Delayed Static Output Feedback Control of a Network of Double Integrator Agents [⋆]

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Abstract

This paper considers a network of vehicles moving in a two dimensional plane. The overall network, described by a collection of double integrator dynamics, is controlled by a novel distributed static output feedback methodology to maintain a desired formation. The distributed control architecture stabilizes the network using static output feedback of position information only, by exploiting delays in communication of the relative information. An optimization algorithm, based on Linear Matrix Inequalities together with the DIRECT search algorithm, is used to synthesize the controller gains and the delay.

Key words: Static output feedback, Delay, Cooperative control

1 Introduction

Recently, consensus and coordinated control have been widely studied using single and double integrator dynamics [1]-[2]. In the literature, see for example [3]-[5] to name but a few, state feedback has been used to obtain consensus. Consensus algorithms using double integrator dynamics, representing position and velocity information, have also been applied for formation control; see [4] - [6] and the references therein. However the measurement of all the states of a system is not viable in many practical problems. Output feedback control then becomes necessary in such scenarios. Although (static) output feedback control is a well studied problem, no complete solution has been found [7]. Furthermore, of particular relevance to this paper, it is well known that double integrator systems cannot be asymptotically stabilized via static output feedback. One solution is to create a dynamical feedback control law - possibly based on observers; however, in [8], it is shown that certain systems stabilize due to delays. Necessary conditions for the existence of stabilizing static output feedback controllers with multiple delays are developed in [9] and in particular, the stabilization of a double integrator by using delays is briefly described. In [10] it is illustrated (by example) that the introduction of a delay may improve the convergence rate in consensus problems for multi-agent systems. Consensus problems in 2nd order systems using delayed state (position and velocity) information is reported in [11]. In [12] frequency domain methods are employed for examining the robustness of consensus properties and the allowable delay margin.

In this paper static output feedback using only position measurements, but making use of a fixed delay, is studied. Precalculated offsets are incorporated along with the relative information communicated among vehicles, so that the vehicles can attain a stable formation. A numerical optimization approach based on the so-called DIRECT [13] algorithm in conjunction with a set of linear matrix inequalities (LMIs), arising from the discretized Lyapunov-Krasovskii functional from [14], is used to find the optimal controller gains and the delay simultaneously. Exponential stabilization of the network is achieved with an optimal decay rate, while utilizing the minimum possible control effort and the smallest possible value of the delay.

2 Problem Formulation

Consider a state space representation of a network of N double integrators, each representing an identical vehicle moving in a 2-dimensional (‘$x-y$’) plane, given by:

$$\dot{\xi}_i(t) = A\xi_i(t) + Bu_i(t) \tag{1}$$
$$\zeta_i(t) = C\xi_i(t) \tag{2}$$

where $\xi_i = Col(x_i, \dot{x}_i, y_i, \dot{y}_i)$ and

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$$A=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix},\quad B=\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix},\quad C=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix} \tag{3}$$

In this paper, each vehicle has access only to its local position information, and delayed relative position measurements from neighbouring agents with which each agent interacts. Bidirectional communication is assumed between the agents. Following a graph abstraction as in [5], the collective system is represented as a graph with N vertices (nodes) each representing a vehicle. An edge in the graph represents bidirectional information exchange between two vehicles. A delayed exchange of relative position information is assumed to have a form

$$z_i(t-\tau)=\sum_{j\in\mathcal{J}_i}(\zeta_i(t-\tau)-\zeta_j(t-\tau)),\ \text{for } i=1,\ldots,N \tag{4}$$

and the nonempty set $\mathcal{J}_i \subset \{1,2,\ldots N\}/\{i\}$ represents the set of neighbouring vehicles with which the i^{th} vehicle can establish communication.

It is well known that the two decoupled double integrators associated with (A,B,C) cannot be stabilized by static output feedback (i.e. position information alone). To circumvent this, distributed static output feedback control laws involving delay terms is proposed as follows:

$$u_i(t)=-K_1\zeta_i(t)+K_2\left(\zeta_i(t-\tau)+\beta z_i(t-\tau)-d_i\right) \tag{5}$$

where $K_1=k_1I_2$ and $K_2=k_2I_2$, and k_1, k_2 are non-zero scalars. The scalar $\beta>0$ represents an a priori known scalar weighting for z_i and τ is a fixed (chosen) delay. The term $d_i\in\mathbb{R}^2$ in (5) is the offset in the relative information at each node, so that each agent maintains a desired relative distance from its neighbours.

Remark 1 *In real engineering systems, relative sensing and communication of information will incur delays. Here it is assumed that a minimum delay of* $\tau_{min}>0$ *will be present in relative sensing and communication. Since it is assumed that each node has access to its own output information, it is assumed in addition, that it is possible to store this information and use it in a delayed feedback.*

At a network level, the system given in (1) and (4) can be conveniently represented by

$$\dot{X}(t)=(I_N\otimes A)X(t)+(I_N\otimes B)U(t) \tag{6}$$
$$Z(t)=(\mathcal{L}\otimes C)X(t-\tau) \tag{7}$$

where the augmented state $X(t)=\mathcal{C}ol(\xi_1(t),....,\xi_N(t))$, the augmented inputs $U(t)=\mathcal{C}ol(u_1(t),\ldots,u_N(t))$ and the aggregated delayed position information vector $Z(t)=\mathcal{C}ol(z_1(t),\ldots,z_N(t))$. The square matrix $\mathcal{L}$ is the Laplacian associated with $\mathcal{J}_i$. The Laplacian matrix is a symmetric, positive semi definite matrix. As in references [5,3] (and many others), because it is assumed none of the $\mathcal{J}_i$ are empty, i.e. each vehicle has information about at least one other vehicle, it follows that $rank(\mathcal{L})=N-1$.

At a network level, the control law (5) can be written in a convenient compact form as

$$\begin{aligned}U(t)=&-(I_N\otimes K_1C)X(t)+(I_N\otimes K_2)D\\&+((I_N+\beta\mathcal{L})\otimes K_2C)X(t-\tau)\end{aligned} \tag{8}$$

where $D=\mathcal{C}ol(d_1,\ldots,d_N)$. Substituting (8) in (6), the closed loop system is

$$\dot{X}(t)=\bar{A}_0X(t)+\bar{A}_1X(t-\tau)+(I_N\otimes BK_2)D \tag{9}$$

where the system matrices

$$\bar{A}_0=I_N\otimes(A-BK_1C) \tag{10}$$
$$\bar{A}_1=(I_N+\beta\mathcal{L})\otimes BK_2C \tag{11}$$

Since the system (A,B,C) is not stabilizable by static output feedback, the system in (9) is not stable for $\tau=0$.

3 Design Procedure

The approach is, for a given $\beta>0$, to find the triplet (k_1,k_2,τ) with $\tau>\tau_{min}$ such that the system (9) is stable. First, introduce a transformation $\bar{X}(t)=X(t)-X_f$ where

$$X_f=\mathcal{C}ol\left(x_1^f,\ 0,\ y_1^f,\ 0,\ \ldots\ x_N^f,\ 0,\ y_N^f,\ 0\right) \tag{12}$$

where (x_i^f,y_i^f) represents the final desired steady state position of the i-th agent. In the new coordinate system

$$\begin{aligned}\dot{\bar{X}}(t)=&\bar{A}_0\bar{X}(t)+\bar{A}_1\bar{X}(t-\tau)+(\bar{A}_0+\bar{A}_1)X_f\\&+(I_N\otimes BK_2)D\end{aligned} \tag{13}$$

The aim is to make $(\bar{A}_0+\bar{A}_1)X_f+(I_N\otimes BK_2)D=0$ by choice of D. Suppose K_1 and K_2 are given and the offset vector D is chosen to satisfy

$$(I_N\otimes K_2)D=((I_N\otimes K_1C)-((I_N+\beta\mathcal{L})\otimes K_2C))\,X_f \tag{14}$$

Note that since $K_2=k_2I_2$, $(I_N\otimes K_2)=k_2I_{2N}$, and so provided $k_2\neq0$, equation (14) has the unique solution

$$D=\frac{1}{k_2}((I_N\otimes K_1C)-((I_N+\beta\mathcal{L})\otimes K_2C))X_f \tag{15}$$

Note that from (12) and (3), for any choice of (x_i^f,y_i^f), $(I_N\otimes A)X_f=0$. Thus for any choice of D satisfying (14), left multiplication on both sides of (14) by $(I_N\otimes B)$ and the addition of $(I_N\otimes A)X_f$ to the left hand side yields

$$(I_N\otimes BK_2)D=-(\bar{A}_0+\bar{A}_1)X_f \tag{16}$$

by using the definitions of $\bar{A}_0$ and $\bar{A}_1$ from (10) and (11). Thus with the choice of D as in (15), (13) simplifies to

$$\dot{\bar{X}}(t) = \bar{A}_0\bar{X}(t) + \bar{A}_1\bar{X}(t-\tau) \tag{17}$$

Remark 2 *Note that the offset D in (9) is chosen as in (15) once the gains k_1 and k_2 are designed. The offset D also depends on desired formation encapsulated in X_f.*

Since $\mathcal{L}$ is symmetric positive semi-definite, and $\beta > 0$, $(I_N + \beta\mathcal{L})$ is symmetric positive definite. Hence, by spectral decomposition $(I_N + \beta\mathcal{L}) = V\Lambda V^T$ where V is an orthogonal matrix formed from the eigenvectors of $(I_N+\beta\mathcal{L})$ and $\Lambda = Diag(\lambda_1, \ldots \lambda_N)$ is the matrix of the eigenvalues of $(I_N + \beta\mathcal{L})$. Note that $\lambda_i \geq 1$ for all $i = 1, \ldots, N$ (since $\mathcal{L}$ has an eigenvalue at 0.). Consider an orthogonal state transformation $\bar{X} \mapsto (V^T \otimes I_4)\bar{X} = \tilde{X}$. In the new coordinates (17) becomes

$$\dot{\tilde{X}}(t) = \tilde{A}_0\tilde{X}(t) + \tilde{A}_1\tilde{X}(t-\tau) \tag{18}$$

where $\tilde{A}_0 = \bar{A}_0$ and $\tilde{A}_1 = (\Lambda \otimes BK_2C)$. Because of the special block diagonal forms of $\tilde{A}_0$ and $\tilde{A}_1$ in (18), equation (18) can be represented as

$$\dot{\tilde{\xi}}_i(t) = A_0\tilde{\xi}_i(t) + A_i\tilde{\xi}_i(t-\tau), \quad i = 1, \ldots, N \tag{19}$$

where the states $\tilde{X}(t) = Col(\tilde{\xi}_1(t), \ldots, \tilde{\xi}_N(t))$ and the matrices $A_0 = (A - BK_1C)$ and $A_i = \lambda_i BK_2C$. In order to ensure a level of performance in the closed loop system, as suggested in [15], consider the transformation

$$\tilde{\xi}_{i\alpha}(t) = e^{\alpha t}\tilde{\xi}_i(t), \quad \text{where } \alpha > 0 \tag{20}$$

Asymptotic convergence of the $\tilde{\xi}_{i\alpha}$ coordinates implies exponential convergence of $\tilde{\xi}_i$ with a decay rate α. Further details appear in [15]. With this additional transformation, the system in (19) becomes

$$\dot{\tilde{\xi}}_{i\alpha}(t) = A_{\alpha 0}\tilde{\xi}_{i\alpha}(t) + A_{\alpha i}\tilde{\xi}_{i\alpha}(t-\tau), \quad i = 1, \ldots, N \tag{21}$$

where $A_{\alpha 0} = A_0 + \alpha I_4$ and $A_{\alpha i} = e^{\alpha\tau}A_i$. The stability of system (21) is determined using *Proposition 5.22* in [14], which divides the delay interval $[-\tau,\ 0]$ into M equal partitions and employs a discretized Lyapunov functional. This can be cast as an LMI of the form

$$\mathcal{LMI}(A_{\alpha_0}, A_{\alpha_i}, P_i, S_{pi}, R_{pqi}) < 0 \tag{22}$$

where P_i, S_{pi}, and R_{pqi} for $p, q = 1, 2, \ldots M$, represent decision variables which parameterize the discretized Lyapunov functional from *Proposition 5.22* in [14]. For specific details of the LMIs in (22) see [6]. Provided the gains k_1 and k_2, the delay τ, and the decay rate α are fixed, the LMI in (22) provides a tractable feasibility check for stability for the system in (21).

3.1 Optimization Algorithm

A maximum possible decay rate α, for a set of stabilizing gain values for k_1 and k_2 and an associated minimum delay τ satisfying the LMIs in *Proposition 5.22* [14] is sought. This represents a non-convex optimization problem. Solving such problems are not straightforward, and solution methods often depend on fine gridding of the search space (or a similar technique). However, often there is no guarantee of finding the optimal solution, or even a sub-optimal one. In this paper a solution is obtained by employing a deterministic global optimization algorithm, the so-called Dividing Rectangles (DIRECT) approach [16], which is a derivative free method using a center point sampling strategy. The method was originally developed in [13] as a modification of the classical 1D Lipschitz optimization algorithm known as Schubert's algorithm. A normalised parametric search space is posed as an n-dimensional hypercube $\{\delta \in \mathbb{R}^n : 0 \leq \delta_i \leq 1, \quad \forall i = 1, \ldots, n\}$. The algorithm works in the normalized parametric space, transforming to the actual search space as and when the cost function has to be evaluated. The main idea can be summarized as: while the algorithm proceeds, the search space is partitioned into smaller hypercubes and each hypercube is sampled at the center point. As iterations progress, the algorithm tries to find all the 'potentially optimal' hypercubes in the search space and then partitions them, (see [13] for details and definitions of the potentially optimal hypercubes and the division strategies) thereby obtaining the global solution. The proof of asymptotic convergence is derived in [16].

Since there are multiple minimization objectives, a collective single optimization objective function is defined with appropriate scaling as follows:

$$J(k_1, k_2, \tau, \alpha) := \frac{W_1|k_1| + W_2|k_2| + W_3|\tau|}{\alpha} \tag{23}$$

In (23) the scalars W_i for $i = 1, 2, 3$ are the weights of the optimization variables k_1, k_2, and τ respectively. The objective is to minimize $J(k_1, k_2, \tau, \alpha)$ subject to the feasibility of the LMIs in (22) and the constraints on the optimization variables $k_{1_{min}} \leq k_1 \leq k_{1_{max}}$, $k_{2_{min}} \leq k_2 \leq k_{2_{max}}$, $\tau_{min} \leq \tau \leq \tau_{max}$, and $\alpha_{min} \leq \alpha \leq \alpha_{max}$ The rationale behind this objective function is to obtain the maximum possible decay rate α and the gain set that provides minimum control effort at a minimum possible level of delay. When the LMIs in (22) are not feasible for a specific set of gains k_1 and k_2, a delay τ and decay rate α, the cost $J(k_1, k_2, \tau, \alpha)$ associated with such a candidate design point in the parameter space is penalized by assigning it a large, value over-riding (23).

Remark 3 *Note that the "optimal solution" depends on the choice of M since this affects the LMIs in (22) which are used to establish feasibility.*

Psuedo Code - DIRECT optimisation

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(1) Normalize the domain $(k_1, k_2, \tau, \alpha)$ to be the unit hyperbox with center $c_1$.
(2) Test the LMIs in (22) and subsequently evaluate the cost $J(k_1, k_2, \tau, \alpha)$ at $c_1$: $J_{min} = J_{c_1}(.)$, $i = 0$, $m = 1$.
(3) Evaluate $J(.)$ at $c_1 \pm \delta \mathbf{e}_i$, $1 \leq i \leq n$ and divide the hyperbox where $\delta$ is one third of the side length of the hyperbox and $\mathbf{e}_i$ is the $i^{th}$ unit vector.
(4) while ($i \leq iter_{max}$) and $m \leq eval_{max}$ do
    (a) Identify the set $\mathcal{S}$ of all potential optimal hyperboxes (for details see [13]).
    (b) for all $j \in \mathcal{S}$
        (i) Identify the longest side of the $j^{th}$ hyperbox.
        (ii) Evaluate $J(.)$ at the centre of the new hyperbox, and divide the $j^{th}$ box into smaller ones.
        (iii) Update the optimal solution $J_{min}$ and $(k_1, k_2, \tau, \alpha)_{min}$ and increment counter $m$.
    (c) end for
    (d) Increment counter $i$
(5) end of while
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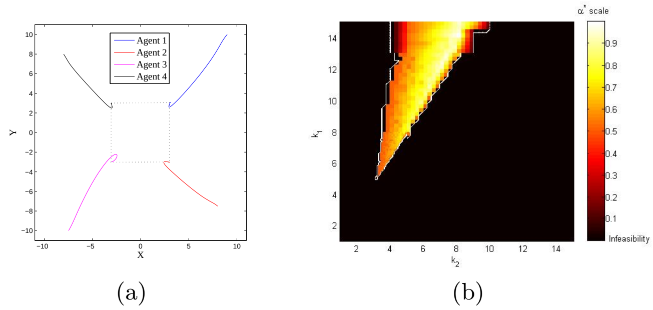
4 Numerical Example

Consider a network of $N = 4$ agents, described by (1)-(3), connected over a nearest neighbour interconnection topology. In the example the weighting β in (5) is considered to be $\beta = 0.1$. The number of partitions of the delay interval for *Proposition 5.22* in [14] was considered to be $M = 1$. The DIRECT algorithm has been employed with the following bounds $k_1 \in [1\ 15]$; $k_2 \in [1\ 15]$; $\tau \in [0.1\ 0.3]$; $\alpha \in [0.5\ 1.5]$. The weights for the cost function $J(k_1, k_2, \tau, \alpha)$ in (23) for this range have been chosen as $W_1 = W_2 = 0.1$ and $W_3 = 1$. The optimal gains k_1 and k_2, delay τ and the maximum possible rate of decay α obtained from within these bounds are $k_1 = 7.9936$; $k_2 = 4.8889$; $\tau_{opt} = 0.2913$ $\alpha = 0.7915$. The desired formation is a square with $(x_i^f, y_i^f) = (\pm 3, \pm 3)$. The offset D is then calculated using (15). The system in (6) has been simulated with the control law in (8) using the values in (4). The initial condition for the delayed output in (9) is set as $X(t) = X(0)$ for the interval $t \in [-\tau_{opt}, 0]$. Figure 1(a) shows the agents settling into a square formation as a function of time. Figure 1(b) plots the optimal values α^* versus k_1 and k_2 where $\alpha^* := \arg\min_\alpha J(k_1, k_2, \tau, \alpha)$ for an a-priori grid of fixed controller gains k_1 and k_2 and the optimal delay τ_{opt}.

5 Conclusions

In this paper, a novel distributed output feedback control law has been proposed to stabilize a formation of multi-agent systems described by double integrator dynamics. The control scheme employs deliberate delays as part of the static output feedback laws to stabilize the network of double integrators. An optimal decay rate is obtained from the numerical minimization of a cost function, which penalizes control effort and the length of the delay, and is numerically solved using a combination of an LMI solver and the DIRECT optimization algorithm.



(a) (b)

Fig. 1. (a) Evolution of the formation of agents with time, (b) Optimal decay rate vs. controller gains

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