Universality of Nash components.

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Abstract

We show that Nash equilibrium components are universal for the collection of connected polyhedral sets. More precisely for every polyhedral set we construct a so-called binary game—a game where all players have two pure strategies and a common utility function with values either zero or one—whose success set (the set of strategy profiles where the maximal payoff of one is indeed achieved) is homeomorphic to the given polyhedral set. Since compact semi-algebraic sets can be triangulated, a similar result follows for the collection of connected compact semi-algebraic sets.

We discuss implications of our results for the strategic stability of success sets, and use the results to construct a Nash component with index k for any fixed integer k.

JEL Codes. C72, D44.
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1 Introduction

In non-cooperative game theory the claim that Nash equilibrium components can have any conceivable shape they can reasonably be expected to have seems to have the status of a "folk conjecture". [1] Few researchers in the field doubt this claim, yet no proof seems to be available, and it is not entirely clear what is meant by "every conceivable shape". In this paper we provide a rigorous proof of a specific version of this "folk conjecture" on the topological structure of Nash equilibrium components.

Concretely, in this paper we establish four facts. Below we first briefly state each separate fact. Next, we explain the precise formulation and interpretation of each of these facts in more detail. In the remainder of the introduction we then discuss some of the implications of our results and briefly explain the organization of the paper.

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[1] We thank the anonymous editor for coining this expression.

THE FACTS. The four facts we prove are

[1] The success set [2] of a binary game is a cubistic set.

[2] Conversely, for every cubistic set C in $\mathbb{R}^n$, there is an n-person binary game for which C equals its success set.

[3] The success set of a binary game is the union of Nash components of that binary game.

[4] Every polyhedral set is homeomorphic to a cubistic set. [3]

Explanation of [1]. A binary game is an n-person strategic form game with the following special features. First, each player has two pure strategies. Second, all players have the same payoff function. Third, for every profile of pure strategies the common payoff to the players is either zero or one.

Since each player has two pure strategies, the strategy space of the mixed extension of a binary game can be identified with the standard hypercube in $\mathbb{R}^n$. The success set of a binary game is the collection of points in that hypercube that correspond to a strategy profile in which the (common) payoff to the players equals one. We argue that the success set C of a binary game is in fact what we call a cubistic set, meaning that

[a] C can be written as the union of a number of faces of the hypercube, and

[b] a face F of the hypercube is a subset of C whenever each vertex of F is an element of C.

Explanation of [2]. Conversely, we show that for every cubistic set C in $\mathbb{R}^n$ there is an n-person binary game such that C is the success set of that particular binary game. In short, [1] and [2] together show that there is a one-to-one match between cubistic subsets of the hypercube in $\mathbb{R}^n$ and success sets of n-person binary games.

Explanation of [3]. Since the success set of a binary game is the collection of strategy profiles with (joint) payoff one—the maximal payoff possible—to the players, it is evident that the success set is a subset of the set of Nash equilibria of the game. We show the stronger statement that the success set of a binary game is the union of Nash components of that binary game. To be precise, given a binary game with success set C, there is an open set U containing C such that every Nash equilibrium in U is also an element of C.

[2] We also thank the anonymous editor for coining this expression.

[3] Two subsets X and Y of $\mathbb{R}^n$ are homeomorphic if there exists a continuous map from X to Y that has a continuous inverse.

Explanation of [4]. Finally we show that cubistic sets are what is often called “universal” in the literature, meaning that for every polyhedral set there is a cubistic set that is homeomorphic with the given polyhedral set. In other words, the collection of cubistic sets encompasses—up to homeomorphisms—every conceivable shape a polyhedral set might have. In again other words, every shape we can possibly construct using polyhedral sets, we can also achieve using cubistic sets.

the consequences. We discuss three direct consequences of our facts.

Our main contribution is the universality of Nash components. To be precise, our results imply that for every connected polyhedral set there is a—binary—game whose success set is a Nash component that is homeomorphic to the given polyhedral set. This main result is a direct consequence of facts [4], [2], and [3], respectively. Thus, a Nash equilibrium component has topologically no additional structure beyond what follows directly from the definition and the fundamental triangulation result of semi-algebraic sets by Llojasiewicz [9].

Secondly, we note that success sets are strict equilibrium sets in the sense of Balkenborg and Schlag [3]. Therefore we obtain from their Theorem 6 the result that every success set with non-zero Euler characteristic contains a stable set in the sense of Mertens [10].

Thirdly, we use our results to provide a simple alternative to the construction described in Govindan, von Schemde and von Stengel [7] of a Nash equilibrium component with index p for any given integer p.

setup of the paper. Section 2 introduces some of the notation we need in this paper. In section 3 we prove facts [1] and [2]. In section 4 we prove fact [3]. In section 5 we prove fact [4]. In section 6 we use facts [2], [3], and [4] to show that Nash equilibrium components are universal. In sections 7 and 8 we discuss the implications of our results for strategic stability, and we provide an alternative for the construction of a Nash component of index p for any given integer p.

related literature. Bubelis [4] shows that, given any real algebraic number a, there exists a 3-person game with rational data in which a is the payoff to at least one player in the unique Nash equilibrium point of that game. He also presents a method which reduces an arbitrary n-person game to a 3-person game. Finally, a game is constructed whose equilibrium set is a differentiable manifold of dimension one, namely a circle, in the space of completely mixed strategy profiles.

Datta [5] shows that every real algebraic variety is isomorphic to the set of totally mixed Nash equilibria of some three-person game, and also to the set of totally mixed Nash equilibria of an n-person game in which each player has two pure strategies. It follows that every compact differentiable manifold can be written as the set of totally mixed Nash equilibria of some game. Moreover, there exist isolated Nash equilibria of arbitrary topological degree.

McKelvey and McLennan [12] provide a tight upper bound on the maximal number of regular totally mixed Nash equilibria for generic finite games in strategic form.

2 Preliminaries

A finite normal form game consists of a finite set of players $N = \{1, \dots, n\}$, and for each player $i \in N$ a finite pure strategy set S_i and a payoff function $u_i \colon S \to \mathbb{R}$ on the set $S := \prod_{i \in N} S_i$ of pure strategy profiles. A mixed strategy σ_i of player i is a vector $(\sigma_i(s_i))_{s_i \in S_i}$ that assigns a probability $\sigma_i(s_i) \geq 0$ to each pure strategy $s_i \in S_i$. The support of a mixed strategy σ_i is the set of all pure strategies s_i with $\sigma_i(s_i) > 0$. The multilinear extension of the payoff function u_i of player i is the function that assigns to each strategy profile $\sigma = (\sigma_i)_{i \in N}$ the expected payoff

$$u_i(\sigma) = \sum_{s \in S} \prod_{j \in N} \sigma_j(s_j) \cdot u_i(s).$$

By $u_i(\sigma \mid s_i)$ we denote the payoff to player i when player i plays pure strategy $s_i \in S_i$ while his opponents adhere to the mixed strategy profile σ. A strategy profile σ is a Nash equilibrium when $u_i(\sigma) \geq u_i(\sigma \mid s_i)$ holds for every player i and every pure strategy s_i of player i.

BINARY GAMES. In this paper we deal with binary games only. A binary game is a finite normal form game $(N, (u_i)_{i \in N})$ such that

[1] $S_i = \{A, B\}$ for every player $i \in N$,

[2] there is a common payoff function u such that $u_1 = \cdots = u_n = u$, and

[3] $u(s) \in \{0, 1\}$ for all strategy profiles $s = (s_1, \dots, s_n)$.

To simplify notation for binary games we write σ_i for the probability that player i plays pure strategy A. So, the probability that player i plays pure strategy B equals $1 - \sigma_i$. We write $\sigma = (\sigma_i)_{i \in N}$ for a strategy profile, and we denote the space of strategy profiles by Σ. So, Σ is in fact the unit hypercube of dimension n.

CUBISTIC SETS. We also need the following terminology. For a set $T \subseteq N$, define the characteristic vector $e_T \in \mathbb{R}^n$ by

$$e_{Ti} = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise.} \end{cases}$$

The (standard) hypercube Σ in $\mathbb{R}^n$ is the convex hull of all characteristic vectors in $\mathbb{R}^n$. A subset $F \subset \Sigma$ of the hypercube is a face of Σ if there are disjoint sets Z and P in N such that

$$F = \{\sigma \in \Sigma \mid \sigma_i = 0 \text{ for all } i \in Z \text{ and } \sigma_j = 1 \text{ for all } j \in P\}.$$

The characteristic vectors that are elements of a face F are called the vertices of F. It is straightforward to check that a face F is the convex hull of its vertices.

A subset C of Σ is called a *cubistic set* if

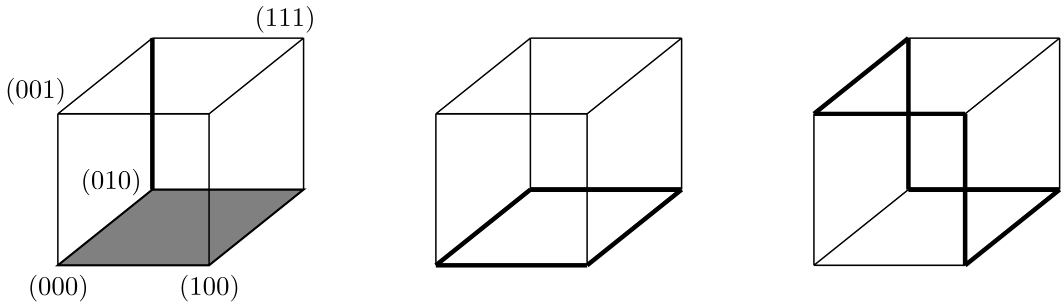
[1] C is the union of faces of Σ

[2] if for a face F of Σ all vertices of F are elements of C, then $F \subseteq C$.

EXAMPLES. We give three examples in $\mathbb{R}^3$. Consider the sets [4]

$$\begin{aligned} C_1 &= \text{ch}\{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\} \bigcup \text{ch}\{(0,1,0),(0,1,1)\} \\ C_2 &= \text{ch}\{(0,0,0),(1,0,0)\} \bigcup \text{ch}\{(1,0,0),(1,1,0)\} \bigcup \\ &\quad \text{ch}\{(1,1,0),(0,1,0)\} \bigcup \text{ch}\{(0,1,0),(0,0,0)\} \\ C_3 &= \text{ch}\{(1,0,0),(1,1,0)\} \bigcup \text{ch}\{(1,1,0),(0,1,0)\} \bigcup \text{ch}\{(0,1,0),(0,1,1)\} \bigcup \\ &\quad \text{ch}\{(0,1,1),(0,0,1)\} \bigcup \text{ch}\{(0,0,1),(1,0,1)\} \bigcup \text{ch}\{(1,0,1),(1,0,0)\} \end{aligned}$$

depicted below as subsets of the unit cube in $\mathbb{R}^3$.



The set C_1 *The set C_2* *The set C_3*

The set C_1 is cubistic. The set C_2 is not cubistic, since the vertices $(000), (100), (010)$, and (110) are elements of C_2, while the convex hull of these four vertices is not a subset of C_2. The set C_3 is again cubistic.

[4] Here the expression ch denotes the convex hull.

POLYHEDRAL SETS. A collection $\mathcal{D}$ of polytopes in $\mathbb{R}^n$ is called a polyhedral complex when

[a] for every polytope $P \in \mathcal{D}$ and every face F of P, F is an element of $\mathcal{D}$, and

[b] for every $P, Q \in \mathcal{D}$, $P \cap Q$ is a face of both P and Q.

The set $D = \bigcup_{P \in \mathcal{D}} P$ is the carrier of the complex $\mathcal{D}$. A set D in $\mathbb{R}^n$ that is the carrier of a polyhedral complex is called a polyhedral set.

3 Binary games and success sets

In this section we prove the first two facts mentioned in the introduction. We show that the success set of a binary game is a cubistic set, and conversely, that for every cubistic set C there is a binary game whose success set equals C.

Consider a binary game (N, u), where u is the payoff function common to all players. As said before, Σ is the set of strategy profiles $\sigma = (\sigma_i)_{i \in N}$, where $0 \leq \sigma_i \leq 1$ for all $i \in N$. The *success set* of the binary game (N, u) is the subset C of Σ defined by

$$C = \{\sigma \in \Sigma \mid u(\sigma) = 1\}.$$

Fix the set of players N. So, also the hypercube Σ of profiles of mixed strategies is fixed. We prove the following theorem.

Theorem 3.1 *For a set $C \subset \Sigma$ the following two statements are equivalent.*

[1] *C is a cubistic set*

[2] *there is a binary game (N, u) that has C as its success set.*

Proof. We show both directions separately.

A. Suppose that C is cubistic. Define $T \subseteq \{0, 1\}^n$ by

$$T = \{s \in \{0, 1\}^n \mid s \in C\}.$$

Let $u \colon \Sigma \to \mathbb{R}$ be the multilinear extension of the function $u \colon \{0, 1\}^n \to \mathbb{R}$ defined by

$$u(s) = \begin{cases} 1 & \text{for} \quad s \in T \\ 0 & \text{for} \quad s \notin T \end{cases}$$

We show that C is the success set of (N, u). Take $\sigma \in C$. Since C is cubistic, there is a face F of Σ with $\sigma \in F \subseteq C$. Since for every vertex $s \in F$ we have $u(s) = 1$, it is straightforward to check that also $u(\sigma) = 1$.

Conversely, take $\sigma \notin C$. Let F be the smallest face of Σ that contains σ. Since $\sigma \notin C$, we know that there is at least one vertex e^T in F with $e^T \notin C$. Then, $u(e^T) = 0$, so that by multilinearity $u(\sigma) < 1$. Hence, C is the success set of (N, u).

B. Suppose that C is the success set of the binary game (N, u). Take $\sigma \in C$. So, $u(\sigma) = 1$. Let F be the minimal face of Σ that contains σ. Then, since $u(s) \in \{0, 1\}$ for every vertex of Σ, necessarily $u(s) = 1$ for every vertex s of F by the multilinearity of u. Hence, C is the union of faces of Σ.

Further, if $u(s) = 1$ for all vertices of a face F of C, then also $u(\sigma) = 1$ for all $\sigma \in F$ by the multilinearity of u. ■

4 Binary games and Nash equilibria

In this section we prove Fact 3 stated in the introduction. We show that the success set of a binary game is the union of Nash components of that game. The proof we provide here relies on the following triangulation theorem of Hironaka. [5] Let $\mathcal{C}$ be the collection of solution sets of inequalities of the form

$$f(x_1, \ldots, x_m) \geq 0$$

where f is a polynomial in variables $x_1, \ldots, x_m$. Any set that can be iteratively obtained by taking finite unions, finite intersections and complements of sets in $\mathcal{C}$ is called semi-algebraic.

Let $(X_\alpha)_{\alpha \in A}$ be a finite collection of bounded semi-algebraic sets in $\mathbb{R}^m$. Then there is a decomposition of $\mathbb{R}^m$ into countably many open simplices [6] $\Delta_1, \Delta_2, \cdots$ and a semi-algebraic automorphism $\kappa \colon \mathbb{R}^m \to \mathbb{R}^m$ such that

[1] each X_α is a finite union of some of the $\kappa(\Delta_a)$

[2] κ induces an analytic isomorphism between each open simplex Δ_a and its image $\kappa(\Delta_a)$.

Decomposition means here that every point in $\mathbb{R}^m$ is in a unique open simplex Δ_a and in the closure of only finitely many open simplices. Moreover, if the open simplex Δ_a intersects the

[5] We know of three different ways to prove Fact 3. First of all, it is a direct consequence of the results in Balkenborg and Schlag [3]. Their arguments though depend on lengthy calculations using Taylor series expansion. Secondly, we have a direct and elementary proof, which is given in the SSRN discussion paper version of this paper. It is, however, not short. Here we provide a very different proof, due to the associate editor.

[6] An open simplex Δ of dimension r in $\mathbb{R}^m$ is a set of all points of the form $\lambda_1 v_1 + \cdots + \lambda_{r+1} v_{r+1}$ with $\sum_i \lambda_i = 1$, and each $\lambda_i > 0$, where $v_1, \ldots, v_{r+1}$ are geometrically independent points in $\mathbb{R}^m$. This differs from the standard definition of a simplex, in that the weights are supposed to be strictly positive instead of merely non-negative.

closure of the open simplex Δ_b, then it is one of its faces. Semi-algebraic automorphism means that κ is a homeomorphism with a semi-algebraic graph.

We continue with our argument. Let (N, u) be a binary game, and let C be its success set. We show that C is a union of Nash components of that game. Since in any strategy profile $\sigma \in C$ every player receives the maximal payoff 1, it is clear that every $\sigma \in C$ is a Nash equilibrium. So, C is contained in the union of all Nash components of the binary game. It suffices to show the following theorem.

Theorem 4.1 *Suppose that $\sigma \in C$, and that τ is a Nash equilibrium in the same Nash component as σ. Then also $\tau \in C$.*

Proof. The proof rests on two separate claims.

Claim A. There are a continuous function $\rho\colon [0,1] \to \Sigma$ from the unit interval $[0,1]$ into the set of Nash equilibria, and a partition $0 = t_0 < \ldots < t_m = 1$ of the unit interval such that on each open interval (t_k, t_{k+1})

[1] each ρ_i has constant support and

[2] each function $t \mapsto \rho_i(t)$ is continuously differentiable.

Proof of claim A. The set of Nash equilibria of the binary game is the solution set to a system of polynomial inequalities and is hence a bounded semi-algebraic set in $\mathbb{R}^n$. Similarly, all faces of the hypercube of all strategy profiles Σ are semi-algebraic in $\mathbb{R}^n$.

Consider the collection consisting of the set X of all Nash equilibria of the binary game, and all faces of Σ. Take countably many open simplices $\Delta_1, \Delta_2, \cdots$ and a semi-algebraic automorphism $\kappa\colon \mathbb{R}^n \to \mathbb{R}^n$ as in the above Theorem by Hironaka. Let Z be the union of all open simplices that map into X. Further, take $x, y \in Z$ with $\kappa(x) = \sigma$ and $\kappa(y) = \tau$. It is an easy exercise to construct a continuous map $p\colon [0,1] \to Z$ and a partition $0 = t_0 < \ldots < t_m = 1$ of the unit interval such that $p(0) = x$, $p(1) = y$, and for each simplex $\Delta_a \subseteq Z$ there is at most one k with

[1] $p(t) \in \Delta_a$ precisely when $t_k < t < t_{k+1}$, and

[2] the function $p\colon (t_k, t_{k+1}) \to \Delta_a$ is linear.

Define $\rho = \kappa \circ p$. It is straightforward to check that ρ and $0 = t_0 < \ldots < t_m = 1$ satisfy the conditions in the above claim. End proof of claim A.

Claim B. $u(\sigma) = u(\tau)$.

Proof of claim B. Consider the map $\rho\colon [0,1] \to \Sigma$ from the previous claim. Take $k \in \{0,1,\ldots,m-1\}$ fixed. Since ρ has constant support on (t_k, t_{k+1}), either $\rho_i(t) \in \{0,1\}$ for all $t \in (t_k, t_{k+1})$, in which case $\frac{d\rho_i}{dt} = 0$, or $0 < \rho_i(t) < 1$ for all $t \in (t_k, t_{k+1})$, in which case $u(\rho(t) \mid A) = u(\rho(t) \mid B)$ for all $t \in (t_k, t_{k+1})$. Since

$$u(\sigma) = \sigma_i \cdot u(\sigma \mid A) + (1-\sigma_i) \cdot u(\sigma \mid B)$$

we know

$$\frac{\partial u}{\partial \sigma_i}(\rho) = u(\rho \mid A) - u(\rho \mid B). \tag{1}$$

Further, notice that $t \mapsto (u \circ \rho)(t)$ is differentiable on the open interval (t_k, t_{k+1}). Thus, using the chain rule and the previous observations we obtain

$$\frac{d(u \circ \rho)(t)}{dt} = \sum_{i=1}^{n} \frac{\partial u}{\partial \sigma_i}(\rho(t)) \cdot \frac{d\rho_i(t)}{dt} = \sum_{i=1}^{n} [u(\rho(t) \mid A) - u(\rho(t) \mid B)] \cdot \frac{d\rho_i(t)}{dt} = 0.$$

We conclude that the composite function $u \circ \rho$ is constant on $[t_k, t_{k+1}]$ because it is continuous and its derivative in $t_k < t < t_{k+1}$ is zero. Hence, since this holds for each $k = 0, \ldots m-1$, $u(\sigma) = u(\rho(0)) = u(\rho(1)) = u(\tau)$. ■

Remarks. A few observations are in order here.

[1] The proof we presented here, which was suggested by the associate editor, is short. However, it is not elementary because it uses the triangulation result of Hironaka [8] for semi-algebraic sets, which itself is a non-trivial application of the Tarski-Seidenberg theorem. [7] A modern and computationally more efficient version of the decomposition result for semi-algebraic sets may be found in Arnon, Collins, and McCallum [1].

[2] Our proof that the common utility is constant on each Nash component easily generalizes to arbitrary games where all players have identical utility functions. So, this still holds even when players do not necessarily have exactly two pure strategies each, or when the common payoff is not either zero or one, but may be any payoff.

[3] Finally, our proof hinges on the observation that for any differentiable function $u\colon \mathbb{R}^n \to \mathbb{R}$ and any continuous path $\sigma\colon [0,1] \to \mathbb{R}^n$ of critical points of u that is differentiable in $(0,1)$, the value of u is constant along the path. This may first seem at odds with the example in the paper by Whitney [13] of a function that is not constant on a connected set of critical points. However, in Whitney's example the set of critical points is not

[7] Hironaka [8] did not aim to prove a new result, but rather give in the setting of semi-algebraic sets a simpler proof than provided earlier by Llojasiewicz [9] for semi-analytic sets.

semi-algebraic and so it is possible to connect different critical points by a continuous, but not by a piecewise differentiable path of critical points. This is the crucial extra piece of information we borrowed from Hironaka in the proof.

5 Cubistic and polyhedral sets

In this section we prove Fact 4 of the introduction. We show that for every polyhedral set there exists a cubistic set that is homeomorphic to the given polyhedral set.

The standard simplex Δ in $\mathbb{R}^n$ is the set of convex combinations of the collection $\{e_i \mid i \in N\}$ of unit vectors in $\mathbb{R}^n$. A polyhedral set in $\mathbb{R}^n$ is called standard if it is a union of faces of the standard simplex Δ. We first need the following observation.

Proposition 5.1 *Any polyhedral set is homeomorphic to a standard polyhedral set.*

Proof. Let S be a polyhedral set, and let $\mathcal{S}$ be any polyhedral complex whose carrier is S. Let V be the collection of vertices (polytopes of dimension zero) in $\mathcal{S}$. For $v \in V$, let $e(v) \in \mathbb{R}^V$ denote the unit vector defined by, for every $w \in V$,

$$e(v)_w = \begin{cases} 1 & \text{if } w = v \\ 0 & \text{if } w \neq v. \end{cases}$$

For every $D \in \mathcal{S}$, let $F(D)$ denote the convex hull of the set

$$\{e(v) | v \text{ is a vertex of } D\}.$$

It is straightforward to check that $\mathcal{T} = \{F(D) | D \in \mathcal{S}\}$ is a polyhedral complex, and that each $F(D)$ is a face of the standard simplex in $\mathbb{R}^V$. Hence, $T = \bigcup_{E \in \mathcal{T}} E$ is a standard polyhedral set. Further, the piecewise linear extension of the map $v \mapsto e(v)$ is a homeomorphism from S to T. ∎

We show that for any standard polyhedral set there is a cubistic set that is homeomorphic to it. For $x = (x_i)_{i \in N} \in \mathbb{R}^n$, define

$$\|x\|_1 = \sum_{i \in N} |x_i| \quad \text{and} \quad \|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

the Manhattan norm—also called ℓ_1 norm—and the supremum norm—also called ℓ_∞ norm—respectively. For $\kappa \in \Delta$, consider the ray

$$L_\kappa = \{\lambda \cdot \kappa \mid \lambda > 0\}$$

it generates. This ray intersects the standard simplex Δ in κ itself. Further, it intersects the boundary of the hypercube Σ in the point $\frac{\kappa}{\|\kappa\|_\infty}$. This relationship defines a homeomorphism Φ between Δ and $\Phi(\Delta) \subseteq \Sigma$ by $\Phi(\kappa) := \frac{\kappa}{\|\kappa\|_\infty}$, with inverse function $\Phi^{-1}(\sigma) := \frac{\sigma}{\|\sigma\|_1}$. Note that $\Phi(\Delta)$ equals the union of those faces of Σ that do not contain the origin. Using the homeomorphism Φ we obtain the following result.

Theorem 5.2 *Every standard polyhedral set is homeomorphic to a cubistic set.*

Proof. Let S be a standard polyhedral set in Δ. Define $C = \Phi(S)$. We show that C is a cubistic set. Clearly, Φ maps each face $F \subseteq S$ of the standard simplex onto a union of faces of the hypercube, so that C is a union of faces of Σ. In order to show that C is cubistic, take a face G in Σ whose vertices are all in C. We show that G is in C. Since G is a face of Σ there are disjoint sets P and Z in N with

$$G = \{\sigma \in \Sigma \mid \sigma_i = 0 \text{ for all } i \in Z \text{ and } \sigma_j = 1 \text{ for all } j \in P\}.$$

Take $\sigma \in G$. We show that $\sigma \in C$. Define $\tau \in \Sigma$ by

$$\tau_i = \begin{cases} 1 & \text{if } \sigma_i > 0 \\ 0 & \text{if } \sigma_i = 0. \end{cases}$$

Then clearly τ is a vertex of G. So, $\tau \in C = \Phi(S)$. So, we can take $\kappa \in S$ with $\Phi(\kappa) = \tau$. Let F be the smallest face of Δ in S that still contains κ. Then, since by the definition of Φ we have that $\kappa_i > 0$ precisely when $\tau_i = 1$ precisely when $\sigma_i > 0$ precisely when $\frac{\sigma}{\|\sigma\|_1} > 0$, we can conclude that $\mu = \frac{\sigma}{\|\sigma\|_1} \in F$. So, $\mu \in F \subseteq S$. Hence, since $\|\sigma\|_\infty = \|\tau\|_\infty = 1$, we obtain $\sigma = \Phi(\mu) \in C$. ■

6 Universality

The first, and main, consequence of our facts is the universality of Nash components. To be precise, a collection $\mathcal{C}$ of sets is universal for a collection $\mathcal{D}$ of sets if

[a] $\mathcal{C}$ is a subset of $\mathcal{D}$, and more importantly

[b] for every set $D \in \mathcal{D}$ there is a set $C \in \mathcal{C}$ that is homeomorphic to D.

Our results imply the following Theorem.

Theorem 6.1 *The collection of success sets is universal for the collection of polyhedral sets. Consequently, the collection of Nash components of binary games is universal for the collection of connected compact semi-algebraic sets.*

Proof. Take a polyhedral set D. So, by Theorems 5.1 and 5.2, the set D is homeomorphic to a cubistic set. Then, by Theorem 3.1, the set D is homeomorphic to the success set of a binary game. This concludes the proof of the first statement.

In order to show the second claim, first note that a Nash component is a connected compact semi-algebraic set. Take a connected, compact semi-algebraic set K. According to the Theorem of Hironaka [8] stated earlier there exists a polyhedral set C that is homeomorphic to K. Then, according to the claim we just showed, there is a success set D of a binary game that is homeomorphic to C, and hence also to K. According to Theorem 4.1 there exists an open set V that contains D such that D equals the set of Nash equilibria in V. Hence, since D—being homeomorphic to K—is connected, it is a Nash component. ∎

REMARK. Our proof in fact establishes a somewhat stronger result than Theorem 6.1. Given any semi-algebraic set, we construct a binary game whose success set is itself a union of Nash components that is homeomorphic to the given semi-algebraic set. The somewhat weaker phrasing in the Theorem is chosen both out of convenience and tradition.

AN EXAMPLE. We construct a binary game with a Nash component homeomorphic to the connected and compact semi-algebraic set

$$S_1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}.$$

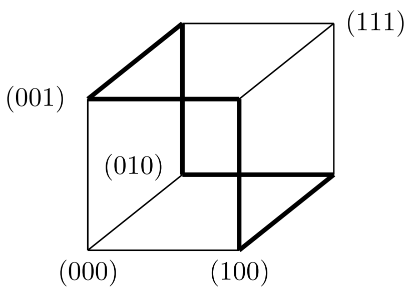
We first construct a standard polyhedral set that is homeomorphic to S_1. Consider the standard simplex

$$\Delta = \{(\sigma_1, \sigma_2, \sigma_3) | \sigma_1 + \sigma_2 + \sigma_3 = 1, \sigma_1 \geq 0, \sigma_2 \geq 0, \sigma_3 \geq 0\}.$$

Its boundary

$$B = \{(\sigma_1, \sigma_2, \sigma_3) \in \Delta | \sigma_1 = 0 \vee \sigma_2 = 0 \vee \sigma_3 = 0\}$$

is a standard polyhedral set that is homeomorphic to S_1. The subset $\Phi(B)$ of the cube Σ is the cubistic set C_3 depicted below.



The set C_3

The resulting 3-player $2 \times 2 \times 2$ binary game is

$$\begin{array}{cc} & A \quad\quad B \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0,0,0 & 1,1,1 \\ 1,1,1 & 1,1,1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & A \quad\quad B \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 1,1,1 & 1,1,1 \\ 1,1,1 & 0,0,0 \end{bmatrix} \end{array}$$

where the left-hand matrix reports the payoffs when player 3 chooses A (corresponding to $\sigma_3 = 1$), and the right-hand matrix reports the payoffs when player 3 chooses B (corresponding to $\sigma_3 = 0$). The set C_3 is the success set of this game, and hence a Nash component of the set of Nash equilibria. However, the game has one more Nash equilibrium $\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{2}$, which forms a singleton Nash component. This shows that the success set of a binary game need not be the entire set of Nash equilibria. Nevertheless, the success set itself does constitute a single Nash component in our construction.

7 Success sets and stability

Let C be the success set of a binary game. Then C is a strict equilibrium set of this game as defined in Balkenborg and Schlag [3]. This means that if $(y_i, x_{-i}) \in C$ and if z_i is a best reply of player i against x_{-i}, then $(z_i, x_{-i}) \in C$. Since all elements of C yield the maximum payoff of 1 to all players, it is clear that C satisfies this condition. Therefore Theorem 6 of Balkenborg and Schlag [3] yields

Theorem 7.1 *The success set C is asymptotically stable and consists of stable restpoints for the replicator dynamics.*

Let $\mathcal{D}$ be a polyhedral complex. For $k \in \mathbb{N}$, let $\mathcal{D}_k$ denote the collection of polytopes $P \in \mathcal{D}$ of dimension k. Since $\mathcal{D}$ is a polyhedral complex in $\mathbb{R}^n$, it is clear that $\mathcal{D}_k = \phi$ for $k > n$. The Euler characteristic $\chi(\mathcal{D})$ of the polyhedral complex is defined by

$$\chi(\mathcal{D}) = \sum_{k=0}^{n} (-1)^k \cdot |\mathcal{D}_k|,$$

where $|\mathcal{D}_k|$ denotes the number of elements of $\mathcal{D}_k$. It can be shown that the Euler characteristic is a topological invariant. [8] In particular, for any two polyhedral complexes $\mathcal{D}$ and $\mathcal{E}$ whose carriers are homeomorphic, we have $\chi(\mathcal{D}) = \chi(\mathcal{E})$. This further implies that for any polyhedral set D, and any two complexes $\mathcal{D}$ and $\mathcal{E}$ whose carrier equals D, we have $\chi(\mathcal{D}) = \chi(\mathcal{E})$. Hence, the Euler characteristic depends only on the carrier, and the Euler characteristic $\chi(D)$ of the polyhedral set D is a well-defined integer number. Now Corollary 1 in Demichelis and Ritzberger [6] yields

[8] It is a known fact that the Euler characteristic is a homotopy invariant, a stronger notion than topological invariance.

Proposition 7.2 *Suppose the success set C has non-zero Euler characteristic. Then it contains a strategically stable set in the sense of Mertens [10].*

8 Success sets and index

In this section we provide an alternative for the constructions in Govindan, von Schemde and von Stengel [7], where for every integer $k \in \mathbb{Z}$ a game with a Nash component of index k is presented.

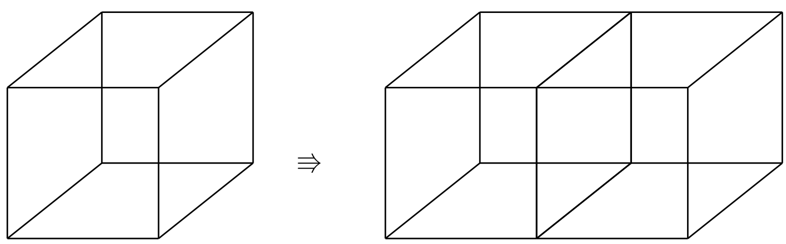
Let $k \in \mathbb{Z}$ be given. We construct a connected polyhedral set of dimension 2 in $\mathbb{R}^3$ with Euler characteristic k. [9]

Take a polyhedral complex $\mathcal{D}$ in $\mathbb{R}^3$ with carrier D. In case that the dimension of D is 2 (which means that $\mathcal{D}_3 = \phi$ and $\mathcal{D}_2 \neq \phi$), the formula for the Euler characteristic $\chi(D)$ simplifies to

$$\chi(D) = V - E + F,$$

where $V = |\mathcal{D}_0|$ is the number of vertices in $\mathcal{D}$, $E = |\mathcal{D}_1|$ is the number of edges in $\mathcal{D}$, and $F = |\mathcal{D}_2|$ is the number of faces (2-dimensional polytopes) in $\mathcal{D}$. We use this formula in the remainder of this section to compute the Euler characteristic of 2-dimensionsal polyhedral sets.

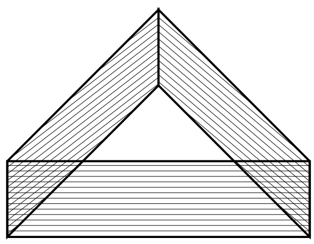
For $k = 2$ we take the boundary of the unit cube in $\mathbb{R}^3$, indicated on the left-hand side in the display below, as our polyhedral set. The associated complex consists of 6 squares (the facets of the unit cube), 12 line segments (the edges of the unit cube), and 8 points, the vertices of the unit cube. Its Euler characteristic is therefore $F - E + V = 6 - 12 + 8 = 2$. For $k = 3$, we glue a second unit cube onto one of the sides of the first cube. The result is indicated on the right-hand side of the display below. Its Euler characteristic is $11 - 20 + 12 = 3$.



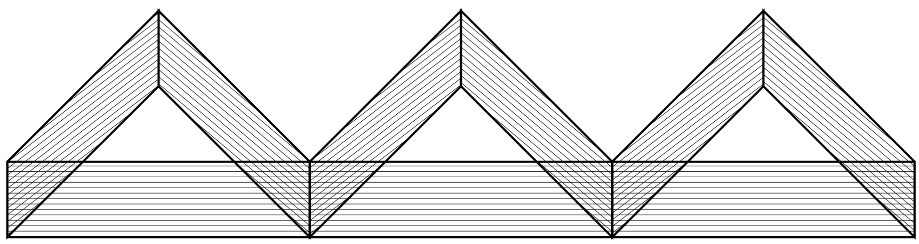
[9] The fact that the polyhedral set we construct is of dimension 2 in $\mathbb{R}^3$ is not essential. We can in general construct a polyhedral set of dimension d in $\mathbb{R}^n$ with Euler characteristic k for any d with $2 \leq d \leq n - 1$. Negative integers can be achieved with polyhedral sets of dimension 1 in $\mathbb{R}^2$.

In general, adding a next cube adds 5 new squares, 8 new edges, and 4 new vertices, so that the Euler characteristic increases by $5-8+4=1$. Hence, a sequence of k cubes glued together in a row has Euler characteristic $k+1$.

For $k = 0$, we take the bracelet (see below) which is a polyhedral set consisting of three rectangles glued together in a tubular shape. The Euler characteristic is $3-9+6=0$.



Gluing bracelets together yields negative Euler characteristics. For example, for $k = -2$ we need three bracelets as indicated below.



The Euler characteristic of the three adjoined bracelets is $9 - 25 + 14 = -2$. The Euler characteristic of two adjoined bracelets is $6 - 17 + 10 = -1$. In general, addition of an extra bracelet adds 3 squares, 8 edges, and 4 vertices. Thus the Euler characteristic changes by $3-8+4=-1$, so a net reduction of 1. Hence, a sequence of $m+1$ bracelets attached along an edge yields Euler characteristic $-m$.

Thus, for every integer k there exists a polyhedral set with Euler characteristic k.

Theorem 8.1 *The Euler characteristic of a success set equals its index. Moreover, for every integer k, there exists a binary game with a connected success set C and $\chi(C) = k$.*

Proof. Note that the success set C of a binary game is a polyhedral set. So the Euler characteristic $\chi(C)$ is defined. Further, Theorem 1 of Demichelis and Ritzberger [6] states that the index of C equals its Euler characteristic $\chi(C)$, whenever there is a Nash flow F_t such that C is asymptotically stable for F_t. Hence, from Theorem 7.1 above it follows that the Euler characteristic of the success set C equals its index. [10]

In order to prove the second claim, take the polyhedral set constructed above with Euler characteristic k. Then, since the Euler characteristic is a topological invariant, the success set associated with this polyhedral set has Euler characteristic k as well. ∎

9 Concluding remarks and open questions

In our construction the number of strategies per player is as small as possible. However, our construction needs as many players as there are vertices in the initial polyhedral complex. For example, any polyhedral set that is homeomorphic to the torus has at least 7 vertices. [11] So, our construction uses at least 7 players, and hence embeds the torus into the hypercube in $\mathbb{R}^7$, whereas the torus can be embedded in $\mathbb{R}^3$. The question remains how far our construction is from the minimum dimension implementation.

Bubelis [4] provides a construction which identifies the set of Nash equilibria for an n-player ($n \geq 3$) game with the set of Nash equilibria in a 3-player game. This suggests that the collection of Nash equilibrium components in 3-player games is universal for the collection of compact and connected semi-algebraic sets. However, Bubelis's construction is one-to-one only on the *interior* of the space of strategy profiles. So, it does not apply to success sets, which are on the boundary of the strategy space. Thus, the above question, or the equivalent question for 2-player games, is currently open.

We showed that every cubistic set is a finite union of Nash equilibrium components of a binary game. However, as we already noted in the example above, typically the game constructed will have additional Nash equilibria. Since the sum of the indices over all components equals one, this is necessarily the case if the Euler characteristic of the success set is not 1. It is an open question however whether in our construction the success set equals the set of Nash equilibria when the Euler characteristic (and hence the index) of the success set equals 1. More generally,

[10] There is a slight subtlety here. A stated above, Balkenborg and Schlag [3] show that C is asymptotically stable with respect to the replicator dynamics. And globally, the replicator dynamics does not induce a Nash flow, since it may have restpoints that are not Nash equilibria. Locally though, close to the Nash component C, the replicator dynamics *is* a Nash flow, which is sufficient for application of the results in Demichelis and Ritzberger [6].

[11] We thank Bernhard von Stengel for drawing our attention to this result.

if a polyhedral set has Euler characteristic 1, can we construct a game such that the polyhedral set is homeomorphic to the set of *all* Nash equilibria in this game?

We considered here only the topological structure. One might more generally ask whether any compact connected semi-algebraic set is diffeomorphic or even algebraically equivalent to a Nash equilibrium component. Finally, another open question is whether similar results can be achieved for the sets of perfect or proper equilibria instead of the set of Nash equilibria.

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