Robustness of λ-tracking and funnel control in the gap metric

Achim Ilchmann* and Markus Mueller**

***Abstract*— For m-input, m-output, finite-dimensional, linear systems satisfying the classical assumptions of adaptive control (i.e., (i) minimum phase, (ii) relative degree one and (iii) positive definite high-frequency gain matrix), two control strategies are considered: the well-known λ-tracking and funnel control. An application of the λ-tracker to systems satisfying (i)–(iii) yields that all states of the closed-loop system are bounded and $|e|$ is ultimately bounded by some prespecified $\lambda > 0$. An application of the funnel controller achieves tracking of the error e within a prescribed performance funnel if applied to linear systems satisfying (i)–(iii). Moreover, all states of the closed-loop system are bounded. The funnel boundary can be chosen from a large set of functions.**

Invoking the conceptual framework of the nonlinear gap metric, we show that the λ-tracker and the funnel controller are robust. In the present setup this means in particular that λ-tracking and funnel control copes with bounded input and output disturbances and, more importantly, may be applied to any system which is "close" (in terms of a "small" gap) to a system satisfying (i)–(iii), and which may not satisfy any of the classical conditions (i)–(iii), as long as the initial conditions and the disturbances are "small".

I. Introduction

In the early 1980s, a novel feature in classical adaptive control was introduced: adaptive control without identifying the entries of the system being controlled. Pioneering contributions to the area include [1], [15], [16], [18], [20] (see, also, the survey [10] and references therein). The classical assumptions on such a system class – rather than a single system – of linear m-input, m-output systems are: (i) minimum phase, (ii) strict relative degree one and (iii) positive-definite high-frequency gain matrix. Then the simple output feedback $u(t) = -k(t)\,y(t)$ stabilizes each system belonging to the above class and $k(\cdot)$ adapted by $\dot{k}(t) = \|y(t)\|^2$. In this work we consider a variation thereof: the so-called λ-tracker, which has the advantage that, if tracking is the control objective, it needs not to be combined with an internal model and, more importantly, is applicable to systems in the presence of any additive input or output L^∞-disturbances.

However, two major drawbacks of the latter strategy are first, the gain $k(t)$ is, albeit bounded, monotonically increasing which might finally become too large whence amplifying measurement noise, and secondly, whilst asymptotic performance is guaranteed, transient behaviour is not taken into account (apart from [17], where the issue of prescribed transient behaviour is successfully addressed).

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*Institute of Mathematics, Ilmenau University of Technology, Weimarer Straße 25, 98693 Ilmenau, DE, achim.ilchmann@tu-ilmenau.de

**Dept. E&E Engineering, The University of Melbourne, Parkville, VIC 3010, Australia, mmuller@unimelb.edu.au

A fundamentally different approach, the so-called "funnel controller", was introduced in [11] in the context of the following output regulation problem: this controller ensures prespecified transient behaviour of the tracking error, has a non-monotone gain, is simpler than the above adaptive controller (actually it is not adaptive in so far the gain is not dynamically generated) and does not invoke any internal model. Funnel control has been applied to a large class of systems described by functional differential equations including nonlinear or/and infinite dimensional systems and systems with higher relative degree [12], it has been successfully applied in experiments controlling the speed of electric devices [13] (see [10] for further applications and a survey), and recently it has be shown that funnel control copes with input constraints if a certain feasibility inequality holds [6].

The contribution of the present paper is to show that λ-tracker and the funnel controller are *robust* in the sense that the control objectives (bounded signals and asymptotic tracking (when applying the λ-tracker) and tracking within a prespecified performance funnel (when applying the funnel controller), resp.) are still met if the λ-tracker and the funnel controller, resp., are applied to any system "close" (in terms of the gap metric) to a system satisfying the classical assumptions (i)–(iii). This will be achieved by exploiting the concept of (nonlinear) gap metric and graph topology from [5], [2]. The results are analogous in structure as these in [7] and [8].

A. System class

We consider the class of linear n-dimensional, m-input m-output systems ($n, m \in \mathbb{N}$ with $n \geq m$)

$$\left.\begin{array}{ll} \dot{x}(t) = A\,x(t) + B\,u_1(t), & x(0) = x^0 \in \mathbb{R}^n, \\ y_1(t) = C\,x(t), & \end{array}\right\} \qquad (1)$$

which satisfy the classical assumptions in high-gain adaptive control, that is minimum phase with relative degree one and positive definite high-frequency gain matrix, i.e. they belong to

$$\widetilde{\mathcal{M}}_{n,m} := \left\{ \begin{array}{l|l} (A,B,C) & CB + (CB)^T > 0\,, \\ \in \mathbb{R}^{n\times n} & \forall\, s \in \overline{\mathbb{C}}_+ \,: \\ \quad \times \mathbb{R}^{n\times m} & \\ \quad \times \mathbb{R}^{m\times n} & \det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0 \end{array} \right\}.$$

The state space dimension $n \in \mathbb{N}$ needs not to be known but only the dimension $m \in \mathbb{N}$ of the input/output space. Most importantly, only structural assumptions are required but the system entries may be completely unknown.

Note that for any $(A, B, C) \in \widetilde{\mathcal{M}}_{n,m}$ with $\det CB \neq 0$ we may choose $V \in \mathbb{R}^{n\times(n-m)}$ with $\operatorname{rk} V = n - m$ and

$\operatorname{im} V = \ker C$; then $T := [B(CB)^{-1}, V]$ is invertible and

$$T^{-1}AT = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} CB \\ 0 \end{bmatrix},$$
$$CT = \begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix}.$$

Moreover, if (A, B, C) is minimum-phase, then A_4 has spectrum in the open left half complex plane $\mathbb{C}_-$. Therefore, we replace $\widetilde{\mathcal{M}}_{n,m}$ by

$$\mathcal{M}_{n,m} := \left\{ \begin{array}{c|l} (A,B,C) & A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ \in \mathbb{R}^{n\times n} & C = [I, 0],\ B_1, A_1 \in \mathbb{R}^{m\times m}, \\ \times \mathbb{R}^{n\times m} & \operatorname{spec}(A_4) \subset \mathbb{C}_-, \\ \times \mathbb{R}^{m\times n} & B_1 + B_1^T > 0 \end{array} \right\},$$

and restrict our attention to systems $(A, B, C) \in \mathcal{M}_{n,m}$ in Byrnes-Isidori normal form, see for example [14, Sec. 4], i.e.

$$\left. \begin{array}{ll} \dot{y}_1 = A_1 y_1 + A_2 z + CB\, u_1, & y_1(0) = y_1^0 \in \mathbb{R}^m, \\ \dot{z} = A_3 y_1 + A_4 z, & z(0) = z^0 \in \mathbb{R}^{n-m}. \end{array} \right\} \quad (2)$$

We will study the initial value problem (1) or (2) as *plant* P mapping the interior input signal u_1 to the interior output signal y_1, in conjunction with the *controller* C (the λ-tracker (4) or funnel controller (5) in our setup, resp.), mapping the interior output-signal y_2 to the interior input signal u_2, and in the presence of additive input/output disturbances u_0, y_0 so that

$$u_0 \;=\; u_1 + u_2, \qquad y_0 \;=\; y_1 + y_2\,, \qquad (3)$$

as depicted in Figure 1.

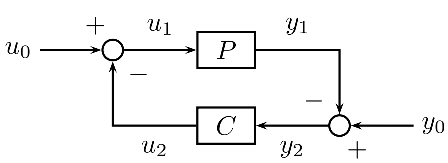

Fig. 1. The closed-loop system $[P, C]$

B. λ-tracker

For $\lambda > 0$, $k^0 \in \mathbb{R}$, the adaptive λ-tracker

$$\left. \begin{array}{l} \dot{k}(t) = \operatorname{dist}(y_2(t), [-\lambda, \lambda]) \cdot \|y_2(t)\|, \quad k(0) = k^0, \\ u_2(t) = -k(t) y_2(t), \end{array} \right\} \quad (4)$$

where, for $e \in \mathbb{R}^m$, $\operatorname{dist}(e, [-\lambda, \lambda]) := \max\{0, \|e\| - \lambda\}$, has been introduced by [9], and will be applied to (1) or (2). The λ-tracker overcomes the shortcomings of the "classical" adaptive controller $u(t) = -k(t)\, y(t)$, $\dot{k}(t) = \|y(t)\|^2$, see [1], namely no internal model is required when tracking is the control objective and, more importantly, that the classical controller fails stabilizing systems in the presence of additive arbitrarily small input or output L^∞-disturbances, see [4].

C. Performance funnel and funnel controller

For funnel control, the control objective, defined in the following sub-section, will be captured in terms of the *performance funnel*

$$\mathcal{F}_\varphi := \{(t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1\}\,,$$

determined by $\varphi(\cdot)$ belonging to

$$\Phi := \left\{ \varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{l} \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}), \\ \varphi(0) = 0,\ \forall\, t > 0 : \varphi(t) > 0, \\ \liminf_{t\to\infty} \varphi(t) > 0, \\ \forall\, \varepsilon > 0 : \varphi|_{[\varepsilon,\infty)}(\cdot)^{-1} \text{ is} \\ \text{globally Lipschitz continuous} \end{array} \right\}.$$

Note that the funnel boundary is given by $1/\varphi(t)$, $t > 0$; see Figure 2. The concept of performance funnel had been introduced by [11]. There it is not assumed that $\varphi(\cdot)$ has the Lipschitz condition as given in Φ; we incorporate this mild assumption for technical reasons. The assumption $\varphi(0) = 0$ allows to start with arbitrarily large initial conditions x_0 and output disturbances y_0. If for special applications the initial value and y_0 are known, then $\varphi(0) = 0$ may be relaxed by $\varphi(0)\|y_0(0) - Cx^0\| < 1$, see also the simulations in Example 3.4.

The funnel controller, for prespecified $\varphi(\cdot) \in \Phi$, is given by

$$u_2(t) = -k(t) y_2(t), \qquad k(t) = \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|} \qquad (5)$$

and will be applied to (1) or (2). Note that the funnel controller (5) is actually not an adaptive controller in the sense that it is not dynamic. The gain $k(t)$ is the reciprocal of the distance between $y_2 = y_0 - y_1$ (i.e. the difference of a reference signal y_0 and the output of (1)) and the funnel boundary $\varphi(t)^{-1}$; and, loosely speaking, if the error approaches the funnel boundary, then $k(t)$ becomes large, thereby exploiting the high-gain properties of the system and precluding boundary contact.

D. Control objectives

We will study properties of the closed-loop system generated by the application of the λ-tracker (4) or funnel controller (5), resp., to systems (1) of class $\mathcal{M}_{n,m}$ or of class $\mathcal{P}_{n,m}$ (see below) in the presence of disturbances (u_0, y_0) (from signal spaces specified below) satisfying the interconnection equations (3).

If, for prespecified $\lambda > 0$, the λ-tracker (4) is applied to any system (1), belonging to the class $\mathcal{M}_{n,m}$, in the presence of disturbances $(u_0, y_0) \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations (3), then the closed-loop system (2), (4), (3) is supposed to meet the following control objectives:

- all signals are bounded;
- the output error $y_2(t) = y_0(t) - y_1(t)$ of the output disturbance and the output of the linear system satisfies

$$\limsup_{t\to\infty} \operatorname{dist}(y_2(t), [-\lambda, \lambda]) = 0.$$

Alternatively, if, for prespecified $\varphi \in \Phi$ determining the funnel boundary, the funnel controller (5) is applied to any system (1), belonging to the class $\mathcal{M}_{n,m}$, in the presence of disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations (3), then the closed-loop system (2), (5), (3) is supposed to meet the control objectives:

- all signals are bounded;
- the output error $y_2(t) = y_0(t) - y_1(t)$ of the output disturbance and the output of the linear system evolves in the funnel, in other words

$$\forall\, t \geq 0 \;:\; (t, y_2(t)) \in \mathcal{F}_\varphi = \\ \{(t,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}\,.$$

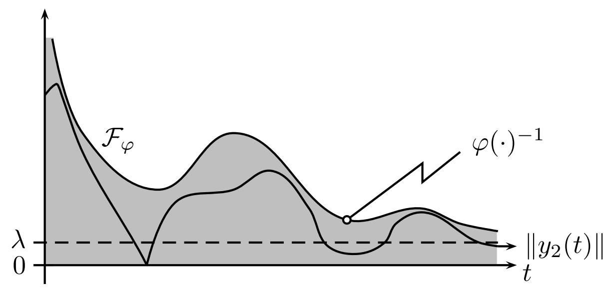

Fig. 2. Funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi$ and $\inf_{t>0} \varphi(t)^{-1} = \lambda$

E. Main result: robustness

The main result of the present paper is to show robustness of the λ-tracker and funnel controller in the following sense: The control objectives should still be met if $(A, B, C) \in \mathcal{M}_{n,m}$ is replaced by some system $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ belonging to the system class

$$\mathcal{P}_{q,m} := \left\{ \begin{array}{l|l} (A,B,C) & (A,B,C) \text{ is} \\ \in \mathbb{R}^{q\times q} & \text{stabilizable} \\ \quad\times \mathbb{R}^{q\times m} & \text{and detectable} \\ \quad\times \mathbb{R}^{m\times q} & \end{array} \right\} \supsetneq \mathcal{M}_{q,m}$$

where $q, m \in \mathbb{N}$ with $q \geq m$, and $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is close (in terms of the gap metric) to a system belonging to $\mathcal{M}_{n,m}$ and the initial conditions and the disturbances are "small".

For the purpose of illustration, we will further show that a minimal realization $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ of the transfer function

$$s \mapsto \frac{N(M-s)}{(s-\alpha)(s+N)(s+M)}\,, \qquad \alpha, N, M > 0\,, \tag{6}$$

(which obviously does not satisfy any of the classical assumptions since it is not minimum phase, has relative degree 2 and negative high-frequency gain) is the closer to a system in $\mathcal{M}_{n,m}$ the larger N and M.

II. Adaptive λ-tracking and funnel control

In this section we show that the λ-tracker (4) and the funnel controller (5) applied to any linear system (A, B, C) of class $\mathcal{M}_{n,m}$ achieves, in presence of input/output disturbances (u_0, y_0) in $W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, in case of λ-tracking, or in $L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, in case of funnel control, resp., the control objectives: $\limsup_{t\to\infty} \operatorname{dist}(y_2(t), [-\lambda, \lambda]) = 0$, in case of λ-tracking, or y_2 evolves within a performance funnel $\mathcal{F}_\varphi$ for prespecified $\varphi \in \Phi$, in case of funnel control, and (in both cases) all signals and states of the closed-loops (2), (3), (4) and (2), (3), (5), resp., remain essentially bounded. Moreover, it is shown that the derivatives of the output signals y_1, y_2 and the state $\left(\begin{smallmatrix} y_1 \\ z \end{smallmatrix}\right)$ are essentially bounded, too.

Write, for $n, m \in \mathbb{N}$ with $n \geq m$,

$$\begin{aligned} \mathcal{D}^{\mathcal{L}}_{n,m} &:= \mathcal{M}_{n,m} \times (\mathbb{R}^m \times \mathbb{R}^{n-m}) \times \mathbb{R} \\ &\quad\times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \\ \mathcal{D}^{\mathcal{F}}_{n,m} &:= \mathcal{M}_{n,m} \times (\mathbb{R}^m \times \mathbb{R}^{n-m}) \times \Phi \\ &\quad\times L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m). \end{aligned}$$

Proposition 2.1: Let $n, m \in \mathbb{N}$ with $n \geq m$ and $\lambda > 0$. Then there exists a continuous map $\nu_{\mathcal{L}} \colon \mathcal{D}^\lambda_{n,m} \to \mathbb{R}_{\geq 0}$ such that, for all tuples $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), k^0, u_0, y_0\right) \in \mathcal{D}^{\mathcal{L}}_{n,m}$, the associated closed-loop initial value problem (2), (3), (4) satisfies

$$\|(u_2, y_2, z, k)\|_{W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{m+n+1})} \leq \nu_{\mathcal{L}}(d) \tag{7}$$

and

$$\limsup_{t\to\infty} \|y_2(t)\| \leq \lambda. \tag{8}$$

Furthermore, there exists a function $\nu_{\mathcal{F}} \colon \mathcal{D}^{\mathcal{F}}_{n,m} \to \mathbb{R}_{\geq 0}$, such that, for all tuples $e = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right) \in \mathcal{D}^{\mathcal{F}}_{n,m}$, the associated closed-loop initial value problem (2), (3), (5) satisfies

$$\|(k, u_2, y_2, \eta)\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^{1+m}) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^n)} \leq \nu_{\mathcal{F}}(e), \tag{9}$$

and

$$\forall\, t \geq 0 \;:\; (t, y_2(t)) \in \mathcal{F}_\varphi = \\ \{(t,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}\,. \tag{10}$$

Proof: The first statement, i.e. λ-tracking, is proved by [7, Prop. 2.1]. A proof for the second statement, i.e. funnel control, can be found in [8, Prop. 2.1]. ■

III. Robustness of λ-tracking and funnel control

In this section we show that the λ-tracker (4) and the funnel controller (5) are robust in the sense that one may apply these controllers to any stabilizable and detectable system which is "close" (in terms of a "small" gap) to any system in $\mathcal{M}_{m,n}$, as long as the initial conditions and the disturbances are "small".

A. The concept of the gap metric

We refer the reader to [3, Sec. 2], [8, Sec. 3] and mainly [19, Ch. 6] for a detailed outline of all required definitions for extended and ambient spaces, well posedness, the nonlinear gap, gain-functions and gain-function stability, which are required for the results on robust stability in Section III.

However, we recall some basic concepts which are required for the robustness analysis in this section. For signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, recall the definitions for local, global and regular well posedness: Assume that, for plant and controller operators $P\colon \mathcal{U}_a \to \mathcal{Y}_a$, $u_1 \mapsto y_1$, and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$, $y_2 \mapsto u_2$, resp., the closed-loop

$$[P, C] \,:\, y_1 = P u_1,\; u_2 = C y_2,\;\; \begin{array}{l} u_0 = u_1 + u_2 \\ y_0 = y_1 + y_2 \end{array} \tag{11}$$

corresponding to the closed-loop shown in Figure 1, has the existence and uniqueness property. For each $w_0 \in \mathcal{W}$, define $\omega_{w_0} \in (0, \infty]$, by the property

$$[0, \omega_{w_0}) := \bigcup_{(\hat{w}_1, \hat{w}_2) \in \mathcal{X}_{w_0}} \operatorname{dom}(\hat{w}_1, \hat{w}_2)$$

and $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with $\operatorname{dom}(w_1, w_2) = [0, \omega_{w_0})$, by the property: $(w_1, w_2)|_{[0,t)}$ solves (11) for all $t \in [0, \omega_{w_0})$. This construction induces the closed-loop operator

$$H_{P,C}\colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a,\;\; w_0 \mapsto (w_1, w_2).$$

The closed-loop system $[P, C]$, given by (11), is said to be:

- *locally well posed* if, and only if, it has the existence and uniqueness properties and the operator $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$, $w_0 \mapsto (w_1, w_2)$, is causal;
- *globally well posed* if, and only if, it is locally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W}_e \times \mathcal{W}_e$;
- *regularly well posed* if, and only if, it is locally well posed and

$$\forall\, w_0 \in \mathcal{W} \,:\, \Big[\omega_{w_0} < \infty \implies \big\| (H_{P,C} w_0)|_{[0,\tau)} \big\|_{\mathcal{W}_\tau \times \mathcal{W}_\tau} \to \infty \text{ as } \tau \to \omega_{w_0} \Big]. \tag{12}$$

To measure the distance between two plants P and P_1 it is necessary to find sets associated with the plant operators within some space where one may define a map which identifies the gap. These sets are the *graphs* of the operators: for the plant operator $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ define the *graph* $\mathcal{G}_P$ as

$$\mathcal{G}_P := \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \,\middle|\, u \in \mathcal{U},\, Pu \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

The essence of this section is the study of robust stability of λ-tracking and funnel control in a specific control context. Robust stability is the property that the stability properties of a globally well posed closed-loop system $[P, C]$ persists under "sufficiently small" perturbations of the plant. In other words, robust stability is the property that $[P_1, C]$ inherits the stability properties of $[P, C]$, when the plant P is replaced by any plant P_1 sufficiently "close" to P. In the present context, plants P and P_1 are deemed to be close if, and only if, their respective graphs are *close* in the gap sense of [5]. The nonlinear gap is defined as follows: Let, for signal spaces $\mathcal{U}$ and $\mathcal{Y}$, $\Gamma(\mathcal{U}, \mathcal{Y}) := \{ P\colon \mathcal{U}_a \to \mathcal{Y}_a \mid P \text{ is causal} \}$ and, for $P_1, P_2 \in \Gamma$, define the (possibly empty) set

$$\mathcal{O}_{P_1,P_2} := \{ \Phi\colon \mathcal{G}_{P_1} \to \mathcal{G}_{P_2} \mid \Phi \text{ is causal, surjective, and } \Phi(0) = 0 \}.$$

The *directed nonlinear gap* is given by

$$\vec{\delta}\colon \Gamma(\mathcal{U}, \mathcal{Y}) \times \Gamma(\mathcal{U}, \mathcal{Y}) \to [0, \infty],$$
$$(P_1, P_2) \mapsto \inf_{\Psi \in \mathcal{O}_{P_1,P_2}} \sup_{\substack{x \in \mathcal{G}_{P_1} \setminus \{0\} \\ \tau > 0}} \frac{\| T_\tau (\Psi - I)|_{\mathcal{G}_{P_1}}(x) \|_{\mathcal{W}}}{\| T_\tau x \|_{\mathcal{W}}},$$

with the convention that $\vec{\delta}(P_1, P_2) := \infty$ if $\mathcal{O}_{P_1,P_2} = \emptyset$.

We close this sub-section with an example. Define, for $\alpha, N, M > 0$, $x^0 \in \mathbb{R}$, $\tilde{x}^0 \in \mathbb{R}^3$ and any signal spaces for λ-tracking or funnel control (see Prop. 2.1), the plant operator

$$P_\alpha\colon \mathcal{U}_e \to \mathcal{Y}_e,$$
$$u_1 \mapsto y_1 = x,\; \dot{x} = \alpha x + u_1,\; x(0) = x^0, \tag{13}$$

and, for a minimal realization $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ of (6), the plant operator

$$P_{N,M,\alpha}\colon \mathcal{U}_e \to \mathcal{Y}_e,$$
$$\widetilde{u}_1 \mapsto \widetilde{y}_1 = \widetilde{c}\, x,\; \dot{x} = \widetilde{A}\, x + \widetilde{b}\, \widetilde{u}_1,\; x(0) = \widetilde{x}^0. \tag{14}$$

In [7, Sec. 3] it is shown that, for sufficiently large $M > 0$ and $N = 2M$, P_α is close to $P_{N,M,\alpha}$ in the sense

$$\limsup_{M \to \infty} \vec{\delta}(P_\alpha, P_{2M,M,\alpha}) = 0. \tag{15}$$

B. Well posedness of λ-tracking and funnel control

For $n, m \in \mathbb{N}$ with $n \geq m$, we may consider $\mathcal{M}_{n,m}$ and $\mathcal{P}_{n,m}$ as subspaces of the Euclidean space $\mathbb{R}^{n^2+2nm}$ by identifying a plant $\theta = (A, B, C)$ with a vector θ consisting of the elements of the plant matrices, ordered lexicographically. With normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$ and $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$, where x^0 is the initial value of a linear system (1), we associate the causal plant operator

$$P(\theta, x^0)\colon \mathcal{U}_a \to \mathcal{Y}_a, \quad u_1 \mapsto P(\theta, x^0)(u_1) := y_1\,, \tag{16}$$

where, for $u_1 \in \mathcal{U}_a$ with $\operatorname{dom}(u_1) = [0, \omega)$, we have $y_1 = cx$, x being the unique solution of (1) on $[0, \omega)$. Consider, for $\lambda > 0$, $k^0 \in \mathbb{R}$ and $\varphi \in \Phi$, the control strategies (4) and (5), resp., and associate the causal control operators, parameterized by λ and the initial value k^0 in case of the λ-tracker, and parameterized by φ in case of the funnel controller, resp., i.e.

$$C_{\mathcal{L}}(\lambda, k^0)\colon \mathcal{Y}_a \to \mathcal{U}_a, \quad y_2 \mapsto C_{\mathcal{L}}(\lambda, k^0)(y_2) := u_2. \tag{17}$$
$$C_{\mathcal{F}}(\varphi)\colon \mathcal{Y}_a \to \mathcal{U}_a, \quad y_2 \mapsto C_{\mathcal{F}}(\varphi)(y_2) := u_2. \tag{18}$$

Next we show that the closed-loop systems $[P(\theta, x^0), C_{\mathcal{L}}(\lambda, k^0)]$ and $[P(\theta, x^0), C_{\mathcal{F}}(\varphi)]$ of any plant $\theta \in \mathcal{P}_{n,m}$ of the form (1) and initial value $x^0 \in \mathbb{R}^n$ (with associated operator $P(\theta, x^0)$) and controller (4) (with associated operator $C_{\mathcal{L}}(\lambda, k^0)$ for $\lambda > 0$ and $k^0 \in \mathbb{R}$) or (5) (with associated operator $C_{\mathcal{F}}(\varphi)$ for $\varphi \in \Phi$) are regularly well posed. Furthermore we show that, for $\theta \in \mathcal{M}_{n,m}$, the closed-loop systems $[P(\theta, x^0), C_{\mathcal{L}}(\lambda, k^0)]$ and $[P(\theta, x^0), C_{\mathcal{F}}(\varphi)]$ are globally well posed and $(\mathcal{U} \times \mathcal{Y})$-stable, where we consider signal spaces $\mathcal{U} \times \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ for λ-tracking, and $\mathcal{U}\times\mathcal{Y} = L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$ for funnel control, resp.

Proposition 3.1: Let $n,m\in\mathbb{N}$ with $n\geq m$, $\lambda>0$, $k^0\in\mathbb{R}$, $\varphi\in\Phi$ and $(\theta,x^0)\in\mathcal{M}_{n,m}\times\mathbb{R}^n$. Then, for $\mathcal{U}\times\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$, plant operator $P(\theta,x^0)$ and λ-tracking operator $C_{\mathcal{L}}(\lambda,k^0)$, given by (16) and (17), resp., the closed-loop initial value problem $[P(\theta,x^0),C_{\mathcal{L}}(\lambda,k^0)]$, given by (2), (3), (4), is globally well posed and moreover $[P(\theta,x^0),C_{\mathcal{L}}(\lambda,k^0)]$, is $\mathcal{U}\times\mathcal{Y}$-stable. Furthermore, for $\mathcal{U}\times\mathcal{Y} = L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$ and funnel control operator $C_{\mathcal{F}}(\varphi)$, given by (18) the closed-loop initial value problem $[P(\theta,x^0),C_{\mathcal{F}}(\varphi)]$, given by (2), (3), (5), is globally well posed and moreover $[P(\theta,x^0),C_{\mathcal{F}}(\varphi)]$, is $\mathcal{U}\times\mathcal{Y}$-stable.

Proof: The proposition is a direct consequence of Prop. 2.1. ∎

We will show that an application of the λ-tracker or controller to any stabilizable and detectable linear system (A,B,C) yields a closed-loop system which is regularly well posed. This is required for the robustness analysis in the next Sub-section C, namely the application of [19, Th. 6.5.3 and Th. 6.5.4]. Note that, for $(A,B,C)\in\mathcal{P}_{n,m}$, $x^0\in\mathbb{R}^n$, $\lambda>0$, $k^0\in\mathbb{R}$ and $\varphi\in\Phi$, the closed-loop initial value problems (1), (3), (4) and (1), (3), (5) may be written as

$$\left.\begin{array}{l}
\dot x(t) = Ax(t)+B[u_0(t)-u_2(t)], \quad x(0)=x^0,\\
\text{either}\\
\dot k(t) = \operatorname{dist}(y_2(t),[-\lambda,\lambda])\,|y_2(t)|, \quad k(0)=k^0,\\
\text{for } \lambda\text{-tracking, or}\\
k(t) = \dfrac{\varphi(t)}{1-\varphi(t)\|y_2(t)\|}, \text{ for funnel control,}\\
y_2(t) = y_0(t)-Cx(t),\\
u_2(t) = -k(t)y_2(t).
\end{array}\right\} \quad (19)$$

Proposition 3.2: Let $n\in\mathbb{N}$ with $n\geq m$, $\lambda>0$, $k^0\in\mathbb{R}$, $\varphi\in\Phi$, $(\theta,x^0)\in\mathcal{P}_{n,m}\times\mathbb{R}^n$. Then, for plant operator $P(\theta,x^0)$ and λ-tracking operator $C_{\mathcal{L}}(\lambda,k^0)$, given by (16) and (17), resp., and for $\mathcal{U}\times\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$, the closed-loop initial value problem $[P(\theta,x^0),C_{\mathcal{L}}(\lambda,k^0)]$, given by (19), has the following properties:

(i) there exists a unique solution $x\colon[0,\omega)\to\mathbb{R}^n$, for some $\omega\in(0,\infty]$, and the solution is maximal in the sense that for every compact $\mathcal{K}\subset\mathbb{R}_{\geq 0}\times\mathbb{R}^n$ exists $t\in[0,\omega)$ such that $(t,x(t))\notin\mathcal{K}$;

(ii) if $k\in W^{1,\infty}([0,\omega)\to\mathbb{R})$, then $\omega=\infty$;

(iii) if $y_2\in W^{1,\infty}([0,\omega)\to\mathbb{R}^m)$, then $\omega=\infty$;

(iv) $[P(\theta,x^0),C_{\mathcal{L}}(\lambda,k^0)]$ is regularly well posed.

Furthermore, for the funnel control operator $C_{\mathcal{F}}(\varphi)$, given by (18) and for $\mathcal{U}\times\mathcal{Y} = L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$, the closed-loop initial value problem $[P(\theta,x^0),C_{\mathcal{F}}(\varphi)]$, given by (19), has the properties:

(v) there exists a unique solution $x\colon[0,\omega)\to\mathbb{R}^n$, for some $\omega\in(0,\infty]$, and the solution is maximal;

(vi) if $(u_2,y_2)\in L^\infty([0,\omega)\to\mathbb{R}^m)\times W^{1,\infty}([0,\omega)\to\mathbb{R}^m)$, then $\omega=\infty$, $k\in L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R})$ and y_2 is uniformly bounded away from the funnel boundary $1/\varphi(\cdot)$;

(vii) $[P(\theta,x^0),C_{\mathcal{F}}(\varphi)]$ is regularly well posed.

Proof: Statements (i)–(iv) are proved by [7, Prop. 4.2] and statements (v)–(vii) are proved by [8, Prop. 4.2] ∎

C. Robustness

In Prop. 3.1 we have established that, for $(\theta,x^0)\in\mathcal{M}_{n,m}\times\mathbb{R}^n$, where $n,m\in\mathbb{N}$ with $n\geq m$, and $\lambda>0$, $k^0\in\mathbb{R}$ and $\varphi\in\Phi$ and corresponding signal spaces $\mathcal{U}$ and $\mathcal{Y}$, the closed-loop systems $[P(\theta,x^0),C_{\mathcal{L}}(\lambda,k^0)]$ (λ-tracking) and $[P(\theta,x^0),C_{\mathcal{F}}(\varphi)]$ (funnel control) are globally well posed and have certain stability properties.

The purpose of this sub-section is to determine conditions under which these properties are maintained when the plant $P(\theta,x^0)$ is perturbed to a plant $P(\widetilde\theta,\widetilde x^0)$ where $(\widetilde\theta,\widetilde x^0)\in\mathcal{P}_{q,m}\times\mathbb{R}^q$ for some $q\in\mathbb{N}$, $q\geq m$, in particular when $\widetilde\theta\notin\mathcal{M}_{q,m}$. Prop. 3.2 shows that the closed-loop systems $[P(\widetilde\theta,\widetilde x^0),C_{\mathcal{L}}(\lambda,k^0)]$ and $[P(\widetilde\theta,\widetilde x^0),C_{\mathcal{L}}(\varphi)]$ are regularly well posed. This provides the basis for our main result: Thm. 3.3 shows that stability properties of the λ-tracker and the funnel controller persist if (a) the plants $P(\widetilde\theta,0)$ and $P(\theta,0)$ are sufficiently close (in the gap sense) and (b) the initial data $\widetilde x^0$ and disturbance $w_0=(u_0,y_0)$ are sufficiently small. As a consequence $(\widetilde A,\widetilde B,\widetilde C)=\widetilde\theta\in\mathcal{P}_{q,m}$ may not even satisfy any of the classical assumptions: minimum phase, relative degree one and positive high-frequency gain.

Theorem 3.3: Let $n,q,m\in\mathbb{N}$ with $n,q\geq m$, $\theta\in\mathcal{M}_{n,m}$ and $\mathcal{W}=\mathcal{U}\times\mathcal{Y}$ for signal spaces $\mathcal{U}$ and $\mathcal{Y}$ specified in due course. For $(\widetilde\theta,\widetilde x^0)\in\mathcal{P}_{q,m}\times\mathbb{R}^q$ consider the associated operator $P(\widetilde\theta,\widetilde x^0)\colon\mathcal{U}_a\to\mathcal{Y}_a$ defined by (16).

Consider, for $\mathcal{U}\times\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$, and the λ-tracking operator $C_{\mathcal{L}}(\lambda,k^0)\colon\mathcal{Y}_a\to\mathcal{U}_a$ defined by (17), where $\lambda>0$ and $k^0\in\mathbb{R}$, the closed-loop initial value problem (1), (3), (4). Then there exist a continuous function $\eta_{\mathcal{L}}\colon(0,\infty)\to(0,\infty)$ and a functions $\psi_{\mathcal{L}}\colon\mathcal{P}_{q,m}\to(0,\infty)$ such that the following holds:

$$\forall\,(\widetilde\theta,\widetilde x^0,w_0,r)\in\mathcal{P}_{q,m}\times\mathbb{R}^q\times\mathcal{W}\times(0,\infty)\ :$$
$$\left.\begin{array}{r}\psi_{\mathcal{L}}(\widetilde\theta)\|\widetilde x^0\|\\ +\|w_0\|_{\mathcal{W}}\leq r\\ \vec\delta\big(P(\theta,0),P(\widetilde\theta,0)\big)\\ \leq\eta_{\mathcal{L}}(r)\end{array}\right\}\ \Rightarrow\ \left\{\begin{array}{l}\limsup_{t\to\infty}\|y_2(t)\|\leq\lambda,\\ k\in W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}),\\ x\in W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^q),\end{array}\right.$$

where (x,k) and y_2 satisfy (19) in case of λ-tracking.

Consider, for $\mathcal{U}\times\mathcal{Y} = L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$, and the funnel control operator $C_{\mathcal{F}}(\varphi)\colon\mathcal{Y}_a\to\mathcal{U}_a$ defined by (17), where $\varphi\in\Phi$, the closed-loop initial value problem (1), (3), (5). Then there exist a continuous function $\eta_{\mathcal{F}}\colon(0,\infty)\to(0,\infty)$ and a function $\psi_{\mathcal{F}}\colon\mathcal{P}_{q,m}\to(0,\infty)$ such that the following holds:

$$\forall\,(\widetilde\theta,\widetilde x^0,w_0,r)\in\mathcal{P}_{q,m}\times\mathbb{R}^q\times\mathcal{W}\times(0,\infty)\ :$$
$$\left.\begin{array}{r}\psi_{\mathcal{F}}(\widetilde\theta)\|\widetilde x^0\|\\ +\|w_0\|_{\mathcal{W}}\leq r\\ \vec\delta\big(P(\theta,0),P(\widetilde\theta,0)\big)\\ \leq\eta_{\mathcal{F}}(r)\end{array}\right\}\ \Rightarrow\ \left\{\begin{array}{l}\forall\,t\geq 0\ :\ (t,y_2(t))\in\mathcal{F}_\varphi,\\ k\in L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}),\\ x\in W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^q)\,,\end{array}\right.$$

where (x, k) and y_2 satisfy (19) in case of funnel control.

Proof: The first part, namely robustness of λ-tracking, is shown in [7, Thm. 4.5]. The second statement, namely robustness of funnel control, is shown in [8, Thm. 4.5]. ∎

Note that Thm. 3.3 is proved by first showing gain-function stability of the so-called augmented closed-loops $[\widetilde{P}, \widetilde{C}_{\mathcal{L}}]$ and $[\widetilde{P}, \widetilde{C}_{\mathcal{F}}]$, see [7, Prop. 4.3] and [8, Prop. 4.3] for details, and secondly utilizing the robust stability results [19, Th. 6.5.3 and Th. 6.5.4] to show $\mathcal{U}\times\mathcal{Y}$-stability of the closed-loop systems $[P(\widetilde{\theta}, \widetilde{x}^0), C_{\mathcal{L}}(\lambda, k^0)]$ and $[P(\widetilde{\theta}, \widetilde{x}^0), C_{\mathcal{L}}(\varphi)]$ for a system $\widetilde{\theta}$ belonging to the system class $\mathcal{P}_{q,m}$ if, for a system θ belonging to $\mathcal{M}_{n,m}$, the gap between $P(\widetilde{\theta}, 0)$ and $P(\theta, 0)$, the initial value $\widetilde{x}^0 \in \mathbb{R}^q$ and the input/output disturbances $w_0 = (u_0, y_0)$ are sufficiently small, see [7, Prop. 4.4] and [8, Prop. 4.4].

Example 3.4: Finally, we revisit the example systems (13) and (14). We have already shown that for zero initial conditions the gap between the system $(\widetilde{A}, \widetilde{b}, \widetilde{c}) \in \mathcal{P}_{3,1} \setminus \mathcal{M}_{3,1}$ and $(\alpha, 1, 1) \in \mathcal{M}_{1,1}$ tends to zero as $N = 2M$ and M tends to infinity, see (15). Now we visualize the above theoretical result. Let $\lambda = 0.1$ and specify the funnel boundary $1/\varphi(\cdot)\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ by

$$t \mapsto 1/\varphi(t) = \begin{cases} 15.31 - 7.8\,t + t^2, & \text{if } t \in [0, 3.9) \\ \lambda, & \text{if } t \geq 3.9. \end{cases}$$

Then, for initial values $x^0 = 1$ for system (13) and $\widetilde{x}^0 = (0.1, 0.1, 0.08)^T$ for system (14) and input/output disturbances $u_0 = y_0 \equiv 0$, Figures 3 and 4 indicate that the λ-tracker and the funnel controller are applicable to a system which is "close" to a system which satisfies the classical assumptions for adaptive control, namely relative degree one, minimum phase and positive high-frequency gain.

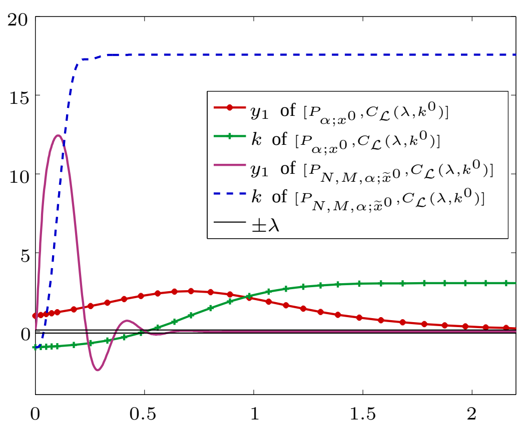

Fig. 3. $[P_{\alpha;x^0}, C_{\mathcal{L}}(\lambda, k^0)]$ and $[P_{N,M,\alpha;\widetilde{x}^0}, C_{\mathcal{L}}(\lambda, k^0)]$ with $k^0 = -1$ and $u_0 = y_0 \equiv 0$

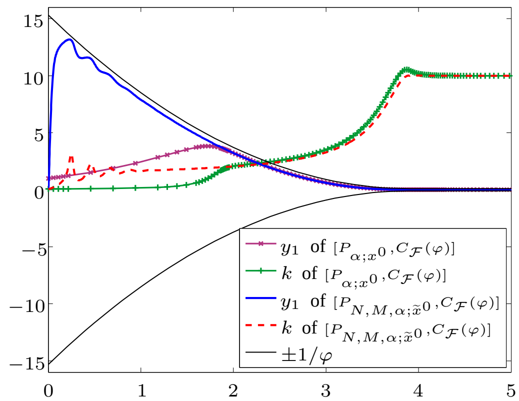

Fig. 4. $[P_{\alpha;x^0}, C_{\mathcal{F}}(\varphi)]$ and $[P_{N,M,\alpha;\widetilde{x}^0}, C_{\mathcal{F}}(\varphi)]$ with $u_0 = y_0 \equiv 0$

REFERENCES

[1] C. I. Byrnes and J. C. Willems, "Adaptive stabilization of multivariable linear systems," in *Proc. 23rd IEEE Conf. Decis. Control*, 1984, pp. 1574–1577.

[2] M. French, "Adaptive control and robustness in the gap metric," *IEEE Trans. Autom. Control*, vol. 53, no. 2, pp. 461–478, 2008.

[3] M. French, A. Ilchmann, and M. Mueller, "Robust stabilization by linear output delay feedback," *SIAM J. Control Optim.*, vol. 48, no. 4, pp. 2533–2561, 2009.

[4] M. French, A. Ilchmann, and E. P. Ryan, "Robustness in the graph topology of a common adaptive controller," *SIAM J. Control Optim.*, vol. 45, no. 5, pp. 1736–1757, 2006.

[5] T. T. Georgiou and M. C. Smith, "Robustness analysis of nonlinear feedback systems: An input-output approach," *IEEE Trans. Autom. Control*, vol. 42, no. 9, pp. 1200–1221, 1997.

[6] N. Hopfe, A. Ilchmann, and E. P. Ryan, "Funnel control with saturation: linear MIMO systems," 2009, provisionally accepted, http://www.tu-ilmenau.de/fakmn/5980+M54099f70862.0.html.

[7] A. Ilchmann and M. Mueller, "Robustness of λ-tracking in the gap metric," *SIAM J. Control Optim.*, vol. 47, no. 5, pp. 2724–2744, 2008.

[8] ——, "Robustness of funnel control in the gap metric," 2009, provisionally accepted, http://www.tu-ilmenau.de/fakmn/5592+M54099f70862.0.html.

[9] A. Ilchmann and E. P. Ryan, "Universal λ-tracking for nonlinearly-perturbed systems in the presence of noise," *Automatica*, vol. 30, no. 2, pp. 337–346, 1994.

[10] ——, "High-gain control without identification: a survey," *GAMM Mitt.*, vol. 31, no. 1, pp. 115–125, 2008.

[11] A. Ilchmann, E. P. Ryan, and C. J. Sangwin, "Tracking with prescribed transient behaviour," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 7, pp. 471–493, 2002.

[12] A. Ilchmann, E. P. Ryan, and P. Townsend, "Tracking with prescribed transient behavior for nonlinear systems of known relative degree," *SIAM J. Control Optim.*, vol. 46, no. 1, pp. 210–230, 2007.

[13] A. Ilchmann and H. Schuster, "PI-funnel control for two mass systems," *IEEE Trans. Autom. Control*, vol. 54, no. 4, pp. 918–923, 2009.

[14] A. Isidori, *Nonlinear Control Systems*, 3rd ed., ser. Communications and Control Engineering Series. Berlin: Springer-Verlag, 1995.

[15] I. M. Y. Mareels, "A simple selftuning controller for stably invertible systems," *Syst. Control Lett.*, vol. 4, no. 1, pp. 5–16, 1984.

[16] B. Mårtensson, "The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization," *Syst. Control Lett.*, vol. 6, no. 2, pp. 87–91, 1985.

[17] D. E. Miller and E. J. Davison, "An adaptive controller which provides an arbitrarily good transient and steady-state response," *IEEE Trans. Autom. Control*, vol. 36, no. 1, pp. 68–81, 1991.

[18] A. S. Morse, "Recent problems in parameter adaptive control," in *Outils et Modèles Mathématiques pour l'Automatique, l'Analyse de Systèmes et le Traitment du Signal*, I. D. Landau, Ed. Paris: Éditions du Centre National de la Recherche Scientifique (CNRS), 1983, vol. 3, pp. 733–740.

[19] M. Mueller, "Output feedback control and robustness in the gap metric," Ph.D. dissertation, Faculty of Mathematics and Natural Sciences, Ilmenau University of Technology, 2009.

[20] J. C. Willems and C. I. Byrnes, "Global adaptive stabilization in the absence of information on the sign of the high frequency gain," in *Analysis and Optimization of Systems, Proc. of the 6th INRIA Conference, Nice, France*, ser. Lecture Notes in Control and Information Sciences, A. Bensoussan and J. L. Lions, Eds. Berlin: Springer-Verlag, 1984, no. 62, pp. 49–57.