Acoustic Metamaterials: ‘The Influence of Boundary Effects on Resonant Acoustic Transmission through a Thin Slit’

Submitted by Ruth Kathleen Lovelock to the University of Exeter as a dissertation for the degree of Master of Philosophy in Physics, September 2014

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I certify that all material in this dissertation which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

Supervisors: Professor J. Roy Sambles (University of Exeter), Professor Alastair Hibbins (University of Exeter) and Dr. John. D. Smith (DSTL)
Abstract

Resonant transmission of sound through a single, open-ended slit in an aluminium plate is explored. The experimental results demonstrate a significant reduction in the resonant frequency as the slit is narrowed to below ~1% of the fundamental free-space wavelength. This is in agreement with a little referenced study of Lord Rayleigh [Phil. Mag. 1, 3, (1901)] concerning how viscous and thermal effects at the slit walls significantly reduces the sound velocity within the slit, for slit widths substantially greater than the thickness of the boundary layer. The experimental results are in full accord with Lord Rayleigh's original theory and with numerical modelling.
Acknowledgements

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Chapter 1: Outline and Acoustic Metamaterial Review

1.1 Introduction

Pressure variations in the air around us are detected by our ear drum, transferred to the cochlea and interpreted as sound. Sound waves have been studied extensively for decades and, other than in ordinary communication, are used in applications such as ultrasound imaging, music production, navigation, underwater detection and novel levitation [1]; this latter technique is now being used commercially for chemical mixing [2]. All creatures create sound, whether as consequence of movement or to communicate with others; detecting these sound waves allow animals to be identified acoustically. For example, the clicks made by sperm whales can be analysed to calculate the size of the creature [3]; the clicks include multiple pulses and the time delay between them correspond to the distance between the air sacs in the whales head. Seismic waves can also be understood by the application of similar physics, thus understanding how to manipulate acoustic waves could lead to earthquake safe structures [4]. Speculations suggest that infrasound, frequencies below 2Hz, was used during the First World War to induce nausea in soldiers as a form of defence but that the authorities rapidly realised that the effects were felt by the Allies as well; there is no evidence that infrasound has this affect however! It is true that infrasound is vital in the Animal Kingdom with many animals, for example elephants, using infrasound to communicate over large distances [53] [54].

A sound wave is a pressure wave and thus the properties of a system which affect the pressure would be assumed to affect the speed of sound propagation in that medium as well. Material properties which do not directly affect the pressure are therefore not attributed to modifying the sound propagation. For example, viscous and thermal effects at the boundary between a rigid wall and a fluid in acoustic systems are often assumed negligible in acoustic metamaterial systems, and are generally omitted in the analysis. This report however, shows experimentally that these contributions play a vital role in the resonant frequency of thin systems. At slit widths ~ 1% of the acoustic wavelength, between two rigid bodies, these boundary effects appear to be the origin of the significant reduction observed in the speed of sound of the system. Much research into the acoustic response of tubes has been published however the response of a single slit has been left untouched. In 1901, Lord Rayleigh [5] suggested that the acoustic theory for tubes and slits was essentially the same, stating that the only substitution necessary in the mathematics was to replace the tube radius with the slit-width.

1.2 Review of the field

A substantial amount of work has been done in the field of acoustic metamaterials. A metamaterial is a material that exhibits properties, due to its structure, which are not usually found in nature. These materials are often resonant devices which exhibit exotic properties such as negative acoustic refraction
[20] and, as such, often only work in a narrow frequency band. Fabricating equivalent broadband devices, i.e. devices with exotic properties that work over a large frequency range, is an on-going field of research which will make these devices of greater commercial interest.

Electromagnetic metamaterial research is a well-established field, and with the previous experience of the Electromagnetic Materials research group at the University into electromagnetic metasurfaces, one obvious question was how far the similarities between acoustics and electromagnetism could be taken.

There are indeed many similarities between electromagnetism and acoustics. An electromagnetic wave is an oscillation of electric and magnetic fields, and it’s propagation through a medium is dictated by the permittivity and permeability of the material. For an acoustic wave travelling through a material, the analogous parameters are the elastic modulus and mass density of the medium. An acoustic wave is composed of areas of low pressure, rarefactions, and high pressure, compressions. These form a propagating wave through the gas or fluid, forming a longitudinal, or compression, wave. Unlike electromagnetic waves which are transverse in nature, sound waves cannot be polarised as they are longitudinal. This obvious difference has implications on the permitted propagating modes.

Since acoustic waves are mechanical, in a solid they can also be transverse in nature due to the shear stresses experienced by the wave. These are known as shear waves. As such, in low symmetry crystals, they may have rather complex dispersion and polarisation. In principle, viscous effects also allow shear acoustic waves in a gas or liquid [55], however the shear wavelength is much shorter than the bulk wavelength, so shear wave propagation is generally of negligible significance by comparison to the longitudinal components.

The analogies between electromagnetic and acoustic variables that can be drawn, in 2D, are shown in the table below.

<table>
<thead>
<tr>
<th>Acoustic variable</th>
<th>Electromagnetic variable</th>
<th>Analogy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$ acoustic pressure</td>
<td>$H_z$ magnetic field</td>
<td>$p$ and $H_z$</td>
</tr>
<tr>
<td>$v$ acoustic velocity</td>
<td>$E$ electric field</td>
<td>$v_z$ and $E_z$, $v_x$ and $-E_x$</td>
</tr>
<tr>
<td>$\rho_e(\omega)$ dynamic density</td>
<td>$\varepsilon$ complex permittivity</td>
<td>$\rho_e$ and $\varepsilon$</td>
</tr>
<tr>
<td>$B(\omega)$ dynamic bulk modulus</td>
<td>$\mu$ permeability</td>
<td>$B$ and $1/\mu$</td>
</tr>
</tbody>
</table>

Figure 1: Table showing the equivalent electromagnetic and acoustic variables.

These analogies have implications for the boundary conditions of each system. The acoustic equivalent of a perfect conductor in electromagnetism is a perfectly rigid body i.e. at the boundary the normal component of the particle velocity and the electric field go to zero. At the surface of a perfect conductor the magnetic and electric fields are out of phase and similarly for a perfectly rigid body pressure and velocity are out of phase; the acoustic pressure must be a maximum at the boundary between a tangential fluid and rigid body. The analogy is dependent on the boundary conditions chosen and for a pressure-
release boundary i.e. $P = 0$ the acoustic pressure is associated with the electric field not the magnetic field as it will be zero at the boundary. One difference between the two situations is that for a real electrical conductor skin effects, which will impact how a wave propagates through a structure, in electromagnetism decay into the walls of the metal bounding a waveguide, whereas in acoustics there are boundary effects inside the medium the sound wave is propagating through e.g. the air.
1.2.2 Acoustic Wave Equation

Acoustics is a first-order approximation of the fluid dynamic equations, which neglect non-linear terms. The fluid and medium are seen as a continuum that allows for the omission of molecular influences. The conservation laws in acoustics are given by conservation of mass and momentum. The first is given by the diffusion equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = m$$  \hspace{1cm} (1)

where \( \rho \) is the density and \( \mathbf{v} \) is the velocity of the fluid, generally \( m \), rate of increase of mass per unit volume, equals zero when conservation of mass is assumed. Conservation of momentum is given by

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \nabla P + \rho \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{f}$$  \hspace{1cm} (2)

where \( P \) is the pressure and \( \mathbf{f} \) is the force density acting on the fluid.

Manipulation of equations 1, 2 and the ideal gas law equation of state leads to the one dimensional acoustic wave equation

$$\frac{d^2 P}{dx^2} = \frac{1}{c^2} \frac{d^2 P}{dt^2}$$  \hspace{1cm} (3)

where \( c \) is the speed of sound in the medium. It is worth noting that, unlike the electromagnetic wave equation, the wave motion is dependent on a scalar rather than a vector quantity.

The nonlinear Navier-Stokes equation for fluid flow is given by equation 4, and is derived from Newton’s second law of motion [6]. The stress tensor becomes the viscosity term, \( \mu \nabla^2 \cdot \mathbf{v} \), for incompressible fluids, including air at room temperature.

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla P + \left( \frac{4}{3} \mu + \mu_B \right) \nabla(\nabla \cdot \mathbf{v}) - \mu \nabla \times \nabla \times \mathbf{v} + \mathbf{f}$$  \hspace{1cm} (4)

Here \( \mu \) is the kinematic viscosity, \( \mu_B \) is the bulk viscosity, \( \mathbf{v} \) is the velocity and \( \mathbf{f} \) is the driving force acting on a unit volume of the fluid.

This equation takes into account convective acceleration and the divergence of stress (i.e. shear stress), which is relevant when looking at sound waves travelling through tubes and slits as they have a spatial rather than a temporal dependence. For example, convective acceleration is the velocity change due to a narrowing of a tube etc.

These equations are looked at in greater detail in Chapter 3.
1.2.3 Speed of Sound in a Gas

The speed of sound in a gas is dependent on the pressure, temperature and humidity of the medium it is travelling through [10]. These variables affect the compressibility and therefore the elastic modulus of the medium which dictates the rapidity with which a disturbance is passed between particles. When Sir Isaac Newton calculated the speed of sound in 1687, as the field of thermodynamics had not then been fully developed, he used isothermal considerations rather than adiabatic ones. This meant that Newton’s calculated value was missing a factor of $\sqrt{\gamma}$, where $\gamma$ is the ratio of specific heat capacities. This omission was realised by Laplace and the two different values for the speed of sound understood [11]. Thus unlike the speed of light, which is invariant, the speed of sound has both an adiabatic and an isothermal value. In a single medium, depending on the boundary conditions of the system, there can be regions of isothermal and adiabatic motion; the resultant speed of sound observed may then be a combination of the two different speeds.

This thesis project concerns the propagation of sound waves in air, therefore the intricacies of the bulk speed of sound in a solid and its shear modulus dependence are not discussed in detail, however the formula (equation 5) is included for completeness [12].

$$c^2 = (B + \frac{4}{3}S)/\rho_0$$  \hspace{1cm} (5)

which reduces to equation 6 for a fluid.

$$c^2 = B/\rho_0$$  \hspace{1cm} (6)

Here the bulk (elastic) and shear moduli of the medium are given by $B$ and $S$ respectively, and $\rho_0$ is the equilibrium density. The bulk modulus term of the speed of sound, given by equation 7, takes into account the compression of the material.

$$B = -V \frac{\partial p}{\partial V}$$  \hspace{1cm} (7)

Here $V$ is the volume of the material. For an anisotropic solid, an elasticity tensor must be used and its inclusion adds complexity to the calculation.

All Newtonian fluids support compression waves, but cannot facilitate shear waves due to the lack of elastic recovery in fluids. Regardless of this, at the boundary between a fluid and a rigid surface, shear effects are present as the particle velocity reduces to zero at the walls, creating a graded velocity boundary layer.

The bulk and shear viscosities affect the attenuation of a sound wave propagating in a system. The shear viscosity is a consequence of the diffusion of momentum by molecular collisions and arises when layers
of different net velocities move over each other. This can cause absorption in the longitudinal motion and has been experimentally shown to be solely dependent on temperature [13]. Likewise, the bulk viscosity takes into account the energy conversion in the system on a molecular level and is often called the volume viscosity.

1.2.4 Cut-off Frequency

Within acoustics, for rigid wall boundary conditions, the lack of a cut-off frequency for propagating modes in an infinitely long hollow waveguide is a fundamental departure from electromagnetic theory. In electromagnetism, there is a cut-off frequency for TE (transverse electric) and TM (transverse magnetic) modes. However, by using a co-axial waveguide, which changes the boundary conditions, TEM (transverse electromagnetic) modes, which exhibit no cut-off, can be supported.

The cut-off is due to momentum conservation and the boundary conditions at the walls; tangential electric field (or magnetic field) must be zero, which quantises the wave across the waveguide (it has to be one of the transverse eigenmodes). This then sets a lower limit frequency for a propagating mode in the waveguide; below this frequency the waves do not propagate but are purely evanescent as $k_z^2 (z$ is the propagation direction) is negative,

$$k_0^2 = k_x^2 + k_y^2 + k_z^2$$  \hspace{1cm} (8)

If $k_x^2 + k_y^2 > k_0^2$, $k_z$ must be imaginary and consequently a propagating wave cannot be supported. The incident wave does not have sufficient momentum to excite the first waveguide mode quantised across the guide. Assuming a channel of constant cross-section with a rigid boundary condition, in acoustics, there is no cut-off frequency due to the scalar nature of the field. However, if the cross-section varies along its length a cut-off can occur.

Furthermore, an effective cut-off frequency can be introduced into an acoustic waveguide by the presence of membranes, periodically placed along the guide. Each identical membrane has a specific resonant frequency, and so introduces a quantisation into the system. This mode can only be excited at particular frequencies and at frequencies below this condition the fields may be evanescent i.e. a cut-off has been introduced [15][16][17].

1.3 What is a Metamaterial?

Metamaterials were pioneered in order to achieve a desired response from a naturally occurring material. To create a specific acoustic response, new material properties are required which are not inherent to the natural structure. Through the use of metamaterials, which endow a material with ‘effective’ properties through structural manipulation, the desired responses can be obtained. By
texturing a surface (creating a metasurface) with periodic sub-wavelength features, such as grooves or holes, the incident sound wave ‘sees’ an effective density and elastic modulus which is different to the untextured material. The periodicity of the perforation must be $\leq \frac{\lambda}{10}$. This can lead to novel properties such as negative refraction as proposed by Veselago in the 1960s [18] for electromagnetic radiation in which he hypothesised that a 3D material with simultaneously negative permeability and permittivity should possess a negative refractive index. It was not until Pendry et al revisited this proposal and then suggested the design for such a material that this phenomenon was shown experimentally [19]. In electromagnetism the texturing of a metal affects the effective impedance (permeability and permittivity) of the material. The subsequent behaviour is dependent on the ratio of surface feature size to incident wavelength thereby only permitting metamaterials to operate within a certain frequency band. In acoustics, the elastic modulus, $B$, and density, $\rho$, dictate the acoustic refractive index, $n$, of a material as they give the speed of sound. The refractive index is a dimensionless quantity which represents the ratio of the velocity of sound in air and the velocity in the chosen medium, thus showing how much the velocity increases or decreases in the new medium. The refractive index is given by equation 9

$$n = \sqrt{\frac{B_0 \rho_1}{\rho_0 B_1}},$$

(9)

where $\rho_0$ is the density of air and $\rho_1$ is the density of the second medium.

Consequently, owing to the ability to manipulate the density and modulus of a material, phenomena like negative refraction are indeed possible, but resonantly. When dealing with ultrasonic frequencies in water, wavelengths are of the order of centimetres so the necessary sub-wavelength features are of the order of millimetres. As the speed of sound in air is less than that of water the dimensions need to be scaled down by approximately a factor of four, making the fabrication of such structures more difficult for ultrasound in air, but not of course for lower frequencies.

1.3.2 Negative Refraction

A pre-requisite for negative refraction is negative group velocity; the wave energy must propagate in the opposite direction to the phase. This occurs naturally in anisotropic materials due to band folding at the Brillouin zone boundary. This is a unit cell which is repeated across reciprocal space when discussing dispersion relations. At the point which the dispersion repeats the group velocity becomes negative whilst the phase is still positive. However, to artificially re-create this and exploit negative refraction, an isotropic medium with this property is necessary. This can be achieved through negative index metamaterials. A propagating wave in a medium must have a real wavevector, $k$, and consequently a real acoustic refractive index.
\[ k = |n| \frac{c_0}{v} \]  

(10)

where \( n \) is given by equation 12. This condition can be achieved in two ways; either both the elastic modulus and mass density are positive, or both must be negative [20]. The two situations give positive and negative refraction respectively as shown in Figure 2. One negative parameter will give a purely imaginary, or evanescent, wave.

![Figure 2: Diagram showing the direction of a wave under negative and positive refraction [21].](image)

Due to the backward wave propagation associated with negative refraction this phenomenon can be used to direct waves around an object, leading to the possibility of superlensing. The challenge is fabricating materials with these negative parameters as they do not occur naturally.

**1.3.3 Negative Index Acoustic Materials**

For acoustic materials to exhibit a negative index they must exhibit both a negative bulk modulus or mass density. The concept of negative bulk modulus and mass density is counter-intuitive and is a resonant phenomenon. A negative modulus material will expand on compression and a negative density material will move in the opposite direction to the driving force. These properties can be achieved through a variety of methods that have been both postulated and tested in the literature. Li *et al* show that rubber spheres immersed in water exhibit double negativity over a range of frequencies [20]. The response is due to coexisting monopole and dipole resonances which create a negative bulk modulus and mass density respectively. The volume dilation of a sphere is out of phase with the hydrostatic pressure field creating the negative modulus observed. Similarly, negative density is due to a resonant condition; the displacement of the sphere’s centre of mass oscillates out of phase with the incident pressure field causing negative density. Double negativity is achieved when the group velocity becomes negative. Negative group velocity does however occur naturally at a Brillouin zone boundary in a periodic lattice, due to Bragg scattering. Li *et al* increased the concentration of spheres and observed an increase in the double negative region to ensure the phenomena observed was independent of Bragg scattering.
Arrays of subwavelength Helmholtz resonators have also been shown to have a negative modulus response [22]. Assuming the periodicity of the array is less than the longitudinal wavelength the system can be viewed as an effective medium with a frequency dependent bulk modulus given by

\[
B_{\text{eff}}^{-1}(\omega) = B_0^{-1} \left[1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma\omega}\right]
\]

where \( F \) is a geometrical factor, \( B_0 \) is the bulk modulus of the medium, \( \omega_0 \) is the resonant angular frequency and \( \Gamma \) is dissipation loss in the elements. The angular frequency depends on the dimensions of the resonator, \( \omega_0 \propto \sqrt{\frac{S}{V}} \); \( S \) is the cross sectional area of the neck, \( V \) is the volume of the cavity and \( L \) is the neck length.

At resonance the displacement of the air in the neck becomes large and over many cycles energy is stored in the cavity. This stored energy causes the air in the cavity to continue to oscillate. As the driving frequency is swept through resonance the instantaneous displacement of mass in the neck becomes out of phase with the driving field leading to a negative response. The elastic response can be further tuned by varying the dimensions of the Helmholtz resonators. Negative modulus can also create collective surface states allowing the coupling of the evanescent fields from the subwavelength features into the ultrasonic metamaterial [22]. The evanescent enhancement could compensate evanescent decay in free space and lead to acoustic superlensing, which allows non diffraction limited resolution. The limiting factor in this scenario would be viscous effects at the walls.

An example of a composite structure possessing a negative acoustic index consists of a tube with an array of membranes and side holes alternating at an equidistant separation. This forms a type of Helmholtz resonator as shown in figure 3 [23]. The side holes create a negative modulus for similar reasons to those discussed above and the membranes facilitate negative density.

![Diagram of composite structure](image-url)
The effective density can be determined from the ratio between the pressure gradient and acceleration of the fluid due to this gradient. For negative density, phase difference between the pressure gradient and acceleration must give $cos \theta < 0$, where $\theta$ is the phase difference between the fluid displacement and the pressure gradient [23].

1.3.4 Acoustic Sub-wavelength Imaging

Usually, the resolution of an image is limited by diffraction. If the detail to be resolved is smaller than the imaging wavelength, the feature will be blurred or even missed; this is due to the rapid decay of the evanescent fields. Small spatial distances correspond to large momentum space values (above the available momentum of the incident radiation) which result in this information being carried by the evanescent fields. Metamaterials can be used to exploit negative refraction consequently allowing the evanescent fields to be recovered which leads to acoustic superlensing; imaging beyond the diffraction limit. These metamaterials have a flat lens property meaning they can converge a diverging incident sound wave [24].

A negative index medium sandwiched between positive index media creates a superlens. In-plane momentum must be conserved and the wave propagation in the central medium is reversed as shown in figure 4. Power flow, as described by the Poynting vector, flows in positive $x$-direction and thus is anti-parallel to the wave propagation in the negative media. The strength of the evanescent field enhancement is dependent on the coupling between each side of the metamaterial slab thus the resolution of the lens is dependent on the thickness of the slab and the source-slab/slab-image distance. The focus in the metamaterial acts to recover the evanescent decay; the negativity of the metamaterial dictates how tight the focus is so the negativity will also affect the resolution.

Figure 4: A ray diagram showing the direction of power flow (blue arrows) through a negative index material (brown slab) and the direction of wave (phase) propagation in the slab (red arrows).
One method of superlensing requires the beam to be focussed twice. Firstly, focusing the propagating wave through a flat negative refraction material allows the recovery of the evanescent fields. The final focus, outside the negative index material, creates an image with subwavelength resolution. Theoretically, an image spot of $\lambda/5$ can be achieved. This was demonstrated for sound in 2012 [25] using an anisotropic metamaterial consisting of layers of hole arrays embedded in a sound-hard medium to create a lens with a focus both outside and within the medium. This double focus allows for subwavelength imaging. The role of the evanescent waves in the image sensitivity is evident from figure 5 where the minimum spectral spot size corresponds to the maximum transmission of evanescent waves.

Figure 5: a) Graph illustrating that the spectral spot is a minimum when evanescent wave transmission is a maximum. b) Pressure diagram showing the wave travelling from the source, through the metamaterial and focussing at the image spot [25].

Superlensing has been experimentally proven through negative refraction focussing created by a 2D array of Helmholtz resonators. For this lens, objects with a size equivalent to $\lambda/2$ could be resolved. The experimental system used is shown in figure 6 [26].

Figure 6: Schematic of experimental set up for a superlens consisting of Helmholtz resonators [26].
1.3.5 Enhanced Acoustic Transmission (EAT)

The phenomenon of extraordinary optical transmission (EOT) is the transmission of light through a metal dressed with a periodic, subwavelength structure. This is ‘extraordinary’ as the structures are subwavelength, and subsequently should not transmit light as they have a cut off frequency beyond the visible. The observed transmission is due to the coupling of surface modes present on each side of the periodic structure. The importance of the coupling between the surface plasmons on either side of the metal plate was proposed by Porto et al in 1999 [27]. A similar phenomenon occurs in acoustics for transmission through subwavelength apertures, for example the single fishnet structure which consists of a plate with a periodic array of holes. In the acoustic case, due to the lack of a cut off frequency, this transmission is merely ‘enhanced’ rather than ‘extraordinary’ as transmission will occur regardless. At the resonant mode of the system evanescent surface acoustic modes on either side of the plate couple strongly together resulting in enhanced transmission through the structure. This behaviour is due to the coupling of guided modes inside the holes with the induced evanescent surface waves present on either side of the plate [28]. The surface waves occur due to diffractive coupling at the end of each hole; an artefact of the periodicity of the structure. This diffraction is strongest at the resonance of the system; \( \lambda = 2d \), where \( d \) is the plate thickness [29]. EAT has been experimentally shown through 1D gratings with very narrow apertures [28] in which the authors attribute the increased transmission to strong coupling between the diffracted surface wave and the waveguide modes. The authors also investigate the dependence of the transmission on incident angle as shown in figure 7.
Figure 7: (a) and (b) transmission as a function of the ratio of wavelength to length for a variety of incident angles. Graphs show theoretical and experimental acoustic data respectively. c) Graph (a) magnified to show the region in which the cavity mode, diffraction edge and surface mode occur. d) Showing the geometry; d = 4.5mm [28].

Figure 7 shows how the transmission varies for a range of incident wavelengths. As wavelength is increased the cavity mode appears followed by the diffraction edge, which occurs at $\lambda/d = 1$, causing the transmission to drop to zero due to propagating diffractive modes. Closely bound to the diffraction edge is the surface mode at which point transmission reaches 100%. The figure shows the dependence of transmission mode on incident angle. One peak remains unaffected by angle whereas the mode close to the Wood’s anomaly rapidly decreases with incident angle. This paper [28] suggests that this angle dependence shows that the enhanced transmission is due to the diffractive coupling of surface modes. The literature also states that another path to EAT is through the excitation of high order evanescent modes that cause an additional acoustic reactance, through the elasticity of the transmitting medium, and consequently changes the resonant behaviour of the system [30]. EAT has also been observed using inverse fishnet structures; an array of narrow slits made from cylinders as shown in figure 8 [31].
Figure 8: Diagram showing the geometry of the inverse fishnet system which allows EAT. The sphere are steel cylinders [31].

It was shown that for a small $R/a$ ratio, where $R$ is the radius of the cylinders and $a = 20$ cm, high transmission is achieved for the entire long wavelength region (20 cm - 50 cm) as the wave sees an effective medium. This is shown in Figure 9 (a) by the flat response.

Figure 9: a) Zero order transmission for an acoustic wave propagating through a steel cylinder grating in air for various cylinder sizes. b) The equivalent graph to (a) but for $R/a=0.48$ with different background media. Steel grating in water (dotted line), water cylinder grating in mercury background (solid line) [31].

However, when $R/a = 0.48$ transmission resonances can be seen. Similar dependence on incident angle is shown for these larger ratios as in optical transmission. Adding multiple layers of cylinders does not change the transmission properties of the system other than adding an additional resonant peak due to the coupling between the layers. This paper suggests that for the inverse fishnet structure, the origin of EAT is in the excitation of transverse resonant modes which form due to the coupling of local resonant modes around the cylinders as illustrated in figure 10. If the distance between the cylinders is such that the resonant modes around them can influence each other, transverse coupled acoustic modes are
formed. Acoustic waves of the correct wavelength incident on this grating are then coupled to this transverse mode and EAT is induced.

Figure 10: Pressure intensity diagram showing the coupling of transverse waves around the cylinders [31].

1.3.6 Acoustic Double Fishnet (ADF) Structure

The ADF structure is formed of two plates with a periodic array of holes and has been shown to have a broad transmission stop band [32]. The sets of holes are aligned and the plates separated by a gap, $d_g$, as shown in figure 11. When the driving frequency coincides with the frequency of the gap mode, $f_g$, the imaginary part of the impedance is dominant and there is no propagating wave through the structure i.e. a stop band appears and reflection is a maximum. At the gap resonance, the input impedance is zero due to the impedance mismatch at the boundary between the holes and the gap which gives the total impedance of the system to be zero and thus it cannot be governed by the real propagating component. This implies the ADF is a single negative medium as $\lambda \rho$ must be negative for the impedance to be imaginary. The gap between the ADF plates causes a difference in behaviour between the odd and even transmission modes of the system due to a pressure leakage into the gap. Even modes have a node in the gap and therefore are largely unaffected by its presence; however odd modes have an antinode in the gap which results in a leakage of pressure into the gap region. This leakage causes a change in resonant frequency from that expected due to conservation of volume flow in the system. At $f_g$ the odd and even modes destructively interfere giving a pressure profile as shown in figure 12 E. The diagram shows a region of zero pressure which corresponds to a transmission minimum through the structure. A pressure wave incident on a boundary experiences a 180˚ phase change at the interface.
Figure 11: Diagram showing nine units cells of the ADF structure [32].

Figure 12: Pressure profiles for even and odd modes, showing the discontinuity at the gap for the odd modes and the subsequent zero transmission at the gap frequency [32].

1.4 Acoustic Wave Behaviour at an Interface

As mentioned previously metamaterials allow bulk material properties to be adapted enabling the impedance of the ‘effective material’ to be tuned to give a desired acoustic response. There are many applications for a material that is electromagnetically impedance matched with air as it opens up the possibility for cloaking devices. The same can be said for the acoustic regime.

Impedance matching allows maximum power transmission and no reflection. Using the definition of acoustic impedance
\[ Z = \frac{P}{v} \quad (12) \]

where \( P \) is the pressure and \( v \) is the particle velocity, gives

\[ R = \frac{Z_1 - Z_2}{Z_1 + Z_2} \quad (13) \]

where \( R \) is the reflection coefficient. When \( R = 0 \), this gives \( Z_1 = Z_2 \) [33]. Acoustic impedance is given by \( z = \sqrt{\rho \lambda} \) where \( \lambda \) is the bulk modulus, \( B \), for most media.

There are many applications for this commercially and regarding stealth technology. Reflections from objects are vital for detection and therefore impedance matching allows two materials to appear acoustically similar allowing the concealment of objects.

### 1.4.2 Acoustic Cloaking

Metamaterials allow the refractive index of a material to be tuned and thus a wave to be directed around an object. The ability to cloak an object, both electromagnetically and acoustically, has been a topic of interest in recent years. Pendry et al postulated the properties of a microwave cloak in 2006 (19). There are two paths to cloaking an object; either with an exterior or an interior cloak [34]. An exterior cloak uses complementary media that has equal and opposite properties to the region to be cloaked; the result of this is the wave sees a continuum of the background media and thus scattering is prevented. Figure 13 shows the acoustic results of Popa et al in 2011 who use this method [35].

![Figure 13](image-url)

**Figure 13:** External ground cloak, left hand pressure diagrams show the theoretical response without (top) and with the cloak (bottom). The right hand diagrams show the experimental equivalents [35].

An interior cloak encircles the object and, using the method of graded refractive index which aims to impedance match the cloak with the background medium, refraction directs the wave around this region as shown in figure 14. One experimental cloak is shown in figure 15 [36]; the layers have different
polymer filling fractions which result in a varying refractive index and impedance matching. The disadvantage of the latter method is the object cannot ‘hear’ or communicate with the surrounding environment as all waves are steered around it.

Figure 14: Pressure diagram showing the propagation of waves around the object without (a) and with (b) the interior cloak present. Note how the waves bend inside the cloak and do not penetrate the region to be concealed (green circle) [34].

Figure 15: Experimental pressure diagrams of the cloak shown on the right hand side [36].
Creating a temperature gradient to achieve wave speeds larger than in air to ensure that the waves travelling around the cloaked object arrive in phase with the waves propagating through the background is another method of cloaking that has been investigated [37]. This cloak consisted of layers of cylinders which could have their individual temperature controlled.

The challenge for cloaking is fabricating metamaterials with the desired properties. The method of transformation acoustics, which is the analogue of transformation optics which uses the invariance of Maxwell’s equations under coordinate transformation [38], exploits the invariance of the acoustic wave equation. For a specific coordinate transformation one can set infinity at the centre which can be ‘pushed out’ and objects hidden within. Such transformations give the necessary modulus and mass density for the metamaterial to ensure that the wave experiences this coordinate transformation [39]. Without derivation, which is the theoretician’s domain, the required mass density, \( \rho \), and modulus, \( \lambda \), for elements under this transformation are given by equation 16 [37].

\[
\rho_r = \frac{r}{r - R_1}, \quad \rho_\theta = \frac{r - R_1}{r}, \quad \lambda^{-1} = \left( \frac{R_2}{R_2 - R_1} \right)^2 \frac{r - R_1}{r} \tag{14}
\]

All these parameters are positive, \( r > R_1 \) and \( R_2 > R_1 \), one consequence of which is broadband acoustic cloaking which is not achievable in electromagnetism. As in electromagnetism, all of the parameters governing the propagation of an acoustic wave are dispersive (i.e. have a frequency dependence) and knowing this dependence and how it changes the propagation of waves is vital.

### 1.5 Outline of Dissertation

As previously discussed metamaterials are generally resonant structures in order to obtain the desired properties. Therefore factors which affect the behaviour of a structure at resonance e.g. losses, are important to understand, and it is essential to have a fundamental grasp of their consequences and how to manipulate them.

The work presented in this thesis explores one of the very simplest systems which has been largely overlooked and yet may form a key element of many structures. That is a simple single slit and how the sound behaviour within it is affected as the slit is narrowed to much less than the wavelength. Two loss channels are explored, which are usually overlooked and thought to have a negligible effect due to the geometry of the systems. However, as the results show, this may be far from the truth. Understanding the viscous and thermal contributions to the propagation of sound through the channel and how it is affected by channel width forms the core of this study.

In the following chapter I will present the background theory which is the key to understanding my experimental results which are presented in Chapter 4. This is followed by a discussion of the modelling method used and a breakdown of the different finite element method models solved for this
investigation. Chapter 4 lays out the experimental technique and results, before the presentation of the analytical form for this system in Chapter 5.
Chapter 2: Background theory

Quantifying the influence of viscous and thermal effects in thin structures will aid researchers’ understanding of the acoustic behaviour in subwavelength structures. The ever-increasing knowledge of the origin of acoustic responses in the metamaterial regime will allow the informed design of new devices.

2.1 End Effects

We begin by examining tubes which have been extensively studied. For a single, open-ended tube, one loss channel which is well known and accepted is that due to diffraction at each end (end effects). This is a radiative loss channel and gives the tube an effective length which is longer than its physical length. The end correction causes the resonant frequency to be lower than naively expected from the Fabry-Perot frequency given by equation 15.

\[ f_n = \frac{nc}{2L} \]  

(15)

Here \( f_n \) is the frequency of the nth order mode and \( L \) is the tube length. Rayleigh in his book investigated this correction and said the first order end correction is proportional to the tube radius [40] taking the form,

\[ f_n = \frac{nc}{2(L+\beta r)} \]  

(16)

where \( r \) is the tube radius and \( \beta \) has a value of 0.84. Rayleigh calculated the end correction using two methods [40] which give the limits of \( \beta \); 0.79 \( \leq \beta \leq \) 0.83. The lower limit arises from using a model based upon calculating the acoustic conductivity of the channel and the resistance of the channel which gives the end effects. The upper limit uses a piston model to calculate the coefficient. The radiation from the tube end is not purely spherical in form and tracing it back the wave seems to radiate from a point that is just outside the tube; the distance between the radiation point and the tube end is the end correction. Conceptually, this can be thought of as the distance taken for the impedance the wave sees on exit of the tube to become the free space impedance i.e. the distance over which the presence of the tube walls still influences the way in which the air in that region moves. The end correction modifies the length of the tube while keeping the bulk speed of sound constant, resulting in a reducing resonant frequency for increasing radii.

However, there is another loss factor which acts to reduce the bulk speed of sound within the tube, which is now discussed.

2.2 Viscous and Thermal Effects
Sound waves are impeded by viscous effects unlike in electromagnetism, and so the speed of sound is not invariant for a given media and can be modified by influences indirectly related to sound propagation i.e. viscosity and thermal conductivity. This is one very obvious difference between acoustic waves and electromagnetic waves.

There are two types of viscosity in a fluid.

The kinematic viscosity, which is the shear viscosity coefficient divided by the fluid density, quantifies the fluid’s resistance to shearing flows. This creates a boundary layer in the vicinity of the tube wall. For an acoustically rigid material bounding a tube, the boundary condition at the wall forces the tangential particle velocity to zero. Associated with this non-slip boundary condition there is a viscous (momentum) boundary layer with characteristic thickness, \( \delta_v \sim \sqrt{\mu/\omega} \) [41], where \( \mu \) is the kinematic fluid viscosity.

The second type of viscosity is the bulk viscosity, or volume viscosity. This quantifies the mechanical energy lost by compression/dilatation. Only fluids which are not monatomic have a finite bulk viscosity as its origin lies in the losses that are not translational in nature. This is not a shearing process and thus is not confined to a boundary; it is an inherent form of attenuation for most gases.

At the tube walls, as well as a pressure and velocity boundary condition, there must be a thermal boundary condition as temperature variation is also a source of sound propagation; temperature fluctuations are proportional to pressure fluctuations. The thermal boundary condition at the wall must either be adiabatic or isothermal. Due to the no-slip boundary condition, the corresponding thermal condition is isothermal. This is because the collision between the particles and the boundary occurs slowly enough for the particles to transfer energy to the boundary in the form of heat. Therefore the motion in the tube can no longer be considered strictly adiabatic, as this will never be true at the walls.

This gives rise to a thermal boundary layer with its own characteristic thickness, \( \delta_K \sim \sqrt{\alpha/\omega} \), where \( \alpha \) is the fluid’s thermal diffusivity. Since for an ideal gas both thermal effects and viscous effects arise from molecular motion, these two characteristic lengths are expected to be similar, and for air the Prandtl number, \( \sigma = \delta_v^2/\delta_K^2 \), which is the ratio of the viscous and thermal boundary layers, is around 0.7.

If the tube was fabricated from pressure-release material (sound soft), rather than a rigid body, the boundary conditions would be reversed. There is now a velocity maximum at the walls and a pressure minimum. The thermal boundary condition will change to being adiabatic at the walls if the tube material has similar acoustic properties to the fluid, air, inside.

2.3 Previous work
The viscous loss mechanisms discussed in section 2.2 are usually neglected in air as the order of thickness of the boundary layers are \(~25 \mu m\) at 4.95 kHz. For a system with dimensions orders of magnitude greater than \(~25 \mu m\), the boundary layer is a tiny fraction of the tube diameter and is expected to have no substantial effect on sound propagation. However, over some hundred years ago, Helmholtz began to investigate the velocity reduction due to viscosity, and later Kundt, while trying to verify Helmholtz’s equation using a dust-tube experiment, found the sound velocity decrease to be larger than predicted and suggested heat conduction as a possible reason. In 1868 Kirchoff presented the complete theory of velocity reduction due to thermal and viscous effects in tubes. Kirchoff’s theory solely focuses on what he deemed the most important case for ‘wide’ tubes. In 1953 Weston extended Kirchoff’s theory to determine the subsequent equations for ‘narrow’ and also ‘very wide’ tubes. Both Weston [42] and Yazaki [41] give a brief overview of Kirchoff’s work. Kirchoff’s theory assumes the following: an indefinitely long tube; a non-slip boundary condition at the tube walls; a constant fluid temperature at the walls i.e. an isothermal boundary condition and a much slower thermal and viscous wave propagation compared to the bulk acoustic wave. The infinite nature of the system, present in all the above theories, allows corrections due to diffraction at the tube ends to be disregarded. These theories purely take into account the effects of the thermal conductivity and viscosity on the speed of sound [42]. Kirchoff’s theory gives the modified speed of sound for the ‘wide’ case as [42],

\[
c' = c \left[ 1 - \frac{\gamma'}{r_w^2 \omega^2} \right] \tag{17}
\]

where \(r_w\) is the tube radius, \(\omega\) is the angular frequency and \(\gamma'\) is Kirchoff’s constant which has contributions from both the viscosity and thermal conductivity. Equation 17 implies that as the tube radius reduces the speed of sound reduces when taking into account viscous and thermal effects.

\[
\gamma' = \sqrt{\mu + (\gamma - 1)\left(\frac{\mu}{\gamma}\right)^{1/2}} \tag{18}
\]

where \(\gamma\) is the ratio of the specific heat capacities. The equations from Weston’s paper are discussed in greater detail in Chapter 5.

Weston in his paper presents velocity and phase profiles for a ‘wide’ tube as shown in figure 16. He says that the enhanced axial particle velocity close to the walls is an artefact of Kirchoff’s theory. The literature also refers to the maximum close to the walls as ‘Richardson’s Annular Effect’ [43][44]. This attributes the origin to a transverse decaying wave in the tube, where the ‘wave’ referred to is not a physical wave but an oscillating diffusion effect. This is due to the viscous boundary layer and can be described as a form of viscous wave. Figure 16 below shows an axial particle velocity profile which describes what the sound wave is doing and a radial velocity profile which illustrates boundary layer effects but carries no knowledge of the dilatation and the phase-front through the tube.
For finite systems, end corrections are present and thus there is a ‘developing’ distance over which the flow stabilises, as illustrated in figure 17. Profiles must be taken where the flow is fully developed so that any perturbations due to end effects are minimised.
In the study of fluid flow, when the boundary layer flow is fully developed it is referred to as Poiseille flow. This characteristic curved wavefront occurs when the boundary layers are overlapping.

So far this chapter has discussed viscous and thermal effects in tubes. However, the system of interest in this investigation is a single air-filled slit in an aluminium block. There appears to be only one substantive comment about the effect of viscosity and thermal conductivity in sound propagation through a slit and that was given by Lord Rayleigh in 1901 [5]. Rayleigh states that the theory for tubes can be replicated exactly for slits if the substitution of tube radius, $r$, for slit-width, $w$, is made. The analytic theory in Chapter 5, and the good fit to FEM data, confirms Rayleigh’s simple substitution between the two different geometries.
Chapter 3: Modelling

Finite Element Method (FEM) modelling software, COMSOL, was used for the numerical modelling [46]. The modelled 2D system is a single resonant slit-cavity of width, \(w\), in an acoustically rigid material. The variation of the cavity resonance as a function of slit-width is the result of interest. Two different modelling modules had to be used in order to accurately represent the physical system. The ‘pressure acoustic’ module (PA) solves equation 19 which is an inhomogeneous Helmholtz equation [47]. It is a form of the acoustic wave equation for pressure variation i.e. it just takes into account pressure fluctuations. This means that no loss channels, other than diffraction at the two open-ends of the slit, are accounted for.

\[
\nabla \cdot \left( -\frac{1}{\rho_c} (\nabla p - q_d) \right) - \frac{\omega^2 p}{\rho_c c_c^2} = Q_m
\]

Here \(\rho_c\) and \(c_c\) are the complex density and speed of sound respectively, having both a real and imaginary part, \(\omega\) is the angular frequency, \(Q_m\) is the rate of change of a specific volume (zero in this single slit system), \(q_d\) is the external force (zero in this single slit system) and the pressure, \(p\), has the form \(p(\mathbf{r}) = p(x,y)e^{-ikz}\). In most cases the out of plane wavevector, \(k_z\), is zero.

This PA model accurately predicts how the end correction contribution becomes negligible as the slit-width is narrowed and the frequency of the modes supported tend to the Fabry-Perot frequencies. However this model omits the effect of viscosity and thermal conductivity in the slit.

An additional module is needed to take into account the thermal and viscous properties of the gas; the ‘thermo-acoustic’ module (TA) in COMSOL. This model solves the linearised Navier-Stokes equation for a viscous fluid with no-slip boundary conditions at rigid walls, and allows heat transfer. The motion of a viscous, compressible Newtonian fluid (air in this case) is governed by the following equations [48];

The Continuity Equation:

\[
\frac{d\rho}{dt} + \rho (\nabla \cdot \mathbf{v}) = 0
\]

where \(\rho\) is the fluid density and \(\mathbf{v}\) is the particle velocity. The first term is the mass flow into the system and the second term is the mass flow out due to velocity gradients (i.e. how quickly the mass is carried away).

The Energy Equation:
\[ \rho C_p \left( \frac{dT}{dt} \right) = -\nabla \cdot (-K \nabla T) + \beta T \frac{dp}{dt} + \varphi(\mu, \mu_B, \nu) + Q \]  

(21)

where \( K \) is the thermal conductivity of the fluid, \( C_p \) is the specific heat at constant pressure, \( \mu \) is the kinematic viscosity (losses due to shearing), \( \mu_B \) is the bulk viscosity (losses due to contraction and expansion of the fluid) and \( Q \) is a heat source and can be set to zero. The left-hand-side of the equation represents the quantity of heat energy needed to be put into the system in order to increase the temperature by \( dT \). The first term on the right-hand-side is a conduction term; representing how easily heat is conducted through a material. The second term is a thermal expansion term; if a gas is compressed it heats up, thus a sound wave will have regions of high and low temperature and the pressure change, therefore, has a dependency on the thermal expansion coefficient, \( \beta, \beta = -\frac{1}{\rho} \frac{\partial \rho}{\partial T} \). The third term is a viscous dissipation function. This function relates the shear stress tensor and rate of strain tensor (velocity gradient with respect to the radial direction). The viscous dissipation function vanishes when the equation is linearised, if the mean particle velocity is zero. If this function was non-zero it would act as an oscillating sink/source term.

The Navier-Stokes equation:

\[ \rho \frac{du}{dt} = \nabla \cdot \left( -pI + \mu (\nabla v + \nabla v^T) - \left( \frac{2\mu}{3} - \mu_B \right) (\nabla \cdot v)I \right) + F \]  

(22)

where \( I \) is the unit matrix and \( F \) is an external volume force. The Navier-Stokes equation comes from Newton’s second law.

\[ \rho \frac{Dv}{Dt} = \rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = \nabla \cdot \sigma_{ij} = -\nabla P + \nabla \cdot \tau + F \]  

(23)

Here \( \sigma_{ij} \) is the stress tensor and \( \tau_{ij} \) is the deviatoric tensor (explained below). The left-hand-side of equation 25 describes the acceleration of the flow while the right-hand-side is a summation of body forces and the divergence of stress; these give Newton’s second law. The additional term, \( v \cdot \nabla v \), in the standard Newtonian form denotes convective acceleration, which is a spatial effect and reduces to the gradient of surface forces, \( \nabla \cdot \tau \). For a system undergoing deformation this tensor is split into a pressure term, \( P \), which is made up of the diagonal elements and results in a change in volume, and the deviatoric tensor, \( \tau_{ij} \), which results in a change of shape. For a body undergoing deformation, the off-diagonal elements are the shear components and these lead to the change. The deviatoric tensor deforms the body and therefore has to be composed of the stress tensor and an additional factor to quantify the size of the deformation at each point. This factor is the rate of strain tensor, \( \epsilon_{ij} \). The deviatoric and rate of strain tensor are linked through the viscosity and after manipulation the \( \tau \) term can be written as

\[ \mu (\nabla v + (\nabla v)^T) - \mu (\nabla \cdot v)I \]  

(24)
The unit matrix, \( I \), selects the main diagonal components i.e. \( \tau_{xx}, \tau_{yy} \) and \( \tau_{zz} \). The off diagonal terms are related to the vorticity and are not relevant in our linear system.

A third module may also be used, ‘thermo-elastic’ (TE), which takes into account the elastic properties of the material (i.e. the walls in the slit case). However, as aluminium is being utilised, which is in effect rigid, this module is not called upon to obtain the results. It does, if one were using a different material, allow for coupling of the temperature and particle velocity between the fluid and solid which is useful for some systems. This will be important when trying to model membranes in tubes, or other systems with sound soft boundaries and an elastic response.

At large slit widths boundary effects become negligible and so the PA and TE/TA models are expected to converge. However, as figure 18 shows, this is not the case. This is because even at \( w = 7 \) mm there is still an effect from the boundary layers which influences the resonant frequency. This is confirmed in the analytical theory presented in Chapter 5. The pressure and temperature can be manually set in the TA model which allows the exact experimental conditions to be replicated in the model.

Models are split into domains which can have different modules assigned to them. This allows computational time to be minimised, while still solving all the necessary physics in the model to achieve a good representation of the physical system i.e. the areas that are not affected by viscous losses can be run as PA models and the cavity and areas around it can be run as full TA models (PA is less computationally expensive compared to TA). This means one can change parameters, turn off viscous and thermal losses and compare the responses, see figure 18. On observing figure 18 it is evident that the viscous and thermal losses reduce the frequency causing a ‘turn down’. The slight deviation from a smooth trace is due to the resolution of the models (±5 Hz) and the broad nature of some of the resonances. The density of meshing tetrahedra was increased to ensure the models had converged and all the features seen were real.

Please note that throughout this report the abbreviations PA and TE are used to describe the loss-less model, just end corrections accounted for, and the full viscous/thermal model respectively.
FIG. 18. Predictions of the resonant frequency vs slit-width in air (ComsolCOMSOL) showing the comparison between PA and TE models when the loss channels are varied. Increasing slit-width shows increasing end effects compared to smaller slit-widths which show increasing viscous losses.

The final 2D model used for this investigation encompassed all three of the modules mentioned above. The modules solved in each domain are shown in figure 19. The model is surrounded by perfectly matched layers (PMLs) which, as far as the wave is concerned, extends the medium to infinity (i.e. will rapidly damp any fields in the PML) therefore preventing any reflections.
FIG. 19. Schematic representation of the COMSOL model, indicating the module solved in each domain. The green domains represent the aluminium block, the remaining domains are air.

The boundary layer thickness is inversely proportional to $\sqrt{\omega}$ therefore higher order modes will experience a turn down in resonant frequency at smaller slit-widths as the boundary layer will be smaller.
Chapter 4: Experimental Study

4.1 Experimental Technique

In 2004, Suckling et al [49] investigated how the finite conductance of the metal forming a single slit in part governs its resonant frequency in the microwave regime. For this regime, a significant reduction in resonant frequency of the fundamental Fabry-Perot mode in the slit was found for slit widths less than 2% of the radiation wavelength. This phenomenon was attributed to the finite conductivity of the metal, which leads to a skin depth of the fields into the cavity walls. Due to the invariance of the speed of light in air and the field matching conditions in the system, the total wavevector is less than that expected from a simple Fabry-Perot cavity. An imaginary wavevector perpendicular to the cavity walls is required to fulfil the field matching conditions and this consequently reduces the wavevector.

Due to the growing interest in exploring the extent of the electromagnetic and acoustic equivalence the question of whether there is a similar reduction in resonant frequency with reducing slit-cavity width, $w$, in the acoustic regime was raised. Air-borne experiments in this capacity had not been undertaken at the University of Exeter before so this chapter also touches on some of the lessons learnt before successful results were finally achieved.

The acoustic sample consists of two aluminium slabs (202 mm × 243 mm × 35 mm) which were machined to ensure smooth surfaces and sharp right-angled edges. Mylar spacers were used to separate the slabs by the required slit-width, $w$. The spacers have a length of ~35 mm and a thickness of either 0.5 mm or 0.1 mm. To guarantee a constant cavity length for each width, metal dowels were drilled into each side of the slab, as shown in figure 20, to ensure the slabs were placed directly on top of each other for every measurement. A small offset would mean a different cavity length and thus a different intrinsic resonant frequency.

![Image of the aluminium blocks showing the dowel and spacers.](image)
Cutting the spacers to the length of the cavity and drilling a hole in the centre the size of the dowel ensured the spacers were positioned in the same place for each measurement and did not protrude outside of the cavity. Such differences could cause variable diffraction from the edges and consequently affect the sound transmitted through the slit-cavity. Initially, continuous wave measurements were taken. However, unlike with pulse measurements, reflections from the surroundings could not be time-gated out and therefore multiple standing waves were superimposed onto the data. To minimise reflections, the sample was embedded in a screen of acoustic absorber i.e. an acoustic baffle. A comparison between the reflectivity of the acoustic and microwave absorber (also tested for sound absorption) is given in figure 21.

![Graph comparing the reflectivity of the acoustic and microwave absorber. Microwave absorber is shown by the blue trace. The other traces are acoustic absorber; flat sheet (green); square patches (cyan); pyramids (red). Measurements taken by Hugh Willman, a work experience student, and Natalie Whitehead, a summer student; July 2013.](image)

Due to these results an amalgamation of both absorbers was utilised. Figure 22 shows the transmission through the three types of acoustic absorber used in the final experimental configuration. These transmission measurements were taken using a source and detector either side of the acoustic absorber. A straight through measurement was taken so that the materials absorbing properties could be analysed.
The transmission is calculated in decibels using the ratio of the transmitted signal with, and without, the absorber present.
FIG. 22. Graphs of transmission through the 3 types of acoustic absorber, and the microwave absorber. The blue traces show transmission through a single sheet, the red traces show transmission through two absorber sheets placed back-to-back. The bottom graphs (narrow pyramids) are the microwave absorber. The absorber profile is shown by the image next to the corresponding graphs. Measurements taken by Toby Weston, a work experience student, July 2013.

A functional pulse measurement LabView code had been developed which enabled reflections to be time-gated out. Performing a Fast Fourier Transform (FFT) on the detected time-domain data allowed the transmitted frequency spectrum through the slit-cavity to be obtained. The waveform used was a Gaussian single cycle pulse, figure 23, as there is a smooth transition before and after the pulse unlike a single sinewave cycle which would have had a step function and a consequential sinc envelope in the Fourier Transform. The microphone response spectrum exhibited a minimum in the region of interest, 3 kHz – 6 kHz, thus the pulse is centred at 4.95 kHz to counteract this.

FIG. 23. The Gaussian single cycle pulse shape used in the pulse experiment.

To time-gate out any reflections a timebase, TB, of 30 was used. One pulse measurement provides 10,000 frequency data points. A TB of 30 is equivalent to a total data acquisition time of 4.48 ms and time between samples, $\Delta t$, of 448 ns as given by equation 25,
$$\frac{2(TB - 2)}{125 \times 10^6} = \Delta t.$$ (25)

To prevent the reflections from surrounding objects interfering with the signal of interest, an area with a radius of at least 0.8 m was cleared around the experimental set up. This was to ensure that reflections from the walls would not be able to travel to the detector within the given data acquisition time. One data set consisted of 25 repeats. To ensure the reflections from the wall did not enter the subsequent repeats, a burst period, i.e. the time elapsed between pulses, of 1.5 s was used. This allowed enough time for any reverberations left from the previous pulse to have attenuated before the next pulse is sent. For reference, a list of parameters entered into the LabView code is recorded in the table below.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time base</td>
<td>30</td>
</tr>
<tr>
<td>Number of data points</td>
<td>10000</td>
</tr>
<tr>
<td>Pulse centre frequency</td>
<td>4.95 kHz</td>
</tr>
<tr>
<td>Repeats</td>
<td>25</td>
</tr>
<tr>
<td>Burst period</td>
<td>1.5 s</td>
</tr>
<tr>
<td>Number of cycles</td>
<td>1</td>
</tr>
</tbody>
</table>

FIG. 24. Table of LabView parameters.

When performing the FFT the data acquisition time dictates the maximum frequency carried within the pulse and the resolution is dictated by the number of data points. Extending the data with 10,000 zeros either side of the measured data means that the resolution is doubled with no detriment to the features of interest. A total of 30,000 data points gives a resolution of 68 Hz. Figure 25 shows the resolution comparison for an extended and raw data set. The increased resolution improves the accuracy of the final steps in the data analysis.
FIG. 25. Normalised experimental data for a slit-width of 0.5mm with increasing number of data points either side of the measured data; the only change is the resolution of the resonant peak. Data taken on 23/6/13. The transmission values are arbitrary as they depend on width of the beam and sample size.

To minimise signal leakage around the sample, it is embedded in an acoustic baffle and placed between a speaker and microphone (Brüel & Kjær 4190) resulting in the experimental set up shown in figures 26 and 27. As stated above the slit-cavity width, $w$, is varied using mylar spacers at both sides of the cavity, positioned outside of the illuminated area. The slabs are placed such that the acoustic beam is incident on the centre of the cavity. The beam profile is approximately planar at the surface of the sample shown through transmission experiments. A signal generator creates a single-oscillation pulse centred at 4.95 kHz and the data is recorded by a detector connected to a PC-based oscilloscope (picoscope). To correct for background noise, the transmission with the slit fully closed is subtracted from the $w \neq 0$ mm data in the time domain. This took into account any signal being transmitted around the sample. The corrected data is then normalised in the frequency domain by dividing by a reference signal (also corrected for background) recorded without a sample in the baffle. Microwave absorber was placed in the sample holder during the ‘no sample’ reference, figure 27e, to prevent reflections from the base of the sample holder. A T-piece, placed above the sample, prevented signal leakage over the top of the sample as it rests flush against the top of the sample regardless of the slit-
Acoustic absorber and blankets were positioned on the floor between the source and detector to maximise the absorption of stray reflections, figure 27b.

FIG. 26. Schematic of the experimental set up.
FIG. 27. The experimental set up. a) Sample in acoustic baffle. b) Experimental structure. c) Side on view of set up. d) The loaded speaker to achieve a flatter source response. e) Reference set up, with foam in the sample holder to prevent reflections.

Clamp stands were used to secure the speaker and detector. To prevent additional vibrations from these structures interfering with the response, the instruments are embedded in microwave absorber in order to rapidly damp any oscillations. A source of standing wave resonance to be aware of is between the source and detector; this will be apparent in the reference and therefore will be superimposed on the normalised signal. This problem, while performing continuous wave experiments, was overcome by positioning thin disks of absorber in front of the speaker and microphone. As the output signal intensity is much greater than the reflected wave, the absorber disks do not significantly impede the incident signal.

Ideally, the reference signal should have a flat response in the frequency domain. In this case, unlike the loudspeaker, the microphone had a flat response. To ensure that any transmission minima in the frequency response around the cavity’s resonant frequency was avoided, the speaker was loaded, as
shown in figure 28d, in order to flatten the response around 4.5 kHz, figure 28. The Fabry-Perot resonance of this slit-cavity is at 4.9 kHz and, from figure 18, the region of interest is ~4.5 kHz. Due to the range and response of the instruments the signal drops off below ~2.5 kHz and above 10 kHz. This provided appropriate range of this experiment which was investigating the region between 3 kHz – 6 kHz.

FIG. 28. Frequency response of source; with (left) and without (right) loading. The y-axis scale is the same for all graphs.

Originally, instead of extending the data with zeros, data with a slightly different timebase was taken so that each data point was amongst a cluster all slightly offset from each other. This manually mapped out the resonance with a higher frequency resolution, as shown in figure 29. However, this process was laborious and not as effective as padding the data with zeros. Figure 30 shows the normalised data for a slit-width of 0.5 mm as the timebase is varied. For larger time bases (i.e. longer data acquisition time) reflections from the walls can be seen to enter into the data and create an oscillation. For a longer timebase to be used, an appropriate radius must be cleared around the sample to prevent reflections from the walls being detected in the data.
FIG. 29. The graphs show, for a slit of width 2 mm, data taken for three different timebases. The data points are slightly offset from each other; compiling the data sets allows for better resolution to be achieved. This data was taken before the experimental set up was optimised which is the reason for the poor quality of data.

FIG. 30. Comparison between low and high timebases, for a slit-width of 0.5 mm, using the optimised experimental configuration. Above TB 30 reflections from the walls begin to enter the data set giving the jagged traces. Measurements taken by a work experience student, Toby Weston, July 2013.
The data acquisition time needs to be such that the loudspeaker has time to completely ‘ring down’ after an electrical pulse has been sent so there is no vibrational residue, but not too long that reflections from the surrounding environment are detected. Figure 31 shows the ‘no sample’ pulse, the pulse for \( w = 0.5 \) mm and \( w = 0 \) mm, and the corresponding normalised frequency domain transmission measurement.

*FIG. 31. Pulse graph showing the ring down and the contributions to the normalisation from the three measurements. The insert shows the final normalised resonant peak for \( w = 0.5 \) mm. Data was taken on 23/6/13.*

### 4.2 Experimental Results

The following experimental results were taken during weekends, or in the evening, to ensure that the building was quiet and there were no sporadic noises that could influence the individual measurements. One needs to be confident that the conditions in which the reference signal is taken is the same as the sample measurements. This can be tricky when performing audible acoustic experiments as a full data set can take ~3 hours to complete. As the background noise level varies a reference signal was taken so the transmission data could be normalised to correct for this. A weather station which records humidity, pressure and temperature was placed in the laboratory to monitor the conditions in the room so that these conditions could be recorded at the beginning and end of each data set to monitor the fluctuation.
in the speed of sound. As these conditions vary day to day, a full data set (0.1 mm ≤ w ≥ 7 mm) had to be acquired within a short time period. For example, as figure 32 shows, for w = 0.5 mm, there is a difference of 9 Hz in the resonant position between the first (4514 Hz) and last (4523 Hz) measurements. This data was taken over the course of three hours.

FIG. 32. Graph showing the variation between the first and last measurements of a data set; w = 0.5 mm. Fitting a Lorentzian to the resonance using graphical analysis package gives the peak positions to be 4514 Hz and 4523 Hz for the first and last measurement respectively. This implies an error in frequency due to the speed of sound fluctuation of ± 5 Hz. Data taken on 23/6/13 at 9pm.

These conditions and their influence on the speed of sound and thus the resonance is a major source of error and can account for some of the discrepancy between experimental data and the COMSOL predictions. For example, a difference in the speed of sound of 0.5 ms⁻¹ is equivalent to a change in the expected Fabry-Perot resonant frequency of ~ 7 Hz.

Slit width data was taken in 0.5 mm ± 0.0075 mm intervals for 0.5 mm ≤ w ≤ 7 mm, 0.2 mm ± 0.004 mm intervals for 2 mm ≤ w ≤ 3 mm and 0.1 mm ± 0.002 mm for 0.1 mm ≤ w ≤ 2 mm. A graph of resonant frequency, of the fundamental mode, as a function of slit-cavity width is shown in figure 33.
FIG. 33. Frequency of the fundamental resonant peak as a function of slit-cavity width. The turn down occurs at $w = 0.6$ mm. The solid line is the numerically predicted response without viscous and thermal effects; the dashed line is the TE results. The data points are experimental data taken at average room temperature 292.17 K, pressure 101.89 kPa and humidity 46.5%.

On inspection of the figure it is noticeable that there is a difference between the predicted TE FEM data and the experimental data. The graph shows an error for each slit-width. The resonant peak position was determined by fitting a Lorentzian to the data using graphical analysis software and the error quoted arises from the uncertainty in the fitting. The uncertainty is greater both at wider ($\geq 3.5$ mm) and narrower ($\leq 0.34$ mm) slit-widths as it is in these regimes that the mode broads substantially and thus the fitting is less precise, as shown in figure 34.
There are also potential systematic errors in the system, which have not been included in the error bars, and these may account for the shift in the experimental data from that predicted. This can be attributed to the error in cavity length and the fluctuation in the speed of sound. The modelled cavity length is 35 mm compared to the experimentally measured length of 34.8 mm, this corresponds to a change in frequency of 28 Hz. A difference in the speed of sound of 0.5 m s$^{-1}$ corresponds to a frequency change of 7 Hz. These systematic errors are shown in figure 3 from the three experimental data sets. Each data set was taken on a different day with different atmospheric conditions which accounts for the offset of the data sets from each other.
FIG. 35. Graph showing the different experimental data sets and the COMSOL predictions with (solid line), and without (dash-dot line), viscous and thermal effects. The red data points show data taken on 23/5/13 at a temperature of 292.15 K; the green data points show data taken on 23/6/13 with average conditions of 100.56 kPa, 293.15 K and 48.5% humidity; the blue data points show data taken on 25/6/13 with average conditions of 101.90 kPa, 292.17 K and 46.5% humidity.

Taking data at smaller intervals in slit width (12 µm) around the turn down enabled the widths around which the behaviour of the frequency is reversed to be tracked at a higher resolution as shown in figure 36. The resonances for $0.1 \text{ mm} \leq w \geq 2.5 \text{ mm}$ are shown in figure 37 which shows the general trend of the turn down occurring, as well as the narrowing and broadening of the modes.
FIG. 36. Showing the turn down for $412 \mu m \leq w \leq 636 \mu m$.

FIG. 37. Graph showing the reduction in resonant frequency and broadening of the modes as the width is reduced; widths top trace to bottom trace span the range $0.1 \ mm \leq w \leq 2.8 \ mm$. 
From these figures it can be observed that, instead of the modes narrowing and sharpening with decreasing slit width, as would be expected if solely the end correction (PA) behaviour is considered due to the reduced damping, the modes shallow and broaden showing a reduction in the Q factor. This implies another form of loss dominates the systems behaviour at small widths. Figure 38 illustrates the broadening and shallowing of the modes around the Fabry Perot resonant frequency as the sit-cavity width is increased in the PA model, and the comparison for the TA model where the damping is increased again by the addition of viscous and thermal effects.

FIG. 38. Graph showing the loss factor, X, as a function of frequency. a) The behaviour expected from a system in which the loss is steadily reduced. b) A system with an additional loss mechanism becoming dominant at small widths causes the behaviour to return to that of a heavily damped system. This occurs below 0.5 mm.
As mentioned earlier, unlike in the electromagnetic case, the reduced resonant frequency behaviour seen for narrow slits is attributed to a modified speed of sound in the slit-cavity. This modification comes from viscous and thermal effects in the slit-cavity. Adapting the FEM modelling to investigate the effect that viscosity and thermal conductivity of the air individually have on the turn down, as shown in figure 39, confirmed that it is dependent on both the viscosity of the fluid and its thermal conductivity. Our modelling confirms that this behaviour is not dependent on the thermal conductivity of the walls. It is also observed in the work of Weston [42] and Yazaki [41] in their investigations of sound in tubes which ignore the thermal conductivity of the wall. These studies assume that any temperature gradient within the wall can be ignored and the wall is treated as an isothermal boundary. The presence of the ‘turn down’ is not reliant on the viscosity of the air which simply introduces an additional loss term and therefore the ‘turn down’ occurs at a larger $w$. This implies, as realised by Kirchoff [42][41] that both the thermal conductivity and the viscosity contribute to the modified speed of sound and they are subsumed within a single constant with the exact nature of amalgamation dependent on the slit width. This implies the system could be degenerate, i.e. fluids with different thermal conductivities and viscosities which sum together to give the same overall loss constant give therefore the same behaviour.

**FIG. 39.** Illustrates, from COMSOL modelling, the presence of the turn down, despite either the viscosity or thermal conductivity of the air set to zero, suggesting the effect is due to an amalgamation of the two contributions.
The velocity profiles for tubes are well known and understood [42]. The axial velocity profiles are related to the distribution of sound (as they are in the direction of wave propagation) and the radial velocity profiles, having no bearing on the sound propagation, are influenced by the boundary layers and the vorticity. Similar velocity profiles are expected for a slit and these are plotted in figure 40. There are three waveforms in the slit geometry – thermal wave, viscous wave and the acoustic wave. The acoustic wave is given by the axial velocity and the loss waves (thermal and viscous) are represented by the radial velocity. The profiles are taken from the FEM modelling on a resonant Fabry Perot mode. The profiles are taken mid-way down the slit-cavity length to ensure that the end effect perturbation is a minimum and the flow has ‘fully developed’. In other words, because this is a finite system, some end effects (i.e. Lord Rayleigh stated that the velocity must go to infinity at the edges of the cavity) may perturb the actual profiles expected from solely boundary layer effects. Plotting the axial particle velocity profiles across the centre of the slit-cavity for a range of thicknesses, figure 40b, shows a maximum in velocity near the walls of the slit. This maximum becomes more tightly confined to the walls (i.e. is a smaller percentage of the slit width) as the width is increased, or the frequency increased, which implies that it is an artefact of the boundary layers which reduce with increasing frequency and have less influence at large slit-widths. The reason for the velocity maxima near the walls is a viscous wave propagating in the boundary layer (Chapter 2), decaying away from the walls [42]. The combined effect of isothermal propagation and viscous drag at the walls cause the wavefronts to diverge [5] and this is encompassed in the radial velocity profiles. This influences the entire wavefront and results in increased attenuation and a lower effective phase velocity along the slit thus decreasing the resonant frequency. Surprisingly, these thermal and viscous effects are found to have a strong influence at slit widths much larger than the decay length. As sound is also a pressure wave, the pressure profiles give information about sound propagation. The pressure and velocity are out of phase, thus it is unsurprising that at the walls there is a pressure maximum and a velocity minimum (the rigidity of the walls dictate the velocity boundary condition which is then related to the pressure).
FEM modelling also confirms that the similarities between tubes and slits extends to the phase-fronts predicted by Weston for a tube [42]. The evolution of the phase fronts can be seen in figure 41. For wide slits, the wavefronts are predominantly planar with a boundary layer effect near the walls. As the width reduces, the boundary effects penetrate more of the slit until these two effective layers overlap and create a curved wavefront, with reduced velocity.
4.3 Discussion of Experimental Results

Our results show a significant divergence from the accepted, and well understood, end-correction model for pipes. There are three contributions to the behaviour observed. Firstly, corrections at each end due to diffraction give the cavity an effective length which is longer than the physical length. This reduces the frequency, but as the correction is proportional to the slit-width it tends to the Fabry Perot frequency as the slit-width tends to zero; the diffraction at the entrance/exit to the slit becomes more confined and has less of an effect as $w \to 0$. This correction, to first order, is found to be $\sim 0.84\pi$, where 0.84 is the circular tube end correction (see Chapter 5 for the analytic theory). For large widths, $w \geq 2$ mm, this diffraction correction dominates. Below this width the resonant frequency begins to diverge from the PA predictions.

The second consideration is the viscosity. In this system there are no slip boundary conditions on the walls i.e. the tangential velocity is zero. This introduces a boundary layer which can be visualised as layers of different velocities moving over each other, in a similar way to shear waves in a solid. The larger the viscosity of the fluid, the slower the rate of change of particle velocity away from the walls. This effect is accentuated when the width is reduced as the viscous layers become a larger proportion...
of the total width compared to a wide slit-cavity. This can be seen in the axial velocity profiles; the maximum near the walls are artefacts of the boundary layers and the decaying waves these create. As the width is reduced this feature becomes more prominent in the profile. The presence of the maxima also gives an idea of how far the influence of the boundary layers extends in comparison to $\delta_{v,K}$.

The final consideration is the thermal conductivity of the fluid and the thermal boundary condition at the walls. In papers which discuss the viscous and thermal effects on cavity wave propagation [41][42] mention is ever made of the thermal conductivity of the walls, suggesting this has no influence on the behaviour observed. Note both Weston [42] and Yazaki [41] in their investigations of tubes ignore the thermal conductivity of the wall. This assumes that any temperature gradient within the wall can be ignored thus treating the wall as an isothermal boundary. The good agreement between numerical and experimental results in [41] validates this assumption. COMSOL modelling also confirms this. Changing the thermal conductivity of the walls between 0 W(mK)$^{-1}$ and 1.6 kW(mK)$^{-1}$ has no effect on the resonant positions; figure 42 shows that the full TE model and the TE model with $K_{aluminium}$ turned off have a very similar acoustic response.
FIG. 42. a) Showing the response for a system with no \( K_{\text{aluminium}} \) present is the same as the full TE model. b) same as (a) but focusing on the turn down region.

For wide cavities the wave is travelling close to the adiabatic speed of sound, \( c_A \), and for these systems the thermal conductivity of the fluid does not significantly affect the propagation. However, due to the no-slip boundary condition in the cavity, near the walls the particles are travelling with a slow enough particle velocity to allow heat exchange; the speed of sound can no longer take the adiabatic value. Instead the bulk speed of sound tends to the isothermal speed of sound, \( c_T \). The isothermal speed of sound is less than the adiabatic speed of sound:

\[
c_A > c_T.
\]

The sound waves propagating in the narrowest slit cavities investigated in this report do not travel at the isothermal speed of sound, this occurs when the boundary layers are overlapping, but instead they travel with a modified adiabatic speed of sound. The tubes, or slits, in which the boundary layers overlap, are referred to by Weston as ‘narrow’ as discussed below.

In 1953, Weston, in his paper on the propagation of plane sound waves in circular tubes, categorised the tubes as follows [42]:
• Wave propagation through a ‘narrow tube’ is isothermal and is determined mainly by viscous, not inertial, forces.
• The wave motion in a ‘wide tube’ is adiabatic but thermal and viscous processes lead to attenuation and loss.
• In a ‘very wide tube’ the motion is similar to the wide tube, however the sound energy is confined to a channel near the tube wall.

The single slit-cavities discussed in this report are in the wide/transition to narrow categories. The turn down is in the ‘transition’ regime, \( w \leq 0.6 \) mm, and is seen where the frequency begins to deviate from the ‘end effect’ behaviour. The ‘wide tube’ behaviour is for widths \( 0.6 \) mm \( \leq w \geq 7.5 \) mm.

These definitions split figure 42a into three sections; an isothermal regime, a viscous regime, and an end effect regime. In the latter regime, the speed of sound is adiabatic and the modification in frequency comes from the effective cavity length, rather than a reduced speed of sound. At these widths the viscous and thermal boundary layers are small enough to be assumed negligible.

Yazaki numerically and experimentally plots out the modification of the speed of sound with tube radius using dimensionless parameters, shown in figure 43 [41]. Comparing this to a similar graph by Weston shows the same results. Yazaki’s graph is more accurate as it is numerical compared to Weston’s, which is analytical and hand drawn. If any information is needed from Weston’s graph (figure 2 from reference [42]) I would recommend re-plotting the graph; the equations he uses are stated in his paper.
FIG. 43. Yazaki’s comparison of experimental propagation constants with theoretical solutions [41]. The red and blue lines show the corresponding position for 3 mm and 0.1 mm slit width if only the boundary layer effects are taken into account (this is was the case for Yazaki and Weston, no end-effect calculations were included). Equation 2.1 [41] is the Kirchhoff solution to the propagation constant for sound in a rigid circular tube as seen on page 2856 of the reference.

As can be seen from figure 43, the slit-widths used in this report do not enter the ‘narrow’ region which is for $\omega \tau \leq 1$ where the rapid reduction in the speed of sound occurs and full isothermal motion achieved. The region around $1 \leq \omega \tau \leq 10$ is the transition region referred to by Weston [42]; the transition between wide and narrow tubes. Above $\omega \tau \sim 50$ the wide regime is entered and the very wide region beyond this. The majority of the single slit experimental data lies in the transition region. In the transition region, the boundary layer effects begin to dominate the acoustic response of the system.

As can be seen from figure 42, individually eliminating the different loss channels confirms the “turn down” to be independent of viscosity. The “turn down” occurs at a narrower slit-width as one source of attenuation has been removed, but the same behaviour is still observed.

The axial velocity profiles across the slit-cavity (figure 16) become increasingly curved as the width is reduced; showing velocity maxima initially confined to the walls, for large widths, and becoming a larger fraction of the total slit-width as the width is reduced. Although this behaviour seems counter-intuitive, axial velocity profiles in infinite tubes show similar behaviour to the single slit [42]. This profile is surprising given the expected parabolic velocity profile for a tube with no slip boundary.
conditions [44] which becomes more planar as the slit width is increased. The similarity between the axial velocity profiles from COMSOL for a single slit and those for Weston’s infinite tube imply that this profile is ‘correct’ (figure 3 in [42]).

Only the axial particle velocity profile reflects the acoustic behaviour of the system as a sound wave is longitudinal i.e. propagates along the axis. The radial velocity illustrates the effects of the boundary layers and therefore is dominated by the curl of the velocity, however does not provide information about the dilatation. The change in radial velocity is not confined to the boundary and, as it is dominated by the curl, does not reflect the distribution of sound, despite this, it could be important for some applications.

Weston states that for a very wide, infinitely long tube, the sound energy is concentrated near the wall of the tube. In the vicinity of the wall the sound velocity is reduced, and in terms of electromagnetism this would act like a graded refractive index medium with a high index region near the walls. In this situation, the high index medium channels the power into the high index layer, with the wavevector in the propagation direction being matched across the entire slit width by having an imaginary $k_z$ component, a decay amplitude, towards the centre of the slit. In this system the high index regions are the boundary layers.
Chapter 5: Analytic Theories

5.1 End effects

A paper by Christensen et al [30], using a modal matching method, gives an analytical theory for the end corrections for a single slit. This is plotted alongside the FEM end correction prediction in figure 44. Christensen et al’s theory did not take into account boundary effects because the apertures in their system were not sufficiently narrow for this perturbation to become significant; the rapid reduction in frequency occurs when the slit-width is ~ 1% of the fundamental wavelength. Although there is good agreement between COMSOL and Christensen in figure 44, the deviation from COMSOL shows that Christensen’s method is incomplete as the viscous effects are not taken into account.

![Comparison between Christensen’s modal-matching method analysis and the Comsol predictions.](image)

Omitting all loss channels, apart from diffractive losses, the modified Fabry Perot resonant frequency equation for a tube, is given by,

\[ f_1 = \frac{c}{2(L + 0.84r)} \]  

(26)
where \( r \) is the tube radius.

The equivalent expression for a slit cavity, of width \( w \) is unknown by the author. To determine this expression a polynomial was fitted to the PA FEM data.

Assuming, to first order, the end correction for a slit is proportional to its width, as for a tube, the equation takes the form,

\[
\frac{f_1}{c} = \frac{2}{2(L + \alpha w)}
\]

(27)

where \( w \) is the slit-width and \( \alpha \) is a constant.

With a second and third order correction term this equation becomes,

\[
\frac{f_1}{c} = \frac{2}{2(L + \alpha w + \beta w^2 + \gamma w^3)}
\]

(28)

Inverting equation 30 and taking the derivative, with respect to \( w \), eliminates the known constant, \( L \), and allows a polynomial to be fitted.

\[
\frac{1}{f_1} = \frac{2(L + \alpha w + \beta w^2 + \gamma w^3)}{c}
\]

(29)

\[
\frac{d}{dw}\left(\frac{1}{f_1}\right) = \frac{2(\alpha + 2\beta w + 3\gamma w^2)}{c}
\]

(30)

From the polynomial fit, the unknown coefficients in equation 32 are calculated. Origin, a graphical analysis software, was used to plot equations 31 and 32 as shown in figures 45 and 46.
FIG. 45. Graph of inverse frequency against slit-width from COMSOL pressure-acoustic modelling (equation 31).

FIG. 46. Graph of equation 32 with a second order polynomial fit (red trace), allowing the coefficients in equation 32 to be calculated. This returns coefficients of; $\alpha = 2.7; \beta = -248; \gamma = 11,100$.

Using MatLab and plotting the equation for the end correction for slits to third order, equation 33, gives the fit shown in figure 47.
Therefore, to third order, the end correction for slits is given analytically by the following equation,

\[ f_1 = \frac{c}{2(L + 2.7w - 248w^2 + 11100w^3)} \]  \hspace{1cm} (31)

Note that the first order correction coefficient for a single slit is simply \( \sim 0.84\pi \), where 0.84 is the circular tube correction term [40].

### 5.2 Boundary Layer Effects

To complete the analytic description of the single slit system the boundary layer contribution must be determined. Throughout the following analytical theory Lord Rayleigh's substitution of slit-width, \( w \), for tube radius, \( r \), [5] is utilised.

Weston [42] defines the viscous and thermal contributions as follows (parameters taken from source [50]):

\[ \gamma' = \sqrt{\nu} + (\gamma - 1) \left( \frac{\nu'}{\gamma} \right) \]  \hspace{1cm} (32)
\[ \nu = \text{viscous contribution} = \frac{\mu}{\rho} = \frac{18.27 \times 10^{-6}}{1.2} = 1.5 \times 10^{-5} \text{ pas} \]

\[ \nu' = \text{thermal contribution} = \frac{K}{\rho c_v} = \frac{0.0257}{1.2 \times 0.718} = 2.7 \times 10^{-5} \text{ m}^2\text{s}^{-1} \]

\[ \gamma = \text{ratio of the specific heats} = 1.4 \]

Rayleigh and Weston give the modified speed of sound (and therefore frequency) to be:

\[ f' = f_p \frac{c'}{c} \quad (33) \]

where the modified speed of sound is:

\[ c' = c \left[ 1 - \frac{\gamma'}{w\sqrt{2\omega}} \right] \quad (34) \]

Treating the system as a piston, the oscillatory term, in its argument, has both a propagating and an attenuation term. The second order correction to the modified frequency comes from minimising this argument, picking out the resonance, [51] and the equation becomes,

\[ f_{\text{res}} \approx f_p \frac{c'}{c} \left( 1 - c \frac{dm'm'L}{d\omega \pi} \right) \quad (35) \]

where the attenuation \( m' \) is given by Weston [42]:

\[ m' = \left( \frac{\gamma'}{cw} \right) \sqrt{\omega/2} \quad (36) \]

Figure 48 shows the comparison between the first and second order boundary layer frequency equations (equation 33 and 35). Note that the second order correction term only adds to the reduction in frequency for widths less than 1 mm, for widths wider than this the attenuation term is negligible. One can infer from this that as the slits become even narrower \( w \ll 1 \text{ mm} \) this contribution becomes more important.
Combining equation 31 and 35 gives the total contribution from the three loss channels present in a resonant single slit. From these equations the behaviour observed can easily be explained, especially the dominance of the losses in the narrow slit-width region. The individual contributions and the combined effect, using equation 31 and equation 35, are shown in figure 49 below. The graph also shows the expected result from the full COMSOL model. The deviation from the FEM prediction for $1.7 \text{ mm} \leq w \leq 6 \text{ mm}$ is where equation 31 does not exactly fit the PA model (as can be seen in figure 47). The analytic theories fit the turn down exactly however, which implies that the first order correction for the end effect and the description for the boundary layers are accurate. Higher order terms may be needed to more accurately describe the end correction for a single slit.
FIG. 49. Analytic theory showing the individual contributions for the modified speed of sound due to the boundary layers (equation 35) and the end effects (equation 31). The fundamental frequency for a Fabry Perot cavity of 35 mm is given by the horizontal line.
Chapter 6: Conclusion and Future Work

In conclusion, the optimal experimental configuration for acoustic single slit transmission experiments has been developed. Clean, experimental data for resonant transmission through a wide range of slit-widths has been achieved for a single 35 mm long slit cavity in an aluminium slab. It is found that for large slit widths there is significant deviation from the simply predicted Fabry-Perot resonant frequency due to diffraction at the ends. More interestingly, for slit-widths $w \leq 0.6$ mm there is a significant reduction in the resonant frequency due to wall effects reducing the velocity of sound in the slit. This is, as Kirchoff predicted [42], due to viscous and thermal effects at the boundary of the slit. This report also confirms that Lord Rayleigh’s modified version of equations describing the modified speed of sound in a tube [5], using the slit-width rather than the tube radius, is correct. These loss contributions reduce the speed of sound in the cavity and thus, since the wavevector is fixed by the Fabry-Perot standing wave condition, the resonant frequency is reduced. These boundary-layer-effects significantly perturb the speed of sound at slit-widths much greater than the boundary layer thickness, which is not intuitive. These effects in air start to become significant when the width is $\sim 10\delta_k$.

The similarity between the velocity and pressure profiles for slits and tubes has been confirmed. The axial velocity shows a maximum near the walls which becomes a larger fraction of the slit width as the width is reduced; this is due to the increased influence of the viscous effects in the boundary layer. Phase-fronts are plotted from FEM modelling showing the gradual curvature of the wavefronts as the boundary layers (the thickness of the boundary layer effect; this is greater than the physical boundary length size) begin to overlap. For widths approaching the turn down in frequency, $w \sim 0.4$ mm to 0.5 mm, the boundary layers seen in the phase-fronts begin to communicate. These boundary layers show complete overlap when the gradient of the frequency vs slit-width graph is positive and the resonant frequency then begins to fall.

An extension of this geometry would be to compare the acoustic response of a single slit with an array of slits; creating a more ‘metamaterial-like’ system. One could also insert a slat into the slit-cavity and investigate phase resonances as has been done in the microwave regime [52], varying the size and shape of the gaps either side of the slat. Performing the same single slit experiment in water, with a foam-filled slit would possibly open up fresh studies of novel acoustic materials for use underwater.

The single slit experiment could be used as a controllable way to characterise the speed of sound in materials without needing to know the bulk modulus of the unknown material. This would prove useful for characterising metamaterials. A fluid-filled single slit may be fully characterised and comparing the response of the fluid-filled slit with one filled with an unknown material would allow the speed of sound in the material to be determined.
Further experiments could be undertaken with the slits themselves patterned with ‘teeth’ along their sides so that the contact between the boundary layer and the walls is reduced. It would also be interesting to explore how patterning the entrance and exit faces with grooves of suitable spacing, depth and width may change the end effects to avoid the fall in resonant frequency seen with broader slits. Another aspect worthy of exploration is to consider how one might use the channelling of sound towards the sides of the slit to advantage sound absorption technologies. For example, would a set of narrow slits running along the sound propagation direction along both sides of the primary slit cause strongly enhanced sound attenuation? Is this a possible route to ventilation ducts which allow air flow but strongly inhibit sound transmission?

One final thought is to consider the effect of reducing the widths even further. This will of course further reduce the sound velocity and increase the loss, but is there more understanding to be gained as the gas molecules mean free path approach the slit width?
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