

Three Essays on Bank Profitability, Fragility, and Lending

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Abstract

We present three chapters on theoretical issues of banking. These deal with bank runs, risk sharing, lending and profitability. In the first chapter, we examine the agency problem in the bank-depositor relationship. Depositors are the principals and banks are the agents. Banks choose investment portfolios and are subject to moral hazard in that they have incentive to take on more risk than desirable to depositors because they are residual claimants. We study an incentive-compatible mechanism that prompts banks to follow a safe investment policy. This mechanism leaves the bank a profit margin in a similar manner to a CEO being paid a bonus by a company.

In the second chapter, we extend Allen and Gale (1998) by adding a long-term riskless investment opportunity to the original portfolio of a short-term liquid asset and a long-term risky illiquid asset. Through portfolio diversification, we identify the risk-sharing deposit contract in a three-period model that maximizes the ex-ante expected utility of depositors. Unlike Allen and Gale, there are no information-based bank runs in equilibrium. In addition, our model can improve consumers' welfare over the Allen and Gale model. I also show that the bank will choose to liquidate the cheaper investments, in terms of the gain-loss ratios for the two types of existing long-term assets, when there is liquidity shortage in some cases. Such a policy reduces the liquidation cost and enables the bank to meet the outstanding liability to depositors without large liquidation losses.

In the third chapter, we study the role of banks in providing loans to borrower firms. This paper extends the theory of designing optimal loan contracts (for profits) in the Bolton and Scharfstein (1996) model to a setting where asymmetry of information exists. Based on the verifiability of information structure, we analyze complete and incomplete contracts. Through this analysis,

optimal, incentive-compatible loan contracts that maximize the expected profit of the bank are characterized. Our analysis suggests that a bank could be induced to liquidate a borrower's project under specific conditions. Furthermore, we identify implementable mechanisms for the renegotiation game given the bargaining power between a borrower and a bank.

Chapter 1 Introduction

Banks play a critical role in the economy. They are important for investment projects, economic growth, and welfare. We study an economy where banks raise funds from depositors (liabilities) and provide loans to borrower firms (assets). By doing so, banks engage in maturity transformation by converting short-term liabilities into long-term assets. Hence, maturity mismatch between assets and liabilities can arise.

Historically and especially during the U.S. mortgage crisis of 2007-9, banking and financial crises are related to macroeconomic and liquidity problems. These problems caused financial distress, fire sales of bank assets, losses of GDP and welfare, and bank failures. Besides, these problems have destructive effects on the whole economy. We aim to analyze the effect of these problems on banking and learn some lessons for the future. To alleviate the bank destabilization, we provide some effective mechanisms as we shall see later.

Our work is divided into three main chapters in addition to the introductory chapter. In chapters 2 and 3, we attempt to describe the contractual relationships between the banks and their depositors then a bank-borrower setting is studied in chapter 4.

All three chapters use the same fundamental model of a bank which models taking deposits, lending, and investment as the core activities of a bank. Deposits are the source of funds and loans make use of these funds. Banks make profits equal the difference between revenues and costs. There are three periods and two states of nature, high and low.

In the first chapter, we investigate the conflict of interests between the bank and its risk-averse depositors. The starting point of our analysis is that there is an agency problem between the bank and depositors. Banks have incentive to invest in risky projects that pay high returns for them even if these returns occur with a small probability while depositors could suffer a large loss if the

investments fail. This is because depositors receive a fixed payment on their deposits and no share of any extra profit.

Risk-taking by banks can potentially cause bank failure. Thereby, we need to mitigate the agency problem.

The primary contribution of our study is to provide a simple incentive scheme that alleviates the agency problem. Namely, a bonus given to the bank for choosing a safe investment strategy.

Under full commitment, we find that the bank offers deposit contracts that maximize the expected utility of depositors and obtains a zero expected profit. If lack of commitment exists, the bank can make a strictly positive profit while the contracts are still accepted by depositors. Further, two theoretical interpretations are also provided in which the bank may or not exist.

Banking crises and bank runs are often triggered by economic downturn and uncertainty about the future of the economy. Yet, bank runs can further exacerbate the situation that creates them by causing real economic damage. Stopping bank runs can be difficult as we saw that several bank runs in the banking system despite the existence of lender of last resort, deposit insurance, and the banking regulations.

Banks are depository institutions that provide liquidity insurance to risk-averse depositors against uncertain liquidity needs. In the second chapter, we investigate the relationship between the bank and depositors. Depositors concern about the future prospects of the bank investments. We need to determine the suitable mechanism to improve the stability of the bank and the depositor welfare.

We extend Allen and Gale (1998) model by adding a long-term riskless asset to the original portfolio that includes two assets: short-term liquid and long-term risky illiquid. Risky asset returns are subject to macroeconomic conditions. Thereby, they cannot be controlled. Adding a riskless asset can have a significant effect on the depositor behavior and bank runs. This is because

it can accomplish diversification in the bank portfolio and strengthen the depositors confidence about the performance of portfolio returns.

The bank maximizes the expected utility of depositors subject to resource constraints and incentive constraints. From ex-ante perspective, we endogenously derive the optimal allocation of resources. The consumption allocation depends on risky asset returns.

We contribute to literature in the following three points. First, we characterize the optimal consumption allocation and risk sharing deposit contract of a diversified investment portfolio of three distinct assets. In contrast with Allen and Gale (1998), there are no information-based bank runs in equilibrium.

Second, our investment portfolio can enable the bank to provide additional liquidity to meet the liquidity needs quickly and at a low cost. That is, the bank will choose to liquidate a riskless asset in the high state and a risky asset in the low state. Hence, fire-sale losses can be avoided. This can maintain the depositors' confidence in the bank investments. In this case, the payments to late withdrawers still exceed the payments to early withdrawers.

Third, we provide a welfare analysis between our model and Allen and Gale model. We show that our model increases ex-ante welfare of depositors. In addition, it provides a higher risk sharing and better liquidity insurance over Allen and Gale model in the low state.

In the third chapter, we study a theory of lending and optimal loan contracts (for profits). Our theoretical framework provides the necessary conditions to show that these contracts are incentive-compatible to prevent the strategic behavior of borrowers.

Our analysis involves that borrower firms require funds from banks to finance their investment projects. We generalize and extend Bolton and Scharfstein (1996) by studying the role of symmetric and asymmetric information on lending process and contracting.

To achieve equilibrium, the bank maximizes its expected payoff under the borrower's limited

liability and incentive-compatibility constraints to not strategically behave.

We discuss two cases. First, in a world of symmetric information, we analyze the complete and incomplete contracting. Contracts are complete if states of nature are observable and verifiable. The third party (court) can enforce the contract. Hence, there is no role for liquidation or renegotiation. However, contracts are incomplete where the states of nature are non-contractible because they are observable but not verifiable by the third party. Hence, the bank has the right to liquidate the project or renegotiate the contract in period 1 following a low repayment by the borrower.

Second, we investigate incomplete contracts with asymmetric information between the borrower and the bank. The key feature is that the borrower knows the state of nature in period 1, while the bank does not. Incomplete contracts can be enforced if exogenous transaction costs are paid by the bank. Hence, the state of nature becomes observable and verifiable. Our analysis suggests that the bank has the right to liquidate the project if and only if the verification reveals the borrower's strategic behavior.

Costly verification can be considered as costly auditing. This auditing is stochastic. It aims to both provide a credible threat to borrowers by inducing them to report the cash flows truthfully in period 1 and reduce the deadweight auditing costs.

We also explore that ex-post renegotiation rather than liquidation can occur if the outcome of renegotiation is better to both parties.

In the following, we begin our contribution to the literature with the first of our three chapters.

Chapter 2 A Theory of Bank Profit

2.1 Introduction

Banking crises and bank failure are often caused by agency problems between banks and their depositors. The agency problem reflects the commitment problem on the side of the bank after signing the deposit contract.

The aim of this paper is to study the agency problem arising from a bank-depositor relationship. Depositors are the principals and banks are the agents. The agency problem is derived from investing in risky assets by a bank. Banks have incentives to take on more risk than desirable to depositors because they are residual claimants and their potential gains are considerable. Depositors would be more sensitive to the bank action and risk associated with deposits where a bank defaults on its obligations.

If a bank chooses a risky investment strategy, there is a possibility of earning higher returns that may benefit the depositors as well. However, there is also the probability of losing the deposits. Consequently, this increases the risk of bank failure. Notably, the global financial crisis is a result of excessive risk taking by banks. The key question then is how agency problems can be controlled to prevent the excessive risk-taking that led to the recent financial crisis?

Presumably there is no external enforcement mechanism to fully eliminate the agency problem and control the bank behavior. Thereby, the bank can be incentivized to act in the interest of the depositors by obtaining some economic rent or profit margin.

The initial margin is intended to motivate the bank to follow a safe investment policy when investing in the long-term assets. Such a design is similar to a bonus payment scheme or a reward performance related bonus being paid to a CEO by a company. This can act as a disciplinary device to alleviate the agency problem.

Our primary contribution relative to the literature is to provide incentive-compatible constraints to induce the bank to choose a safe investment strategy. This strategy can leave the bank with a profit margin. The result of this model shows that, for any utility function associated with depositors, the bank will make arrangements to protect deposits and choose only a safe portfolio under reasonable assumptions. We also provide conditions for the existence of the bank in our model.

Initially, we analyze two different contracts. Standard deposit contracts that provide fixed payments independent of the state and state-contingent contracts whose future payoffs are contingent on the realization of two states of the world.

In relation to literature, our model shares some aspects with the following models. One strand is characterized by the information set. Holmstrom (1979) mentions that, in a principal-agent relationship, moral hazard or incentive problems are results of asymmetric information among individuals which makes their actions unobservable and non-contractible. The informed party is able to generate some rents from the uninformed party. He also finds that monitoring any additional information about the agent's action can be employed to improve the welfare of both the principal and the agent.

Regarding the market structure, Shy and Stenbacka (2004) have shown that the banks' incentives for risk-taking are invariant to a change of the market structure in the banking industry from duopoly to monopoly. They present that competition can improve social welfare by producing higher deposit interest rates. They also find that the bank will sacrifice the higher expected returns of the risky portfolio to invest in a less risky portfolio if it faces sufficiently risk-averse depositors. Our analysis is consistent with this viewpoint by demonstrating that aspects of the competition in the market could not lead to riskier portfolios to provide higher interest rates to risk-averse depositors. But we differ in that depositors cannot observe the riskiness in the

portfolio at the time of depositing.

This model addresses some issues related to the portfolio structure. Fishburn and Porter (1976) show that risk-averse expected utility maximizing investors choose portfolios of safe and risky assets. Increases in the safe asset returns encourage investors to increase their holdings of these assets if the absolute risk aversion is nondecreasing or if proportional risk aversion never exceeds one. Bhattacharya (1981) shows that controlling interest rates paid to depositors can produce the efficient choice of the portfolio asset risk for an intermediary firm if the choice of the risk is unobservable in the presence/absence of deposit insurance.

This model is similar in vein to the idea of the models that give some benefits to the informed party for good behavior. i.e. safe investment decisions. This can have a positive effect on the depositor's utility. This has been studied by three related papers. Bisin, Gottard and Rampini (2008) show that the firm's shareholders provide incentives "bonuses or compensations" to the managers based on the good performance of the firms. Our model shows that the bank enjoys a rent if it chooses a safe investment strategy. Similarly, Shavell (1979) designs a Pareto optimal fee schedule by which the principal pays a fee to his agent to extract more effort. Greater effort by the agent will have a positive first order effect on the expected utility of the principal and hence a fee can be left to the agent to make them better. Vallascas and Hagendorff (2013) show that increases in CEO bonus payments decrease the default risk of a bank for a sample of U.S. and European banking firms.¹

Our analysis is also in line with the two related papers in terms of designing optimal incentive contracts. Yanelle (1997) presents a study of the transaction costs associated with the two transaction technologies in a world of uncertainty and asymmetry of information. In the first technology, creditors can eliminate asymmetry of information by paying monitoring costs that are

¹ See also financial stability board (2009) on the principles and standards for compensation schemes that reduce risk taking by banks and hence reduce the risk of bank failure.

very expensive. The second technology involves offering incentive-compatible contracts from creditors to debtors and imposing penalties on them to cover the difference between the initial value of debt and the actual repayment. We have incentive-deposit contracts, similar to the second technology, that reduce/eliminate the transaction costs. Biais et al. (2010) study the optimal dynamic contract in an agent-principal relationship that provides an incentive to the agent, who is protected by limited liability, to exert more efforts to mitigate the loss risk in a dynamic moral hazard problem. To do so, they use two tools: positive payments as compensation policy to the agent and firm size dynamics (downsizing and liquidation decisions). By contrast, we do not examine the impact of project size management in our model.²

Our analysis, in this context, is different from other strands of literature. The following papers model the problem of risky investment strategies and renegotiation. Mathews (2003) shows that more risk-averse agents will optimally hold a large proportion of the portfolio in the riskier asset of two risky assets which can lower the portfolio risk. Our work also contrasts with Hermalin and Katz (1991) who focus on the role of renegotiation in the standard principal-agent problem in which renegotiation can improve welfare. The agent takes an action and the principal receives an unverifiable signal of this action. Therefore, renegotiation is valuable because it reduces the cost of implementing any action to its first-best cost. The analogy with Fudenberg and Tirole (1990) is that they study the principal-agent problem with moral hazard. Conversely, they design optimal contracts that give the agent a positive rent after the renegotiation stage which occurs after the action is taken by the agent and before observing the consequences.

We do not model the influence of market structure on the bank risk taking unlike Dam and Castillo (2006), who consider the impact of market concentration on the banks' risk-taking behavior in the presence of deposit insurance. With highly competitive banking, banks tend to

² See also Holmstrom and Tirole (1997). They study an incentive model of financial intermediation in which firms and intermediaries are capital constrained.

invest in risky assets and take less risk when the market concentration is high.

Our study is methodologically different from other papers and has different results. For instance, Matutes and Vives (2000) show that limited liability causes maximal risk in an economy without deposit insurance if the portfolio risk is unobservable (moral hazard problem). Henceforth, both deposit limits (deposit rate ceilings) and direct asset restrictions are required when the social cost of failure is high and competition is severe. Boyd and De Nicoló (2005) study an environment in which banks' risk incentives increase as the market becomes more concentrated and the bank will choose riskier portfolios to extract monopoly rents.³

We also differ from the following models in their ways to prevent the agency problem and moral hazard problem. Andolfatto and Nosal (2007) study the agency problem between one bank and where the number of depositors exceeds two. The bank in their model, unlike ours, is both a profit-maximizing and a self-interested agent with a technological advantage in record-keeping over depositors. The consumption allocation will depend on the private information associated with the recorded history of depositors' types by the bank. However, depositors cannot remember their date 1 communications with the bank or date 1 consumption. Thus, banks can be incentivised to not make false reports to consumers to extract more profits.

Calomiris and Kahn (1991) show that demandable debt contracts can provide an incentive for disciplining the banker in the presence of costly information. In their model, the bank is a monopolist and it offers profit-maximizing contracts which yield to the depositors at least their reservation value S . More specifically, there is an agency problem caused by absconding and fraudulent behavior by the banker under which depositors can prevent absconding by choosing

³ Another important aspect of our model is that moral hazard problem can be avoided if depositors are risk-neutral unlike Harris and Raviv (1976) who show that moral hazard problem can be avoided if the agent is risk-neutral. We are also different from Poblete and Spulber (2012) in risk neutrality assumption. They study optimal agency contracts with moral hazard and limited liability in which both the principal and the agent are risk neutral. They introduce a "critical ratio" that shows the payments that provide the agent with incentives for exerting effort in each state. The form of the contract is debt (a capped bonus) when this ratio is increasing (decreasing) in the state. Holmstrom (1999) analyzes moral hazard problems associated with managerial risk-taking and provides incentive schemes to prevent them.

to liquidate the bank contingent on costly signals. The number of monitors and the threshold of liquidation are chosen optimally to minimize the corresponding costs of liquidation, absconding, and monitoring.

Diamond (1984) studies a costly monitoring by a financial intermediary, in an ex-post asymmetry of information, who collects funds from many lenders (depositors) promising them specific returns and provides loans to entrepreneurs. Optimal debt contracts are written between the intermediary and depositors. These contracts must be incentive-compatible to incentivise the intermediary to monitor the information about the outcomes of entrepreneurs' projects which are not observed by depositors to make sufficient payments to attract depositors. These incentives are costly but diversification of loans to many independent projects can reduce the cost of delegated monitoring. In our setting, the bank plays the role of both intermediary and entrepreneur and can be incentivised to act in the depositor's interest when given a minimum payment which exceeds the payment of a risky strategy.

The conflicts of interest in a bank among three different claimants "equity holders (managers), depositors and senior and subordinated debt holders" are described in Sheng (1991). Hence, managers would take riskier activities to maximize their profits while depositors would monitor the bank managers in order to transfer their deposits into safe banks as a disciplining device. Subordinated debt holders would also include restrictions on risk-taking in bond covenants. While in our model, we focus on the specific effect of a bank-depositors relationship only.

We do not model the impact of the government intervention and banking regulations explicitly.⁴ We do not feature the effect of the banking regulations unlike Yoon and Mazumdar (1996) who show that the riskiness in the banking industry can be reduced by imposing stricter capital requirements by the regulator to improve the quality of the bank loan portfolio. Similarly, we

⁴ Introducing deposit insurance tends to increase deposit rates substantially and subsequently it can increase the bank's risk taking when competing for deposits (Niinimä (2004)). Green and Lin (1999) show that the riskless asset can reduce the need for deposit insurance, but it abstracts from the choice of risk in the bank portfolio.

do not model capital structure and deposit insurance in comparison with Marini (2007) who demonstrates that a capitalized financial intermediary can provide liquidity to depositors at a low cost and invest more efficiently. Our work contrasts with the result of Giamarino, Lewis, and Sappington (1993), who find that imposing regulations on banks such as capital adequacy constraint, monitoring, and restrictions on lending can motivate banks to hold high quality portfolios.

In contrast to our setting, Diamond and Rajan (1999) argue that the banker's rents are driven to zero when the bank has the right amount of deposits outstanding. Hence, any attempt by the banker to extract a rent from depositors by threatening not to collect loans, by using its specific collection skills, will be met by a run.

Our approach is to address the above findings as follows. We apply the basic set-up of Diamond and Dybvig (1983) and we introduce the agency problem within its set-up. We study deposit contracts that incentivise banks, and attract deposits. These contracts have two types: standard contracts that give fixed repayments regardless of the realized state and state-contingent contracts whose repayments depend on the two states of nature. We show that these contracts are incentive-compatible for banks that prevent them from engaging in risky activities in addition to achieving optimal allocations for depositors. Hence, depositors will not monitor the bank to invest safely as a way of reducing the agency costs of monitoring and enforcement that affect the depositor's utility. Neither delegated monitoring nor renegotiation is captured in the features of our model.

The paper is organized as follows. We present the model in section 2. Deposit contracts in addition to some assumptions are provided in sections 3 and 4 respectively. In section 5, we introduce the specification of the bank portfolio and optimal allocation. Section 6 includes discussion. State-contingent contracts are established in section 7. Section 8 summarizes the main

results and the conclusion of the paper. Appendix can be found in section 9.

2.2 Model

The basic model is similar to Diamond and Dybvig (1983) and Allen and Gale (1998). However, we will have some significant differences by modelling the impact of the agency problem on both the bank and the depositors.

The time-line of the model consists of three discrete dates $t = \{0, 1, 2\}$. There is a continuum of risk-averse depositors with measure one and a competitive banking system. There is no money in the economy. Rather, there is a single homogeneous good that serves as a numeraire for consumption and investment in the economy.

Each depositor is endowed with one unit of the good at date 0. At date 0, depositors have uncertain liquidity needs. They are impatient with probability μ or patient with probability $1 - \mu$. Here, there is no aggregate uncertainty about the liquidity needs and depositor types. Depositors decide to place their endowments in the bank to obtain the future consumption and utility. There are two possible states of nature at date 1 indexed by H, L .

A bank collects the deposits or the units of the consumption good ω and invests in a portfolio of assets. The bank does not hide part of the endowment for itself and claims that it has been given to other depositors. The bank invests in short-term liquid assets I_s and long-term illiquid assets: riskless I_ℓ and risky I_r .⁵ Short-term assets pay one unit for each unit invested at either date 1 or 2. A riskless asset pays $R_\ell > 1$ at date 2 or $V_\ell < 1$ if it is liquidated at date 1. While a risky asset pays at date 2 $R_H > R_\ell$ with probability π or $R_L < 1$ with probability $1 - \pi$, but its liquidation value is $V_r < R_L$ at date 1. Uncertainty of risky asset returns is due to business cycle risk and changes in market conditions. In this context, we assume that $R = \pi R_H + (1 - \pi)R_L \leq R_\ell$.

⁵ The only way the bank can avoid the run and default if the bank invests everything in the liquid assets. However, if the bank invests all the funds in the short-term assets, the bank's expected profit = 0 as they need to pay the whole amount to depositors. However, the expected profit will be strictly positive if the bank invests a proportion of the fund in the long-term assets. In equilibrium, the optimal portfolio will consist of both short and long-term assets (Acharya, Shin, and Yorulmazer (2011))

Although their returns are greater than short-term asset returns, long-term assets cannot dominate short-term assets because of costly liquidation. For simplicity, there is no role for liquidation in this model.

At the beginning of date 1, each depositor receives liquidity shock and the state is realized. Uncertainty about the types is resolved. Still, types are private information of depositors. Suppose that the utility functions represent depositors' preferences. The utility of consumption is $U(C_t)$. $U(\cdot)$ is twice differentiable, strictly increasing, concave and satisfies the other Inada conditions. Being impatient means that a depositor derives their utility from date 1 consumption or being patient means deriving their utility from date 2 consumption.

2.3 Deposit Contracts

In the first part of our analysis, we study deposit contracts in which the consumption levels are fixed. The contracts that can be written here are similar to the real world deposit contracts.

In the second part, we study state-contingent contracts in which the consumption allocation depends on the future realized returns of the long-term risky assets.

2.4 Assumptions

- (1) Depositors place their funds in the bank which invests on their behalf.
- (2) At date 0, all depositors are identical. Each faces a privately observed, uninsurable risk of being impatient or patient. The bank offers the deposit contract that provide liquidity insurance to depositors who face uncertain liquidity intertemporal consumption preferences. The bank also chooses the investment portfolio to provide the consumption to impatient depositors at date 1 and patient depositors at date 2. At date 1, uncertainty is resolved in which depositors privately discover their types and the state is realized and it becomes common knowledge.
- (3) There is no bank equity capital.
- (4) The discount rate is zero.
- (5) There is no arbitrage opportunity in the economy “no free lunch” that yields a risk free profit for the bank.

2.5 Specification of Bank Portfolio and Optimal Allocation

2.5.1 First-Best Allocation and Bank Profit

The bank chooses a portfolio of short-term and long-term assets to maximize the expected

utility of depositors. On the basis of this analysis we assume that the bank can choose a risky investment strategy, a mix of risky and riskless investments or a safe investment strategy. Then we compare the results of these three sets of investment strategies from a welfare perspective. We present the following two lemmas before showing the main result in proposition 1.

2.5.1.1 Risky Investment Strategy

The bank chooses a portfolio of both liquid assets and long-term risky assets to maximize the expected utility of depositors. The objective function can be written as follows

$$\max_{C_1, C_2} E[\mu U(C_1) + (1 - \mu)U(C_2)]$$

subject to

$$\mu C_1 \leq I_s$$

$$\mu C_1 + (1 - \mu)C_2 \leq I_s + RI_r$$

$$I_s + I_r \leq \omega$$

$$C_1 \leq C_2$$

The first two constraints are the resource constraints. Specifically, the first constraint implies that early consumption must not exceed the short-term asset returns. While the second constraint implies that long-term risky asset returns and the remaining short-term asset returns must be sufficient to meet the date 2 consumption. The third constraint is the budget constraint. The last constraint is incentive-compatibility constraint for patient depositors to not withdraw early. The non-negativity constraints are $C_1 \geq 0$ and $C_2 \geq 0$.

Lemma 1 *The optimal consumption allocation of a portfolio of short-term and long-term risky assets satisfies that $C_1 > 1$ and $C_2 < R$.*

Proof. See Appendix. ■

2.5.1.2 Mix of Risky and Riskless Investments

Following a similar approach to that used above, the bank chooses a combination of three

different assets in the portfolio: short-term and long-term riskless and risky.

The objective function can be written

$$\max_{C_1, C_2} E [\mu U(C_1) + (1 - \mu)U(C_2)]$$

subject to

$$\mu C_1 \leq I_s$$

$$\mu C_1 + (1 - \mu)C_2 \leq I_s + I_\ell \cdot R_\ell + R I_r$$

$$I_s + I_\ell + I_r \leq \omega$$

$$C_1 \leq C_2$$

The only difference from the first case is the second constraint. It implies that the two long-term asset returns and the remaining short-term asset returns must be sufficient to meet patient depositors.

Lemma 2 *When there is a portfolio of three assets: short-term liquid and two long-term illiquid riskless and risky, the optimal consumption allocation will exceed the consumption allocation of a risky investment strategy since $R < R_\ell$.*

Proof. See Appendix. ■

2.5.1.3 Safe Investment Strategy (main result)

We need to endogenously derive the portfolio choice of a bank maximizing the expected utility of depositors in the presence of exogenous liquidity shock. Here, the bank invests in a portfolio of short-term and long-term riskless assets. Hence, there is agency problem (full commitment).

The objective function can be written as

$$\max_{C_1, C_2} E [\mu U(C_1) + (1 - \mu)U(C_2)]$$

subject to

$$\mu C_1 \leq I_s$$

$$\mu C_1 + (1 - \mu)C_2 \leq I_s + I_\ell R_\ell$$

$$I_s + I_\ell \leq \omega$$

$$C_1 \leq C_2$$

The bank maximizes the expected utility of depositors subject to similar constraints to the above situations. The only difference from these cases is investing in riskless assets. Hence, the implication of the second constraint refers to the availability of long-term riskless asset returns and the remaining short-term asset returns must be sufficient to meet date 2 consumption.

The condition for the optimal portfolio is $U'(C_1) = R_\ell U'(C_2)$. That is, the marginal utility equals marginal productivity. The solution also implies that $C_1 < C_2$ since $R_\ell > 1$. This also implies that the optimal consumption levels satisfy $C_1 > 1$ and $C_2 < R_\ell$. This requires the degree of relative risk aversion to be greater than one.

The deposit contracts can provide insurance against uncertain liquidity needs at date 1. In equilibrium, it is optimal for impatient depositors to withdraw at date 1 and patient depositors to wait until date 2 to withdraw. This equilibrium achieves optimal risk sharing among depositors.⁶

Proposition 1 *For a given condition $\pi R_H + (1 - \pi)R_L \leq R_\ell$, the consumption allocation under a portfolio of short-term and long-term riskless assets can dominate any other allocation under other investment strategies.*

Proof. See Appendix. ■

It can be noted that the benefit for depositors from a safe strategy exceeds the expected benefits from other investment strategies. The bank makes zero profit. There is no agency problem (full

⁶ This analysis is consistent with Diamond and Dybvig (1983).

commitment). In addition, the bank never invests in risky assets because there is no frictions in the market.

2.6 Discussion

We derive the optimal investment portfolios of the bank by maximizing the expected utility of depositors. The above utility maximization problems can be used to describe which of the aforementioned portfolios dominates in terms of welfare. In all cases, the bank earns exactly zero expected profits in equilibrium.

Note that the condition $R_\ell = [\pi R_H + (1 - \pi)R_L]$ should hold to choose a mix of two types of long-term assets in the portfolio by a bank. As a result, a risky investment strategy can be weakly dominated by a combination of three assets in the portfolio. The expected utility of the portfolio of three assets exceeds the expected utility of the risky portfolio.

Since the bank maximizes the depositor's expected utility in these cases, it will tend to invest in the safe strategy if and only if $[\pi R_H + (1 - \pi)R_L] \leq R_\ell$. The benefit of the safe strategy is captured by the fact the welfare of depositors will be higher. The bank will be motivated to choose the safe portfolio even if the choice of the bank assets is unobservable to depositors. The safe portfolio appears more likely to increase welfare if R_H is not too high. Likewise, if π is too small.

Our analysis can provide a rationale for the safe portfolio as an effective mechanism to improve welfare. Thus, the safe strategy can be considered as a dominant strategy (first-best) among other investment strategies.

Next, we extend our analysis to study the agency problem between the bank and the depositors. Accordingly, we design incentive-compatible contracts for the bank to maximize the expected utility of depositors.

2.6.1 Second-Best Allocation and Bank Profit

In this subsection, we examine the existence of the agency problem or lack of commitment in the bank-depositor relationship. This can open up the possibility for such rent seeking behavior

by a bank. The bank chooses the investment portfolio at date 0. The bank can choose to invest in risky assets because it is a residual claimant.

Depositors cannot observe the investment strategy of a bank ex-ante, but they can ex-post. Initially, depositors can neither enforce the bank to choose the investment strategy that would suit their interests in terms of risk tolerance and return nor impose an ex-post penalty on the bank for investing in risky projects due to the limited liability of the bank.

The bank should be given incentives to choose a safe investment strategy that alleviates the impact of the agency problem.

2.6.1.1 The Bank's Objective Function

In an environment without commitment, the banking authority or the economy has to offer incentives to the bank to maximize the expected utility of a representative depositor. Since depositors are ex-ante identical, the maximization problem for the depositor's utility is

$$\max_{C_1, C_2} \mu U(C_1) + (1 - \mu)U(C_2)$$

subject to

$$\mu C_1 \leq I_s$$

$$\mu C_1 + (1 - \mu)C_2 \leq I_s + I_\ell \cdot R_\ell$$

$$I_s + I_\ell \leq \omega$$

$$I_s + I_\ell R_\ell - \mu C_1 - (1 - \mu)C_2 \geq \pi(\tilde{I}_s + \tilde{I}_r R_H + \tilde{I}_\ell R_\ell - \mu C_1 - (1 - \mu)C_2) + (1 - \pi)(\max\{0, \tilde{I}_s + \tilde{I}_r R_L + \tilde{I}_\ell R_\ell - \mu C_1 - (1 - \mu)C_2\})$$

$$C_1 \leq C_2$$

$$C_1, C_2, I_s, I_\ell \geq 0$$

$$\forall \tilde{I}_s, \tilde{I}_r, \tilde{I}_\ell \geq 0 \text{ s.t. } \tilde{I}_s + \tilde{I}_r + \tilde{I}_\ell \leq \omega \text{ and } \mu C_1 \leq \tilde{I}_s$$

The first two constraints are the resource constraints. The third constraint is the budget constraint. The fourth constraint is the bank's incentive constraint. The main feature of this

constraint is characterized by the fact that the bank must be better off under the safe investment strategy than the other investment strategy (a combination of three different assets: short-term, long-term riskless and risky). This constraint will change the risk attitude and the investment strategy of the bank since depositors will not extract all rents from the bank through their consumption allocation. It also guarantees that the bank will not deviate from the contract. The fifth constraint is the incentive-compatibility constraint to depositors to ensure that late consumers will not misrepresent their types and withdraw early. Taking into consideration this optimization problem leads to the following proposition.

Proposition 2 *Under an agency problem in the bank-depositor relationship, a bank has an incentive to take a safe investment strategy if it produces a higher profit than any other strategy.*

Proof. See Appendix. ■

The results imply that the bank holds sufficient liquid assets to insure the depositors against uncertain liquidity needs. In other words, the payment to impatient depositors must be equal to I_g . That is, the constraint does indeed bind.

It is easy to see that the optimality condition can be derived as follows

$$\frac{U'(C_1)}{U'(C_2)} = \frac{(1 - \mu)C_2}{\omega - \mu C_1}$$

First order conditions are sufficient to give solutions at maximum since $U(C_t)$ is a concave function. The optimal allocation is obtained by the tangency of the objective function. This solution has the property that the ratio $\frac{R_\ell - \pi R_H}{1 - \pi} < R$ since $R_H > R_\ell$. The sign of $\frac{R_\ell - \pi R_H}{1 - \pi}$ is important to characterize the optimal consumption allocation. Further, it can capture the cost of the moral hazard problem associated with investing in long-term risky assets. Hence, two cases can be considered.

Case (1) It is easy to show that the bank exists if and only if $\frac{R_\ell - \pi R_H}{1 - \pi} \geq 1 \implies C_1 \leq C_2$ regardless of the utility function since ($U' > 0$, $U'' < 0$ from the concavity of U).

This yields the following result

$$C = C_1 = C_2 = \frac{(R_\ell - \pi R_H) \omega}{\mu(R_\ell - \pi R_H) + (1 - \mu)(1 - \pi)}$$

This case provides insight for the existence of the bank by providing positive returns on the bank portfolio in addition to the liquidity insurance against unobservable liquidity shock.

In equilibrium, the optimal consumption allocation is determined by the bank portfolio choice and how much will be held in each asset that satisfies FOCs

$$\begin{aligned} I_s &= \mu C_1 = \frac{(R_\ell - \pi R_H) \cdot \mu \cdot \omega}{\mu(R_\ell - \pi R_H) + (1 - \mu)(1 - \pi)} \\ I_\ell &= \omega - I_s \end{aligned}$$

It follows immediately that the bank's profit is given by the following equation

$$\Pi = \omega \cdot R_\ell - \mu(R_\ell - 1)C > 0$$

Provided that the short-term asset returns enough to pay impatient depositors and the second constraint is nonbinding to generate some profits for the bank at date 2 equal to $I_\ell R - (1 - \mu)C_2$, the bank will be motivated to invest in a safe portfolio only. However, the bank profit will drop to zero if the constraint binds.

Under the standard demand deposit contract, the bank is successfully induced to invest in a safe strategy given a strictly positive profit in equilibrium. Hence, we claim that investing in the safe investment strategy can be regarded as a dominant strategy for the bank under the specification of this model.

Case (2) Now consider if $\frac{R_\ell - \pi R_H}{1 - \pi} < 1$ then the solution has the following properties $C_1 + C_2 < \omega$ so there is no bank because deposit contracts can neither provide insurance against the liquidity risk nor produce positive returns on the bank portfolio. Accordingly, the depositors will not participate in such a bank and they would rather invest their funds in money market instruments if they existed.

In the next section, we analyze state-contingent contracts and a risky investment strategy of a bank.

2.7 Risky Portfolio and State-Contingent Contracts

It is known that the returns on risky assets deteriorate during the economic downturns. If banks invest in long-term risky assets and offer standard deposit contracts, they are more likely subject to default in the low state. However, the risk of bank failure and agency problem can be hypothetically eliminated or reduced under state-contingent contracts.

We should note that standard deposit contracts can be dominated by state-contingent contracts if these contracts yield higher welfare to depositors than the standard contracts.⁷

The bank portfolio is chosen at date 0 and the consumption allocation will depend on this realization of the state to efficiently share risk between depositors. The state is realized at date 1 and it becomes common knowledge.

The bank offers the consumption allocation $\{C_1^*(L), C_2^*(L)\}$ for the low state and $\{C_1^*(H), C_2^*(H)\}$ for the high state. The optimal consumption and portfolio allocation are defined by the following optimization problem.

2.7.1 Optimization Problem under State-Contingent Contracts

The bank invests in a portfolio of assets: short-term and long-term risky and offers state-contingent contracts. The bank can maximize the ex-ante welfare of depositors subject to the resource constraints, a budget constraint and incentive-compatibility constraints for depositors as follows

$$\max_{C_1(\cdot), C_2(\cdot)} \pi [\mu U(C_1(H)) + (1 - \mu)U(C_2(H))] + (1 - \pi) [\mu U(C_1(L)) + (1 - \mu)U(C_2(L))]$$

⁷ Diamond and Rajan (2005) show that offering non-state contingent deposit contracts can lead to liquidity shortage, solvency problem and bank failure.

subject to

$$\mu C_1(L) \leq I_s$$

$$\mu C_1(H) \leq I_s$$

$$\mu C_1(L) + (1 - \mu)C_2(L) \leq I_s + I_r R_H$$

$$\mu C_1(H) + (1 - \mu)C_2(H) \leq I_s + I_r R_L$$

$$I_s + I_r \leq \omega$$

$$C_1(L) \leq C_2(L)$$

$$C_1(H) \leq C_2(H)$$

Proposition 3 *Under state-contingent contracts, the optimal consumption allocation across the two states of nature must satisfy the*

$$\text{conditions} \left\{ \begin{array}{l} C_1(L) = C_1(H) \\ C_2(H) = \frac{R_H}{R_L} C_2(L) \end{array} \right. \text{ if } I_s < I_r R_H \text{ or } \left\{ \begin{array}{l} C_1(H) = C_2(H) \\ C_1(L) = C_2(L) \end{array} \right. \text{ if } I_s \geq I_r R_L$$

Proof. The proof provides the solutions of the problem which are given by solving the optimization problem of a representative bank. Check the Appendix. ■

The intuition for this result is as follows. The bank's expected profit under the state-contingent contracts equals zero. The bank will be able to meet the demand for withdrawals in both states of the world.

2.8 Concluding Remarks

Banking and financial systems have experienced large losses during the recent financial crisis 2007-2009. These losses are mainly attributed to the sharp decline in the bank investments (real estate assets). The crisis emphasizes the need to make banks safer.

The agency problem associated with the bank's risk-taking behavior can lead to bank failure. Banks should be given incentives to not take excessive risks (gamble) with the depositors' funds when they make their investment decisions.

We have developed a simple theoretical framework for bank portfolio choice. One implication

of this model is that the bank will obtain zero profit under the full commitment and depositors will obtain the first-best allocation in the utility maximization problem with a safe investment strategy.

Another important implication is that we show that the bank will obtain a strictly positive profit that is driven by the agency problem associated with lack of commitment on the side of the bank, while depositors will obtain the second-best allocation.

Alternatively, the bank expects to make zero profit on a risky investment strategy and state-contingent contracts. These contracts might produce higher welfare to depositors, but they are hardly realistic. One possible extension in future research is to study the effect of bank regulations on the bank portfolio.

2.9 Appendix

2.9.1 The Proof of Lemma 1

The bank maximizes the expected utility of a depositor by choosing a portfolio of short-term and long-term risky assets

$$\begin{aligned}
& \max_{C_1, C_2} \mu U(C_1) + (1 - \mu)U(C_2) \\
& - \lambda_1 [\mu C_1 - I_s] \\
& - \lambda_2 [\mu C_1 + (1 - \mu)C_2 - I_s - RI_r] \\
& - \lambda_3 [I_s + I_r - \omega]
\end{aligned}$$

The non-negativity constraint $C_1 \geq 0$ and $C_2 \geq 0$.

The first-order conditions of the underlying utility function are given as

$$\frac{\delta \mathcal{L}}{\delta C_1} = \mu U'(C_1) - \mu \lambda_1 - \mu \lambda_2 = 0$$

$$U'(C_1) = \lambda_1 + \lambda_2 \tag{A1}$$

$$\frac{\delta \mathcal{L}}{\delta C_2} = (1 - \mu)U'(C_2) - (1 - \mu)\lambda_2 = 0$$

$$U'(C_2) = \lambda_2 \tag{A2}$$

$$\frac{\delta \mathcal{L}}{\delta I_s} = \lambda_1 + \lambda_2 - \lambda_3 = 0$$

$$\lambda_1 + \lambda_2 = \lambda_3 \quad (\text{A3})$$

$$\frac{\delta \mathcal{L}}{\delta I_r} = R\lambda_2 - \lambda_3 = 0$$

$$R\lambda_2 = \lambda_3 \quad (\text{A4})$$

(A3) and (A4) imply that

$$\lambda_1 + \lambda_2 = R\lambda_2$$

Now substitute λ_1, λ_2 to find out that

$$U'(C_1) = RU'(C_2) \quad (\text{A5})$$

This implies that $C_1 < C_2$ since $R > 1$. It also implies that the optimal consumption allocation satisfies $C_1 > 1$ and $C_2 < R$.

2.9.2 The Proof of Lemma 2

Solving the bank's problem is similar to solving the previous problem under the risky assets.

The Lagrangian

$$\begin{aligned} & \max_{C_1, C_2} \mu U(C_1) + (1 - \mu)U(C_2) \\ & - \lambda_1 [\mu C_1 - I_s] \\ & - \lambda_2 [\mu C_1 + (1 - \mu)C_2 - I_s - I_\ell R_\ell - RI_r] \\ & - \lambda_3 [I_s + I_\ell + I_r - \omega] \end{aligned}$$

The first-order conditions for the maximization problem are given as

$$\frac{\delta \mathcal{L}}{\delta C_1} = \mu U'(C_1) - \mu\lambda_1 - \mu\lambda_2 = 0$$

$$U'(C_1) = \lambda_1 + \lambda_2 \quad (\text{B1})$$

$$\frac{\delta \mathcal{L}}{\delta C_2} = (1 - \mu)U'(C_2) - (1 - \mu)\lambda_2 = 0$$

$$U'(C_2) = \lambda_2 \quad (\text{B2})$$

$$\frac{\delta \mathcal{L}}{\delta I_s} = \lambda_1 + \lambda_2 - \lambda_3 = 0$$

$$U'(C_1) = \lambda_3 \quad (\text{B3})$$

$$\frac{\delta \mathcal{L}}{\delta I_r} = R\lambda_2 - \lambda_3 = 0$$

$$R\lambda_2 = \lambda_3 \quad (\text{B4})$$

$$\frac{\delta \mathcal{L}}{\delta I_\ell} = \lambda_2 R_\ell - \lambda_3 = 0$$

$$R_\ell \lambda_2 = \lambda_3 \quad (\text{B5})$$

(B4) and (B5) imply that

$$R = R_\ell \quad (\text{B6})$$

$$U'(C_1) = RU'(C_2) = R_\ell U'(C_2) \quad (\text{B7})$$

2.9.3 The Proof of Proposition 1

First we construct the utility maximization and Lagrangian function

$$\begin{aligned} & \max_{C_1, C_2} \mu U(C_1) + (1 - \mu)U(C_2) \\ & - \lambda_1 [\mu C_1 - I_s] \\ & - \lambda_2 [\mu C_1 + (1 - \mu)C_2 - I_s - I_\ell \cdot R_\ell] \\ & - \lambda_3 [I_s + I_\ell - \omega] \end{aligned}$$

The first-order conditions for the maximization problem are given as

$$\frac{\delta \mathcal{L}}{\delta C_1} = \mu U'(C_1) - \mu \lambda_1 - \mu \lambda_2 = 0$$

$$U'(C_1) = \lambda_1 + \lambda_2 \quad (\text{C1})$$

$$\frac{\delta \mathcal{L}}{\delta C_2} = (1 - \mu)U'(C_2) - (1 - \mu)\lambda_2 = 0$$

$$U'(C_2) = \lambda_2 \quad (\text{C2})$$

$$\frac{\delta \mathcal{L}}{\delta I_s} = \lambda_1 + \lambda_2 - \lambda_3 = 0$$

$$\lambda_1 + \lambda_2 = \lambda_3 \quad (\text{C3})$$

$$\frac{\delta \mathcal{L}}{\delta I_\ell} = R_\ell \lambda_2 - \lambda_4 = 0$$

$$R_\ell \lambda_2 = \lambda_3 \quad (\text{C4})$$

This implies that

$$U'(C_1) = \lambda_1 + \lambda_2 = \lambda_3 = R_\ell \lambda_2 \quad (\text{C5})$$

Now substitute $\lambda_2 = U'(C_2)$, the following condition must be satisfied for the optimal portfolio

$$U'(C_1) = R_\ell U'(C_2) \quad (\text{C6})$$

With binding constraints, the solution to the above system of equations is unique. To obtain a unique portfolio of the bank

$$I_s = \mu C_1$$

$$I_\ell = \omega - I_s$$

Consumption allocations can be expressed as

$$C_1 = \frac{I_s}{\mu}$$

$$(1 - \mu)C_2 = I_\ell \cdot R_\ell \implies C_2 = \frac{I_\ell \cdot R_\ell}{1 - \mu}$$

Hence, the bank's profit = 0.

2.9.4 The Proof of Proposition 2

IC can be written as

$$I_s + I_\ell R - \mu C_1 - (1 - \mu)C_2 \geq \pi(\tilde{I}_s + \tilde{I}_r R_H + \tilde{I}_\ell R - \mu C_1 - (1 - \mu)C_2)$$

$$+(1 - \pi)(\tilde{I}_s + \tilde{I}_r R_L + \tilde{I}_\ell R - \mu C_1 - (1 - \mu)C_2)$$

By substituting $\tilde{I}_s = I_s = \mu C_1$, IC can be reduced to

$$I_\ell R \geq \pi(\tilde{I}_r R_H + \tilde{I}_\ell R) + (1 - \pi)(\tilde{I}_r R_L + \tilde{I}_\ell R)$$

We can keep increasing C_1, C_2 to reach the point where the expression $(1 - \pi)(\tilde{I}_s + \tilde{I}_r R_L + \tilde{I}_\ell R - \mu C_1 - (1 - \mu)C_2)$ equals zero. Rather, we can increase C_1, C_2 in the objective function

which can increase the utility function to obtain a similar result.

The bank's problem becomes

$$\max_{C_1, C_2} \mu U(C_1) + (1 - \mu)U(C_2)$$

s.t.

$$\mu C_1 = I_s$$

$$I_s + I_\ell R - \mu C_1 - (1 - \mu)C_2 = \pi(I_s + I_r R_H + I_\ell R - \mu C_1 - (1 - \mu)C_2)$$

The bank will invest in short and long-term risky assets only in a portfolio of three different assets and will not invest in the long-term riskless assets because $R_H > R$. Then we drop the long-term riskless asset $I_l = 0$ from the right hand side of the second equation. This in turn implies that

$$I_s + I_\ell R - \mu C_1 - (1 - \mu)C_2 = \pi(I_s + I_r R_H + I_\ell R - \mu C_1 - (1 - \mu)C_2)$$

$$\implies I_s + I_\ell R - \mu C_1 - (1 - \mu)C_2 = \pi(I_s + I_r R_H - \mu C_1 - (1 - \mu)C_2)$$

Then by substituting $\mu C_1 = I_s$ and $I_r = I_\ell$, one can see that

$$I_s + I_\ell R - \mu C_1 - (1 - \mu)C_2 = \pi(I_s + I_r R_H - \mu C_1 - (1 - \mu)C_2)$$

$$\implies I_\ell R - (1 - \mu)C_2 = \pi(I_\ell R_H - (1 - \mu)C_2)$$

Now substitute $I_\ell = \omega - \mu C_1$

$$(\omega - \mu C_1)R - (1 - \mu)C_2 = \pi [(\omega - \mu C_1)R_H - (1 - \mu)C_2]$$

$$\implies (\omega - \mu C_1)(R - \pi R_H) = (1 - \pi)(1 - \mu)C_2$$

Then the bank's problem in a relaxed form can be stated as follows

$$\max_{C_1, C_2} \mu U(C_1) + (1 - \mu)U(C_2)$$

s.t.

$$(\omega - \mu C_1)R - (1 - \mu)C_2 = \pi [(\omega - \mu C_1)R_H - (1 - \mu)C_2]$$

Lagrangian function becomes

$$\mathcal{L} = \mu U(C_1) + (1 - \mu)U(C_2) + \lambda [(\omega - \mu C_1)(R - \pi R_H) - (1 - \pi)(1 - \mu)C_2]$$

First order conditions FOCs:

$$\frac{\delta \mathcal{L}}{\delta C_1} = \mu U'(C_1) - \lambda \mu (R - \pi R_H) = 0$$

$$U'(C_1) = \lambda (R - \pi R_H)$$

$$\frac{\delta \mathcal{L}}{\delta C_2} = (1 - \mu)U'(C_2) - \lambda (1 - \mu)(1 - \pi) = 0$$

$$U'(C_2) = \lambda (1 - \pi)$$

$$\frac{\delta \mathcal{L}}{\delta \lambda} = (\omega - \mu C_1)(R - \pi R_H) - (1 - \pi)(1 - \mu)C_2 = 0$$

$$(\omega - \mu C_1)(R - \pi R_H) = (1 - \pi)(1 - \mu)C_2$$

By dividing $\frac{U'(C_1)}{U'(C_2)}$, we find that

$$\frac{U'(C_1)}{U'(C_2)} = \frac{R - \pi R_H}{1 - \pi}$$

Where $\frac{U'(C_1)}{U'(C_2)}$ represents the ratio of expected marginal utilities of consumption at dates 1 and 2.

The equation $(\omega - \mu C_1)(R - \pi R_H) = (1 - \pi)(1 - \mu)C_2$ implies that

$$\frac{R - \pi R_H}{1 - \pi} = \frac{(1 - \mu)C_2}{\omega - \mu C_1}$$

Then by substituting $C_1 = C_2 = C$ into the maximization problem it becomes

$$\max_{C_1, C_2} \mu U(C_1) + (1 - \mu)U(C_2) = \mu U(C) + (1 - \mu)U(C) = U(C)$$

We can now rewrite the above condition as follows

$$\frac{R - \pi R_H}{1 - \pi} = \frac{(1 - \mu)C_2}{\omega - \mu C_1} \implies \frac{R - \pi R_H}{1 - \pi} = \frac{(1 - \mu)C}{\omega - \mu C}$$

The bank profit:

$$\begin{aligned}
\Pi &= I_s + I_\ell R_\ell - \mu C_1 - (1 - \mu)C_2 \\
&= I_\ell R_\ell - (1 - \mu)C_2 \\
&= (\omega - I_s)R_\ell - (1 - \mu)C_2 \\
&= (\omega - \mu C_1)R_\ell - (1 - \mu)C_1 \text{ by plugging } C_1 = C_1 = C \\
&= \omega \cdot R_\ell - \mu(R_\ell - 1)C > 0
\end{aligned}$$

2.9.5 The Proof of Proposition 3

To prove this, we define the standard Lagrangian optimization problem as:

$$\begin{aligned}
\max_{C_1, C_2} \pi [\mu U(C_1(H)) + (1 - \mu)U(C_2(H))] + (1 - \pi) [\mu U(C_1(L)) + (1 - \mu)U(C_2(L))] \\
- \lambda_{1L} [\mu C_1(L) - I_s] \\
- \lambda_{1H} [\mu C_1(H) - I_s] \\
- \lambda_{2L} [\mu C_1(L) + (1 - \mu)C_2(L) - I_s - I_r R_L] \\
- \lambda_{2H} [\mu C_1(H) + (1 - \mu)C_2(H) - I_s - I_r R_H] \\
- \lambda_3 (I_s + I_r - \omega)
\end{aligned}$$

$$\lambda_{1L}, \lambda_{1H}, \lambda_{2L}, \lambda_{2H}, \lambda_3 \geq 0$$

We need to solve for $C_1(L), C_2(L), C_1(H), C_2(H), I_s, I_r$.

The first-order conditions are

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta C_1(L)} = (1 - \pi)\mu U'(C_1(L)) - \lambda_{1L}\mu - \lambda_{2L}\mu = 0 \\
(1 - \pi)U'(C_1(L)) = \lambda_{1L} + \lambda_{2L} \tag{D1}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta C_2(L)} = (1 - \pi)(1 - \mu)U'(C_2(L)) - \lambda_{2L}(1 - \mu) = 0 \\
(1 - \pi)U'(C_2(L)) = \lambda_{2L} \tag{D2}
\end{aligned}$$

$$\frac{\delta \mathcal{L}}{\delta C_1(H)} = \pi \mu U'(C_1(H)) - \lambda_{1H} \mu - \lambda_{2H} \mu = 0$$

$$\pi U'(C_1(H)) = \lambda_{1H} + \lambda_{2H} \quad (\text{D3})$$

$$\frac{\delta \mathcal{L}}{\delta C_2(H)} = \pi(1 - \mu)U'(C_2(H)) - \lambda_{2H}(1 - \mu) = 0$$

$$\pi U'(C_2(H)) = \lambda_{2H} \quad (\text{D4})$$

$$\frac{\delta \mathcal{L}}{\delta I_s} = \lambda_{1L} + \lambda_{1H} + \lambda_{2L} + \lambda_{2H} - \lambda_3 = 0$$

$$\lambda_{1L} + \lambda_{1H} + \lambda_{2L} + \lambda_{2H} = \lambda_3 \quad (\text{D5})$$

$$\frac{\delta \mathcal{L}}{\delta I_l} = \lambda_{2L}R_L + \lambda_{2H}R_H - \lambda_3 = 0$$

$$\lambda_{2L}R_L + \lambda_{2H}R_H = \lambda_3 \quad (\text{D6})$$

Substitute λ_{2L} in (D1)

$$\lambda_{1L} = (1 - \pi) [U'(C_1(L)) - U'(C_2(L))]$$

Substitute λ_{2H} in (D3)

$$\lambda_{1H} = \pi (U'(C_1(H)) - U'(C_2(H)))$$

Now substitute (D1), (D3) in (D5)

$$\pi U'(C_1(H)) + (1 - \pi)U'(C_1(L)) = \lambda_3$$

By plugging (D2), (D4) in (D6)

$$(1 - \pi)U'(C_2(L))R_L + \pi U'(C_2(H))R_H = \lambda_3$$

This implies that

$$\pi U'(C_1(H)) + (1 - \pi)U'(C_1(L)) = (1 - \pi)U'(C_2(L))R_L + \pi U'(C_2(H))R_H \quad (\text{D7})$$

If the constraints bind

$$\mu C_1(L) = I_s$$

$$\mu C_1(H) = I_s$$

$$(1 - \mu)C_2(L) = I_r R_L$$

$$(1 - \mu)C_2(H) = I_r R_H$$

This implies that

$$C_1(L) = C_1(H)$$

$$(1 - \mu)C_2(L) = (\omega - \mu C_1(L)) R_L$$

$$(1 - \mu)C_2(H) = (\omega - \mu C_1(L)) R_H$$

To solve these linear equations we multiply $(1 - \mu)C_2(L) = (\omega - \mu C_1(L)) R_L$ by $\frac{R_H}{R_L}$ then we

subtract the first equation from the second $(1 - \mu) \left[C_2(H) - \frac{R_H}{R_L} C_2(L) \right] = 0$

Consumption allocations should satisfy $\begin{cases} C_1(L) = C_1(H) \\ C_2(H) = \frac{R_H}{R_L} C_2(L) \end{cases}$ if $I_s < I_r R_H$ (high state)

We know that

$$I_s = \mu C_1(L)$$

$$I_r = \omega - I_s$$

Optimally, I show that banks have effectively met their obligations by holding enough resources in each period to fulfill their promises with binding resource constraints. Otherwise, patient depositors will receive less than impatient depositors and incentive constraints for consumers will not hold and solutions cannot be optimal.

If the first two constraints do not bind: $\lambda_{1L} = \lambda_{1H} = 0, \lambda_3 > 0, \lambda_{2L}; \lambda_{2H} > 0$.

F.O.Cs become

$$(1 - \pi)U'(C_1(L)) = \lambda_{2L}$$

$$(1 - \pi)U'(C_2(L)) = \lambda_{2L}$$

This implies that

$$U'(C_1(L)) = U'(C_2(L))$$

$$C_1(L) = C_2(L)$$

Similarly,

$$\pi U'(C_1(H)) = \lambda_{2H}$$

$$\pi U'(C_2(H)) = \lambda_{2H}$$

This implies that

$$U'(C_1(H)) = U'(C_2(H))$$

$$C_1(H) = C_2(H)$$

Consumption allocations should satisfy $\begin{cases} C_1(L) = C_2(L) = I_s + I_r R_L \\ C_1(H) = C_2(H) = I_s + I_r R_H \end{cases}$ if $I_s \geq I_r R_H$ (low state)

If the last two constraints do not bind: $\lambda_{2L} = \lambda_{2H} = 0, \lambda_3 > 0, \lambda_{1L}; \lambda_{1H} > 0$.

F.O.Cs become

$$(1 - \pi)U'(C_1(L)) = \lambda_{1L}$$

$$(1 - \pi)U'(C_2(L)) = 0$$

Similarly,

$$\pi U'(C_1(H)) = \lambda_{1H}$$

$$\pi U'(C_2(H)) = 0$$

D5, D6 become

$$\lambda_{1L} + \lambda_{1H} = \lambda_3$$

$$0 = \lambda_3$$

λ_3 cannot be zero because $\lambda_{1L}; \lambda_{1H} > 0$ by assumption. Hence, there is no solution in this case.

If $\lambda_{1L} = \lambda_{1H} = \lambda_{2L} = \lambda_{2H} = 0, \lambda_3 > 0$.

F.O.Cs become

$$(1 - \pi)U'(C_1(L)) = 0$$

$$(1 - \pi)U'(C_2(L)) = 0$$

Similarly,

$$\pi U'(C_1(H)) = 0$$

$$\pi U'(C_2(H)) = 0$$

D5, D6 become

$$0 = \lambda_3$$

λ_3 cannot be zero because it is greater than 0 by assumption. Hence, there is no solution in this case.

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Chapter 3 The Design of Deposit Contracts, Liquidity Provision, and Investment Portfolios

3.1 Introduction

Banks take deposits and use them to finance their investment portfolios with different maturities. They pool their financial resources ex-ante to fund the potential investment projects and allow depositors to withdraw ex-post, contingent on their liquidity needs. Banks have to plan to meet liquidity demands for early and late withdrawals. Banks can face the possibility of massive unexpected withdrawals of deposits. The potential for panic withdrawals leads to bank runs.

The essential question is: What are the causes of bank runs and banking panics? Historical analysis shows that banking fragility and bank runs may be caused by macroeconomic conditions such as business cycle downturns, recessions, or exogenous liquidity shocks.

Depositors rush to withdraw their deposits early if there is negative information about future return of the bank asset portfolio. Thereby, we need to determine the suitable mechanism to improve the stability of the bank and the depositor welfare.

Our objective is to develop a simple model of Allen and Gale (1998), henceforth AG, which captures risk sharing but with the bank vulnerable to runs. This model has inspired a large amount of banking research recently. The model has three dates and features the bank-depositor framework under which the economy is subject to random liquidity shock. I will extend the model that consists of two assets: a short-term liquid asset and a long-term risky asset by adding a long-term riskless asset which would smooth consumption and ultimately contribute to the stability of the bank.

A long-term riskless asset in the portfolio is important for three reasons. (1) It can prevent overinvestment in the short-term assets. (2) It can also act as a buffer against severe deterioration

in risky asset returns in the economic downturn. (3) It can provide liquidity and stability in the bank portfolio when there is excess liquidity demand. Our analysis is consistent with previous studies. Franck and Krauz (2007) state that holding more cash by a bank to face liquidity shortages is a waste of resources. In addition, it decreases the bank's profitability since the amount of loans provided has decreased. Thus, they indicate that overinvestment in short-term assets can be prevented when both a lender of last resort and a securities market exist. Wagner (2007) shows that an increase in asset liquidity makes capital requirements less effective in ensuring the bank stability. Diamond (2007) shows that liquidity of bank assets can provide depositors with insurance against costly liquidation.

It should be noted that a good way to provide a stable bank is to set the rules and instruments ex-ante such that there is no bank run in the future. We start with a benchmark case for an optimal risk sharing when the fraction of early consumer is fixed. We shall determine the bank holding of three different types of assets in equilibrium to obtain the optimal consumption allocation.

However, a potential liquidity mismatch between the short-term liabilities and the resources available at date 1 may occur. Thus, the bank needs to liquidate long-term assets. The logic is how banks can respond to the excess liquidity demands with no need to liquidate the assets at fire-sale prices.

This paper contributes to the growing literature in a fairly tractable way by considering the following standpoints. First, the bank specifies the deposit contract, allocation of resources, and liquidation decision that lead to less fragile bank and better consumption risk sharing that is conditional on uncertain risky asset returns.

Second, our model has a positive effect on depositor welfare. Welfare increases over AG, regardless of the degree of relative risk aversion, by reducing its holding of short-term assets and investing more in the productive long-term riskless asset that absorbs losses.

Third, the stability of the bank has increased. We obtain a unique equilibrium that alleviates the run risk.

Fourth, we show that there exists a second-best risk sharing mechanism if there is an excess demand for liquidity at date 1. The bank will compare the liquidation costs of two long-term assets and then it will choose to liquidate a risky asset in the low state and a riskless asset in the high state. This reduces the liquidation costs and smooths consumption.

We shall examine the underlying mechanisms of the aforementioned results in the following sections. The remainder of this paper is organized as follows. Section 2 provides a quick review of the relevant literature with emphasis on a bank-depositor relationship. Section 3 characterizes the underlying environment. Section 4 refers to a benchmark case. Section 5 analyzes excess demand for liquidity and liquidation. Section 6 presents the discussion. Section 7 includes the banking panics. Section 8 contains concluding remarks and the Appendix is provided in section 9.

3.2 Related Literature

Our paper is related to a number of strands of the theoretical literature on the determinants of bank runs and banking fragility based on the investment-consumption problem. We derive the optimal allocation of resources by solving the related utility-maximization problem of the bank deposits. Broadly speaking, we relate our paper to the literature by analyzing the following areas.

First, we describe some causes of bank runs. Diamond and Dybvig (1983), henceforth DD, show that there are multiple equilibria and self-fulfilling prophecies at date 1. A good equilibrium is where only impatient depositors would withdraw and a bad equilibrium (a bank run) is where all depositors will withdraw early if a run is feared. Diamond and Rajan (2001) show that a bank with a fragile capital structure is subject to runs. Hence, bank fragility can induce the bank to create liquidity, allow depositors to withdraw on demand as well as provide a buffer for borrowers against liquidity shock. Chari and Jagannathan (1988) analyze the impact of fears of bank insolvency after observing a large number of withdrawers on precipitating the possibility

of bank runs. Jacklin and Bhattacharya (1988) show that information-based runs are attributed to asymmetry of information. On the one hand, the bank cannot observe the actual needs of liquidity for depositors. On the other hand, depositors may be unaware of the bank investments. Allen and Gale (1998) consider that bank runs are a clear result of the standard deposit contract. Calomiris and Kahn (1991) show that the bank run serves as a disciplinary tool to prevent the banker absconding.

However, some ways to prevent banking panics and bank runs can be addressed. Ennis and Keister (2010) consider that self-fulfilling run in DD can be prevented by relaxing sequential service constraint. Green and Lin (1999) find that truth-telling equilibrium (by depositors about their types) can yield efficient allocation in the version of DD without leading to bank runs. Postlewaite and Vives (1987) show that bank runs should not be observed in equilibrium since no one would deposit anticipating such a run. Instead, depositors can deposit their fund in the bank provided that the possibility of a run is very low.

Second, the bank performs risk sharing between depositors. AG show that a run occurs if and only if the signal of the business cycle at date 1 indicates that date 2 return is zero. Hence, impatient and patient depositors receive equal shares of the liquid assets at date 1 assuming that risky assets cannot be liquidated. Moreover, given this assumption, a partial run can occur if the signal at date 1 implies that date 2 return is small enough because only a fraction of patient depositors, who believe that the bank will be unable to fully meet their needs, withdraws at date 1 and others will wait because there is always something left to pay for late consumers. Then the possibility of equilibrium bank runs causes the bank to hold the efficient risk sharing deposit contract and the optimal portfolio. Our work is in a similar vein with the basic set-up of AG in designing risk sharing deposit contracts.

Our model has similar features to most of the above models which help us explore the causes

of the banking crises and bank runs as well as find tools for preventing such crises. In comparison, our first result shows that there is a unique good equilibrium and it rules out an information-based bank run in a risk sharing contract without aggregate uncertainty about the depositors' types even in the presence of sequential service constraint.

Third, we explain the prospect of excess liquidity demand and liquidation. This work is also consistent with Gale and Yorulmazer (2011) in studying liquidity management through investigating the impact of random liquidity shocks on liquid and illiquid banks. This considerably influences illiquid banks by enforcing them to sell their assets at fire-sale prices while liquid banks are not exposed to such a risk. Shin (2009) considers a Northern Rock bank run in 2007 and the role of the Bank of England as a lender of last resort in providing emergency liquidity assistance. It is easy to see that this problem emphasizes the need for an adequate liquidity to meet deposit withdrawals. In their analysis of the U.S. Great Depression, Anari et al. (2005) show that banking crises and liquidation of banks are pertaining to the general economic performance. Safe banks achieve the optimal allocation of resources and diversify their portfolios in different investment opportunities to prevent bank runs and banking crises. However, our model contributes to the literature by showing that there is no information-based bank runs in equilibrium when there is excess demand for liquidity.

Our analysis differs from other papers in the literature. Lazopoulos (2012) compares between two types of contracts, in the presence of intrinsic aggregate uncertainty about the liquidity demand, demand deposit and default free equity contracts in terms of welfare. He also shows that deposit contracts can dominate under corner preferences since they offer more liquidity insurance to risk-averse consumers. However, our work does not exhibit features of equity contracts. Rather, we compare our findings to the basic set-up of AG.

Further, we examine how different models in literature use different methods and techniques

when resolving banking crises and bank runs. It is shown in Diamond and Rajan (2001) that the bank stability obtained under such procedures as deposit insurance, lender of last resort LLR and suspension of convertibility is costly because they decrease “the bank commitment power provided by a run prone capital structure.” Oh and Wrase (1990) design a deposit contract that prevents a speculative bank run by suspending the payments. Specifically, this suspension may not prevent fundamental bank runs. In contrast, we do not consider the role of deposit insurance, LLR, capital structure or suspension of convertibility in our model. Cheung and Hailiang Yang (2007) describe a model of investment-consumption problem for a risk-averse investor under which the return of risky assets depends on the state of the economy by using a Markov chain with an absorbing state for bankruptcy.

In short, these models help us better understand bank runs, but they have quite different methodologies for identifying the optimal investment and consumption allocation to ours.

This paper analyzes the desirable effects of adding the long-term riskless asset on risk sharing and depositors’ welfare. In such an environment, there is a unique equilibrium under which depositors engage in risk sharing contracts to prevent fundamental bank runs.

We shall next analyze the portfolio diversification in addition to the balance sheet of banks.

3.3 Portfolio Diversification and Bank’s Balance Sheet

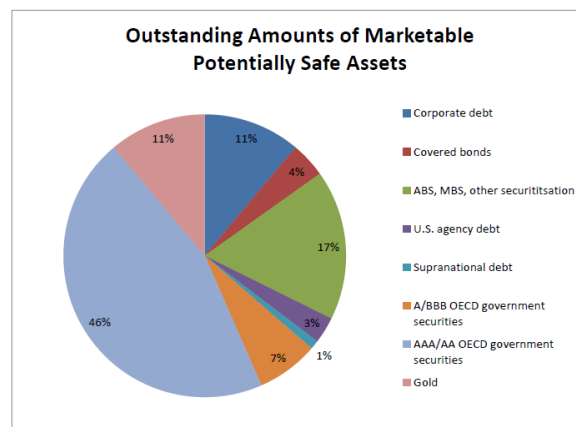
The bank can use deposits to build-up a diversified portfolio from different investment opportunities to reduce the liquidity risk. Suppose the following balance sheet of a bank

Assets	Liabilities
Long-term assets (risky and riskless)	Deposits
Short-term assets	

On the asset side of the balance sheet, treasury bonds are an example of riskless assets while loans or securities can be considered as risky assets. Treasury bills or cash can represent short-term assets.

At end-2011, AAA-rated and AA-rated OECD government securities accounted for \$33.2

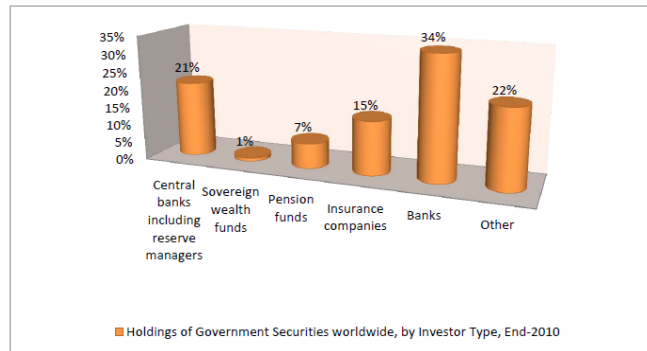
trillion out of \$74.4 trillion or 45 percent of the total supply of potentially safe assets as follows



Source: Bank for International Settlements; Dealogic; the European Covered Bond Council, the Securities Industry and Financial Markets Association; Standard & Poor's, World Gold Council; and IMF staff estimates.⁸

Safe assets “long-term government bonds” are important, as explained recently by the International Monetary Fund, in the international financial markets because they have the following characteristics. (1) They provide a steady stream of income over time while preserving portfolio values. (2) They serve as high-quality collateral to secure the transactions. (3) They are also integral to prudential regulations. (4) They are often considered as performance benchmarks where the yields on government bonds are reference rates for the hedging, asset pricing, and valuation of risky assets. (5) They are a good part of liquidity operations adopted by major central banks in response to the crisis.

⁸ Data for government and corporate debt are as of 2011:Q2; supranational debt and gold, as of end-2011; covered bonds, as of end-2010; and U.S. agency debt and securitization, as of 2011:Q3, ABS=asset-backed securities, MBS=mortgage-backed securities; OECD=Organization for Economic Cooperation and Development.



Source: Bank for International Settlements; Bankscope; Organization for Economic Cooperation and Development; and IMF staff estimates.⁹

The Bank of England stated in its July 2015 Financial Stability Report that UK banks are well prepared to face liquidity shocks by increasing their holdings of liquid assets and high quality unencumbered securities (riskless assets). UK banks have trebled their holdings of these assets since the recent financial crisis to perfectly match Liquidity Coverage Ratio (LCR) requirements of 100% (of stressed outflows).

We shall next present a description of the underlying structure of the model that can make the bank more stable.

3.4 The Environment

In this section, we present the model that includes the timeline, consumption good, banking sector, depositors and endowments, the contract and optimization problem, and feasibility constraints.

3.4.1 The Model Framework and the Main Characteristics of the Model

We introduce a class of the important features of this model within this framework.

⁹ Banks include commercial, investment, and development banks; data for pension funds include only direct holdings; SWF holdings are an IMF staff estimate; reserve manager holdings are an IMF staff estimate based on a representative allocation of total official reserves to government securities and own government bond holdings by the Federal Reserve, Bank of England, and Bank of Japan. Other is estimated as a remainder based on BIS data on total outstanding government securities worldwide.

3.4.1.1 Timeline and Consumption Good

To keep things simple, let us assume that the economy has three discrete dates 0, 1, and 2 and a single homogeneous consumption good.

3.4.1.2 Banking Sector and Technology Set

Consider a competitive banking sector with free entry. We assume that there is a representative bank. The bank has no initial capital. The bank raises funds ω from depositors to invest on their behalf.

At date 0, the bank deposits are invested in constant-returns-to-scale investment technologies called assets. There are three types of assets available to the bank: a short-term asset I_o , a long-term riskless asset I_s , and a long-term risky asset I_r . An investment of one unit of the date 0 good in the short-term asset yields one unit at date 1 or one unit at date 2. An investment of one unit at date 0 in the long-term riskless asset yields $R_s > 1$ at date 2, or $L_s < 1$ if it is liquidated at date 1. Also, an investment of one unit at date 0 in the long-term risky asset yields in expectation $E[R_r] > R_s$ at date 2 as compensation for this risk, or $L_r < L_s$ if it is liquidated at date 1. Thus, the bank will not invest all of its deposits in the long-term assets due to costly liquidation.

There are two possible realizations of the state of nature: high and low. The probability of a high state is π , while the probability of a low state is $1 - \pi$. The realization of the state is common knowledge at date 1. The return of the risky investment depends on the economic state. It takes h in the high state and ℓ in the low state. i.e. $R_r = \{h, \ell\}$.

The bank follows the sequential service constraint in responding to early withdrawal requests and is protected by limited liability.

3.4.1.3 Depositors and Endowments

The economy has a continuum with measure one of ex-ante identical depositors. Each depositor is endowed with one unit of time 0 good and none at subsequent dates. Depositors place their endowments in the bank at date 0 in exchange for consumption at dates 1, 2.

There are two types of depositors: patient and impatient. There is no aggregate uncertainty over the types of depositors. We assume that a law of large numbers holds so that depositors know that they are impatient with probability μ and patient with probability $1 - \mu$. Impatient depositors derive their utility only from consuming at date 1 while patient consumers derive their utility from date 2 consumption. Henceforth, depositors become consumers.

The economy is hit by liquidity shock at date 1. Each consumer faces a privately observed uninsurable risk of being impatient or patient. Impatient consumers are affected by liquidity shock. The uncertainty is resolved at date 1 and consumers learn their types whether they are impatient or patient.

Note that the utility function of a consumer is von Neumann-Morgenstern utility $U(C)$, where C represents the consumption level. We assume that the utility function has the following form

$$U(C) = \frac{C^{1-\rho}}{1-\rho}$$

The utility function reflects the consumers' attitude toward the risk. Note that $U'(C) > 0$ and $U''(C) < 0$. Thus, this function is indeed concave and increasing. Without loss of generality, we assume that $U(0) = 0$. We assume the coefficient of the relative risk aversion for the agents is greater than unity and consumers are not too risk-averse.

We first assume that the bank will hold sufficient resources to meet early consumption needs in the basic set-up then we relax this assumption later to introduce that the bank has insufficient liquid assets to meet all of its commitments at date 1.

In what follows, I will summarize the specification and timing of the model at these three dates:

- Date 0: Depositors (consumers) place their funds into the bank and then the bank chooses the consumption and asset allocation decisions between short-term liquid and long-term illiquid assets.
- Date 1: In the beginning of date 1, all consumers know their types and they also realize the state of nature. The bank may need to liquidate some of the long-term assets to meet excess early demand.
- Date 2: The payoffs from long-term assets are used to meet the consumption demand of late

consumers.

Figure 1 presents the outline of the model

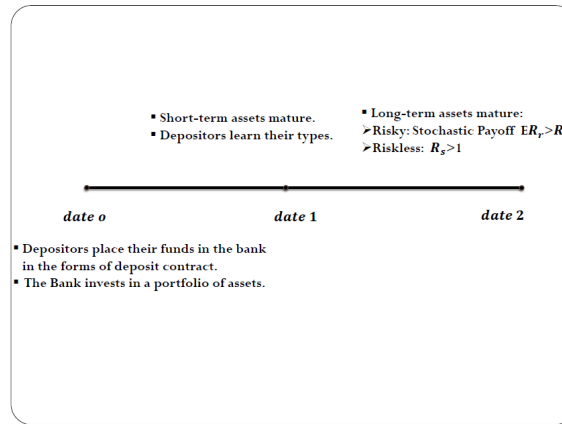


Figure 1: The time-line

We will mention, in the context of this work, some measures that can be undertaken to support the stability of the liability side of the balance sheet (the bank deposits) and the bank as a whole.

3.4.2 The Contract

The standard deposit contracts with fixed payments have a potential undesirable equilibrium where all consumers run on the bank if they perceive a low state expecting that the bank will be unable to meet the contractual promises.

Our model analyzes risk sharing contracts that are contingent on the realization of the risky asset returns. Note that banks offer risk sharing contracts when consumers are uncertain about their liquidity needs. Moreover, the consumption allocation will depend on the date of withdrawal not on the consumers' types which are unobservable. Denote the deposit contracts by the pairs $\{C_1^*(\ell), C_2^*(\ell)\}$ for the low state or $\{C_1^*(h), C_2^*(h)\}$ for the high state.

The contingency of risk sharing contracts on the long-term risky asset returns are superior to the standard demand deposit contracts because the amount of consumption a consumer receives will depend on the state of nature and the associated returns. This allows the bank to meet the depositor liquidity needs in both states by holding sufficient resources.

3.4.3 Assumptions

- (1) Consumers cannot invest directly in the asset market.
- (2) The consumers place their entire endowments in the bank at date 0 to obtain the future consumption and utility.¹⁰
- (3) There are a continuum of ex-ante identical depositors, whose measure is normalized to 1. Each faces idiosyncratic uncertainty about his preference type (early or late consumers). The bank offers the deposit contract and chooses the investment portfolio to provide the consumption to early consumers at date 1 and late consumers at date 2. All uncertainty is resolved at date 1. Consumers privately discover their types and the state is realized and it becomes common knowledge.
- (4) Ignore the market discount rate.
- (5) Consumers will not withdraw funds from banks to buy higher-yielding securities “opportunity cost of holding securities” since there are no financial markets.
- (6) Since borrowing from other banks or financial sources is not possible, the bank is forced to liquidate long-term assets when there is excess liquidity demand.
- (7) All assets are divisible.
- (8) Partial liquidation is possible.
- (9) Liquidation value of a risky asset is less than a low state return $L_r < l$. If $l < L_r$, an arbitrage opportunity can be exploited by immediately liquidating the asset for a higher return rather than causing a welfare loss if it is kept.
- (10) There are no transaction costs.

3.4.4 Optimization Problem and Feasibility Constraints

The consumers care about maximizing the expected utility of consumption that reflects their preferences as follows

$$E[\mu U(C_1(R_r)) + (1 - \mu)U(C_2(R_r))]$$

$C_1(R_r)$ and $C_2(R_r)$ are the quantities consumed by an agent at dates 1, 2 respectively.

Intuitively, the bank chooses a set of admissible investment strategies to maximize the expected utility of a representative consumer subject to feasibility constraints at $t=0$:

$$\max \pi [\mu U(C_1(h)) + (1 - \mu)U(C_2(h))] + (1 - \pi)[\mu U(C_1(l)) + (1 - \mu)U(C_2(l))] \quad (3.1)$$

¹⁰ Banks can provide both liquidity and higher returns. This combination is better than that of a depositor who makes his own investment decision and invests by himself (Allen and Gale, 2002). Allen and Gale (2004) assume that the bank has access to long-term assets while depositors do not. Hence, it can invest in a portfolio of short-term and long-term assets that provide a better combination of returns and liquidity than individual consumers.

$$\mu C_1(\ell) \leq I_o \quad (3.2)$$

$$\mu C_1(h) \leq I_o \quad (3.3)$$

$$\mu C_1(\ell) + (1 - \mu)C_2(\ell) \leq I_o + I_s R_s + I_r \ell \quad (3.4)$$

$$\mu C_1(h) + (1 - \mu)C_2(h) \leq I_o + I_s R_s + I_r h \quad (3.5)$$

The starting point is to show how to efficiently allocate the resources. The fraction of impatient consumers is μ , while the fraction of patient consumers is $1 - \mu$. Where $0 < \mu < 1$. Furthermore, $C_1(\ell)$ and $C_2(\ell)$ are the quantities consumed by an agent at dates 1, 2 respectively if a low state occurs, while $C_1(h)$ and $C_2(h)$ are the quantities consumed at dates 1, 2 respectively if a high state occurs.

The first two constraints imply that short-term asset returns must be sufficient to meet date 1 consumption. If the bank chooses to hold excess liquidity (the return of the short-term investments exceeds date 1 consumption), then it can be costlessly stored and transferred to late consumers.

The third and the fourth constraints imply that the long-term investment returns and the remaining short-term asset returns must be sufficient to provide consumption for late consumers.

The bank invests I_o , I_s , I_r in short-term, long-term riskless and risky investments respectively. Let ω denote the initial endowment of the consumption good. The following constraint says that the investment portfolio should satisfy the budget constraint

$$I_o + I_s + I_r \leq \omega \quad (3.6)$$

The marginal utility of substitution of early consumption for late consumption for a

representative consumer is

$$MRS_{c_1, c_2} = \frac{\mu MU_{c_1}}{(1 - \mu) MU_{c_2}}$$

Early consumers cannot imitate late consumers but late consumers can pretend that they are impatient and obtain $\{C_1(\ell); C_1(h)\}$ at date 1 and then they can store them to consume at date 2. It is optimal to do this unless

$$\begin{aligned} C_1(\ell) &\leq C_2(\ell) \\ C_1(h) &\leq C_2(h) \end{aligned} \tag{3.7}$$

The late consumers must be at least as well off as the early consumers.

The above conditions can make the problem nontrivial and ensure incentive-compatibility for depositors and the contract is a run-proof once types have been known.

Finally, consumption must be non-negative

$$C_1(\ell); C_1(h); C_2(\ell); C_2(h) \geq 0$$

We next study the properties of risk sharing and equilibrium under two cases: (1) Benchmark case (without excess liquidity demand). (2) Liquidity needs and liquidation.

3.5 Benchmark Case (without Excess Demand for Liquidity)

This section presents a version of risk sharing and utility maximization for the case in which the bank has a correct belief about the fraction of impatient depositors who decide to withdraw early. The analysis characterizes the consumption of both early and late consumers with respect to risky asset returns.

3.5.1 Characterization of the Optimal Risk Sharing Contract

We start with a case where the bank has sufficient liquid resources to meet early withdrawals. Hence, there is no need to liquidate long-term assets. The bank maximizes (1) subject to the constraints (2), (3), (4), (5), (6), (7). Thus, we solve the bank's optimization problem for the consumption and investment at date 0 and we derive the first-best solution. The results are

summarized in proposition 1.

Proposition 1 *A risk sharing deposit contract given that the bank's belief about the fraction of impatient depositors matches μ satisfies the following. (a) The optimal consumption allocation is*

$$C_2(h) = \frac{\pi^{\frac{1}{\rho}} R_s (h - \ell)}{\mu \left[\frac{(R_s - \ell)^{\frac{1}{\rho}}}{[R_s(h - \ell)]^{\frac{1-\rho}{\rho}}} \right] + (1 - \mu) \left[\frac{\pi^{\frac{1}{\rho}} (h - R_s)^{\frac{1-\rho}{\rho}} (R_s - \ell) + (1 - \pi)^{\frac{1}{\rho}} (R_s - \ell)^{\frac{1}{\rho}}}{(h - R_s)^{\frac{1-\rho}{\rho}}} \right]} \cdot \omega$$

$$C_2(\ell) = \left[\frac{(1 - \pi)(R_s - \ell)}{\pi(h - R_s)} \right]^{\frac{1}{\rho}} C_2(h) \leq C_2(h)$$

$$C_1 = \left[\frac{R_s - \ell}{\pi R_s (h - \ell)} \right]^{\frac{1}{\rho}} C_2(h) \geq 1$$

(b) *Bank investments are*

$$I_o = \mu \left[\frac{R_s - \ell}{\pi R_s (h - \ell)} \right]^{\frac{1}{\rho}} C_2(h)$$

$$I_r = \frac{1 - \mu}{h - \ell} \left\{ \frac{[\pi(h - R_s)]^{\frac{1}{\rho}} - [(1 - \pi)(R_s - \ell)]^{\frac{1}{\rho}}}{[\pi(h - R_s)]^{\frac{1}{\rho}}} \right\} C_2(h)$$

$$I_s = \omega - I_o - I_r$$

Proof. The proof is in Appendix.¹¹ ■

Proposition 1 defines the benchmark case. The main result of this section characterizes the socially optimal allocation and equilibrium that gives the consumers the highest possible consumption and utility. We see that the bank faces the same constraints as a social planner. That is, the bank plays a role of a social planner as if it is a risk-averse expected utility maximizer. Then we show that the bank earns zero-profit in equilibrium.

Thus, we can define the bank problem as a social planning problem which achieves a socially optimal solution. By using this, the risk sharing contract can yield a socially optimal consumption allocation determined by the portfolio choice.

Investing in riskless assets can act as a buffer against a run on the bank in the low state unlike AG who show that a partial or a full-scale bank run can occur if a low state is realized. Our

¹¹ Using power utility function allows depositors to share risk (return loss ex-post) efficiently. The solution of the power utility maximization problem demonstrates the impact of risk sharing on consumption. We have also explored that if the consumers' preferences are corresponding to the log utility function (which is a special case of the power utility), there will be no risk sharing among depositors.

investment policy can reduce the fear of receiving a low return (or even zero) at date 2 attributed to a severe deterioration in risky asset returns in the economic downturn in case of no riskless assets.

The degree of risk sharing provided by this contract to exploit the whole potential benefits of risk sharing and asset diversification is

$$\frac{C_2(\ell)}{C_1} = \left[\frac{\pi R_s (h - \ell)}{(h - R_s)} \right]^{\frac{1}{\rho}}$$

$$\frac{C_2(h)}{C_1} = \left[\frac{\pi R_s (h - \ell)}{R_s - \ell} \right]^{\frac{1}{\rho}}$$

Our analysis of optimal risk sharing demonstrates how different degrees of contingency on the risky assets will influence bank runs and what the optimal policy would be to hold the right degree of contingency. This first-best allocation can fairly yield the desirable degree of contingency with respect to the risky asset returns for providing the so-called optimal risk sharing allocation.

In an optimal risk sharing deposit contract, both types of depositors can consume more. Further, all other required conditions for a nonnegative consumption of good are satisfied.

We can see that date 1 consumption is the same in both states of nature. Moreover, it is not possible for early consumers to consume more than I_o . In effect, it is not technologically possible to carry consumption back from date 2 to date 1 but you can carry it forward through time.

Corollary *The optimal allocation of consumption across the two realizations of the risky asset returns can be written as follows* $\begin{cases} C_1 = C_1(\ell) = C_1(h) \\ C_2(\ell) < C_2(h) \end{cases}$

To illustrate the operation of risk sharing deposit contract in the model, we consider the following example.

3.5.2 Numerical Example

Given that $\pi = 0.5, \mu = 0.5, \rho = 4, R_s = 1.26$. The feasibility of the allocation requires that incentive constraints and non-negativity constraints are satisfied.

- **Case 1:** $h = 2.5, \ell = 0.5$

Consumption: $C_2(h) = 1.2359, C_1 = 1.0892, C_2(\ell) = 1.0935$

Investments: $I_0 = 0.50, I_1 = 0.46, I_2 = 0.04$

- **Case 2:** $h = 3, \ell = 0$

Consumption: $C_2(h) = 1.2072, C_1 = 1.0908, C_2(\ell) = 1.1133$

which in turn implies investments: $I_1 = 0.39, I_s = 0.59, I_r = 0.02$

Even with $\ell = 0$, we can obtain an optimal risk sharing contract which is incentive-compatible.

Obviously, there is no information-based bank run at date 1.

In what follows, we will use the optimal consumption allocation to compare between our model and AG in terms of welfare, risk sharing and liquidity insurance.

3.5.3 The Effect on Welfare: A Comparative Analysis

This model has important implications for the welfare properties. I will compare my work to Allen and Gale (1998) in terms of liquidity insurance and risk sharing. In both AG and my work, the consumption allocations are contingent on the realization of the risky asset returns. Now suppose the two utility functions of consumers U, V in the present model and AG respectively.

The expected utility of consumers in our model

$$E [\mu U(C_1(R_r)) + (1 - \mu)U(C_2(R_r))]$$

The expected utility of consumers in AG model

$$E [\mu V(C_1(R_r)) + (1 - \mu)V(C_2(R_r))]$$

For a formal analysis, we find that

$$EU \succ EV$$

The present model increases welfare for both consumers regardless of the degree of relative risk aversion. This is because there is a unique equilibrium that alleviates the run risk. The bank can also reduce its holding of short-term assets and invest more in the productive long-term riskless asset. Thus, our model welfare dominates their model. Moreover, our model would, ad hoc, be more desirable for two reasons. First, risk-averse consumers can protect themselves from the risk of reducing risk sharing gains arising from earning low or zero returns on their deposits in the low state. Second, banks, in turn, can mitigate future losses and runs.

Correspondingly, it is easy to check that $U(C_1(R_r)) = V(C_1(R_r)) = U(\frac{I_0}{\mu})$ for both models, but they differ in the utility derived from date 2 consumption when a low state is realized as follows

$$\begin{aligned}
U(C_2(\ell)) - V(C_2(\ell)) &> 0 \\
U\left(\frac{I_s R_s + I_r \ell}{1 - \mu}\right) - V\left(\frac{I_r \ell}{1 - \mu}\right) &> 0 \\
U\left(\frac{a + b}{1 - \mu}\right) - V\left(\frac{b}{1 - \mu}\right) &> 0
\end{aligned}$$

Defining $b = I_r \ell$, $a = I_s R_s > 0$ for some a (they took $a = 1$). In our model, this mathematical description presents a nonzero date 2 consumption and must be at least the same as date 1 consumption in the low state. In such a situation, a rational decision for patient depositors is to wait until date 2.

In contrast, AG model implies that the consumption at date 2 will be equal to zero when $\ell = 0$ or small enough $\ell = \epsilon$ when date 2 return is sufficiently low $\epsilon > 0$. This means that the bank will experience a full-scale bank run when $\ell = 0$ or a partial bank run when $\ell = \epsilon$. This analysis has an interesting implication that the equilibrium in our model exhibits better characteristics of the utility of consumption in the low state without runs.

In the following table we compare the consumption allocations in this model relative to the corresponding consumption levels in AG.

AG		Our Model
$C_2(h) - C_1(h)$	<	$C_2(h) - C_1$
$C_2(\ell) - C_1(\ell)$	>	$C_2(\ell) - C_1$
$\frac{C_2(h)}{C_1(h)}$	<	$\frac{C_2(h)}{C_1}$
$\frac{C_2(\ell)}{C_1(\ell)}$	>	$\frac{C_2(\ell)}{C_1}$

We proceed in our discussion to demonstrate the following characteristics. From an ex-ante perspective, on the one hand, risk-averse consumers prefer AG model because it is associated with increased risk sharing (higher utility) in terms of offering a higher degree of risk sharing and more

liquidity insurance in the high state than our model. On the other hand, the present model may be particularly effective in providing more liquidity insurance and it has subsequently facilitated the consumption smoothing between periods and offered better risk sharing in the low state relative to AG. This implies that our model is more valuable to risk-averse consumers and more plausible results are obtained in the low state.

In the following discussion, we relax the assumption that the bank has a correct belief about the fraction of impatient depositors to a more realistic case where the bank's initial belief about the proportion of early consumers is lower than the actual μ . This advances insufficient resources to meet early consumption. We stress that the bank needs to liquidate some of its assets to cover ex-post liquidity needs. Hence, we would investigate some quantitative and qualitative properties of the stochastic withdrawals and liquidation in the model studied above in order to draw a transparent picture about how we should liquidate in an optimal way.

3.6 The Case of Liquidity Shortage (Early Withdrawals exceed Short-term Assets)

In this section, we investigate the case where early liquidity needs exceed the resources available [$\mu C_1(\cdot) > I_o$] and the bank needs to liquidate some of the long-term assets.

3.6.1 Liquidity Needs and Liquidation Decision

Generally, the bank has a correct belief about the fraction of impatient depositors who decide to withdraw early. In a more realistic setting, the bank can make errors in its beliefs about the fraction of impatient depositors causing misallocation of resources between short-term and long-term assets especially if this fraction is lower than the actual μ .

We can use the same analysis developed above with modification to obtain the desired result in this characterization. Explicitly, we consider the liquidation decision at date 1 in case the demand of early consumers exceeds the resources available. In such a case, the bank has to transfer some resources within dates to meet the potential shortage of liquidity by liquidating some of the long-term assets.

3.6.2 Optimality Conditions

We explore in the following lemma which of the two long-term assets can be liquidated through the comparison of gain-loss ratios. i.e. the ratio of asset return to its liquidation value.

Lemma *If early liquidity demand exceeds the resources available at date 1, the bank will optimally choose to liquidate either a risky investment if the realized state is low where $\frac{\ell}{L_r} < \frac{R_s}{L_s}$ (less costly) or a riskless investment if the realized state is high where $\frac{R_s}{L_s} < \frac{h}{L_r}$.*

Proof. See Appendix. ■

The proof implies that the bank is forced to liquidate some long-term assets before maturity in order to obtain liquidity. There is no “pecking order” when liquidating long-term assets. Liquidating productive long-term assets deteriorates the welfare of consumers. Moreover, costly asset liquidation can lead to bank runs. Thereby, banks will choose the most effective source of liquidation (cheaper investments) in order to maximize the expected utility of consumers and prevent bank runs by comparing the cost of obtaining liquidity of these two long-term investments. The difference in payoffs, $\{R_s - L_s\}$; $\{h - L_r, \ell - L_r\}$, are the costs of liquidation.

In essence, the lower the gain-loss ratio, the more liquid a bank asset is. In a similar analysis, Lazopoulos (2012) compares demand deposit and equity contracts in the presence of aggregate uncertainty and he finds that costly liquidation of productive long-term assets degenerates the welfare of risk-averse consumers relative to equity contracts.

In the following, we study excess liquidity demand and two types of asset liquidation. When there is excess demand, the bank will choose to liquidate risky assets in the low state and riskless assets in the high state.

3.6.3 The Equilibrium Liquidation Dynamics and Optimal Allocation

If assets are to be liquidated, both liquidation values and gain-loss ratios should be considered. Obviously, the bank cannot achieve the first-best allocation because of costly liquidation. The question is how can a bank respond to excess liquidity demand in an effective way?

To answer this question, the bank will choose to liquidate risky or riskless assets that yield higher welfare along with risk sharing. This implies that asset liquidation depends on the realization of the state and cost of liquidation.

If the bank chooses to liquidate a proportion of risky assets α , there is a loss to be incurred from early liquidation. This loss is sufficiently low in the low state (compared to a riskless asset liquidation) and the maximum amount of liquidity a bank can raise is $L_r I_r < I_r$. This can be shown in the following proposition.

Proposition 2 *If liquidity needs exceed the available liquid resources, the risk sharing contract that the bank will offer involves liquidation of the risky investments in the low state.*

Proof. The proof is divided into two parts. We characterize the derivation of the optimal consumption allocation in part 1. Then we present, in part 2, our theoretical results related to the mathematical proof.

Part 1: Finding the second best risk-sharing allocation for consumers.

$$C_2(h) = \frac{\left[\frac{R_s(h-\ell)(1-\alpha)}{(1-\alpha L_r)R_s - (1-\alpha)\ell} \right]}{\mu \left\{ \left(\frac{1}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1-\alpha L_r)R_s - (1-\alpha)\ell}{(1-\alpha)(h-\ell)R_s} \right]^{\frac{1-\rho}{\rho}} \right\} + (1-\mu) \left\{ 1 + \left(\frac{1-\pi}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1-\alpha L_r)R_s - (1-\alpha)\ell}{(1-\alpha)h - (1-\alpha L_r)R_s} \right]^{\frac{1-\rho}{\rho}} \right\}} \cdot \omega$$

$$C_2(\ell) = \left(\frac{1-\pi}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1-\alpha L_r)R_s - (1-\alpha)\ell}{(1-\alpha)h - (1-\alpha L_r)R_s} \right]^{\frac{1}{\rho}} C_2(h)$$

$$C_1 = \left[\frac{(1-\alpha L_r)R_s - (1-\alpha)\ell}{\pi(1-\alpha)(h-\ell)R_s} \right]^{\frac{1}{\rho}} C_2(h)$$

For full proof see Appendix .

Part 2: In theory, the proof is done in several steps. The first step is the realization of the state of nature at date 1. Assume that the realized state is low.

The second step is to prove that this solution is feasible, given a low state, if and only if the liquidation loss of a risky asset is sufficiently low comparing to a riskless asset as shown $\frac{\ell}{L_r} < \frac{R_s}{L_s}$.

The third step is liquidating a risky asset over a riskless asset. ■

From these three steps, one can conclude that the underlying liquidation of a risky asset will

have a significant influence in the low state since the bank liquidates a lower return asset. This can provide better liquidity insurance for consumers.

The degree of risk sharing provided

$$\frac{C_2(h)}{C_1} = \left[\frac{\pi(1-\alpha)(h-\ell)R_s}{(1-\alpha L_r)R_s - (1-\alpha)\ell} \right]^{\frac{1}{\rho}}$$

$$\frac{C_2(\ell)}{C_1} = \left[\frac{(1-\pi)(1-\alpha)(h-\ell)R_s}{(1-\alpha)h - (1-\alpha L_r)R_s} \right]^{\frac{1}{\rho}}$$

In the low state, the bank will choose to liquidate a risky investment if early liquidity demand exceeds the resources available at date 1. By doing so, in this setting, second best allocations are obtained.

Optimizing banks will liquidate only sufficient assets to meet early withdrawals. Liquidating too many assets may negatively affect the beliefs of remaining consumers and provide a self-fulfilling prophecy for a bank run by consumers. This can be driven by the fact that the bank will be unable to fully meet late consumer promises according to insufficient resources which are left for date 2.

Similarly, we specify that the bank will choose to liquidate a proportion of riskless assets β in the high state to meet early liquidity needs because it is less costly to do so. The riskless asset liquidation can be summarized in the following proposition.

Proposition 3 *If liquidity needs exceed the available liquid resources, the risk sharing contract that the bank will offer involves liquidation of the riskless investments in the high state.*

Proof. Following a similar proof to proposition 1. We present both quantitative and theoretical results demonstrating our approach. One can show that the solution to the optimization problem is

$$C_2(h) = \frac{\left[\frac{(1-\beta)(h-\ell)R_s}{(1-\beta)R_s - (1-\beta L_s)\ell} \right]}{\left\{ \mu \left(\frac{1}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1-\beta)R_s - (1-\beta L_s)\ell}{(1-\beta)(h-\ell)R_s} \right]^{\frac{1-\rho}{\rho}} + (1-\mu) \left[1 + \left(\frac{1-\pi}{\pi} \right)^{\frac{1}{\rho}} \left(\frac{(1-\beta)R_s - (1-\beta L_s)\ell}{(1-\beta L_s)h - (1-\beta)R_s} \right)^{\frac{1-\rho}{\rho}} \right] \right\}} \cdot \omega$$

$$C_2(\ell) = \left(\frac{1 - \pi}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1 - \beta)R_s - (1 - \beta L_s)\ell}{(1 - \beta L_s)h - (1 - \beta)R_s} \right]^{\frac{1}{\rho}} C_2(h)$$

$$C_1 = \left[\frac{(1 - \beta)R_s - (1 - \beta L_s)\ell}{\pi(1 - \beta)(h - \ell)R_s} \right]^{\frac{1}{\rho}} C_2(h)$$

The analytical proof is based on the simple observation that the loss of liquidating a risky asset is sufficiently high in the high state $\frac{R_s}{L_s} < \frac{h}{L_r}$. In order for the bank to meet the excess withdrawal at date 1, it is ex-post efficient to liquidate the riskless asset. Early liquidation results in a loss equal to $R_s - L_s$, per unit. Hence, the maximum amount of liquidity that can be raised is $L_s I_s < I_s$. This is why the bank should avoid excessive liquidation because it can reduce the consumption available to late consumers. See Appendix for mathematical calculations. ■

The proof of this proposition yields a sufficient condition that is consistent with the features of our logistic analysis described above. In fact, the result would otherwise show less efficiency and weaker stability if productive risky assets were liquidated in the high state to meet liquidity needs.¹²

The degree of risk sharing in this case

$$\frac{C_2(h)}{C_1} = \left[\frac{\pi(1 - \beta)(h - \ell)R_s}{(1 - \beta)R_s - (1 - \beta L_s)\ell} \right]^{\frac{1}{\rho}}$$

$$\frac{C_2(\ell)}{C_1} = \left[\frac{(1 - \pi)(1 - \beta)(h - \ell)R_s}{(1 - \beta L_s)h - (1 - \beta)R_s} \right]^{\frac{1}{\rho}}$$

It is convenient to summarize the main results of this section in the following. First, we characterize second-best allocations without runs. Second, we show that these allocations can support the existence of risk sharing among consumers. Third, we identify the maximum level of utility that consumers can obtain in case of liquidity shortage.

The intuition behind this result is quite straightforward. The bank will be forced to liquidate a

¹² Tsomocos (2003) defines that financial fragility is attributed to both liquidity shortages and banking industry vulnerability. While, Allen and Gale (2002) establish that financial fragility is “an extreme case of excess sensitivity to small shocks.”

risky asset if the riskless asset does not exist. It is very costly to liquidate the risky asset in the high state. This can be interpreted as a fire-sale price. This in turn fuels an excessive liquidation for the bank assets at date 1 to meet the demand of early consumers. Hence, the remaining consumers will run on the bank at date 1 if they believe that the bank is unable to meet their needs at date 2 after liquidating an excessive amount of long-term risky investments.

To avoid such a cost, the presence of riskless assets is essential to smooth consumption since the risky investment returns are influenced by macroeconomic downturns and business cycle fluctuations.

3.7 Discussion

Our model presents three main findings. First, it shows standard risk sharing deposit contracts and portfolio diversification that provide optimal allocations in equilibrium. In the absence of any arbitrage opportunities, the fund could be invested in the short-term liquid assets only. However, even a risk-averse consumer would prefer to hold long-term risky and riskless assets in the portfolio since they can provide higher returns than short-term asset returns. A long-term asset cannot dominate the liquid asset because its return is less than one if it is interrupted at date 1.

In particular, the bank will essentially exploit arbitrage opportunities available between short-term and long-term assets for the interest of consumers. The bank will hold a diversified portfolio of short-term, long-term risky and riskless assets to maximize the expected utility of consumers. Such a portfolio is less susceptible to the changes in macroeconomic conditions.

Second, if there is liquidity shortage at date 1, the bank will be forced to liquidate some of the long-term assets. Liquidating long-term assets is costly. Costly liquidation of long-term assets can cause a bank run. The bank can reduce the cost of liquidation by choosing to liquidate a risky asset in the low state or a riskless asset in the high state. Ennis and Keister (2006) show that the bank will hold liquid assets that are equal to early withdrawals if a run does not occur and there is no precautionary or excess liquidity held. However, the bank aims to reduce the probability of a

run by investing less and holding more reserves due to high liquidation costs.

Liquidating a risky asset at date 1 yields a low payment to consumers in the high state according to its substantial loss. It also triggers more liquidation of productive risky assets to meet its short-term obligations. This will then lead to continued stress in the bank, seizing more of the assets and tightening the available resources. For this reason, risky assets should be held until termination not only to benefit consumers from the productive asset returns but also to avoid sending adverse signals to the consumers because of insufficient resources to cover early withdrawal. This view is consistent with Cooper and Ross (1998).

Third, the optimality of deposit contracts requires that all feasibility resource constraints hold with equality to cover a possible liquidity shortage.

In addition, asset liquidation should not exceed liquidity needs. Liquidating a large amount of the long-term assets makes the bank assets exhausted and it can reduce the date 2 consumption.

In the following, we provide insight into panic-bank runs.

3.8 Panic-Based Bank Runs

Although an informational bank run can not occur in our model in the low state, the bank is still vulnerable to a run for non-economic or fundamental reasons. The run here is rather panic which is considered as a worst-case scenario where banks struggle to survive. The main reason for this run is an overly pessimistic impression as a reflection of liquidity problems.

In particular, all consumers rush to withdraw early when they lose confidence in the bank. For example, the collapse of Bear Stearns was a clear result of a sudden loss of confidence attributed to adverse news and rumors.

The model just characterized may be subject to panics arising from the rational behavior.¹³ It may be rational for a consumer to panic and withdraw early if they believe that other consumers

¹³ See for more detailed information Kindleberger (1978) who shows that banking panics are caused by “mob psychology” or “mass hysteria”. Likewise, these panics can be attributed to sunspots as in Diamond and Dybvig (1983) model or to “Some omen, rumor, prognostication, or universally felt animal spirit” which can induce patient depositors to withdraw early as discussed by Freeman (1988) or if the depositors are too risk-averse.

are going to do so making some of them worse off than if they had not participated in such a panic.

The natural justification for a panic-based run is the coordination problem which arises among consumers in which all consumers attempt to withdraw their deposits simultaneously at date 1.¹⁴ Hence, consumers arriving at the bank late will receive nothing in case of a run because the bank follows a sequential service constraint. To prevent panic-based runs in our model, we can just close the bank to stop runs as with Diamond-Dybvig (1983).

A key factor to resolve panic-based runs is to build confidence to incentivise late consumers to not withdraw early. However, some authors view bank runs as an equilibrium outcome as in Diamond and Dybvig (1983).

Note that the bank, with limited liability, has to fully liquidate its investments in a run. As mentioned above, liquidation is used to transfer resources between two dates 1, 2. For concreteness, we will make the follow-up conjecture about hedging with financial derivatives.

We conjecture that, in the presence of futures market, the risk of low interest rate and low returns on investments can be hedged by engaging in financial futures contracts by taking a long position in the risky assets and then taking a short position in these assets at a specific future date for an agreed sale price. Initially, this process can hedge both the investment risk and interest rate risk and produce some profit to the bank from the difference in the purchase and selling prices.

3.9 Conclusions

Destabilization and banking fragility may be caused by macroeconomic conditions such as business cycle downturns, recessions, or exogenous liquidity shocks. Throughout this paper, we have extended Allen and Gale (1998) by adding a long-term riskless asset to the bank portfolio that consists of two assets: short-term and long-term risky. We consider that the bank has a

¹⁴ Nevertheless, Goldstein and Pauzner (2005) endogenously derive the ex-ante probability of panic-based bank runs in demand deposit contract of DD. Further, one observation of the underlying mechanism of modern banking about the multiple equilibria of DD model is that one in which coordination of depositors around the preferred equilibrium of waiting yields optimal returns, while coordination around the inferior equilibrium of withdrawing breaks the bank and perhaps the whole economy as in Ahdieh (2010).

comparative advantage in providing liquidity insurance to depositors by offering standard risk sharing deposit contracts and modifying the portfolio risk to be tolerant by all consumers.

This paper presents theoretical foundations about two key points. First, we specify deposit contracts to suitably share the risk between early and late consumers contingent on risky asset returns. We then solve the constrained utility maximization problem to derive the optimal risk sharing allocation (first-best). We show that the bank will choose a combination of three assets with positive proportions in the portfolio to prevent bank runs.

Second, we shed some light on the concept of liquidation costs and any deviation from first-best allocation. When there is liquidity shortage, the bank will choose to liquidate a riskless asset in the high state or a risky asset in the low state.

The riskless asset has a positive impact on stability since this asset can be transferred into liquidity quickly and with a small loss. This mechanism can prevent the main reason of fragility and stabilize the bank. There are no information-based runs because of the bank's ability to meet its commitments to late consumers.

It is useful to compare our work to the existing literature. Allen and Gale (1998) show that a bank run is the result of the informational signals about the future returns where depositors run on the bank when the realized state of nature is low. While in ours, a run on the bank in the low state is eliminated because the late consumers will still receive riskless asset returns in addition to the low returns of risky assets. The existence of a riskless asset can ensure that payments for late withdrawals will not fall below the payments promised on early withdrawals.

We also discuss the implications of bank portfolio choice on welfare. As I mentioned above, the intuition of our results is that this optimal contract can achieve welfare-superior allocations or welfare dominance over AG.

The benefits of investing in the riskless assets stem from two perspectives. First, there is a

clear benefit to holding the riskless assets in the low state because late consumers will not request early deposit withdrawal and receive nothing at date 2. Alongside these structural changes in the bank portfolio, this work shows that the bank has a comparative advantage over AG in securing a greater stability and avoiding information based runs when the realized state is low.

Second, even if there is liquidity shortage at date 1, date 2 consumption can still exceed date 1 consumption and risk sharing allocation is implemented which may also dominate AG allocation in terms of liquidation. Eventually, this is because the bank can avoid the fire-sale prices of the productive risky assets. This finding is consistent with Lazopoulos (2012) who shows that the costly liquidation of productive long-term investments can decrease the welfare performance in the deposit contracts relative to equity contracts.

One can suggest how an increase in the level of the riskless assets in the portfolio can reinforce the area of strength in the bank assets and prevent a bank run especially if liquidity shortage is caused by misallocation of the resources between dates 1, 2. But, holding riskless assets may have a negative effect by decreasing welfare because of earning lower returns than the expected returns of risky assets.

It would also be interesting to relax some restrictive assumptions to illustrate their effects on our model and find a proper measure of bank stability. The role of interbank market is also important. Moreover, it is useful in this sense to determine prudential supervision and regulation to make the banking crises and bank runs less costly and less likely. Yet, these issues can be investigated in future research.

In essence, our results are quite intuitive and the proofs are straightforward. Finally, we hope that this argument has illustrated the subtle issues in the bank-depositor setting and will inspire further research in the field of banking.

3.10 Appendix: Proofs

The proofs include the benchmark case and liquidation choice.

3.10.1 The Benchmark Case

3.10.1.1 The Properties of the CRRA Utility Function

The utility function reflects the depositor's attitude toward the risk. The power utility function (isoelastic utility function) exhibits a constant relative risk aversion CRRA $R_A(C_t) = \rho$ and the elasticity of intertemporal substitution is

$$\epsilon = \frac{1}{\rho} U(C_t) = \frac{C_t^{1-\rho}}{1-\rho}$$

$$U'(C_t) = C_t^{-\rho}$$

$$U''(C_t) = -\rho C_t^{-\rho-1}$$

Arrow-Pratt absolute risk aversion coefficient:

$$R_A(C_t) = - \left[\frac{-\rho C_t^{-\rho-1}}{C_t^{-\rho}} \right] = \frac{\rho}{C_t}$$

$$\frac{dR_A(C_t)}{dC_t} = -\frac{\rho}{C_t^2} < 0$$

Arrow-Pratt relative risk aversion coefficient:

$$R_R(C_t) = - \left[\frac{-\rho C_t^{-\rho-1}}{C_t^{-\rho}} \right] C_t = \rho$$

If $\rho = 1$, it can be simplified to the case of log utility

$$\lim_{\rho \rightarrow 1} \frac{C_t^{1-\rho}}{1-\rho} = \ln C_t$$

3.10.1.2 The Proof of Proposition 1

Optimal Consumption Levels in the Benchmark Case The aim is to compute the expected utility of a representative agent in which case the utility function is nonlinear.

By taking the expectation we can now define our maximization problem as follows:

$$\max \pi \left[\mu \frac{C_1^{1-\rho}(h)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(h)}{1-\rho} \right] + (1-\pi) \left[\mu \frac{C_1^{1-\rho}(\ell)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(\ell)}{1-\rho} \right]$$

$$\mu C_1(\ell) \leq I_o$$

$$\mu C_1(h) \leq I_o$$

$$\mu C_1(\ell) + (1 - \mu)C_2(\ell) \leq I_o + I_s R_s + I_r \ell$$

$$\mu C_1(h) + (1 - \mu)C_2(h) \leq I_o + I_s R_s + I_r h$$

$$I_o + I_s + I_r \leq \omega$$

Optimal risk sharing implies that incentive constraint can be ignored and the necessary condition of Kuhn-Tucker can ensure that IC will always be satisfied.

The standard Lagrangian optimization problem is:

$$\pi \left[\mu \frac{C_1^{1-\rho}(h)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(h)}{1-\rho} \right] + (1-\pi) \left[\mu \frac{C_1^{1-\rho}(\ell)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(\ell)}{1-\rho} \right]$$

$$-\lambda_{1\ell} [\mu C_1(\ell) - I_o]$$

$$-\lambda_{1h} [\mu C_1(h) - I_o]$$

$$-\lambda_{2\ell} [\mu C_1(\ell) + (1-\mu)C_2(\ell) - I_o - I_s R_s - I_r \ell]$$

$$-\lambda_{2h} [\mu C_1(h) + (1-\mu)C_2(h) - I_o - I_s R_s - I_r h]$$

$$-\lambda_3 [I_o + I_s + I_r - \omega]$$

$$\lambda_{1\ell}, \lambda_{1h}, \lambda_{2\ell}, \lambda_{2h}, \lambda_3 \geq 0$$

We need to solve for $C_1(\ell), C_2(\ell), C_1(h), C_2(h), I_o, I_s, I_r$.

First derivatives:

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta C_1(\ell)} &= (1 - \pi)\mu C_1^{-\rho}(\ell) - \lambda_{1\ell}\mu - \lambda_{2\ell}\mu = 0 \\ (1 - \pi)C_1^{-\rho}(\ell) &= \lambda_{1\ell} + \lambda_{2\ell}\end{aligned}\quad (3.8)$$

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta C_2(\ell)} &= (1 - \pi)(1 - \mu)C_2^{-\rho}(\ell) - \lambda_{2\ell}(1 - \mu) = 0 \\ (1 - \pi)C_2^{-\rho}(\ell) &= \lambda_{2\ell}\end{aligned}\quad (3.9)$$

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta C_1(h)} &= \pi\mu C_1^{-\rho}(h) - \lambda_{1h}\mu - \lambda_{2h}\mu = 0 \\ \pi C_1^{-\rho}(h) &= \lambda_{1h} + \lambda_{2h}\end{aligned}\quad (3.10)$$

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta C_2(h)} &= \pi(1 - \mu)C_2^{-\rho}(h) - \lambda_{2h}(1 - \mu) = 0 \\ \pi C_2^{-\rho}(h) &= \lambda_{2h}\end{aligned}\quad (3.11)$$

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta I_o} &= \lambda_{1\ell} + \lambda_{1h} + \lambda_{2\ell} + \lambda_{2h} - \lambda_3 = 0 \\ \lambda_{1\ell} + \lambda_{1h} + \lambda_{2\ell} + \lambda_{2h} &= \lambda_3\end{aligned}\quad (3.12)$$

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta I_s} &= \lambda_{2\ell}R_s + \lambda_{2h}R_s - \lambda_3 = 0 \\ (\lambda_{2\ell} + \lambda_{2h})R_s &= \lambda_3\end{aligned}\quad (3.13)$$

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta I_r} &= \lambda_{2\ell}\ell + \lambda_{2h}h - \lambda_3 = 0 \\ \lambda_{2\ell}\ell + \lambda_{2h}h &= \lambda_3\end{aligned}\quad (3.14)$$

Substituting (3.9) in (3.8) yields

$$\begin{aligned}\lambda_{1\ell} &= (1 - \pi)C_1^{-\rho}(\ell) - (1 - \pi)C_2^{-\rho}(\ell) \\ \lambda_{1\ell} &= (1 - \pi) [C_1^{-\rho}(\ell) - C_2^{-\rho}(\ell)]\end{aligned}\quad (3.15)$$

Similarly, by plugging (3.11) into (3.10)

$$\begin{aligned}\lambda_{1h} &= \pi C_1^{-\rho}(h) - \pi C_2^{-\rho}(h) \\ \lambda_{1h} &= \pi [C_1^{-\rho}(h) - C_2^{-\rho}(h)]\end{aligned}\quad (3.16)$$

Equation (3.12) implies that

$$(1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h) = \lambda_3 \quad (3.17)$$

Equation (3.13) implies that

$$[(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] R_s = \lambda_3 \quad (3.18)$$

Equation (3.14) implies that

$$(1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h = \lambda_3 \quad (3.19)$$

By simple algebraic manipulation for the last three equations, we can see that the necessary Kuhn-Tucker condition to achieve incentive-compatible constraint and to support our model is

$$[(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] R_s = (1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h = (1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h) \quad (3.20)$$

The Kuhn-Tucker conditions bring marginal utility into line with marginal productivity. These conditions satisfy IC and the optimal allocation should satisfy Kuhn-Tucker conditions. In what follows I will focus on interior solutions at optimum.

Case 1 Our base case is chosen such that all the constraints are binding let $\lambda_{1\ell} > 0, \lambda_{1h} > 0, \lambda_{2\ell} > 0, \lambda_{2h} > 0, \lambda_3 > 0$. Our findings suggest that the form of constraints with strict equality has crucial importance for the optimal consumption plan

$$\mu C_1(\ell) = I_o$$

$$\mu C_1(h) = I_o$$

$$\mu C_1(\ell) + (1 - \mu)C_2(\ell) = I_o + I_s R_s + I_r \ell$$

$$\mu C_1(h) + (1 - \mu)C_2(h) = I_o + I_s R_s + I_r h$$

We also know the budget constraint binds assuming the bank uses all deposits in investments

$$I_o + I_s + I_r = \omega$$

Or equivalently, it is easy to see that

$$\mu C_1(\ell) = I_o$$

$$\mu C_1(h) = I_o$$

$$(1 - \mu)C_2(\ell) = I_s R_s + I_r \ell$$

$$(1 - \mu)C_2(h) = I_s R_s + I_r h$$

It follows that $C_1(\ell) = C_1(h) = C_1$. Substitute $I_r = \omega - I_s - \mu C_1$ into the last two equations

$$(1 - \mu)C_2(\ell) = I_s R_s + [\omega - I_s - \mu C_1] \cdot \ell$$

$$(1 - \mu)C_2(h) = I_s R_s + [\omega - I_s - \mu C_1] \cdot h$$

Rearranging

$$(1 - \mu)C_2(\ell) = I_s (R_s - \ell) + [\omega - \mu C_1] \cdot \ell$$

$$(1 - \mu)C_2(h) = I_s (R_s - h) + [\omega - \mu C_1] \cdot h$$

To solve these two linear equations we multiply the first equation by $\frac{R_s - h}{R_s - \ell}$ then we subtract the first one from the second

$$\begin{aligned} \frac{R_s - h}{R_s - \ell} (1 - \mu)C_2(\ell) &= I_s (R_s - \ell) \cdot \frac{R_s - h}{R_s - \ell} + [\omega - \mu C_1(\ell)] \cdot \ell \cdot \frac{R_s - h}{R_s - \ell} \\ (1 - \mu)C_2(h) &= I_s (R_s - h) + [\omega - \mu C_1] \cdot h \end{aligned}$$

I_s can be eliminated and outcome can be simplified to

$$\begin{aligned} (1 - \mu) \left[C_2(h) + \frac{h - R_s}{R_s - \ell} C_2(\ell) \right] &= [\omega - \mu C_1] \left[h + \frac{h - R_s}{R_s - \ell} \cdot \ell \right] \\ (1 - \mu) \left[C_2(h) + \frac{h - R_s}{R_s - \ell} C_2(\ell) \right] &= [\omega - \mu C_1] \left(\frac{h - \ell}{R_s - \ell} \right) R_s \end{aligned} \quad (3.21)$$

The following condition implies

$$\begin{aligned} R_s [(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] &= (1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h \\ (1 - \pi)C_2^{-\rho}(\ell)(R_s - \ell) &= \pi C_2^{-\rho}(h)(h - R_s) \\ C_2^{-\rho}(\ell) &= \frac{\pi(h - R_s)}{(1 - \pi)(R_s - \ell)} C_2^{-\rho}(h) \\ C_2(\ell) &= \left[\frac{(1 - \pi)(R_s - \ell)}{\pi(h - R_s)} \right]^{\frac{1}{\rho}} C_2(h) \end{aligned} \quad (3.22)$$

By plugging $C_1(\ell) = C_1(h) = C_1$, $C_2^{-\rho}(\ell) = \frac{\pi(h-R_s)}{(1-\pi)(R_s-\ell)}C_2^{-\rho}(h)$ into

$$(1-\pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h = (1-\pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h)$$

It implies

$$\begin{aligned} C_1^{-\rho} &= (1-\pi) \cdot \frac{\pi(h-R_s)}{(1-\pi)(R_s-\ell)} C_2^{-\rho}(h) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h \\ C_1^{-\rho} &= \left[\frac{\pi(h-R_s)}{R_s-\ell} \cdot \ell + \pi \cdot h \right] C_2^{-\rho}(h) \\ C_1^{-\rho} &= \left[\frac{\pi \cdot R_s(h-\ell)}{R_s-\ell} \right] C_2^{-\rho}(h) \\ C_1 &= \left[\frac{R_s-\ell}{\pi R_s(h-\ell)} \right]^{\frac{1}{\rho}} C_2(h) \end{aligned} \quad (3.23)$$

Now substitute $C_1, C_2(\ell)$ into (3.21)

$$\begin{aligned} & (1-\mu) \left\{ C_2(h) + \frac{h-R_s}{R_s-\ell} \left[\frac{(1-\pi)(R_s-\ell)}{\pi(h-R_s)} \right]^{\frac{1}{\rho}} C_2(h) \right\} \\ &= \left\{ \omega - \mu \left[\frac{R_s-\ell}{\pi R_s(h-\ell)} \right]^{\frac{1}{\rho}} C_2(h) \right\} \cdot \left[\frac{R_s(h-\ell)}{R_s-\ell} \right] \\ & (1-\mu) C_2(h) \left[1 + \left(\frac{1-\pi}{\pi} \right)^{\frac{1}{\rho}} \left(\frac{R_s-\ell}{h-R_s} \right)^{\frac{1-\rho}{\rho}} \right] \\ &= \left[\frac{R_s(h-\ell)}{R_s-\ell} \right] \omega - \mu \frac{1}{\pi^{\frac{1}{\rho}}} \left[\frac{R_s-\ell}{R_s(h-\ell)} \right]^{\frac{1-\rho}{\rho}} C_2(h) \\ & C_2(h) \left\{ (1-\mu) \left[1 + \left(\frac{1-\pi}{\pi} \right)^{\frac{1}{\rho}} \left(\frac{R_s-\ell}{h-R_s} \right)^{\frac{1-\rho}{\rho}} \right] + \mu \frac{1}{\pi^{\frac{1}{\rho}}} \left[\frac{R_s-\ell}{R_s(h-\ell)} \right]^{\frac{1-\rho}{\rho}} \right\} = \left[\frac{R_s(h-\ell)}{R_s-\ell} \right] \omega \\ & \frac{C_2(h)}{\pi^{\frac{1}{\rho}}} \left\{ (1-\mu) \left[\frac{\pi^{\frac{1}{\rho}}(h-R_s)^{\frac{1-\rho}{\rho}} + (1-\pi)^{\frac{1}{\rho}}(R_s-\ell)^{\frac{1-\rho}{\rho}}}{(h-R_s)^{\frac{1-\rho}{\rho}}} \right] + \mu \left(\frac{R_s-\ell}{R_s(h-\ell)} \right)^{\frac{1-\rho}{\rho}} \right\} = \left[\frac{R_s(h-\ell)}{R_s-\ell} \right] \omega \\ & C_2(h) = \frac{\pi^{\frac{1}{\rho}} R_s(h-\ell)}{(1-\mu) \left[\frac{\pi^{\frac{1}{\rho}}(h-R_s)^{\frac{1-\rho}{\rho}}(R_s-\ell) + (1-\pi)^{\frac{1}{\rho}}(R_s-\ell)^{\frac{1-\rho}{\rho}}}{(h-R_s)^{\frac{1-\rho}{\rho}}} \right] + \mu \left[\frac{(R_s-\ell)^{\frac{1-\rho}{\rho}}}{[R_s(h-\ell)]^{\frac{1-\rho}{\rho}}} \right]} \cdot \omega \end{aligned} \quad (3.24)$$

$$C_2(\ell) = \left[\frac{(1-\pi)(R_s-\ell)}{\pi(h-R_s)} \right]^{\frac{1}{\rho}} C_2(h) \quad (3.25)$$

$$C_1 = \left[\frac{R_s - \ell}{\pi R_s (h - \ell)} \right]^{\frac{1}{\rho}} C_2(h) \quad (3.26)$$

Find I_o, I_s, I_r of the Investment Portfolio: Bank investments are endogenously determined by the deposits received. Substituting the optimal consumption allocation gives the investment strategy of the bank.

The following expressions imply

$$(1 - \mu)C_2(\ell) = I_s R_s + I_r \ell$$

$$(1 - \mu)C_2(h) = I_s R_s + I_r h$$

$$I_r = \frac{1 - \mu}{h - \ell} [C_2(h) - C_2(\ell)]$$

The bank's investment strategy:

$$\begin{aligned} I_o &= \mu C_1 = \mu \left[\frac{R_s - \ell}{\pi R_s (h - \ell)} \right]^{\frac{1}{\rho}} C_2(h) \\ I_r &= \frac{1 - \mu}{h - \ell} \left\{ 1 - \left[\frac{(1 - \pi)(R_s - \ell)}{\pi(h - R_s)} \right]^{\frac{1}{\rho}} \right\} C_2(h) \\ &= \frac{1 - \mu}{h - \ell} \left\{ \frac{[\pi(h - R_s)]^{\frac{1}{\rho}} - [(1 - \pi)(R_s - \ell)]^{\frac{1}{\rho}}}{[\pi(h - R_s)]^{\frac{1}{\rho}}} \right\} C_2(h) \\ I_s &= \omega - I_o - I_r \end{aligned}$$

It shows that there exists equilibrium in the optimal risk sharing deposit contract.

Case 2 Constraints can also be written in the following form

$$\mu C_1(\ell) < I_o$$

$$\mu C_1(h) < I_o$$

$$\mu C_1(\ell) + (1 - \mu)C_2(\ell) = I_o + I_s R_s + I_r \ell$$

$$\mu C_1(h) + (1 - \mu)C_2(h) = I_o + I_s R_s + I_r h$$

By plugging $\lambda_{1\ell} = 0, \lambda_{2\ell} > 0$, (3.8) and (3.9) imply that

$$\begin{aligned} (1 - \pi)C_1^{-\rho}(\ell) &= (1 - \pi)C_2^{-\rho}(\ell) = \lambda_{2\ell} \\ C_1(\ell) &= C_2(\ell) \end{aligned}$$

Similarly, by plugging $\lambda_{1h} = 0, \lambda_{2h} > 0$, (3.10) and (3.11) imply that

$$\begin{aligned} \pi C_1^{-\rho}(h) &= \pi C_2^{-\rho}(h) = \lambda_{2h} \\ C_1(h) &= C_2(h) \end{aligned}$$

The two constraints above turn out to

$$\begin{aligned} C_1(\ell) = C_2(\ell) &= I_o + I_s R_s + I_r \ell \\ C_1(h) = C_2(h) &= I_o + I_s R_s + I_r h \end{aligned}$$

The following conditions are not satisfied (they imply that $R_s = 1$ which is incorrect by definition)

$$[(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] R_s = (1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h = (1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h)$$

Therefore, there is no possible solution in this case. Next, we present a parallel result regarding the existence of equilibrium with liquidation.

By substituting the consumption levels back into the expected utility

$$\begin{aligned} EU(C_t) &= (1 - \pi) \left[\frac{\mu}{1 - \rho} \left(\frac{I_0}{\mu} \right)^{1-\rho} + \frac{1 - \mu}{1 - \rho} \left(\frac{I_s R_s + I_r \ell}{1 - \mu} \right)^{1-\rho} \right] \\ &\quad + \pi \left[\frac{\mu}{1 - \rho} \left(\frac{I_0}{\mu} \right)^{1-\rho} + \frac{1 - \mu}{1 - \rho} \left(\frac{I_s R_s + I_r h}{1 - \mu} \right)^{1-\rho} \right] \\ &= \frac{\mu}{1 - \rho} \left(\frac{I_0}{\mu} \right)^{1-\rho} + \frac{1 - \mu}{1 - \rho} \left[(1 - \pi) \cdot \left(\frac{I_s R_s + I_r \ell}{1 - \mu} \right)^{1-\rho} + \pi \cdot \left(\frac{I_s R_s + I_r h}{1 - \mu} \right)^{1-\rho} \right] \end{aligned}$$

To compare between AG and our model, we can assign numbers as shown in the above example then we compute the expected utility of the two models. The comparison implies that our model

produces higher welfare than AG.

3.10.2 The Equilibrium Liquidation Dynamics (Partial Liquidation)

3.10.2.1 The Proof of Lemma

Consider first the two long-term assets: riskless I_s and risky I_r . We can write the comparison as $\frac{\ell}{L_r} < \frac{R_s}{L_s} < \frac{h}{L_r}$. The bank will choose to liquidate the cheaper investments (the lower the ratio) in order to reduce the effect of costly liquidation. Thus, with liquidity shortage, the bank will choose to liquidate a risky asset in the low state and a riskless asset otherwise. One can show that any deviation from the efficient liquidation can violate the incentive-compatibility constraints.

3.10.2.2 The Proof of Proposition 2

Optimization Problem and Resource Constraints In this case, a similar mathematical calculation is applied in which the bank chooses to liquidate a fraction of the risky asset $0 < \alpha < 1$.

We will again apply a similar method to this aforementioned case.

$$\max \pi \left[\mu \frac{C_1^{1-\rho}(h)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(h)}{1-\rho} \right] + (1-\pi) \left[\mu \frac{C_1^{1-\rho}(\ell)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(\ell)}{1-\rho} \right]$$

s.t.

$$\mu C_1(\ell) \leq I_o + \alpha I_r L_r$$

$$\mu C_1(h) \leq I_o + \alpha I_r L_r$$

$$\mu C_1(\ell) + (1-\mu) C_2(\ell) \leq I_o + \alpha I_r L_r + (1-\alpha) I_r \ell + I_s R_s$$

$$\mu C_1(h) + (1-\mu) C_2(h) \leq I_o + \alpha I_r L_r + (1-\alpha) I_r h + I_s R_s$$

$$I_o + I_s + I_r \leq \omega$$

Recall that the Lagrangian problem is explicitly characterized by:

$$\pi \left[\mu \frac{C_1^{1-\rho}(h)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(h)}{1-\rho} \right] + (1-\pi) \left[\mu \frac{C_1^{1-\rho}(\ell)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(\ell)}{1-\rho} \right]$$

$$-\lambda_{1\ell} [\mu C_1(\ell) - I_o - \alpha I_r L_r]$$

$$-\lambda_{1h} [\mu C_1(h) - I_o - \alpha I_r L_r]$$

$$-\lambda_{2\ell} [\mu C_1(\ell) + (1 - \mu) C_2(\ell) - I_o - \alpha I_r L_r - (1 - \alpha) I_r \ell - I_s R_s]$$

$$-\lambda_{2h} [\mu C_1(h) + (1 - \mu) C_2(h) - I_o - \alpha I_r L_r - (1 - \alpha) I_r h - I_s R_s]$$

$$-\lambda_3 [I_o + I_s + I_r - \omega]$$

First order conditions are:

$$\frac{\delta \mathcal{L}}{\delta C_1(\ell)} = (1 - \pi) \mu C_1^{-\rho}(\ell) - \lambda_{1\ell} \mu - \lambda_{2\ell} \mu = 0$$

$$(1 - \pi) C_1^{-\rho}(\ell) = \lambda_{1\ell} + \lambda_{2\ell} \quad (\text{A1})$$

$$\frac{\delta \mathcal{L}}{\delta C_2(\ell)} = (1 - \pi) (1 - \mu) C_2^{-\rho}(\ell) - \lambda_{2\ell} (1 - \mu) = 0$$

$$(1 - \pi) C_2^{-\rho}(\ell) = \lambda_{2\ell} \quad (\text{A2})$$

$$\frac{\delta \mathcal{L}}{\delta C_1(h)} = \pi \mu C_1^{-\rho}(h) - \lambda_{1h} \mu - \lambda_{2h} \mu = 0$$

$$\pi C_1^{-\rho}(h) = \lambda_{1h} + \lambda_{2h} \quad (\text{A3})$$

$$\frac{\delta \mathcal{L}}{\delta C_2(h)} = \pi (1 - \mu) C_2^{-\rho}(h) - \lambda_{2h} (1 - \mu) = 0$$

$$\pi C_2^{-\rho}(h) = \lambda_{2h} \quad (\text{A4})$$

$$\frac{\delta \mathcal{L}}{\delta I_o} = \lambda_{1\ell} + \lambda_{1h} + \lambda_{2\ell} + \lambda_{2h} - \lambda_3 = 0$$

$$\lambda_{1\ell} + \lambda_{1h} + \lambda_{2\ell} + \lambda_{2h} = \lambda_3 \quad (\text{A5})$$

$$\frac{\delta \mathcal{L}}{\delta I_s} = \lambda_{2\ell} R_s + \lambda_{2h} R_s - \lambda_3 = 0$$

$$(\lambda_{2\ell} + \lambda_{2h}) R_s = \lambda_3 \quad (\text{A6})$$

$$\frac{\delta \mathcal{L}}{\delta I_r} = \lambda_{1\ell} \alpha L_r + \lambda_{1h} \alpha L_r + \lambda_{2\ell} [\alpha L_r + (1 - \alpha) \ell] + \lambda_{2h} [\alpha L_r + (1 - \alpha) h] - \lambda_3 = 0$$

$$(\lambda_{1\ell} + \lambda_{1h} + \lambda_{2\ell} + \lambda_{2h}) \alpha L_r + (1 - \alpha) (\lambda_{2\ell} \ell + \lambda_{2h} h) = \lambda_3 \quad (\text{A7})$$

Substituting (A2) in (A1) yields $(1 - \pi)C_1^{-\rho}(\ell) = \lambda_{1\ell} + \pi C_2^{-\rho}(\ell)$

$$\begin{aligned}\lambda_{1\ell} &= (1 - \pi)C_1^{-\rho}(\ell) - (1 - \pi)C_2^{-\rho}(\ell) \\ \lambda_{1\ell} &= (1 - \pi) [C_1^{-\rho}(\ell) - C_2^{-\rho}(\ell)]\end{aligned}\tag{A8}$$

Plugging (A4) into (A3) yields $\pi C_1^{-\rho}(h) = \lambda_{1h} + \pi C_2^{-\rho}(h)$

$$\begin{aligned}\lambda_{1h} &= \pi C_1^{-\rho}(h) - \pi C_2^{-\rho}(h) \\ \lambda_{1h} &= \pi [C_1^{-\rho}(h) - C_2^{-\rho}(h)]\end{aligned}\tag{A9}$$

Equation (A5) implies that

$$(1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h) = \lambda_3\tag{A10}$$

Equation (A6) implies that

$$R_s [(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] = \lambda_3\tag{A11}$$

Equation (A7) implies that

$$[(1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h)] \cdot \alpha L_r + (1 - \alpha) [(1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h] = \lambda_3\tag{A12}$$

The optimality conditions:

$$\begin{aligned}&(1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h) \\ &= R_s [(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] \\ &= [(1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h)] \cdot \alpha L_r + (1 - \alpha) [(1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h]\end{aligned}\tag{A13}$$

Case 1 Start with binding constraints

$$\mu C_1(\ell) = I_o + \alpha I_r \cdot L_r$$

$$\mu C_1(h) = I_o + \alpha I_r L_r$$

$$\mu C_1(\ell) + (1 - \mu)C_2(\ell) = I_o + \alpha I_r L_r + (1 - \alpha)I_r \ell + I_s R_s$$

$$\mu C_1(h) + (1 - \mu)C_2(h) = I_o + \alpha I_r L_r + (1 - \alpha)I_r h + I_s R_s$$

This implies that

$$\mu C_1(\ell) = I_o + \alpha I_r L_r$$

$$\mu C_1(h) = I_o + \alpha I_r L_r$$

$$(1 - \mu)C_2(\ell) = (1 - \alpha)I_r \ell + I_s R_s$$

$$(1 - \mu)C_2(h) = (1 - \alpha)I_r h + I_s R_s$$

It follows that $C_1(\ell) = C_1(h) = C_1$. Now, substitute $I_s = \omega - I_r - \mu C_1 + \alpha I_r L_r$ into the last two equations

$$(1 - \mu)C_2(\ell) = (1 - \alpha)I_r \ell + [\omega - I_r - \mu C_1 + \alpha I_r L_r] R_s$$

$$(1 - \mu)C_2(h) = (1 - \alpha)I_r h + [\omega - I_r - \mu C_1 + \alpha I_r L_r] R_s$$

Rearranging

$$(1 - \mu)C_2(\ell) = I_r [(1 - \alpha)\ell - (1 - \alpha L_r)R_s] + [\omega - \mu C_1] R_s$$

$$(1 - \mu)C_2(h) = I_r [(1 - \alpha)h - (1 - \alpha L_r)R_s] + [\omega - \mu C_1] R_s$$

To solve these two linear equations we multiply the first equation by $\left[\frac{(1-\alpha)h - (1-\alpha L_r)R_s}{(1-\alpha)\ell - (1-\alpha L_r)R_s} \right]$ then we subtract the first equation from the second. Hence, I_r can be eliminated

$$(1 - \mu) \left[C_2(h) + \frac{(1 - \alpha)h - (1 - \alpha L_r)R_s}{(1 - \alpha L_r)R_s - (1 - \alpha)\ell} C_2(\ell) \right] = \left[\frac{R_s(h - \ell)(1 - \alpha)}{(1 - \alpha L_r)R_s - (1 - \alpha)\ell} \right] [\omega - \mu C_1]$$

Now recall optimality conditions (A13)

First:

$$\begin{aligned}
[(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] R_s &= [(1 - \pi)C_1^{-\rho} + \pi C_1^{-\rho}] \cdot \alpha L_r + (1 - \alpha) [(1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h] \\
[(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] R_s &= \alpha L_r \cdot C_1^{-\rho} + (1 - \alpha) [(1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h] \\
C_1^{-\rho} &= \frac{(1 - \pi) [R_s - (1 - \alpha) \cdot \ell] C_2^{-\rho}(\ell) - \pi [(1 - \alpha)h - R_s] C_2^{-\rho}(h)}{\alpha L_r}
\end{aligned}$$

Second:

$$\begin{aligned}
(1 - \alpha) [(1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h] &= (1 - \alpha L_r) C_1^{-\rho} \\
C_1^{-\rho} &= \frac{(1 - \alpha) [(1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h]}{(1 - \alpha L_r)}
\end{aligned}$$

This implies

$$\begin{aligned}
(1 - \pi)C_2^{-\rho}(\ell) [(1 - \alpha L_r) R_s - (1 - \alpha)\ell] &= [(1 - \alpha)h - (1 - \alpha L_r) R_s] \pi C_2^{-\rho}(h) \\
C_2^{-\rho}(\ell) &= \left(\frac{\pi}{1 - \pi} \right) \left[\frac{(1 - \alpha)h - (1 - \alpha L_r) R_s}{(1 - \alpha L_r) R_s - (1 - \alpha)\ell} \right] C_2^{-\rho}(h) \\
C_2(\ell) &= \left(\frac{1 - \pi}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1 - \alpha L_r) R_s - (1 - \alpha)\ell}{(1 - \alpha)h - (1 - \alpha L_r) R_s} \right]^{\frac{1}{\rho}} C_2(h) \tag{A14}
\end{aligned}$$

Third:

$$\begin{aligned}
[(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] R_s &= C_1^{-\rho} \\
C_1^{-\rho} &= \pi R_s \left[\frac{(1 - \alpha)h - (1 - \alpha L_r) R_s}{(1 - \alpha L_r) R_s - (1 - \alpha)\ell} \right] C_2^{-\rho}(h) + \pi R_s C_2^{-\rho}(h) \\
C_1^{-\rho} &= \frac{\pi(1 - \alpha)(h - \ell)R_s}{(1 - \alpha L_r) R_s - (1 - \alpha)\ell} C_2^{-\rho}(h) \\
C_1 &= \left[\frac{(1 - \alpha L_r) R_s - (1 - \alpha)\ell}{\pi(1 - \alpha)(h - \ell)R_s} \right]^{\frac{1}{\rho}} C_2(h) \tag{A15}
\end{aligned}$$

Now substitute in $C_1(\ell)$, $C_2(\ell)$ in (A15)

$$\begin{aligned}
(1 - \mu) \left[C_2(h) + \left(\frac{1 - \pi}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1 - \alpha)h - (1 - \alpha L_r)R_s}{(1 - \alpha L_r)R_s - (1 - \alpha)\ell} \right] \left[\frac{(1 - \alpha L_r) R_s - (1 - \alpha)\ell}{(1 - \alpha)h - (1 - \alpha L_r) R_s} \right]^{\frac{1}{\rho}} C_2(h) \right] &= \\
\left[\frac{R_s(h - \ell)(1 - \alpha)}{(1 - \alpha L_r)R_s - (1 - \alpha)\ell} \right] \left[\omega - \mu \left[\frac{(1 - \alpha L_r) R_s - (1 - \alpha)\ell}{\pi(1 - \alpha)(h - \ell)R_s} \right]^{\frac{1}{\rho}} C_2(h) \right] &= \\
C_2(h) = \frac{\left[\frac{R_s(h - \ell)(1 - \alpha)}{(1 - \alpha L_r)R_s - (1 - \alpha)\ell} \right]}{(1 - \mu) \left\{ 1 + \left(\frac{1 - \pi}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1 - \alpha L_r)R_s - (1 - \alpha)\ell}{(1 - \alpha)h - (1 - \alpha L_r)R_s} \right]^{\frac{1 - \rho}{\rho}} \right\} + \mu \left\{ \left(\frac{1}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1 - \alpha L_r)R_s - (1 - \alpha)\ell}{(1 - \alpha)(h - \ell)R_s} \right]^{\frac{1 - \rho}{\rho}} \right\}} \cdot \omega & \tag{A16}
\end{aligned}$$

Case 2

$$\mu C_1(\ell) < I_o + \alpha I_r L_r$$

$$\mu C_1(h) < I_o + \alpha I_r L_r$$

$$\mu C_1(\ell) + (1 - \mu)C_2(\ell) = I_o + \alpha I_r L_r + (1 - \alpha)I_r \ell + I_s R_s$$

$$\mu C_1(h) + (1 - \mu)C_2(h) = I_o + \alpha I_r L_r + (1 - \alpha)I_r h + I_s R_s$$

This implies that $\lambda_{1\ell} = 0$, $\lambda_{2\ell} > 0$, $\lambda_{1h} = 0$, $\lambda_{2h} > 0$.

$$\lambda_{2\ell} = (1 - \pi)C_1^{-\rho}(\ell) = (1 - \pi)C_2^{-\rho}(\ell)$$

$$C_1(\ell) = C_2(\ell) = I_o + \alpha I_r L_r + (1 - \alpha)I_r \ell + I_s R_s$$

$$\lambda_{2h} = \pi C_1^{-\rho}(h) = \pi C_2^{-\rho}(h)$$

$$C_1(h) = C_2(h) = I_o + \alpha I_r L_r + (1 - \alpha)I_r h + I_s R_s$$

This contradicts the optimality conditions (A13) because $R_s > 1$. Therefore, there is no possible solution when the optimal conditions are being violated.

3.10.2.3 The Proof of Proposition 3

Optimization Problem and the Resource Constraints The optimization problem can be expressed by the following formulation of the expected utility with replacing L_r by L_s . Partial liquidation can be expressed by a fraction $0 < \beta < 1$.

$$\max \pi \left[\mu \frac{C_1^{1-\rho}(h)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(h)}{1-\rho} \right] + (1-\pi) \left[\mu \frac{C_1^{1-\rho}(\ell)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(\ell)}{1-\rho} \right]$$

s.t.

$$\mu C_1(\ell) \leq I_o + \beta I_s L_s$$

$$\mu C_1(h) \leq I_o + \beta I_s L_s$$

$$\mu C_1(\ell) + (1 - \mu)C_2(\ell) \leq I_o + \beta I_s L_s + (1 - \beta)I_s R_s + I_r \ell$$

$$\mu C_1(h) + (1 - \mu)C_2(h) \leq I_o + \beta I_s L_s + (1 - \beta)I_s R_s + I_r h$$

$$I_o + I_s + I_r \leq \omega$$

Lagrangian function:

$$\pi \left[\mu \frac{C_1^{1-\rho}(h)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(h)}{1-\rho} \right] + (1-\pi) \left[\mu \frac{C_1^{1-\rho}(\ell)}{1-\rho} + (1-\mu) \frac{C_2^{1-\rho}(\ell)}{1-\rho} \right]$$

$$-\lambda_{1\ell} [\mu C_1(\ell) - I_o - \beta I_s L_s]$$

$$-\lambda_{1h} [\mu C_1(h) - I_o - \beta I_s L_s]$$

$$-\lambda_{2\ell} [\mu C_1(\ell) + (1-\mu)C_2(\ell) - I_o - \beta I_s L_s - (1-\beta)I_s R_s - I_2 \ell]$$

$$-\lambda_{2h} [\mu C_1(h) + (1-\mu)C_2(h) - I_o - \beta I_s L_s - (1-\beta)I_s R_s - I_2 H]$$

$$-\lambda_3 [I_o + I_1 + I_2 - \omega]$$

We now derive the first order conditions F.O.Cs:

$$\frac{\delta \mathcal{L}}{\delta C_1(\ell)} = (1-\pi)\mu C_1^{-\rho}(\ell) - \lambda_{1\ell}\mu - \lambda_{2\ell}\mu = 0$$

$$(1-\pi)C_1^{-\rho}(\ell) = \lambda_{1\ell} + \lambda_{2\ell} \quad (\text{B1})$$

$$\frac{\delta \mathcal{L}}{\delta C_2(\ell)} = (1-\pi)(1-\mu)C_2^{-\rho}(\ell) - \lambda_{2\ell}(1-\mu) = 0$$

$$(1-\pi)C_2^{-\rho}(\ell) = \lambda_{2\ell} \quad (\text{B2})$$

$$\frac{\delta \mathcal{L}}{\delta C_1(h)} = \pi\mu C_1^{-\rho}(H) - \lambda_{1h}\mu - \lambda_{2h}\mu = 0$$

$$\pi C_1^{-\rho}(h) = \lambda_{1h} + \lambda_{2h} \quad (\text{B3})$$

$$\frac{\delta \mathcal{L}}{\delta C_2(h)} = \pi(1 - \mu)C_2^{-\rho}(h) - \lambda_{2h}(1 - \mu) = 0$$

$$\pi C_2^{-\rho}(h) = \lambda_{2h} \quad (\text{B4})$$

$$\frac{\delta \mathcal{L}}{\delta I_o} = \lambda_{1\ell} + \lambda_{1h} + \lambda_{2\ell} + \lambda_{2h} - \lambda_3 = 0$$

$$\lambda_{1\ell} + \lambda_{1h} + \lambda_{2\ell} + \lambda_{2h} = \lambda_3 \quad (\text{B5})$$

$$\frac{\delta \mathcal{L}}{\delta I_s} = \lambda_{1\ell}\beta L_s + \lambda_{1h}\beta L_s + \lambda_{2\ell}[\beta L_s + (1 - \beta)R_s] + \lambda_{2h}[\beta L_s + (1 - \beta)R_s] - \lambda_3 = 0$$

$$(\lambda_{1\ell} + \lambda_{1h} + \lambda_{2\ell} + \lambda_{2h})\beta L_s + (\lambda_{2\ell} + \lambda_{2h})(1 - \beta)R_s = \lambda_3 \quad (\text{B6})$$

$$\frac{\delta \mathcal{L}}{\delta I_r} = \lambda_{2\ell}\ell + \lambda_{2h}h - \lambda_3 = 0$$

$$\lambda_{2\ell}\ell + \lambda_{2h}h = \lambda_3 \quad (\text{B7})$$

Substituting (B2) in (B1) yields $\pi_\ell C_1^{-\rho}(\ell) = \lambda_{1\ell} + \pi_\ell C_2^{-\rho}(\ell)$

$$\lambda_{1\ell} = (1 - \pi)C_1^{-\rho}(\ell) - \pi_\ell C_2^{-\rho}(\ell)$$

$$\lambda_{1\ell} = (1 - \pi)[C_1^{-\rho}(\ell) - C_2^{-\rho}(\ell)] \quad (\text{B8})$$

By plugging (B4) into (B3), it yields $\pi C_1^{-\rho}(h) = \lambda_{1h} + \pi_h C_2^{-\rho}(h)$

$$\lambda_{1h} = \pi C_1^{-\rho}(h) - \pi_h C_2^{-\rho}(h)$$

$$\lambda_{1h} = \pi [C_1^{-\rho}(h) - C_2^{-\rho}(h)] \quad (\text{B9})$$

Equation (B5) implies that

$$(1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h) = \lambda_3 \quad (\text{B10})$$

Equation (B6) implies that

$$[(1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h)] \cdot \beta L_s + [(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] \cdot (1 - \beta)R_s = \lambda_3 \quad (\text{B11})$$

Equation (B7) implies that

$$(1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h = \lambda_3 \quad (\text{B12})$$

The optimality conditions:

$$\begin{aligned} & [(1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h)] \cdot \beta L_s + [(1 - \pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] (1 - \beta) \cdot R_s \\ &= (1 - \pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h = (1 - \pi)C_1^{-\rho}(\ell) + \pi C_1^{-\rho}(h) \end{aligned} \quad (\text{B13})$$

Case 1 If the constraints bind

$$\mu C_1(\ell) = I_o + \beta I_s L_s$$

$$\mu C_1(h) = I_o + \beta I_s L_s$$

$$\mu C_1(\ell) + (1 - \mu)C_2(\ell) = I_o + \beta I_s L_s + (1 - \beta)I_s R_s + I_r \ell$$

$$\mu C_1(h) + (1 - \mu)C_2(h) = I_o + \beta I_s L_s + (1 - \beta)I_s R_s + I_r h$$

This implies

$$\mu C_1(\ell) = I_o + \beta I_s L_s$$

$$\mu C_1(h) = I_o + \beta I_s L_s$$

$$(1 - \mu)C_2(\ell) = (1 - \beta)I_s R_s + I_r \ell$$

$$(1 - \mu)C_2(h) = (1 - \beta)I_s R_s + I_r h$$

It follows that $C_1(\ell) = C_1(h) = C_1$. Now, substitute $I_r = \omega - I_s - \mu C_1 + \beta I_s L_s = \omega - \mu C_1 - (1 - \beta L_s) I_s$ into the last two equations

$$(1 - \mu)C_2(\ell) = (1 - \beta)I_s R_s + [\omega - I_s - \mu C_1 + \beta I_s L_s] \cdot \ell$$

$$(1 - \mu)C_2(h) = (1 - \beta)I_s R_s + [\omega - I_s - \mu C_1 + \beta I_s L_s] \cdot h$$

Rearranging

$$(1 - \mu)C_2(\ell) = I_s [(1 - \beta)R_s - (1 - \beta L_s)\ell] + [\omega - \mu C_1] \cdot \ell$$

$$(1 - \mu)C_2(h) = I_s [(1 - \beta)R_s - (1 - \beta L_s)h] + [\omega - \mu C_1] \cdot h$$

To solve these two linear equations we multiply the first equation by $\left[\frac{(1-\beta)R_s-(1-\alpha L_s)h}{(1-\alpha)R_s-(1-\alpha L_s)\ell}\right]$ then we subtract the first equation from the second. Hence, I_s can be eliminated and it can be simplified to

$$(1-\mu) \left[C_2(h) + \frac{(1-\beta L_s)h - (1-\beta)R_s}{(1-\beta)R_s - (1-\beta L_s)\ell} C_2(\ell) \right] = [\omega - \mu C_1] \cdot \left[\frac{(1-\beta)(h-\ell)R_s}{(1-\beta)R_s - (1-\beta L_s)\ell} \right] \quad (\text{B14})$$

Optimality conditions (B13) imply the following

First: The following condition implies that

$$[(1-\pi)C_1^{-\rho} + \pi C_1^{-\rho}] \cdot \beta L_s + [(1-\pi)C_2^{-\rho}(\ell) + \pi C_2^{-\rho}(h)] \cdot (1-\beta)R_s = (1-\pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h$$

Now substitute $C_1^{-\rho} = (1-\pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h$

$$(1-\pi)C_2^{-\rho}(\ell) \cdot \ell \beta L_s + (1-\pi)C_2^{-\rho}(\ell)(1-\beta)R_s - (1-\pi)C_2^{-\rho}(\ell) \cdot \ell = \pi C_2^{-\rho}(h) \cdot h - \pi C_2^{-\rho}(h) \cdot h \beta L_s - \pi C_2^{-\rho}(h)(1-\beta)R_s$$

$$(1-\pi)C_2^{-\rho}(\ell) [\ell \beta L_s + R_s(1-\beta) - \ell] = [h - h \beta L_s - (1-\beta)R_s] \pi C_2^{-\rho}(h)$$

$$C_2^{-\rho}(\ell) = \left(\frac{\pi}{1-\pi} \right) \left[\frac{h - h \beta L_s - (1-\beta)R_s}{\ell \beta L_s + (1-\beta)R_s - \ell} \right] C_2^{-\rho}(h)$$

$$C_2(\ell) = \left(\frac{1-\pi}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1-\beta)R_s - (1-\beta L_s)\ell}{(1-\beta L_s)h - (1-\beta)R_s} \right]^{\frac{1}{\rho}} C_2(h) \quad (\text{B15})$$

Second: Substitute (B14) and (B16) in the following equation

$$C_1^{-\rho} = (1-\pi)C_2^{-\rho}(\ell) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h$$

$$C_1^{-\rho} = (1-\pi) \left(\frac{\pi}{1-\pi} \right) \left[\frac{(1-\beta L_s)h - (1-\beta)R_s}{(1-\beta)R_s - (1-\beta L_s)\ell} \right] C_2^{-\rho}(h) \cdot \ell + \pi C_2^{-\rho}(h) \cdot h$$

$$C_1^{-\rho} = \left[\frac{\pi R_s(1-\beta)(h-\ell)}{(1-\beta)R_s - (1-\beta L_s)\ell} \right] C_2^{-\rho}(h)$$

$$C_1 = \left[\frac{(1-\beta)R_s - (1-\beta L_s)\ell}{\pi(1-\beta)(h-\ell)R_s} \right]^{\frac{1}{\rho}} C_2(h) \quad (\text{B16})$$

Now substitute $C_1, C_2(\ell)$ in the following in (B15)

$$\begin{aligned} & (1-\mu) \left\{ C_2(h) + \left[\frac{(1-\beta L_s)h - (1-\beta)R_s}{(1-\beta)R_s - (1-\beta L_s)\ell} \right] \left(\frac{1-\pi}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1-\beta)R_s - (1-\beta L_s)\ell}{(1-\beta L_s)h - (1-\beta)R_s} \right]^{\frac{1}{\rho}} C_2(H) \right\} \\ &= \left\{ \omega - \mu \left[\frac{(1-\beta)R_s - (1-\beta L_s)\ell}{\pi(1-\beta)(h-\ell)R_s} \right]^{\frac{1}{\rho}} C_2(h) \right\} \cdot \left[\frac{(1-\beta)(h-\ell) \cdot R_s}{(1-\beta)R_s - (1-\beta L_s)\ell} \right] \end{aligned}$$

$$\begin{aligned}
& \left\{ \begin{aligned} & (1 - \mu) \left[1 + \left(\frac{1-\pi}{\pi} \right)^{\frac{1}{\rho}} \left(\frac{(1-\beta)R_s - (1-\beta L_s)\ell}{(1-\beta L_s)h - (1-\beta)R_s} \right)^{\frac{1-\rho}{\rho}} \right] \\ & + \\ & \mu \left(\frac{1}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1-\beta)R_s - (1-\beta L_s)\ell}{R_s(1-\beta)(h-\ell)} \right]^{\frac{1-\rho}{\rho}} \end{aligned} \right\} C_2(h) = \left[\frac{(1-\beta)(h-\ell) \cdot R_s}{(1-\beta)R_s - (1-\beta L_s)\ell} \right] \cdot \omega \\
C_2(h) = & \frac{\left[\frac{(1-\beta)(h-\ell) \cdot R_s}{(1-\beta)R_s - (1-\beta L_s)\ell} \right]}{\left\{ \mu \left(\frac{1}{\pi} \right)^{\frac{1}{\rho}} \left[\frac{(1-\beta)R_s - (1-\beta L_s)\ell}{(1-\beta)(h-\ell)R_s} \right]^{\frac{1-\rho}{\rho}} + (1 - \mu) \left[1 + \left(\frac{1-\pi}{\pi} \right)^{\frac{1}{\rho}} \left(\frac{(1-\beta)R_s - (1-\beta L_s)\ell}{(1-\beta L_s)h - (1-\beta)R_s} \right)^{\frac{1-\rho}{\rho}} \right] \right\}} \cdot \omega \\
& \tag{B17}
\end{aligned}$$

Case 2

$$\mu C_1(\ell) < I_o + \beta I_s L_s$$

$$\mu C_1(h) < I_o + \beta I_s L_s$$

$$\mu C_1(\ell) + (1 - \mu)C_2(\ell) = I_o + \beta I_s L_s + (1 - \beta)I_s R_s + I_r \ell$$

$$\mu C_1(h) + (1 - \mu)C_2(h) = I_o + \beta I_s L_s + (1 - \beta)I_s R_s + I_r h$$

This implies that $\lambda_{1\ell} = 0, \lambda_{2\ell} > 0, \lambda_{1h} = 0, \lambda_{2h} > 0$.

$$\lambda_{2\ell} = (1 - \pi)C_1^{-\rho}(\ell) = (1 - \pi)C_2^{-\rho}(\ell)$$

$$C_1(\ell) = C_2(\ell) = I_o + \beta I_s L_s + (1 - \beta)I_s R_s + I_r \ell$$

$$\lambda_{2h} = \pi C_1^{-\rho}(h) = \pi C_2^{-\rho}(h)$$

$$C_1(h) = C_2(h) = I_o + \beta I_s L_s + (1 - \beta)I_s R_s + I_r h$$

By checking the optimality conditions (B13), there is a contradiction because $h > 1$ and $\ell < 1$.

Thus, there is no possible solution when the optimal conditions are being violated.

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Chapter 4 Lending Process and Optimal Loan Contracts

4.1 Introduction

Banks play a crucial role in the economy by providing loans to borrower firms for investment projects. Such projects are important for economic growth and development.

The goal of this paper is to study the lending process and loan contracts. We focus on lending to borrower firms. We also analyze how the design of the bank loan contracts affects the borrower behavior. We focus on some key tools of the bank such as liquidation and renegotiation to credibly threaten the borrower. Our analysis provides a theoretical analysis of a bank-borrower relationship by generalizing and extending the model of Bolton and Scharfstein (1996).

Bolton and Scharfstein's model characterizes optimal incomplete debt contracts. This model studies a borrowing firm which has an investment project of two assets that requires capital K of a creditor. The firm is a limited liable (has no initial wealth). The project produces a random cash flow of x with probability θ or zero with probability $1 - \theta$ in period 1. If the project continues until period 2, the cash flow will be y . The creditor has the right to liquidate the firm's assets in period 1, but this will decrease the firm value due to costly liquidation.

We aim to introduce modifications to Bolton and Scharfstein as follows. With probability θ , a good state occurs and the project yields x_g, y_g in periods 1 and 2 respectively. With probability $1 - \theta$, a bad state occurs and cash flows are x_b, y_b in periods 1 and 2 respectively. We obtain simple results by generalizing Bolton and Scharfstein's model to a more realistic framework where the cash flow in the state b is deterministic but not zero and period 2 payoffs are different y_g, y_b . These are important for two reasons. (1) They allow us to distinguish between liquidity default and earning low returns. (2) If the project yields a strictly positive repayment to the bank in the bad state, the bank will have less incentive to liquidate the project. This is because the bank

will be able to recover the initial loan.

More specifically, we study how the information structure and the associated costs can have a significant impact on loan contracts. We focus on three cases. First, symmetric information and complete contracts. Second, symmetric information and incomplete contracts. Third, asymmetric information and incomplete contracts.

Our primary contribution lies in three directions. First, unlike Bolton and Scharfstein, we provide a model with symmetric information and complete contracts that can serve as a benchmark. In this case, there are no transaction costs or any other costs in the economy.

Second, writing ex-ante complete contracts is very costly. Therefore, we analyze incomplete contracts. We show that the main result of Bolton and Scharfstein, given incompleteness, does hold in a more general setting as described in the previous paragraph.

Third, more realistically, we differ from Bolton and Scharfstein by exploring the influence of asymmetry of information on incomplete contracts. In this case, as we shall see later, the borrower has private information regarding the state of nature, but the bank needs to pay some cost to verify the state. It is useful to show how the asymmetry of information and verification costs can change the structure of the model.

Throughout this analysis, I incorporate possible ex-post actions such as liquidation and renegotiation that are much closer to the real world contracts. I also examine issues like efficiency and profitability associated with optimal contracts. We shall examine these issues in greater detail later in this chapter.

We are not the first to study loan contracts in the bank lending literature. There is an extensive literature describing the optimality of loan contracts. We share some similarities with the following models. The structure of information has a critical role in the lending process and lending standards (see, e.g., Aigner and Sprengle (1968) and Dell'Araccia and Marquez (2006)).

Our approach is similar to Hart and Moore (1988) in terms of modelling incomplete contracts.

We describe incentive contracts that prevent strategic behavior of borrowers under asymmetry of information as in Webb (1992) and Gale and Hellwig (1985).

Besides, we shed light on renegotiation and liquidation (see, e.g., Hart and Moore (1998)). Hart and Moore show that the creditor has the right to renegotiate or liquidate the assets if the debtor defaults. In particular, the creditor will renegotiate the debt contract if the liquidation value is low. However, there are some important differences between Hart and Moore's (1998) work and ours. Hart and Moore study debt contracts in a world of symmetric information only. While in our analysis we analyze the effects of asymmetric information and symmetric information on loan contracts.

Similar to most of the previous work, our analysis focuses on the following characteristics: the role of information, strategic behavior, incompleteness, renegotiation and liquidation.

Our theory is different from three strands of literature. One strand of literature shows that the presence of the third party, within an incomplete contract, can improve efficiency and secure the realization of the project expected returns (Menichini (2008)). In contrast, we focus on economic measurements in our analysis rather than a third party involvement.

Our work also departs from another strand of lending literature in characterizing the role of monitoring in loan contracts. Diamond (1984) considers a financial intermediary who raises funds from depositors, lends to risk neutral entrepreneurs, and spends resources on monitoring the entrepreneurs' information and enforcing loan contracts. He depicts that these delegated monitoring contracts (with covenants) are less costly than those without monitoring. Carletti (2004) shows that banks can control the borrower firm's behavior through costly monitoring and they will intervene if the firm misbehaves. Anhy and Thi (2010) have pointed out that monitoring technology can be used by the bank to control the borrower's behavior. Marjit and Mallick (2004)

show that bank lending is positively related to collateral monitoring and regulatory mechanism under asymmetry of information. Fischel (1989) argues that when a lender provides a loan to a borrower, both the default probability of the borrower and the likelihood of recovery after default should be considered. Nevertheless, the lender will implement various monitoring and bonding mechanisms to reduce the risk of exogenous events or the borrower misbehavior. In these models, costly monitoring is used to control the borrower firm behavior. In contrast, we do not use monitoring technology in our analysis to avoid any unnecessary costs.

In the third strand, we have not investigated the use of collateralized debt and covenants in debt contracts which are suggested in the following models. Benmelech et al. (2005) present the impact of collateral quality on decreasing the cost of borrowing. They also show that liquidation values and asset redeployability can have a significant impact on incomplete loan contracts under which higher liquidation values imply larger loans with longer maturities and lower interest rates. Berlin and Mester (1991) include restrictive covenants in debt contracts to control agency problems then these contractual covenants can be renegotiated. Garleanu and Zwiebel (2009) show that the ex-post renegotiation of covenants in debt contracts under asymmetric information gives control rights and ownership to the uninformed lender. Although loan covenants as monitoring devices can control the strategic behavior of borrowers, they are very costly because they are difficult to include in incomplete contracts.

We aim to develop a theory of loan contracts with two characteristics: complete and incomplete by generalizing and extending Bolton and Scharfstein's 1996 model. We first start by introducing a model with symmetric information under which complete and incomplete contracts can be written. Then we derive sufficient conditions for an optimal contract and we provide a basic explanation for a case with asymmetric information and costly state verification. We also aim to provide incentives for borrowers to truthfully report their project cash flows.

The rest of this paper is organized as follows. The next section refers to the basic model. Section 3 refers to contracts under symmetric information environment (complete and incomplete). Section 4 analyzes the impact of asymmetry of information and costly state verification on the model. Section 5 includes conclusions. Appendix is contained in section 6.

4.2 The Basic Model

This study will chiefly focus on the basic framework of Bolton and Scharfstein's (1996) model. We study a lending relationship between a bank and a borrower firm. The model consists of three discrete periods 0, 1, and 2.

In the initial period 0, a contract between a bank and a borrower firm is written in which the bank provides a loan K to the borrower to initiate a project. Both parties know exactly how the contract will work in each state and all the possibilities.

We assume that there are two states of the world: good g and bad b with probabilities θ and $1 - \theta$ respectively. The project produces random cash flows contingent upon the realized state in periods 1, 2 as follows: x_g, y_g in the good state or x_b, y_b in the bad state.

In period 1, the borrower has to repay the bank R_g if the state is g or R_b if the state is b . Consequently, the bank has the right to liquidate the project with proportions β_g, β_b at given states g, b respectively. Assume that $R_b < K < R_g$ and $R_g \leq y_g, R_b \leq y_b$.

The borrower can strategically behave and steal some of the cash flows by transferring R_b instead of R_g if the realized state is good. Therefore, liquidation threat is required to prevent such strategic behavior of the borrower.¹⁵ Ex-post liquidation values are L_g, L_b contingent on the state. They are common knowledge and exogenously given. If the project is liquidated in period 1, it will yield $L_g < x_g$ in a state g or $L_b < x_b$ in a state b .

¹⁵ See for example Gorton and Kahn (2000), who find that the bank can mitigate moral hazard problem on the part of borrowers by either liquidating the project or forgiving some debt as incentive to the borrowers. Qian and Strahan (2007) show that a lender can control the borrower behavior if they are given the right to seize the collateralized assets or to credibly threaten to do so in the event of default. DeMarzo and Fishman (2007) describe that liquidation is a threatening device by the investor to induce the agent to pay out the project cash flows. Parlour and Rajan (2001) show that a borrower has a marginal incentive to default since their consumption increases.

If the project continues until period 2, the borrower will get the resulting cash flows in period 2 y_g, y_b as a reward and an incentive. Thus, the borrower is sufficiently well compensated for their effort and can virtually have the residual cash flow after making the required payment to the bank.

Note that Bolton and Scharfstein has $y_g = y_b$ and $x_b = 0$.

4.2.1 Assumptions

This model is studied under the following assumptions:

- (1) The borrower has no initial wealth so there is no equity participation from the side of the borrower.
- (2) Discount rate is zero which means that all cash flows are already discounted.
- (3) Liquidation is not costless.
- (4) Suppose the bank and the borrower are risk neutral and rational.¹⁶ Under which each solely aims to maximize its benefit.
- (5) Partial liquidation is not possible.
- (6) Borrowers do not have access to any sources of financing other than banks.
- (7) The bank has no managerial experience of running the project.
- (8) No depreciation for the tangible project.

In the following, we shed some light on the symmetry of information and complete contracts.

4.3 Contracts under Symmetric Information Environment

In a world of symmetric information, states are common knowledge and the bank can observe the behavior of the borrower. In reality, asymmetric information and uncertainty exist. We shall examine next complete contingent contracts and contract enforcement under symmetric information.

4.3.1 Complete Contracts

This case is particularly simple. There is uncertainty about the true state of nature at the time of contracting and it will be revealed in period 1. We do not use standard loan contracts (non-contingent contracts). Rather, a state-contingent contract, upon the realization of the state and the cash flow, is written. In consequence, the borrower should be able to meet her contractual obligations in the two states of the world. The loan contract is $\{R_{s=g,b}, K\}$.

¹⁶ We do not study the risk sharing mechanism between the borrower and the bank in this analysis.

$$R_b \leq x_b$$

The expected profit of the bank IR

$$\theta R_g + (1 - \theta)R_b - K \geq 0$$

Where θR_g denotes the bank's payoff in the good state and $(1 - \theta)R_b$ denotes the bank's payoff in the bad state.

4.3.1.2 Ex-ante Efficiency Considerations

Proposition 1 *A: Participation constraint for the borrower PC*

$$\theta [x_g - R_g + y_g] + (1 - \theta) [x_b - R_b + y_b] \geq 0$$

and

$$x_g + y_g \geq R_g, x_b + y_b \geq R_b$$

B: Individually Rational constraint for the bank IR

$$\theta R_g + (1 - \theta)R_b \geq K$$

C: Efficiency

$$\theta (x_g + y_g) + (1 - \theta) (x_b + y_b) \geq K$$

Consider the following statements: (1) If A and B are true \implies C is true. (2) C \implies There exists R_g, R_b st. A and B hold.

Proof. See Appendix. ■

The contract can achieve the first-best efficiency provided the borrower hands over all the cash flows to the bank in period 1. Moreover, there is no strategic behavior by the borrowers since the state of the world and the associated payoffs are observable and verifiable. The court can ensure that the obligations of both parties have been fulfilled.

Contracts are usually incomplete in the real world because of transaction costs associated with writing complete contracts. We next study symmetric information and incomplete contracts.

4.3.2 Incomplete Contracts

In the initial period, the bank and the borrower sign incomplete state-contingent contracts.¹⁷ It is worth observing that incompleteness arises because writing ex-ante complete contracts requires

¹⁷ In fact, most real world contracts are incomplete. This incompleteness is also consistent with Grossman and Hart (1986). As emphasized in Tirole (1999), there exist three reasons for incompleteness: unforeseen contingencies, the costs of writing contracts and the costs of enforcing contracts by a court. However, in practice, writing complete contingent contracts is expensive and difficult in a complex setting as spelled out by Sharpe (1990).

high transaction costs. These exogenous costs are associated with the processing of information to include sufficient details about the state of the world in the contract as in Hart and Moore (1988). Hence, in this case the states are observable ex-post by both parties, but not verifiable to the third party (court).¹⁸

The bank chooses the terms of the loan contract characterized by $\{(R_s, \beta_s)_{s=g,b}, K\}$ where $R = \{R_g; R_b\}$ the repayment, $\beta_s = \{\beta_g; \beta_b\}$ the probability of liquidation and K the initial loan. The bank chooses to liquidate the project to reduce the incentive for the strategic behavior of borrowers. If there is no liquidation threat, the borrower has no incentive to pay out any cash flows. It is worth noting that this type of contract is non-enforceable. Moreover, any dispute should be resolved between the two parties according to non-verifiability.¹⁹

4.3.2.1 The Expected Payoffs

Within the incomplete contracts framework, the participation constraints are given by the following expressions²⁰

For the borrower firm

$$\theta [x_g - R_g + (1 - \beta_g)y_g] + (1 - \theta) [x_b - R_b + (1 - \beta_b)y_b] \geq 0$$

With probability θ , a good state occurs and cash flow x_g is realized. Hence, the borrower has to pay out R_g to the bank and keeps the remaining cash flow. The borrower will also obtain the period 2 payoff y_g if no liquidation takes place in period 1. Similarly, with probability $1 - \theta$, a bad state occurs and cash flow x_b is realized. Hence, the borrower has to pay out R_b to the bank and keeps the remaining cash flow. In addition, the borrower will get y_b if no liquidation takes place in period 1.

¹⁸ Battigalli and Maggi (2002) explain that writing complete contracts is very costly because all contingencies and policies are required to be specified ex-ante and verified ex-post.

¹⁹ For more information, see Figure 2 in Appendix.

²⁰ The reservation values are normalized to zero

For the bank

$$\theta [R_g + \beta_g L_g] + (1 - \theta) [R_b + \beta_b L_b] - K \geq 0$$

In the good state of the world which occurs with probability θ , the bank receives in period 1 the fixed payment R_g as well as the payoff from liquidation after receiving the payment. Liquidation payoff in the good state is given by liquidation value L_g multiplied by the probability of liquidation β_g . Likewise, in the bad state of the world which occurs with probability $1 - \theta$, the bank receives the payment R_b and the liquidation payoff. Liquidation payoff in the bad state is given by liquidation value L_b multiplied by the probability of liquidation β_b . Finally, the borrowed amount is deducted from the total amount received (the repayments and liquidation proceeds).

The emergence of strategic behavior implies that the borrower can obtain a private benefit from repaying the bank R_b rather than R_g when the cash flow is x_g .²¹ Therefore, incentives should be provided to prevent the strategic behavior by the borrower. Specifically, the repayment mechanism $\{R_g, R_b\}$ is incentive-compatible if and only if it satisfies the following constraint

$$x_g - R_g + (1 - \beta_g)y_g \geq x_g - R_b + (1 - \beta_b)y_g \implies (\beta_b - \beta_g)y_g \geq R_g - R_b$$

That is, the incentive constraint to the borrower requires that the borrower is better off under good behavior than under strategic behavior.

The following two constraints are limited liability constraints that indicate that repayments to the bank should not exceed the available cash flows in both states of nature

$$R_g \leq x_g$$

$$R_b \leq x_b$$

4.3.2.2 The Objective Function and Optimization Problem

The following proposition summarizes the features of the loan contracts when the contract is incomplete.

²¹ Note that Jaffee-Russel (1984) model shows that a dishonest borrower will choose to default when the cost of default is low enough.

Proposition 2 *The loan contracts are profit maximizing for the bank and incentive-compatible for borrowers if they have the following properties: (a) $\beta_g = 0$, (b) $R_b = x_b$ if $0 < \beta_b \leq 1$ or $R_b < x_b$ if $\beta_b = 0$.*

Proof. See Appendix. ■

Our contract is consistent with Pecchenino (1988) who shows that a bank would offer a contract that maximizes its profit and gives a minimum payoff to the firm which is consistent with truth-telling and individual rationality. In the following, we will briefly discuss the results of the proposition.

4.3.2.3 Discussion

The intuitive description of proposition 2 is specified as follows. It is shown that $\beta_g = 0$ which implies that the bank will never choose to liquidate the project when the borrower repays R_g to the bank. If β_g is strictly positive, then the borrower's incentive constraint will be relaxed and hence they will rather transfer R_b in the good state.

Here we have demonstrated that the threat of liquidation can support the existence of incentive-compatibility constraint to the borrower.

Establishing that $R_b = 0$ also maximizes the contractual surplus (the joint gains) and provides incentives to the borrower. The alternative solution is to set $R_b < 0$ by leaving the borrower with something in the bad state. But it would be better to increase R_b and decrease β_b .

There are two features of the state-contingent contract efficiency. The first feature characterizes the first-best efficiency

$$\theta(x_g + y_g) + (1 - \theta)(x_b + y_b) - K$$

This outcome captures the efficiency in complete contingent contracts.

While the second feature captures the expected efficiency loss associated with contractual incompleteness

$$(1 - \theta)\beta_b(y_b - L_b)$$

The expression $(1 - \theta)(y_b - L_b)\beta_b > 0$ since $y_b > L_b$.

It is worth noting that the inefficiency arises from the contractual incompleteness and it is triggered by costly liquidation. In such a case, the borrower's expected payoff is decreasing in β_b , while the bank's profit is increasing in β_b .

The bank has the right to liquidate the project with a probability β_b which is given by the following expression

$$\beta_b = \beta_b^* \equiv \frac{K - x_b}{\theta y_g + (1 - \theta)L_b}$$

Since $0 < \beta_b^* \leq 1 \Leftrightarrow K - x_b \leq \theta y_g + (1 - \theta)L_b$

$$K \leq x_b + \theta y_g + (1 - \theta)L_b$$

If $K \leq x_b + \theta y_g + (1 - \theta)L_b \implies$ the project will be financed. If $K > x_b + \theta y_g + (1 - \theta)L_b \implies$ the project will not be financed.

In the following, we add renegotiation to our analysis.

4.3.2.4 Renegotiation and Liquidation Decisions

The loan contract implies that the bank can liquidate the project or renegotiate the contract if the repayment of the loan by the borrower is R_b .²² The choice between renegotiation and liquidation will depend not only on the profitability of the strategy to the bank, but also on the borrower's incentives to engage in such renegotiation.

If the bank chooses to liquidate the project, it will be sold to the outsider by using a second price auction. We assume that the arising funds from the borrower's strategic behavior $[x_g - R_b]$ are insufficient to purchase the project. Hence, the borrower will take the money and run.

If liquidation does not occur, then the bank will renegotiate the loan contract with the borrower.

The renegotiation mechanism is reflected in the terms of the contract. If both parties agree to

²² Two propositions summarizing the bargaining game between the two parties are explained in Hart (1995). Proposition 1 is based on chapter 7 in the US or liquidation in the UK which refers here to selling (liquidating) the project by making a cash auction. Proposition 2 is set under chapter 11 in US or UK administration (structured bargaining).

renegotiate the contract in period 1, they will benefit from the period 2 payoff. What needs to be mentioned is that the borrower will participate in such renegotiation if their payoff is greater than their outside option. Henceforth, both parties will write a new contract that splits the period 2 payoff 50:50 between them based on the Nash bargaining game model.

Bargaining Structure and Renegotiation Game The bank can announce whether there is renegotiation in period 1 after observing the state of nature and receiving the payment. We assume that the bargaining power is divided equally in renegotiation. That is, the period 2 payoff is divided 50:50 between the bank and the borrower under renegotiation. However, the contract cannot include a provision stating that there is no renegotiation in the good state. This is because states are ex-post observable by both contracting parties but non-verifiable.

To see why ex-post renegotiation will never occur if the repayment is R_g , we need to analyze the influence of renegotiation on both parties (the borrower and the bank). In case of a renegotiation, the expected payoff for the borrower is $[x_g - R_b + \frac{y_g}{2}]$ if they choose to strategically behave and transfer R_b in the good state. While, if the borrower does not lie and renegotiation occurs following a truthful repayment R_g in period 1, they will get $[x_g - R_g + \frac{y_g}{2}]$. From the borrower's perspective, the incentive to transfer R_g in the good state is violated

$$x_g - R_g + \frac{y_g}{2} \not\geq x_g - R_b + \frac{y_g}{2} \text{ since } R_g > R_b$$

The borrower will never transfer R_g if renegotiation takes place. Therefore, the bank cannot extract more gains as expressed by $[R_g + \frac{y_g - c}{2}]$ by choosing to renegotiate. Specifically, it can be specified in the contract that the contract is never renegotiated after the payment R_g has been made as long as

$$R_b + \frac{y_g}{2} \leq R_g$$

In addition, the bank will choose to liquidate or renegotiate if the borrower pays out R_b rather

than R_g in the good state. If the project is liquidated, the bank payoff will be $[R_b + L_g]$. If the contract is renegotiated, then the bank payoff is $[R_b + \frac{y_g}{2}]$.

Thus, the contract will be renegotiated if and only if

$$R_b + L_g \leq R_b + \frac{y_g}{2} \implies L_g \leq \frac{y_g}{2}$$

In the good state, the bank will renegotiate rather than liquidate if the renegotiation outcome is greater than the project's liquidation value.

Moreover, the bank will choose to liquidate or renegotiate the contract with the borrower in the bad state since liquidation or renegotiation payoff is always greater than R_b . If the bank chooses to liquidate the project, then the bank will receive $[R_b + L_b]$. While if the bank chooses to renegotiate the contract, it will obtain $[R_b + \frac{y_b}{2}]$.

However, a contract is rather renegotiated provided that

$$R_b + L_b \leq R_b + \frac{y_b}{2} \implies L_b \leq \frac{y_b}{2}$$

We explore whether ex-post renegotiation is more desirable than liquidation if it yields higher mutual benefit to both parties as well as reduces the potential losses of liquidation as the following proposition shows.

Proposition 3 *The bargaining between the bank and the borrower will take place in the bad state if the gain (continuation payoffs) from the bargaining is higher than the liquidation value of the project $\frac{y_b}{2} \geq L_b$.*

Proof. See Appendix. ■

Throughout, the bank profit in the presence of liquidation or renegotiation in the state b can be written as

$$\Pi = R_b + \gamma L_b + (1 - \gamma)\left(\frac{y_b}{2}\right)$$

Where γ is the probability of liquidation and $1 - \gamma$ is the probability of renegotiation.

The interpretation of this proposition is that it is better to continue with the initial project after

bargaining over the new contractual terms if liquidation yields less favorable outcomes for a bank.

Thus far, the discussion implies two things. First: the loan repayments R_g, R_b are contingent on the two states g, b in period 1. Second: if the bank finds liquidation is costly, then the bank would rather choose to renegotiate.

Our analysis can be extended to introduce asymmetry of information which is different from Bolton and Scharfstein (1996). Asymmetry of information arises from the borrower's private information about the true state and the associated cash flow in period 1. In the presence of asymmetric information between banks and their borrowers, costly auditing can reveal some information about the state and the realized cash flows. This is the case we shall investigate more thoroughly in the next section.

4.4 Asymmetric Information and Costly State Verification (CSV)

This section describes the structure of the market, contract, costly auditing, game, equilibrium and optimality.

4.4.1 Framework Adjustments

In the basic version of the model, we studied both complete contingent and incomplete contingent contracts under symmetric information.

This version of the model contrasts with Bolton and Scharfstein in the following two aspects. First, the borrower firm has informational advantage about the realized state over the bank. Notice that this asymmetry of information may cause potential financial loss to the bank because the borrower can derive some benefit at the expense of the bank.

Second, our analysis also differs from Bolton and Scharfstein by identifying that the bank can only have the right to liquidate the project if a costly state verification reveals borrower deception.²³ The state is observed by the borrower without costs, while the bank can observe the state at some fixed costs $\mu > 0$. The bank cannot make the costs borne by the borrower. These

²³ See Figure 3 in Appendix for the timing of events.

costs stem from hiring outside or independent auditors. Henceforth, costly state verification can be interpreted as a costly auditing.²⁴

Having described the problem of asymmetry of information, let us analyze the mechanism to restrict undesirable actions by the borrowers.

4.4.2 The Contract

The initial contract (R, K, ρ) involves the repayment, initial loan, and the probability of auditing. The ownership of the project will be transferred to the bank if the auditing reveals borrower deception. Then the bank can choose either to liquidate or renegotiate.

In period 1, the borrower announces the state and the associated payment. The bank solely announces whether there is an auditing contingent on the borrower's announcement. If the borrower announces a bad state, then the bank chooses to audit with probability ρ or not to audit with probability $1 - \rho$.²⁵

The reasons behind threatening the borrower by stochastic auditing are generally borne not only by decreasing the resource cost of verification but also inducing honesty (Townsend, 1979).²⁶ The bank choice of stochastic auditing may provide a stronger incentive to the borrower to report cash flows truthfully.

4.4.3 Assumptions

- (1) We disregard any additional profit opportunities or alternative source of investment for the bank other than lending, but the borrower can rely on other banks to finance the project. Therefore, the bank cannot impose its condition on the borrower.
- (2) Our analysis has been centred on loans as the sole source of funds to borrowers leaving out the existence of other alternative sources such as the issuance of shares.
- (3) Credit rationing is not present in this model.
- (4) We ignore the role of moral hazard and adverse selection which might arise from asymmetry of information.

²⁴ In Renou (2008), obtaining additional information about a borrower's risky project requires paying some costs by the lender.

See figure 3 in Appendix.

²⁵ Stochastic verification may dominate the deterministic verification depending on the cost-benefit analysis as in Townsend (1979) and Dowd (1992). This is largely attributed to money saving advantage if there is no verification.

²⁶ Townsend (1979) shows that when one party has private information, non contingent debt contracts will often be the only possible solution. Monnet and Quintin (2005) design optimal contracts in a dynamic costly state verification model with stochastic monitoring.

4.4.4 Auditing Rule and Penalties for Misbehavior

It is particularly apparent that the borrower will lose the project, as a pledged collateral, to the bank if auditing reveals deception.

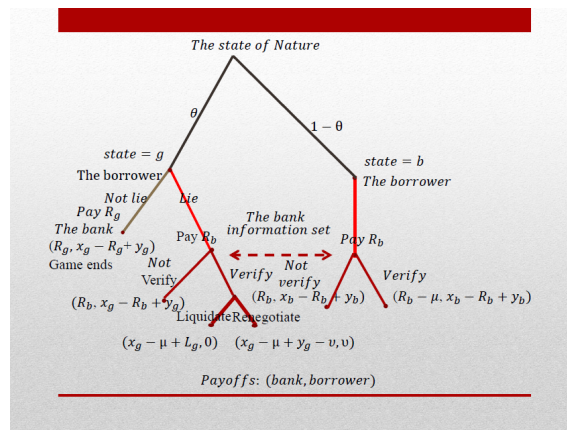
If the auditing reveals deception, then, the bank has the right to punish the borrower. The borrower must pay out the maximum amount x_g . That is, the bank will get all cash flows; after this the bank will choose to liquidate or renegotiate. While if the borrower pays R_g or after auditing no deception is found, there will be no punishment, liquidation or renegotiation.

4.4.5 The Structure of the Game

The probability of auditing is denoted by ρ . We assume that the borrower chooses whether to deceive or not with probabilities π , $1 - \pi$ respectively.

The borrower gains from deception especially if there is no auditing by the bank. By auditing, the bank recovers all cash flows which significantly exceed the audit costs provided that $x_g - \mu + L_g \geq 0$.

Now we can briefly describe the structure of the game as follows:



4.4.6 Pure versus Mixed Strategies

To test whether the two parties (the bank and the borrower) will play a pure strategy or a mixed strategy, we need to compute π , ρ as follows.

4.4.6.1 Pure Strategies

Analyzing the Effect of π

Case 1

If the borrower does not deceive, then the bank will never audit since $\mu > 0$.

Hence, the borrower's expected payoff by telling the truth in the good state is

$$x_g - R_g + y_g$$

If the borrower lies, they will get

$$x_g - R_b + y_g$$

By comparing the two expected payoffs

$$x_g - R_b + y_g > x_g - R_g + y_g$$

The borrower benefit from deceiving the bank when the state is good in the first period (RHS) is higher since $R_b < R_g$. This analysis implies that the borrower should lie because there exists a more profitable outcome from deception.

Case 2

Similarly, if the borrower deceives, then the bank will always audit.

The borrower payoff by telling the truth is

$$x_g - R_g + y_g$$

The borrower payoff by lying is 0.

It tells us clearly that

$$x_g - R_g + y_g > 0$$

The borrower's expected payoff under truth-telling is significantly greater than that under lying. Hence, the borrower will not deceive.

Analyzing the Effect of ρ By providing a similar analysis for the probability of auditing by the

bank.

Case 1

If it is established that the bank does not audit, then the borrower will always lie.

The payoff for a bank by auditing is

$$x_g - \mu + L_g$$

By not auditing, the bank gets R_b

where

$$R_b < x_g - \mu + L_g$$

Thus, the bank should audit.

Case 2

If the bank audits, the borrower will not lie.

Compare the bank payoffs with auditing (RHS) and without auditing (LHS)

$$R_b > R_b - \mu$$

Then it implies that the bank should not always audit since $\mu > 0$. As a result, there is no equilibrium in pure strategies and we need to look for a mixed strategy Nash equilibrium.

4.4.6.2 Mixed Strategy Equilibrium

There is a unique mixed strategy for this game. In equilibrium, randomization is possible only if each party is indifferent between two pure strategies.

Lemma 1 *For a bank's choice of ρ , the expected payoff of the borrower will be indifferent between deception and honesty if and only if*

$$\rho = \frac{R_g - R_b}{x_g - R_b + y_g}$$

Proof. See Appendix. ■

Lemma 2 *The ex ante probability of the borrower deception π that equates the bank's payoff,*

under two pure strategies with auditing or not, is

$$\pi = \frac{1 - \theta}{\theta} \frac{\mu}{(x_g - R_b - \mu + L_g)}$$

Proof. See Appendix. ■

Thus, the unique mixed strategy Nash equilibrium of this game is

$$(\pi, \rho) = \left(\frac{1 - \theta}{\theta} \frac{\mu}{(x_g - R_b - \mu + L_g)}, \frac{R_g - R_b}{x_g - R_b + y_g} \right)$$

Selecting a mixed strategy means that a party is indifferent about the two pure strategies. Still, the use of a mixed strategy is often criticized because people do not take random actions in the real world.

4.4.7 The Expected Payoffs and Equilibrium Conditions

The expected payoff for the borrower firm

$$\theta \{(1 - \pi)(x_g - R_g + y_g) + \pi[(1 - \rho)(x_g - R_b + y_g) + \rho \cdot 0]\} + (1 - \theta)(x_b - R_b + y_b) \geq 0$$

With probability θ , the borrower can obtain $x_g - R_g + y_g$ if they are honest (reporting the truth) or $(1 - \rho)(x_g - R_b + y_g) + \rho \cdot 0$ if they are dishonest. While the borrower can earn $x_b - R_b + y_b$ with probability $1 - \theta$.

The expected profit of the bank IR

$$\theta \{(1 - \pi)R_g + \pi[(1 - \rho)R_b + \rho(x_g - \mu + L_g)]\} + (1 - \theta)[(1 - \rho)R_b + \rho(R_b - \mu)] - K \geq 0$$

This expression represents the sum of expected payoffs of the bank conditional on the state of nature minus the initial loan K . With probability θ , the bank can obtain R_g without borrower deception or $(1 - \rho)R_b + \rho(x_g - \mu + L_g)$ if there is fraudulent or deceptive practices by a borrower. Then $(1 - \rho)R_b + \rho(x_g - \mu + L_g)$ captures the expected payoff of the bank, given that the borrower acts deceptively, without and with auditing respectively. With probability $1 - \theta$, the bank can earn $(1 - \rho)R_b + \rho(R_b - \mu)$ without and with auditing respectively.

In fact, the borrower is protected by limited liability

$$R_g \leq x_g$$

$$R_b \leq x_b$$

Incentive-compatibility constraints for truthful reporting by the borrower

$$x_g - R_g + y_g \geq (1 - \rho)(x_g - R_b + y_g) + \rho \cdot 0 \implies \frac{R_g - R_b}{x_g - R_b + y_g} \leq \rho$$

As a reward for the borrower effort, other incentives should be satisfied

$$y_b \geq R_b$$

$$y_g \geq R_g$$

These constraints reflect that the borrower payoffs in period 2 should be greater than the payout to the bank in period 1.

4.4.8 Characterization of the Optimal Loan Contracts

Some implications of this contract can be summarized in the following proposition.

Proposition 4 *An optimal loan contract is individual rational for the bank and the borrower and it is incentive-compatible to the borrower if and only if: (1) the probability of an audit is $\rho = \frac{R_g - R_b}{x_g - R_b + y_g}$, (2) the repayment is $R_b = x_b$ since $0 < \rho \leq 1$.*

Proof. See Appendix. ■

In the Appendix, we solve the optimization problem in detail. This proposition establishes that, with a positive probability of an audit, an incentive compatible loan contract is attained and a bank's expected payoff is maximized.

Any unilateral deviation from truth-telling, which can be verified, could result in suboptimal outcomes for the borrower. In this case, the bank has the right to control the project and then all bargaining power is given to the bank as a maximum punishment should liquidation or

renegotiation begin.²⁷

We assume that partial liquidation is not possible. Due to costly liquidation, the bank may prefer to renegotiate the pre-existing contract if its payoff under renegotiation is higher than under liquidation.

Here, we consider a two party bargaining problem under full information. In addition, the borrower will participate in the bargaining game as long as their outside option is zero. The borrower and the bank will divide y_g according to Nash bargaining. The bank proposes a split (bank, borrower) = $(y_g - v, v)$.

Ex post renegotiation could lead to ex post efficient continuation equilibrium. On the one hand, if the borrower rejects the offer, their expected payoff will be zero. Therefore, it is better to accept and cooperate with the bank since $v \geq 0$. On the other hand, the bank will offer v if the expected payoff is maximized.

To summarize, the borrower will be better off under truth-telling than under deception. Therefore, if we have incomplete contracts but allow for costly auditing, we can get incentive-compatible contracts that provide incentives for borrowers to report cash flows truthfully.

4.5 Conclusions

Throughout this paper we have presented a simple theoretical model of optimal loan contracts that maximize the bank's expected payoff and attract borrowers. The model has generalized and extended Bolton and Scharfstein (1996). In their model, they study optimal incomplete debt contracts. They also show that the creditor has a credible threat to liquidate the borrower's project in order to fully recover the initial loan.

In our model we discuss a bank-borrower relationship in a three-period model under symmetric and asymmetric information. We also show contracts that govern the relationship between both

²⁷ The entrepreneur will continue to run the project as long as they meet their commitments towards the investor. While, they will lose their control rights to the investor if they default or go bankrupt (Aghion, and Bolton (1992)).

parties. In particular, banks write state-contingent loan contracts with borrowers.

We analyze the design of complete and incomplete state-contingent contracts under symmetric information. We argue that this is closely related to the observability and verifiability of the information structure. The contract is complete provided that the states of the world are observable to both parties and verifiable to the third party (court). If the states of the world are observable but not verifiable, then writing a complete contract is costly. Rather, the contract is incomplete. The reason for incompleteness is associated with the existence of transaction costs to include sufficient details about the state. Further, the contract should be incentive-compatible to the borrower. A threat of liquidation can also be used to ensure good behavior on the part of the borrower.

To produce a better understanding of the real world contracts, alternatively, we proceed by analyzing a model of a lending relationship under asymmetric information. Asymmetric information implies that the borrowers have private information regarding the realized state in period 1. Thus, they can derive private benefits from the bank. To resolve this issue, costly auditing is undertaken. The bank needs to pay some costs to verify the state of the world. In this regard, the auditing is contractible and it gives rise to legal action.

In our analysis, the borrower (the informed party) who deceives will not be punished by liquidation or renegotiation until the bank (the uninformed party) verifies this deception through a stochastic audit. The threat of an audit induces the borrower to tell the truth in the first period. Thus, the objective of this audit is to provide incentives for truth-telling by the borrower on the one hand and save the deadweight verification cost to the bank on the other hand.

Yet, this result shows the optimal contracts and equilibrium outcomes under costly auditing. In addition, other determinants are calculated such as the probabilities of verification and the deception of the borrower. Moreover, we show that the borrowers are better off under truth-telling than under liquidation or renegotiation.

We can extend our model by assuming that the states of the world take the four possible values (g, g) , (g, b) , (b, g) and (b, b) . In a more general setting, a moral hazard problem can arise from the project. Another direction to be considered is that the borrower has initial wealth ω , but it is not enough to finance the project K . Therefore, the borrower will obtain the difference $K - w$ from the bank as in Holmstrom and Tirole (1997) who study an incentive scheme in which firms are capital constrained. It is also worthwhile to explore the effect of maximum equity participation in the model. However, these issues are left for further study.

4.6 Appendix

4.6.1 The Proof of Proposition 1

To show the proof of this proposition; first, we will sketch the following proof of part 1.

(A) implies:

$$\theta(x_g + y_g) + (1 - \theta)(x_b + y_b) - [\theta R_g + (1 - \theta)R_b] \geq 0$$

$$\theta(x_g + y_g) + (1 - \theta)(x_b + y_b) \geq \theta R_g + (1 - \theta)R_b$$

(B) implies:

$$\theta R_g + (1 - \theta)R_b \geq K$$

We obtain, by transitivity, the following:

$$\theta(x_g + y_g) + (1 - \theta)(x_b + y_b) \geq K$$

Thus (1) is satisfied.

Second, we will present a proof of part 2 to check if IR for the bank holds.

(C) implies that

$$\theta(x_g + y_g) + (1 - \theta)(x_b + y_b) \geq K$$

Let the following expressions be true

$$\widetilde{R}_g := x_g + y_g \text{ and } \widetilde{R}_b := x_b + y_b$$

Then by plugging them into (C), one can find that IR for the bank satisfied

$$\theta \widetilde{R}_g + (1 - \theta) \widetilde{R}_b \geq K$$

We will continue on the proof of part 2 in order to see whether IR for the borrower holds.

(A) implies that

$$\theta [x_g - R_g + y_g] + (1 - \theta) [x_b - R_b + y_b] \geq 0$$

Then by plugging

$$\widetilde{R}_g := x_g + y_g \text{ and } \widetilde{R}_b := x_b + y_b$$

into (A), it appears to be

$$\theta [x_g + y_g - (x_g + y_g)] + (1 - \theta) [x_b + y_b - (x_b + y_b)] \geq 0$$

Which after simplifying always holds for

$$x_g + y_g \geq R_g \text{ and } x_b + y_b \geq R_b$$

Thus (2) is also satisfied. Efficiency can be achieved.

borrower IC. (3) limited liability constraints for the borrower LLC,

$$\max \theta [R_g + \beta_g L_g] + (1 - \theta) [R_b + \beta_b L_b] - K$$

s.t.

$$\theta [x_g - R_g + (1 - \beta_g)y_g] + (1 - \theta) [x_b - R_b + (1 - \beta_b)y_b] \geq 0$$

$$(\beta_b - \beta_g)y_g \geq R_g - R_b$$

$$R_g \leq x_g$$

$$R_b \leq x_b$$

We first construct the Lagrangian:

$$\begin{aligned} \mathcal{L} = & \theta [R_g + \beta_g L_g] + (1 - \theta) [R_b + \beta_b L_b] - K \\ & - \lambda_1 [\theta (x_g - R_g + (1 - \beta_g)y_g) + (1 - \theta) (x_b - R_b + (1 - \beta_b)y_b)] \\ & - \lambda_2 [R_g - R_b - (\beta_b - \beta_g)y_g] \\ & - \lambda_3 [R_g - x_g] \\ & - \lambda_4 [R_b - x_b] \end{aligned}$$

Where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ are Lagrangian multipliers or Kuhn-Tucker multipliers.

The first order partial derivatives of Lagrangian equal zero.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial R_b} &= (1 - \theta) + (1 - \theta)\lambda_1 + \lambda_2 - \lambda_4 = 0 \\ \frac{\partial \mathcal{L}}{\partial R_g} &= \theta + \lambda_1\theta - \lambda_2 - \lambda_3 = 0 \end{aligned}$$

$$(1 - \theta)(1 + \lambda_1) = \lambda_4 - \lambda_2$$

$$\theta(\lambda_1 + 1) = \lambda_2 + \lambda_3$$

Next, we need to check the following constraints

$$\lambda_2 [R_g - R_b - (\beta_b - \beta_g)y_g] = 0$$

$$\lambda_3 [-R_g + x_g] = 0$$

$$\lambda_4 [-R_b + x_b] = 0$$

The incentive constraint must be binding. If not, it would be optimal to set $\beta_b = 0$ then the borrower would pay out $R_b < x_b$.

It is straightforward to show that the optimal solution for this problem is

$$\lambda_2 > 0, \lambda_3 = 0, \lambda_4 > 0 \implies \beta_g = 0, R_b = x_b \text{ if } 0 < \beta_b \leq 1 \text{ or } R_b < x_b \text{ if } \beta_b = 0$$

Setting $\beta_g = 0$ and $R_b = x_b$ if $0 < \beta_b \leq 1$ is optimal because they can maximize the expected joint surplus from the contract and provide the borrower with incentives to pay out cash flow.

By plugging the optimal solutions $\beta_g = 0$, $R_b = x_b$ and rearranging, IC at the optimal point becomes

$$R_g \leq x_b + \beta_b y_g$$

Incentive-compatibility constraints are binding under the optimal contract

$$R_g = x_b + \beta_b y_g$$

Then we can proceed by substituting

$$R_g = x_b + \beta_b y_g$$

$$\beta_g = 0$$

$$R_b = x_b$$

into the expected payoffs of the borrower and the bank.

First, the expected payoff of the bank becomes

$$\begin{aligned} & \theta [R_g + \beta_g L_g] + (1 - \theta) [R_b + \beta_b L_b] - K = 0 \\ \implies & \theta [x_b + \beta_b y_g] + (1 - \theta) [x_b + \beta_b L_b] - K = 0 \\ \implies & \beta_b [\theta y_g + (1 - \theta) L_b] + x_b - K = 0 \\ \implies & \beta_b \theta y_g = K - x_b - (1 - \theta) \beta_b L_b \end{aligned}$$

Second, the borrower's expected payoff becomes

$$\theta(x_g + y_g - \beta_b y_g - x_b) + (1 - \theta)(1 - \beta_b)y_b$$

Substituting the bank's expected payoff into the borrower's expected payoff yields

$$\theta(x_g + y_g - x_b) - [K - x_b - (1 - \theta)\beta_b L_b] + (1 - \theta)(1 - \beta_b)y_b$$

This expression simplifies to

$$\theta(x_g + y_g) + (1 - \theta)(x_b + y_b) - K - (1 - \theta)\beta_b(y_b - L_b)$$

Third, the optimal solution for β_b at the minimum possible level is endogenously derived from the equation

$$\begin{aligned} & \beta_b [\theta y_g + (1 - \theta) L_b] + x_b - K = 0 \\ & \beta_b = \frac{K - x_b}{\theta y_g + (1 - \theta) L_b} \end{aligned}$$

$0 < \beta_b \leq 1$. The last equation can be rewritten as

$$0 < K - x_b \leq \theta y_g + (1 - \theta) L_b$$

Therefore, the bank's maximum total profit is

$$K \leq x_b + \theta y_g + (1 - \theta) L_b$$

Here the project will be financed.

While if

$$K > x_b + \theta y_g + (1 - \theta)L_b$$

This expression indicates that if the initial loan is greater than the maximum total profit of the bank, the project will not be financed.

4.6.4 The Proof of Proposition 3

Following a renegotiation, the cash flow y_b is split equally (50:50) between the two parties according to the Nash bargaining solution. Hence, it yields a cooperative outcome $(\frac{y_b}{2}, \frac{y_b}{2})$ since any other proposition such as $(y_b, 0)$, $(0, y_b)$ is not possible and better than the disagreement outcome $(0, 0)$.²⁸

4.6.5 The Proof of Lemma 1

To find a mixed strategy, we can determine the optimal audit probability of a bank by using the expected payoff for the borrower firm

$$\begin{aligned} x_g - R_g + y_g &= (1 - \rho)(x_g - R_b + y_g) \\ \rho &= \frac{R_g - R_b}{x_g - R_b + y_g} \end{aligned}$$

4.6.6 The Proof of Lemma 2

The bank randomizes between these two strategies if the expected profit is equal

$$\begin{aligned} \theta\pi(x_g - \mu + L_g) + (1 - \theta)(R_b - \mu) &= \theta\pi R_b + (1 - \theta)R_b \\ \theta\pi(x_g - R_b - \mu + L_g) &= (1 - \theta)\mu \\ \pi &= \frac{1 - \theta}{\theta} \frac{\mu}{(x_g - R_b - \mu + L_g)} \end{aligned}$$

4.6.7 The Proof of Proposition 4

Finding an optimal incentive-compatible contract is equivalent to maximizing the expected

²⁸ The bargaining is costless and the bargaining cost is normalized to zero. However, without loss of generality, our results can still hold in a world with bargaining costs equal to c . We assume that the bank loses liquidation market once it chooses to renegotiate.

payoff for the bank subject to the feasibility of three constraints. (1) Participation constraint for the borrower firm. (2) Incentive compatibility constraint for the borrower IC. (3) Limited liability constraint for the borrower LLC. The standard problem is as follows:

$$\max \theta \{(1 - \pi)R_g + \pi [(1 - \rho)R_b + \rho(x_g - \mu + L_g)]\} + (1 - \theta) [(1 - \rho)R_b + \rho(R_b - \mu)] - K$$

s.t.

$$\theta \{(1 - \pi)(x_g - R_g + y_g) + \pi [(1 - \rho)(x_g - R_b + y_g) + \rho \cdot 0]\} + (1 - \theta)(x_b - R_b + y_b) \geq 0$$

$$\rho \geq \frac{R_g - R_b}{x_g - R_b + y_g}$$

$$R_g \leq x_g$$

$$R_b \leq x_b$$

The standard procedure for solving the corresponding maximization problem is as follows: We first construct the Lagrangian .

$$\begin{aligned} \mathcal{L} = & \theta \{(1 - \pi)R_g + \pi [(1 - \rho)R_b + \rho(x_g - \mu + L_g)]\} + (1 - \theta) [(1 - \rho)R_b + \rho(R_b - \mu)] - K \\ & - \lambda_1 [\theta \{(1 - \pi)(x_g - R_g + y_g) + \pi [(1 - \rho)(x_g - R_b + y_g) + \rho \cdot 0]\} + (1 - \theta)(x_b - R_b + y_b)] \\ & - \lambda_2 \left[-\rho + \frac{R_g - R_b}{x_g - R_b + y_g} \right] \\ & - \lambda_3 [R_g - x_g] \\ & - \lambda_4 [R_b - x_b] \end{aligned}$$

Where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ are Lagrangian multipliers or Kuhn-Tucker multipliers.

The first order partial derivatives of Lagrangian equal zero.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial R_b} &= \theta\pi(1-\rho) + (1-\theta)[(1-\rho) + \rho] + \pi\theta\lambda_1(1-\rho) + \lambda_1(1-\theta) + \lambda_2(1-\rho) - \lambda_4 = 0 \\
&= \theta\pi(1-\rho) + (1-\theta)(1+\lambda_1) + (\pi\theta\lambda_1 + \lambda_2)(1-\rho) - \lambda_4 = 0 \\
&= (\theta\pi + \pi\theta\lambda_1 + \lambda_2)(1-\rho) + (1-\theta)(1+\lambda_1) - \lambda_4 = 0 \\
\frac{\partial \mathcal{L}}{\partial R_g} &= \theta(1-\pi) + \theta(1-\pi)\lambda_1 + \lambda_2 - \lambda_3 = 0
\end{aligned}$$

$$\lambda_4 = (\theta\pi + \pi\theta\lambda_1 + \lambda_2)(1-\rho) + (1-\theta)(1+\lambda_1)$$

$$\lambda_3 = \theta(1-\pi)(1+\lambda_1) + \lambda_2$$

Next, we need to check the following constraints

$$\begin{aligned}
\lambda_2 \left[\rho - \frac{R_g - R_b}{x_g - R_b + y_g} \right] &= 0 \\
\lambda_3 [-R_g + x_g] &= 0 \\
\lambda_4 [-R_b + x_b] &= 0
\end{aligned}$$

It is straightforward to show that the optimal solution for this problem is

$$\lambda_2 > 0, \lambda_3 = 0, \lambda_4 > 0 \implies \rho = \frac{R_g - R_b}{x_g - R_b + y_g}, R_b = x_b \text{ since } 0 < \rho \leq 1$$

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