geqslant s\}$. Hence $s(\hat{\alpha}) = \sup\{s : f(s, \hat{\alpha}) \geqslant s\} \geqslant \sup\{s : f(s, \tilde{\alpha}) \geqslant s\} = s(\tilde{\alpha})$ by Lemma B.1. ∎

B.2.2 Constant Contamination Capacities

Below we define a special case of JP-capacities which arise naturally when considering pure equilibria in games.

Definition B.2 (Constant Contamination) *A capacity,* $\nu_i^{\tilde{s}}\left(\alpha_i, \delta_i, \varsigma_i\right)$, *on* S_{-i} *is said to display* constant contamination *(henceforth CC) if it may be written in the form* $\nu_i^{\tilde{s}}\left(\alpha_i, \delta_i, \varsigma_i\right) = \nu_i^{\tilde{s}_{-i}}\left(A, \alpha_i, \delta_i, \varsigma_i\right) = (1 - \delta_i)\,\pi_i^{\tilde{s}_{-i}}\left(A\right) + \delta_i\left[\alpha_i\varsigma_i\left(A\right) + (1 - \alpha_i)\,\bar{\varsigma}_i\left(A\right)\right]$, *where* $\pi_i^{\tilde{s}_{-i}}$ *denotes the probability distribution[23] on* S_{-i}, *which assigns probability 1 to* $\tilde{s}_{-i}$ *and* ς_i *is a convex capacity with* $\operatorname{supp}\varsigma_i = \emptyset$. *To simplify notation we shall suppress the arguments* $\left(\alpha_i, \delta_i, \varsigma_i\right)$ *when it is convenient.*

We interpret the capacity $\nu_i\left(\varsigma_i, \delta_i, \alpha_i\right)$ as describing a situation where player i 'believes' that his opponents will play the pure strategy profile $\tilde{s}_{-i}$ but lacks confidence in this belief. The CC-capacity embodies a separation between beliefs represented by π_i, ambiguity-attitude represented by α_i and ambiguity represented by ς_i and δ_i. The parameter δ_i determines the weight the individual gives to ambiguity. Higher values of δ_i correspond to more ambiguity. The capacity ς_i determines which strategy profiles the player regards as ambiguous. The following result finds the support of a CC capacity.

Lemma B.4 *Let* $\nu_i = (1 - \delta_i)\,\pi^{\tilde{s}_{-i}}\left(A\right) + \delta_i\left[\alpha_i\varsigma_i\left(A\right) + (1 - \alpha_i)\,\bar{\varsigma}_i\left(A\right)\right]$ *be a CC capacity. Then* $\operatorname{supp}_{JP}\nu_i = \{\tilde{s}_{-i}\}$.

Proof. If we define a convex capacity $\mu_i\left(A\right)$ by $\mu_i = (1 - \delta_i)\,\pi^{\tilde{s}_{-i}} + \delta_i\varsigma_i\left(A\right)$ then $\nu_i = \alpha_i\mu_i + (1 - \alpha_i)\,\bar{\mu}_i$. By definition $\operatorname{supp}_{JP}\nu_i = \operatorname{supp}\mu_i$. If $p \in \mathcal{C}\left(\mu_i\right)$ then $p\left(\tilde{s}_{-i}\right) \geqslant \mu_i\left(\tilde{s}_{-i}\right) = (1 - \delta_i)\,\pi^{\tilde{s}_{-i}}\left(\tilde{s}_{-i}\right) + \delta_i\varsigma_i\left(\tilde{s}_{-i}\right) = (1 - \delta_i)$, since $\operatorname{supp}\varsigma_i = \varnothing$, which implies $\varsigma_i\left(\tilde{s}_{-i}\right) = 0$. Thus $\forall p \in \mathcal{C}\left(\mu_i\right), p\left(\tilde{s}_{-i}\right) > 0$, which implies $\tilde{s}_{-i} \in \operatorname{supp}\mu$.

To show $\{\tilde{s}_{-i}\} = \operatorname{supp}\mu$, suppose if possible, there exists $\bar{s}_{-i} \in \operatorname{supp}\mu$ such that $\tilde{s}_{-i} \neq \bar{s}_{-i}$. Then $\mu_i\left(\bar{s}_{-i}\right) = (1 - \delta_i)\,\pi^{\tilde{s}_{-i}}\left(\bar{s}_{-i}\right) + \delta_i\varsigma_i\left(\bar{s}_{-i}\right) = 0$ since $\pi^{\tilde{s}_{-i}}\left(\bar{s}_{-i}\right) = 0$ and $\operatorname{supp}\varsigma_i = \varnothing$. If $q = \operatorname{argmin}_{p \in \mathcal{C}(\mu)} p\left(\bar{s}_{-i}\right)$, then $q\left(\bar{s}_{-i}\right) = 0$. Thus, $\bar{s}_{-i} \notin \operatorname{supp}q$ and consequentially $\bar{s}_{-i} \notin \operatorname{supp}\mu_i =$

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[23]This distribution is usually denoted by $\delta_{\tilde{s}_{-i}}$. However we are using the symbol δ elsewhere to denote degree of ambiguity.

$\bigcap_{p \in \mathcal{C}(\mu_i)} \operatorname{supp} p$. However, this is a contradiction, which establishes that $\operatorname{supp} \mu_i \subseteq \{\tilde{s}_{-i}\}$. The result follows. ∎

The following lemma shows that any capacity which describes the equilibrium of a game is a CC-capacity. This provides a useful characterization of equilibrium beliefs.

Lemma B.5 *Let Γ be a game with positive externalities and let $\hat{\nu}$ be a singelton equilibrium in JP-capacities of Γ with equilibrium strategy profile $\hat{s}$. Then $\hat{\nu}$ is a profile of CC-capacities, i.e. there exist convex capacities $\zeta_i, 1 \leqslant i \leqslant n$, with $\operatorname{supp} \zeta_i = \varnothing$ and $\delta_i, 1 \leqslant i \leqslant n$, such that if we define $\mu_i = \delta_i \varsigma_i + (1 - \delta_i) \pi_i^{\hat{s}_{-i}}$ then $\nu_i = \alpha \mu_i + (1 - \alpha_i) \bar{\mu}_i$ for $1 \leqslant i \leqslant n$. Moreover $\lambda(\mu) = (1 - \delta_i) \lambda(\zeta)$ and $\gamma(\mu) = (1 - \delta_i) \gamma(\zeta)$.*

Proof. Since $\hat{\nu}$ is an equilibrium in JP-capacities, we may write the equilibrium beliefs of individual i in the form $\hat{\nu}_i = \alpha_i \mu_i + (1 - \alpha_i) \bar{\mu}_i$ for some convex capacity μ_i. Define a capacity ζ_i by $\zeta_i = \frac{\mu_i - \delta_i \pi_i^{\hat{s}_{-i}}}{1 - \delta_i}$, where $\delta_i = \mu_i(\hat{s}_{-i})$. Then $\hat{\nu}_i = \delta_i \pi_i^{\hat{s}_{-i}}(A) + (1 - \delta_i) [\alpha_i \varsigma_i(A) + (1 - \alpha_i) \bar{\varsigma}_i(A)]$. We claim that ζ_i is convex. To prove this we need to show $\zeta_i(A \cup B) \geqslant \zeta_i(A) + \zeta_i(B) - \zeta_i(A \cap B)$ for all $A, B \subseteq S_{-i}$. There are four cases to consider.

If $\hat{s}_{-i} \in A$ and $\hat{s}_{-i} \in B$, then $\zeta_i(A \cup B) + \zeta_i(A \cap B) = \frac{1}{1 - \delta_i}(\mu_i(A \cup B) + \mu_i(A \cap B) - 2\delta_i) \geqslant \frac{1}{1 - \delta_i}(\mu_i(A) + \mu_i(B) - 2\delta_i)$ by convexity of μ_i. Since $\zeta_i(A) + \zeta_i(B) = \frac{1}{1 - \delta_i}(\mu_i(A) + \mu_i(B) - 2\delta_i)$ the claim is proved in this case.

If $\hat{s}_{-i} \notin A$ and $\hat{s}_{-i} \notin B$, then the claim follows from convexity of μ_i, since $\zeta_i = \frac{1}{1 - \delta_i} \mu_i$ for all four sets.

If $\hat{s}_{-i} \in A$ and $\hat{s}_{-i} \notin B$, $\zeta_i(A \cup B) + \zeta_i(A \cap B) = \frac{1}{1 - \delta_i}(\mu_i(A \cup B) - \delta_i) + \frac{1}{1 - \delta_i} \mu_i(A \cap B) \geqslant \frac{1}{1 - \delta_i} \mu_i(A) + \frac{1}{1 - \delta_i} \mu_i(B) - \frac{\delta_i}{1 - \delta_i}$, by convexity of μ_i. Since $\zeta_i(A) + \zeta_i(B) = \frac{1}{1 - \delta_i}(\mu_i(A) + \mu_i(B) - \delta_i)$ this proves convexity in this case. The remaining case can be established by similar reasoning.

Since $\operatorname{supp}_{JP} \nu_i = \hat{s}_{-i}$, $\operatorname{supp} \mu_i = \hat{s}_{-i}$. Hence for $\tilde{s}_{-i} \neq \hat{s}_{-i}$, $\mu_i(\tilde{s}_{-i}) = 0$, which implies that for all $s_{-i} \in S_{-i}, \zeta_i(s_{-i}) = 0$. Since ζ_i is convex, it follows from Proposition A.1 that $\operatorname{supp} \zeta_i = \varnothing$.

Now consider $A \subsetneq S_{-i}$. Assume without loss of generality $\hat{s}_{-i} \in A$. Then

$1 - \zeta_i(A) - \zeta_i(A^c) = 1 - \frac{1}{1 - \delta_i}(\mu_i(A) - \delta_i) - \frac{1}{1 - \delta_i}(\mu_i(A^c))$

$= \frac{1}{1 - \delta_i}[1 - \delta_i - \mu_i(A) + \delta_i - \mu_i(A^c)] = \frac{1}{1 - \delta_i}[1 - \mu_i(A) - \mu_i(A^c)]$, which implies $(1 - \delta_i) \lambda(\zeta) =$

$\lambda\left(\mu\right)$ and $\left(1-\delta_{i}\right)\gamma\left(\zeta\right)=\gamma\left(\mu\right).$ ■

B.2.3 Increasing/decreasing Differences

Recall that a game, $\Gamma=\langle N;\left(S_{i}\right),\left(u_{i}\right):1\leqslant i\leqslant n\rangle$, has *positive aggregate externalities* if $u_{i}\left(s_{i},s_{-i}\right)=u_{i}\left(s_{i},f_{i}\left(s_{-i}\right)\right)$, for $1\leqslant i\leqslant n$, where u_{i} is increasing (resp. decreasing) in f_{i} and $f_{i}:S_{-i}\rightarrow\mathbb{R}$ is increasing in all arguments. Since S_{-i} is finite, we may enumerate the possible values of f_{i}, $f_{i}^{0}<...<f_{i}^{M}$. Since f is increasing $f_{i}^{0}=f\left(\underline{s}_{1},...,\underline{s}_{n}\right)$ and $f_{i}^{M}=f\left(\bar{s}_{1},...,\bar{s}_{n}\right)$. The *Choquet integral* of $u_{i}\left(s_{i},s_{-i}\right)$ with respect to capacity ν_{i} on S_{-i} may be written in the form

$$V_{i}\left(s_{i}\right)=\int u_{i}\left(s_{i},s_{-i}\right)d\nu_{i}=u_{i}\left(s_{i},f_{M}\right)\nu_{i}\left(H_{M}\right)+\sum_{r=0}^{M-1}u_{i}\left(s_{i},f_{r}\right)\left[\nu_{i}\left(H_{r}\right)-\nu_{i}\left(H_{r+1}\right)\right],$$

where H_{r} denotes the event $\left\{s_{-i}\in S_{-i}:f\left(s_{-i}\right)\geqslant f_{r}\right\}$.

Define $W_{i}\left(s_{i},\tilde{s}_{-i},\alpha_{i},\delta_{i},\varsigma_{i}\right)$ to be player i's (Choquet) expected payoff given that his beliefs are represented by the capacity $\nu_{i}^{\tilde{s}-i}\left(A,\alpha_{i},\delta_{i},\varsigma_{i}\right)$ i.e.

$$W_{i}\left(s_{i},\tilde{s}_{-i},\alpha_{i},\delta_{i},\varsigma_{i}\right)=\int u_{i}\left(s_{i},s_{-i}\right)d\nu_{i}^{\tilde{s}-i}\left(A,\alpha_{i},\delta_{i},\varsigma_{i}\right).$$

Lemma B.6 *If $u_{i}\left(s_{i},s_{-i}\right)$ satisfies increasing differences in $\langle s_{i},s_{-i}\rangle$ so does $W_{i}\left(s_{i},s_{-i},\alpha_{i},\delta_{i},\varsigma_{i}\right)$.*

Proof. Suppose $s_{-i}'>s_{-i}''$, then $W_{i}\left(s_{i},s_{-i}',\alpha_{i},\delta_{i},\varsigma_{i}\right)-W_{i}\left(s_{i},s_{-i}'',\alpha_{i},\delta_{i},\varsigma_{i}\right)$

$=\alpha_{i}\delta_{i}\int u_{i}\left(s_{i},s_{-i}\right)d\varsigma_{i}+\left(1-\alpha_{i}\right)\delta_{i}\int u_{i}\left(s_{i},s_{-i}\right)d\bar{\varsigma}_{i}+\left(1-\delta_{i}\right)u_{i}\left(s_{i},s_{-i}'\right)-\alpha_{i}\delta_{i}\int u_{i}\left(s_{i},s_{-i}\right)d\varsigma_{i}$

$-\left(1-\alpha_{i}\right)\delta_{i}\int u_{i}\left(s_{i},s_{-i}\right)d\bar{\varsigma}_{i}-\left(1-\delta_{i}\right)u_{i}\left(s_{i},s_{-i}''\right)=\left(1-\delta_{i}\right)\left[u_{i}\left(s_{i},s_{-i}'\right)-u_{i}\left(s_{i},s_{-i}''\right)\right]$, which is increasing in s_{i} since u_{i} has increasing differences in $\langle s_{i},s_{-i}\rangle$. ■

Lemma B.7 *The function $W_{i}\left(s_{i},s_{-i},\alpha_{i},\delta_{i},\varsigma_{i}\right)$ has decreasing differences in $\langle s_{i},\alpha_{i}\rangle$.*

Proof. Suppose $s_{i}'>s_{i}''$, then $W_{i}\left(s_{i}',s_{-i},\alpha_{i},\delta_{i},\varsigma_{i}\right)-W_{i}\left(s_{i}'',s_{-i},\alpha_{i},\delta_{i},\varsigma_{i}\right)$

$=\alpha_{i}\delta_{i}\int u_{i}\left(s_{i}',s_{-i}\right)d\varsigma_{i}+\left(1-\alpha_{i}\right)\delta_{i}\int u_{i}\left(s_{i}',s_{-i}\right)d\bar{\varsigma}_{i}+\left(1-\delta_{i}\right)u_{i}\left(s_{i}',s_{-i}\right)$

$-\alpha_{i}\delta_{i}\int u_{i}\left(s_{i}'',s_{-i}\right)d\varsigma_{i}-\left(1-\alpha_{i}\right)\delta_{i}\int u_{i}\left(s_{i}'',s_{-i}\right)d\bar{\varsigma}_{i}-\left(1-\delta_{i}\right)u_{i}\left(s_{i}'',s_{-i}\right)$

$=\alpha_{i}\delta_{i}\left\{\int\left[u_{i}\left(s_{i}',s_{-i}\right)-u_{i}\left(s_{i}'',s_{-i}\right)\right]d\varsigma_{i}-\int\left[u_{i}\left(s_{i}',s_{-i}\right)-u_{i}\left(s_{i}'',s_{-i}\right)\right]d\bar{\varsigma}_{i}\right\}$

$+\delta_{i}\int\left[u_{i}\left(s_{i}',s_{-i}\right)-u_{i}\left(s_{i}'',s_{-i}\right)\right]d\bar{\varsigma}_{i}+\left(1-\delta_{i}\right)\left[u_{i}\left(s_{i}',s_{-i}\right)-u_{i}\left(s_{i}'',s_{-i}\right)\right].$

We have used the fact that since there are positive aggregate externalities, all four integrands in the curly brackets are comonotonic. It is sufficient to show that the coefficient of α_i is positive. This is equal to

$$
\begin{aligned}
&\left[u_i\left(s_i', f_M\right) - u_i\left(s_i'', f_M\right)\right] \varsigma_i\left(s_{-i}^M\right) + \sum_{r=0}^{M-1}\left[u_i\left(s_i', s_{-i}^r\right) - u_i\left(s_i'', s_{-i}^r\right)\right]\left[\varsigma_i\left(H_r\right) - \varsigma_i\left(H_{r+1}\right)\right] \\
&- \left[u_i\left(s_i', f_M\right) - u_i\left(s_i'', f_M\right)\right] \bar{\varsigma}_i\left(s_{-i}^M\right) \\
&- \sum_{r=0}^{M-1}\left[u_i\left(s_i', s_{-i}^r\right) - u_i\left(s_i'', s_{-i}^r\right)\right]\left[\bar{\varsigma}_i\left(H_r\right) - \bar{\varsigma}_i\left(H_{r+1}\right)\right].
\end{aligned} \tag{3}
$$

By increasing differences, $u_i\left(s_i', s_{-i}\right) - u_i\left(s_i'', s_{-i}\right) > 0$ and is an increasing function of s_{-i}. Equation (3) is the difference of two weighted sums of $u_i\left(s_i', s_{-i}\right) - u_i\left(s_i'', s_{-i}\right)$. The first k weights in the first sum add up to: $\varsigma_i\left(s_{-i}^M\right) + \sum_{r=M-k+1}^{M-1}\left[\varsigma_i\left(H_r\right) - \varsigma_i\left(H_{r+1}\right)\right] = \varsigma_i\left(H_{M-k+1}\right)$. Similarly the first k weights in the second sum in total are equal to: $\bar{\varsigma}_i\left(H_{M-k+1}\right)$. Since ς_i is convex, $\varsigma_i\left(H_{M-k+1}\right) \leqslant \bar{\varsigma}_i\left(H_{M-k+1}\right)$, hence the weights in the first sum are first order stochastically dominated by those in the second. Thus, the first sum is smaller which makes the overall expression negative. This establishes that $W_i\left(s_i', s_{-i}, \alpha_i\right) - W_i\left(s_i'', s_{-i}, \alpha_i\right)$ is a decreasing function of α_i. ∎

Lemma B.8 *The best response correspondence of player i, $B_i\left(\hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$, defined by*

$B_i\left(s_{-i}, \alpha_i, \delta_i, \varsigma_i\right) = \operatorname{argmax}_{s_i \in S_i} W_i\left(s_i, s_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$ *is increasing in s_{-i} and decreasing in α_i.*

Proof. To show $B_i\left(s_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$ is increasing in s_{-i}, assume $\hat{s}_{-i} > \tilde{s}_{-i}$. Consider $y \in B_i\left(\tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right), z \in B_i\left(\hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$ and let $m = \min\{y, z\}$ and $M = \max\{y, z\}$. Now $W_i\left(y, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right) \geqslant W_i\left(z, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$ which implies $W_i\left(M, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right) \geqslant W_i\left(z, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$. By increasing differences, $W_i\left(M, \hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right) \geqslant W_i\left(z, \hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$, hence $M \in B_i\left(\hat{s}_{-i}, \alpha_i, \varsigma_i\right)$.

Since $W_i\left(y, \hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right) - W_i\left(z, \hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right) \leqslant 0$, increasing differences implies $W_i\left(y, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right) - W_i\left(z, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right) \leqslant 0$. Thus $W_i\left(m, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right) \geqslant W_i\left(y, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$ and hence $m \in B_i\left(\tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$. This establishes that $B_i\left(s_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$ is increasing in s_{-i}. We may establish that $B_i\left(s_{-i}, \alpha_i, \delta_i, \varsigma_i\right)$ is decreasing in α_i by a similar argument. ∎

Definition B.3 *The maximal and minimal best response correspondences of player i are defined*

respectively by

$$\bar{B}_i\left(s_{-i}, \alpha_i, \gamma_i, \lambda_i, \delta_i\right) = \max_{\zeta_i}\left\{B_i\left(s_{-i}, \alpha_i, \delta_i, \zeta_i\right); \forall A \subsetneqq S_{-i}, \frac{\lambda_i}{1-\delta_i} \geqslant \bar{\zeta}_i\left(A\right) - \zeta_i\left(A\right) \geqslant \frac{\gamma_i}{1-\delta_i}\right\},$$

$$\underline{B}_i\left(s_{-i}, \alpha_i, \gamma_i, \lambda_i, \delta_i\right) = \min_{\zeta_i}\left\{B_i\left(s_{-i}, \alpha_i, \delta_i, \zeta_i\right); \forall A \subsetneqq S_{-i}, \frac{\lambda_i}{1-\delta_i} \geqslant \bar{\zeta}_i\left(A\right) - \zeta_i\left(A\right) \geqslant \frac{\gamma_i}{1-\delta_i}\right\}.$$

It follows from Lemma B.5 that the maximal (resp. minimal) best response correspondence is the greatest (resp. least) best response to all beliefs whose support is the pure strategy s_{-i} with minimal (resp. maximal) degree of ambiguity is at least γ (resp. at most λ).

Proof of Theorem 4.2 We shall prove the result for the highest equilibrium strategy. The lowest equilibrium strategy can be covered by a similar argument. Lemma B.5 establishes that if $\hat{s}$ is an equilibrium strategy profile when the minimal (resp. maximal) degree of ambiguity is γ (resp. λ), then there exist ζ_i with $\frac{\lambda_i}{1-\delta_i} \geqslant \bar{\zeta}_i\left(A^c\right) - \zeta_i\left(A\right) \geqslant \frac{\gamma_i}{1-\delta_i}$ such that $\hat{s}_i \in B_i\left(s_{-i}, \alpha_i, \zeta_i\right)$ for $1 \leqslant i \leqslant n$. Thus any given equilibrium, satisfying these constraints, is smaller than the largest fixed point of the maximal best response correspondence $\bar{B}_i\left(s_{-i}, \alpha_i, \gamma_i, \lambda_i\right)$.

Therefore since $\bar{s}\left(\alpha\right)$ is the profile of greatest equilibrium strategies it is the largest fixed point of the maximal best response function, i.e. $\bar{s}\left(\alpha\right) \in \bar{B}\left(\bar{s}, \bar{\alpha}\right)$ and $\underline{s}\left(\alpha\right) \in \underline{B}\left(\underline{s}, \underline{\alpha}\right)$. By Lemma B.8, $\bar{B}_i\left(s_{-i}, \alpha_i, \gamma_i, \lambda_i\right)$ is increasing in s_{-i} and decreasing in α_i. It follows from Lemma B.3 that $\bar{s}\left(\alpha\right)$ is decreasing in α. ∎

B.2.4 Multiple Equilibria

In this section we show that equilibrium is unique if there is sufficient ambiguity.

Lemma B.9 *Consider a game, Γ, of positive externalities and increasing differences. There exists $\bar{\gamma}$ such that if the minimal degree of ambiguity is $\gamma\left(\mu_i\right) \geqslant \bar{\gamma}$, then in any equilibrium $\nu = \langle\nu_1, ..., \nu_n\rangle$, $\operatorname{supp}\nu_i \subseteq \mathcal{A}$, where $\mathcal{A}$ denotes the set $\operatorname{argmax}_{s_i \in S_i}\left\{\alpha_i u_i\left(s_i, \underline{s}_{-i}\right) + \left(1 - \alpha_i\right) u_i\left(s_i, \bar{s}_{-i}\right)\right\}$, for $1 \leqslant i \leqslant n$.*

Proof. Suppose $\hat{s}_i \in \mathcal{A}, \tilde{s}_i \notin \mathcal{A}$. Number the strategy profiles of the opponents so that $u_i\left(\hat{s}_i, s_{-i}^1\right) \geqslant u_i\left(\hat{s}_i, s_{-i}^2\right) \geqslant ... \geqslant u_i\left(\hat{s}_i, s_{-i}^R\right)$ and $u_i\left(\tilde{s}_i, \sigma_{-i}^1\right) \geqslant u_i\left(\tilde{s}_i, \sigma_{-i}^2\right) \geqslant ... \geqslant u_i\left(\tilde{s}_i, \sigma_{-i}^R\right)$. Although in general $\sigma_{-i}^r \neq s_{-i}^r$, positive externalities implies that $s_{-i}^1 = \sigma_{-i}^1 = \bar{s}_{-i}$ and $s_{-i}^R = $

$\sigma_{-i}^R = \underline{s}_{-i}$. Suppose that the beliefs of individual i may be represented by a JP-capacity, $\nu_i = \alpha_i \mu_i + (1 - \alpha_i)\bar{\mu}_i$. If i plays strategy $\hat{s}_i$, (s)he receives utility:

$V_i(\hat{s}_i) = \alpha_i \int u_i(\hat{s}_i, s_{-i}) \, d\mu_i + (1 - \alpha_i) \int u_i(\hat{s}_i, s_{-i}) \, d\bar{\mu}_i$

$= \alpha_i u_i(\hat{s}_i, \bar{s}_{-i}) \mu_i(\bar{s}_{-i}) + \alpha_i \sum_{r=2}^{R-1} u_i(\hat{s}_i, s_{-i}^r) \left[\mu_i(s_{-i}^1, ..., s_{-i}^r) - \mu_i(s_{-i}^1, ..., s_{-i}^{r-1}) \right]$

$+ \alpha_i u_i(\hat{s}_i, \underline{s}_{-i}) \left[1 - \mu_i(S_{-i} \backslash \underline{s}_{-i}) \right] + (1 - \alpha_i) u_i(\hat{s}_i, \bar{s}_{-i}) \left[1 - \mu_i(S_{-i} \backslash \bar{s}_{-i}) \right]$

$+ (1 - \alpha_i) \sum_{r=2}^{R-1} u_i(\hat{s}_i, s_{-i}^r) \left[\mu_i(s_{-i}^r, ..., s_{-i}^R) - \mu_i(s_{-i}^{r+1}, ..., s_{-i}^R) \right] + (1 - \alpha_i) u_i(\hat{s}_i, \underline{s}_{-i}) \mu_i(\underline{s}_{-i}).$

Similarly if i plays strategy $\tilde{s}_i$ (s)he receives utility:

$V_i(\tilde{s}_i) = \alpha_i u_i(\tilde{s}_i, \bar{s}_{-i}) \mu_i(\bar{s}_{-i}) + \alpha_i \sum_{r=2}^{R-1} u_i(\tilde{s}_i, \sigma_{-i}^r) \left[\mu_i(\sigma_{-i}^1, ..., \sigma_{-i}^r) - \mu_i(\sigma_{-i}^1, ..., \sigma_{-i}^{r-1}) \right]$

$+ \alpha_i u_i(\tilde{s}_i, \underline{s}_{-i}) \left[1 - \mu_i(S_{-i} \backslash \underline{s}_{-i}) \right] + (1 - \alpha_i) u_i(\tilde{s}_i, \bar{s}_{-i}) \left[1 - \mu_i(S_{-i} \backslash \bar{s}_{-i}) \right]$

$+ (1 - \alpha_i) \sum_{r=2}^{R-1} u_i(\tilde{s}_i, \sigma_{-i}^r) \left[\mu_i(\sigma_{-i}^r, ..., \sigma_{-i}^R) - \mu_i(\sigma_{-i}^{r+1}, ..., \sigma_{-i}^R) \right] + (1 - \alpha_i) u_i(\tilde{s}_i, \underline{s}_{-i}) \mu_i(\underline{s}_{-i}).$

In the limit, as $\bar{\gamma}$ tends to 1, all the terms involving μ_i tend to 0. Hence $V_i(\hat{s}_i)$ tends to $\alpha_i u_i(\hat{s}_i, \underline{s}_{-i}) + (1 - \alpha_i) u_i(\hat{s}_i, \bar{s}_{-i})$ and $V_i(\tilde{s}_i)$ tends to $\alpha_i u_i(\tilde{s}_i, \underline{s}_{-i}) + (1 - \alpha_i) u_i(\tilde{s}_i, \bar{s}_{-i})$. Since $\hat{s}_i \in \mathcal{A}, \tilde{s}_i \notin \mathcal{A}, \alpha_i u_i(\hat{s}_i, \underline{s}_{-i}) + (1 - \alpha_i) u_i(\hat{s}_i, \bar{s}_{-i}) - \left[\alpha_i u_i(\tilde{s}_i, \underline{s}_{-i}) + (1 - \alpha_i) u_i(\tilde{s}_i, \bar{s}_{-i}) \right] > 0$. It follows that $\tilde{s}_i$ will not be played when $\bar{\gamma}$ is sufficiently high. ∎

Proof of Proposition 4.1 By Lemma B.9, if the minimal degree of ambiguity is sufficiently high, $\text{supp}\,\nu_i \subseteq \text{argmax}_{s_i \in S_i} \left\{ \alpha_i u_i(s_i, \underline{s}_{-i}) + (1 - \alpha_i) u_i(s_i, \bar{s}_{-i}) \right\}$, for $1 \leqslant i \leqslant n$. If α_i is also sufficiently high (resp. low) then $\text{supp}\,\nu_i \subseteq \text{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i})$ (resp. $\text{supp}\,\nu_i \subseteq \text{argmax}_{s_i \in S_i} u_i(s_i, \bar{s}_{-i})$). By Theorem 4.2, the resulting equilibrium is smaller (resp. greater) than the lowest (resp. highest) equilibrium without ambiguity. ∎

B.3 The Centipede Game

In this appendix, we illustrate our model by applying the EUA concept to a simplified version of the Centipede game. Two players, $I = \{1, 2\}$, with actions c ("continue") and e ("exit"), $A := \{c, a\}$, alternate in choosing their actions. We shall consider the case of three periods represented in Figure B.3. Denote by ce the strategy of Player 1 to continue at $t = 1$ and to exit at $t = 3$. Similarly, cc indicates the strategy to continue both at $t = 1$ and $t = 3$. In strategic

form the game is represented by the following table[24].

<div align="center">

Player 2

		e	c
	e	1, 0	1, 0
Player 1	ce	0, 6	11, 5
	cc	0, 6	10, 10

</div>

In the unique NE, has Player 1 exits immediately and Player 2 chooses an arbitrary mixed strategy. Given the attractive pay-offs after continuation, it is not surprising that most experimental studies of the centipede game show that players continue for a sequence of moves, though not to the end of the game[25].

The following result provides conditions on the ambiguity-attitude parameters (α_1, α_2) and players' beliefs represented by JP-capacities (μ_1, μ_2) for an equilibrium in beliefs under ambiguity, where Player 1 continues for one stage, even though Player 2 will exit in stage 2.

Proposition B.1 *Suppose* $10 - 11 \cdot \alpha_1 (1 - \varepsilon_c) > 0$ *and* $(1 - \alpha_2) [9 - 5 \cdot \gamma_{e,ce}] - \alpha_2 \gamma_{cc,ce} > 0$,

---

[24]The table contains a slight abuse of notation. Strictly we should distinguish between the strategies *ec* exit then continue and *ee* exit at both nodes for player 1. However since these two strategies always yield the same pay-off we have combined them into a single strategy *e*.

[25]See, e.g., McKelvey and Palfrey (1992) and Nagel and Tang (1998).

then the beliefs

$$\mu_1(E) := \begin{cases} 0 & \textit{for} \quad E = \{e\} \\ \varepsilon_c & \textit{for} \quad E = \{c\} \end{cases} \quad , \qquad \mu_2(E) := \begin{cases} 0 & \textit{for} \quad E = \{e\} \\ \gamma_{ce} & \textit{for} \quad E = \{ce\} \\ 0 & \textit{for} \quad E = \{cc\} \\ \gamma_{e,ce} & \textit{for} \quad E = \{e, ce\} \\ 0 & \textit{for} \quad E = \{e, cc\} \\ \gamma_{cc,ce} & \textit{for} \quad E = \{cc, ce\} \end{cases} \quad ,$$

with $\operatorname{supp} \mu_1 = \{c\}$ *and* $\operatorname{supp} \mu_2 = \{ce\}$ *are an EUA.*

Notice that the conditions of the claim require only a small degree of ambiguity for both players about the behaviour of the opponent paired with a little optimism $\alpha_1 < 0$ and $\alpha_2 < 1$. Behaviour in this equilibrium resembles the behaviour observed in laboratory experiments of the centipede game. Both players will be better off (ex-post) given their behaviour in this EUA than in the unique NE.

Denoting beliefs by

$$\mu_1(E) := \begin{cases} \varepsilon_e & \text{for} \quad E = \{e\} \\ \varepsilon_c & \text{for} \quad E = \{c\} \end{cases} \quad \text{and} \quad \mu_2(E) := \begin{cases} \gamma_e & \text{for} \quad E = \{e\} \\ \gamma_{ce} & \text{for} \quad E = \{ce\} \\ \gamma_{cc} & \text{for} \quad E = \{cc\} \\ \gamma_{e,ce} & \text{for} \quad E = \{e, ce\} \\ \gamma_{e,cc} & \text{for} \quad E = \{e, cc\} \\ \gamma_{cc,ce} & \text{for} \quad E = \{cc, ce\} \end{cases}$$

Player 1's CEU payoff from actions e; ce, and cc can be written as

$$\begin{aligned} V_1(e; \alpha_1, \mu_1) &= 1, \\ V_1(ce; \alpha_1, \mu_1) &= 11 \left[\alpha_1 \varepsilon_c + (1 - \alpha_1)(1 - \varepsilon_e) \right], \\ V_1(cc; \alpha_1, \mu_1) &= 10 \left[\alpha_1 \varepsilon_c + (1 - \alpha_1)(1 - \varepsilon_e) \right]. \end{aligned}$$

Similarly, one obtains Player 2's CEU payoff:

$$V_2(e; \alpha_2, \mu_2) = 6\alpha_2 \gamma_{cc,ce} + (1 - \alpha_2) [1 - \gamma_e],$$

$$V_2(c; \alpha_2, \mu_2) = 5 \left[\alpha_2 \left(\gamma_{cc} + \gamma_{cc,ce} \right) + (1 - \alpha_2) \left(2 - \gamma_{e,ce} - \gamma_e \right) \right].$$

Let

- $\varepsilon_c > 0$ $\qquad\qquad$ $\varepsilon_e = 0$

- $\gamma_{e,ce} = \gamma_{cc,ce} = \gamma_{ce} > 0$ $\quad$ $\gamma_E = 0$ otherwise.

Proof of Proposition B.1 $\quad$ We need to show that $\operatorname{supp} \mu_1 \subseteq R_2(\alpha_2, \mu_2)$, and $\operatorname{supp} \mu_2 \subseteq R_1(\alpha_1, \mu_1)$, or equivalently, that (i) $V_1(ce; \alpha_1, \mu_1) \geqslant V_1(cc; \alpha_1, \mu_1)$, (ii) $V_1(ce; \alpha_1, \mu_1) \geqslant V_1(e; \alpha_1, \mu_1)$ and (iii) $V_2(c; \alpha_2, \mu_2) \geqslant V_2(e; \alpha_2, \mu_2)$ hold.

Straightforward calculations yield: $V_1(e; \alpha_1, \mu_1) = 1$, $V_1(ce; \alpha_1, \mu_1) = 11 [\alpha_1 \varepsilon_c + (1 - \alpha_1)]$, and $V_1(cc; \alpha_1, \mu_1) = 10 [\alpha_1 \varepsilon_c + (1 - \alpha_1)]$. Since ce dominates cc, (i) is always satisfied with strict inequality. Moreover, (ii) holds for $10 - 11\alpha_1 (1 - \varepsilon_c) \geqslant 0$. For Player 2, one computes $V_2(e; \alpha_2, \mu_2) = 6\alpha_2 \gamma_{cc,ce} + (1 - \alpha_2)$ and $V_2(c; \alpha_2, \mu_2) = 5\alpha_2 \gamma_{cc,ce} + (1 - \alpha_2) \left(10 - 5\gamma_{e,ce} \right)$. Hence, (iii) holds for $(1 - \alpha_2) \left[9 - 5\gamma_{e,ce} \right] - \alpha_2 \gamma_{cc,ce} \geqslant 0$. ∎

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