VARIATION OF FUNDAMENTAL GROUPS OF CURVES IN POSITIVE CHARACTERISTIC

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ABSTRACT. In this paper we prove that given a non-isotrivial family of hyperbolic curves in positive characteristic, the isomorphism type of the geometric fundamental group is not constant on the fibres of the family.

CONTENTS

§0. Introduction. Let k be an algebraically closed field and X a proper, smooth, and connected algebraic curve over k of genus $g \geq 2$. The structure of the étale fundamental group $\pi_1(X)$ of X is well understood, if $\mathrm{char}(k) = 0$, thanks to Riemann's Existence Theorem. Namely, $\pi_1(X)$ is isomorphic to the profinite completion Γ_g of the topological fundamental group of a compact orientable topological surface of genus g. In particular, the isomorphism type of $\pi_1(X)$ is constant and depends only on g in this case. In the case where $\mathrm{char}(k) = p > 0$, the structure of the full $\pi_1(X)$ is still mysterious and is far from being understood. One only knows the structure of certain quotients of $\pi_1(X)$ in this case. More precisely, let $\pi_1(X)^p$ (resp. $\pi_1(X)^{p'}$) be the maximal pro-p (resp. maximal pro-prime-to-p) quotient of $\pi_1(X)$. Then it is well-known that there exists a surjective continuous specialisation homomorphism $\mathrm{Sp} : \Gamma_g \twoheadrightarrow \pi_1(X)$, which induces an isomorphism $\mathrm{Sp}' : \Gamma_g^{p'} \xrightarrow{\sim} \pi_1(X)^{p'}$ between the maximal pro-prime-to-p parts (cf. [SGA1]). Moreover, $\pi_1(X)^p$ is a free pro-p group on r-generators where r is the p-rank of (the Jacobian of) the curve X (cf. [Sh]). The full structure of $\pi_1(X)$ is not known for a single example of a curve X of genus $g \geq 2$ in characteristic $p > 0$.

In order to understand the complexity of the geometric fundamental group π_1 of hyperbolic curves in positive characteristic, it is natural to investigate the variation of the structure of π_1 when curves vary in their moduli. Let $\mathcal{M}_{g,\mathbb{F}_p}$ be the coarse moduli space of proper, smooth, and connected curves of genus g in characteristic $p > 0$. Given a point $x \in \mathcal{M}_{g,\mathbb{F}_p}$, choose a geometric point $\bar{x}$ above x and let $C_{\bar{x}}$ be a curve corresponding to the moduli point $\bar{x}$ (well-defined up to isomorphism).

Then the isomorphism type of the (geometric) étale fundamental group $\pi_1(C_{\bar{x}})$ is independent of the choice of $\bar{x}$ and $C_{\bar{x}}$ (and the implicit base point on $C_{\bar{x}}$ used to define $\pi_1(C_{\bar{x}})$). (See [S], §4 for more details). An important tool in studying fundamental groups in positive characteristic is the specialisation theory of Grothendieck (cf. [SGA1]). Given points $x, y \in \mathcal{M}_{g, \mathbb{F}_p}$, such that $x \in \overline{\{y\}}$ holds, there exists a continuous surjective **specialisation homomorphism** $\mathrm{Sp}_{y,x} : \pi_1(C_{\bar{y}}) \to \pi_1(C_{\bar{x}})$. Concerning this specialisation homomorphism we have the following fundamental specialisation theorem (cf. Theorem 2.1), which is proven in [T3].

Theorem A. *Let $x, y \in \mathcal{M}_{g, \mathbb{F}_p}$ be distinct points of $\mathcal{M}_{g, \mathbb{F}_p}$ such that $x \in \overline{\{y\}}$ holds. Assume that x is a **closed** point of $\mathcal{M}_{g, \mathbb{F}_p}$. Then the specialisation homomorphism $\mathrm{Sp}_{y,x} : \pi_1(C_{\bar{y}}) \to \pi_1(C_{\bar{x}})$ is not an isomorphism.*

It is quite plausible that Theorem A may hold in general without the extra assumption that x is a closed point of $\mathcal{M}_{g, \mathbb{F}_p}$. It is also plausible that the isomorphism type of $\pi_1(C_{\bar{x}})$ tends to depend on the moduli point $x \in \mathcal{M}_{g, \mathbb{F}_p}$. In the spirit of Grothendieck's anabelian geometry this would suggest the possibility that strong anabelian phenomena for curves over algebraically closed fields of positive characteristic may occur, in contrast to the situation in characteristic 0 where the geometric fundamental group carries only topological informations.

A slightly weaker approach to the above specialisation theorem is the following. Let k be a field of characteristic $p > 0$ and set $\mathcal{M}_{g,k} \overset{\text{def}}{=} \mathcal{M}_{g, \mathbb{F}_p} \times_{\mathbb{F}_p} k$. Let $\mathcal{S} \subset \mathcal{M}_{g,k}$ be a subvariety. We say that the (geometric) fundamental group π_1 is **constant** on $\mathcal{S}$ if, for any two points y and x of $\mathcal{S}$, such that $x \in \overline{\{y\}}$ holds, the specialisation homomorphism $\mathrm{Sp}_{y,x} : \pi_1(C_{\bar{y}}) \to \pi_1(C_{\bar{x}})$ is an isomorphism. We say that π_1 is not constant on $\mathcal{S}$ if the contrary holds (cf. Definition 3.1). In [S] was raised the following question.

Question. Does $\mathcal{M}_{g,k}$ contain any subvariety of positive dimension on which π_1 is constant?

When k is an algebraic closure of $\mathbb{F}_p$, the answer to this question is negative by Theorem A. For general k, the following result is proven in [S] (cf. [S], Theorem 4.4).

Theorem B. *Let k be a field of characteristic $p > 0$ and $\mathcal{S} \subset \mathcal{M}_{g,k}$ a **complete** subvariety of $\mathcal{M}_{g,k}$ of positive dimension. Then the (geometric) fundamental group π_1 is not constant on $\mathcal{S}$.*

The aim of this paper is to remove the completeness assumption in Theorem B and give a negative answer to the above question in general. The main result of this paper is the following (cf. Theorem 3.6).

Theorem C. *Let k be a field of characteristic $p > 0$ and $\mathcal{S} \subset \mathcal{M}_{g,k}$ a (not necessarily complete) subvariety of $\mathcal{M}_{g,k}$ of positive dimension. Then the (geometric) fundamental group π_1 is not constant on $\mathcal{S}$.*

Note that the validity of Theorem A in general, i.e., without the extra assumption that x is a closed point, would immediately imply Theorem C. Our proof of Theorem C is quite different from the proof of Theorem B in [S], and relies on Theorem A and Raynaud's theory of theta divisors. We also prove certain variants of Theorem C for curves which are not necessarily proper (cf. Theorems 3.12 and 3.13).

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Next, we briefly review the contents of each section. In §1 we review basic facts on the theta divisor of the sheaf of locally exact differentials on a curve in characteristic $p > 0$. In §2 we review some key facts (proven in the course of proving Theorem A in [T3]) which are used in this paper. In §3 we state the main theorems, and in §4 we proceed to their proof.

§1. Review of the Sheaf of Locally Exact Differentials in Characteristic $p > 0$ and its Theta Divisor.

In this paper p denotes a (fixed) prime number. In this section, we will review Raynaud's theory of theta divisors in characteristic p, initiated in [R].

Let S be an $\mathbb{F}_p$-scheme. We denote by F_S the absolute Frobenius endomorphism $S \to S$. For an S-scheme X, we define X_1 to be the pull-back of X by F_S. Thus, we have a cartesian square

$$
\begin{array}{ccc}
X_1 & \longrightarrow & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{F_S} & S
\end{array}
$$

The absolute Frobenius endomorphism $F : X \to X$ induces in a natural way an S-morphism $F_{X/S} : X \to X_1$, the relative Frobenius morphism, which is an integral radicial morphism. Next, assume that X is a proper and smooth (relative) **S-curve** of genus g, i.e., the morphism $X \to S$ is proper and smooth and its fibres are (geometrically connected) curves of (constant) genus g. Then X_1 is also a proper and smooth S-curve of genus g, and $F_{X/S} : X \to X_1$ is finite locally free of degree p. The canonical differential

$$(F_{X/S})_* d : (F_{X/S})_* \mathcal{O}_X \to (F_{X/S})_* \Omega^1_{X/S}$$

is a morphism of $\mathcal{O}_{X_1}$-modules. Its image

$$B_X \overset{\mathrm{def}}{=} \mathrm{Im}((F_{X/S})_* d)$$

is the sheaf of locally exact differentials. We have a natural exact sequence

$$0 \to \mathcal{O}_{X_1} \to (F_{X/S})_* \mathcal{O}_X \to B_X \to 0,$$

and B_X is a locally free $\mathcal{O}_{X_1}$-module of rank $p - 1$.

Let J (resp. J_1) denote the relative Jacobian $\mathrm{Pic}^0_{X/S}$ of X (resp. $\mathrm{Pic}^0_{X_1/S}$ of X_1) over S, which is an abelian scheme of relative dimension g over S (cf. [BLR], 9.4, Proposition 4). Then, étale-locally on S, there exists a universal degree 0 line bundle $\mathcal{L}_1$ on $X_1 \times_S J_1$. We define $\Theta = \Theta_X$ to be the closed subscheme of J_1 defined by the 0-th Fitting ideal of $R^1(\mathrm{pr}_{J_1})_*(\mathrm{pr}^*_{X_1}(B_X) \otimes \mathcal{L}_1)$, where pr_{X_1} and pr_{J_1} denote the projections $X_1 \times_S J_1 \to X_1$ and $X_1 \times_S J_1 \to J_1$, respectively. Since Θ is independent of the choice of $\mathcal{L}_1$ (cf. [T2], Proposition 2.2(i)), we can define Θ not only étale-locally but also globally on S. By definition, the formation of Θ commutes with any base change of S. The following result is essentially due to Raynaud.

Theorem 1.1. Θ *is a relative Cartier divisor on J_1/S.*

Proof. See [T3], Theorem (5.1). $\square$

In this paper we will refer to the divisor Θ as the (relative) **Raynaud theta divisor**.

From now on, we assume $S = \mathrm{Spec}(k)$, where k is an algebraically closed field of characteristic $p > 0$.

Definition 1.2. Let M be an abelian group and M_{tor} the subgroup of torsion elements of M.

(i) For each element $x \in M_{\text{tor}}$ we define the subset $\text{sat}(x)$ (which we call the **saturation** of x) of M to be the set of elements in the form $i \cdot x$, where i is an integer prime to the order N_x of x. (Thus, $\sharp(\text{sat}(x)) = \varphi(N_x)$.)

(ii) For each subset X of M_{tor} we define the subset $\text{sat}(X)$ (which we call the **saturation** of X) of M to be the union of $\text{sat}(x)$ for all $x \in X$. (Note that $\text{sat}(X) \subset M_{\text{tor}}$.) Moreover, we say that X is saturated if $\text{sat}(X) = X$.

Recall that given a scheme S, a cyclic group G of order N which is invertible on S, and an abelian S-scheme $\mathcal{A}$ endowed with a G-action, one defines naturally the **"new part"** $\mathcal{A}_{\text{new}}$ of $\mathcal{A}$ with respect to this action (cf. [T3], §4 for more details). The following result relates the geometry of the Raynaud theta divisor to fundamental groups.

Proposition 1.3. *Let N be a positive integer prime to p. Let x be a torsion point of $J_1(k)$ of order N, and $Y_1 \to X_1$ the μ_N-torsor associated to x. Then, $\text{sat}(x) \cap \Theta(k) = \varnothing$ if and only if $Y_1 \to X_1$ is new-ordinary in the sense that $(J_{Y_1})_{\text{new}}$ is an ordinary abelian variety.*

Proof. See [T3], Proposition (5.2). □

2. The Specialisation Theorem for Fundamental Groups. In this section k_0 denotes an algebraic closure of the prime field $\mathbb{F}_p$ of characteristic $p > 0$. Let S be an $\mathbb{F}_p$-scheme, s and t points of S such that $s \in \overline{\{t\}}$ holds. We denote by $\bar{s}$ and $\bar{t}$ geometric points above s and t, respectively. Let X be a proper and smooth S-curve of genus g. Write $X_{\bar{s}} \stackrel{\text{def}}{=} X \times_S \bar{s}$, $X_{\bar{t}} \stackrel{\text{def}}{=} X \times_S \bar{t}$ for the geometric fibres of X above s and t, respectively. Then we have a **specialisation homomorphism** (cf. [SGA1], Exposé IX, 4 and Exposé XIII, 2.10)

$$\text{Sp} : \pi_1(X_{\bar{t}}) \to \pi_1(X_{\bar{s}}),$$

which is surjective and induces an isomorphism $\pi_1(X_{\bar{t}})^{(p')} \stackrel{\sim}{\to} \pi_1(X_{\bar{s}})^{(p')}$ between the maximal pro-prime-to-p quotients.

Recall that a curve over a field containing $k_0 = \overline{\mathbb{F}}_p$ is **constant**, if it descends to a curve over k_0. The following result is fundamental.

Theorem 2.1. *Assume $g \geq 2$. Assume that $X_{\bar{s}}$ is constant and that $X_{\bar{t}}$ is not constant. Then the specialisation homomorphism $\text{Sp} : \pi_1(X_{\bar{t}}) \to \pi_1(X_{\bar{s}})$ is not an isomorphism.*

Proof. See [T3], Theorem (8.1). □

Theorem 2.1 follows easily from the following (cf. [T3], Theorem (6.1)).

Theorem 2.2. *Let R be a complete discrete valuation ring isomorphic to $k_0[[t]]$, and set $S \stackrel{\text{def}}{=} \text{Spec}(R) = \{\eta, s\}$, where η (resp. s) stands for the generic (resp. closed) point of S. Let X be a proper and smooth S-curve of genus $g \geq 2$, and assume that $X_{\bar{\eta}}$ is not constant. Then the specialisation homomorphism $\text{Sp} : \pi_1(X_{\bar{\eta}}) \to \pi_1(X_{\bar{s}})$ is not an isomorphism.*

For an abelian variety A over an algebraically closed field k, we write $A\{p'\}$ for the union of $A[N](k)$ for all positive integers N prime to p, and for a subscheme Z of A we write $Z\{p'\} \stackrel{\text{def}}{=} Z(k) \cap A\{p'\}$.

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Let $X \to S = \mathrm{Spec}(R) = \{\eta, s\}$ be as in Theorem 2.2. (In the following discussion including Lemma 2.3, R may be an arbitrary discrete valuation ring of equal characteristic $p > 0$.) Let X_1 be the Frobenius twist of X over S (cf. §1) and J_1 the (relative) Jacobian of X_1 over S. This is an abelian scheme over S and can be identified with the Néron model of $J_{1,\eta} \overset{\mathrm{def}}{=} J_1 \times_S \eta$. By the Néron property, we have a natural specialisation isomorphism

$$J_{1,\bar{\eta}}\{p'\} \to J_{1,\bar{s}}\{p'\},$$

where $J_{1,\bar{\eta}} \overset{\mathrm{def}}{=} J_1 \times_S \bar{\eta}$ and $J_{1,\bar{s}} \overset{\mathrm{def}}{=} J_1 \times_S \bar{s}$. Identifying these two abelian groups with each other by this specialisation isomorphism, we will write

$$J_1\{p'\} \overset{\mathrm{def}}{=} J_{1,\bar{\eta}}\{p'\} = J_{1,\bar{s}}\{p'\}.$$

(Thus, $J_1\{p'\}$ is a mere abelian group). Moreover, we have the Raynaud theta divisor Θ in J_1, and under the above identification we have

$$\Theta_{\bar{\eta}}\{p'\} \subset \Theta_{\bar{s}}\{p'\} \text{ (in } J_1\{p'\}).$$

The following is a crucial observation (cf. [T3], Lemma (6.2)).

Lemma 2.3. *If* Sp *is an isomorphism, then* $\mathrm{sat}(\Theta_{\bar{\eta}}\{p'\}) = \mathrm{sat}(\Theta_{\bar{s}}\{p'\})$ *must hold in* $J_1\{p'\}$.

In the course of proving Theorem 2.2, the following more precise statement is proven (cf. [T3], §7).

Proposition 2.4. *Under the assumptions of Theorem 2.2 ($R \simeq k_0[[t]]$), there exists a finite étale cover* $Y \to X$ *whose Galois closure is of degree prime to* p, *such that*

$$\mathrm{sat}(\Theta_{Y,\bar{\eta}}\{p'\}) \subsetneq \mathrm{sat}(\Theta_{Y,\bar{s}}\{p'\})$$

holds in $J_{Y_1}\{p'\}$. *Here,* J_{Y_1} *and* $\Theta_Y \subset J_{Y_1}$ *denote the (relative) Jacobian of* Y_1 *over* S *and the (relative) Raynaud theta divisor for* $Y \to S$, *respectively.*

Remark 2.5. In fact, the finite étale cover $Y \to X$ in Proposition 2.4 can be chosen to factorise as $Y = X_3 \to X_2 \to X_1 \to X_0 = X$, where $X_i \to X_{i-1}$ is a μ_{N_i}-torsor for a suitable positive integer N_i prime to p, $i \in \{1, 2, 3\}$.

§3. Families of Curves with a Given Fundamental Group in Positive Characteristic.
In this section we state our main results. We will start with the following elementary definition.

Definition 3.1. Let S be a set and assume that for each $s \in S$, a profinite group Π_s is given. We denote by Π the map from S to the set of isomorphism classes of profinite groups that assigns to each $s \in S$ the isomorphism class of Π_s.
(i) We say that Π is **constant** on S if, for any $s, t \in S$, one has $\Pi_s \simeq \Pi_t$.
(ii) Assume moreover that S is the underlying (vertex) set of an oriented graph, and that for each oriented edge $t \rightarrowtail s$ linking vertices t and s of the graph S, a surjective (continuous) homomorphism $\mathrm{Sp}_{t,s} : \Pi_t \to \Pi_s$ is given. Then we say that Π is **Sp-constant** on S if for any oriented edge $t \rightarrowtail s$ of the graph S, the homomorphism $\mathrm{Sp}_{t,s} : \Pi_t \to \Pi_s$ is an isomorphism.

Lemma 3.2. *In the situation of Definition 3.1 (ii), consider the following conditions: (i) Π is constant; and (ii) Π is Sp-constant. If S is connected (as a graph), then one has (ii) $\Longrightarrow$ (i). If Π_s is finitely generated (as a profinite group) for each $s \in S$, then one has (i) $\Longrightarrow$ (ii).*

Proof. The first assertion is clear. The second assertion follows from the fact that a finitely generated profinite group is Hopfian (cf. [FJ], Proposition 15.3). $\square$

In the rest of this section, let p be a prime number and g an integer ≥ 2 (unless otherwise stated), and set $\mathcal{M}_g \overset{\text{def}}{=} \mathcal{M}_{g,\mathbb{F}_p}$, the **coarse moduli space** of proper, smooth and geometrically connected curves of genus g in characteristic p. Let k be a field of characteristic p, and $\mathcal{M}_{g,k} \overset{\text{def}}{=} \mathcal{M}_g \times_{\mathbb{F}_p} k$, which turns out to be the coarse moduli space of proper, smooth and geometrically connected curves of genus g over k-schemes. Given a point $x \in \mathcal{M}_{g,k}$, choose a geometric point $\bar{x}$ above x and let $C_{\bar{x}}$ be a curve corresponding to the moduli point $\bar{x}$ (well-defined up to isomorphism). Then the isomorphism type of the (geometric) étale fundamental group $(\pi_1)_x \overset{\text{def}}{=} \pi_1(C_{\bar{x}})$, which is a finitely generated profinite group (cf. [SGA1], Exposé X, Théorème 2.6), is independent of the choice of $\bar{x}$ and $C_{\bar{x}}$ (and the implicit base point on $C_{\bar{x}}$ used to define $\pi_1(C_{\bar{x}})$). If x and y are points in $\mathcal{M}_{g,k}$ such that $x \in \overline{\{y\}}$ holds, then Grothendieck's specialisation theory for fundamental groups implies the existence of a (continuous) surjective specialisation homomorphism $\mathrm{Sp}_{y,x} : \pi_1(C_{\bar{y}}) \twoheadrightarrow \pi_1(C_{\bar{x}})$. (See [S], §4 for more details.)

Similarly, let S be a scheme of characteristic p and $f : X \to S$ a proper and smooth S-curve of genus g. Given a point $s \in S$, choose a geometric point $\bar{s}$ above s and let $X_{\bar{s}}$ be the geometric fibre of f at $\bar{s}$. Then the isomorphism type of the (geometric) étale fundamental group $(\pi_1)_s \overset{\text{def}}{=} \pi_1(X_{\bar{s}})$, which is a finitely generated profinite group, is independent of the choice of $\bar{s}$ (and the implicit base point on $X_{\bar{s}}$ used to define $\pi_1(X_{\bar{s}})$). If s and t are points in S such that $s \in \overline{\{t\}}$ holds, then Grothendieck's specialisation theory for fundamental groups implies the existence of a (continuous) surjective specialisation homomorphism $\mathrm{Sp}_{t,s} : \pi_1(X_{\bar{t}}) \twoheadrightarrow \pi_1(X_{\bar{s}})$.

Note that any topological space T (e.g., any subset of a scheme) can be regarded as an oriented graph by assigning a vertex to each point of T and an oriented edge $t \rightarrowtail s$ to each pair (s,t) of distinct points of T such that $s \in \overline{\{t\}}$ holds.

Lemma 3.3. *(i) Let $\mathcal{S}$ be a connected subscheme of $\mathcal{M}_{g,k}$. Then π_1 is constant on $\mathcal{S}$ if and only if π_1 is Sp-constant on $\mathcal{S}$ (cf. Definition 3.1).*

(ii) Let k be a field of characteristic p, S a connected k-scheme of finite type, $f : X \to S$ a proper and smooth S-curve of genus g and $h : S \to \mathcal{M}_{g,k}$ the (coarse) classifying morphism for f. Then the following are all equivalent:
(a) π_1 is constant on S;
(b) π_1 is Sp-constant on S;
(c) π_1 is constant on $h(S)$.

Proof. This follows from Lemma 3.2, together with the (easily verified) fact that a noetherian scheme is connected as a scheme (or, equivalently, as a topological space) if and only if the associated graph is connected. $\square$

In [S] was raised the following question.

Question 3.4. Let k be a field of characteristic p. Does $\mathcal{M}_{g,k}$ contain any subvariety of positive dimension on which π_1 is constant?

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The following result is proven in [S] (cf. [S], Theorem 4.4.).

Theorem 3.5. *Let k be a field of characteristic p and $\mathcal{S} \subset \mathcal{M}_{g,k}$ a **complete** subvariety of $\mathcal{M}_{g,k}$ of positive dimension. Then π_1 is not constant on $\mathcal{S}$.*

The main result of this paper is the following theorem, where we remove the assumption in Theorem 3.5 that the subvariety $\mathcal{S}$ is complete.

Theorem 3.6. *Let k be a field of characteristic p, and $\mathcal{S} \subset \mathcal{M}_{g,k}$ a (not necessarily complete and even not necessarily closed) subvariety of $\mathcal{M}_{g,k}$ of positive dimension. Then π_1 is not constant on $\mathcal{S}$.*

Theorem 3.6 follows from Theorem 3.9 below.

In the rest of this section, let k be a field of characteristic p, S a connected and reduced k-scheme of finite type, and $f : X \to S$ a proper and smooth S-curve of (constant) genus g.

Definition 3.7. We say that f is **isotrivial**, if there exist a finite extension k'/k, a connected k'-scheme S', a finite étale k'-morphism $S' \to S \times_k k'$, and a proper and smooth k'-curve X_0', such that $X \times_S S'$ is isomorphic to $X_0' \times_{k'} S'$ over S'.

Lemma 3.8. *Write $h : S \to \mathcal{M}_{g,k}$ for the (coarse) classifying morphism for f. Then the following conditions are all equivalent.*
 (i) f is isotrivial.
 (ii) The image of h consists of a single closed point of $\mathcal{M}_{g,k}$.
 (ii') The image of h consists of a single point of $\mathcal{M}_{g,k}$ (i.e., $h : S \to \mathcal{M}_{g,k}$ is set-theoretically constant).
 (iii) For each generic point η of S, $h(\eta)$ is a closed point.
 (iv) For each 1-dimensional irreducible, reduced, closed subscheme C of S, the C-curve $f_C : X \times_S C \to C$ is isotrivial.
 (iv') For each irreducible component W of S (regarded as a reduced closed subscheme of S), there exists a closed point s of W, such that, for each 1-dimensional irreducible, reduced, closed subscheme C of W passing through s, the C-curve $f_C : X \times_S C \to C$ is isotrivial.

Proof. Standard. $\square$

Theorem 3.9. *Assume that $f : X \to S$ is non-isotrivial. Then π_1 is not constant on S.*

In the process of proving Theorem 3.9 we prove the following more precise result.

Theorem 3.10. *Assume that $f : X \to S$ is non-isotrivial. Then there exist a connected finite étale cover $S' \to S$, a connected finite étale cover $Y' \to X' \overset{\mathrm{def}}{=} X \times_S S'$ whose Galois closure is geometrically connected over S' and of degree prime to p over $X \times_S S'$, a generic point $\eta' \in S'$ and a point $v' \in \overline{\{\eta'\}} \subset S'$, such that the p-rank of the geometric fibre $Y'_{\bar{v}'}$ of Y' above v' is smaller than the p-rank of the geometric fibre $Y'_{\bar{\eta}'}$ of Y' above η' (thus, $v' \neq \eta'$ necessarily).*

Theorem 3.9 admits certain variants (Theorems 3.12 and 3.13) for curves which are not necessarily proper. To state them, let X be a proper and smooth S-curve of (constant) genus $g \geq 0$ and $D \subset X$ a relatively étale divisor of (constant) degree $n \geq 0$. Given a point $s \in S$, choose a geometric point $\bar{s}$ above s and let $(X_{\bar{s}}, D_{\bar{s}})$ be the geometric fibre of $(X, D)/S$ at $\bar{s}$. Then the isomorphism type of the (geometric)

tame fundamental group $(\pi_1^t)_s \overset{\text{def}}{=} \pi_1^t(X_{\bar{s}} - D_{\bar{s}})$ (resp. the (geometric) fundamental group $(\pi_1)_s \overset{\text{def}}{=} \pi_1(X_{\bar{s}} - D_{\bar{s}})$) is independent of (resp. (if $n > 0$) dependent on) the choice of $\bar{s}$.

We say that $(X, D)/S$ is **isotrivial**, if there exist a finite extension k'/k, a connected k'-scheme S', a finite étale k'-morphism $S' \to S \times_k k'$, a proper and smooth k'-curve X_0' and a relatively étale divisor D_0' of X_0', such that $(X, D) \times_S S'$ is isomorphic to $(X_0', D_0') \times_{k'} S'$ over S'.

Remark 3.11. (i) As in the proper case, if s and t are points in S such that $s \in \overline{\{t\}}$ holds, then Grothendieck's specialisation theory for tame fundamental groups implies the existence of a (continuous) surjective specialisation homomorphism $\mathrm{Sp}_{t,s}^t : \pi_1^t(X_{\bar{t}} - D_{\bar{t}}) \twoheadrightarrow \pi_1^t(X_{\bar{s}} - D_{\bar{s}})$. Thus, by Definition 3.1, we have the notion of Sp^t-constancy of $(X, D)/S$ as well, which is equivalent to the notion of constancy of $(X, D)/S$ by Lemma 3.2. (Note that the tame fundamental group of affine curves is finitely generated (cf. [SGA1], Exposé XIII, Corollaire 2.12).) Note that no such specialisation homomorphisms are available for (full) fundamental groups, if $n > 0$.

(ii) Assume that $2 - 2g - n < 0$. Then, as in the proper case (cf. Lemma 3.8), $(X, D)/S$ is isotrivial if and only if the (coarse) classifying morphism $h : S \to \mathcal{M}_{g,[n],k}$ for $(X, D)/S$ is set-theoretically constant. Here, $\mathcal{M}_{g,[n],k}$ ($= \mathcal{M}_{g,[n],\mathbb{F}_p} \times_{\mathbb{F}_p} k$) is the coarse moduli space of proper, smooth and geometrically connected curves of genus g equipped with a relatively étale divisor of degree n over k-schemes.

Theorem 3.12. *Assume that $2 - 2g - n < 0$ and that $(X, D)/S$ is non-isotrivial. Then π_1^t is not constant on S.*

Theorem 3.13. *Assume that $2 - 2g - n < 0$ and that $(X, D)/S$ is non-isotrivial. Then π_1 is not constant on S.*

§**4. Proof of the Main Theorems.** In this section we prove the main results: Theorems 3.6, 3.9, 3.10, 3.12 and 3.13. First, we work with the assumptions in Theorems 3.9 and 3.10. In particular, k is a field of characteristic $p > 0$, S is a connected and reduced k-scheme of finite type, and $f : X \to S$ is a non-isotrivial proper and smooth S-curve of genus $g \geq 2$.

Proof of Theorem 3.10.

Lemma 4.1. *Let s be a point of S and $X_{\bar{s}}$ the geometric fibre of $f : X \to S$ at a geometric point $\bar{s}$ above s. Let $Y_{\bar{s}} \to X_{\bar{s}}$ be a finite étale cover whose Galois closure is of degree prime to p over $X_{\bar{s}}$. Then there exist a connected finite étale cover $S' \to S$ and a connected finite étale cover $Y' \to X' \overset{\text{def}}{=} X \times_S S'$ whose Galois closure is geometrically connected over S' and of degree prime to p over $X \times_S S'$, a point $s' \in S'$ above $s \in S$ and a geometric point $\bar{s}'$ above s' which dominates $\bar{s}$, such that the cover $(Y')_{\bar{s}'} \to (X')_{\bar{s}'}$ is isomorphic to the pull-back of $Y_{\bar{s}} \to X_{\bar{s}}$ to $\bar{s}'$.*

Proof. This follows from the fact (cf. [St], Proposition 2.3) that the natural sequence of profinite groups

$$1 \to \pi_1(X_{\bar{s}})^{p'} \to \pi_1(X)^{(p')} \to \pi_1(S) \to 1$$

is exact, where $\pi_1(X_{\bar{s}})^{p'}$ stands for the maximal pro-prime-to-p quotient of $\pi_1(X_{\bar{s}})$ and $\pi_1(X)^{(p')}$ stands for the maximal quotient of $\pi_1(X)$ in which the image of $\mathrm{Ker}(\pi_1(X) \to \pi_1(S))$ is pro-prime-to-p. $\square$

We may reduce the problem to the case where S is geometrically connected and geometrically reduced over k. Indeed, first, take a connected component of $S \times_k k^{\mathrm{sep}}$ (regarded as a scheme over k^{sep}), which descends to a scheme S_1 over a finite separable extension k_1 of k. Next, consider the reduced closed subscheme $(S_1 \times_{k_1} k_1^{\mathrm{perf}})_{\mathrm{red}}$ of $S_1 \times_{k_1} k_1^{\mathrm{perf}}$ (regarded as a scheme over k_1^{perf}), which descends to a scheme S_2 over a finite purely inseparable extension k_2 of k_1. Then S_1 is geometrically connected over k_1, and S_2 is geometrically connected and geometrically reduced over k_2. As S_1 is a connected finite étale cover of S, we may replace $X \to S \to \mathrm{Spec}(k)$ by $X \times_S S_1 \to S_1 \to \mathrm{Spec}(k_1)$. As the morphisms $S_2 \to S_1$ and $X \times_S S_2 \to X \times_S S_1$ preserve the categories of finite étale covers, we may replace $X \times_S S_1 \to S_1 \to \mathrm{Spec}(k_1)$ by $X \times_S S_2 \to S_2 \to \mathrm{Spec}(k_2)$.

So, from now on, we will assume that S is geometrically connected and geometrically reduced over k.

Lemma 4.2. *There exist a finitely generated $\mathbb{F}_p$-subalgebra R of k, a connected scheme $\mathcal{S}$ flat, of finite type over R with geometrically connected and geometrically reduced fibres, and a proper and smooth $\mathcal{S}$-curve $\tilde{f} : \mathcal{X} \to \mathcal{S}$ of (constant) genus g, such that we have a commutative diagram with cartesian squares*

$$
(*) \qquad
\begin{array}{ccccc}
\mathcal{X} & \xrightarrow{\tilde{f}} & \mathcal{S} & \longrightarrow & \mathrm{Spec}(R) \\
\uparrow & & \uparrow & & \uparrow \\
X & \xrightarrow{f} & S & \longrightarrow & \mathrm{Spec}(k)
\end{array}
$$

where the right vertical map is the natural morphism induced by the inclusion $R \subset k$.

Proof. As k is the direct limit of finitely generated $\mathbb{F}_p$-subalgebras, it follows from [EGA IV], Théorème (8.8.2) that there exists a finitely generated $\mathbb{F}_p$-subalgebra R of k, schemes $\mathcal{S}$ and $\mathcal{X}$ of finite type over R, and an R-morphism $\tilde{f} : \mathcal{X} \to \mathcal{S}$, such that we have a commutative diagram $(*)$ with cartesian squares. Replacing R by a finitely generated $\mathbb{F}_p$-subalgebra R' of k containing R and $\tilde{f} : \mathcal{X} \to \mathcal{S}$ by $\tilde{f}_{R'} = \tilde{f} \times_R R' : \mathcal{X} \times_R R' \to \mathcal{S} \times_R R'$, we may assume that $\mathcal{S}$ is flat over R ([EGA IV], Théorème (11.2.6)) with geometrically connected and geometrically reduced fibres ([EGA IV], Théorème (9.7.7)) and that $\mathcal{X}$ is proper and smooth over $\mathcal{S}$ ([EGA IV], Théorème (8.10.5) and Proposition (17.7.8)). As $\mathcal{S} \to \mathrm{Spec}(R)$ is flat of finite type, it is (universally) open ([EGA IV], Théorème (2.4.6)), hence its generic fiber, which is (geometrically) connected, is dense. This implies that $\mathcal{S}$ is connected. As $\mathcal{X} \to \mathcal{S}$ is proper and smooth and has geometrically connected fibres on the image of S in $\mathcal{S}$, we conclude (by observing the Stein factorisation) that $\mathcal{X} \to \mathcal{S}$ must have geometrically connected fibres everywhere (cf. [EGA III], Remarque (7.8.10) and [SGA1], Exposé X, Proposition 1.2). Finally, as $\mathcal{X} \to \mathcal{S}$ is proper and flat and $\mathcal{S}$ is connected, the dimension and the (arithmetic) genus of the fibres are constant. Thus, $\mathcal{X} \to \mathcal{S}$ is a proper and smooth $\mathcal{S}$-curve of constant genus g, as desired. $\square$

Next, set $T \overset{\mathrm{def}}{=} \mathrm{Spec}(R)$ and let ξ be the generic point of T. For each $t \in T$, set $\mathcal{S}_t \overset{\mathrm{def}}{=} \mathcal{S} \times_T t$ and $\mathcal{X}_t \overset{\mathrm{def}}{=} \mathcal{X} \times_T t$. We have a commutative diagram with cartesian squares

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$$\begin{array}{ccccc}
\mathcal{X} & \xrightarrow{\tilde{f}} & \mathcal{S} & \longrightarrow & T = \mathrm{Spec}(R) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{X}_t & \xrightarrow{\tilde{f}_t} & \mathcal{S}_t & \longrightarrow & t = \mathrm{Spec}(\kappa(t))
\end{array}$$

where $\mathcal{S}_t \to \mathrm{Spec}(\kappa(t))$ is of finite type, geometrically connected and geometrically reduced, and $\tilde{f}_t : \mathcal{X}_t \to \mathcal{S}_t$ is a proper and smooth $\mathcal{S}_t$-curve of (constant) genus g.

Lemma 4.3. *There exists a closed point $t \in T$ such that the $\mathcal{S}_t$-curve $\tilde{f}_t : \mathcal{X}_t \to \mathcal{S}_t$ is non-isotrivial.*

Proof. Let $h : S \to \mathcal{M}_{g,k}$ and $\tilde{h} : \mathcal{S} \to \mathcal{M}_{g,\mathbb{F}_p}$ be the (coarse) classifying morphisms for $f : X \to S \ (\to \mathrm{Spec}(k))$ and $\tilde{f} : \mathcal{X} \to \mathcal{S} \ (\to \mathrm{Spec}(\mathbb{F}_p))$. Let $\tilde{\beta} : \mathcal{S} \to T$ be the structure morphism and set $\tilde{h}_T \overset{\mathrm{def}}{=} (\tilde{h}, \tilde{\beta}) : \mathcal{S} \to \mathcal{M}_{g,\mathbb{F}_p} \times_{\mathbb{F}_p} T$. For each $t \in T$, let $\tilde{h}_t : \mathcal{S}_t \to \mathcal{M}_{g,\kappa(t)}$ be the (coarse) classifying morphism for $\tilde{f}_t : \mathcal{X}_t \to \mathcal{S}_t \ (\to \mathrm{Spec}(\kappa(t)))$. Then we have $\tilde{h}_t = \tilde{h}_T \times_T t$. Further, we have $\tilde{h}_T \times_T k = (\tilde{h}_\xi) \times_{\kappa(\xi)} k = h$.

By Lemma 3.8, there exists a generic point η of S such that $h(\eta)$ is not a closed point of $\mathcal{M}_{g,k}$. Let $\tilde{\eta}$ be the image of η in $\mathcal{S}$, which is a generic point of $\mathcal{S}$. Then $\tilde{\beta}(\tilde{\eta}) = \xi$, and $\tilde{h}_\xi(\tilde{\eta})$ is not a closed point of $\mathcal{M}_{g,\kappa(\xi)}$. Note that there exists an open neighbourhood $\mathcal{W}$ of $\tilde{\eta}$ which is irreducible. (For example, remove all the irreducible components of $\mathcal{S}$ but $\overline{\{\tilde{\eta}\}}$.) Set $\alpha \overset{\mathrm{def}}{=} \tilde{h}_T|_\mathcal{W}$ and $\beta \overset{\mathrm{def}}{=} \tilde{\beta}|_\mathcal{W}$. By [EGA IV], Corollaire (9.2.6.2), there exists a non-empty open subset $\mathcal{U}$ of $\mathcal{W}$ such that $\dim(\alpha^{-1}(\alpha(s)))$ and $\dim(\beta^{-1}(\beta(s)))$ are constant for $s \in \mathcal{U}$. Finally, for each $t \in T$, set $\mathcal{W}_t \overset{\mathrm{def}}{=} \mathcal{W} \times_T t$.

Now, take any closed point s of $\mathcal{U}$ and set $t \overset{\mathrm{def}}{=} \beta(s)$, which is a closed point of T. As $\tilde{h}_\xi|_{\mathcal{W}_\xi}$ (or, equivalently, $\tilde{h}|_{\mathcal{W}_\xi}$) is non-constant and $\mathcal{W}_\xi$ is irreducible, we have

$$\dim(\alpha^{-1}(\alpha(\tilde{\eta}))) < \dim(\beta^{-1}(\beta(\tilde{\eta}))) = \dim(\mathcal{W}_\xi),$$

hence

$$\dim(\alpha^{-1}(\alpha(s)) < \dim(\beta^{-1}(\beta(s))) = \dim(\mathcal{W}_t),$$

which implies that $\tilde{h}$ (or, equivalently, $\tilde{h}_t$) is non-constant on each irreducible component of $\mathcal{W}_t$ of maximal dimension ($= \dim(\mathcal{W}_t)$). As $\mathcal{M}_{g,\kappa(t)} \to \mathcal{M}_{g,\mathbb{F}_p}$ is finite, this implies that $\tilde{h}_t|_{\mathcal{S}_t}$ is non-constant, as desired. $\square$

Next, consider the (relative) curve $\mathcal{X}_1 \to \mathcal{S}$ where $\mathcal{X}_1$ is the Frobenius twist of $\mathcal{X}$ (cf. §1), $\mathcal{J}_1 \overset{\mathrm{def}}{=} \mathrm{Pic}^0(\mathcal{X}_1/\mathcal{S})$ the (relative) Jacobian of $\mathcal{X}_1$ over $\mathcal{S}$ which is an $\mathcal{S}$-abelian scheme, and $\Theta \hookrightarrow \mathcal{J}_1$ the (relative) Raynaud theta divisor for $\mathcal{X} \to \mathcal{S}$ (cf. §1). Let t be a closed point of T such that the relative curve $\mathcal{X}_t \to \mathcal{S}_t$ is non-isotrivial (cf. Lemma 4.3). Set $\mathcal{J}_{1,t} \overset{\mathrm{def}}{=} \mathcal{J}_1 \times_T t$ and $\Theta_t \overset{\mathrm{def}}{=} \Theta \times_T t$, which are the (relative) Jacobian of $\mathcal{X}_{1,t} \overset{\mathrm{def}}{=} \mathcal{X}_1 \times_T t$ over $\mathcal{S}_t$ and the (relative) Raynaud theta divisor for $\mathcal{X}_t \to \mathcal{S}_t$, respectively.

Let $\mathcal{S}^{\mathrm{sm}} \subset \mathcal{S}$ and $\mathcal{S}_t^{\mathrm{sm}} \subset \mathcal{S}_t$ be the smooth loci for $\mathcal{S} \to T$ and $\mathcal{S}_t \to t$, respectively. As $\mathcal{S} \to T$ is of finite type, $\mathcal{S}^{\mathrm{sm}} \subset \mathcal{S}$ and $\mathcal{S}_t^{\mathrm{sm}} \subset \mathcal{S}_t$ are open. As $\mathcal{S} \to T$ is flat, we have $(\mathcal{S}^{\mathrm{sm}})_t = \mathcal{S}_t^{\mathrm{sm}}$. As $\mathcal{S} \to T$ has geometrically reduced fibres, $\mathcal{S}^{\mathrm{sm}} \subset \mathcal{S}$ is dense in each fibre, and, in particular, $\mathcal{S}_t^{\mathrm{sm}}$ is dense in $\mathcal{S}_t$. Let $\mathcal{S}_t^0$ be the (disjoint)

union of connected (or, equivalently, irreducible) components V of $\mathcal{S}_t^{\mathrm{sm}} \subset \mathcal{S}_t$ for which the V-curve $\tilde{f}_V = (\tilde{f}_t)_V : \mathcal{X} \times_T V = \mathcal{X}_t \times_t V \to V$ is non-isotrivial. By Lemma 4.3, $\mathcal{S}_t^0$ is non-empty.

Take any component V of $\mathcal{S}_t^0$ and any closed point s of V. Then, by Lemma 3.8, there exists a 1-dimensional, irreducible, closed subscheme C of V passing through s such that the C-curve $\tilde{f}_C = (\tilde{f}_V)_C : \mathcal{X} \times_T C \to C$ is non-isotrivial. Let γ and u be the generic points of C and V, respectively. Then it follows from Theorem 2.1 that the specialisation homomorphism $\mathrm{Sp} : \pi_1(\mathcal{X}_{t,\bar{\gamma}}) \to \pi_1(\mathcal{X}_{t,\bar{s}})$ is not an isomorphism. Here, $\bar{\gamma}$ (resp. $\bar{s}$) is a geometric point above γ (resp. s), and $\mathcal{X}_{t,\bar{\gamma}} \stackrel{\mathrm{def}}{=} \mathcal{X}_t \times_t \bar{\gamma}$ (resp. $\mathcal{X}_{t,\bar{s}} \stackrel{\mathrm{def}}{=} \mathcal{X}_t \times_s \bar{s}$).

Now, it follows from Proposition 2.4 and Lemma 4.1 that the following hold. There exist a finite étale cover $\mathcal{S}' \to \mathcal{S}$, a finite étale cover $\mathcal{Y}' \to \mathcal{X}' \stackrel{\mathrm{def}}{=} \mathcal{X} \times_\mathcal{S} \mathcal{S}'$ whose Galois closure is of degree prime to p, a (generic) point $u' \in \mathcal{S}_t' \stackrel{\mathrm{def}}{=} \mathcal{S}' \times_T t$ above u, a point $\gamma' \in \overline{\{u'\}} \subset \mathcal{S}_t'$ above γ and a (closed) point $s' \in \overline{\{\gamma'\}} \subset \mathcal{S}_t'$ above s, such that $\mathrm{sat}(\Theta'_{\bar{u}'}\{p'\}) \subset \mathrm{sat}(\Theta'_{\bar{\gamma}'}\{p'\}) \subsetneq \mathrm{sat}(\Theta'_{\bar{s}'}\{p'\})$ holds in $\mathcal{J}_1'\{p'\} \stackrel{\mathrm{def}}{=} \mathcal{J}_{1,\bar{u}'}'\{p'\} = \mathcal{J}_{1,\bar{\gamma}'}'\{p'\} = \mathcal{J}_{1,\bar{s}'}'\{p'\}$, where $\mathcal{J}_1' \stackrel{\mathrm{def}}{=} \mathrm{Pic}^0(\mathcal{Y}_1'/\mathcal{S}')$ is the (relative) Jacobian of $\mathcal{Y}_1'$ over $\mathcal{S}'$, $\Theta' \hookrightarrow \mathcal{J}_1'$ is the (relative) Raynaud theta divisor for $\mathcal{Y}' \to \mathcal{S}'$, and $\bar{u}'$ (resp. $\bar{\gamma}'$, resp. $\bar{s}'$) is a geometric point above u' (resp. γ', resp. s'). In particular, there exists $x \in \mathcal{J}_1'\{p'\}$ such that $\mathrm{sat}(x) \cap \Theta'_{\bar{u}'}\{p'\} = \varnothing$ and $\mathrm{sat}(x) \cap \Theta'_{\bar{s}'}\{p'\} \neq \varnothing$. Let $z \in \mathrm{sat}(x) \cap \Theta'_{\bar{s}'}\{p'\}$, and $N \stackrel{\mathrm{def}}{=} \mathrm{ord}(z)$. Let $\mathcal{J}_1'[N] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\mathcal{J}_1' \xrightarrow{[N]} \mathcal{J}_1')$ be the kernel of multiplication by N on the abelian scheme $\mathcal{J}_1'$. Thus, $\mathcal{J}_1'[N]$ is a finite étale commutative $\mathcal{S}'$-group scheme which is étale-locally isomorphic to $(\mathbb{Z}/N\mathbb{Z})^{2g}$. After possibly passing to a finite étale cover of $\mathcal{S}'$ which "trivialises" $\mathcal{J}_1'[N]$, we can assume (without loss of generality) that there exists a section $\sigma : \mathcal{S}' \to \mathcal{J}_1'[N]$ of the natural projection $\mathcal{J}_1'[N] \twoheadrightarrow \mathcal{S}'$, with image $\mathcal{Z} \stackrel{\mathrm{def}}{=} \mathrm{Im}(\sigma)$, such that $\mathcal{Z}_{\bar{s}'} \stackrel{\mathrm{def}}{=} \mathcal{Z} \times_{\mathcal{S}'} \bar{s}' = z$ holds. Write $\mathcal{Z}.\Theta'$ for the scheme-theoretic intersection of $\mathcal{Z}$ and Θ' inside $\mathcal{J}_1'$. By definition, we have $\sigma(s') \in \mathcal{Z}.\Theta' \neq \varnothing$. (By a slight abuse of notation, we write $z = \sigma(s')$.) We have natural morphisms $\mathcal{Z}.\Theta' \to \mathcal{S}' \to \mathcal{S} \to T$. The following is the main technical ingredient of our proof.

Proposition 4.4. *The morphism $\mathcal{Z}.\Theta' \to T$ is flat at $z \in \mathcal{Z}.\Theta'$.*

Proof. We have natural morphisms $\mathcal{O}_{T,t} \to \mathcal{O}_{\mathcal{S},s} \to \mathcal{O}_{\mathcal{S}',s'} = \mathcal{O}_{\mathcal{Z},z}$. Note that $\mathcal{O}_{\mathcal{S}',s'}$ is $\mathcal{O}_{T,t}$-flat, since $\mathcal{S}'$ is étale over $\mathcal{S}$ and $\mathcal{S}$ is flat over T. Moreover, there exists a natural surjective homomorphism $\mathcal{O}_{\mathcal{J}_1',z} \twoheadrightarrow \mathcal{O}_{\mathcal{Z},z}$. Let $\mathcal{I}_{\Theta',z} = (f) \subset \mathcal{O}_{\mathcal{J}_1',z}$ be the ideal defining the theta divisor Θ' (cf. Theorem 1.1). We will show that $\mathcal{O}_{\mathcal{Z}.\Theta',z} = \mathcal{O}_{\mathcal{Z},z}/(f)$ is flat over $\mathcal{O}_{T,t}$.

Let $\kappa(t)$ be the residue field of T at t, and g the image of f in $M \stackrel{\mathrm{def}}{=} \mathcal{O}_{\mathcal{Z},z} \otimes_{\mathcal{O}_{T,t}} \kappa(t)$. By [EGA IV], Chapitre 0, Proposition (15.1.16), $c) \implies a)$, it suffices to show that g is M-regular. Furthermore, by [EGA IV], Chapitre 0, Proposition (16.3.6) and (Proposition) 16.5.5, $b) \implies a)$, to show the latter it suffices to show that $\dim(M/gM) < \dim M$. Observe that $M = \mathcal{O}_{\mathcal{Z},z} \otimes_{\mathcal{O}_{T,t}} \kappa(t) = \mathcal{O}_{\mathcal{Z}_t,z}$ and that $M/gM = \mathcal{O}_{(\mathcal{Z}.\Theta)_t,z}$, where $\mathcal{Z}_t \stackrel{\mathrm{def}}{=} \mathcal{Z} \times_T t$. Now, since $(\mathcal{Z}.\Theta)_{t,\bar{u}'} = \varnothing$ (cf. discussion before Proposition 4.4), we have $\dim(M/gM) < \dim M$, as desired. $\square$

Set $S_0 \stackrel{\mathrm{def}}{=} \mathcal{S} \times_T \xi$, $S_0' \stackrel{\mathrm{def}}{=} \mathcal{S}' \times_T \xi$, $X_0 \stackrel{\mathrm{def}}{=} \mathcal{X} \times_T \xi$, and $X_0' \stackrel{\mathrm{def}}{=} X_0 \times_{S_0} S_0' = (\mathcal{X} \times_\mathcal{S} \mathcal{S}') \times_T \xi$. Let η' be the generic point of S_0'. Then $\mathrm{sat}(z) \cap \Theta'_{\bar{\eta}'} = \varnothing$ since

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sat$(z) \cap \Theta'_{\bar{u}'} = \varnothing$ and η' specialises into u'. Next, we claim that there exists a point $v' \in S'_0$ such that sat$(z) \cap \Theta'_{\bar{v}'} \neq \varnothing$, which would imply the assertion of Theorem 3.10 (cf. Proposition 1.3). As the morphism $\mathcal{Z}.\Theta' \to T$ is flat at $z \in \mathcal{Z}.\Theta'$ (which is above $t \in T$), its image contains the image of the natural morphism $\operatorname{Spec}(\mathcal{O}_{T,t}) \to T$, hence, in particular, contains the generic point $\xi \in T$. So, let $y \in \mathcal{Z}.\Theta'$ be any point above $\xi \in T$, and $v' \in \mathcal{S}'$ the image of y in $\mathcal{S}'$. Then by assumption it holds that $z \in \operatorname{sat}(z) \cap \Theta'_{\bar{v}'} \neq \varnothing$ as claimed.

This finishes the proof of Theorem 3.10. $\quad\square$

Proof of Theorem 3.9. Theorem 3.9 follows easily from Theorem 3.10. Indeed, notations as in Theorem 3.10, write η and v for the images of η' and v' in $\mathcal{S}$, respectively. Then we have the following commutative diagram

$$
\begin{array}{ccc}
\pi_1(Y'_{\bar{\eta}'}) & \twoheadrightarrow & \pi_1(Y'_{\bar{v}'}) \\[1em]
\cap & & \cap \\[1em]
\pi_1(X'_{\bar{\eta}'}) & \twoheadrightarrow & \pi_1(X'_{\bar{v}'}) \\[1em]
\| & & \| \\[1em]
\pi_1(X_{\bar{\eta}}) & \twoheadrightarrow & \pi_1(X_{\bar{v}}),
\end{array}
$$

where the horizontal arrows are specialisation homomorphisms. The assertion of Theorem 3.10 implies that $\operatorname{Sp} : \pi_1(Y'_{\bar{\eta}'}) \twoheadrightarrow \pi_1(Y'_{\bar{v}'})$ is not injective, which implies that $\operatorname{Sp} : \pi_1(X_{\bar{\eta}}) \twoheadrightarrow \pi_1(X_{\bar{v}})$ is not injective, as desired. $\quad\square$

Proof of Theorem 3.6. (cf. [S], Theorem 4.4.) Theorem 3.6 follows easily from Theorem 3.9. Indeed, there exist a connected and reduced k-scheme $\mathcal{S}'$ of finite type, a finite surjective k-morphism $\mathcal{S}' \to \mathcal{S}$ and a proper and smooth $\mathcal{S}'$-curve $f : \mathcal{X}' \to \mathcal{S}'$ of (constant) genus g, such that the (coarse) classifying morphism $\mathcal{S}' \to \mathcal{M}_{g,k}$ for f coincides with the composite of $\mathcal{S}' \to \mathcal{S} \subset \mathcal{M}_{g,k}$ (cf. [S], Proof of Proposition B.1). Suppose that π_1 is constant on $\mathcal{S} \subset \mathcal{M}_{g,k}$. Then π_1 is constant on $\mathcal{S}'$ for $\mathcal{X}' \to \mathcal{S}'$. Now, by Theorem 3.9, $\mathcal{X}' \to \mathcal{S}'$ is isotrivial, which implies that $\mathcal{S} \subset \mathcal{M}_{g,k}$ consists of a single point. This is absurd, as $\dim(\mathcal{S}) > 0$. $\quad\square$

Next, we work with the assumptions in Theorems 3.12 and 3.13. In particular, k is a field of characteristic $p > 0$, S is a connected and reduced k-scheme of finite type, $f : X \to S$ is a non-isotrivial proper and smooth S-curve of genus g, and $D \subset X$ is a relatively étale divisor of degree n, such that $2 - 2g - n < 0$.

Proof of Theorem 3.12. (cf. [S], Theorem 4.8.) Theorem 3.12 follows easily from Theorem 3.9. Indeed, suppose that π_1^t is constant on S for $(X, D)/S$. By using the tame version of Lemma 4.1 (cf. [St], Proposition 2.3), we may construct a tame Galois cover $(X', D')/S'$ of $(X, D)/S$ which is geometrically of degree prime to p and ramified at every point of D, such that the genus of (the fibres of) X' is ≥ 2 (cf. [T3], Theorem (8.1)). By construction, π_1^t is constant on S' for $(X', D')/S'$, which implies that π_1 is constant on S' for $X' \to S'$. (This can be proved by either considering the tame version of the specialization homomorphism (cf. Remark 3.11(i)) or resorting to [T2], Theorem (5.2).) Now, by Theorem 3.9, $X' \to S'$ is isotrivial, which implies that $(X, D)/S$ is isotrivial, as in [T3], Proof of Theorem (8.1). This completes the proof. $\quad\square$

Proof of Theorem 3.13. Theorem 3.13 follows from Theorem 3.12, together with [T1], Corollary 1.5. □

<div align="center">References.</div>

[BLR] Bosch, S., Lütkebohmert, W., Raynaud, M., Néron models, Ergebnisse der Mathematik und ihrer Grenzegebiete (3), 21, Springer-Verlag, 1990.

[EGA III] Grothendieck, A., Élements de géométrie algébrique, Étude cohomologique des faisceaux cohérents, Publications mathématiques IHES, 11 and 17, 1961 and 1963.

[EGA IV] Grothendieck, A., Élements de géométrie algébrique, Étude locale des schémas et des morphismes de schémas, Publications mathématiques IHES, 20, 24, 28 and 32, 1964, 1965, 1966 and 1967.

[FJ] Fried, M. D., Jarden, M. F., Field arithmetic, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 11, Springer-Verlag, 1986.

[R] Raynaud, M., Sections des fibrés vectoriels sur une courbe, Bull. Soc. Math. France 110 103–125 (1982).

[S] Saïdi, M., On complete families of curves with a given fundamental group in positive characteristic, Manuscripta Math. 118, 425–441 (2005).

[SGA1] Grothendieck, A., Revêtements étales et groupe fondamental, Lecture Notes in Math. 224, Springer, Heidelberg, 1971.

[Sh] Shafarevich, I., On p-extensions, A.M.S translation, Serie 2, Vol.4 (1965).

[St] Stix, J., A monodromy criterion for extending curves, Internat. Math. Res. Notices 2005, No.29, 1787–1802.

[T1] Tamagawa, A., On the fundamental groups of curves over algebraically closed fields of characteristic > 0, Internat. Math. Res. Notices 1999, No.16, 853–873.

[T2] Tamagawa, A., On the tame fundamental groups of curves over algebraically closed fields of characteristic > 0, in Galois groups and fundamental groups, Math. Sci. Res. Inst. Publ., 41 (L. Schneps, ed.), Cambridge University Press, 2003, pp. 47–105.

[T3] Tamagawa, A., Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups, J. Algebraic Geometry, 13 (2004) 675–724.

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