

On the Solvability of the Constrained Lyapunov Problem

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Abstract

This paper considers system theoretic conditions for the solvability of the so-called Constrained Lyapunov Problem *for non-square systems*. These problems commonly appear in the control systems literature. Both a static output feedback problem and an observer problem are considered. The basis for the work described here is a new canonical form which simplifies the analysis and deals with the equality constraint in a simple way.

I. INTRODUCTION

A common approach in the literature for the design of controllers and observers for nonlinear systems is to treat the system of interest as being composed of a linear system in feedback with a nonlinear element – a classical Lure system. One common strategy for demonstrating stability of the system is to synthesize a Lyapunov function based on the linear system element in such a fashion that stability can be proved for the nonlinear system. The so-called Popov and Circle criterion are well known examples of such an approach [9]. The Circle criteria employs a quadratic form as the Lyapunov function whilst the Popov criterion augments the quadratic term with a nonlinear one which depends on an integral of the nonlinear elements. In terms of a controller *synthesis* problem, the use of a Popov-type Lyapunov function invariably leads to an intractable problem and so is usually used more often for analysis than design. In considering a design problem to establish stability with respect to a quadratic Lyapunov function (so-called quadratic stabilizability) a problem occurs which was termed by Galimidi & Barmish [7] as a *Constrained Lyapunov Problem* (CLP). It commonly occurs in uncertain linear systems where the so-called *matching condition* is assumed to be satisfied and when full state availability does not exist. Subsequently this problem has appeared widely in several guises in the control systems

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literature over several decades: for example, in problems involving robust static output feedback [7], adaptive observers [14], sliding mode observers [13] and decentralized control [16]. The solvability of constrained Lyapunov equations is therefore an interesting problem of practical significance. Many authors have considered this problem but almost all the published work has focused on square systems. However, systems involving constrained Lyapunov equations are in most cases non-square [5], [12], [4]. Therefore to consider the solvability of constrained Lyapunov equations for non-square systems is important and meaningful. The Constrained Lyapunov Problem was posed and solved in [7] for both square and non-square systems in the sense that necessary and sufficient algebraic conditions were given to enable its solution. The conditions in [7] are given in algebraic terms and there is no suggestion as to when they are solvable in system theoretic terms. More recently, for square systems, Kim & Park [10] drew parallels between the CLP and the robust output feedback work of Gu [8]. The work presented in this paper can be viewed as an extension of the work of Kim & Park [10] for non-square systems as well as an observer design formulation. The notation used throughout is quite standard. For a square matrix $\lambda(\cdot)$ represents the spectrum and for a given symmetric matrix, $\lambda_{max}(\cdot)$ is the largest eigenvalue; $\mathcal{N}(\cdot)$ represents the null-space of a matrix.

II. PROBLEM FORMULATION

Two specific controller/observer theory related examples will be considered for a given system triple (A, B, C) where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ where C has full row rank and B has full column rank. Here it will be assumed that $p > m$; the square case where $p = m$ has been considered recently by Kim & Park [10]. The case when $p > m$ is a typical situation where more sensors are available than actuators; the additional outputs are used to assist in the development of control schemes to enhance the performance of a subset of ‘controlled outputs’. For a given triple (A, B, C) two specific situations will be considered:

- a) The problem of finding a static output feedback gain K such that

$$P(A - BKC) + (A - BKC)^T P < 0 \quad (1)$$

where $P \in \mathbb{R}^{n \times n}$ is s.p.d and subject to the linear constraint

$$B^T P = FC \quad (2)$$

where $F \in \mathbb{R}^{m \times p}$. In this problem K , P and F will be treated as variables. This output feedback control problem arises for example in [6], [7], [8].

b) The problem of finding a gain G such that

$$P(A - GC) + (A - GC)^T P < 0 \quad (3)$$

where $P \in \mathbb{R}^{n \times n}$ is s.p.d and subject to the linear constraint (2). Here G , P and F will be treated as variables. This essentially is an observer problem which has appeared in [6], [7], [13], [14], [15].

Remark 1: These two problems were originally posed and solved in a more abstract form in [7]. They are associated with a nominal linear triple (A, B, C) but are usually linked to an overarching problem involving both linear and nonlinear terms since in a) the triple $(A - BKC, B, FC)$ is rendered passive whilst in b) the triple $(A - GC, B, FC)$ is rendered passive [1].

Kim & Park [10] recently showed that for the square case when $p = m$, necessary and sufficient conditions to solve the first problem are that $\det(CB) \neq 0$ and none of the $n - m$ invariant zeros of the triple (A, B, C) lie in \mathbb{C}_+ . These conditions are system theoretic and are independent of the state-space representation. They amount to the nominal system being minimum phase and relative degree one. This paper shows that the natural extension of these two conditions are necessary for the non-square case also. Specifically, it will be assumed that the following restrictions on the triple (A, B, C) hold:

A1) $\text{rank}(CB) = m$

A2) no invariant zeros of the triple (A, B, C) lie in \mathbb{C}_+

It will be assumed throughout that the pair (A, B) is controllable. No assumptions will be made directly on the pair (A, C) .

Remark 2: In the square case, the assumption that $\det(CB) \neq 0$ ensures the triple (A, B, C) has exactly $n - m$ zeros. In the non-square case, the triple (A, B, C) does not necessarily have invariant zeros, and indeed it can be argued that, typically, unless specific structures exist within the system, non-square systems tend not to possess any invariant zeros [11]. Thus, typically, for non-square systems such as those considered in the paper, A2 is trivially satisfied and the strongest constraint on the class of systems arises from the relative degree one requirement A1.

III. MAIN RESULTS

Both the output feedback and the observer problems discussed earlier will be treated separately.

Remark 3: It is easy to check that if the triple $(A, B, C) \mapsto (\mathcal{T}A\mathcal{T}^{-1}, \mathcal{T}B, C\mathcal{T}^{-1}) := (\tilde{A}, \tilde{B}, \tilde{C})$ via a nonsingular coordinate change associated with an invertible matrix \mathcal{T} , then P solves the constraint (2) if and only if $\tilde{P} := (\mathcal{T}^{-1})^T P \mathcal{T}^{-1}$ solves $\tilde{B}^T \tilde{P} = F\tilde{C}$, i.e. the solvability of problems a) and b) are independent of the coordinate system and hence are system properties.

A. The Output Feedback Problem

First consider the static output feedback problem associated with the problem of finding a K , P and F to satisfy (1)-(2). In order to tackle this problem, a useful lemma will first be stated and proved which introduces a canonical form to help solve the problem of interest.

Lemma 1: Let (A, B, C) be a linear system with $p > m$ and $\text{rank}(CB) = m$. Then a change of coordinates exists so that the triple in the new coordinate system has the following structure:

1) The system matrix can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (4)$$

where $A_{11} \in \mathbb{R}^{(n-m) \times (n-m)}$ and when partitioned has the structure

$$A_{11} = \begin{bmatrix} A_{1111} & A_{1112} & A_{1113} \\ 0 & A_{1122} & A_{1123} \\ 0 & A_{1132} & A_{1133} \end{bmatrix} \quad (5)$$

where $A_{1111} \in \mathbb{R}^{r \times r}$, $A_{1122} \in \mathbb{R}^{(n-p-r) \times (n-p-r)}$ and $A_{1132} \in \mathbb{R}^{(p-m) \times (n-p-r)}$ for some $r \geq 0$ and the pair (A_{1122}, A_{1132}) is completely observable. Furthermore, the eigenvalues of A_{1111} are the invariant zeros of (A, B, C) .

2) The input distribution matrix has the form

$$B = \begin{bmatrix} 0 & B_2^T \end{bmatrix}^T \quad (6)$$

where $B_2 \in \mathbb{R}^{m \times m}$ and is nonsingular.

3) The output distribution matrix has the form

$$C = \begin{bmatrix} 0 & T \end{bmatrix} \quad (7)$$

where $T \in \mathbb{R}^{p \times p}$ and is orthogonal.

Proof A pair of linear transformations will be demonstrated which bring about the required canonical form. Without loss of generality assume $C = \begin{bmatrix} 0 & I_p \end{bmatrix}$ and the input distribution matrix is partitioned in a compatible way so that

$$B^T = \begin{bmatrix} B_{c1}^T & B_{c2}^T \end{bmatrix} \quad \text{where } B_{c1} \in \mathbb{R}^{(n-p) \times m} \text{ and } B_{c2} \in \mathbb{R}^{p \times m}$$

In this coordinate system $CB = B_{c2}$, and so by assumption $\text{rank}(B_{c2}) = m$. Hence in particular the left pseudo-inverse $B_{c2}^\dagger = (B_{c2}^T B_{c2})^{-1} B_{c2}^T$ for B_{c2} exists. Also there exists an orthogonal matrix $T \in \mathbb{R}^{p \times p}$ such that

$$T^T B_{c2} = \begin{bmatrix} 0 & B_2^T \end{bmatrix}^T \quad (8)$$

where $B_2 \in \mathbb{R}^{m \times m}$ and is nonsingular (QR-decomposition). The coordinate transformation

$$T_b = \begin{bmatrix} I_{n-p} & -B_{c1}B_{c2}^\dagger \\ 0 & T^T \end{bmatrix} \quad (9)$$

is nonsingular and with respect to the new coordinates the input and output distribution matrices are in the form of (6) and (7) respectively. Partition the new system matrix as

$$T_b A T_b^{-1} = \left[\begin{array}{c|c} A_{111} & A_{112} \\ \hline A_{211} & A_{222} \\ A_{212} & \end{array} \right]$$

where $A_{111} \in \mathbb{R}^{(n-p) \times (n-p)}$, $A_{211} \in \mathbb{R}^{(p-m) \times (n-p)}$ and $A_{222} \in \mathbb{R}^{p \times p}$. Let $T_{obs} \in \mathbb{R}^{(n-p) \times (n-p)}$ be any matrix which puts (A_{111}, A_{211}) into the stair-case observability canonical form [3]¹. It can easily be verified that changing coordinates with respect to the nonsingular transformation $T_a = \text{diag}\{T_{obs}, I_p\}$ provides a basis in which the system triple satisfies properties 1), 2) and 3) in the lemma statement once the system matrix is re-partitioned conformably with (4). By direct computation from the Rosenbrock system matrix it can be shown the eigenvalues of A_{1111} are the invariant zeros of (A, B, C) . ■

Using this lemma the following will be proved for the output feedback problem.

Proposition 1: For a given triple (A, B, C) there exists a static output feedback gain K and a s.p.d. matrix P such that (1)-(2) holds where $F \in \mathbb{R}^{m \times p}$ if and only if A1 and A2 hold and the fictitious triple (A_{11}, A_{12}, C_1) is static output feedback stabilizable where A_{11} and A_{12} are defined in (4) and

$$C_1 := \begin{bmatrix} 0_{(p-m) \times (n-p)} & I_{p-m} \end{bmatrix} \quad (10)$$

Proof

(Necessity) Suppose there exist matrices K , P and F such that (1)-(2) hold. Then multiplying (2) on the right by B yields $B^T P B = F C B$. Because P is s.p.d, $\text{rank}(B^T P B) = m$ and so $\text{rank}(F C B) = m$. Since $\text{rank}(F C B) \leq \min\{\text{rank}(F), \text{rank}(C B)\}$ it follows that $\text{rank}(C B) = m$ i.e. assumption A1 holds. By changing coordinates if necessary it can be assumed the triple (A, B, C) is in the form of Lemma 1. Let the s.p.d. matrix P have a structure

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

¹This transformation is available in most design packages such as MATLAB.

which is commensurate with the partition in (4). Now change coordinates $x \mapsto T_p x$ where

$$T_p := \begin{bmatrix} I_{n-m} & 0 \\ P_{22}^{-1} P_{12}^T & I_m \end{bmatrix}$$

In the new coordinates, if $(A, B, C, P) \mapsto (A_p, B_p, C_p, P_p)$ then

$$P_p = (T_p^{-1})^T P T_p^{-1} = \begin{bmatrix} P_{11} - P_{12} P_{22}^{-1} P_{12}^T & 0 \\ 0 & P_{22} \end{bmatrix}$$

and $B_p = T_p B = B$ i.e. the input distribution matrix is invariant. Let $FT = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$ where $F_1 \in \mathbb{R}^{m \times (p-m)}$ and $F_2 \in \mathbb{R}^{m \times m}$ and T is the orthogonal matrix in (7) then

$$FC_p = FCT_p^{-1} = \begin{bmatrix} 0 & F_1 & | & F_2 \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ -P_{22}^{-1} P_{12}^T & I_m \end{bmatrix} = \begin{bmatrix} F_1 C_1 - F_2 P_{22}^{-1} P_{12}^T & F_2 \end{bmatrix} \quad (11)$$

where C_1 is from (10). As a result of the change in coordinates, (2) becomes $B_p^T P_p = FC_p$ then comparing the expression for

$$B_p^T P_p = \begin{bmatrix} 0 & B_2^T P_{22} \end{bmatrix}$$

with (11), in order that $B_p^T P_p = FC_p$ holds, it follows that $F_1 C_1 - F_2 P_{22}^{-1} P_{12}^T = 0$. Consequently since $\det F_2 \neq 0$ (because $F_2 = B_2^T P_2$) it follows that

$$P_{22}^{-1} P_{12}^T = F_2^{-1} F_1 C_1 \quad (12)$$

After the change of coordinates

$$A_p = T_p A T_p^{-1} = \begin{bmatrix} A_{11} - A_{12} P_{22}^{-1} P_{12}^T & A_{12} \\ * & * \end{bmatrix}$$

where the $*$'s are matrices which play no part in the following argument. It is easy to verify

$$P_p (A_p - B_p K C_p) + (A_p - B_p K C_p)^T P_p = \begin{bmatrix} P_1 (A_{11} - A_{12} P_{22}^{-1} P_{12}^T) + (A_{11} - A_{12} P_{22}^{-1} P_{12}^T)^T P_1 & * \\ * & * \end{bmatrix}$$

where $P_1 := P_{11} - P_{12} P_{22}^{-1} P_{12}^T$ and again the $*$'s represent (different) matrices which play no part in the argument. The matrix inequality (1) together with the expression above implies

$$P_1 (A_{11} - A_{12} P_{22}^{-1} P_{12}^T) + (A_{11} - A_{12} P_{22}^{-1} P_{12}^T)^T P_1 < 0$$

and consequently from standard Lyapunov theory the matrix $(A_{11} - A_{12} P_{22}^{-1} P_{12}^T)$ is stable. Using the expression for $P_{22}^{-1} P_{12}^T$ from (12) it follows that $(A_{11} - A_{12} F_2^{-1} F_1 C_1)$ is stable i.e. the triple

(A_{11}, A_{12}, C_1) is output feedback stabilizable as claimed. From the definition of C_1 from (10) and A_{11} from (5) it follows that

$$(A_{11} - A_{12}F_2^{-1}F_1C_1) = \begin{bmatrix} A_{1111} & A_{1112} & * \\ 0 & A_{1122} & * \\ 0 & A_{1132} & * \end{bmatrix}$$

where the $*$ represent matrix sub-blocks which play no part in the argument. As a consequence, $\sigma(A_{1111}) \subset \sigma(A_{11} - A_{12}F_2^{-1}F_1C_1)$. The preceding argument has shown that $(A_{11} - A_{12}F_2^{-1}F_1C_1)$ is stable and therefore the submatrix A_{1111} must be stable. From Lemma 1 the spectrum of A_{1111} precisely corresponds to the invariant zeros of (A, B, C) and so A2 must hold. This shows that a necessary requirement for solvability is that the system triple (A, B, C) is minimum phase.

‡

(Sufficiency) Suppose A1 and A2 hold then without loss of generality the triple (A, B, C) can be assumed to be in the form of Lemma 1. Under the assumptions of the proposition there exists a matrix $M \in \mathbb{R}^{m \times (p-m)}$ such that $A_{11} - A_{12}MC_1$ is stable. Define

$$F = F_2 \begin{bmatrix} M & I_m \end{bmatrix} T^T \quad (13)$$

where T is the orthogonal matrix in (7), $F_2 \in \mathbb{R}^{m \times m}$ and $\det F_2 \neq 0$ is a design parameter.

Change coordinates according to $x \mapsto T_M x$ where

$$T_M := \begin{bmatrix} I_{n-m} & 0 \\ MC_1 & I_m \end{bmatrix} \quad (14)$$

In the new coordinate system

$$\tilde{A} = T_M A T_M^{-1} := \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \quad \tilde{B} = T_M B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = B \quad (15)$$

where $\tilde{A}_{11} = A_{11} - A_{12}MC_1$ and so by construction is stable. Also

$$F\tilde{C} = FCT_M^{-1} = \begin{bmatrix} 0 & F_2 \end{bmatrix} \quad (16)$$

where F_2 is the parameter from (13). The expression in (16) follows from (7) and (13) since

$$FC = F_2 \begin{bmatrix} MC_1 & I_m \end{bmatrix}$$

Define $\tilde{P} := \text{diag}(\tilde{P}_1, I_m)$ where $\tilde{P}_1 \in \mathbb{R}^{(n-m) \times (n-m)}$. Notice that if $F_2 := B_2^T$, then by construction, $\tilde{B}^T \tilde{P} = F\tilde{C}$ and so (2) holds. Let $K :=: \gamma B_2^{-1} (B_2^T)^{-1} F = \gamma B_2^{-1} \begin{bmatrix} M & I_m \end{bmatrix} T^T$ where γ is a positive design scalar then $K\tilde{C} = \gamma B_2^{-1} (B_2^T)^{-1} F\tilde{C} = \begin{bmatrix} 0 & \gamma B_2^{-1} \end{bmatrix}$ and

$$\tilde{A} - \tilde{B}K\tilde{C} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} - \gamma I_m \end{bmatrix}$$

and therefore

$$\tilde{P}(\tilde{A} - \tilde{B}K\tilde{C}) + (\tilde{A} - \tilde{B}K\tilde{C})^T\tilde{P} = \begin{bmatrix} \tilde{P}_1\tilde{A}_{11} + \tilde{A}_{11}^T\tilde{P}_1 & \tilde{P}_1\tilde{A}_{12} + \tilde{A}_{21}^T \\ \tilde{A}_{12}^T\tilde{P}_1 + \tilde{A}_{21} & \tilde{A}_{22} + \tilde{A}_{22}^T - 2\gamma I_m \end{bmatrix} \quad (17)$$

Since \tilde{A}_{11} is stable, \tilde{P}_1 can be chosen so that $\tilde{P}_1\tilde{A}_{11} + \tilde{A}_{11}^T\tilde{P}_1 < 0$. Then from the Schur complement the right hand side of (17) can be made negative provided

$$\gamma > \max \left\{ 0, \frac{1}{2} \lambda_{\max} \left((\tilde{A}_{22} + \tilde{A}_{22}^T) - (\tilde{A}_{12}^T\tilde{P}_1 + \tilde{A}_{21})(\tilde{P}_1\tilde{A}_{11} + \tilde{A}_{11}^T\tilde{P}_1)^{-1}(\tilde{P}_1\tilde{A}_{12} + \tilde{A}_{21}^T) \right) \right\} \quad (18)$$

Thus A1, A2 and the stabilizability of the triple (A_{11}, A_{12}, C_1) are sufficient conditions. ■

Remark 4: In the original paper describing the Constrained Lyapunov Problem [7], the necessary and sufficient conditions for its solution are given in terms of A1 and the stabilizability of (A_*, B_*, C_*) where the matrices $A_* = \Theta A^T \Theta^T - \Theta^T C^T C B (B^T C^T C B)^{-1} B^T A^T \Theta$, $B_* = \Theta^T C^T \Lambda$ and $C_* = B^T A^T \Theta$; $\Theta \in \mathbb{R}^{n \times (n-m)}$ is rank $n - m$ and formed from orthogonal vectors which span $\mathcal{N}(B^T)$; and $\Lambda \in \mathbb{R}^{p \times (p-m)}$ is rank $p - m$ and formed from the orthogonal vectors which span the null space of $(CB)^T$. After some algebra and using the canonical form in Lemma 1 it can be shown $A_* = A_{11}^T$, $B_* = C_1^T$ and $C_* = B_2^T A_{12}^T$. It follows the conditions are the same but:

- it is straightforward from the canonical form in Lemma 1 to show that the pair (A_{11}, A_{12}) is controllable iff (A, B) is controllable. This was never explicitly addressed in [7];
- lack of detectability of (A_{11}, C_1) , and hence lack of stabilizability of the fictitious triple (A_{11}, A_{12}, C_1) , follows from the presence of invariant zeros of the original triple (A, B, C) lying in the RHP. The relationship between the detectability of (A_{11}, C_1) and the invariant zeros was never identified.

Remark 5: The difficult part of Proposition 1 is to establish the stabilizability by static output feedback of the triple (A_{11}, A_{12}, C_1) . This of course is still an open problem [12]. Nevertheless there are some significant advantages to the approach proposed in this paper:

- the CLP is reduced to a standard static output feedback problem and any of the wealth of existing methods and literature can be used;
- whereas the original system (A, B, C) has n states, p outputs and m inputs, the static output feedback problem to be studied is of reduced order: (A_{11}, A_{12}, C_1) has $n - m$ states, $p - m$ outputs and m inputs. Sometimes this reduced order problem is more amenable to solution. For example, in systems with one input and two outputs, the CLP problem reduces to a classical ‘root-locus’ investigation;
- restrictions on n, p, m can be imposed so that the Kimura-Davison conditions [12] are satisfied for (A_{11}, A_{12}, C_1) . This dimensionality inequality together with A1) and A2) then represent sufficient conditions for the CLP to be solved;

- if $n - m \leq m$ and $\text{rank}(A_{12}) = m$ the output feedback problem ‘collapses’ to a state-feedback problem for the pair (A_{11}^T, C_1^T) (see the example in Section IV).

Remark 6: For a given M which makes $A_{11} - A_{12}MC_1$ stable, the problem of finding a P and K to satisfy (1) and (2) is convex. In the coordinates associated with (14) and (15) if $K := [K_1 \ K_2]T^T$ where $K_1 \in \mathbb{R}^{m \times (p-m)}$ then

$$\tilde{A} - \tilde{B}K\tilde{C} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} - B_2K_1C_1 & \tilde{A}_{22} - B_2K_2 \end{bmatrix}$$

If $\tilde{P} = \text{diag}(\tilde{P}_1, \tilde{P}_2)$ then

$$\tilde{P}(\tilde{A} - \tilde{B}K\tilde{C}) + (\tilde{A} - \tilde{B}K\tilde{C})^T\tilde{P} < 0$$

is an LMI with respect to Y_1, Y_2, \tilde{P}_1 and \tilde{P}_2 where $Y_1 := \tilde{P}_2B_2K_1$ and $Y_2 := \tilde{P}_2B_2K_2$. The equality $\tilde{B}^T\tilde{P} = F\tilde{C}$ is satisfied provided F is chosen as in (13) where $F_2 := B_2^T\tilde{P}_2$. Once a feasible solution to the LMI with respect to Y_1, Y_2, \tilde{P}_1 and \tilde{P}_2 has been found (one is guaranteed to exist by the proof of Proposition 1) then K_1 and K_2 can be recovered from Y_1 and Y_2 respectively since B_2 and \tilde{P}_2 are nonsingular.

Remark 7: In the situation of an over-actuated system i.e. when $m > p$, the constraint in (2) cannot be satisfied since the right hand side will be rank m whilst the left hand side can have at most rank p . The method described above can be used on the dual system (A^T, C^T, B^T) . The dual system fits into the framework described above and so a control law $u_d = -Ky_d$ can be synthesized so that $(A^T - C^TKB^T)$ is stable. It follows that $(A - BK^TC)$ is stable and $u = -K^Ty$ is a controller for the original plant. Now the structural constraint becomes $P^{-1}C^T = BF$ [7].

B. The Observer Problem

Another lemma introducing a specific canonical form will now be quoted. It is similar to Lemma 1 but for clarity and ease of exposition later on it will be given in its entirety

Lemma 2: Let (A, B, C) represent a non-square system with $p > m$ and suppose $\text{rank}(CB) = m$. Then a change of coordinates exists so that (A, B, C) has the following structure:

1) The system matrix can be written as

$$A = \left[\begin{array}{c|c} A_{11}^o & A_{12}^o \\ \hline A_{21}^o & A_{22}^o \end{array} \right] \quad \text{and} \quad A_{21}^o = \begin{bmatrix} A_{211}^o \\ A_{212}^o \end{bmatrix} \quad (19)$$

where $A_{11}^o \in \mathbb{R}^{(n-p) \times (n-p)}$, $A_{211}^o \in \mathbb{R}^{(p-m) \times (n-p)}$ and when partitioned have the structure

$$A_{11}^o = \begin{bmatrix} A_{1111}^o & A_{1112}^o \\ 0 & A_{1122}^o \end{bmatrix} \quad \text{and} \quad A_{211}^o = \begin{bmatrix} 0 & A_{1132}^o \end{bmatrix} \quad (20)$$

where $A_{1111}^o \in \mathbb{R}^{r \times r}$ and $A_{1132}^o \in \mathbb{R}^{(p-m) \times (n-p-r)}$ for some $r \geq 0$ and the pair (A_{1122}^o, A_{1132}^o) is completely observable. Furthermore, the eigenvalues of A_{1111}^o are the invariant zeros of (A, B, C) .

2) The input distribution matrix has the form

$$B = \begin{bmatrix} 0 & B_2^T \end{bmatrix}^T \quad (21)$$

where $B_2 \in \mathbb{R}^{m \times m}$ is nonsingular.

3) The output distribution matrix has the form

$$C = \begin{bmatrix} 0 & T \end{bmatrix} \quad (22)$$

where $T \in \mathbb{R}^{p \times p}$ and is orthogonal

Proof This is similar to Lemma 1 ■

Remark 8: Whilst it has been assumed that (A, B) is controllable, no assumptions have been made concerning the observability of (A, C) . However using the Popov-Belevitch-Hautus test it can be easily shown from the canonical form in Lemma 2 that conditions A1 and A2 imply the pair (A, C) is detectable (and if (A, B, C) has no invariant zeros, then (A, C) is observable).

Proposition 2: For a given triple (A, B, C) there exists a gain matrix G and a s.p.d. matrix $P \in \mathbb{R}^{n \times n}$ such that (2)-(3) holds where $F \in \mathbb{R}^{m \times p}$ if and only if A1 and A2 hold.

Proof

(Necessity) Suppose there exist matrices G, P and F such that (2)-(3) hold. As in the proof of Proposition 1 because (2) is assumed to hold, it follows that $\text{rank}(CB) = m$ i.e. assumption A1 holds. By changing coordinates if necessary it can be assumed the triple (A, B, C) is in the form of Lemma 2. Let the s.p.d. matrix P have a partition

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

where $P_{11} \in \mathbb{R}^{(n-p) \times (n-p)}$ which is commensurate with the partition in (19). Now change coordinates $x \mapsto T_o x$ where

$$T_o := \begin{bmatrix} I_{n-p} & P_{11}^{-1} P_{12} \\ 0 & I_p \end{bmatrix}$$

In the new coordinates assume that $(A, B, C, G, P) \mapsto (A_o, B_o, C_o, G_o, P_o)$ and it follows that

$$P_o = (T_o^{-1})^T P T_o^{-1} = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} - P_{12}^T P_{11}^{-1} P_{12} \end{bmatrix}$$

Notice that as a result of the transformation $C_o = CT_o^{-1} = C$ and

$$B_o = T_o B = \begin{bmatrix} P_{11}^{-1} P_{12} B_{o2} \\ B_{o2} \end{bmatrix} \quad \text{where} \quad B_{o2} := \begin{bmatrix} 0_{(p-m) \times m} \\ B_2 \end{bmatrix} \quad (23)$$

and B_2 is defined in (21). In order that $B_o^T P_o = FC_o$ holds, $P_{12} B_{o2} = 0$ or equivalently $P_{11}^{-1} P_{12} B_{o2} = 0$ must hold. Furthermore

$$P_{11}^{-1} P_{12} B_{o2} = 0 \quad \Rightarrow \quad P_{11}^{-1} P_{12} = \begin{bmatrix} L & 0_{(n-p) \times m} \end{bmatrix} \quad (24)$$

where $L \in \mathbb{R}^{(n-p) \times (p-m)}$ because of the structure of B_{o2} in (23) and the fact that $\det(B_2) \neq 0$. After the change of coordinates

$$A_o = T_o A T_o^{-1} = \begin{bmatrix} A_{11}^o + P_{11}^{-1} P_{12} A_{21}^o & * \\ A_{21}^o & * \end{bmatrix}$$

where the *'s are matrices which play no part in the subsequent analysis. Because of the partition of C_o it follows that

$$A_o - G_o C_o = \begin{bmatrix} A_{11}^o + P_{11}^{-1} P_{12} A_{21}^o & * \\ A_{21}^o & * \end{bmatrix}$$

The fact that only the last p columns of A_o are affected by the output injection follows from the structure of C_o from (22). Consequently

$$P_o(A_o - G_o C_o) + (A_o - G_o C_o)^T P_o = \begin{bmatrix} P_{11}(A_{11}^o + P_{11}^{-1} P_{12} A_{21}^o) + (A_{11}^o + P_{11}^{-1} P_{12} A_{21}^o)^T P_{11} & * \\ * & * \end{bmatrix}$$

since $P_o(A_o - G_o C_o) + (A_o - G_o C_o)^T P_o < 0$ it follows

$$P_{11}(A_{11}^o + P_{11}^{-1} P_{12} A_{21}^o) + (A_{11}^o + P_{11}^{-1} P_{12} A_{21}^o)^T P_{11} < 0$$

As a result the matrix $(A_{11}^o + P_{11}^{-1} P_{12} A_{21}^o)$ is stable. From the structure of A_{11}^o and A_{21}^o from (20) it follows that

$$(A_{11}^o + L A_{21}^o) = \begin{bmatrix} A_{1111}^o & * \\ 0 & * \end{bmatrix}$$

where the * represent matrix sub-blocks which play no part in the argument. Consequently $\sigma(A_{1111}^o) \subset \sigma(A_{11}^o + L A_{21}^o)$ and since from the argument above $(A_{11}^o + L A_{21}^o)$ is stable, the sub-block A_{1111}^o must be stable. From Lemma 2 the invariant zeros of (A, B, C) are precisely the eigenvalues of A_{1111}^o and so A2 must hold. This shows that a necessary requirement for solvability is that the system triple (A, B, C) is minimum phase. $\#$

(Sufficiency) Now suppose A1 and A2 hold then, without loss of generality, change coordinates according to Lemma 2 and establish the matrices A_{11}^o and A_{21}^o as in equation (20). From

Lemma 2 the undetectable modes of the pair (A_{11}^o, A_{211}^o) are the invariant zeros of (A, B, C) and so consequently by assumption the pair (A_{11}^o, A_{211}^o) is detectable. Let

$$L = \begin{bmatrix} L_o & 0_{(n-p) \times m} \end{bmatrix} \quad \text{where} \quad L_o \in \mathbb{R}^{(n-p) \times (p-m)} \quad (25)$$

such that $A_{11}^o + L_o A_{211}^o$ is stable. Change coordinates $x \mapsto T_L x$ according to

$$T_L = \begin{bmatrix} I_{n-m} & L \\ 0 & I_p \end{bmatrix}$$

As a result of this change of coordinates $\bar{C} = C T_L^{-1} = C$ and

$$\bar{B} = T_L B = \begin{bmatrix} L \bar{B}_2 \\ \bar{B}_2 \end{bmatrix} \quad \text{where} \quad \bar{B}_2 := \begin{bmatrix} 0_{(p-m) \times m} \\ B_2 \end{bmatrix} \quad (26)$$

Because of the special structure of \bar{B}_2 from (26) and the structure of L in (25) $L \bar{B}_2 = 0$ and so $\bar{B} = T_L B = B$. The system matrix

$$\bar{A} = T_L A T_L^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \quad (27)$$

where $\bar{A}_{11} = A_{11}^o + L_o A_{211}^o$ and is therefore stable. Let $\bar{A}_s \in \mathbb{R}^{p \times p}$ be a symmetric negative definite matrix and define

$$\bar{G} = \begin{bmatrix} \bar{A}_{12} \\ \bar{A}_{22} - \bar{A}_s \end{bmatrix} T^T \quad \Rightarrow \quad \bar{A} - \bar{G} \bar{C} = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_s \end{bmatrix} \quad (28)$$

Notice $\lambda(\bar{A} - \bar{G} \bar{C}) = \lambda(\bar{A}_{11}) \cup \lambda(\bar{A}_s)$ and therefore $\bar{A} - \bar{G} \bar{C}$ is stable. Choose $\bar{P} = \text{diag}(\bar{P}_1, \alpha I_p)$ where $\bar{P}_1 \in \mathbb{R}^{(n-p) \times (n-p)}$ is s.p.d and α is a positive scalar. Since \bar{A}_{11} is stable, \bar{P}_1 can be chosen to make $\bar{P}_1 \bar{A}_{11} + \bar{A}_{11}^T \bar{P}_1 < 0$. Notice that $B^T \bar{P} = F C$ provided $F := \alpha \bar{B}_2^T T^T$ where \bar{B}_2 is defined in (26). It can be verified that

$$\bar{P}(\bar{A} - \bar{G} \bar{C}) + (\bar{A} - \bar{G} \bar{C})^T \bar{P} = \begin{bmatrix} \bar{P}_1 \bar{A}_{11} + \bar{A}_{11}^T \bar{P}_1 & \alpha \bar{A}_{21}^T \\ \alpha \bar{A}_{21} & 2\alpha \bar{A}_s \end{bmatrix} \quad (29)$$

Then from the Schur complement, the right-hand-side of (29) is negative definite if and only if

$$2\bar{A}_s < \alpha \bar{A}_{21} (\bar{P}_1 \bar{A}_{11} + \bar{A}_{11}^T \bar{P}_1)^{-1} \bar{A}_{21}^T \quad (30)$$

This can always be satisfied for small enough α , since by definition \bar{A}_s is symmetric negative definite. Consequently G , P and F can be found to satisfy (2) and (3). ■

Remark 9: The solvability conditions from [7] are that the pair (A_*, B_*) are stabilizable where $A_* := \Psi^T A^T \Psi - \Psi^T C^T C B (B^T C^T C B)^{-1} B^T A^T \Psi$ and $B_* := \Psi A^T C^T \Lambda$ where $\Psi \in \mathbb{R}^{n \times (n-p)}$ is a rank $n-p$ matrix formed from the orthogonal vectors which span $\mathcal{N}(C)$ and $\Lambda \in \mathbb{R}^{p \times (p-m)}$ is a

rank $p-m$ matrix formed from the orthogonal vectors which span the null space of $(CB)^T$. Again after some algebra, from the canonical form in Lemma 2, it can be shown that $A_* = (A_{11}^o)^T$ and $B_* = (A_{211}^o)^T$ and so the stabilizability condition for the pair (A_*, B_*) ties in with the results of Proposition 2. Furthermore

- Proposition 2 provides additional insight and concludes that the lack of stabilizability of (A_*, B_*) follows from the presence of invariant zeros of (A, B, C) lying in the RHP.
- The problem discussed in Proposition 2, can, through a change of variables, be transformed into a convex optimization problem. Specifically if a new variable $L = PG$ is introduced then (2)-(3) are Linear Matrix Inequalities (LMIs) [2] in terms of the decision variable L , P and F . This is not the case for the static output feedback problem. However the equality constraint (2) cannot be directly handled by several commonly used LMI solvers. The approach embedded in the proof of Proposition 2 can be used to circumvent this. Instead of choosing \bar{G} as in (28) allow the decision variable $G = T_L^{-1}\bar{G}$ to have a more general form. In the coordinates of Lemma 2 if

$$P := \begin{bmatrix} P_1 & -P_1L \\ -L^TP_1 & P_2 + L^TP_1L \end{bmatrix} \quad (31)$$

where L is given in (25), then for $F := \bar{B}_2^T P_2 T^T$ the constraint (2) is satisfied for all L , and s.p.d. matrices P_1 and P_2 . Consequently making the change of variables $P_{11} := P_1$, $P_{12} = -P_1L$ and $P_{22} = P_2 + L^TP_1L$ and $Y = PG$ where P is given in (31), a simpler convex problem appears in terms of P_{11} , P_{12} , P_{22} and Y . For given values of these variables, P_1 , P_2 and L , (i.e. P) and finally $G = P^{-1}Y$ can be obtained. The number of scalar decision variables associated with P_{11} , P_{12} , P_{22} and Y may be significantly less than those associated with P , G and F and in addition the equality constraint has been removed.

IV. EXAMPLE

Consider the following system which represents the longitudinal dynamics of a passenger aircraft

$$A_p = \begin{bmatrix} -0.6803 & 0.0002 & -1.0490 & 0 \\ -0.1463 & -0.0062 & 5.1216 & -9.7942 \\ 1.0050 & -0.0006 & -0.5717 & 0 \\ 1.0000 & 0 & 0 & 0 \end{bmatrix} \quad B_p = \begin{bmatrix} -1.5539 & 0.0154 \\ 0 & 1.3287 \\ -0.0398 & -0.0007 \\ 0 & 0 \end{bmatrix} \quad C_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The states represent pitch rate (rad/s), true airspeed (m/s), angle of attack (rad) and pitch angle (rad) respectively. The inputs are elevator deflection (rad) and thrust (10^5 N). Using the change

of coordinates $x \mapsto \mathcal{T}x$ where

$$\mathcal{T} = \begin{bmatrix} 0.0256 & -0.0008 & -1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 1.0000 & 0 & 0 \\ 1.0000 & 0 & 0 & 0 \end{bmatrix} \quad (32)$$

the system in the regular form of Lemma 1 can be represented by the triple

$$A = \begin{bmatrix} -0.5407 & 0.0080 & 0.0002 & -1.0085 \\ 0 & 0 & 0 & 1.0000 \\ -5.1216 & -9.7942 & -0.0104 & -0.0151 \\ 1.0490 & 0 & 0.0011 & -0.7072 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1.3287 \\ -1.5539 & 0.0154 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

It follows from Lemma 1

$$A_{11} = \begin{bmatrix} -0.5407 & 0.0080 \\ 0 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0.0002 & -1.0085 \\ 0 & 1.0000 \end{bmatrix} \quad C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The pair (A_{11}, C_1) is not observable and so -0.5407 is an invariant zero of (A, B, C) . Choosing $M^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$ implies

$$A_{11} - A_{12}MC_1 = \begin{bmatrix} -0.5407 & 1.0165 \\ 0 & -1.0000 \end{bmatrix}$$

and $\lambda(A_{11} - A_{12}MC_1) = \{-0.5407, -1\}$. It can be shown that $P_1 = I_2$ is a Lyapunov matrix for $\tilde{A}_{11} = A_{11} - A_{12}MC_1$ and after performing the change of coordinates in (14) to obtain \tilde{A} and its sub-matrices, it can be shown that γ from (18) must be greater than 114.3230.

From an observer perspective, from the canonical form in Lemma 2, $A_{11}^o = -0.5407$ and

$$(A_{21}^o)^T = \begin{bmatrix} 0 & -5.1216 & 1.0490 \end{bmatrix} \quad \Rightarrow A_{211}^o = 0$$

Consequently

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \left[\begin{array}{c|cccc} -0.5407 & 0.0080 & 0.0002 & -1.0085 \\ \hline 0 & 0 & 0 & 1.0000 \\ -5.1216 & -9.7942 & -0.0104 & -0.0151 \\ 1.0490 & 0 & 0.0011 & -0.7072 \end{array} \right]$$

If $\bar{A}_s = -I_3$ then from (28)

$$\bar{G} = \begin{bmatrix} -1.0085 & 0.0002 & 0.0080 \\ 1.0000 & 0 & 1.0000 \\ -0.0151 & 0.9896 & -9.7942 \\ 0.2928 & 0.0011 & 0 \end{bmatrix} \quad (33)$$

which means $\lambda(\bar{A} - \bar{G}\bar{C}) = \{-0.5407, -1, -1, -1\}$. It follows that $\bar{P}_1 = 1$ is an appropriate choice of Lyapunov matrix for \bar{A}_{11} and so from (30), $\alpha < 1/12.6380$ is a valid choice.

V. CONCLUSIONS

This paper has considered conditions for the solvability of the so-called Constrained Lyapunov Problem *for non-square systems*. Both a static output feedback problem and an observer problem have been considered. Necessary and sufficient conditions have been given based on system theoretic properties rather than the algebraic ones which appeared in the original work by Galimidi & Barmish [7]. The viewpoint adopted here is more akin to the recent work of Kim & Park [10] which has been extended in this paper to more general non-square systems. The basis for the work in this paper is a canonical form which simplifies the analysis and deals with the equality constraint in the CLP problem in a simple way. The advantages from the standpoint of convex representations (of the observer problem particularly) have also been given.

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