One dimensional electromagnetic waves on flat surfaces

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We show that one dimensional electromagnetic waves can be constrained to propagate along a join between two thin sheets when one surface supports TM polarised surface waves and the other supports TE polarised surface waves. We calculate the dispersion relation of these modes and show that they are exceptionally tightly confined to the join, with characteristic decay lengths an order of magnitude smaller than the surface waves supported by each individual surface. We give an example of a metasurface implementation where low frequency instances of such waves may be observed.

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Electromagnetic waves can be constrained to propagate in two dimensions through tailoring the properties of an interface between two bulk media. The surface plasmon polariton (SPP) [1] is the most common example, and is a wave that is bound to an interface between media with opposite signs of permittivity. In a similar manner, the magnetic analog of these waves—surface magnon polaritons [2]—are confined to the interface between media with permeability of opposite signs. In recent years, there has been interest in structuring interfaces on a sub–wavelength scale, creating a so–called metasurface [3, 4]. Such structured surfaces can also support bound waves that behave much like plasmon or magnon polaritons, even though the constituent media have neither negative permeability or permittivity [5, 8, 9]. Sievenpiper's structured surface [10], for example, is ideally composed of perfectly conducting elements, but can support surface waves of both magnetic and electric polarization, depending on the operating frequency.

In this work we investigate a new kind of electromagnetic mode that is constrained to propagate along a line on a flat surface. We demonstrate that there exists a solution to Maxwell's equations that is bound to the line where three media meet, shown schematically by the red wavy line in figure 1. The solution has some similarity with the one dimensional propagating modes that arise on wedge shaped surfaces [11–13], but in our case the surface remains flat and potentially thin compared to the free space wavelength. In particular we we will show that the aforementioned mode is supported when we have a top half space that is vacuum, and two surfaces joined along $x=0$ that respectively support TE and TM surface waves. Whilst naturally occurring materials with the required properties exist, it may be a simpler matter to form them artificially and we suggest a metamaterial design that may be appropriate.

Consider the situation sketched in the main panel of figure 1. We have a surface that is translationally invariant along the z axis, so that all fields can be taken to have only an $\exp(ik_z z)$ dependence along this axis.

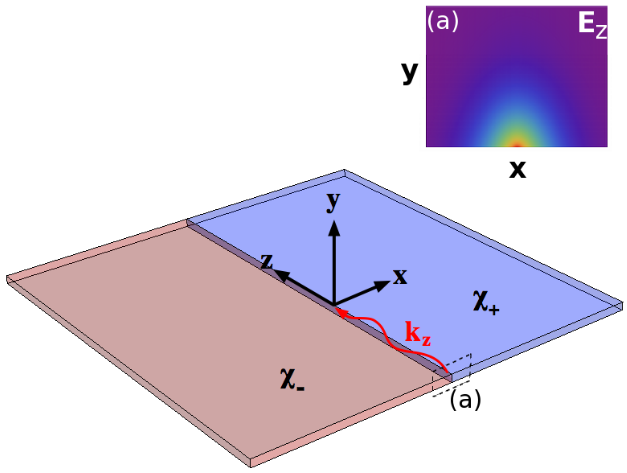

FIG. 1: The diagram shows a join between two surfaces with different surface reactances, χ_- and χ_+. If the reactances are of opposite sign then a wave can be bound to the region of space surrounding the join, with the limit of $ck_z/\omega\to\infty$ occurring when $\chi_-=-\chi_+$. The inset shows the electric field distribution, E_z in the $ck_z/\omega\to\infty$ limit (computed from (6)) in the region indicated (a).

Maxwell's equations allow us to write the fields in the $x-y$ plane entirely in terms of the field components along the z axis [15],

$$\boldsymbol{E}_{\parallel}=\frac{i(k_z\boldsymbol{\nabla}_{\parallel}E_z-\frac{\eta_0\omega}{c}\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_{\parallel}H_z)}{\frac{\omega^2}{c^2}-k_z^2}$$
$$\boldsymbol{H}_{\parallel}=\frac{i(k_z\boldsymbol{\nabla}_{\parallel}H_z+\frac{\omega}{\eta_0 c}\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_{\parallel}E_z)}{\frac{\omega^2}{c^2}-k_z^2}. \quad (1)$$

where a subscript $\parallel$ is used to indicate a vector lying in the $x-y$ plane, and $\eta_0=\sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. Equation (1) greatly simplifies our calculation, as we need only work in terms of two unknowns rather than six. We characterize the surface in terms of a surface impedance, Z, which is a good description for some metasurfaces (for example see [16]),

$$\left[\boldsymbol{E}-\hat{\boldsymbol{y}}(\boldsymbol{E}\cdot\hat{\boldsymbol{y}})=Z\hat{\boldsymbol{y}}\times\boldsymbol{H}\right]_{y=0}. \quad (2)$$

Such a characterization can be a valid alternative to using

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ϵ and μ, although it is only in cases when the effective index of the underlying medium is large that we may consider Z to be a function of frequency alone [17]. For instance, metamaterial surfaces supporting spoof surface plasmon polaritons in the manner described in [5] are very well described in terms of surface impedance. The influence of surface discontinuities and dissipation on the validity of this boundary condition is discussed for example by Wang [6], and Senior's text also provides a first principles treatment [7]. In our case the impedance boundary condition greatly simplifies the mathematical treatment, and we will show at the end of the paper that the same results can be obtained when the impedance boundary condition is replaced with a dielectric medium.

Working in the limit of zero loss ($Z(\omega) = i\chi(\omega)$, χ real) we combine (2) with equation (1). To simplify matters further we also take the electrostatic limit [14] $|k_z| \gg |\omega/c|$, finding

$$\left[E_z = -\frac{\chi(x)}{k_z}\frac{\partial H_z}{\partial x}\right]_{y=0}$$
$$\left[H_z = -\frac{1}{\chi(x)k_z}\frac{\partial E_z}{\partial x}\right]_{y=0}. \tag{3}$$

For the configuration shown in figure 1 the reactance is given by,

$$\chi(x) = \begin{cases}\chi_- & x < 0\\ \chi_+ & x > 0\end{cases}. \tag{4}$$

The two differential equations (3) determine the field on the surface, and in this limit the two polarizations are entirely cross coupled by the boundary conditions.

When the two reactances are equal in magnitude and opposite in sign, $\chi_+ = -\chi_- = \chi$, equation (3) has a solution which is bound to the line $x = y = 0$,

$$E_z(y=0) = E_0 e^{-|k_z||x|}e^{ik_z z}$$
$$H_z(y=0) = E_0\frac{\text{sign}(k_z)}{\chi}e^{-|k_z||x|}e^{ik_z z}, \tag{5}$$

This is somewhat reminiscent of the electrostatic limit of a surface plasmon, where the field decays at an equal rate on each side of a planar interface between two media with permittivity of equal magnitude and opposite sign [1], although we emphasize that in this case the decay is away from the line $x = y = 0$.

The full spatial dependence of the field can be reconstructed from (5) by writing it as an expansion of the following form,

$$\boldsymbol{E} = e^{ik_z z}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\mathcal{E}(k)\hat{\boldsymbol{e}}e^{-\kappa(k)y}e^{ikx}$$
$$\boldsymbol{H} = \frac{\text{sign}(k_z)}{\chi}\boldsymbol{E} \tag{6}$$

where $\kappa(k) = \sqrt{k^2 + k_z^2}$ and $\hat{\boldsymbol{e}} = (c/\omega)[k\hat{\boldsymbol{x}} + i\kappa(k)\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}}]$. Equating (6) with (5) at $y = 0$ then enables us to find the expansion coefficient,

$$\mathcal{E}(k) = \frac{2\omega E_0/c}{k^2 + k_z^2}. \tag{7}$$

The inset of figure 1 shows a plot of E_z in this limit. The energy per unit length contained in the mode above the surface is then given by,

$$\frac{E}{L} = \frac{\epsilon_0}{4}\int_0^{\infty}dy\int_{-\infty}^{\infty}dx\left[|\boldsymbol{E}|^2 + c^2|\boldsymbol{B}|^2\right]$$
$$= \frac{\epsilon_0 E_0^2}{\pi k_z^2}\left(1 + \frac{\eta_0^2}{\chi^2}\right)$$

which is finite despite the fact that the field (6) diverges as one approaches the line where the two surfaces meet. Through changing the magnitude of χ one changes the balance of energy in the mode. For small χ the energy is almost entirely contained in the magnetic field, and for large χ the energy is mostly in the electric field. As we are working in the limit of $k_z \gg \omega/c$, the momentum per unit length above the surface is zero, which can be seen immediately from (6), because the electric and magnetic fields are proportional to one another.

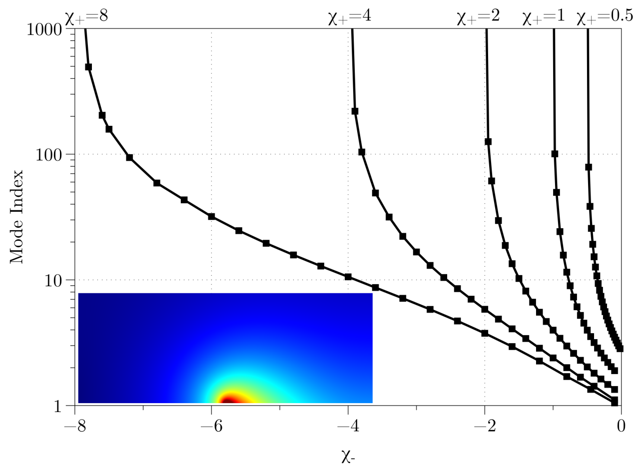

FIG. 2: The numerically calculated mode index of the one-dimensional bound mode supported at the join between surfaces with opposite signs of reactance for various combinations of χ_- and χ_+. Inset: the time averaged E-field for the mode when $\chi_- = -0.05$ and $\chi_+ = 2$.

The above theoretical argument indicates the existence of a mode that is bound to the line where two surfaces of opposite reactance meet, with a dispersion relation that is such that $k_z \to \infty$ as $\chi_+ \to -\chi_-$. To verify this theoretical argument we performed an eigenmode analysis of the above configuration using Comsol multiphysics [18]. In figure 2 we show the mode index ($n = k_z c/\omega$) of these one-dimensional bound waves for various combinations of χ_+ and χ_- and demonstrate that as $k_z \to \infty$ the dispersion of the mode asymptotically approaches the condition $\chi_- = -\chi_+$ as predicted. Also shown inset in figure 2 is the time-averaged electric field of the mode in the

$x-y$ plane when we are away from this asymptotic limit $\chi_-(\omega) = -0.05$ and $\chi_+(\omega) = 2$, corresponding to a mode index of 1.3. The mode is clearly still bound to the join between the surfaces but extends further in space, and there is now an asymmetry in the positive and negative x-directions which arises whenever $\chi_- \neq -\chi_+$.

Analysis of the fields also indicates the extreme degree of confinement exhibited by these modes. For the field plot shown inset in figure 2 the decay lengths (defined as the distance over which the field magnitude falls by a factor e) are $L_{x_-} = \lambda_0/83$, $L_{x_+} = \lambda_0/22$, and $L_y = \lambda_0/60$ where λ_0 is the free-space wavelength given by $2\pi c/\omega$. Thus the field confinement of these one-dimensional bound modes can be an order of magnitude greater than their two-dimensional equivalents, which typically decay over length scales of the order of $\lambda_0/2$ for similar values of impedance.

We also examined the effect of dissipation on this mode, finding that when the field is poorly confined to the join the effect of dissipation is relatively small. For example when $\chi_+ = 2.0 + 0.1i$ and $\chi_- = -0.05 + 0.0025i$ (arbitrarily choosing $\mathrm{Im}[\chi] = \mathrm{Re}[\chi]/20$), we obtain a mode index of $n = 1.296 + 0.02i$. However, if we choose $\chi_- = -1.5 + 0.075i$) such that we are closer to the asymptotic limit and the mode is more confined to the join, the mode index becomes $n = 9.158 + 3.17i$, indicating a far greater degree of dissipation. This is similar to the dispersion of SPPs, where absorption also prevents the mode from propagating as the asymptotic limit is approached. Another similarity to SPPs is that the dispersion relation of the mode is rather sensitive to the bounding dielectric. For example taking vacuum as the surrounding medium and $\chi_- = -0.2 + 0.01i$ we obtain $n = 1.473 + 0.042i$, but even increasing the permittivity of the dielectric by just 0.1 increases the mode index to $n = 1.530 + 0.043i$. This degree of sensitivity is similar to that of SPPs, but the decreased modal volume may make these systems of interest to the sensing community. We also note that this sensitivity is much more pronounced close to the asymptotic limit, but that in this region the losses will tend to be large.

It is simple to translate the description of these surfaces in terms of impedance into one in terms of ϵ and μ in the case where $\epsilon_{yy} = \mu_{yy} = \infty$ (no power propagation below the surface), $\epsilon_{xx} = \epsilon_{zz} = \epsilon_\parallel$, and $\mu_{xx} = \mu_{zz} = \mu_\parallel$. For a large surface thickness the surface impedance is given by $Z = \sqrt{\mu_\parallel/\epsilon_\parallel}$. A positive or negative reactance then implies respectively media with either negative $\epsilon_\parallel$/positive $\mu_\parallel$ or negative $\mu_\parallel$/positive $\epsilon_\parallel$. One can obtain $\epsilon_\parallel < 0$ and $\mu_{yy} = \epsilon_{yy} = \infty$ at microwave frequencies or below through drilling deep, sub–wavelength holes in a metal [5]. The opposite case of $\mu_\parallel < 0$ and $\mu_{yy} = \epsilon_{yy} = \infty$ can be obtained (for example) through filling these holes with a material with $\mu < 0$ [5] (see below). We also note that these impedance boundaries can be tolerably approximated by isotropic media with large negative values of permittivity and permeability.

We now demonstrate that these modes can be supported on a relatively thin structured material. In this case the form of the relative impedance as a function of $\epsilon_\parallel$ and $\mu_\parallel$ becomes a little more complicated, but can be simplified by placing the material slabs onto perfectly conducting ground planes, in which case the relationship becomes,

$$Z_i(\omega) = -i\sqrt{\frac{\mu_\parallel}{\epsilon_\parallel}}\tan\left(\sqrt{\mu_\parallel\epsilon_\parallel}\,\omega d/c\right) \qquad (8)$$

where d is the thickness of the slab.

Equation (8) indicates that the impedance of a grounded slab of the previously described material depends critically upon its thickness, and we might expect the modes supported by such systems to be significantly modified when that thickness becomes small. This is

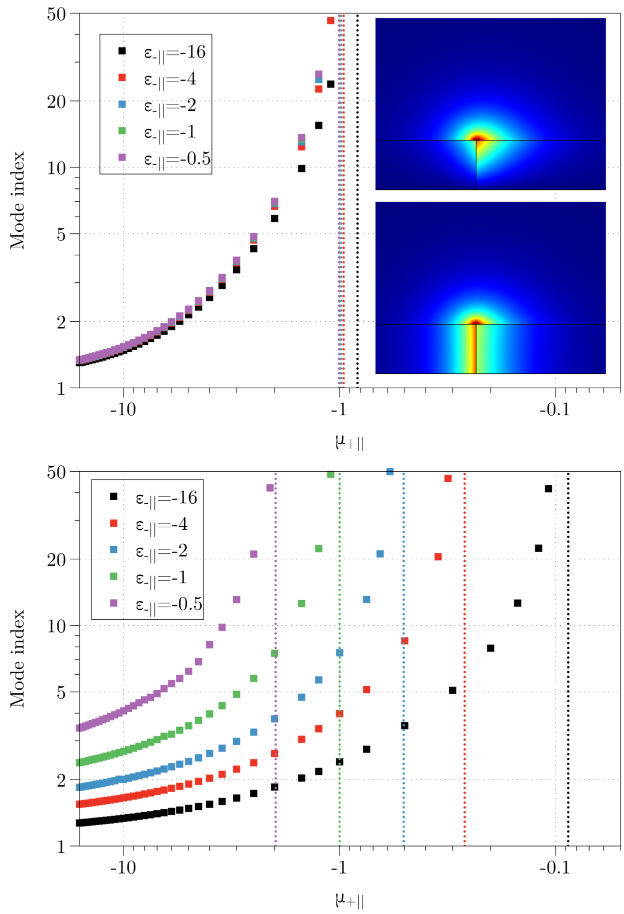

FIG. 3: The numerically calculated mode index of the one-dimensional bound mode supported at the join between grounded anisotropic media where $\epsilon_\perp = \mu_\perp = 1 \times 10^6$ and $\mu_{-\parallel} = \epsilon_{+\parallel} = 1$, for various combinations of $\epsilon_{-\parallel}$ and $\mu_{+\parallel}$. The dotted lines indicate the expected asymptotic limits when $Z_- = -Z_+$ as given by (8). Top: $d = \lambda_0/30$ (inset: the time-averaged E-field (top) and time-averaged H-field (bottom) when $\epsilon_{-\parallel} = -0.5$ and $\mu_{+\parallel} = -2$). Bottom: $d = 2\lambda_0/3$.

indeed the case, and we demonstrate this for the one-dimensional mode by plotting the numerically calculated mode index for various combinations of $\epsilon_{-\|}$ and $\mu_{+\|}$ (where the sign in the subscript indicates the medium on either side of $x = 0$), and two thicknesses (figure 3). When the slab thickness is much smaller than the free space wavelength the high mode index limits (indicated by the dotted lines in the figure) tend to $\mu_{+\|} = -1$, regardless of the value of $\epsilon_{-\|}$. However, when the thickness is increased, and the influence of the tan term in (8) becomes negligible, $Z \approx \sqrt{\mu_\|/\epsilon_\|}$ and the asymptotic limits occur when $\epsilon_{-\|} = 1/\mu_{+\|}$ as anticipated from the argument in terms of surface impedance.

Also shown inset in figure 3 are the time-averaged E- and H-field profiles in the $x - y$ plane of the mode supported at the join between grounded layers of thickness $d = \lambda_0/30$ and $\epsilon_{-\|} = -0.5$ and $\mu_{+\|} = -2$. Whilst the H-field profile demonstrates the distribution one might expect for an impedance boundary description (the fields are invariant in the z-direction below the boundary), the influence of the conducting ground plane on the E-field distribution is clear. However, it is important to note that one only requires a slab thickness of the order of the free space wavelength for the character of the mode to mimic that of the impedance boundary model. It is clear from this that the use of slabs of appropriately designed metamaterials, such as the hole arrays described previously, may be one potential route to observing these bound one-dimensional modes. We note however that it will only be possible to explore the dispersion of the mode up to the limit where the metamaterial structure begins to become comparable in size to the decay length. For example, if we were to realise the reactance through drilling $a \times a$ holes per unit area A in a metal and fill some of them with a magnetic material, the reactance would be [19] $\chi = -(a^2/A)\mu_0\mu\omega/\sqrt{\pi^2/a^2 - \mu\omega^2/c^2}$. The sign of χ only depends on the sign of the permeability of the magnetic material filling the holes. For low frequencies one may therefore be able to realise small reactances of opposite signs that support this mode, with the effective medium approximation remaining valid up until some point when $\chi_+ \sim -\chi_-$, when the decay length becomes comparable to a.

We have demonstrated the existence of a new type of bound one-dimensional electromagnetic wave localised at the join between two surfaces that are described by reactances of opposite sign. The mode can be exceptionally tightly confined, with characteristic decay lengths an order of magnitude smaller than those of the two-dimensional bound waves supported by the individual surfaces. We have also calculated the dispersion characteristics of these modes and described a low frequency, thin layer implementation in which the media are described by effective properties similar to those of common metamaterials. Although the investigation of this mode will be difficult at large wavevectors, it may be possible to experimentally investigate at low frequencies far away from the asymptotic limit.

Acknowledgments

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[18] We set up a 2D model bounded by perfectly matched layers, set the impedance boundary condition (4) on a surface at $z = 0$, and used an eigenmode analysis to solve for the out of plane wave–vector k_z. The mesh size was reduced until the results stopped changing significantly.

[19] This expression was found through taking the ratio of the

in plane $\boldsymbol{E}$ and $\boldsymbol{H}$ fields for the lowest order mode within an infinitely long waveguide, and multiplying by a factor a^2/A to account for the number of holes per unit area.