Non–reflecting permittivity profiles and the spatial Kramers–Kronig relations
(Supplementary Material)

A. Non–reflecting profiles and the spatial Kramers–Kronig relations

Here we give an alternative analytic argument to relate non–reflecting permittivity profiles to the spatial Kramers–Kronig relations. This argument does not rely on the convergence of the series discussed in the main text.

Suppose we have a TE polarized electromagnetic wave incident onto a planar medium. As in the main text we take incidence at an angle defined by a parallel wave–vector \( k_y \) and the \( x \)–dependence of the field is determined by

\[
\frac{d^2}{dx^2} + k_0^2 \epsilon(x) \right] E_z(x) = 0 \tag{1}
\]

where without loss of generality we have set \( k_y = 0 \) (changing \( k_y \) is equivalent to modifying \( \epsilon_b \)). At this point we assume that the values of \( k \) are discrete and spaced by \( \Delta k \)

\[
E_z(x) = e^{i\sqrt{\epsilon_b k_0 x}} \sum_{n=-\infty}^{\infty} E_n e^{in\Delta k x} \epsilon(x) = \epsilon_b + \sum_{n=-\infty}^{\infty} \alpha_n e^{in\Delta k x}. \tag{2}
\]

This is with the intention of taking the limit \( \Delta k \to 0 \) at the end of the calculation. We also assume that \( \alpha_0 = 0 \), this being the average (real) value of \( \alpha(x) \) included in \( \epsilon_b \). Using the expansion (2) equation (1) can be written as a matrix equation

\[
\sum_{m=-\infty}^{\infty} M_{nm} E_m = 0 \tag{3}
\]

where

\[
M_{nm} = \delta_{nm} \left[ k_0^2 \epsilon_b - (\sqrt{\epsilon_b k_0} + n\Delta k)^2 \right] + k_0^2 \alpha_{n-m}
\]

The solutions to the differential equation (1) correspond to eigenvectors of the matrix \( M_{nm} \) with eigenvalue zero. We now assume the discrete analogue of the analyticity conditions given in the main text, \( \alpha_{n<0} = 0 \), in which case the matrix \( M \) takes the form

\[
M = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & M_{-2,-2} & 0 & 0 & 0 & 0 \\
\cdots & M_{-1,-2} & M_{-1,-1} & 0 & 0 & 0 \\
\cdots & M_{0,-2} & M_{0,-1} & M_{0,0} & 0 & 0 \\
\cdots & M_{1,-2} & M_{1,-1} & M_{1,0} & M_{1,1} & 0 \\
\cdots & M_{2,-2} & M_{2,-1} & M_{2,0} & M_{2,1} & M_{2,2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \tag{4}
\]
For such a triangular matrix, the $m^{th}$ eigenvector is,

$$E^{(m)} = \begin{pmatrix} \cdots \\ 0 \\ 0 \\ E_m \\ E_{m+1} \\ E_{m+2} \\ \cdots \end{pmatrix}$$ \hspace{1cm} (5)

To see that the eigenvectors are of this form note that all the equations $\sum_p M_{n<p,m} E_p^{(m)} = 0$ are identically true, and we are left with a number of non-trivial equations equal to the number of unknowns in (5). The corresponding eigenvalue $\lambda_m$ to each of (5) is

$$\lambda_m = M_{mm} = k_0^2 \epsilon_b - (\sqrt{\epsilon_b k_0} + m \Delta k)^2$$ \hspace{1cm} (6)

Making the choice of discretization $\Delta k = \sqrt{\epsilon_b k_0}/N$, two of these eigenvalues equal zero and occur at the following value of $m$

$$m = N (-1 \pm 1)$$

These two solutions of (3) are the solutions of (1). Given the form of (5), the solution with $m = 0$ contains only right-going waves, and in the limit $N \to \infty$ can be written in the form

$$E_z(x) = \int_0^\infty \frac{dk}{2\pi} \tilde{E_z}(k)e^{ikx}$$ \hspace{1cm} (7)

which are the functions discussed in the main text. Therefore, when $\epsilon(x) - \epsilon_b$ is an analytic function in the upper half position plane, which tends to zero as $|x| \to \infty$ then there are solutions within such a profile that take the form (7), and do not reflect for incidence from the left. An identical argument applies to the TM polarization and, given the form of the TM Helmholtz equation, it is in this case $1/\epsilon(x) - 1/\epsilon_b$ that must be analytic in the upper half plane. To further verify that our results do not rely on a smallness condition on $\alpha(x)$, figure 1 shows the field obtained through numerical integration of the wave equation (1) as the index contrast is increased by an order of magnitude, and actually covers the case in which the real part of the permittivity goes negative.

**B. Two profiles with exact solutions**

We now given two example profiles that illustrate the above non-reflecting behaviour, but can be treated exactly. The two profiles are

$$\epsilon_1(x) = 1 - \frac{1}{x/\xi + i}$$ \hspace{1cm} (8)

and

$$\epsilon_2(x) = 1 + \frac{1}{(x/\xi + i)^2}.$$ \hspace{1cm} (9)
FIG. 1: Panels (a)–(b) show the real and imaginary parts of the permittivity \( \epsilon(x) = 1 - A/(x/\xi + i) \) for \( \xi = 0.06\lambda \). Panels (c)–(e) show wave propagation within these profiles, for incidence from the left. The blue and green lines indicate the real and imaginary parts of the function, while the dashed red line denotes the absolute value. The lack of oscillations in the absolute value of the field indicate that the profile does not reflect radiation incident from the left, despite the fact that the real part of the permittivity becomes negative.

where \( \xi > 0 \). The first of which is entirely lossy, and the second of which is PT–symmetric around \( x = 0 \). The wave equation for the TE polarization in each case is given by (1) (now including a non–zero \( k_y \)). An examination of the differential equations for the Confluent Hypergeometric function (NIST Digital Library of Mathematical Functions http://dlmf.nist.gov/\$13) shows that in both cases (8) and (9) there are exact solutions

\[
E_{z1}(x) = A_1M_{\kappa,\mu} \left( \frac{\xi^2}{2} \right)^{\frac{1}{2}} \left( -2i\sqrt{k_0^2 - k_y^2(x + i\xi)} \right) + B_1W_{\kappa,\mu} \left( \frac{\xi^2}{2} \right)^{\frac{1}{2}} \left( -2i\sqrt{k_0^2 - k_y^2(x + i\xi)} \right)
\]

and

\[
E_{z2} = A_2M_{0,\sqrt{\frac{1}{4} - \xi^2k_0^2}} \left( -2i\sqrt{k_0^2 - k_y^2(x + i\xi)} \right) + B_2W_{0,\sqrt{\frac{1}{4} - \xi^2k_0^2}} \left( -2i\sqrt{k_0^2 - k_y^2(x + i\xi)} \right)
\]

where \( M_{\kappa,\mu}(z) \) and \( W_{\kappa,\mu}(z) \) are Whittaker’s functions, obeying Whittaker’s equation: \( W_{\kappa,\mu}'' + \frac{\kappa}{z} + \frac{1}{4} - \mu^2)z^2 - \frac{1}{4} \right)W_{\kappa,\mu} = 0 \).

The profiles (8–9) are analytic in the upper half position plane and therefore ought to be non–reflecting for waves incident from the left. To verify this we examine the asymptotic behaviour of (10–11) when \( |x| \) is large. The asymptotic expansions of the \( W_{\kappa,\mu} \) function can
FIG. 2: Comparison between the exact solutions (10–11) and numerical integration of the wave equation (1). Panels (a) and (b) respectively show the numerically integrated solution and the exact solution (10) for wave propagation in the permittivity profile (8), for the choices $\xi = 0.1\lambda$ and $k_y = 0$. The numerical integration reproduces the exact solution, and in both cases the lack of oscillations in the absolute value of the field indicate that the profile does not reflect radiation incident from the left. Panels (c) and (d) show the same comparison for wave propagation in the permittivity profile (9), with $\xi = 0.48\lambda$ and $k_y = 0$. Panels (e) and (f) show the permittivity profiles (8) and (9), with the blue line denoting the real part of the permittivity and the dashed red line the imaginary part.
be found in *ibid* §13.19 and to leading order is

\[ W_{\kappa,\mu} \sim e^{-\frac{1}{2}z^2}z^\kappa \quad \text{for} \quad |\arg(z)| \leq \frac{3}{2}\pi - \delta \quad (12) \]

where \( \delta \) is an arbitrary small positive constant. Setting \( A_1 = A_2 = 0 \) and dropping overall factors, we find the asymptotic expressions for waves incident from the left

\[ E_{z1}(x) \sim B_1 e^{i\sqrt{k_0^2 - k^2}(x+i\xi)} - \frac{ik_0^2}{2 \sqrt{k_0^2 - k^2}} \log(x+i\xi) \]

\[ E_{z2}(x) \sim B_2 e^{i\sqrt{k_0^2 - k^2}(x+i\xi)}. \quad (13) \]

Neither solution exhibits any reflection, independent of the value of \( k_y \) (although we are assuming that \( k_y < k_0 \)). Interestingly, the solution within the profile (9) is the same on both the far left and far right of the profile, meaning that asymptotically this index profile has no effect on the wave. Taking the complex conjugate of (13) we find the solution to the complex conjugate of (1), which involves changing the direction of incidence from left to right, while swapping the analyticity of \( \epsilon_{1,2}(x) \) from the upper to lower half position plane. This is in accordance with our prediction that profiles analytic in the upper half plane do not reflect waves incident from the left, and profiles analytic in the lower half plane to not reflect waves incident from the right. Figure 2 shows a comparison between the exact solutions (10–11) evaluated with \( A_1 = A_2 = 0 \), and the numerically integrated solutions, which we have found to agree to within an arbitrary accuracy.

**C. Further numerical investigation**

Here we add some numerical examples to those considered in the main text. In each case the 1D wave equations for TE and TM radiation (see main text) are numerically integrated with the same method used in figures 1 and 2. Figure 3 shows wave propagation in the permittivity profiles listed in table I. In all cases we find that there is zero reflection for incidence from the left.

<table>
<thead>
<tr>
<th>Panels</th>
<th>Permittivity profile ( \epsilon(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a), (b) &amp; (c)</td>
<td>( 2 - \frac{2}{x/\xi + 1} - \frac{1}{2[(x-1)/\xi + 55/10] - 2[(x+1)/\xi + 12/10]} )</td>
</tr>
<tr>
<td>(d), (e) &amp; (f)</td>
<td>( 3 + \frac{1}{4} [1 + \text{erf}(x/\xi)] [1 + \text{erf}((20 - x)/\xi)] + i\hat{\mathcal{H}}[...] )</td>
</tr>
<tr>
<td>(g), (h) &amp; (i)</td>
<td>( 1 + \frac{1}{4} [1 + \text{erf}(x/\xi)] [1 + \text{erf}((30\xi - x)/\xi)] \sin^2(4\pi x/17\xi) + i\hat{\mathcal{H}}[...] )</td>
</tr>
<tr>
<td>(j), (k) &amp; (l)</td>
<td>( \frac{3}{2} + i\text{erfc}((1/20 - ix)/\xi)e^{((1/20 - ix)/\xi)^2} + i\text{erfc}((1/10 - (x - 2))/\xi)e^{((1/10 - i(x - 2))/\xi)^2} )</td>
</tr>
</tbody>
</table>

**TABLE I:** Permittivity profiles for which wave propagation is calculated in figure 3. In all cases the scale of the profile is fixed by \( \xi = 0.1\lambda \). The notation \( \hat{\mathcal{H}}[...] \) implies the Hilbert transform of the preceding part of the function minus the background value of the permittivity. This Hilbert transform was calculated numerically over a spatial range roughly 30 times larger than shown in the plots.
D. The effect of truncating an infinite profile

We have found that based upon the analytic properties of a permittivity profile $\epsilon(x)$ in the complex position plane, one can ensure zero reflection from one side of a planar medium where the permittivity varies over the cross section. Strictly speaking these profiles are of infinite extent, even though as is clear in figures 2 and 3 they tend to the value of the background medium as the distance from the medium is increased. If we wanted to build such an index profile then these infinite tails must be truncated. In this section we show the effect of introducing such a truncation.

Suppose the imaginary part of the permittivity is a known function of position $\text{Im}[\epsilon(x)]$ that is only non zero over the range $x \in [0, L]$ (the discussion below also applies if the real part is known and is only non–zero over a certain interval). If the reflection is zero from the left of the medium, then the real part of the profile $\text{Re}[\epsilon(x)]$ is given by eq. (12) in the main text which in this case is

$$\text{Re}[\epsilon(x)] = \epsilon_b + \frac{1}{\pi} \text{P} \int_0^L \frac{\text{Im}[\epsilon(x')]}{x' - x} dx'.$$

For positions outside of the region $[0, L]$, this can be expanded as a series

$$\text{Re}[\epsilon(x)] = \epsilon_b - \sum_{n=0}^{\infty} \frac{1}{\pi x} \int_0^L \left( \frac{x'}{x} \right)^n \text{Im}[\epsilon(x')] dx'.$$

The leading term in the series goes as $1/x$, and is proportional to the net absorption in the profile $\int_0^L \text{Im}[\epsilon(x')] dx'$. For profiles exhibiting PT–symmetry, this term vanishes and the real part of the permittivity approaches the background value as $1/x^2$ or faster, making it much easier to truncate.

Figure 4 shows the reflection that results from truncating one of these non–reflecting profiles at a finite distance from the origin, and how this reflection varies as a function of $k_y$. In this particular case the truncation of the profile is achieved through multiplying by a Gaussian envelope as follows:

$$\epsilon(x) = \begin{cases} 
1 - \frac{2e^{-\frac{(x/\xi)^2}{2}}}{x/\xi + i} & |x| \leq 2.5\lambda \\
1 & |x| > 2.5\lambda.
\end{cases}$$ (14)
FIG. 3: Wave propagation in the permittivity profiles listed in table I. The left hand panels show the spatial profiles of the permittivity, the middle panels show wave propagation for incidence from the left, and the right hand panels show wave propagation for incidence from the right (the meaning of the line colours and styles is the same as in figure 2). In all cases we set \( k_y = 0 \), although changing this quantity is equivalent to changing the background value of the permittivity, which is not the same in the four cases (see table I). The presence of oscillations in the absolute value of the field in regions where the permittivity approaches the background value indicates that the profile reflects radiation. While the first three cases (a–j) were calculated for the TE wave polarization, the final case (k–l) is calculated for the TM wave polarization.
FIG. 4: The permittivity profile shown in panel (f) is that given in (14), truncated to a value of 1 at the vertical dashed lines shown in both panels (f) and (a). Panels (a)–(e) show wave propagation in this profile for angles of incidence approaching grazing ($\theta \to \pi/2$). As the angle of incidence approaches $\pi/2$ the effective wavelength in the $x$ direction increases, and the effect of the truncation of the profile becomes evident as increased reflection.