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Title
Flow-chart proofs with open problems as scaffolds for learning about geometrical proofs

Concise and informative title
Flow-chart proofs with open problems as scaffolds for learning

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Abstract
Recent research on the scaffolding of instruction has widened the use of the term to include forms of support for learners provided by, amongst other things, artefacts and computer-based learning environments. This paper tackles the important and under-researched issue of how mathematics lessons in junior high schools can be designed to scaffold students' initial understanding of geometrical proofs. In order to scaffold the process of understanding the structure of introductory proofs, we show how flow-chart proofs with multiple solutions in 'open problem' situations are a useful form of scaffold. We do this by identifying the 'scaffolding functions' of flow-chart proofs with open problems through the analysis of classroom-based data from a class of Grade 8 students (aged 13-14 years old) and quantitative data from three classes. We find that using flow-chart proofs with open problems support students' development of a structural understanding of proofs by giving them a range of opportunities to connect proof assumptions with conclusions. The implication is that such scaffolds are useful to enrich students' understanding of introductory mathematical proofs.

Keywords
scaffolding, proving, flow-chart proof, open problem, geometry

1. Introduction

The notion of instructional scaffolding is generally traced back to the work of Wood, Bruner and Ross (1976). They describe scaffolding as a process where "the 'adult' [controls] those elements of the task that are initially beyond the learner's capacity, thus permitting him [*sic*] to concentrate upon and complete only those elements that are within his [*sic*] range of competence" (p. 90). Since its introduction, this idea has been playing one of the very important roles in educational practice. Research has studied scaffolding in various contexts, from support provided by an expert to support provided by artefacts or computer-based learning environments. In the literature (e.g. Yelland and Masters, 2007; Smit, van Eerde and Bakker, 2013; Belland, 2014), scaffolding has been discussed within another influential notion, that of the 'zone of proximal development' (ZPD), originally conceptualised by Vygotsky (1978) as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers" (p. 86). These two notions are generally recognised as something intertwined with each other as they collectively suggest that we need to identify learning progressions which novice learners have to take in order to master certain concepts, and that we need to design instructional interventions in order to scaffold such progressions.

In a comprehensive review, Belland (2014, p. 514) conclude that there remain a number of unanswered questions about scaffolding, including the extent to which scaffolds that are specific to the knowledge context that are helpful to the learner. In this paper, we explore this issue in the context of introductory proof lessons in junior high schools as students often experience difficulties in understanding and constructing formal proofs (e.g. Hanna and de Villiers, 2012; Mariotti, 2006). The issue that we intend to address in this paper is how to identify the ZPD and what scaffolds could be made available in order to support students in developing their capability to undertake the construction of formal proofs independently. This issue remains under-researched. Bieda (2010), for example, states that in order to improve the current situation we need to seek "a best-case scenario: classrooms using a curriculum with rich opportunities for students to justify and prove, taught by teachers who are experienced with the curriculum and attending ongoing professional development" (p. 353). In relation to rich opportunities for students, Otten *et al.* (2014) suggest that the intellectual necessity of proof, "transitioning from more informal ways of reasoning to formal mathematical proof" (p. 108), should be explicitly presented in the early on in the proof learning process by asking students to consider what it means to construct geometrical proofs in mathematics.

In researching the ZPD and scaffolding of proof learning, the logical structure of proofs needs to be considered seriously because it is one of the most severe difficulties students experience when learning to construct geometrical proofs (Durand-Guerrier, et al, 2012). In Miyazaki and Fujita (2010), we show that in order for students to actively engage in proving in geometry they need to understand the structure of proofs, which consists of the following elements: singular propositions (premises, conclusions, intermediate conditions, etc.), universal propositions (theorems, definitions etc.), and forms of deductive reasoning. We argue that it is in the initial stage of learning about proofs that the foundation of the understanding of the structure of proofs needs to be established. Here, our pedagogical idea is that flow-chart proof tasks in 'open problem' situations allow learners to construct multiple solutions by making the necessary assumptions and intermediate propositions to deduce a given conclusion in a flow-chart format. One topic where this approach could be particularly valuable is proofs that involve the use of the conditions for congruent triangles, as this topic is often used to introduce formal proofs in geometry in junior high schools in Japan (Jones and Fujita, 2013).

In using flow-chart proof tasks involving 'open problem' situations designed especially for the understanding of the structure of proofs, our focus is on the way in which this can be regarded as a form of domain-specific 'scaffold' within the context of learning geometrical proofs. The purpose of this paper is to provide an analysis to demonstrate how the provision of flow-chart proofs with open problems functions to scaffold students' understanding of formal proofs in school geometry. It is important to focus on scaffolding functions because these are indicators of the effectiveness of scaffolds (e.g. Wood, et al, 1976; Sherin, Reiser and Edelson, 2004). In particular we address the following research questions:

- To what extent does the provision of flow-chart proofs with open problems help to scaffold the structural understanding of formal proofs in junior high school geometry?

- What scaffolding functions can be identified when teaching introductory geometrical proofs by flow-chart proofs with open problems?

In what follows we first identify key ideas relevant to scaffolding, and then conceptualise the ZPD in the context of the structure of proofs by introducing the levels of understanding based on our related study (Miyazaki and Fujita, 2010). These levels of understanding are used to identify the status of novice learners' understanding and why flow-chart proofs with open problems might function to scaffold their learning. We then apply the 'scaffolding analysis' framework proposed by Sherin *et al.* (2004) as a way to identify "how the additional features of the scaffolded situation lead to changes in performance along a particular dimension" (p. 388). This analytical framework is particularly suitable for addressing our research questions because, as Sherin *et al.* argue, scaffolded situations can be analysed in terms of 'implicit comparison', 'consistency', and 'an analysis of function'. We use the framework to identify functions of flow-chart proofs with open problems in scaffolding the structural understanding of formal proofs. We then examine this scaffolding empirically using qualitative data from our classroom teaching experiments with Grade 8 students (13-14 years old).

2. Scaffolds for domain-specific knowledge

Since Wood, Bruner and Ross (1976) began using the notion of scaffolding, it has been used in a range of educational context to conceptualise ways to support learning. Compared to its original definition, recent views utilise a wider view of instructional scaffolds to include features of technology-based learning environments. For example, Saye and Brush (2001) define scaffolds as "tools, strategies, and guides which support students in attaining a higher level of understanding; one which would be impossible if students worked on their own" (p. 334). Yelland and Masters (2007) distinguish scaffolds that are 'cognitive' (e.g. the use of questions, modelling, assisting with making plans, etc.), 'technical' (e.g. working with computers), and 'affective' (e.g. encouraging higher order thinking). Molenaar *et al.* (2012) distinguish scaffolds that are 'static' ("one may provide a list of instructions that helps users to perform a learning activity") or 'dynamic' ("one can monitor the progress of the student and provide scaffolds when needed in the learning process") (p. 516). In particular, they suggest that compared to scaffolds that are 'static', 'dynamic' scaffolds have a positive effect on students' learning performance but do not have an impact on their domain-specific knowledge of geography.

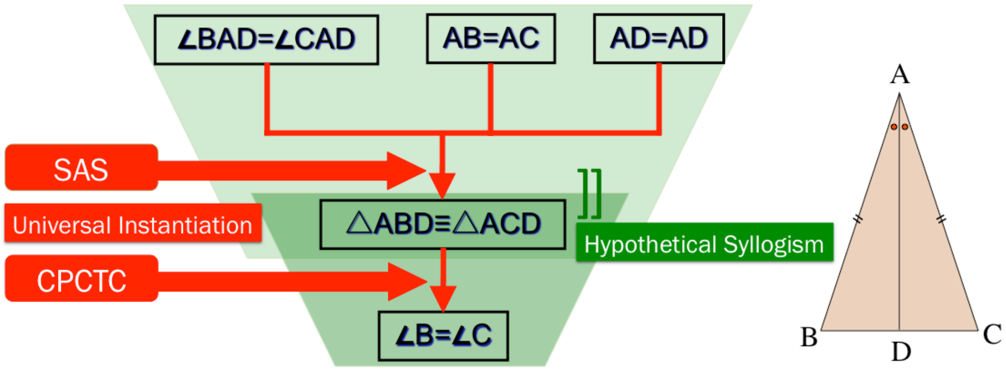
Sharma and Hannafin (2007) consider scaffolding to be "a two-step process of supporting the learner in assuming control of learning and task completion" (p.29). In the first of the two steps, the learner is provided with "appropriate support to identify strategies for accomplishing individually-unattainable learning goals or tasks" (ibid); in the second step the assistance gradually fades as the learner becomes increasingly competent. We mainly follow this notion of scaffolding in this paper, that is, we take scaffolding to be a 'two-step process' that includes 'cognitive', 'technical', and 'affective' supports in order to support learners to achieve their goals. In particular, based on Belland's (2014) comments regarding the need for further research in scaffolds that are specific to the knowledge context, we investigate the features can be used to design scaffolds that provide effective support for students' domain-specific knowledge about introductory proofs in junior high school mathematics.

3. ZPD of students' understanding of the structure of proofs

In order to seek strategies for accomplishing individually-unattainable learning goals, the idea of the ZPD is useful in relation to domain-specific knowledge. Amongst numerous studies into the ZPD, Rowlands (2003) argue that the ZPD can be seen as Vygotsky's method of "ascending from the abstract to the concrete." (p. 164). In the context of learning to geometrical proofs in junior high schools, one of the abstract ideas is the logical structure of proofs. Thus in this section we start from elucidating the structure of proofs and then we introduce the levels of understanding in order to conceptualise the distance between the novice learners' understanding and the goals of their learning. In Miyazaki and Fujita (2010) we provide detailed theoretical arguments for the levels of understanding, accompanied by suitable empirical evidence. In this paper, we concentrate on selected elements of our theoretical ideas in the context of ZPD and scaffolding.

3.1. Structure of proofs

Duval (2002) propose that it is essential to distinguish between premises, conclusions, and theorems in the construction of proofs. In particular, Duval argue that learners need to "[become] aware of the discrepancy between a valid reasoning and a non-valid reasoning" (p. 63). Students need to begin by recognising a proof as a structural 'object' (Miyazaki and Yumoto, 2009). Essentially, seeing proofs as an object enables students to appreciate the elements of a proof, the inter-connections between these elements, and the roles these elements play in the structure of a proof. Figure 1 illustrates a geometrical proof that is commonly found in junior high schools: if AB=AC in ∆ABC, then ∠ABD=∠ACD. This problem requires students to use universal instantiations and hypothetical syllogisms to demonstrate the congruency of two angles of an isosceles triangle.



1) A singular proposition, (i), 'If AB=AC, AD=AD, and ∠BAD=∠CAD, then ΔABD≡ΔACD', is deduced by universal instantiation of the congruency theorem (SAS, a universal proposition)
2) A singular proposition, (ii), 'if ΔABD≡ΔACD then ∠ABD=∠ACD', is deduced by universal instantiation of the universal proposition 'In congruent triangles all corresponding interior angles are equal' (CPCTC);
3) These two propositions (i) and (ii) are connected by a hypothetical syllogism, and we obtain 'If AB=AC, AD=AD, and ∠BAD=∠CAD, then ∠ABD=∠ACD', which is equivalent to the singular proposition to be proved.

Fig. 1: Proof of 'the base angles of isosceles triangles are equal'

We can see that in a formal proof, singular and universal propositions are connected with two types of reasoning: universal instantiations and hypothetical syllogisms. Considering this, we define the 'structure of a proof' as the relational network that combines singular and universal propositions with these two types of deductive reasoning.

3.2 Levels of understanding of proof structures

In order to understand the structure of a proof, students need to pay attention to the elements of the proof and their inter-relationships. Yet questions remain about the process they would undergo in order to develop their understanding of proofs with highly symbolic complex structures. In the context of students reading formal proofs found in textbooks, Lin and Yang (2008) propose the model of Reading Comprehension of Geometry Proofs (RCGP). This model hypothesizes four levels of reading comprehension of geometric proofs (Yang and Lin, 2008, p. 63). By refining and adjusting the ideas in the RCGP, our model of levels of understanding of proof structures considers the progression of understanding of proof structures from initially recognising individual elements to later on recognising their inter-relationships. In this way, students first need to pay attention to the elements of a proof (such as the premises, the conclusions, and the singular propositions to be used), then the inter-relationships between these elements, and eventually they gradually grasp the relational network of the structure of simple proofs (such proofs being ones suitable for high schools).

In our model, the ‘Pre-structural’ level is the most primitive status in terms of understanding the structure of a proof. At this level, students regard proofs as ‘clusters’ of meaningless symbolic objects. As such, when they construct a proof, they fail to see within the structure of the proof that singular propositions are those which are universally instantiated from universal propositions, and that hypothetical syllogisms are necessary to connect singular propositions, etc. When students at this level are asked a question about universal propositions, for example, ‘What do we need to deduce about $\Delta ABD \equiv \Delta ACD$ in proving the theorem ‘base angles of isosceles triangles are equal’?’ (Fig. 1), they would not understand what they are being asked, or they may simply answer with a singular proposition ‘$BD=CD$?’.

Once students begin to pay attention to each of the elements then we consider them to be at the Partial-structural Elemental sub-level (where ‘elemental’ refers to the elements of proof). This level echoes the ‘Recognizing elements’ level of the RCGP. However, being able to recognise elements of proofs is not enough to construct valid proofs; a student at this level still needs to recognize the logical relationships between the elements of a proof (c.f. Yang and Lin, 2008, p. 63). In order to understand the relationships, we argue that both universal instantiations and hypothetical syllogisms are important, something which is not fully acknowledged in the RCGP. Once students begin to pay attention to these forms of reasoning, then we consider them to be at the Partial-structural Relational sub-level. For example, if a student understands universal instantiations, then when he/she is asked a question such as ‘In ΔABD and ΔACD (Fig. 1), $AB=AC$ is already assumed; what additional premises should be made to prove $\Delta ABD \equiv \Delta ACD$?’, the student is able to answer by stating, for example, ‘In order to use the condition of congruent triangles, $\angle BAD=\angle CAD$ and $\angle ABD=\angle ACD$ are needed.’

At the Partial-structural Relational sub-level, there exist students who understand only either hypothetical syllogism or universal instantiations. At this level, students may be able to use theorems and specify each element of proofs (universal instantiations), but may construct or accept a proof with logical circularity due to their insufficient understanding of hypothetical syllogisms.. Conversely, some students may understand the syllogisms but not universal instantiations. For example, suppose a teacher notices that a student writes a proof but does not show which congruency condition (SSS, ASA and SAS) is used, asks the student to prove another property of the triangle. If this student cannot specify the condition he/she used, then we take this to mean that the student is able to use hypothetical syllogisms but unable to use universal instantiations.

Once a student sees a proof as 'whole' (c.f. Yang and Lin, 2008, p. 63), where premises and conclusions are logically connected via universal instantiations and hypothetical syllogisms, we consider them to have reached the 'Holistic-structural' level. After reaching the 'Holistic-structural' level, students can start reconstructing proofs that they have been shown, become aware of the hierarchical relationships between theorems, be able to construct their own proofs, and so on. Our overall framework is illustrated in Figure 2.

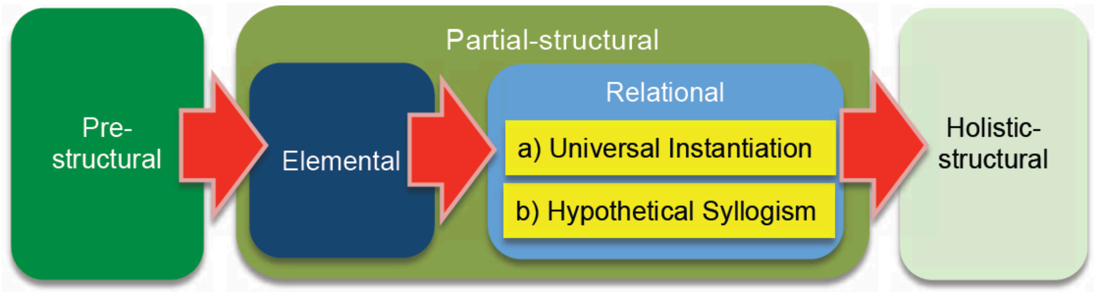


Figure 2: Framework of learner understanding of the structure of proofs

3.3 Necessity of scaffolds to enhance the understanding of proof structures

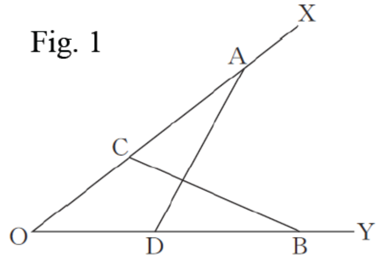
National assessments in Japan have repeatedly made clear that even when junior high school students have been taught mathematical proofs, half or more cannot construct simple formal proofs (e.g. MEXT 2014). For example, Figure 3 shows one of the advanced problems in Maths B.

Takuya is trying to solve the following problem.

Problem

In fig. 1, let us take points A, B, C and D on OX and OY of ΔXOY so that OA = OB and OC = OD. When A and D, and B and C are connected, prove AD = BC.

Fig. 1

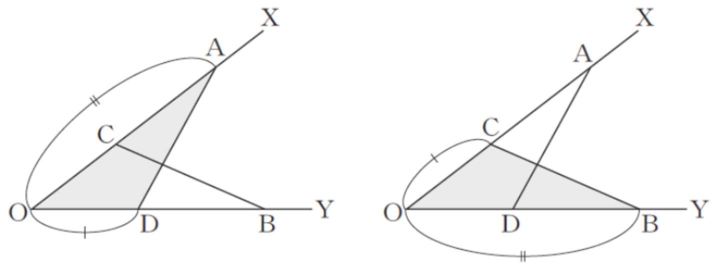


Takuya described his plan to prove it as follows.

Takuya's memo

#1 To prove AD = BC, it is enough to show ΔAOD ≡ΔBOC.

#2 When I see ΔAOD and ΔBOC of Fig.1 more clearly, I can divide it into two parts and show what are assumed as follows.



#3 Based on #2, I think I can prove ΔAOD ≡ΔBOC.

(1) Which property should be used to say ‘in order to prove AD=BC, it is enough to show ΔAOD ≡ΔBOC’ as seen in #1 of Takuya's memo? Choose from a)-d).

a) In congruent figures, corresponding sides are equal.
b) In congruent figures, corresponding angles are equal.
c) In congruent figures, perimeters are equal.
d) In congruent figures, areas are equal.

(2) Prove AD＝BC of **Problem**.

Figure 3 Advanced geometry problems for Grade 8 students

For this problem, Question 1 checks if students can reason backwards from the conclusion AD=BC or not, and Question 2 asks students to construct a formal proof by reference to ‘*Takuya’s memo*’ in Figure 3. This plan shows that ΔAOD≡ΔBOC is adequate to deduce the conclusion AD=BC (#1), that each of two pairs of sides (as the given conditions) are equal (#2), and that it might be possible to deduce ΔAOD≡ΔBOC from the given conditions (#3). The national survey reveals that while 63.3% of students answer Q1 correctly, only 34.2% of them could write a correct proof, despite the information given in #1-3. From the levels of the structure of proofs point of view, one possible interpretation is that about 40% of students seem to reach, but remain at, the Partial-structural Elemental sub-level because they do not fully understand how to use ΔAOD≡ΔBOC to deduce the conclusion. Furthermore, about 65% of students fail to reach the Holistic levels to utilise the given information to construct formal proofs in geometry. One implication of this is that many students need further support in order to advance from the Partial-structural to Holistic level in their proof learning.

In order to improve the situation, it would be helpful for students and teachers if forms of instructional scaffold are developed to enable students to shift from the Partial-structural to the Holistic level. This shift in levels implies the need for a scaffold that can assist students in identifying universal instantiation and hypothetical syllogism correctly. As previously discussed, we consider scaffolding as a two-step process which includes cognitive, technical, and affective supports in order to support learners to achieve their goals (Sharma and Hannafin, 2007; Yelland and Masters, 2007). Therefore, we need to identify the kinds of strategies to help students accomplish individually-unattainable learning goals or tasks, and to identify the process to withdraw such supports from the learning process. Rittle-Johnson and Koedinger (2005, p. 342) claim that the following three design suggestions can be useful: a) the use of story contexts, b) the use of visual representations and c) intermediating procedural steps and then removing the scaffolding. We utilise these suggestions when designing suitable scaffolds in our research project.

4. Scaffolds to support the structural understanding of proofs in geometry

In this section we introduce flow-chart proofs with open problems and explain why this is a promising form of instructional scaffold for learning to construct geometrical proofs. First we describe flow-chart proofs with open problems as a scaffold and then give a detailed analysis of the scaffolding functions using the approach proposed by Sherin *et al.* (2004).

4.1. Flow-chart proofs with open problems

For the understanding of the structure of proofs, one key idea is to use flow-chart proofs that shows a ‘story line’ of the proof by visualising the structure, beginning with the kinds of assumptions from which the conclusion is deduced, including the kinds of theorems being used, deciding how the assumptions and conclusion are connected, and so on. As McMurray (1978) and others suggest, flow-chart proofs can be introduced to students before they learn the more formal ‘two column proof’ format.

We add another important pedagogical idea for formulating questions for students using open problems, where students can construct multiple solutions by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion.

For example, the problem in Figure 4 is intentionally designed so that students can freely choose which assumptions they use to draw the conclusion ∠B=∠C. After drawing a line AO, for instance, students might think backwards from the conclusion to decide which triangles should be congruent to show ∠B=∠C, and what condition of congruent figures should be used. Then, they might show that ∠B=∠C by using the theorem, 'If two figures are congruent, then corresponding angles are equal'. However, other solutions are also possible. One alternative solution might be to use the fact that they have already found AO=AO as the same line and hence ΔABO≡ΔACO can be shown by assuming AB=AC and BO=CO using the SSS condition. As students can construct more than one suitable proof, we refer to this type of problem as 'open'.

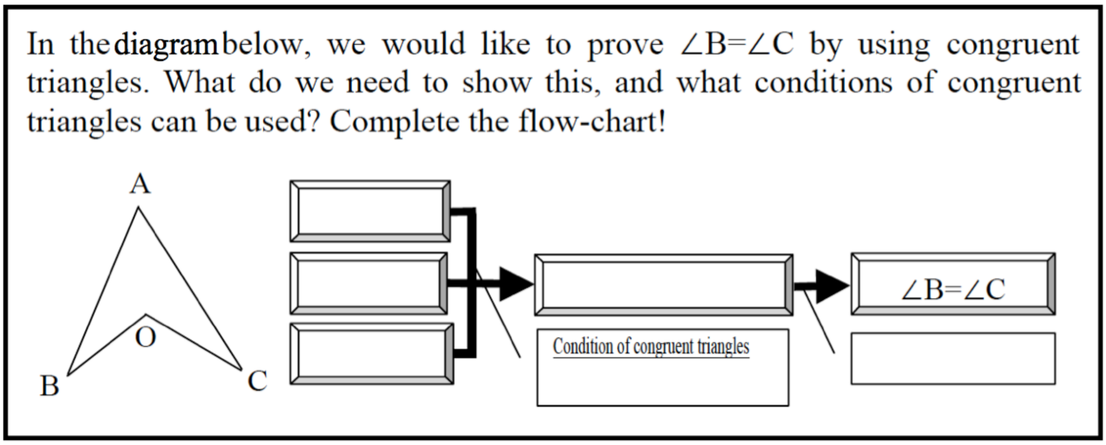


Figure 4: An example of flow-chart proofs with open problems

Now we need to consider why the use of flow-chart proofs with open problems provides scaffolding of understanding of the structure of proofs, in particular supporting the transition from the Part-structural to the Holistic level of thinking.

4.2. Scaffolding functions of flow-chart proofs with open problems to understand the structure of proofs

While evidence noted above points to many students having difficulty in constructing formal proofs, these students are able to identify some elements of a proof, such as relevant sides, angle, their equalities, etc. This indicates that these students reach the Partial-structural Elemental sub-level of understanding of proof structure. At the Relational level, they start to understand universal instantiations and hypothetical syllogisms. We consider that, by using flow-chart proofs with open problems, students' learning of the structure of formal proofs can be enhanced by providing ways of visualising two kinds of deductive reasoning (viz: universal instantiations and hypothetical syllogisms) and their combination in a flow-chart proof format. Additionally, open problems would encourage students to seek different proofs and this is likely to promote students to think forward/backward interactively when constructing a proof using the flow-chart format.

In order to clarify the features of the use of flow-chart proofs with open problems as a scaffold, we adopt the scaffolding analysis proposed by Sherin *et al.* (2004). They do not view 'scaffolds' as 'features' of artefacts or situations nor do they regard scaffolding as "something that may be occurring (or not) in a given situation that we observe" (p. 388); rather, they argue that their 'scaffolding analysis' should be useful for the analysis of the "design rationale" of research and of quasi-experimental and descriptive "empirical work" (p. 398). By using this framework, we can identify the scaffolding functions of flow-chart proofs with open problems.

Sherin *et al.* (pp. 392-398) explain four attributes of a scaffolding analysis: 1) there is an implicit comparison (comparison with/without open problem tasks); 2) something is held consistent; 3) the task is an expert task, and the support will ultimately fade; and 4) there is an analysis of the functions of the scaffold. Based on these attributes (with the exception of attribute 3), they propose that a scaffolding analysis can be framed by examining the three components: 'Situation with/without scaffolding', 'Target performance', and 'Scaffolding function'. We consider that this framework can be used to scrutinize the scaffold provided by flow-chart proofs with open problems because of the following reasons. First, learning with flow-chart proofs with open problems is very different from 'ordinary' lessons which do not use open problems. Second, the use of flow-chart proofs with open problems has a clear learning goal - the understanding of formal proofs in geometry. Finally, our research aims to identify scaffolding functions that bridge the gap between students' levels of thinking from Partial-structural to Holistic levels.

Using the scaffolding analysis framework, we analyse the use of flow-chart proofs with open problems for the understanding of the structure of proofs in junior high school mathematics. In this analysis we use the 'Target Performance' from the national 'Course of Study' in Japan for Grade 8 Geometry to identify what is expected in terms of the structure of formal proofs and the way to construct proofs. Concerning 'Situations without scaffolding', students in Japan usually learn how to construct (simple) formal proofs by following the teaching sequence suggested by an approved textbook. In such teaching there is usually little chance to utilise flow-chart proofs with any open problems. As such, learning is generally restricted to how to construct a formal proof within closed problems (e.g. ordinary proof tasks with only one answer) without using flow-chart proofs. Considering that even ordinary proof tasks have a fairly complex logical structure and that students' performance can be the somewhat deficient (as shown by the National Survey data from Japan, presented in section 3.3 of this paper), 'Situations without scaffolding' have a limited impact on students' understanding.

In contrast, concerning 'Situations with scaffolding', students learn how to construct (simple) formal proofs through constructing flow-chart proofs in open problems. By considering the features of the use of flow-chart proofs with open problems we find the following 'scaffolding functions':

- F1: Enhancing the structural understanding of formal proofs because it is expected that flow-chart format will visualise structural aspects of proofs in geometry, in particular universal instantiations and syllogisms.
- F2: Encouraging thinking backward/forward interactively by using flow-chart proofs with open problems because learners not only deduce a conclusion from given assumptions but also freely choose assumptions to prove the conclusion.

As a result, we can strongly expect that students supported with 'situations with scaffolding' can shift from an elemental level to a relational one by using flow-chart proofs with open problems. In the next section we investigate this issue by reporting an analysis of some of our classroom-based data.

5. Research design, context and methodology

5.1. Learning progression with flow-chart proofs with open problems

In a widely-used Japanese 8th Grade textbook (for 14 year olds) authorized by the Ministry of Education, there are three main sections of geometry: 1) properties of parallel lines and angles, properties of congruent figures, and conditions of congruent triangles through informal proofs; 2) what is a formal proof and how to construct it; and 3) properties of triangles and quadrangles by using formal proofs.

In terms of the scaffolding functions of flow-chart proofs with open problems, we need to consider how to design 'situation with scaffolding' so that students can first engage in individually-unattainable learning goals or tasks with scaffolding, and then these scaffolding supports are gradually removed. We develop the lesson design for the introductory lessons that focus on proof structures and construction in Grade 8 (students aged 14) using flow-chart proofs. The lesson design has three learning phases: 1) constructing flow-chart proofs with open problems, 2) constructing formal proofs with closed problems with reference to flow-chart proofs, and 3) refining formal proofs with closed problems by placing them into a flow-chart proof format. We explain the reasons for these phases below. Thus, the scaffoldings (using flow-chart proofs with open problems) that are used in the first phase are gradually removed during the second and third phase.

In the first phase, using flow-chart proofs with open problems provides scaffolding to understand the structure of proofs. Students are expected to construct flow-chart proofs with open problems. Since students at this very early stage of learning about proofs might see a formal proof as a rather meaningless set of symbols about the properties of geometric shapes, students may not understand why they should engage in such mathematical arguments. In particular, they may have difficulty in connecting the assumptions to the conclusion in a deductive fashion. Through their activities in the first phase of our proposed learning progression, they are expected to learn how to think forward/backward between assumptions/conclusions as they construct their proofs. They are also encouraged to organise their thinking in order to connect assumptions and conclusions. Thus we expect this phase can support them to understand how to 'assemble' a proof as a structural entity, which in turn support students to move from a Partial-structural to Holistic level.

In the second phase during which flow-chart proofs with open problems fades, the main target is a shift in proof construction from a flow-chart format to a paragraph format. Students are expected to first construct a flow-chart proof with a closed problem (similar to the typical form of proof problems that appear in textbooks). Next, they construct a formal proof through transposing a flow-chart proof into a paragraph proof. At this stage, because of their learning experiences in the earlier phase (where they constructed flow-chart proofs), they have a richer understanding of proof structures and how to compose the elements of a proof, and have developed the capability to think forward/backward between assumptions and conclusions.

Finally, in the third phase students first construct paragraph proofs in closed problems, and then refine their paragraph proofs with the use of their own flow-chart proofs if necessary. During this phase, students should be able to construct paragraph proofs by themselves with little support from flow-chart proofs because students have gradually become familiar with constructing paragraph proofs by the end of the second phase.

Aligned with the above design, we planned nine lessons (within the nationally suggested teaching hours allocated for mathematics in Grade 8) taking into account open/closed problems, varying steps of deductive reasoning, and different problems and contexts. Our lesson plans, developed in cooperation with expert mathematics teachers, include detailed teaching guidelines and worksheets for students' activities.

Phase	Activity	
1st	Constructing flow-chart proofs with open problems	4 lessons
2nd	Constructing formal proofs with closed problems with reference to flow-chart proofs	2 lessons
3rd	Refining formal proofs with closed problems by placing them into a flow-chart proof format	3 lessons

Table 1: Outline of lesson sequence

5.2 Classroom teaching experiments and analysis procedures

In order to investigate qualitatively the effects of flow-chart proofs with open problems as a scaffold, we conducted a series of classroom teaching experiments. Here, our data are taken from one of our lesson implementations in which a teacher with 18 years of teaching experience conducted the set of the nine Grade 8 lessons in a university-attached junior high school in Japan during October 2013. The qualitative data from observing these lessons are important as they enable us to investigate the effects of the form of instructional scaffold in the context of teacher-students interactions.

The nine lessons were video-recorded. First, we sought some 'critical events' (Maher and Martino, 1996) from the lessons which might elucidate the effects of the scaffold in terms of the identified scaffolding functions F1 and F2 (see section 4.2 and our theoretical framework of the understanding of structure of proof in section 3.2). After our preliminary examinations, we particularly noticed that the fourth lesson was the most interesting as the scaffolding functions were explicitly observable. This was because prior to the fourth lesson the students had used a one-step flow-chart proof to prove that two given triangles were congruent, but in the fourth lesson they tackled the proof problem in Figure 3. This proof consists of two steps of deductive reasoning; deducing the congruency of triangles from the assumptions and deducing the equivalence of angles from the triangle congruency.

From the transcript of the lesson, we used *Nvivo* to help us extract 158 interactions between the teacher and students. We coded these interactions in terms of F1 and F2 with levels of understanding of the structure of proofs as F-T and F-C codes:

- F1-T: the teacher's interventions to scaffold students' understanding of universal instantiation and syllogism
- F1-C: the students' reactions to the interventions in terms of universal instantiation and syllogism.
- F2-T: the teacher's interventions regarding thinking forward/backward
- F2-C: the students' reactions to thinking forward/backward

The first author initially conducted this analysis and then the second author checked the results. In total, we identified 50 interactions that are related either to the scaffolding functions or to the levels of understanding. An example of the analysis is presented in Table 2.

Protocol	Coding [with comment in brackets]
48. T: OK, please stop working. I think you are struggling to fill the box (of a flow-chart to say why we can deduce ∠B=∠C). You are really wondering why? Let us see this together.	F1-T Enhancing the structural understanding, universal instantiation [This is an intervention from the teacher, and the use of flow-chart helped the teacher notice many students were struggling to fill in the box; also this indicates that many students were still at the Elemental level of understanding.]
49. T: SA, can you tell us what did you put in the box? Your word to explain why (you can deduce ∠B=∠C).	F1-T [The teacher took an example from a student who successfully filled in the box in order to make universal instantiation explicit to all students in the class.]
50. SA: Because of ΔAOB≡ΔAOC	F1-C [Flow-charts allowed SA (the student) to visualise the structure of a proof and it functioned well for him to see why we can deduce ∠B=∠C.]

Table 2: Analysis example

6. Findings from classroom teaching experiments

In reporting our findings from the fourth lesson, first we show the students' levels of thinking at this stage, in particular their incomplete understanding of universal instantiations. Then we show how learning with flow-chart proofs with open problems functioned as an instructional scaffold to support students to understand the structure of proofs.

6.1. Enhancing the structural understanding of formal proofs: universal instantiations

A purpose of the fourth lesson was to make students aware of the importance of universal instantiation (which deduces a singular proposition from a universal proposition). The teacher oriented the students to confirm the need to use supplementary line AC to deduce ∠B=∠C by using the congruency of ΔABO and ΔACO, and wrote “ΔABO≡ΔACO” on the flow-chart on the board. Thereafter, students started to complete the flow-chart proof by themselves. After a suitable time the teacher asked student SA what he would put in the flow-chart box to describe the properties of congruent figures. SA answered “Because of ΔAOB≡ΔAOC” and the teacher wrote this answer on the blackboard. Next, the teacher directed two other students to show their answer. One of them said, “Due to congruent triangles, angles are congruent”, and another said, “In congruent triangles the corresponding angles are equivalent.” The teacher also wrote these answers on the blackboard. Figure 5 shows SA’s proof.

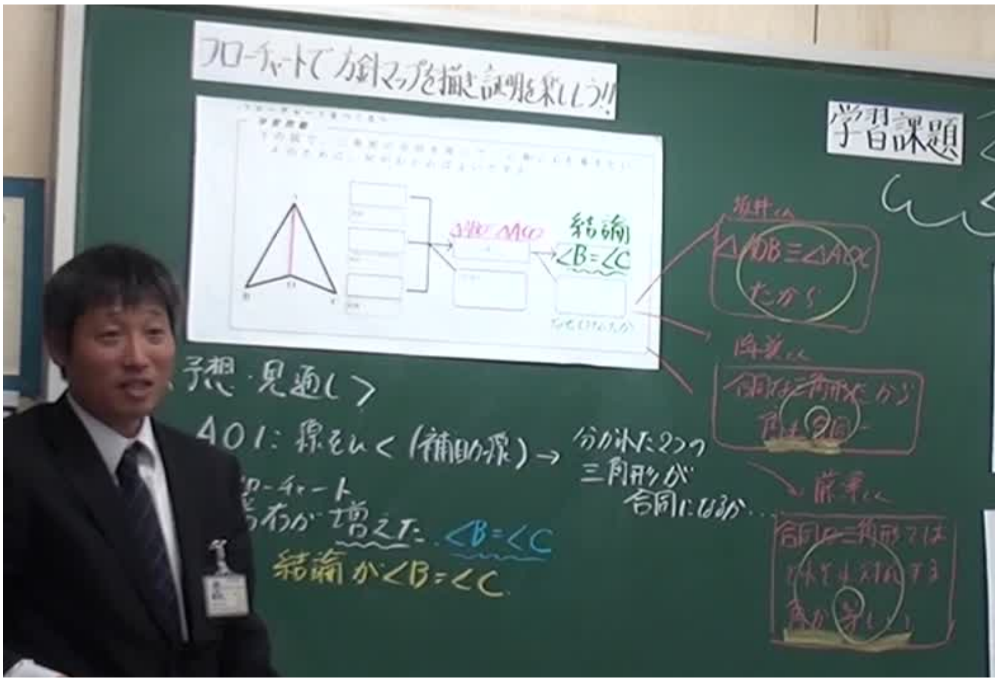


Figure 5: SA’s proof

At this time the teacher compared these three answers, and asked SA to explain more; their dialogue is shown as follows.

57 T: SA, can you tell us why you wrote this?

58 SA: Umm, I considered why the angles are equal; then I found an arrow is drawn. And I put ‘it’.

59 T: What is ‘it’?

60 SA: ΔABO and ΔACO are congruent.

61 T: OK, if we can say these two are congruent, then we can use the arrow. So, SA, if two triangles are congruent, what can we show?

62 SA: Angles are also equal.

63 T: Good, angles are also equal? Anything else?

64 SA: Sides are equal, too.

65 T: Yes, sides are equal too. So, umm, in this case our conclusion is to say the angles are equal, so it is OK. But in general if two triangles are congruent, it can be angles but also sides as well, so we should add information generally about angles such as 'because angles are congruent or equal'.

Given that prior to this lesson most students were able to find the appropriate conditions of triangle congruency and write these into the theorem box (universal proposition) in the one-step flow-chart proof, it was expected that during this lesson the students would reach the Partial-structural Elemental sub-level (by paying attention to elements of proofs). Beyond this, the lessons were designed such that some students might start reaching the Partial-structural Relational sub-level (by understanding both universal instantiations and hypothetical syllogisms) through examining the properties of congruent figures using triangle congruency.

As it transpired, during the early parts of this lesson it was evident that only a small proportion of the students reached the Partial-structural Relational sub-level. In fact, about half the students did not correctly write universal propositions into the two theorem boxes in the flow chart, each of which requested the condition of congruent triangles and the properties of congruent figures. Others only wrote a singular proposition such as 'because of $\Delta ABO \equiv \Delta ACO$' into the theorem boxes (just as student SA said in excerpt above). We infer that these students did not understand that a singular proposition can be deduced by the universal instantiation of a universal proposition. Therefore, concerning the understanding of proof structure we conclude that these students remained at the *elemental* sub-level, and did not reach the *relational* one.

At this point in the lesson, the teacher, in order to resolve SA's lack of understanding, compared SA's answer with other answers in which universal propositions were correctly used (the Partial-structural Relational sub-level) to show that it was necessary to express the property of congruent figures generally because it was being used to deduce the equivalence of angles in this case (although it could also be used to deduce the equivalence of both angles and sides, line 57). This resolution managed by the teacher supported the students by enhancing their understanding of the universal instantiation that deduces a singular proposition with a universal proposition (lines 60 and 62). This, in turn, promoted the transition from the elemental to the relational level.

6.2. Encouraging thinking forward/backward interactively by using flow-chart proofs with open problems

After discussing the incompleteness of SA's proof, students again worked individually with some group interactions. The teacher then selected three students' answers, each of which used different conditions of congruent triangles (this was possible because of the open problems). The teacher checked with the class if three pairs of angle/sides were necessary to deduce ΔABO≡ΔACO, the congruent conditions used, and then the reasons why they chose these pairs on the basis of what was written in the box below by each of the three pairs. For example, as shown in Figure 6, student KA used the ASA condition and the teacher asked him why he chose to use 'AO=AO', '∠BAO=∠CAO', and '∠AOB=∠AOC' in the flow-chart.

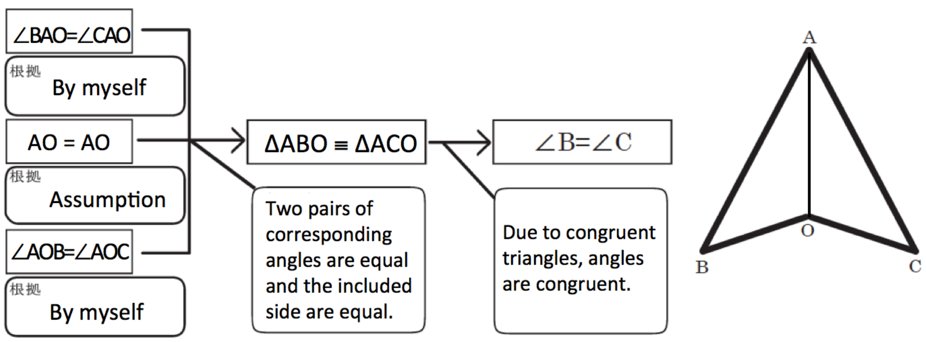


Figure 6: one of the flow-chart proofs by KA on the blackboard

The student's explanation was as follows:

80	KA:	Because we can see AO=AO from the given figure.
81	T:	Can see it from the given figure?
82	KA:	And it is an assumption.
83	T:	OK.
84	KA:	I assumed by myself ∠BAO=∠CAO, and also ∠AOB=∠AOC as well. And then we can show △AOB≡△AOC, and the condition is 'Two pairs of corresponding angles are equal and the included sides equal'. Due to congruent triangles, corresponding angles are equal and therefore ∠B=∠C.

As we can see from the dialogue and the flow-chart proof by KA in Figure 6, KA wrote "Assumption" as the reason for "AO=AO" and explained that this was apparent by means of the given figure (see line 80 KA). In contrast, KA wrote "By myself" as reasons for "∠BAO=∠CAO" and "∠AOB=∠AOC" and explained that he decided by himself that these angles equal to each other (see line 84 KA). These excerpts demonstrate KA's two ways of thinking. The first way is thinking forward. In order to find the conditions for ΔABO≡ΔACO, KA focused on the corresponding angles/sides of these triangles and decided that "AO=AO" could be one of the conditions. The second way is thinking backward. KA chose to use ASA among three congruent conditions of triangles and then looked for the other conditions (in this case, "∠BAO=∠CAO" and "∠AOB=∠AOC") in order to satisfy the ASA condition. Finally he decided to use these two conditions and wrote "By myself" in the box as an evidence of thinking backward. Open problems request learners to find the conditions necessary for the theorems in the proof. As a result learners are encouraged to use these two types of thinking interactively.

Our data show that learning with flow-chart proofs with open problems made it possible for KA to use these ways of thinking interactively accompanied by relational understanding of the structure of proof to help him understand the deductive connection of universal instantiation between three conditions (AO=AO, ∠BAO=∠CAO, ∠AOB=∠AOC) and the ASA congruent theorem. Furthermore, KA wrote two more types of flow-chart proofs in hiw worksheet using SSS and SAS as conditions for congruency. Similar to the case presented, KA determined the assumptions that were necessary to deduce the congruent triangles and wrote "By myself" as a reason to complete these proofs.

Likewise, most other students in the class used two types of thinking interactively to construct three different proofs because they also wrote "By myself" as reasons to complete the proof. This shows that flow-chart proofs with open problems can enhance most students' skills to think forward and backward interactively.

6.3. Summary of evidences from classroom teaching experiments

When we examined individual students' proof notes after the lesson, we saw an improvement in students' reasoning skills. At the end of the lesson, most students who wrote the wrong answers in the theorem box (similar to SA, these students were at the Partial-structural Elemental sub-level) correctly answered the additional flow-chart problem using statements about which theorems should be used to deduce the conclusions (the Partial-structural Relational sub-level).

From the episode in 6.1, we can identify one of the scaffolding functions of flow-chart proofs with open problems, 'F1: Enhancing the students' structural understanding of formal proofs', by providing a way of visualising two kinds of deductive reasoning (universal instantiations and hypothetical syllogisms) separately and together in a flow-chart proof format. With this visualised format, students were supported effectively to focus on the characteristics of the two kinds of deductive reasoning by checking the expression of theorems and confirming their meaning and/or roles.

As previously discussed (in section 6.2), the second scaffolding function of flow-chart proofs with open problems, 'F2: Encouraging thinking backward/forward interactively by using open problems', was apparent. This scaffolding function encourages thinking forward/backward interactively, accompanied by relational understanding of the structure of proofs. The amplification of thinking backward, in particular, can be triggered by the open problem. Moreover, the flow-chart proof format supports students not only to associate two modes of forward/backward thinking visually, but also to keep their relational understanding of proof structures by distinguishing between singular propositions and universal propositions. This systematic learning process with thinking forward/backward interactively and relational understanding is useful for the learners' planning of formal proof that usually precedes its construction (Tsujiyama, 2012). Thus, learning geometrical proofs using flow-chart proofs with open problems in the first phase of introductory lessons of formal proving can be preparatory to the planning of formal proofs in a 'closed problem' situation.

7. Additional evidence of the effectiveness of flow-chart proofs with open problems

In order to provide further evidence of the effectiveness of flow-chart proofs with open problems as a scaffold, we now report quantitative data taken from another study of ours; one that we conducted in a junior high school in the suburbs of a medium-sized city in Japan between December 2010 and January 2011. In this study, three mathematics teachers implemented our sequence of nine geometry lessons in accordance with our learning progression (one of the three teachers is a key person who closely collaborated with us to develop the nine lesson plans). Each teacher taught one class; in total, 94 students were taught. The students' attainment in mathematics was in line with Grade 8 junior high school students based on the result of national assessment in 2011.

After teaching the nine lessons, the teachers taught the usual follow-up lessons on the properties of triangles and quadrangles using a textbook that is widely used in Japan. Our assessment of student learning was conducted in May 2011, approximately four months after the nine lessons (by this time the students were learning a topic in algebra). We used test items from the Japanese National Survey that was conducted for all students in Japan in April 2009 (see Figure 3 for an example test item). The National Survey consisted of two sets of problems: Maths A (basic knowledge and skills) and Maths B (advanced mathematical thinking). In order to compare our results with the National Survey, we used both sets of questions and allocated the same amount of time for our students to answer these questions. Furthermore, to ensure the quality of the assessment, the marking of our survey was conducted by the same organization that marked the National Survey.

The impact of our learning progression using flow-chart proofs with open problems is shown by the results the students obtained on the advanced problems in Maths B (see Figure 3 for the test questions, and Table 3 for the results).

<table>
<tr><td rowspan="3"></td><td>Question 1</td><td colspan="4">Question 2</td></tr>
<tr><td rowspan="2">Correct (%)</td><td colspan="3">Correct (%)</td><td rowspan="2">No answer (%)</td></tr>
<tr><td>Complete proof</td><td>Incomplete proof</td><td>Total</td></tr>
<tr><td>Our sample</td><td>73.4</td><td>44.7</td><td>4.3</td><td>48.9</td><td>21.3</td></tr>
<tr><td>National survey</td><td>63.3</td><td>34.2</td><td>9.1</td><td>43.3</td><td>28.6</td></tr>
</table>

Table 3: Results of the advanced geometry problems for Grade 8 students

In the Maths B test items (see Figure 3), Question 1 checked if students can reason backwards from the conclusion AD=BC. As the data in Table 5 show, 73.4% of the students in our sample answered correctly. This result shows that 10.1% more students in our sample can identify what would be necessary to deduce the conclusion as seen in section 1 of 'Takuya's memo' in Figure 3. We consider that this positive result is due to our students' experience with flow-chart proofs with open problems in the first phase of our learning progression. In this phase, students learn to complete a flow-chart proof and experience thinking forward/backward between assumptions and conclusions. In this way, the students in our teaching experiment gain experience in planning a proof by finding which properties can be used as assumptions in open problems.

Question 2 asked students to construct a formal proof with reference to 'Takuya's memo' in Figure 3. As the data in Table 3 show, 48.9% of the students in our sample answered the question correctly compared to the national average of 43.3%. Furthermore, 21.3% of our sample gave no answer, which is 7.3% lower than the national average of 28.6%.

When we examined the quality of answers to Question 2 in Figure 3 more closely, the correct answers (summarised in Table 3) are divided into two categories. Category 1 includes complete answers that provided correct reasons (e.g. OA=OB because this is an assumption) and used appropriate theorems to support these reasons (e.g. congruent conditions of triangles). Category 2 includes the correct answers without these details. The data in Table 3 show that the proportion of students who answered Question 2 (in Figure 3) in full was greater in our sample (at 44.7%) than in the National survey (at 34.2%).

These results indicate that use of scaffolds in terms of flow-chart proofs with open problems has an effect on increasing the quality of students' proof constructions as they can express more precisely the reasons and theorems required to complete a proof.

8. Discussion

Geometrical proofs in junior high schools have been recognised as difficult topics to teach and to learn. One possible approach to tackle this issue is to prepare instructional scaffolds to support students' understanding of geometrical proofs. Based on our review of existing studies, in particular Sharma and Hannafin (2007) and Yelland and Masters (2007), we take scaffolding as a two-step process which includes cognitive, technical, and affective supports in order to support learners to achieve their goals. From this point of view, we devised flow-chart proofs with open problems to help students develop their understanding of the structure of proofs, in particular the transition between Partial-structural and Holistic levels.

Based on using flow-chart proofs with open problems, we provide a 'scaffolding analysis' (derived from the framework by Sherin et al., 2004) of the introductory proof lessons that we designed. The analysis was undertaken in terms of the scaffolding functions that support students' development in understanding formal proofs in geometry.

Within our focus on students' understanding of the structure of proofs, we identified the scaffolding functions of flow-chart proofs with open problems. One of these functions is that using flow-chart proofs in 'flow-chart proofs with open problems' can enhance the transition towards relational understanding of the structure of formal proofs by allowing students to visualise the connection between singular propositions by hypothetical syllogisms and the connection between a singular proposition and the necessary universal proposition by universal instantiations. Another function of flow-chart proofs with open problems is to encourage students to think forward/backward interactively, accompanied by relational understanding of the structure of proofs.

Our findings contribute to improving the teaching and learning of geometrical proofs, in particular the need to organise effective teaching interventions at the early stage of proof learning (Hanna and de Villiers, 2012; Otten et al, 2014), and to provide rich opportunities for students to justify and construct proofs (Bieda, 2010). Our study offers a new insight for providing rich learning opportunities in how the flow-chart format enables students to visualise the structure of proofs. This proof format with open problems enables students to find necessary conditions and combine them in order to connect assumptions with conclusions. The latter in particular has, to date, not been thoroughly studied and suggested in existing studies, but it is this situation of connecting assumptions with conclusions that requires students to engage in systematic learning with thinking forward/backward interactively in order to make the planning of formal proofs. While we are aware that comparing national averages with results from a single class is known to be problematic, students who experienced our flow-chart proving lessons scored 10.5% better on the full construction of advanced proof problems compared to the national average is an indication that the scaffolding functions we identified in this paper contribute to scaffolding students' understanding of introductory proofs.

9. Concluding remarks

In this paper we show that flow-chart proofs with open problems help to scaffold the structural understanding of formal proof by means of two scaffolding functions and we demonstrate how flow-chart proofs with open problems functions as a scaffold of domain-specific knowledge that make the introductory lessons of formal proofs more effective.

Belland (2014) questions the extent to which scaffolds that are specific to the knowledge context are helpful to the learner, while Molenaar *et al.*'s study (2012) suggests that 'dynamic' scaffolding might not affect students' domain knowledge. In our study, we find evidence that flow-chart proofs with open problems contribute to supporting students' understanding of domain-specific knowledge of mathematical proofs. Rittle-Johnson and Koedinger (2005) suggest using stories, visualisations and intermediating procedural steps in order to design effective scaffolds, and these suggestions worked well in our case. Our findings imply that for some topics in mathematics, in particular those many students find difficult to understand, the use of such scaffolds might be beneficial to enriching students' understanding of mathematics.

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