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Local and global stability indices for a riddled basin attractor of a piecewise linear map

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We consider a piecewise expanding linear map with a Milnor attractor whose basin is riddled with the basin of a second attractor. To characterize the local geometry of this riddled basin, we calculate a stability index for points within the attractor as well as introducing a global stability index for the attractor as a set. Our results show that for Lebesgue almost all points in attractor the index is positive and we characterise a parameter region where some points have negative index. We show there exists a dense set of points for which the index is not converge. Comparing to recent results of Keller, we show that the stability index for points in the attractor can be expressed in terms of a global stability index for the attractor and Lyapunov exponents for this point.

Keywords: Skew product, Milnor attractor, stability index, riddled basins of attraction

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1. Introduction

1.1. *Motivation*

For a dynamical system, an attractor can be characterized by a basin of attraction, i.e. the set of initial points whose orbits are attracted to the attractor. This paper considers so-called *riddled basins* of attraction that have positive measure but that contain no open sets [1]. A riddled basin has a complicated geometric structure in that each open set that intersects the attractor also intersects its complement in a set of positive measure. Such basins have been studied, for example, in [2–9]. While most previous work has focussed on the global structure of riddled basins and bifurcations that create riddled basins, in this paper we turn our focus to the local geometry of a riddled basin.

More precisely, suppose $F : X \to X$ is a continuous map on X, where X is a compact n-dimensional manifold and ℓ is Riemannian measure on X. Let $A \subset X$ be a compact invariant set, we define its basin of attraction:

$$\mathcal{B}(A) = \{x \in X : \omega(x) \subset A\}, \tag{1}$$

where $\omega(x) = \cap_{N>0}\overline{\{F^n(x) \,:\, n > N\}}$, and if A is an attractor in the sense of Milnor,

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then $\ell(\mathcal{B}(A)) > 0$ [10]. We say that the basin of the attractor A is *riddled* with the basin of another attractor C if for any open $U \subset X$ we have $\ell(\mathcal{B}(A) \cap U) > 0$ and $\ell(\mathcal{B}(C) \cap U) > 0$ [1].

The stability index we consider was introduced by Podvigina and Ashwin [11] in order to characterize the local geometry of basins of attraction for heteroclinic cycles. More recently, Lohse [12] and Castro and Lohse [13] have developed such index to understand stability of simple heteroclinic networks in $\mathbb{R}^4$. The index can however be applied to understand more general situations. Keller [14] uses a stability index to characterize the structure and regularity of an invariant graph that is an attractor for a chaotically driven concave map. The current work is inspired by Keller's paper [14] where he formulates the stability index for a point on the graph using methods from the thermodynamic formalism. We used explicit constructive methods which allow us to obtain several results in the context of a single example.

The paper is organized as follows: in Section 1.2, we define the stability index at a point as proposed in [11], discuss basic properties and introduce a global stability index for the attractor as a set. In section 2, we consider a class of piecewise expanding linear skew product maps and give the main result on the stability index for both a point as well as the global stability index for the attractor; we discuss some necessary and sufficient conditions for the convergence of the stability index to fail. We show the proofs for these results in Section 3 and conclude the paper with a brief discussion in Section 4.

1.2. *The stability index of an attractor at a point and the global stability index*

Consider $F : \mathbb{R}^n \to \mathbb{R}^n$ a continuous map and $\ell = \ell_n$ Lebesgue measure on $\mathbb{R}^n$. We recall the definition of the (local) stability index from [11, Definition 5] for a point x in an attractor A with basin $\mathcal{B}(A)$ (the definitions can clearly be extended to maps on smooth n-dimensional manifolds by considering a Riemannian measure).

Definition 1.1: For a point $x \in \mathbb{R}^n$ and $\varepsilon > 0$, define

$$\Sigma_\varepsilon(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}(A))}{\ell(B_\varepsilon(x))}, \tag{2}$$

where $B_\varepsilon(x) = \{y \in \mathbb{R}^n |\ d(x,y) < \varepsilon\}$ is the ε-neighbourhood of x. The (local) *stability index* of A at $x \in A$ is defined to be

$$\sigma(A, x) := \sigma_+(x) - \sigma_-(x), \tag{3}$$

where

$$\sigma_-(x) := \lim_{\varepsilon \to 0} \left[\frac{\log(\Sigma_\varepsilon(x))}{\log \varepsilon} \right], \quad \sigma_+(x) := \lim_{\varepsilon \to 0} \left[\frac{\log(1 - \Sigma_\varepsilon(x))}{\log \varepsilon} \right],$$

as long as these limits converge.

We use the convention that $\sigma_-(x) = \infty$ if there is $\varepsilon > 0$ such that $\Sigma_\varepsilon(x) = 0$ and $\sigma_+(x) = \infty$ if there is $\varepsilon > 0$ such that $\Sigma_\varepsilon(x) = 1$, or if the limits are infinite: we allow $\sigma(A, x) \in [-\infty, \infty]$. Note that the result [11, Theorem 2.2] can be used to show that $\sigma(A, x)$ is an invariant along trajectories if the map is a local diffeomorphism.

The index is related to the local geometry of the basins of attraction of A in the following sense. If $\sigma(A,x) > 0$, this means that there is an increasingly large proportion of points that are attracted to A as the neighbourhood $B_\varepsilon(x)$ shrinks, i.e. $\Sigma_\varepsilon(x)$ goes to 1 as $\varepsilon \to 0$. On the other hand, if $\sigma(A,x) < 0$, this means that there is a decreasingly small proportion of points that are attracted to A as $B_\varepsilon(x)$ shrinks, i.e. $\Sigma_\varepsilon(x)$ goes to 0 as $\varepsilon \to 0$ [13]. We now give a strengthened version of [11, Lemma 2.2] using *exponentially asymptotically tight bounds*: let $g : (0,E) \to \mathbb{R}$ for some $E > 0$. We define

$$\tilde{\Theta}(g) := \left\{ f : (0,E) \to \mathbb{R} \ : \ \begin{array}{l} \forall\, \delta > 0, \exists c_1, c_2, \epsilon_0 > 0, \forall\, 0 < \epsilon \leq \epsilon_0, \\ 0 \leq c_1 \epsilon^{\delta} g(\epsilon) \leq f(\epsilon) \leq c_2 \epsilon^{-\delta} g(\epsilon) \end{array} \right\}$$

and write $f(\epsilon) = \tilde{\Theta}(g(\epsilon))$ to mean $f \in \tilde{\Theta}(g)$. In this case we say g is an *exponentially asymptotically tight bound* for f (cf [15]).

Lemma 1.2: *Suppose that $\sigma(A,x)$ is defined for some $x \in \mathbb{R}^n$, then the following hold:*

(a) If $\sigma_+(x)$ (respectively $\sigma_-(x)$) converges to a positive value, then $\sigma_-(x)$ (respectively $\sigma_+(x)$) converges to 0 (i.e. only one of $\sigma_+(x)$ and $\sigma_-(x)$ can be non-zero).
(b) If $\sigma(A,x) = c > 0$, then $1 - \Sigma_\varepsilon(x) = \tilde{\Theta}(\varepsilon^c)$ (in particular $\Sigma_\varepsilon(x) \to 1$ as $\varepsilon \to 0$).
(c) If $\sigma(A,x) = -c < 0$, then $\Sigma_\varepsilon(x) = \tilde{\Theta}(\varepsilon^c)$ (in particular $\Sigma_\varepsilon(x) \to 0$ as $\varepsilon \to 0$).

Proof. For case (a), note that if $\sigma_-(x) > 0$ then $\lim_{\varepsilon\to 0} \Sigma_\varepsilon(x) = 0$. This implies that $1 - \Sigma_\varepsilon(x)$ converges to 1 as $\varepsilon \to 0$ and so $\sigma_+(x) = 0$. A similar argument shows that $\sigma_+(x) > 0$ implies $\sigma_-(x) = 0$. Case (b) and (c) can be proven similarly - we only prove (b) which follows from noting that $c = \sigma(A,x) = \sigma_+(x) > 0$ and $\sigma_-(x) = 0$. Therefore

$$\lim_{\varepsilon \to 0} \frac{\log(1 - \Sigma_\varepsilon(x))}{\log \varepsilon} = c.$$

Then for all $\delta > 0$, there exists $0 < \varepsilon_0 < 1$ such that

$$c - \delta < \frac{\log(1 - \Sigma_\varepsilon(x))}{\log \varepsilon} < c + \delta,$$

if and only if

$$\varepsilon^{c-\delta} > 1 - \Sigma_\varepsilon(x) > \varepsilon^{c+\delta}.$$

Therefore for all $\delta > 0$, there exist constants $c_1 > 0, c_2 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ we have

$$c_1 \varepsilon^{\delta} \varepsilon^{c} < 1 - \Sigma_\varepsilon(x) < c_2 \varepsilon^{-\delta} \varepsilon^{c},$$

as $\varepsilon \to 0$, and so $1 - \Sigma_\varepsilon(x) = \tilde{\Theta}(\varepsilon^c)$. □

The stability index above is defined for an individual point within an attractor. We introduce a global stability index for an attractor as it turns out that the stability index at points can be related to this global index.

Definition 1.3: Let $A \subset \mathbb{R}^n$ be an attractor and let $\varepsilon > 0$. We define

$$\Sigma_\varepsilon(A) := \frac{\ell(B_\varepsilon(A) \cap \mathcal{B}(A))}{\ell(B_\varepsilon(A))},$$

where $B_\varepsilon(A) = \cup_{x \in A} B_\varepsilon(x)$. Then the *global stability index* of A is defined to be

$$\sigma(A) := \sigma_+(A) - \sigma_-(A),$$

which exists when the following converge:

$$\sigma_-(A) := \lim_{\varepsilon \to 0} \frac{\log(\Sigma_\varepsilon(A))}{\log \varepsilon}, \quad \sigma_+(A) := \lim_{\varepsilon \to 0} \frac{\log(1 - \Sigma_\varepsilon(A))}{\log \varepsilon}.$$

Note we write $\sigma(A)$ to denote the global stability index while $\sigma(A, x)$ denotes the (local) stability index at the point $x \in A$.

2. Attractors for a piecewise linear skew product map

We consider a piecewise linear skew product system $F : [0,1]^2 \to [0,1]^2$, similar to that in Ott *et al.* [8] except that F is now defined on the unit square. The model has two parameters and for an open set of these parameters there are two coexisting Milnor attractors A_0 and A_1 such that basin of the A_0 is riddled with basin of the second attractor A_1. Suppose that $(\theta, x) \in [0,1]^2$ and define

$$F(\theta, x) := (T_s(\theta), h(\theta, x)) \tag{4}$$

where the base map

$$T_s(\theta) := \begin{cases} \theta/s & \text{if } 0 \le \theta < s, \\ (\theta - s)/(1-s) & \text{if } s < \theta \le 1, \end{cases} \tag{5}$$

is the skewed (asymmetric) doubling map and the fibre map is

$$h(\theta, x) := \begin{cases} \min(x/\delta, 1) & \text{if } 0 \le \theta < s \text{ and } 0 \le x < 1, \\ \delta x & \text{if } s < \theta \le 1 \text{ and } 0 \le x < 1, \\ 1 & \text{if } x = 1, \end{cases} \tag{6}$$

for $0 < s < 1$ and $0 < \delta < 1$. Note that $1/\delta$ and δ in (6) above are the expanding and shrinking rates respectively. We define $A_0 := [0,1] \times \{0\}$ and $A_1 := [0,1] \times \{1\}$ and note that these are disjoint compact invariant sets whose basins are $B_0 := \mathcal{B}(A_0)$ and $B_1 := \mathcal{B}(A_1)$ respectively. It was proved in [16] that for $0 < s < 1$, A_1 is always an attractor. When $0 < s < 1/2$, A_0 is an attractor such that its basin B_0 is riddled with B_1. In this paper, we will compute the stability index for the point and attractor in B_0 since the riddled basin only occurs within range $0 < s < 1/2$.

We use the Markov nature of the map to give a partition $[0,1]^2 = A_0 \cup \bigcup_{k=1}^{\infty} X_k$, where

$$X_k := X_{k,1} \dot\cup X_{k,2}, \ X_{k,1} := [0, s) \times (\delta^k, \delta^{k-1}], \ X_{k,2} := [s, 1] \times (\delta^k, \delta^{k-1}],$$

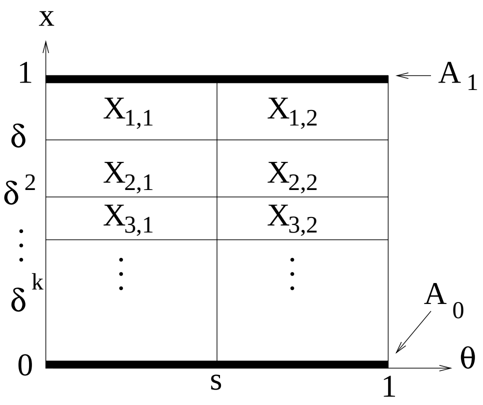


Figure 1. Schematic diagram of the partition for the map F (4).

and $\dot{\cup}$ denotes the union is disjoint; this partition is shown in Figure 1. We note

$$F(X_{k,1}) = X_{k-1} \text{ for } k \geq 2, \tag{7}$$

$$F(X_{1,1}) = A_1, \tag{8}$$

$$F(X_{k,2}) = X_{k+1} \text{ for } k \geq 1. \tag{9}$$

From (5), we consider $[s,1] \subset [0,1]$ and investigate how frequently the orbit of a point $\theta \in [0,1]$ under the skewed doubling map $T_s(\theta)$ visit the right interval $[s,1]$. Let $\mathcal{E}(T_s)$ denote the set of ergodic probability measures for T_s. We note that μ is any ergodic measures in $\mathcal{E}(T_s)$ and the Lebesgue measure ℓ_1 is ergodic for T_s. Let us define

$$n_k(\theta) := \begin{cases} 0 & \text{if } T_s^k(\theta) < s, \\ 1 & \text{if } T_s^k(\theta) \geq s, \end{cases} \tag{10}$$

for $k = 0, \ldots, N-1$. We define

$$i_N(\theta) := \sum_{k=0}^{N-1} n_k(\theta), \tag{11}$$

to denote the number of the first N points in the orbit of θ that lie within $[s,1]$. Hence

$$\lim_{N\to\infty} \frac{i_N(\theta)}{N} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} n_k(\theta) \tag{12}$$

denotes the frequency with which the orbit of θ lie in $[s,1]$. Since $\ell = \ell_1$ is ergodic for T_s, we apply Birkhoff's ergodic theorem which says that for ℓ-almost all $\theta \in [0,1]$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} n_k(\theta) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \chi_{[s,1]}(T_s^k(\theta)) = \int_0^1 \chi_{[s,1]}(y) d\ell(y) = 1-s. \tag{13}$$

The Lyapunov exponents for the map F at a point $(\theta, 0) \in A_0$ can be computed explicitly using in the following lemma.

Lemma 2.1: *Let $\mu \in \mathcal{E}(T_s)$ be given. Define $t := \int_0^s d\mu(\theta)$. Then*

$$\lambda_{\|}(\theta, 0) = -t \log s - (1-t)\log(1-s), \quad \lambda_{\perp}(\theta, 0) = (1-2t)\log\delta,$$

respectively in the base and in the fibre.

Proof. Note that as μ is ergodic, for μ-almost all θ

$$\lambda_{\|}(\theta) = \int \log \frac{dT_s}{d\theta}(\theta)\ d\mu(\theta) = -t\log s - (1-t)\log(1-s), \tag{14}$$

while

$$\lambda_{\perp}(\theta) = \int \log \frac{dh}{d\theta}(\theta)\ d\mu(\theta) = (1-2t)\log\delta. \tag{15}$$

□

2.1. *Local and global stability indices for the attractor*

The first result gives the (local) stability index of A_0 at points $(\theta, 0) \in A_0$ in terms of the parameters and an ergodic measure $\mu \in \mathcal{E}(T_s)$ for the base transformation. The proofs of the results below are detailed in Section 3.

Theorem 2.2: *Suppose that $0 < s < 1/2$ and $0 < \delta < 1$ and $\mu \in \mathcal{E}(T_s)$. Let $\sigma(\theta) := \sigma(A_0, (\theta, 0))$ denote the stability index of A_0 at the point $(\theta, 0) \in A_0$ for the map (4) and define $t := \int_{\theta=0}^s d\mu(\theta)$*

(i) *If $-t\log s - (1-t)\log(1-s) - (1-2t)\log\delta > 0$, then for μ-almost all θ*

$$\sigma(\theta) = \frac{\log s - \log(1-s)}{\log\delta}\left(\frac{-t\log s - (1-t)\log(1-s) - (1-2t)\log\delta}{-t\log s - (1-t)\log(1-s)}\right) > 0.$$

(ii) *If $-t\log s - (1-t)\log(1-s) - (1-2t)\log\delta < 0$, then for μ-almost all θ*

$$\sigma(\theta) = \frac{-t\log s - (1-t)\log(1-s) - (1-2t)\log\delta}{-t\log s - (1-t)\log(1-s)} < 0.$$

The first part of the next result follows as a special case of Theorem 2.2 using $\mu = \ell$, the Lebesgue measure for T_s for any $0 < s < 1$, while the second part follows by considering Theorem 2.2 for measures that give extreme values of t.

Theorem 2.3: *Suppose that $0 < s < 1/2$ and $0 < \delta < 1$ and $\sigma(\theta)$ is as in the previous theorem.*

(i) *For ℓ-almost all θ, we have θ with positive stability index, i.e. $\sigma(\theta) > 0$,*
(ii) *There exists a θ with negative stability index (i.e. $\sigma(\theta) < 0$) if and only if $\delta < s$.*

The global stability index for A_0 can be computed as follows:

Theorem 2.4: *For $0 < s < 1/2$, any $0 < \delta < 1$ the global stability index of the attractor*

A_0 *is*

$$\sigma(A_0) = \frac{\log s - \log(1-s)}{\log \delta}. \tag{16}$$

Comparing the results for Theorem 2.2, 2.4, the expressions from Lemma 2.1 and the general results from Keller [14], note that one can write the stability index for A_0 at a point $(\theta, 0)$ in terms of Lyapunov exponents and the stability index for the attractor as follows: let $\sigma(A_0)$ is the global stability index of A_0 (16) while $\lambda_{||}(\theta)$, $\lambda_{\perp}(\theta)$ are the Lyapunov exponents for the map in the base and fibre directions respectively.

Corollary 2.5: *For any $\mu \in \mathcal{E}(T_s)$ and for the map (4) with $0 < s < 1/2$ and $0 < \delta < 1$, for μ-almost all θ we have*

$$\sigma(\theta) = \begin{cases} \sigma(A_0) \cdot \frac{\lambda_{\|}(\theta) - \lambda_{\perp}(\theta)}{\lambda_{\|}(\theta)} > 0 & \text{if } \lambda_{\|}(\theta) - \lambda_{\perp}(\theta) > 0, \\ \frac{\lambda_{\|}(\theta) - \lambda_{\perp}(\theta)}{\lambda_{\|}(\theta)} < 0 & \text{if } \lambda_{\|}(\theta) - \lambda_{\perp}(\theta) < 0. \end{cases} \tag{17}$$

Comparing (17) with [14, Theorem 2.5] we see that $\sigma(A_0)$ corresponds to Loynes' exponent.

2.2. *Criteria for non-convergence of the stability index*

In the previous section, we have stated that the stability index is well defined if and only if the sequence $i_N(\theta)/N$ converges for any $0 < s < 1/2$ and $0 < \delta < 1$. However, there are some points for which the limit does not exist; to be precise:

Theorem 2.6: *Suppose that $0 < s < 1/2$, $0 < \delta < 1$ and $\sigma(\theta)$ is the stability index of A_0 for the map (4). Then $\sigma(\theta)$ does not exist if and only if the sequence $i_N(\theta)/N$ does not converge. Moreover, there is a dense set of $\theta \in [0,1]$ for which the stability index does not exist.*

The proof of this result is discussed in Section 3. It uses a result of Jordan *et al.* [17] to construct points with non-convergence for $i_N(\theta)/N$.

3. Proof of main results

3.1. *Proof of Theorem 2.2*

For any point $\theta \in [0,1]$ and $N \in \mathbb{N}$, the following set

$$I_N(\theta) := \{\tilde{\theta} \in [0,1] : n_k(\tilde{\theta}) = n_k(\theta) \text{ for } k = 0, \ldots, N-1\}, \tag{18}$$

is a neighbourhood of θ. Note that $T_s^k(I_N(\theta)) \subset I_{n_k(\theta)}$ for $k = 0, \ldots, N-1$ where $\begin{cases} I_0 &= [0,s), \\ I_1 &= [s,1), \end{cases}$ and after N iterations, $I_N(\theta)$ will be mapped to the whole base space $[0,1]$, i.e.,

$$T_s^N(I_N(\theta)) = [0,1].$$

Note also that $T_s^k|_{I_N(\theta)}$ is invertible for $k = 0, \ldots, N$, i.e.,

$$(T_s^N|_{I_N(\theta)})^{-1}([0,1]) = I_N(\theta). \tag{19}$$

The 1-dimensional Lebesgue measure for any invariant set A under the skewed doubling map is

$$\ell_1(T_s(A)) = \begin{cases} \frac{\ell_1(A)}{s} & \text{if} \quad A \subset I_0, \\ \frac{\ell_1(A)}{1-s} & \text{if} \quad A \subset I_1, \end{cases}$$

So, by using invertibility in (19), it follows that:

$$\begin{aligned} \ell_1(I_N(\theta)) &= s^{N-i_N(\theta)}(1-s)^{i_N(\theta)}\ell_1([0,1]), \\ &= s^{N-i_N(\theta)}(1-s)^{i_N(\theta)}, \end{aligned} \tag{20}$$

where $\ell_1([0,1]) = 1$, $(1-s)^{i_N(\theta)}$ describes the number of times the orbit of θ lies in $[s,1]$ (this follows from (11)) and $s^{N-i_N(\theta)}$ describes the number of times the orbit of θ lies in $[0,s]$.

From (20), for all $\varepsilon > 0$, there exists $N > 1$ such that

$$s^{N-i_N(\theta)}(1-s)^{i_N(\theta)} = 2\varepsilon \tag{21}$$

holds in the θ-direction. We can consider the neighbourhoods of $(\theta, 0)$ by writing

$$U_{N,M}(\theta) := \{(\tilde{\theta}, x) : \tilde{\theta} \in I_N(\theta), x < \delta^M\}, \tag{22}$$

where $U_{N,M}(\theta)$ is a neighbourhood that is approximates $B_\varepsilon(\theta, 0)$ if (21) is satisfied at $\delta^M = \varepsilon$ in x-direction, where $M = \frac{\log \varepsilon}{\log \delta}$. To put it simply, this means that the neighbourhood is 2ε in the θ-direction and ε in the x-direction.

Case (i): After Nth iterates, the neighbourhood $U_{N,M}(\theta)$ expands under the skew product transformation F such that

$$F^N(U_{N,M}(\theta)) = [0,1] \times [0, \delta^{Q_\varepsilon(\theta)}], \tag{23}$$

for some $Q_\varepsilon(\theta)$, where this means that after N iterations, $[0, \delta^M]$ expands to $[0, \delta^{Q_\varepsilon(\theta)}]$. This expansion occurs by considering both expanding and shrinking rates in (6). Hence we have

$$\delta^{Q_\varepsilon(\theta)} = \delta^M \times \delta^{i_N(\theta)} \times \delta^{-N+i_N(\theta)} = \delta^{M+2i_N(\theta)-N},$$

where $\delta^{i_N(\theta)}$ counts the number of times $U_{N,M}(\theta)$ shrinks in the fibre direction and $\delta^{-N+i_N(\theta)}$ counts the number of times $U_{N,M}(\theta)$ expands in the fibre direction, during the first N iterates (this follows from (20)). From the above we consider the case

$$Q_\varepsilon(\theta) = M + 2i_N(\theta) - N \geq 0. \tag{24}$$

Notice that from (23), the 2-dimensional Lebesgue measure of $U_{N,M}(\theta)$ after Nth iterations is

$$\ell_2(F^N(U_{N,M}(\theta))) = 1 \times \delta^{Q_\varepsilon(\theta)}.$$

Now we wish to find the 2-dimensional Lebesgue measure of $U_{N,M}(\theta)$. Note that

$$\ell_2(F^N(U_{N,M}(\theta)) = K\ell_2(U_{N,M}(\theta))$$

where, from the argument above,

$$K = s^{i_N(\theta)-1} \times (1-s)^{-i_N(\theta)} \times \delta^{2i_N(\theta)-N}$$

is the (constant) Jacobian of F^N restricted to $U_{N,M}$. This means that

$$\begin{aligned} \ell_2(U_{N,M}(\theta)) &= 1 \times \delta^{Q_\varepsilon(\theta)} \times s^{N-i_N(\theta)} \times (1-s)^{i_N(\theta)} \times \delta^{-2i_N(\theta)+N}, \\ &= \delta^M \times s^{N-i_N(\theta)} \times (1-s)^{i_N(\theta)}. \end{aligned} \tag{25}$$

In order to compute the stability index at point $(\theta, 0)$, we need to find the proportion of B_0 that is in $U_{N,M}(\theta)$. To compute this, we need the following setting. Suppose that A is any invariant set in $[0,1]^2$ for F. Let $A_{i,j} = A \cap X_{i,j}$ and $L_{i,j} := \ell_2(A_{i,j})$. Thus, from (7), when $A \cap X_{k,1}$ we have

$$\begin{aligned} F(A \cap X_{k,1}) &= A \cap X_{k-1}, \\ &= A \cap (X_{k-1,1} \dot{\cup} X_{k-1,2}), \\ &= (A \cap X_{k-1,1}) \dot{\cup} (A \cap X_{k-1,2}), \\ &= A_{k-1,1} \dot{\cup} A_{k-1,2} \text{ for } k \geq 2. \end{aligned} \tag{26}$$

Meanwhile from (9), when $A \cap X_{k,2}$ we have

$$\begin{aligned} F(A \cap X_{k,2}) &= A \cap X_{k+1}, \\ &= A \cap (X_{k+1,1} \dot{\cup} X_{k+1,2}), \\ &= (A \cap X_{k+1,1}) \dot{\cup} (A \cap X_{k+1,2}), \\ &= A_{k+1,1} \dot{\cup} A_{k+1,2} \text{ for } k \geq 1. \end{aligned} \tag{27}$$

Then we can write (26) and (27) in the form of $L_{i,j}$. Note that $F|_{X_{k,1}}$ stretches by $1/s$ in the θ-direction and expands by $\gamma = 1/\delta$ in the x-direction. Meanwhile, $F|_{X_{k,2}}$ stretches by $1/(1-s)$ in the θ-direction and shrinks by δ in the x-direction. Thus for any invariant set A, if $L_{i,j} = \ell_2(A \cap X_{i,j})$, we have the following:

$$\frac{1}{s\delta} L_{k,1} = L_{k-1,1} + L_{k-1,2} \text{ for } k \geq 2, \tag{28}$$

$$\frac{\delta}{1-s} L_{k,2} = L_{k+1,1} + L_{k+1,2} \text{ for } k \geq 1, \tag{29}$$

where $L_{k,1} = \ell_2(A_{k,1}) = \delta s \ell_2(F(A_{k,1}))$ from the left hand side of (26) and $L_{k,2} = \ell_2(A_{k,2}) = ((1-s)/\delta)\ell_2(F(A_{k,2}))$ from the left hand side of (27).

For the case $s < 1/2$, if we consider $A = B_0$ we obtain the general solutions for $L_{k,1}$ and $L_{k,2}$ respectively as

$$\begin{aligned} L_{k,1} &= \frac{s(1-\delta)}{\delta}\delta^k - \frac{s(1-\delta)}{\tilde{\delta}}\tilde{\delta}^k, \\ &= s(1-\delta)[\delta^{k-1} - \tilde{\delta}^{k-1}], \end{aligned} \tag{30}$$

where $\tilde{\delta} = s\delta/(1-s)$ and

$$L_{k,2} = (1-s)(1-\delta)\delta^{k-1} + \frac{s\delta - \tilde{\delta}}{\delta\tilde{\delta}}(1-\delta)\tilde{\delta}^k. \tag{31}$$

We denote $L_k = \ell_2(X_k \cap B_0)$ where $X_k = X_{k,1} \cup X_{k,2}$. Then we can find L_k thanks to

$$\begin{aligned} L_k &= L_{k,1} + L_{k,2}, \\ &= s(1-\delta)(\delta^{k-1} - \tilde{\delta}^{k-1}) + (1-s)(1-\delta)\delta^{k-1} + \frac{(s\delta - \tilde{\delta})(1-\delta)}{\delta}\tilde{\delta}^{k-1}, \\ &= (1-\delta)\left(\delta^{k-1} - \frac{\tilde{\delta}^k}{\delta}\right). \end{aligned} \tag{32}$$

Then we can find the sum of L_k from level $\delta^{Q_\varepsilon(\theta)}$ up to δ^k (for $k = Q_\varepsilon(\theta)+1, \dots, \infty$) by

$$\begin{aligned} L^+_{Q_\varepsilon(\theta)} = \ell_2(F^N(U_{N,M}(\theta) \cap B_0)) &= \sum_{k=Q_\varepsilon(\theta)+1}^{\infty} L_k = \sum_{k=Q_\varepsilon(\theta)+1}^{\infty} (1-\delta)\delta^{k-1} - \sum_{k=Q_\varepsilon(\theta)+1}^{\infty} \frac{(1-\delta)}{\delta}\tilde{\delta}^k, \\ &= \frac{(1-\delta)\delta^{Q_\varepsilon(\theta)}}{1-\delta} - \frac{(1-\delta)\tilde{\delta}^{Q_\varepsilon(\theta)+1}}{\delta(1-\tilde{\delta})} = \delta^{Q_\varepsilon(\theta)} - \frac{(1-\delta)\tilde{\delta}^{Q_\varepsilon(\theta)+1}}{\delta(1-\tilde{\delta})}. \end{aligned} \tag{33}$$

So we obtain the proportion as

$$\begin{aligned} \Sigma_\varepsilon(\theta, 0) &= \frac{\ell_2(U_{N,M}(\theta) \cap B_0)}{\ell_2(U_{N,M}(\theta))} = \frac{\ell_2(F^N(U_{N,M}(\theta) \cap B_0))}{\ell_2(F^N(U_{N,M}(\theta))} = \frac{L^+_{Q_\varepsilon(\theta)}}{\delta^{Q_\varepsilon(\theta)}}, \\ &= 1 - \left(\frac{1-\delta}{1-\tilde{\delta}}\right)\left(\frac{\tilde{\delta}}{\delta}\right)^{Q_\varepsilon(\theta)+1}. \end{aligned} \tag{34}$$

This means that as $\varepsilon \to 0$ we have that $M \to \infty$, hence $Q_\varepsilon(\theta) \to \infty$ and since $\tilde{\delta} < \delta$, $\lim_{\varepsilon\to 0} \Sigma_\varepsilon(\theta, 0) = 1$ and by Lemma 1.2(b) this implies that $\sigma_-(\theta, 0) = 0$. Meanwhile

$$1 - \Sigma_\varepsilon(\theta, 0) = \left(\frac{1-\delta}{1-\tilde{\delta}}\right)\left(\frac{\tilde{\delta}}{\delta}\right)^{Q_\varepsilon(\theta)+1}. \tag{35}$$

Since $\sigma_-(\theta, 0) = 0$, from Lemma 1.2(b) we want to show that

$$1 - \Sigma_\varepsilon(\theta, 0) = \tilde{\Theta}(\varepsilon^{\sigma_+(\theta,0)}). \tag{36}$$

To prove the above, we need to find the values of $Q_\varepsilon(\theta)$ in (35) where $Q_\varepsilon(\theta)$ is defined in (24). Recall that $M = \log\varepsilon/\log\delta$. Let μ_t be the *Bernoulli measure* such that the frequency of visiting the left interval is t, i.e.

$$t := \int_0^s d\mu_t(\theta) = \int_0^1 \chi_{[0,s)}(\theta) d\mu_t(\theta),$$

where $0 < t < 1$. We know from Birkhoff's Ergodic Theorem in (13) that for μ_t-almost all θ and for large N,

$$i_N(\theta) \approx (1-t)N.$$

By substituting this into (21);

$$\begin{aligned} 2\varepsilon &\approx s^{N-(1-t)N}(1-s)^{(1-t)N}, \\ &\approx s^{Nt}(1-s)^{N(1-t)}. \end{aligned}$$

By taking logs for small ε we have

$$\begin{aligned} \log\varepsilon &\approx Nt\log s + N(1-t)\log(1-s), \\ &\approx N(t\log s + (1-t)\log(1-s)). \end{aligned}$$

Therefore we obtain N as

$$N \approx \frac{\log\varepsilon}{t\log s + (1-t)\log(1-s)}.$$

Now substitute M and N into (24) to give

$$\begin{aligned} Q_\varepsilon(\theta) &:= \frac{\log\varepsilon}{\log\delta} + (1-2t)\frac{\log\varepsilon}{t\log s + (1-t)\log(1-s)}, \\ &:= \log\varepsilon\left(\frac{1}{\log\delta} + \frac{1-2t}{t\log s + (1-t)\log(1-s)}\right), \\ &:= \frac{\log\varepsilon}{\log\delta}\left(\frac{\lambda_\parallel(\theta) - \lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right). \end{aligned} \tag{37}$$

Then from (35), we can define the constant $K = \frac{1-\delta}{1-\tilde{\delta}}$ and write

$$\begin{aligned} 1 - \Sigma_\varepsilon(\theta, 0) &= K e^{\log\left(\frac{\tilde{\delta}}{\delta}\right)^{Q_\varepsilon(\theta)+1}}, \\ &= K e^{(Q_\varepsilon(\theta)+1)(\log\tilde{\delta} - \log\delta)}, \\ &= K e^{(\log\tilde{\delta} - \log\delta)} e^{(\log\tilde{\delta} - \log\delta) Q_\varepsilon(\theta)}, \\ &= \tilde{K} e^{(\log\tilde{\delta} - \log\delta)\frac{\log\varepsilon}{\log\delta}\left(\frac{\lambda_\parallel(\theta) - \lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right)}, \\ &= \tilde{K} e^{\left(\frac{\log\tilde{\delta} - \log\delta}{\log\delta}\right)\left(\frac{\lambda_\parallel(\theta) - \lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right)\log\varepsilon}, \\ &= \tilde{K} e^{\log\varepsilon^{\left(\frac{\log\tilde{\delta} - \log\delta}{\log\delta}\right)\left(\frac{\lambda_\parallel(\theta) - \lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right)}}, \\ &= \tilde{K} \varepsilon^{\left(\frac{\log\tilde{\delta} - \log\delta}{\log\delta}\right)\left(\frac{\lambda_\parallel(\theta) - \lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right)}, \end{aligned} \tag{38}$$

where $\tilde{K} = K e^{(\log\tilde{\delta} - \log\delta)}$. Therefore by comparing the above with (36), we have

$$\sigma_+(\theta, 0) = \left(\frac{\log\tilde{\delta} - \log\delta}{\log\delta}\right)\left(\frac{\lambda_\parallel(\theta) - \lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right), \qquad \sigma_-(\theta, 0) = 0.$$

Thus as long as $Q_\varepsilon(\theta) \geq 0$, the stability index at point $(\theta, 0)$ is

$$\begin{aligned} \sigma(\theta) &= \sigma_+(\theta, 0) - \sigma_-(\theta, 0), \\ &= \left(\frac{\log\tilde{\delta} - \log\delta}{\log\delta}\right)\left(\frac{\lambda_\parallel(\theta) - \lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right), \end{aligned} \tag{39}$$

where $\lambda_\parallel(\theta)$ and $\lambda_\perp(\theta)$ are obtained in (14) and (15) respectively. For this case, this index is always positive since A_0 is an attractor. $Q_\varepsilon(\theta) \to \infty$ as $\varepsilon \to 0$ if and only if from (37) we have that $\lambda_\parallel(\theta) - \lambda_\perp(\theta) > 0$ i.e. $\lambda_\parallel(\theta) > \lambda_\perp(\theta)$.

Case (ii): Assume now that

$$Q_\varepsilon(\theta) = M + 2i_N(\theta) - N < 0.$$

The Nth iterates for $U_{N,M}(\theta)$ now is

$$F^N(U_{N,M}(\theta)) = [0,1]^2.$$

In particular,

$$F^N : I_N(\theta) \times [0, \delta^{M - Q_\varepsilon(\theta)}] \to [0,1] \times [0,1],$$

and

$$F^N : I_N(\theta) \times [\delta^{M - Q\varepsilon(\theta)}, \delta^M] \to [0,1] \times \{1\} = A_1.$$

To compute the proportion of B_0 that is in the $U_{N,M}(\theta)$, we use (33) to compute L_0^+ from level $\delta^0 = 1$ up to δ^k (for $k = 1, \ldots, \infty$);

$$L_0^+ = \ell_2(F^N(U_{N,M}(\theta) \cap B_0)) = \sum_{k=1}^{\infty} L_k = 1 - \left(\frac{1-\delta}{1-\tilde{\delta}}\right)\left(\frac{\tilde{\delta}}{\delta}\right).$$

Then the proportion is

$$\begin{aligned}
\Sigma_\varepsilon(\theta,0) &= \frac{\ell_2(U_{N,M-Q_\varepsilon(\theta)}(\theta) \cap B_0)}{\ell_2(U_{N,M}(\theta))}, \\
&= \frac{\ell_2(U_{N,M-Q_\varepsilon(\theta)}(\theta) \cap B_0)}{\ell_2(U_{N,M-Q_\varepsilon(\theta)}(\theta))} \cdot \frac{\ell_2(U_{N,M-Q_\varepsilon(\theta)}(\theta)}{\ell_2(U_{N,M}(\theta))}, \\
&= \frac{\ell_2(F^N(U_{N,M-Q_\varepsilon(\theta)}(\theta) \cap B_0))}{\ell_2(F^N(U_{N,M-Q_\varepsilon(\theta)}(\theta))} \cdot \frac{I_N(\theta) \times \delta^{M-Q_\varepsilon(\theta)}}{I_N(\theta) \times \delta^M}, \\
&= \frac{L_0^+}{1}\delta^{-Q_\varepsilon(\theta)}, \\
&= L_0^+\delta^{-Q_\varepsilon(\theta)}, \\
&= \hat{K}\delta^{-Q_\varepsilon(\theta)},
\end{aligned} \tag{40}$$

where $\hat{K} = L_0^+ = 1 - \left(\frac{1-\delta}{1-\tilde{\delta}}\right)\left(\frac{\tilde{\delta}}{\delta}\right)$. Note that from the above $\ell_2(U_{N,M-Q_\varepsilon(\theta)}(\theta) \cap B_0) \approx \ell_2(F^N(U_{N,M-Q_\varepsilon(\theta)}(\theta) \cap B_0))$ and $\ell_2(U_{N,M-Q}(\theta)) \approx \ell_2(F^N(U_{N,M-Q_\varepsilon(\theta)}(\theta))$ since F^N is linear and invertible on $U_{N,M}(\theta)$. It is clear from (40) that $\Sigma_\varepsilon(\theta,0)$ does not converge to 1, i.e. we can show that

$$\Sigma_\varepsilon(\theta,0) = \tilde{\Theta}(\varepsilon^{\sigma_-(\theta,0)}).$$

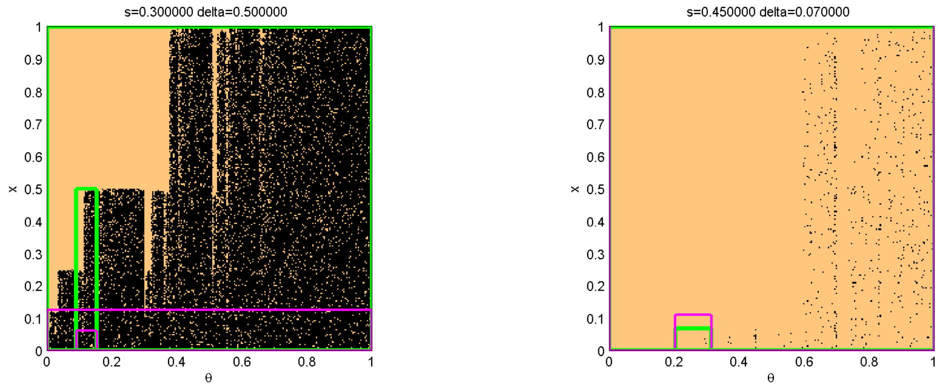
To prove this, we use the values of $Q_\varepsilon(\theta)$ in (37) into (40);

$$\begin{aligned}
\Sigma_\varepsilon(\theta,0) \approx \hat{K}\delta^{-Q_\varepsilon(\theta)} &= \hat{K}e^{\log\delta^{-Q_\varepsilon(\theta)}}, \\
&= \hat{K}e^{-Q_\varepsilon(\theta)\log\delta}, \\
&= \hat{K}e^{-\log\delta\frac{\log\varepsilon}{\log\delta}\left(\frac{\lambda_\parallel(\theta)-\lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right)}, \\
&= \hat{K}e^{-\left(\frac{\lambda_\parallel(\theta)-\lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right)\log\varepsilon}, \\
&= \hat{K}e^{\log\varepsilon^{-\left(\frac{\lambda_\parallel(\theta)-\lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right)}}, \\
&= \hat{K}\varepsilon^{-\left(\frac{\lambda_\parallel(\theta)-\lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right)}.
\end{aligned}$$

Therefore we have

$$\sigma_-(\theta,0) = -\left(\frac{\lambda_\parallel(\theta)-\lambda_\perp(\theta)}{\lambda_\parallel(\theta)}\right).$$

By Lemma 1.2(c)), $1-\Sigma_\varepsilon \to 1$ as $\varepsilon \to 0$ and this implies that $\sigma_+(\theta,0) = 0$. Thus as long



(a) (b)

Figure 2. Numerical approximations of the partition of $[0,1]^2$ into $\mathcal{B}(A_0)$ (dark) and $\mathcal{B}(A_1)$ (light) for two choices of s, δ. (a) shows a typical situation in the case $s < \delta$ ($s = 0.3, \delta = 0.5$). A neighbourhood of a generic point which will have positive stability index: a small square neighbourhood (grey) is mapped onto a strip that samples the bottom of the attractor by an iterate of the map; the rectangle (white) is mapped exactly onto $[0,1]^2$ by the same iterate of the map. (b) shows a typical situation in the case $\delta < s$ ($s = 0.45, \delta = 0.07$). A neighbourhood of a point with negative stability index: a small square neighbourhood (grey) is mapped onto the whole of $[0,1]^2$ by an iterate of the map; the rectangle (white) is mapped exactly onto $[0,1]^2$ by the same iterate of the map.

as $Q_\varepsilon(\theta) < 0$, the stability index at point $(\theta, 0)$ now is

$$\sigma(\theta) = \sigma_+(\theta,0) - \sigma_-(\theta,0) = \left(\frac{\lambda_\|(\theta) - \lambda_\perp(\theta)}{\lambda_\|(\theta)}\right),$$

since $\sigma_+(\theta,0) = 0$. For this case, as $Q_\varepsilon(\theta) \to -\infty$ and as $\varepsilon \to 0$, we have from (37) that $\lambda_\|(\theta) - \lambda_\perp(\theta) < 0$ i.e. $\lambda_\perp(\theta) > \lambda_\|(\theta)$. This index now is always negative.

We show the numerical approximation of the basins for both cases (i) and (ii) along with a small purple square that is mapped onto a “strip” and a small green rectangle that is mapped onto $[0,1]^2$ in Figure 2.

3.2. *Proof of Theorem 2.3*

(i) If $0 < \delta < 1$ and $0 < s < 1/2$, then for ℓ_1-almost all θ we have from (14) and (15) that

$$\lambda_\|(\theta) = -s\log s - (1-s)\log(1-s) > 0, \quad \lambda_\perp(\theta) = (1-2s)\log\delta < 0.$$

So

$$\begin{aligned}\sigma(\theta) &= \frac{\log s - \log(1-s)}{\log\delta}\left(\frac{\lambda_\|(\theta) - \lambda_\perp(\theta)}{\lambda_\|(\theta)}\right),\\ &= \frac{\log s - \log(1-s)}{\log\delta}\cdot\frac{-s\log s - (1-s)\log(1-s) - (1-2s)\log\delta}{-s\log s - (1-s)\log(1-s)} > 0,\end{aligned}$$

where $\frac{\log s - \log(1-s)}{\log\delta} > 0$. Then $\lambda_\|(\theta) - \lambda_\perp(\theta) > 0$ and therefore we will always have $\sigma(\theta) > 0$.

(ii) If $\lambda_{||}(\theta) < \lambda_{\perp}(\theta)$ and $t := \int_0^s d\mu(\theta)$ for some $\mu \in \mathcal{E}(T_s)$, then

$$\begin{aligned}\sigma(\theta) &= \left(\frac{\lambda_{||}(\theta) - \lambda_{\perp}(\theta)}{\lambda_{||}(\theta)}\right), \\ &= \frac{-t\log s - (1-t)\log(1-s) - (1-2t)\log\delta}{-t\log s - (1-t)\log(1-s)}.\end{aligned}$$

We wish to find a μ such that $\sigma(\theta) < 0$ for μ-almost all θ. Note that

$$\inf_{0<t<1}(\lambda_{||}(\theta) - \lambda_{\perp}(\theta)) = -\log s + \log\delta.$$

So, if $-\log s + \log\delta < 0$, we have $\log\delta < \log s$ if and only if $\delta < s$. Then this means that there are θ with $\sigma(\theta) < 0$.

3.3. *Proof of Theorem 2.4*

We consider neighbourhoods of A_0 of the form

$$U_{0,M} = \{(\tilde{\theta}, x) : \tilde{\theta} \in I_0(\theta), x < \delta^M\},$$

where $U_{0,M} \approx B_\varepsilon(A_0)$ which also satisfies at $\delta^M = \varepsilon$, i.e. $M = \frac{\log\varepsilon}{\log\delta}$ and so

$$V_m = \ell_2(B_\varepsilon(A_0)) = \ell_2(U_{0,M}) = 1 \times \delta^M = \delta^M.$$

To determine the measure of B_0 that is in $U_{0,M}$, we use (33) to find the area from level δ^M to δ^k (for $k = M+1, \ldots, \infty$) which gives

$$L_m^+ = \ell_2(B_\varepsilon(A_0) \cap B_0) = \sum_{k=M+1}^{\infty} L_k, = \delta^M - \left(\frac{1-\delta}{1-\tilde{\delta}}\right) \cdot \frac{\tilde{\delta}^{M+1}}{\delta}.$$

Then the proportion is

$$\Sigma_\varepsilon(A_0) = \frac{L_m^+}{V_m} = \frac{\delta^M - \left(\frac{1-\delta}{1-\tilde{\delta}}\right) \cdot \frac{\tilde{\delta}^{M+1}}{\delta}}{\delta^M} = 1 - \left(\frac{1-\delta}{1-\tilde{\delta}}\right)\left(\frac{\tilde{\delta}}{\delta}\right)^{M+1}.$$

From the above, as $M \to \infty$ and since $\tilde{\delta} < \delta$, $\Sigma_\varepsilon(\theta, 0)$ converges to 1 and by Lemma 1.2(b) this implies that $\sigma_-(A_0) = 0$. Therefore

$$1 - \Sigma_\varepsilon(A_0) = \left(\frac{1-\delta}{1-\tilde{\delta}}\right)\left(\frac{\tilde{\delta}}{\delta}\right)^{M+1} = \overline{K}\varepsilon^{\sigma_+(A_0)},$$

where $\overline{K} = \left(\frac{1-\delta}{1-\tilde{\delta}}\right) e^{(\log\tilde{\delta} - \log\delta)}$. We further have

$$\sigma_+(A_0) = \frac{\log\tilde{\delta} - \log\delta}{\log\delta} = \frac{\log s - \log(1-s)}{\log\delta}.$$

Since $\sigma_-(A_0) = 0$ then

$$\sigma(A_0) = \frac{\log s - \log(1-s)}{\log \delta} \tag{41}$$

3.4. *Proof of Theorem 2.6*

Let us define

$$\bar{t} = \limsup_{N\to\infty} \frac{i_N(\theta)}{N}, \quad \underline{t} = \liminf_{N\to\infty} \frac{i_N(\theta)}{N}.$$

According to (38) we have

$$\frac{\log(1-\Sigma(\theta,0))}{\log \varepsilon} = \log \tilde{K} + \frac{Q_\varepsilon(\theta)}{\log \varepsilon}\left(\frac{\log s - \log(1-s)}{\log \delta}\right). \tag{42}$$

Let us denote

$$q(\theta) := \lim_{\varepsilon\to 0} \frac{Q_\varepsilon(\theta)}{\log \varepsilon},$$

where $Q_\varepsilon(\theta)$ is as in (37). If $q(\theta)$ does not exist, then $\sigma(\theta)$ does not exist either. In particular, if we denote

$$\bar{q}(\theta) := \limsup_{\varepsilon\to 0} \frac{Q_\varepsilon(\theta)}{\log \varepsilon}, \quad \underline{q}(\theta) := \liminf_{\varepsilon\to 0} \frac{Q_\varepsilon(\theta)}{\log \varepsilon},$$

then from (42) we have either

$$\limsup_{\varepsilon\to 0} \frac{\log(1-\Sigma(\theta,0))}{\log \varepsilon} = \log \tilde{K} + \bar{q}(\theta)\left(\frac{\log s - \log(1-s)}{\log \delta}\right),$$

or

$$\liminf_{\varepsilon\to 0} \frac{\log(1-\Sigma(\theta,0))}{\log \varepsilon} = \log \tilde{K} + \underline{q}(\theta)\left(\frac{\log s - \log(1-s)}{\log \delta}\right).$$

This means that $\sigma_+(\theta,0)$ oscillates between the lim inf and lim sup. The same property holds for $\sigma_-(\theta,0)$. Thus, $\sigma(\theta)$ does not exist when the $\lim_{N\to\infty} \frac{i_N(\theta)}{N}$ does not exist.

To show the density of points for which the index fails to converge, note that Jordan *et al.* [17] explicitly construct points with non-convergence for $i_N(\theta)/N$: they do this by considering points θ for the doubling map with coding of the form

$$\{n_k(\theta) \ : \ k = 0,1,2,\ldots\} = \{0,1,1,0,0,0,0,1,1,1,1,1,1,1,1,\ldots\}.$$

where 2^{2k-1} zeroes are followed by 2^{2k} ones and then 2^{2k+1} zeros. For this point, the limit set of $i_N(\theta)/N$ is $[1/3, 2/3]$. Clearly the set of preimages of this point is dense set in $[0,1]$ which gives the result.

4. Discussion

This paper investigates the local geometry of the riddled basins of attraction for a piecewise linear skew product example by using the stability index. We explicitly compute the local and global stability index for the riddled basin attractor. For the map that we considered, Theorem 2.3 shows that, depending on the values of parameters δ and s, we have an attractor with a riddled basin where

(i) For Lebesgue almost all points in the attractor, the typical stability index is positive,
(ii) For some parameters there are points in the invariant set that have negative stability index (in particular $\delta < s$).

We show that for some points the stability index does not converge and give necessary and sufficient conditions for the non-convergence of the stability index in Theorem 2.6. Note that Corollary 2.5 states in this case, that the stability index of a point can be computed in terms of Lyapunov exponents and the global stability index of the attractor (where the global stability index plays an analogous role to Loynes' exponent in [14]).

As the computations of stability indices in this paper are for a particular example of a piecewise linear map, it would be of interest to see whether the results can be generalized to understand stability index, for example in non-skew products with riddled basin attractors. We note that, unlike [11] where only eigenvalues are needed to compute the local stability index, for this more general case we also need information about the global stability index.

4.1. *Acknowledgements*

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