On the effective strain tensor in heterogeneous materials

Berger, MA*, Evans, KE*, and Smith, CW*

April 25, 2013

Abstract

This paper considers effective strain tensors within the context of linear elastic equilibrium theory. The elastic properties of structured materials are often averaged over sub-volumes of various scales inside the material. For sub-volumes smaller than a representative volume element, Simple volume averaging of the stress and strain may not preserve the elastic energy. We introduce an averaging process which preserves the energy for all boundary conditions. This averaging process emphasizes the parts of the material which carry the most stress. Here the effective strain is weighted by the local stress, and can be interpreted as an average strain over all paths taken by loads and forces through the volume. This alternative effective strain may be especially appropriate for materials with voids such as foams and granular matter, as the averaging only involves the material itself. For uniform boundary conditions the weighted strain matches the volume averaged strain.

This paper investigates the properties of this weighted strain tensor. First, a simple interpretation of the tensor appears in the principal axis system of the effective stress. For each path taken by loads and forces through the volume we can measure a net length as well as a net extension due to the linear deformation. The weighted effective strain equals the ratio of average length to average extension, where the averaging is over all possible force paths. Thus this method provides a connection to load path analysis.

*CEMPS, U. of Exeter, Exeter EX4 4QF U.K.
m.berger@exeter.ac.uk

Secondly, even when the average rotation within the sub-volume is zero, there may be local fluctuations in the rotation field. These rotations can act like a mechanism, transferring elastic energy between boundaries or degrees of freedom. The effective strain defined here highlights this mechanism effect.

Keywords: linear elasticity, effective tensors, representative volume elements, heterogeneous materials

Contents

1 Introduction

We consider the properties of effective elastic tensors in the context of linear equilibrium theory. In heterogeneous materials, such as granular matter, composites, and foams, it is often desirable to determine the elastic parameters of substructures within the material. The stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\epsilon}$ are averaged over small regions inside a sample, leading to the calculation of effective moduli such as the Young's modulus or Poisson's ratio. In order to most accurately predict the structural properties of the material, the averaging process should faithfully reproduce the correct strain energy function U

inside the region, as well as energy transfers through the region. This may not happen for an arbitrary subvolume, when boundary conditions on the subvolume are not uniform. Representative volume elements (RVEs) are subvolumes which are large enough so that local fluctuations can be smoothed away to a good approximation [Hill, 1963, Nemat-Nasser and Hori, 1998]. For RVEs volume averages of the stress and strain tensors do reproduce the strain energy. However, for smaller subvolumes, fluctuations may introduce significant errors in the energy calculations. In this paper we define effective stress and strain tensors which exactly reproduce the energetics. The effective strain tensor introduced here employs a weighted averaging procedure which emphasizes the volume elements which carry the most stress.

The expressions for weighted effective strain also help to distinguish processes in the material which lead to energy storage from processes which transfer energy between degrees of freedom. The motions corresponding to this energy transfer will contribute to experimental measurements of the strain, even though they do not add to the strain energy density.

In engineering theory, a *mechanism* is a collection of rigid bodies joined together to transform a set of input forces into output forces. Inside a material, however, elements of the material move partly as a rigid body, and partly due to deformation. Mathematically we can decompose the total displacements inside the material into local rotations and strains. These contribute in qualitatively different ways to the structural properties of the material. The mechanism part (local rotations) can transform a load in one direction into forces in another direction. Only the deformation part (local strain) contributes to the internal elastic energy. The methods to be introduced here highlight these differences. Cosserat theory [Cosserat and Cosserat, 1909, Li and Liu, 2009] posits extra internal degrees of freedom corresponding to local rotations. The averaging procedure we describe does not have these internal variables, but has some of the same features, as local rotations receive careful attention.

Linear elasticity theory assumes a linear relation between the effective stress and strain. We can express this relation as $\overline{\boldsymbol{\epsilon}} = \overline{\boldsymbol{S}} \cdot \overline{\boldsymbol{\sigma}}$ where $\overline{\boldsymbol{S}}$ is the compliance tensor. When part of the effective strain arises from mechanism-like motions, interesting phenomena emerge. In particular, as these motions do not contribute to the strain energy function, the form of the effective compliance tensor is less constrained than that of the local compliance tensor. While the compliance tensor as a whole has diagonal (major) symmetry, it can be decomposed into parts which do not have this symmetry. In situations where the mechanism term is dominant (i.e. energy transfer dominates over energy storage) the compliance tensor can behave as though it is nearly anti-symmetric. Presence of the mechanism term may enhance the possibility of producing *auxetic materials*, e.g materials with negative Poisson's ratio [Lakes, 1987, Evans, 1989, Evans et al., 1994, 1995, Grima et al., 2006, Scarpa et al., 2008, Koenders and Gaspar, 2008].

Divergence-free vector fields are ubiquitous in fluid mechanics and electromagnetic theory; however they are less commonly used in solid mechanics. The stress tensor (in the absence of body forces) has zero divergence, which allows the construction of three divergence-free vector fields (e.g. $\boldsymbol{\sigma} \cdot \hat{x}$, $\boldsymbol{\sigma} \cdot \hat{y}$, and $\boldsymbol{\sigma} \cdot \hat{z}$). The integral curves of these fields represent the flow lines of forces through the material. These curves differ from load paths as conventionally defined [Kelley and Elsley, 1995, Berger, 2012], but can be of great utility in structural mechanics [Marhadi and Venkataraman, 2009]. Here we present a different use: the effective strain defined here can be interpreted as a ratio of mean extension to mean length, but where extensions and lengths are averaged over all the paths taken by the force flows.

In Section 2 we discuss the energy preserving effective strain, and how it decomposes into terms involving local strains and local rotations. In section 3 we express the effective strain in terms of averaging over force paths. Section 4 provides an example calculation for regular honeycombs, with conclusions in section 5.

We will sometimes employ Voigt notation where the stress and strain tensors are represented as vectors, e.g. for two dimensional materials the diagonal components are $(\epsilon_1, \epsilon_2) = (\epsilon_{11}, \epsilon_{22})$, and $(\sigma_1, \sigma_2) = (\sigma_{11}, \sigma_{22})$. Also, simple volume averages will be denoted by an overbar, as in $\overline{\epsilon}$. Engineering strains will be denoted by $\overline{\boldsymbol{e}}$, while the effective averages introduced here will be denoted by a curly bar, as in $\widetilde{\epsilon}$.

2 Energy preserving effective stress and strain tensors

2.1 definitions

In linear elasticity theory the strain energy function $U = \frac{1}{2}\overline{\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}}$. The *Hill-Mandel* lemma in homogenization theory [Hill, 1963, Nemat-Nasser and Hori, 1998, Li and Liu, 2009] states that for homogeneous boundary conditions on

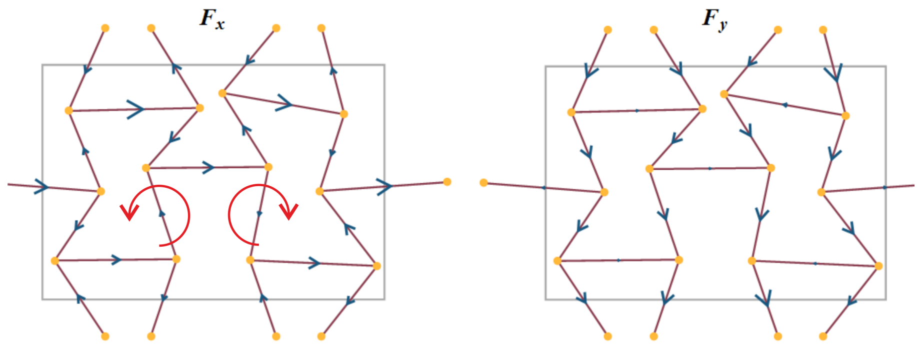

Figure 1: The flow of forces in a disordered honeycomb with Poisson's ratio $\nu_{12} = -0.54$.The figure shows the mean load paths in each rib of a hexagonal asymmetric honeycomb. The load paths also have small scale ripples and loops within each rib of the honeycomb, due to bending forces; this small scale structure is not shown. The honeycomb has fixed boundaries, which have been compressed slightly from the unstressed state (see [Berger, 2012] for details). The left diagram show the directions of $\boldsymbol{F}_{\mathrm{x}} = \hat{x} \cdot \boldsymbol{\sigma}$, while the right diagram show $\boldsymbol{F}_{\mathrm{y}} = \hat{y} \cdot \boldsymbol{\sigma}$. The size of the arrows is proportional to the strength of the forces. The circular arrows show the directions of rotation of two ribs for a vertical compression; note that the sign of rotation correlates with σ_{12} (see equation (11)).

a volume, $\overline{\boldsymbol{\sigma}} \cdot \overline{\boldsymbol{\epsilon}} = \overline{\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}}$; this is not true in general, however. Simple volume averages of the stress and strain do not faithfully reproduce the strain energy function for inhomogeneous boundary conditions, which may be especially relevant to disordered solids. (The product $\overline{\boldsymbol{\sigma}} \cdot \overline{\boldsymbol{\epsilon}}$ can even be negative; for example if $\mathcal{V}$ contains a fibre under compression, $\sigma_1 = -1$, $\epsilon_1 = -2$, and a stiffer fibre under tension, $\sigma_1 = 2$, $\epsilon_1 = 1$.)

These difficulties may be circumvented for certain special boundary conditions. If the boundary forces allow a uniform stress solution or a uniform strain solution, then energy is preserved by volume averaging. Often a *representative volume element* (RVE) is sought which typifies the medium or large scale response of the solid, and for which such special boundary conditions are possible. For disordered solids, however, such RVEs may be difficult to find, and methods must be used to deal with the error in energy averaging. If the disorder has specified statistical properties, then *stochastic volume elements* can be employed, where the effects of non-uniform boundary conditions can be predicted and controlled [Ostoja-Starzewski, 2007].

We seek a different averaging method which automatically gives the correct energy deposition U for *arbitrary boundary conditions* on the volume $\mathcal{V}$. Let us call a pair of effective tensors $\widetilde{\sigma}$ and $\widetilde{\epsilon}$ *energy preserving* if

$$\widetilde{\boldsymbol{\sigma}} \cdot \widetilde{\boldsymbol{\epsilon}} = \overline{\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}}. \tag{1}$$

If we can achieve this, then we can average over small scale structure without affecting energy calculations. In this way, the structural properties of the system will be most closely realized by the effective tensors.

We start with a volume $\mathcal{V}$ where the local stress and strain tensors are known everywhere, and satisfy the linear static equilibrium equations. We first define the *energy transfer tensor*

$$T_{ij} = \frac{1}{2\mathcal{V}} \int_{\mathcal{V}} \sigma_{ik} \frac{\partial u_j}{\partial x_k} \, \mathrm{d}^3 x, \tag{2}$$

where $\mathcal{V} = \mathcal{V}(\tau)$, $\sigma_{ik} = \sigma_{ik}(\tau)$, etc. We then construct an energy preserving pair by simply letting the effective stress be the volume averaged mean stress, $\widetilde{\sigma} \equiv \overline{\sigma}$, then defining an *effective deformation tensor*

$$\widetilde{(\nabla \boldsymbol{u})}_{ij} \equiv 2\widetilde{\sigma}^{-1}_{ik} T_{kj} = \mathcal{V}^{-1} \widetilde{\sigma}^{-1}_{ik} \int_{\mathcal{V}} \sigma_{k\ell} (\nabla u)_{\ell j} \, \mathrm{d}^3 x. \tag{3}$$

The effective strain satisfying equation (1) is the symmetric part

$$\widetilde{\epsilon}_{ij} = \frac{1}{2} \left(\widetilde{(\nabla \boldsymbol{u})}_{ij} + \widetilde{(\nabla \boldsymbol{u})}_{ji} \right). \tag{4}$$

The basic property of energy preservation (equation 1) can be immediately verified.

2.2 Boundary integrals

The effective tensors can be transformed into boundary integrals. As the equilibrium stress is divergence-free in the absence of body forces,

$$T_{ij} = \frac{1}{2\mathcal{V}} \oint_{\partial \mathcal{V}} n_k \sigma_{ik} u_j \, \mathrm{d}^3 x. \tag{5}$$

This tensor can be related to the energy transfer in the buildup to the stressed state, provided that the buildup takes a simple form. We assume here that the evolution is self-similar, in the sense that the time-dependent stress $\sigma(\boldsymbol{x}, t)$ can be separated into the final stress $\sigma(\boldsymbol{x})$ and a time function $g(t)$ (where $g(0) = 0$ and $g(t_{final}) = 1$):

$$\sigma(\boldsymbol{x}, t) = \sigma(\boldsymbol{x})g(t). \tag{6}$$

We will define energy transfer in the principal axis frame of the mean stress $\overline{\sigma}$ (where $\widetilde{\sigma} = \overline{\sigma}$ is diagonal). In this frame, the work done by the x component of the force is given by $W = T_{11}$. While much of this work may go into internal elastic energy $U = T_{ii}$, some may pass through the sample, exiting through the other degree of freedoms. The energy transfer is given by $X = W - U = -(T_{22} + T_{33})$. If σ and ϵ were uniform in $\mathcal{V}$, then we would have

$$X = -\frac{1}{2}\left(\overline{\sigma}_{2k}\overline{\epsilon}_{k2} + \overline{\sigma}_{3k}\overline{\epsilon}_{k3}\right). \tag{7}$$

An important feature of the effective tensors $(\widetilde{\sigma}, \widetilde{\epsilon})$ is that, even with small scale inhomogeneities, the form of equation (7) for the energy transfer is preserved: $X = -\frac{1}{2}\left(\widetilde{\sigma}_{2k}\widetilde{\epsilon}_{k2} + \widetilde{\sigma}_{3k}\widetilde{\epsilon}_{k3}\right)$.

2.3 Local strain and local rotation

Locally, $\nabla u = \boldsymbol{\epsilon} + \boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the rotation tensor. The effective strain $\widetilde{\boldsymbol{\epsilon}}$ decomposes into a 'structural' contribution $\widetilde{\boldsymbol{\epsilon}}_S$ arising from small scale strains, and a 'mechanism' contribution $\widetilde{\boldsymbol{\epsilon}}_M$ arising from small scale rotations: $\widetilde{\boldsymbol{\epsilon}} = \widetilde{\boldsymbol{\epsilon}}_S + \widetilde{\boldsymbol{\epsilon}}_M$, where

$$\mathcal{V}\widetilde{\epsilon}_{S\,ij} = (\widetilde{\sigma}^{-1})_{ik}\int \sigma_{k\ell}\epsilon_{\ell j}\ \mathrm{d}^3x + (\widetilde{\sigma}^{-1})_{jk}\int \sigma_{k\ell}\epsilon_{\ell i}\ \mathrm{d}^3x; \tag{8}$$

$$\mathcal{V}\widetilde{\epsilon}_{M\,ij} = (\widetilde{\sigma}^{-1})_{ik}\int \sigma_{k\ell}\omega_{\ell j}\ \mathrm{d}^3x + (\widetilde{\sigma}^{-1})_{jk}\int \sigma_{k\ell}\omega_{\ell i}\ \mathrm{d}^3x. \tag{9}$$

Note that (because $\sigma_{ij}\omega_{ij} = 0$) only $\widetilde{\boldsymbol{\epsilon}}_S$ contributes to U, i.e.

$$\widetilde{\boldsymbol{\sigma}} \cdot \widetilde{\boldsymbol{\epsilon}}_S = U; \qquad \widetilde{\boldsymbol{\sigma}} \cdot \widetilde{\boldsymbol{\epsilon}}_M = 0. \tag{10}$$

With the two-dimensional honeycomb example presented below, the total rotation is zero; however, the local rotations of each rib correlate with the stress tensor. In particular, quantities like

$$\widetilde{\epsilon}_{M\,11} = \frac{1}{\overline{\sigma}_{11}\mathcal{V}}\int \sigma_{12}\omega_{21}\ \mathrm{d}^3x \tag{11}$$

will be non-zero. Remarkably, local rotations add to the effective strain.

2.4 Uniform boundary conditions

For certain boundary conditions, the weighted effective strain $\widetilde{\boldsymbol{\epsilon}}$ equals the volume averaged strain $\overline{\boldsymbol{\epsilon}}$. First, suppose we apply boundary conditions which allow the material sample to have a uniform stress, i.e. locally $\sigma_{ij} = \overline{\sigma}_{ij}$. Then the stress can be taken out of the integrals in equations (8) and (9). The result is that $\widetilde{\boldsymbol{\epsilon}} = \widetilde{\boldsymbol{\epsilon}}_S = \overline{\boldsymbol{\epsilon}}$ with $\widetilde{\boldsymbol{\epsilon}}_M = 0$. These equations also apply if boundary conditions allow the strain to be uniform. One may note that these resultsare consistent with the Hill-Mandel lemma, as the weighted stress and strain exactly reproduce the elastic energy, but so do the volume averaged stress and strain under these special boundary conditions.

We can also have $\widetilde{\boldsymbol{\epsilon}} = \overline{\boldsymbol{\epsilon}}$ in the presence of significant fluctuations, provided that the fluctuations average to zero, as in an RVE. Interestingly, however, in general $\widetilde{\boldsymbol{\epsilon}}_M \neq 0$ in these situations. Thus the effect of local rotations may be clearly seen in the decomposition $\widetilde{\boldsymbol{\epsilon}} = \widetilde{\boldsymbol{\epsilon}}_S + \widetilde{\boldsymbol{\epsilon}}_M$, providing insight into the mechanisms behind the response of the material to stresses. A honeycomb example displaying this phenomemon is presented in section (4).

2.5 The compliance tensor

For the uniform boundary conditions described above, we can define a linear relation $\widetilde{\boldsymbol{\epsilon}} = \boldsymbol{S} \cdot \widetilde{\boldsymbol{\sigma}}$ in parallel with the usual constitutive equations for the volume averaged tensors, where $\boldsymbol{S}$ is the compliance tensor. Thus we have $U = \frac{1}{2}\widetilde{\boldsymbol{\sigma}} \cdot \boldsymbol{S} \cdot \widetilde{\boldsymbol{\sigma}} = \frac{1}{2}\overline{\boldsymbol{\sigma}} \cdot \boldsymbol{S} \cdot \overline{\boldsymbol{\sigma}}$. The compliance tensor $\boldsymbol{S}$ is symmetric to insure thermodynamic reversibility (e.g. Makinson [1955]). But what about the individual terms $\widetilde{\boldsymbol{\epsilon}}_S$ and $\widetilde{\boldsymbol{\epsilon}}_M$? We caution that linear relations may not be valid for the individual terms, for all possible boundary conditions. However, the zero energy condition for $\widetilde{\boldsymbol{\epsilon}}_M$, equation (10), provides an important constraint on any linear approximation within some subset of boundary conditions.

Suppose we are interested in a particular set of boundary conditions where $\widetilde{\boldsymbol{\epsilon}}_M$ is approximately a linear function of the applied stresses. In particular, we consider some cone-shaped region in the vector space of possible stresses. In this region we define a linearized compliance tensor $\boldsymbol{S}_M$ for $\widetilde{\boldsymbol{\epsilon}}_M$: $\widetilde{\boldsymbol{\epsilon}}_M = \boldsymbol{S}_M \cdot \widetilde{\boldsymbol{\sigma}}$. Then

$$0 = \widetilde{\boldsymbol{\sigma}} \cdot \widetilde{\boldsymbol{\epsilon}}_M = \widetilde{\boldsymbol{\sigma}} \cdot \boldsymbol{S}_M \cdot \widetilde{\boldsymbol{\sigma}}. \tag{12}$$

In order for this to be true for arbitrary stresses within the region of interest in stress space, the compliance tensor $\boldsymbol{S}_M$ must be *antisymmetric*,

$$S_{M\,ji} = -S_{M\,ij}. \tag{13}$$

The tensor $\boldsymbol{S}_S$ can then be written as a mixed tensor so that the total compliance $\boldsymbol{S}$ is symmetric.

Thus in physical regimes where the mechanism term dominates (i.e. $\boldsymbol{S}_S \cdot \widetilde{\epsilon} \approx 0$, the symmetric total compliance tensor $\boldsymbol{S}$ can behave (in terms of its action as a linear operator) as though it is nearly anti-symmetric (see honeycomb example below).

What happens when the boundary conditions are not uniform? For example, suppose we consider a volume too small to be an RVE, but nevertheless are interested in how the mean quantities behave. In this case, the quadratic quantity $\frac{1}{2}\overline{\boldsymbol{\sigma}} \cdot \boldsymbol{S} \cdot \overline{\boldsymbol{\sigma}}$ fails to give the correct energy [Nemat-Nasser and Hori, 1998, Ostoja-Starzewski, 2007]. We then have an additional term $\boldsymbol{\delta}\widetilde{\epsilon}$ arising from the fluctuations:

$$
\begin{aligned}
\widetilde{\boldsymbol{\epsilon}} &= \boldsymbol{S} \cdot \widetilde{\boldsymbol{\sigma}} + \boldsymbol{\delta}\widetilde{\boldsymbol{\epsilon}}; && (14)\\
U &= \frac{1}{2}\widetilde{\boldsymbol{\sigma}} \cdot \boldsymbol{S} \cdot \widetilde{\boldsymbol{\sigma}} + \frac{1}{2}\widetilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\delta}\widetilde{\boldsymbol{\epsilon}}. && (15)
\end{aligned}
$$

Here $\boldsymbol{\delta}\widetilde{\epsilon}$ will be proportional to the correlations between the fluctuations in stress and strain due to the inhomogeneous boundary conditions.

3 Path averages

The effective strain equation (4) can be interpreted in terms of averages over all paths taken by loads and forces through the volume $\mathcal{V}$. Mean strains are commonly expressed as variants of the engineering strain, where, for example,

$$
\overline{e}_{xy} = \frac{\text{average displacement in x direction}}{\text{extension in y direction}}. \tag{16}
$$

We will average the displacements and extensions over all force paths, weighted by the strengths of each path. We show that in the principle axis coordinate system of the mean stress, this weighted engineering strain is given by equation (4). An application to cellular materials is given in [Berger, 2012].

Let $\boldsymbol{F}_{\mathrm{x}} = \hat{x} \cdot \sigma$ be the first row (or column) of the local stress tensor. In the absence of body forces $\boldsymbol{F}_{\mathrm{x}}$ is divergence free, $\nabla \cdot \boldsymbol{F}_{\mathrm{x}} = 0$. To visualize the flow of forces in a material, we can draw the field lines of $\boldsymbol{F}_{\mathrm{x}}$, $\boldsymbol{F}_{\mathrm{y}}$, and $\boldsymbol{F}_{\mathrm{z}}$ just as we can draw magnetic field lines or fluid streamlines. Note that these lines may differ from load paths when shear is present [Kelley and Elsley, 1995]. Figure 1 shows an example for a disordered honeycomb.

Let $\boldsymbol{a}$ be a point on the boundary of $\mathcal{V}$, and consider a boundary area element $\mathrm{d}^2 a$ surrounding $\boldsymbol{a}$. Let $\mathrm{d}\Phi_x(\boldsymbol{a}) = |\boldsymbol{F}_\mathrm{x}(\boldsymbol{a}) \cdot \hat{\boldsymbol{n}}|\ \mathrm{d}^2 a$ be the flux of $\boldsymbol{F}_\mathrm{x}$ through the area element. If $\boldsymbol{F}_\mathrm{x}(\boldsymbol{a}) \cdot \hat{\boldsymbol{n}} < 0$ then the x-force flows into the volume. It then follows a path through the volume to exit at an outflow point $\boldsymbol{b}(\boldsymbol{a})$. The total flux of $\boldsymbol{F}_\mathrm{x}$ into the volume will be called Φ_x.

The net length of the path in the y direction is $L_y(\boldsymbol{a}, \boldsymbol{b}) = b_2 - a_2$. We can calculate the average length of all the $\boldsymbol{F}_\mathrm{x}$ paths as

$$\langle L_y \rangle_x \equiv \frac{1}{\Phi_\mathrm{x}} \int (b_2 - a_2)\ \mathrm{d}\Phi_x(\boldsymbol{a}). \tag{17}$$

We can equally integrate over outflow points $\boldsymbol{b}$, or simply write

$$\begin{aligned} \langle L_y \rangle_x &= \frac{1}{\Phi_\mathrm{x}} \left(\int b_2\ \mathrm{d}\Phi_x(\boldsymbol{b}) - \int a_2\ \mathrm{d}\Phi_x(\boldsymbol{a}) \right) && (18) \\ &= \frac{1}{\Phi_\mathrm{x}} \oint y \boldsymbol{F}_\mathrm{x} \cdot \hat{n}\ \mathrm{d}^2 x. && (19) \end{aligned}$$

By the mean stress theorem, e.g. [Landau and Lifshitz, 1986], the integral in equation (19) equals $\mathcal{V}\overline{\sigma}_{12}$. In general we have

$$\langle L_j \rangle_i = \frac{\mathcal{V}}{\Phi_\mathrm{i}} \overline{\sigma}_{ij}. \tag{20}$$

We can obtain a similar result for displacements. Let the displacements at the entrance and exit points of a path be $\boldsymbol{u}(\boldsymbol{a})$ and $\boldsymbol{u}(\boldsymbol{b})$. The net size in the y direction changes by $\delta L_y = u_2(b) - u_2(a)$. The average path extension is

$$\langle \delta L_y \rangle_x = \frac{1}{\Phi_\mathrm{x}} \oint u_2 \boldsymbol{F}_\mathrm{x} \cdot \hat{n}\ \mathrm{d}^2 x = \frac{2\mathcal{V}}{\Phi_\mathrm{x}} T_{xy}. \tag{21}$$

In general $T_{ij} = \Phi_i \langle \delta L_j \rangle_i / (2\mathcal{V})$.

We can express the effective displacement tensor $\widetilde{\nabla \boldsymbol{u}}$, and hence $\widetilde{\epsilon}$ in terms of these path averages. In the principal frame of $\overline{\sigma}$ we have $\widetilde{\epsilon}_{11} = 2T_{11}/\overline{\sigma}_{11}$, etc. In two dimensions,

$$\widetilde{\nabla \boldsymbol{u}} = \begin{pmatrix} \langle \delta L_x \rangle_x / \langle L_x \rangle_x & \langle \delta L_y \rangle_x / \langle L_x \rangle_x \\ \langle \delta L_x \rangle_y / \langle L_y \rangle_y & \langle \delta L_y \rangle_y / \langle L_y \rangle_y \end{pmatrix}, \tag{22}$$

and similarly for three dimensions. Note that for uniform displacements on each boundary of a rectangular domain, displacements will be the same for each path; thus for this special situation path averaging gives the same result as volume averaging.

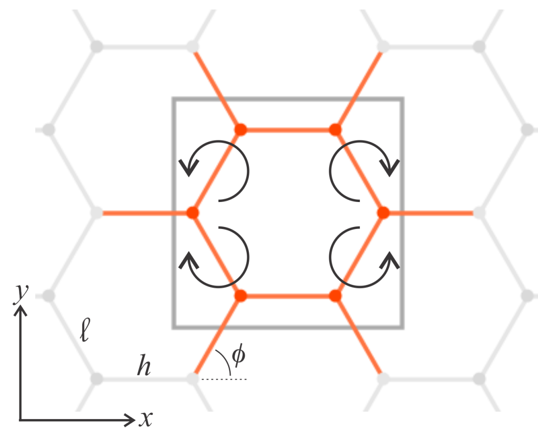

Figure 2: A cell in a two-dimensional honeycomb, consisting of a network of nodes and ribs. The equilibrium stress has nonzero local stress σ_{12} on the slanted ribs. If the honeycomb is compressed from the sides, these ribs will rotate. The product $\sigma_{12}\omega_{12}$ is the same for all of the slanted ribs.

4 A simple honeycomb system

We illustrate with a simple two-dimensional honeycomb network (as in figures 1 and 2). Consider a honeycomb Gibson et al. [1982], Warren and Kraynik [1987], Papka and Kyriakides [1994], Masters and Evans [1996], Gibson and Ashby [1997], Scarpa et al. [2008] consisting of thin ribs joined at nodes. Suppose that the honeycomb deforms due to hinging or flexure near the nodes. To a good approximation the ribs rotate with little change in length. The engineering strain thus arises principally from motions (rotations) which do not add to the strain energy; internal bending, shear, and axial loading of the ribs are present, but only provide only a small part of the observed strain. Here we explore this curious decoupling of mean (engineering) strain from the strain energy.

Because the engineering strain $\overline{\boldsymbol{e}}$ predominantly consists of rotations, loads applied in different directions lead to the same shape of strain tensors. A small squeeze in the x direction will lead to equilibrium boundary displacements in both directions. A small squeeze in the y direction results in the same ratio of boundary displacements. Thus the Poisson's ratio ν_{21} measured by squeezing in the y direction equals $1/\nu_{12}$, measured by squeezing in the x direction. Poisson's ratio can now take any positive or negative value [Lakes, 1987, Evans, 1989, Evans et al., 1994].

If we write $\overline{e_i} = S_{ij}\overline{\sigma}_j$ for $i, j = 1, 2$, then the usual form for the compliance tensor $\boldsymbol{S}$ is (e.g. Torquato [2001])

$$\boldsymbol{S} = \begin{pmatrix} E_1^{-1} & -\nu_{21}E_2^{-1} \\ -\nu_{12}E_1^{-1} & E_2^{-1} \end{pmatrix} \tag{23}$$

where E_1 and E_2 are effective Young's moduli in the x and y directions (generally smaller than the Young's moduli of the material within the ribs). With $\nu_{21} = 1/\nu_{12}$ and the symmetry relation $\nu_{21}E_2^{-1} = \nu_{12}E_1^{-1}$,

$$E_1\boldsymbol{S} = \begin{pmatrix} 1 & -\nu_{12} \\ -\nu_{12} & \nu_{12}^2 \end{pmatrix}. \tag{24}$$

Note that S is singular; the eigenvector with eigenvalue 0 has $\overline{\sigma}_1/\overline{\sigma}_2 = \nu_{12}$; here the stress vector $(\overline{\sigma}_1, \overline{\sigma}_2)$ is perpendicular to the strain vector $(\overline{e}_1, \overline{e}_2)$.

So far these considerations apply for ordered or disordered honeycombs. For a regular honeycomb, one can readily calculate the effective strain due to rotations $\widetilde{\boldsymbol{\epsilon}}_M$ (see Appendix):

$$\widetilde{\epsilon}_{M\,1} = \frac{\overline{\sigma}_2}{\overline{\sigma}_1}\nu_{12}\overline{e}_1; \tag{25}$$

$$\widetilde{\epsilon}_{M\,2} = \overline{e}_2. \tag{26}$$

For uniaxial deformation, $\overline{\sigma}_2 = 0$ and there is zero resistance to strains ϵ_2 in the y direction; thus the energy transfer $X = 0$. But we can also consider the opposite extreme, where the energy input W almost equals the energy transfer X, with internal deposition $U \to 0$ (some internal energy is needed to transfer forces). Suppose the external material resists expansion in the y direction. We apply a stress σ_1 to the left and right sides. On the top and bottom, assume an external resistive force $f_{ext} = -ku_2$ proportional to the displacement u_2 with force constant k. Let the volume have x and y sizes L_x and L_y, and define $\kappa = (k/E)(\nu_{12}L_y/L_x)$. For a material where the effective stiffness E_1 is much smaller then the external resistance k, we have $\kappa \gg 1$.

From equation (24) a simple calculation shows that the engineering strain becomes $E_1\overline{e}_1 = \overline{\sigma}_2/\kappa$, and $E_1\overline{e}_2 = -\nu_{12}\overline{\sigma}_1/(1 + \nu_{12}\kappa)$. From equation (25) and equation (26), this leads to the antisymmetric form

$$E_1\widetilde{\epsilon}_M = D^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\boldsymbol{\sigma}; \qquad D = \kappa + 1/\nu_{12}. \tag{27}$$

After some algebra, we can write $\boldsymbol{e} = \widetilde{\epsilon}$ as the mechanism term plus a second order correction:

$$\overline{e}_1 = \widetilde{\epsilon}_1 = \widetilde{\epsilon}_{M\,1} + \frac{1}{E_1\kappa(1+\nu_{12}\kappa)}; \tag{28}$$

$$\overline{e}_2 = \widetilde{\epsilon}_2 = \widetilde{\epsilon}_{M\,2}. \tag{29}$$

Note that to first order in κ^{-1} the engineering strain is perpendicular to the mean stress, $\boldsymbol{e} \cdot \overline{\boldsymbol{\sigma}} = 0$. The anti-symmetric tensor in equation (27) most simply and naturally expresses the near perpendicularity of $\boldsymbol{e}$ and $\overline{\boldsymbol{\sigma}}$ for high κ. In this regime, axial stresses within the ribs transmit forces, but shear stresses within the ribs are small, as most of the energy input W goes to energy transfer.

5 Conclusions

This paper presents an alternative method of averaging strain in heterogeneous materials. The method preserves the strain energy function for *arbitrary boundary conditions* on the stress and strain. The result can be interpreted as a mean engineering strain, where lengths and displacements are averaged over the paths of forces flowing through the volume. Path averaging emphasizes the parts of the material with the most influence on the distribution of stresses. Stress distributes itself primarily along paths of greatest stiffness, just as electric current favors paths of least resistance. Thus this method may be especially useful for materials with voids or composites with strongly varying elastic properties. This method also readily takes into account the resistance of the external medium to deformations of the boundary.

For some special boundary conditions, e.g. uniform boundary displacements or stresses, the effective strain $\widetilde{\boldsymbol{\epsilon}}$ presented here matches the volume averaged strain $\overline{\epsilon}$. However, conventional volume averaging hides the influence of local rotations, which can strongly affect the structure of heterogeneous materials. The path averaged strain $\widetilde{\boldsymbol{\epsilon}}$ presented here reveals this influence. An important feature of $\widetilde{\boldsymbol{\epsilon}}$ comes with the decomposition into the mechanism term $\widetilde{\boldsymbol{\epsilon}}_M$ and the structural term $\widetilde{\boldsymbol{\epsilon}}_S$. These terms have different physical roles and properties. The structural term $\widetilde{\boldsymbol{\epsilon}}_S$ is associated with elastic energy. The mechanism term $\widetilde{\boldsymbol{\epsilon}}_M$ arises from local rotations. Linearizations of this term can be described with an anti-symmetric compliance tensor. For some materials such as foams and honeycombs placed in an externally resistive medium, the mechanism term can in fact provide the dominant contribution to the experimentally measured strain.

6 Acknowledgements

M Berger gratefully acknowledges funding from the Leverhulme Trust, grant SH-04409.

Appendix

Here we calculate the mechanism term equation (11) for the regular honeycomb system of figure 2. Let $\langle\sigma_{12}\omega_{12}\rangle$ be the mean correlation between σ_{12} and ω_{12} within the ribs. From equation (11),

$$\widetilde{\epsilon}_{M\,1} = -\frac{\langle\sigma_{12}\omega_{12}\rangle}{\overline{\sigma}_1}; \tag{30}$$

$$\widetilde{\epsilon}_{M\,2} = \frac{\langle\sigma_{12}\omega_{12}\rangle}{\overline{\sigma}_2}. \tag{31}$$

We will need to calculate the correlation between σ_{12} and ω_{12}. Suppose during deformation the slanted ribs rotate through angles $\delta\phi = \omega_{12}$. Crucially, the signs of σ_{12} and ω_{12} are perfectly correlated; their product should always be positive inside a rib (and zero outside).

For example, suppose the honeycomb is stretched in the vertical direction. Then the flow of force F_y goes upwards. The diagonal term σ_{12} measure how much this flow moves to the left or right. A rib oriented with $0 < \phi < \pi/2$ will rotate anti-clockwise, making $\sigma_{12}\omega_{12}$ positive. The product is also positive for all other slanted ribs, as well as for compression in the vertical direction.

Let the ribs have thickness t and length ℓ. Then

$$\langle\sigma_{12}\omega_{12}\rangle = \frac{N_x N_y t\ell}{\mathcal{V}}|\sigma_{12}\omega_{12}|. \tag{32}$$

Here, the area $\mathcal{V} = L_x L_y$, where

$$L_x = N_x(h + \ell\cos\phi); \tag{33}$$

$$L_y = N_y \ell \sin\phi. \tag{34}$$

Thus

$$\langle \sigma_{12}\omega_{12}\rangle = \frac{t}{(h+\ell\cos\phi)\sin\phi}|\sigma_{12}\omega_{12}|. \tag{35}$$

Meanwhile, by flux conservation for the divergence-free field $\boldsymbol{F}_{\mathrm{y}}$,

$$\sigma_{12}\frac{t}{\cos\phi} = \pm\overline{\sigma}_2(h+\ell\cos\phi). \tag{36}$$

Also, the total vertical displacement $\delta L_y = N_y\ell\cos\phi\delta\phi$ so

$$\omega_{12} = \delta\phi = \frac{\delta L_y}{N_y\ell\cos\phi}. \tag{37}$$

Thus

$$\langle \sigma_{12}\omega_{12}\rangle = \frac{\delta L_y}{N_y\ell\sin\phi}\overline{\sigma}_2 = \frac{\delta L_y}{L_y}\overline{\sigma}_2. \tag{38}$$

We can now relate the effective strains to the engineering strains:

$$\widetilde{\epsilon}_{M\,1} = -\frac{\overline{\sigma}_2}{\overline{\sigma}_1}\overline{e}_2 = \frac{\overline{\sigma}_2}{\overline{\sigma}_1}\nu_{12}\overline{e}_1; \tag{39}$$

$$\widetilde{\epsilon}_{M\,2} = \overline{e}_2. \tag{40}$$

References

M A Berger. The flow of forces through cellular materials. *arXiv:1201.2259*, 2012.

E Cosserat and F Cosserat. *Theorie des Corps Deformables*. Hermann et Fils, Paris, 1909.

K E Evans. Tensile network microstructures exhibiting negative poisson's ratios. *J Physics D: Applied Phys.*, 22:1870–1876, 1989.

K E Evans, M A Nkansah, and I J Hutchinson. Auxetic foams: Modelling negative poisson's ratios. *Acta Metall Mater*, 42:1289–1294, 1994.

K E Evans, A Alderson, and F R Christian. Auxetic two-dimensional polymer networks. an example of tailoring geometry for specific mechanical properties. *J. Chem. Soc. Faraday Trans.*, 91:2671, 1995.

L J Gibson and M F Ashby. *Cellular Solids: Structure and Properties, 2nd ed.* Cambridge University Press, 1997.

L J Gibson, M F Ashby, G S Schajer, and C I Robertson. The mechanics of two dimensional cellular solids. *Proc. Royal Society London A*, 382:25–42, 1982.

J N Grima, R Gatt, N Ravirala, A Alderson, and K E Evans. Negative poisson's ratios in cellular foam materials. *Materials Science and Engineering A*, 423:214–218, 2006.

R Hill. Elastic properties of reinforced solids: some theoretical principles. *J. Mech. Phys. Solids*, 11:357–372, 1963.

D W Kelley and M Elsley. A procedure for determining load paths in elastic continua. *Engineering Computations*, 12:415–424, 1995.

M A Koenders and N Gaspar. The auxetic properties of a network of bending beams. *Phys. Stat. Sol. (b)*, 245:539–544, 2008.

R Lakes. Foam structure with a negative poisson's ratio. *Science*, 235:1038, 1987.

L D Landau and E M Lifshitz. *Theory of Elasticity.* Pergamon, 1986.

X Li and Q Liu. A version of hill's lemma for cosserat continuum. *Acta Mech Sin*, 25:499–506, 2009.

K R Makinson. Some speculations on possible instability phenomena in the mechanical behaviour of proteins. *Proc. Int. Wool Research Conf., Australia*, pages D54–D61, 1955.

K Marhadi and S Venkataraman. Comparison of quantitative and qualitative information provided by different structural load path definitions. *Int. J. Simul. Multidisci. Des. Optim.*, 3:384–400, 2009.

I G Masters and K E Evans. Formulae for the calculation and estimation of writhe. *Compos. Struct.*, 35:403–422, 1996.

S Nemat-Nasser and M Hori. *Micromechanics: Overall properties of heterogeneous materials.* Elsevier, 1998.

M Ostoja-Starzewski. *Microstructural randomness and scaling in mechanics of materials*. Chapman-Hall/CRC, 2007.

S D Papka and S Kyriakides. In-plane compressive response and crushing of honeycomb. *J Mech. Phys. Solids*, 42:1499, 1994.

F Scarpa, C W Smith, M Ruzzene, and M K Wadee. Mechanical properties of auxetic tubular truss-like structures. *Phys. Stat. Sol. (b)*, 245:584–590, 2008.

S Torquato. *Random Heterogeneous Materials*. Springer, 2001.

W E Warren and A M Kraynik. Foam mechanics: the linear elastic response of two-dimensional spatially preiodic cellular materials. *Mech. Materials*, 6:27–37, 1987.