$\mathcal{L}_2$ optimal decentralised static output feedback stabilisation of a network of dynamical systems

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Abstract

In this paper global $\mathcal{L}_2$ stabilisation of a network of dynamical systems is attained by local decentralised static output feedback control. The proposed synthesis procedure guarantees an upper bound on the achievable $\mathcal{L}_2$ performance in the presence of disturbances. The synthesis of such a controller is posed as an iterative LMI optimisation problem. The algorithm is guaranteed to achieve a local minimum as a result of the iterations. The synthesis of the controller is independent of the number of nodes in the network and depends only on the size of the node level dynamics. A randomly generated academic example with 12 nodes is considered to demonstrate the efficacy of the proposed methodology.

Keywords: Static output feedback, Complex network, Decentralised control, $\mathcal{L}_2$ optimal

1. Introduction

Control and stabilisation of multiple dynamical systems operated over an arbitrary information network has attracted increasing attention from various communities due to its broad range of applications such as cooperative unmanned air vehicles, formation flying of satellite systems, platoons of vehicles, sensor arrays, automated highway systems and multi-agent systems. The research problem associated with such systems is to ensure they operate in agreement i.e., in a synchronised manner, and achieve global stabilisation and performance. Many researchers have contributed to the problem of control of network systems/cooperative control (see Kumar et al. (2005); Ren and Beard (2007) for an overview). State agreement, synchronisation, and consensus, can all be viewed from a similar stabilisation view point (Lin et al. (2007)), and the graph describing the topology of the network, is central to these problems. In the last decade, graph theory has been combined with systems and control ideas to obtain many novel results: see for example Mesbahi and Egerstedt (2010) and the references therein for more details and examples. This perspective has also been adopted in Wu (2007); Wang and Chen (2003, 2002); Ji and Chen (2007); Arcak (2007); Fax and Murray (2004); Olfati-Saber and Murray (2004); Kim and Mesbahi (2006).

A good deal of progress has been made in controlling and stabilising different classes of interconnected dynamical systems over an information network. Fax and Murray (2004) focuses on the stabilisation of formations with linear dynamics, mainly using a full order decentralised state feedback controller. In particular in Fax and Murray (2004), the authors suggested for the first time, the possibility of analysing the stability of a network of N identical systems (the N nodes of the representative graph), by simply studying the stability of a node level system with modifications according to the eigenvalues of the associated graph Laplacian. In Olfati-Saber and Murray (2004), a consensus framework was developed for a network of first order integrators with fixed as well as varying topologies. Subsequently several such protocols and novel variants have been proposed and these ideas were extended to networks of double integrators, which are claimed to be representative of many applications (Ren and Beard (2007)).

Recently researchers have also addressed cooperative control and consensus problems with the aim of achieving a certain level of overall performance, notably: (Massioni and Verhagen (2009), De Castro and Paganini (2004), Gupta et al. (2005), Zelazo and Mesbahi (2010), Menon and Edwards (2010), Li et al. (2011), Zhang et al. (2012)). In Massioni and Verhagen (2009) state feedback, and dynamic output feedback distributed control laws were proposed for a network of identical dynamical systems. A decentralized state feedback control law ensuring consensus with $\mathcal{H}_2$ optimal performance was synthesized in De Castro and Paganini (2004). An LQR synthesis problem is formulated for networked dynamical systems in Gupta et al. (2005) and an optimal design framework for cooperative state and output feedback control is developed for tracking and synchronization problems in Zhang et al. (2012), where unbounded regions of synchronization were derived. In Liu et al. (2009) there has been work to design centralized optimal state feedback regulators for the synchronization problem of a scale free dynamical network model as discussed in Wang and Chen

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(2003). This is in fact also a specific representation of the class of systems discussed in Fax and Murray (2004), and the $\mathcal{L}_2$ -norm of the error dynamics is considered as a performance index of synchronization. Conditions for consensus in multi agent systems with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance, and the existence of such regions, are identified in Li et al. (2011). Note that most of the studies in the literature use a state feedback or dynamic output feedback approach. One exception is the work reported in Menon and Edwards (2010) in which a decentralized static output feedback control law for synchronization of a network of dynamical systems with guaranteed $\mathcal{H}_2$ performance was developed.

The idea in this paper is to design *decentralised static output control* laws which ensure the effects of disturbances on the performance outputs are reduced to a certain acceptable level. A number of static output feedback problems have been studied by many researchers in different contexts, employing many different methods (see Gu and Misra (1994); Kucera and deSouza (1995); Mesbahi (1998); Ghaoui et al. (1997); Cao et al. (1998); Iwasaki and Skelton (1994); Edwards and Spurgeon (1995); Prempain and Postlethwaite (2001, 2005) and the references therein). A survey on static output feedback control in its most general form, which is still an open problem, is provided in Syrmos et al. (1997). In literature pertaining to static output feedback control, many different methods are reported: early attempts in an algebraic Riccati framework were reported by Gu and Misra (1994) and Kucera and deSouza (1995); rank minimisation problems were considered in Mesbahi (1998) and Ghaoui et al. (1997); iterative linear matrix inequalities were investigated by Cao et al. (1998); and linear matrix inequalities exploiting minimum phase properties and associated coordinate transformations (in sliding mode and $\mathcal{H}_\infty$ contexts) are presented in Edwards and Spurgeon (1995); Prempain and Postlethwaite (2001). More recently the Glover-McFarlane loop shaping framework was explored in Prempain and Postlethwaite (2005).

The incorporation of performance measures allows understanding of the rate with which stabilisation or consensus or synchronisation of a network of dynamical systems will be attained. This paper focuses on decentralised static output feedback control of a network of dynamical systems with disturbance attenuation in an $\mathcal{L}_2$ sense. The primary objective of this paper is to stabilize the network with a decentralised static output feedback control strategy. Certain $\mathcal{L}_2$ performance bounds on individual nodes as well as at a network level is guaranteed in the design procedure. The approach relies on algebraic graph theoretical tools, based on the connectivity of the graph Royle and Godsil (2001), are used to represent multiple dynamical systems operating over the network.The main contribution of this paper is a systematic method for the stabilisation of a network of dynamical systems, by means of a decentralised static output feedback control strategy, formulated as a novel iterative linear matrix inequality problem. The efficacy of the proposed method is demonstrated using a numerical example. The resulting decentralised design is also compared with a conventional centralised controller design.

The paper is organised as follows. In Section II, matrix notation and graph theoretic terminology are introduced. Next, the problem formulation is described in Section III. In Section IV, a decentralised output feedback control law is proposed and decentralised $\mathcal{L}_2$ performance aspects are discussed. A numerical example is provided in Section V to demonstrate its effectiveness. Finally conclusions along with some future research directions are given in Section VI.

2. Notation

The notation in the paper is quite standard. The set of real numbers is denoted by $\mathbb{R}$. The set of real-valued vectors with m components is written as $\mathbb{R}^m$. The set of arbitrary real-valued $m \times n$ matrices is given by $\mathbb{R}^{m\times n}$. The expression $Col(.)$ denotes a column vector and $\mathcal{D}iag(.)$ denotes a diagonal matrix. For a symmetric positive definite (s.p.d) matrix $P = P^{\mathrm{T}} > 0$, $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are the minimum and maximum eigenvalues. The symbols $\mathcal{N}(\cdot)$ and $\mathcal{R}(\cdot)$ represent the null space and range space of a matrix respectively. The $\mathcal{L}_2$ norm of a signal is denoted as

$$||x||^2_{\mathcal{L}_2} = \int_0^\infty ||x||^2 dt$$

and the induced $\mathcal{L}_2$ norm for an operator H is written as

$$||H||_2 := \sup_{0\neq x\in\mathcal{L}_2} \frac{||Hx||_2}{||x||_2}$$

The graph theoretic terminology employed is also quite standard. A network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, represents a simple, finite graph consisting of N vertices and k edges. For the graph $\mathcal{G}$, the adjacency matrix $\mathcal{A}(\mathcal{G}) = [a_{ij}]$, is defined by setting $a_{ij} = 1$ if i and j are adjacent nodes, and $a_{ij} = 0$ otherwise. This gives rise to a symmetric matrix. The symbol $\Delta(\mathcal{G}) = [\delta_{ij}]$ represents the degree matrix, and is an $N \times N$ diagonal matrix, where δ_{ii} is the degree of the vertex i. The Laplacian of $\mathcal{G}$, $L(\mathcal{G})$, is defined as the difference $\Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$. The smallest eigenvalue of $L(\mathcal{G})$ is exactly zero and the corresponding eigenvector is given by $\mathbf{1}$ i.e. a vector composed entirely of unit elements. The Laplacian $L(\mathcal{G})$ is always rank deficient and positive semi-definite. Moreover, the rank of $L(\mathcal{G})$ is $n - 1$ if and only if $\mathcal{G}$ is connected.

3. System description

This paper considers a large scale system comprising N identical dynamical systems indexed as $1, 2, ..., N$. A graph theoretic viewpoint will be adopted in which the system is represented as a graph $\mathcal{G}$ with N nodes: each representing an n-dimensional dynamical system. If there is an interconnection between node i and j this constitutes an edge in the graph. The connectivity topology is assumed to be described a-priori by the Laplacian of the graph denoted as L. The dynamics of the i^{th} individual node of the graph $\mathcal{G}$ are given by

$$\dot{x}_i = Ax_i + B_1 w_i + B_2 u_i - c \sum_{j=1}^{N} L_{ij} \Gamma x_j \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state vector, $w_i \in \mathbb{R}^l$ is a disturbance input, and $u_i \in \mathbb{R}^m$ is the control input (Cao et al. (2013)). As described in Section 2, $L \in \mathbb{R}^{N\times N}$ captures the connectivity topology within the network. The matrix $\Gamma \in \mathbb{R}^{n\times n}$ represents the local coupling configuration among the states of the nodes. All the entries of Γ are 1 or 0 and indicate the existence or non-existence of coupling in the respective channels in the network. The real constant $c > 0$ in equation (1) is known as the coupling strength. Here it is assumed the coupling strength is the same throughout the interconnected system.

The measurements $y_i \in \mathbb{R}^p$ associated with the i^{th} node are given by

$$y_i = C_2 x_i + D_{21} w_i + D_{22} u_i \tag{2}$$

and the controlled performance evaluation signal $z_i \in \mathbb{R}^q$ is assumed to have the form

$$z_i = C_1 x_i + D_{11} w_i + D_{12} u_i \tag{3}$$

The following assumption will be made as a basis for the development of the theory:

A 3.1. *For every node, the number of control inputs, m, is equal*[1] *to the number of measured outputs, p.*

A 3.2. *The triple (A, B_2, C_2) is stabilisable, detectable and minimum phase – i.e. the invariant zeros of (A, B_2, C_2) lie in the open LHP.*

A 3.3. *The disturbance input and the control input are not directly fed through to the controlled performance output and measured output channels respectively*[2]*; i.e., $D_{11} = D_{22} = 0$.*

A 3.4. *D_{21} and D_{12} satisfy $D_{12}^T D_{12} = I$ and $D_{21} D_{21}^T = I$.*

A 3.5. *The triple (A, B_2, C_2) is relative degree one: i.e., rank $(C_2 B_2) = m$.*

If Assumptions 3.1 and 3.5 are satisfied then a realization can be found (Edwards et al., 2007) in which

$$B_2 = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 0 & I_p \end{bmatrix} \tag{4}$$

where $B_{22} \in \mathbb{R}^{m\times m}$ and $\det(B_{22}) \neq 0$. It is at this point that the fact that $p = m$ is exploited. This partition underpins all the results and analysis which follows.

A 3.6. *In the coordinate system associated with (4), the local coupling matrix*

$$\Gamma = \mathcal{D}iag\,[0, I_m] \tag{5}$$

implying no coupling in n − m channels.

[1] This assumption will be relaxed later in the paper.

[2] In fact this can be assumed without loss of generality. For details of the loop-shifting transformation to create this representation if the generic system does not inherently have this property see (Green and Limebeer, 1995).

Remark 1. *It should be noted that Assumptions 3.3 and 3.4 are common in the $\mathcal{H}_\infty$ controller synthesis literature (Green and Limebeer, 1995; Skogestad and Postlethwaite, 1996; Doyle et al., 1989). Assumption 3.6 indicates that the interconnections are 'matched' since they act in the input channel of the control signal. The significance of the minimum phase restriction in Assumption 3.2 – which is crucial to the algorithms and methodology which will be developed – is discussed later in the paper.*

3.1. Problem definition

The problem to be addressed in this paper is the design of decentralised static output feedback control laws of the form

$$u_i = -K y_i$$

for $i = 1, ..., N$, for the network in (1)-(2), to minimize the $\mathcal{L}_2$ gain between $w = Col(w_1, ..., w_N)$ and $z = Col(z_1, ..., z_N)$.

4. Decentralised network stabilisation

4.1. Linear static output feedback case

Consider initially a decoupled individual linear dynamical system:

$$\begin{aligned} \dot{x}_i &= A x_i + B_1 w_i + B_2 u_i \\ z_i &= C_1 x_i + D_{12} u_i \\ y_i &= C_2 x_i + D_{21} w_i \end{aligned} \tag{6}$$

The system can be viewed as the special case when the coupling strength $c = 0$ in (1)-(3). It is assumed Assumptions 3.1-3.4 hold for the linear system in (6). Under Assumptions 3.1 and 3.2, using control laws of the form

$$u_i = K y_i \tag{7}$$

for the system in (6), the resulting closed loop is given by

$$\begin{aligned} \dot{x}_i &= A_{cl} x_i + B_{cl} w_i \\ z_i &= C_{cl} x_i + D_{cl} w_i \end{aligned} \tag{8}$$

where $A_{cl} = (A + B_2 K C_2)$, $B_{cl} = (B_1 + B_2 K D_{21})$, $C_{cl} = (C_1 + D_{12} K C_2)$ and $D_{cl} = (D_{12} K D_{21})$. If A_{cl} is Hurwitz, define $G_i(s) = D_{cl} + C_{cl}(sI - A_{cl})^{-1} B_{cl}$ then using the Bounded Real Lemma (Scherer, 1990), $\|G_i(s)\|_\infty < \gamma$ if and only if there exists a s.p.d matrix $X \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A_{cl}^{\mathrm{T}} X + X A_{cl} & X B_{cl} & C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}} X & -\gamma I & D_{cl}^{\mathrm{T}} \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 \tag{9}$$

As in Scherer (1990); Gahinet and Apkarian (1993), separating the terms in A_{cl}, B_{cl}, C_{cl} and D_{cl}, into those that involve the parameter K and those which do not, inequality (9) can be written as

$$\Psi + Q^{\mathrm{T}} K^{\mathrm{T}} P + P^{\mathrm{T}} K Q < 0 \tag{10}$$

where

$$\Psi = \begin{bmatrix} A^{\mathrm{T}}X + XA & XB_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}X & -\gamma I & 0 \\ C_1 & 0 & -\gamma I \end{bmatrix} \tag{11}$$

$$P = \begin{bmatrix} B_2^{\mathrm{T}}X & 0 & D_{12}^{\mathrm{T}} \end{bmatrix} \tag{12}$$

$$Q = \begin{bmatrix} C_2 & D_{21} & 0 \end{bmatrix} \tag{13}$$

Next partition the system matrix A conformably with the partitions of C_2 and B_2 in (4) so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{14}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and $A_{22} \in \mathbb{R}^{m\times m}$. As a consequence of the structure of B_2 and C_2 in (4), it follows that the invariant zeros of (A, B_2, C_2) are the eigenvalues of A_{11} (see for example Edwards et al. (2007)). Consequently from Assumption 3.2, it follows that A_{11} is Hurwitz. This property will be exploited in the algorithms which follow. At the expense of introducing some conservatism, assume the s.p.d matrix X has the block diagonal structure

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \tag{15}$$

where $X_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ and $X_2 \in \mathbb{R}^{m\times m}$. Using (14) and (15)

$$A^{\mathrm{T}}X+XA = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2^{\mathrm{T}} & \phi_3 \end{bmatrix} = \begin{bmatrix} A_{11}^{\mathrm{T}}X_1 + X_1A_{11} & A_{21}^{\mathrm{T}}X_2 + X_1A_{12} \\ A_{12}^{\mathrm{T}}X_1 + X_2A_{21} & A_{22}^{\mathrm{T}}X_2 + X_2A_{22} \end{bmatrix} \tag{16}$$

Also partition the remaining terms in (11) conformably with the partition in (16) so that

$$C_1 = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{17}$$

where $C_{11} \in \mathbb{R}^{(q-m)\times(q-m)}$, $C_{22} \in \mathbb{R}^{m\times m}$, $B_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ with $B_{22} \in \mathbb{R}^{m\times m}$. Substituting (16) and (17) in (11), the matrix Ψ in partitioned form is given by

$$\Psi = \begin{bmatrix} \phi_1 & \phi_2 & \begin{bmatrix} X_1B_{11} & X_1B_{12} \\ X_2B_{21} & X_2B_{22} \end{bmatrix} & \begin{matrix} C_{11}^{\mathrm{T}} & C_{21}^{\mathrm{T}} \\ C_{12}^{\mathrm{T}} & C_{22}^{\mathrm{T}} \end{matrix} \\ \phi_2^{\mathrm{T}} & \phi_3 & & \\ \begin{bmatrix} B_{11}^{\mathrm{T}}X_1 \\ B_{12}^{\mathrm{T}}X_1 \end{bmatrix} & \begin{bmatrix} B_{21}^{\mathrm{T}}X_2 \\ B_{22}^{\mathrm{T}}X_2 \end{bmatrix} & -\gamma I & \begin{matrix} 0 & 0 \end{matrix} \\ C_{11} & C_{12} & 0 & \begin{matrix} -\gamma I & 0 \end{matrix} \\ C_{21} & C_{22} & 0 & \begin{matrix} 0 & -\gamma I \end{matrix} \end{bmatrix} \tag{18}$$

Assume D_{12} and D_{21} have the form

$$D_{12} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & I_m \end{bmatrix} \tag{19}$$

Again this representation is common in the $\mathcal{H}_\infty$ literature (Green and Limebeer, 1995; Skogestad and Postlethwaite, 1996). Exploiting the assumptions on matrices B_2,C_2 and X from (19), the matrices P and Q in (10) have the form

$$P = \begin{bmatrix} 0 & B_{22}^{\mathrm{T}}X_2 & 0 & 0 & I_m \end{bmatrix} \tag{20}$$

$$Q = \begin{bmatrix} 0 & I_m & 0 & I_m & 0 \end{bmatrix} \tag{21}$$

By using the Projection Lemma (Scherer, 1990), inequality (10) is solvable for K if and only if

$$N_P^{\mathrm{T}}\Psi N_P < 0 \tag{22}$$

$$N_Q^{\mathrm{T}}\Psi N_Q < 0 \tag{23}$$

where N_P and N_Q are matrices whose columns span the null spaces of P and Q. The dependency of the matrix P on the s.p.d matrix X can be removed by pre and post multiplying Ψ by $D_X = \mathcal{D}iag(X^{-1}, I_l, I_q)$ and thus the solvability conditions in (22)- (23) are equivalent to

$$N_P^{\mathrm{T}}(D_X\Psi D_X)N_P < 0 \tag{24}$$

$$N_Q^{\mathrm{T}}\Psi N_Q < 0 \tag{25}$$

Exploiting all the partitions and the assumptions, explicit expressions for N_P and N_Q are:

$$N_P = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & -I & 0 & 0 \end{bmatrix} \quad N_Q = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{26}$$

After some algebra, and after pre and post multiplying (24) by $\mathcal{D}iag(X_1, X_2, I)$, the solvability conditions associated with the projection matrices (24)-(25) can be written as

$$\underbrace{\begin{bmatrix} \phi_1 & \phi_2 - C_{21}^{\mathrm{T}} & \begin{bmatrix} X_1B_{11} & X_1B_{12} \\ X_2B_{21} & X_2B_{22} \end{bmatrix} & \begin{matrix} C_{11}^{\mathrm{T}} \\ C_{12}^{\mathrm{T}} \end{matrix} \\ * & \phi_3 - C_{22}X_2 - X_2C_{22}^{\mathrm{T}} - \gamma X_2^2 & & \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{bmatrix}}_{=:\Theta(X_1,X_2,\gamma)-\mathcal{D}iag(0,\gamma X_2^2,0,0)} < 0 \tag{27}$$

and

$$\underbrace{\begin{bmatrix} \phi_1 & \phi_2 - X_1B_{12} & X_1B_{11} & \begin{bmatrix} C_{11}^T & C_{21}^T \\ C_{12}^T & C_{22}^T \end{bmatrix} \\ * & \phi_3 - B_{22}^TX_2 - X_2B_{22} - \gamma I & X_2B_{21} & \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{bmatrix}}_{=:\Xi(X_1,X_2,\gamma)} < 0 \tag{28}$$

The decision variables in (27) and (28) are X_1,X_2 and $\gamma \in \mathbb{R}^+$. Inequality (27) is not affine in X_2 and so (27) and (28) can not be directly solved by LMI tools because of the presence of the term $-\gamma X_2^2$. However $-\gamma X_2^2$ is negative definite since X_2 is s.p.d by construction, and therefore it has a 'stabilising effect' on the inequalities, and thus could be ignored. If ignored, then (27) and (28) are LMIs in the decision variables X_1,X_2 and γ. However, this may lead to conservative results in many cases. In this paper, a different approach will be adopted which does not make this simplification.

Remark 2. *Examining the upper-leftmost sub-blocks of* $\Theta(\cdot)$ *and* $\Xi(\cdot)$ *in (27) and (28) respectively, which must themselves be negative definite in order that* $\Theta(\cdot)$ *and* $\Xi(\cdot)$ *are negative definite, it follows from classical Lyapunov theory that* A_{11} *must be Hurwitz. This is guaranteed by Assumption 3.3.*

To make progress write (27) as

$$\Theta(X_1, X_2, \gamma) - \mathcal{D}iag(0, \gamma X_2^2, 0, 0) < 0 \quad (29)$$

where $\Theta(X_1, X_2, \gamma)$ consists of all terms in (27), other than the quadratic term. It follows by construction that $\Theta(X_1, X_2, \gamma)$ is affine in X_1, X_2 and γ.

Formally the problem to be solved is: Minimize γ w.r.t. X_1, X_2 such that

$$\begin{aligned} \Theta(X_1, X_2, \gamma) - \mathcal{D}iag(0, \gamma(X_2)^2, 0, 0) &< 0 \quad (30) \\ \Xi(X_1, X_2, \gamma) &< 0 \quad (31) \\ X_1 &> 0 \quad (32) \\ X_2 &> 0. \quad (33) \end{aligned}$$

Suppose an initial feasible solution X_1^0, X_2^0, and γ^0 exists for the decision variables X_1, X_2, and γ in the matrix inequalities (30)-(33) at initial step 0. Now consider the addition of a symmetric perturbation Δ term to X_2^0 subject to the condition

$$\Delta + X_2^0 > 0 \quad (34)$$

Substitute $\Delta + X_2^0$ for X_2 in inequality (31), where X_2^0 is considered fixed, to create an inequality

$$\Xi(X_1, X_2^0 + \Delta, \gamma) \leq 0 \quad (35)$$

then replace (30) with

$$\Theta(X_1, X_2^0+\Delta, \gamma) - \mathcal{D}iag(0, \gamma(X_2^0\Delta+\Delta X_2^0+(X_2^0)^2), 0, 0) < 0 \quad (36)$$

Note that unlike (35), inequality (36) is not achieved by direct substitution of X_2^0 by $X_2^0 + \Delta$ which would create a non-affine expression with respect to Δ. Observe that the inequalities (35) and (36) have a feasible solution $X_1 = X_1^0$, $\Delta = 0$, and $\gamma = \gamma^0$ and consider the optimization problem:

Minimize γ with respect to the decision variables X_1 and Δ subject to (35), (36) and (32)-(34).

Suppose the optimal solution to this problem is $(\hat{\gamma}, \hat{\Delta}, \hat{X}_1)$, then clearly $\hat{\gamma} \leq \gamma^0$ (since $X_1 = X_1^0$, $\Delta = 0$, $\gamma = \gamma^0$ is a feasible solution) and

$$\Theta(\hat{X}_1, X_2^0+\hat{\Delta}, \hat{\gamma}) - \mathcal{D}iag(0, \hat{\gamma}(X_2^0\hat{\Delta}+\hat{\Delta}X_2^0+(X_2^0)^2), 0, 0) < 0 \quad (37)$$

Since (37) holds and $-\Delta^2 \leq 0$ it follows

$$\Theta(\hat{X}_1, X_2^0 + \hat{\Delta}, \hat{\gamma}) - \mathcal{D}iag(0, \hat{\gamma}(X_2^0\hat{\Delta}+\hat{\Delta}X_2^0+(X_2^0)^2+\hat{\Delta}^2), 0, 0) < 0$$

which is equivalent to

$$\Theta(\hat{X}_1, X_2^0 + \hat{\Delta}, \hat{\gamma}) - \mathcal{D}iag(0, (\hat{\gamma}(X_2^0 + \hat{\Delta})^2), 0, 0) < 0$$

and consequently $\hat{\gamma}$, $X_2 = X_2^0 + \hat{\Delta}$ and $\hat{X}_1$ is a feasible solution to (30)-(33). Write this new feasible solution as $\gamma^1 = \hat{\gamma}$, $X_2^1 = X_2^0 + \Delta$ and $\hat{X}_1^1 = \hat{X}_1$ and consider a perturbation to X_2^1 as $X_2^1 + \Delta$ and repeat the process. This can be represented as the following iteration based algorithm:

For a given solution X_1^i, X_2^i with an associated value of γ^i, at step $i + 1$:

Minimize γ with respect to the decision variables X_1 and Δ and the inequalities

$$\begin{aligned} \Xi(X_1, X_2^i + \Delta, \gamma) &< 0 \quad (38) \\ \Theta(X_1, X_2^i + \Delta, \gamma) - \mathcal{D}iag(0, \gamma(X_2^i\Delta+\Delta X_2^i+(X_2^i)^2), 0, 0) &< 0 \quad (39) \\ X_1 &> 0 \quad (40) \\ X_2^i + \Delta &> 0 \quad (41) \end{aligned}$$

Write the optimal solution as $\hat{X}_1$ and $\hat{\Delta}$ and define the revised solution (X_1^{i+1}, X_2^{i+1}) as $X_1^{i+1} = \hat{X}_1$ and $X_2^{i+1} = X_2^i + \hat{\Delta}$. The optimal γ at the $i + 1th$ iteration $\hat{\gamma}^{i+1}$ is guaranteed to satisfy $\hat{\gamma}^{i+1} \leq \hat{\gamma}^i$.

Now repeat this process until a given stopping criteria is satisfied: typically $|\hat{\gamma}^{i+1} - \hat{\gamma}^i| < \epsilon$ where ϵ is a user selected tolerance level[3]. Since all the $\gamma^i > 0$ the sequence of γ^i's must converge to a limit and the optimization process is guaranteed to converge to a local minima. The problem of solving (38) and (39) at every iteration can be posed as a generalised eigenvalue problem, and can be solved efficiently using any of the available commercial LMI solvers: for example the optimal γ at each iteration is the value returned by the *gevp* command of the LMI toolbox Gahinet et al. (1995).

If Assumption 3.1 is dropped and replaced instead by the weaker assumption that $p > m$, then the approach described in this section can still be employed albeit in a suboptimal way. The simplest approach is to replace the outputs $y_i \in \mathbb{R}^p$ with a scaled version $\tilde{y}_i \in \mathbb{R}^m$ where $\tilde{y}_i := Fy_i$ where $F \in \mathbb{R}^{m\times p}$. Now with respect to the inputs u_i and outputs $\tilde{y}_i$ the system is square and the optimizations described earlier can be applied. This approach has plenty of precedence in the literature – see for example Saberi and Sannuti (1988), and is sometimes known as 'squaring-down'. The matrix F must be selected so that $rank(FC_2B_2) = m$ (in order to satisfy Assumption 3.2) and to ensure (A, B_2, FC_2) is minimum phase (Assumption 3.5). A prerequisite for (A, B_2, FC_2) to be minimum phase is that (A, B_2, C_2) is minimum phase because any invariant zero of (A, B_2, C_2) is also an invariant zero (A, B_2, FC_2). Some approaches to select F are discussed in (Edwards et al., 2007).

4.2. Decentralized $\mathcal{L}_2$ performance at Network Level

The ideas from the previous section will now be extended to the large scale system in (1)-(2). A decentralised output feedback control law of the form

$$u_i = Ky_i, \quad i = 1, ..., N \quad (42)$$

is proposed where K is the control gain obtained from the optimisation at a single node level as described in the previous section. At node level the objective is to minimise the effect of the w_i on the performance output z_i in an $\mathcal{L}_2$ sense. However, this does not necessarily provide information about the $\mathcal{L}_2$ performance at a network level. To investigate this, the

[3] A related, different, but equally valid stopping criteria, would be to monitor $||\Delta||$ at each time step until the situation in which $||\Delta|| < \epsilon$ occurs.

closed loop dynamics of the large scale system in (1)-(2) can be written as

$$\dot{x}_i = A_{cl}x_i + B_{cl}w_i - c\sum_{j=1}^{N} L_{ij}\Gamma x_j \quad (43)$$

$$z_i = C_{cl}x_i + D_{cl}w_i \quad (44)$$

for $i = 1, ..., N$. In terms of Kronecker products, equations (43)-(44) can be conveniently re-written as

$$\dot{x} = ((I_N \otimes A_{cl}) - c\,(L \otimes \Gamma))\,x + (I_N \otimes B_{cl})\,w \quad (45)$$

$$z = (I_N \otimes C_{cl})\,x + (I_N \otimes D_{cl})\,w \quad (46)$$

where $x = Col(x_1, x_2, ..., x_N)$. Since L is a symmetric matrix, by spectral decomposition, L can be written as

$$L = VDV^{\mathrm{T}} \quad (47)$$

where the orthogonal matrix $V \in \mathbb{R}^{N\times N}$ is formed from the eigenvectors of L, and

$$D := \mathcal{D}iag(d_1, d_2, .., d_i, ..., d_N) \quad (48)$$

with $d_1 \geq d_2 \ldots \geq d_N = 0$ where the d_i constitute the eigenvalues of L.

To write the equations in (43)-(44) in modal form, define a co-ordinate transformation $T : x \mapsto \hat{x} := Tx$, where

$$T := (V^{\mathrm{T}} \otimes I_n) \quad (49)$$

The transformation matrix T is orthogonal since

$$(V^{\mathrm{T}} \otimes I_n)^{\mathrm{T}}(V^{\mathrm{T}} \otimes I_n) = (V \otimes I_n)(V^{\mathrm{T}} \otimes I_n) = (VV^{\mathrm{T}} \otimes I_n) = I_{nN}$$

Applying the transformation $T : x \mapsto \hat{x}$ to the system in (45) and (46), it follows that the new realization is

$$\dot{\hat{x}} = (I_N \otimes A_{cl} - c(D \otimes \Gamma))\,\hat{x} + (V^{\mathrm{T}} \otimes B_{cl})w \quad (50)$$

$$z = (V \otimes C_{cl})\hat{x} + (I_N \otimes D_{cl})w \quad (51)$$

since

$$(V^{\mathrm{T}} \otimes I_N)(I_N \otimes A_{cl})(V^{\mathrm{T}} \otimes I_N)^{\mathrm{T}} = (V^{\mathrm{T}}V \otimes A_{cl}) = (I_N \otimes A_{cl}) \quad (52)$$

and

$$(V^{\mathrm{T}} \otimes I_N)(L \otimes \Gamma)(V^{\mathrm{T}} \otimes I_N)^{\mathrm{T}} = (V^{\mathrm{T}}LV \otimes \Gamma) = (D \otimes \Gamma) \quad (53)$$

To further decouple the input and output signals in (50) and (51), further *orthogonal* transformations can be applied to the signals w and z. Because the proposed transformations are orthogonal, the $\mathcal{L}_2$ gain remains invariant. Define $\hat{z} = (V^{\mathrm{T}} \otimes I_q)z$ and $\hat{w} = (V^{\mathrm{T}} \otimes I_l)w$ then from (50) and (51) it follows

$$\dot{\hat{x}} = (I_N \otimes A_{cl} - c(D \otimes \Gamma))\,\hat{x} + (I_N \otimes B_{cl})\hat{w} \quad (54)$$

$$\hat{z} = (I_N \otimes C_{cl})\hat{x} + (I_N \otimes D_{cl})\hat{w} \quad (55)$$

Since by construction D is diagonal, equation (54)-(55) can be written as:

$$\dot{\hat{x}}_i = (A_{cl} - cd_i\Gamma)\,\hat{x}_i + B_{cl}\hat{w}_i \quad (56)$$

$$\hat{z}_i = C_{cl}\hat{x}_i + D_{cl}\hat{w}_i \quad (57)$$

for $i = 1, ..., N$ where $\hat{x} = Col(\hat{x}_1, ..., \hat{x}_N)$, $\hat{z} = Col(\hat{z}_1, ..., \hat{z}_N)$ and $\hat{w} = Col(\hat{w}_1, ..., \hat{w}_N)$. The advantage of these transformations is that the network level problem has been decomposed into *N decoupled systems*. This forms a suitable basis for the subsequent analysis.

Write the decoupled structure of (54)-(55) as

$$\hat{G}(s) = \mathcal{D}iag(\hat{G}_1(s), \hat{G}_2(s), ..., \hat{G}_N(s)) \quad (58)$$

where the transfer function matrix above is associated with the mapping $\hat{w} \mapsto \hat{z}$ and $\hat{G}_i(s)$ represents the transfer function from $\hat{w}_i \mapsto \hat{z}_i$ with an associated state space representation

$$\hat{G}_i := \left[\begin{array}{c|c} A_{cl} - cd_i\Gamma & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] \quad (59)$$

Now from the approach proposed in Section 4, a matrix K can be synthesised so that

$$\begin{bmatrix} A_{cl}^{\mathrm{T}}X + XA_{cl} & XB_{cl} & C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}}X & -\gamma I & D_{cl}^{\mathrm{T}} \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 \quad (60)$$

is satisfied for some value of γ. From the structures of Γ and X in (5) and (15) it follows

$$X\Gamma + \Gamma^T X = \begin{bmatrix} 0 & 0 \\ 0 & 2X_2 \end{bmatrix}$$

and therefore $-cd_i(X\Gamma + \Gamma^T X) \leq 0$ for $i = 1 \ldots N$. Since

$$\mathcal{D}iag\{-cd_i(X\Gamma + \Gamma^T X), 0, 0\} \leq 0 \quad (61)$$

it follows from (61) that, if (60) is satisfied then

$$\begin{bmatrix} (A_{cl} - cd_i\Gamma)^{\mathrm{T}}X + X(A_{cl} - cd_i\Gamma) & XB_{cl} & C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}}X & -\gamma I & D_{cl}^{\mathrm{T}} \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 \quad (62)$$

and therefore from the bounded real lemma, $||\hat{G}_i(s)||_\infty < \gamma$ for $i = 1 \ldots N$. From the diagonal structure of $\hat{G}(s)$ it follows

$$||\hat{G}(s)||_\infty \leq \max\left\{||\hat{G}_1(s)||_\infty, ||\hat{G}_2(s)||_\infty, \ldots ||\hat{G}_N(s)||_\infty\right\} \leq \gamma$$

since all $||\hat{G}_i(s)||_\infty < \gamma$ for $i = 1 \ldots N$. Hence the $\mathcal{L}_2$ performance of the network is bounded in fact by the $\mathcal{L}_2$ performance of the individual decoupled node level dynamics.

4.3. Centralised design

The aim of this section is, for comparison purposes, to design a *centralised* output feedback control law

$$u = -\mathbb{K}y \quad (63)$$

where $y = Col(y_1, ..., y_N)$, to ensure the network level closed loop system has a given $\mathcal{L}_2$ performance with respect to $w = Col(w_1, ..., w_N)$ and $z = Col(z_1, ..., z_N)$. In the centralised controller synthesis, the dynamics due to the interconnection topology of the graph, L, need to be included explicitly: hence

$$A_{cent} := ((I_N \otimes A) - c(L \otimes \Gamma))$$

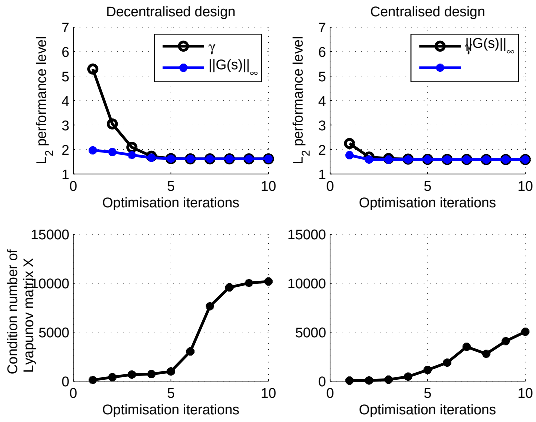

Figure 1: Performance of iterative LMI

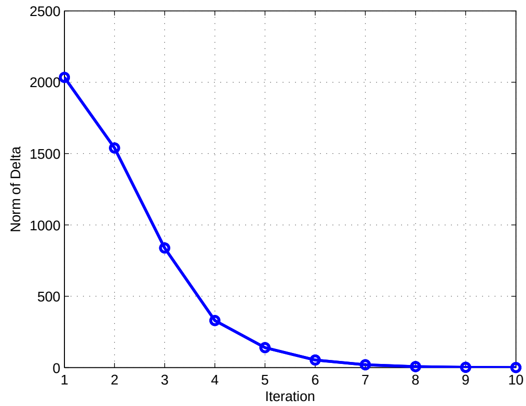

Figure 2: Evolution of $||\Delta||$

is considered for synthesis purposes. If $C_{2,cent} := (I_N \otimes C_2)$ and $B_{2,cent} := (I_N \otimes B_2)$ then $C_{2,cent}B_{2,cent}$ has full rank and $(A_{cent}, B_{2,cent}, C_{2,cent})$ is minimum phase. These properties follow from Assumptions 3.1 and 3.5 at node level. Consequently a transformation always exists providing an identical block partitioned canonical representation to (4). In the centralised control law design case, the iterative optimisation procedure, discussed in the earlier section, can be directly employed and the static output control gain $\mathbb{K}$ can be synthesised providing an $\mathcal{L}_2$ performance for the network system.

5. Design example

In this section, a numerical example is provided to demonstrate the theory developed in this paper. Consider an arbitrary network consisting of 12 identical dynamical systems with 12 interconnections represented as a graph $\mathcal{G}(12, 12)$. The interconnections are arranged according to the nearest neighbour rule as in Massioni and Verhagen (2009). The 12 nodes of the graph represent the identical dynamical systems. The dynamics at individual node level in (1)-(2) are given as follows:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_1 = I_3 \quad B_1 = I_3 \quad D_{12} = \begin{bmatrix} 0 \\ I_2 \end{bmatrix} \quad D_{21} = \begin{bmatrix} 0 & I_2 \end{bmatrix}$$

The example considered is minimum phase and relative degree one and satisfies Assumptions 3.1 - 3.4. The coupling strength c is identical and fixed as 0.5. The local coupling matrix $\Gamma = \mathcal{D}iag[0, 1, 1]$ satisfies Assumptions 3.6. Compared with the usual state feedback policies, only output information will be utilised for stabilizing the network, which is realistic.

A locally decentralised $\mathcal{L}_2$ optimal static output feedback control law is first designed following the iterative LMI procedures described in Section IV. It should be noted that the decentralised static output feedback design requires only local information at the node level. Identical controllers are then used for each node of the network $\mathcal{G}$. The iterative LMI algorithm from Section IV has been employed to synthesise the $\mathcal{L}_2$ optimal output feedback control gains. An initial guess for $X_2^0 \in \mathbb{R}^{2\times 2}$ was chosen as $10I_2$. After a few iterations a static gain with an $\mathcal{L}_2$ bound equal to $\gamma_{decentralised} = 1.616$ was obtained with

$$K = \begin{bmatrix} -0.9999 & 0.0015 \\ 0.0006 & -1.6001 \end{bmatrix}$$

The top left subplot of figure 1 demonstrates the reduction of the $\mathcal{L}_2$ gain bound γ over the course of the iterations, showing the convergence of the proposed iterative LMI algorithm. In the subfigure, a comparison is made between the γ obtained by solving the conditions corresponding to the bounded real lemma in (9) and the actual $||G(s)||_\infty$ value. The left lower subfigure of figure 1 shows the increasing condition number of the Lyapunov matrix X obtained by the algorithm. In a similar way, a *centralised* output feedback controller was designed using the iterative LMI algorithm. The $\mathcal{L}_2$ performance comparison at each iteration, as well as the Lyapunov matrix condition number, is given in the right upper and lower subfigure for comparison purposes. Figure 2 shows a monotonic decrease in the $||\Delta||$ as the iteration proceeds. The $\mathcal{L}_2$ bound for the centralised control design is slightly lower than that of the decentralised controller and is $\gamma_{central} = 1.58$. However, importantly, the iterative optimisation was slow in terms of computation time compared to the decentralised controller. Obviously the increased dimensionality of the centralised plant affects the iterative optimisation procedure. It can be concluded that the decentralised static output feedback control design has the benefits of low dimensionality, an equal level of guaranteed $\mathcal{L}_2$ performance, and computational efficiency.

Figure 3 shows the time response of the closed loop network dynamics with the decentralised feedback controller. For clarity, only a 20 second interval is shown. The state x, a sinusoidal disturbance input w and performance output z in the case of decentralised $\mathcal{L}_2$ optimal feedback, is given in the subfigures of Figure 3. Note that the graph Laplacian matrix, L, containing information about the topology of the graph is not employed in the decentralised control design strategy. As a consequence,

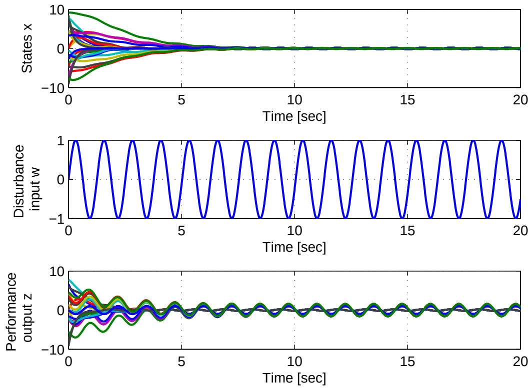

Figure 3: Time response of $\mathcal{L}_2$ optimal decentralised controller at Network level

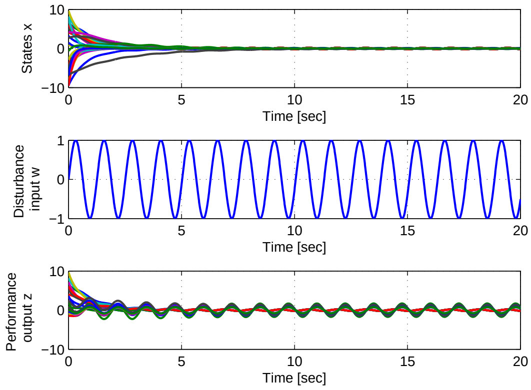

Figure 4: Time response of $\mathcal{L}_2$ optimal centralised controller

loosely speaking, it can be claimed that the proposed decentralised control scheme is robust to edge addition/removal, when the number of nodes is fixed. For comparison, Figure 4 shows the time response of the centralised $\mathcal{L}_2$ optimal controller. As expected, the centralised controller has a slightly quicker response.

6. Conclusions

In this paper, the stabilisation of a network of linear time invariant dynamical systems with a guaranteed upper bound on the $\mathcal{L}_2$ performance is considered. Global stabilization of a complex network has been considered by applying local *decentralized static output feedback control*, under the assumption the individual (decoupled) node level dynamics are minimum phase. A transformation depending on the spectral properties of the network topology is used to achieve a suitable structure for providing the bounds on the $\mathcal{L}_2$ performance level. Disturbance attenuation in an $\mathcal{L}_2$ sense is achieved using the decentralised static output feedback control, which is synthesised using an algorithm involving iterative linear matrix inequalities.

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