RUDNICK AND SOUNDARARAJAN'S THEOREM FOR FUNCTION FIELDS

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Abstract. In this paper we prove a function field version of a theorem by Rudnick and Soundararajan about lower bounds for moments of quadratic Dirichlet L–functions. We establish lower bounds for the moments of quadratic Dirichlet L–functions associated to hyperelliptic curves of genus g over a fixed finite field $\mathbb{F}_q$ in the large genus g limit.

1. Introduction

It is a fundamental problem in analytic number theory to estimate moments of central values of L–functions in families. For example, in the case of the Riemann zeta function the question is to establish asymptotic formulae for

$$M_k(T) := \int_1^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt, \tag{1.1}$$

where k is a positive integer and $T \to \infty$.

A believed folklore conjecture asserts that, as $T \to \infty$, there is a positive constant C_k such that

$$M_k(T) \sim C_k T (\log T)^{k^2}. \tag{1.2}$$

Due to the work of Conrey and Ghosh [3] the conjecture above assumes a more explicit form, namely

$$C_k = \frac{a_k g_k}{\Gamma(k^2+1)}, \tag{1.3}$$

where

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$$a_k = \prod_{p \text{ prime}} \left[\left(1 - \frac{1}{p}\right)^{k^2} \sum_{m \geq 0} \frac{d_k(m)^2}{p^m} \right], \tag{1.4}$$

g_k is an integer when k is an integer and $d_k(n)$ is the number of ways to represent n as a product of k factors.

Asymptotics for $M_k(T)$ are only known for $k = 1$, due to Hardy and Littlewood [7]

$$M_1(T) \sim T \log T, \tag{1.5}$$

and for $k = 2$, due to Ingham [10]

$$M_2(T) \sim \frac{1}{2\pi^2} T \log^4 T. \tag{1.6}$$

Unfortunately the recent technology does not allow us to obtain asymptotics for higher moments of the Riemann zeta function. The same statement applies for the higher moments of other L–functions. However, due to the precursor work of Keating and Snaith [14, 15] and, subsequently, due to the work of Conrey, Farmer, Keating, Rubinstein and Snaith [4], and Diaconu, Goldfeld and Hoffstein [5], there are now very elegant conjectures for moments of L–functions.

The work of Katz and Sarnak [12, 13] associates a symmetry group for each family of L–function and the moments are sensitive and take different forms for each one of these groups. In other words the conjectured asymptotic formulas for the moments of families of L–function depends whether the symmetry group attached to the family is unitary, orthogonal or symplectic. For a recent and detailed discussion about a working definition of a family of L–functions see [21].

We will typify the conjectures above by considering different families of L–functions. For example, the family of all Dirichlet L–functions $L(s, \chi)$, as χ varies over primitive characters (mod q), is an example of a unitary family, and it is conjectured that

$$\sideset{}{^*}\sum_{\chi (\bmod q)} |L(\tfrac{1}{2}, \chi)|^{2k} \sim C_{U(N)}(k) q (\log q)^{k^2}, \tag{1.7}$$

where $k \in \mathbb{N}$ and $C_{U(N)}(k)$ is a positive constant. For a symplectic family of L–functions we consider the quadratic Dirichlet L–functions $L(s, \chi_d)$ associated to the quadratic character χ_d, as d varies over fundamental discriminants. In this case it is conjectured that

$$\sideset{}{^\flat}\sum_{|d| \leq X} L(\tfrac{1}{2}, \chi_d)^k \sim C_{USp(2N)}(k) X (\log X)^{k(k+1)/2}, \tag{1.8}$$

where $k \in \mathbb{N}$ and $C_{USp(2N)}(k)$ is a positive constant. And finally we consider the family of L–functions associated to Hecke eigencuspforms f of weight k for the full modular group $SL(2, \mathbb{Z})$ as f varies in the set H_k of Hecke eigencuspforms. This is an example of an orthogonal family and it is conjectured that

$$\sum_{f \in H_k}^{h} L(\tfrac{1}{2}, f)^r \sim C_{O(N)}(r)(\log k)^{r(r-1)/2}, \tag{1.9}$$

where $C_{O(N)}(r)$ is a positive constant, $k \equiv 0 (\bmod\ 4)$ and

$$\sum_{f \in H_k}^{h} L(\tfrac{1}{2}, f)^r := \sum_{f \in H_k} \frac{1}{\omega_f} L(\tfrac{1}{2}, f)^r, \tag{1.10}$$

with

$$\omega_f := \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \langle f, f \rangle = \frac{k-1}{2\pi^2} L(1, \mathrm{Sym}^2 f), \tag{1.11}$$

where $\langle f, f \rangle$ denotes the Petersson inner product. For more details on Hecke eigencuspforms L–functions see Iwaniec [11].

The conjectures (1.2), (1.7) and (1.8) can be verified for small values of k and the same holds for (1.9), where it can be verified only for small values of r. Ramachandra [17] showed that

$$\int_1^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt \gg T(\log T)^{k^2}, \tag{1.12}$$

for positive integers k. Titchmarsh [24, Theorem 7.19] had proved a smooth version of these lower bound for positive integer k. The work of Heath–Brown [8] extends (1.12) for all positive rational numbers k. Recently Radziwiłł and Soundararajan [16] proved that

$$M_k(T) \geq e^{-30k^4} T(\log T)^{k^2}, \tag{1.13}$$

for any real number $k > 1$ and all large T. For other families of L–functions, as those given above, the lower bounds for moments were proved by Rudnick and Soundarajan in [19, 20] where they have established that

$$\sum_{\substack{\chi(\bmod q) \\ \chi \neq \chi_0}}^{*} |L(\tfrac{1}{2}, \chi)|^{2k} \gg_k q(\log q)^{k^2}, \tag{1.14}$$

for a fixed natural number k and all large primes q. They also proved in [20] that

$$\sum_{f\in H_k}^{h} L(\tfrac{1}{2},f)^r \gg_r (\log k)^{r(r-1)/2}, \tag{1.15}$$

for any given natural number r, and weight $k \geq 12$ with $k \equiv 0(\bmod 4)$. And for the symplectic family they showed that for every even natural number k

$$\sum_{|d|\leq X}^{\flat} L(\tfrac{1}{2},\chi_d)^k \gg_k X(\log X)^{k(k+1)/2}, \tag{1.16}$$

where the sum is taken over fundamental discriminants d. Radziwiłł and Soundararajan [16] pointed out that their method may easily be modified to provide lower bounds for moments to the case of L–functions in families, for any real number $k > 1$.

Recently, in a beautiful paper, Tamam [23] proved the function field analogue of (1.14). In this paper we consider the function field analogue of equation (1.16) for quadratic Dirichlet L–functions associated to a family of hyperelliptic curves over $\mathbb{F}_q$. See next section.

2. Main Theorem

Before we enunciate the main theorem of this paper we need a few basic facts about rational function fields. We start by fixing a finite field $\mathbb{F}_q$ of odd cardinality $q = p^a$ with p a prime. And we denote by $A = \mathbb{F}_q[T]$ the polynomial ring over $\mathbb{F}_q$ and by $k = \mathbb{F}_q(T)$ the rational function field over $\mathbb{F}_q$.

The zeta function associated to A is defined by the following Dirichlet series

$$\zeta_A(s) := \sum_{\substack{f\in A\\ \text{monic}}} \frac{1}{|f|^s} \qquad \text{for } \mathrm{Re}(s) > 1, \tag{2.1}$$

where $|f| = q^{\deg(f)}$ for $f \neq 0$ and $|f| = 0$ for $f = 0$. Surprisingly the zeta function associated to A is a much simpler object than the usual Riemann zeta function and can be showed that

$$\zeta_A(s) = \frac{1}{1-q^{1-s}}. \tag{2.2}$$

Let D be a square–free monic polynomial in A of degree odd. Then we define the quadratic character χ_D attached to D by making use of the quadratic residue symbol for $\mathbb{F}_q[T]$ by

$$\chi_D(f) = \left(\frac{D}{f}\right). \tag{2.3}$$

In other words, if $P \in A$ is monic irreducible we have

$$\chi_D(P) = \begin{cases} 0, & \text{if } P \mid D, \\ 1, & \text{if } P \not| D \text{ and } D \text{ is a square modulo } P, \\ -1, & \text{if } P \not| D \text{ and } D \text{ is a non square modulo } P. \end{cases} \tag{2.4}$$

For more details about Dirichlet characters for function fields see [18, Chapter 3] and [6].

We attach to the character χ_D the quadratic Dirichlet L–function defined by

$$L(s,\chi_D) = \sum_{\substack{f\in A \\ f \text{ monic}}} \frac{\chi_D(f)}{|f|^s} \qquad \text{for } \mathrm{Re}(s) > 1. \tag{2.5}$$

If $D \in \mathcal{H}_{2g+1,q}$, where

$$\mathcal{H}_{2g+1,q} = \{D \in A, \text{ square–free, monic and } \deg(D) = 2g+1\}, \tag{2.6}$$

then the L–function associated to χ_D is indeed the numerator of the zeta function associated to the hyperelliptic curve $C_D : y^2 = D(T)$ and therefore $L(s,\chi_D)$ is a polynomial in $u = q^{-s}$ of degree $2g$ given by

$$L(s,\chi_D) = \sum_{n=0}^{2g} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-ns}. \tag{2.7}$$

(see [18, Propositions 14.6 and 17.7] and [1, Section 3]).

This L–function satisfies a functional equation, namely

$$L(s,\chi_D) = (q^{1-2s})^g L(1-s,\chi_D), \tag{2.8}$$

and the Riemann hypothesis for curves proved by Weil [25] tell us that all the zeros of $L(s,\chi_D)$ have real part equals 1/2.

The main result of this paper is now presented:

Theorem 2.1. *For every even natural number k we have,*

$$\frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D\in\mathcal{H}_{2g+1,q}} L(\tfrac{1}{2},\chi_D)^k \gg_k (\log_q |D|)^{k(k+1)/2}. \tag{2.9}$$

Remark 2.1. *To avoid any misunderstanding concerning the notation and conventions presented in this paper it is necessary a note about the notation used in the theorem above and in the rest of this note. On the formula above the right-hand side of the main lower bound appears $|D| = q^{2g+1}$ while D is the summation variable on the left-hand side of that same formula. This is done because the function $D \mapsto |D|$ is constant within $\mathcal{H}_{2g+1,q}$ and so we can always write*

$$\sum_{D\in\mathcal{H}_{2g+1,q}} |D| = |D| \sum_{D\in\mathcal{H}_{2g+1,q}} 1.$$

In the case the reader feel uncomfortable with the above notation he/she can always remember that $|D| = q^{2g+1}$.

Remark 2.2. *For simplicity, we will restrict ourselves to the fundamental discriminants* $D \in A$, *D monic and* $deg(D) = 2g+1$. *But the calculations are analogous for the even case, i.e.,* $deg(D) = 2g+2$.

Using the same techniques developed by Rudnick and Soundararajan in [19, 20] and extended for function fields in this paper we can also prove the following theorem.

Theorem 2.2. *For every even natural number* k *and* $n = 2g+1$ *or* $n = 2g+2$ *we have,*

$$\frac{1}{\pi_A(n)} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ deg(P)=n}} L(\tfrac{1}{2}, \chi_P)^k \gg_k (\log_q |P|)^{\frac{k(k+1)}{2}}, \tag{2.10}$$

where $\pi_A(n) = \#\{P \in \mathbb{F}_q[T] \text{ monic and irreducible, } deg(P) = n\}$ *and the prime number theorem for polynomials* [18, Theorem 2.2] *says that* $\pi_A(n) = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right)$.

3. Necessary Tools

In this section we present some auxiliary lemmas that will be used in the proof of the main theorem. We start with:

Lemma 3.1 ("Approximate" Functional Equation)**.** *Let* $D \in \mathcal{H}_{2g+1,q}$. *Then* $L(s, \chi_D)$ *can be represented as*

$$L(s,\chi_D) = \sum_{\substack{f_1 \text{ monic} \\ deg(f_1)\leq g}} \frac{\chi_D(f_1)}{|f_1|^s} + (q^{1-2s})^g \sum_{\substack{f_2 \text{ monic} \\ deg(f_2)\leq g-1}} \frac{\chi_D(f_2)}{|f_2|^{1-s}}. \tag{3.1}$$

Proof. The proof of this Lemma can be found in [1, Lemma 3.3]. □

The following lemma is the function field analogue of Pólya–Vinogradov inequality for character sums.

Lemma 3.2 (Pólya–Vinogradov inequality for $\mathbb{F}_q(T)$)**.** *Let* χ *be a non–principal Dirichlet character modulo* $Q \in \mathbb{F}_q[T]$ *such that* $deg(Q)$ *is odd. Then we have,*

$$\sum_{deg(f)=x} \chi(f) \ll |Q|^{1/2}. \tag{3.2}$$

Proof. The proof of this Lemma can be found in [9, Proposition 2.1]. □

The next lemma is taken from Andrade-Keating [1, Proposition 5.2] and it is about counting the number of square–free polynomials coprime to a fixed monic polynomial.

Lemma 3.3. *Let $f \in A$ be a fixed monic polynomial. Then for all $\varepsilon > 0$ we have that*

$$\sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,f)=1}} 1 = \frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|f}} \left(\frac{|P|}{|P|+1} \right) + O\left(|D|^{\frac{1}{2}} |f|^{\varepsilon} \right). \tag{3.3}$$

4. Proof of Theorem 2.1

In this section we prove Theorem 2.1.

Let k be a given even number, and set $x = \frac{2(2g)}{15k}$. We define

$$A(D) = \sum_{\deg(n) \le x} \frac{\chi_D(n)}{\sqrt{|n|}}, \tag{4.1}$$

and let

$$S_1 := \sum_{D \in \mathcal{H}_{2g+1,q}} L(\tfrac{1}{2}, \chi_D) A(D)^{k-1}, \tag{4.2}$$

and

$$S_2 := \sum_{D \in \mathcal{H}_{2g+1,q}} A(D)^k. \tag{4.3}$$

An application of Triangle inequality followed by Hölder's inequality gives us that,

$$\left| \sum_{D \in \mathcal{H}_{2g+1,q}} L(\tfrac{1}{2}, \chi_D) A(D)^{k-1} \right| \le \sum_{D \in \mathcal{H}_{2g+1,q}} |L(\tfrac{1}{2}, \chi_D)| |A(D)|^{k-1}$$

$$\le \left(\sum_{D \in \mathcal{H}_{2g+1,q}} L(\tfrac{1}{2}, \chi_D)^k \right)^{1/k} \left(\sum_{D \in \mathcal{H}_{2g+1,q}} A(D)^k \right)^{\frac{k-1}{k}}. \tag{4.4}$$

From (4.4) we have

$$\begin{aligned}\sum_{D\in\mathcal{H}_{2g+1,q}} L(\tfrac{1}{2},\chi_D)^k &\geq \frac{\left(\sum_{D\in\mathcal{H}_{2g+1,q}} L(\frac{1}{2},\chi_D)A(D)^{k-1}\right)^k}{\left(\sum_{D\in\mathcal{H}_{2g+1,q}} A(D)^k\right)^{k-1}} \\ &= \frac{S_1^k}{S_2^{k-1}}. \end{aligned} \tag{4.5}$$

Hence from (4.5) we can see that to prove Theorem 2.1 we only need to give satisfactory estimates for S_1 and S_2. We start with S_2.

4.1. **Estimating S_2.** We have that

$$A(D)^k = \sum_{\substack{n_1,\dots,n_k \\ \deg(n_j)\leq x \\ j=1,\dots,k}} \frac{\chi_D(n_1\dots n_k)}{\sqrt{|n_1|\dots|n_k|}}. \tag{4.6}$$

So,

$$\begin{aligned} S_2 &= \sum_{D\in\mathcal{H}_{2g+1,q}} A(D)^k \\ &= \sum_{\substack{n_1,\dots,n_k \\ \deg(n_j)\leq x \\ j=1,\dots,k}} \frac{1}{\sqrt{|n_1|\dots|n_k|}} \sum_{D\in\mathcal{H}_{2g+1,q}} \left(\frac{D}{n_1\dots n_k}\right). \end{aligned} \tag{4.7}$$

At this stage we need an auxiliary Lemma. It is called orthogonal relations for quadratic characters and it has appeared in a different form in [1, 2, 6].

Lemma 4.1. *If $n\in A$ is not a perfect square then*

$$\sum_{\substack{D\in\mathcal{H}_{2g+1,q} \\ n\neq\square}} \left(\frac{D}{n}\right) \ll |D|^{1/2}|n|^{1/4}. \tag{4.8}$$

And if $n\in A$ is a perfect square then

$$\sum_{\substack{D\in\mathcal{H}_{2g+1,q} \\ n=\square}} \left(\frac{D}{n}\right) = \frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n}} \left(\frac{|P|}{|P|+1}\right) + O\left(|D|^{\frac{1}{2}}|n|^{\varepsilon}\right), \tag{4.9}$$

for any $\varepsilon>0$.

Remark 4.2. *Equation* (4.8) *can be seen as an improvement on the estimate given in* [6, Lemma 3.1]. *And the same equation* (4.8) *can be used to improve the error term in the first moment of quadratic Dirichlet L–functions over function fields as given in* [1, Theorem 2.1].

Proof. If $n = \square$, then

$$\sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ n=\square}} \left(\frac{D}{n}\right) = \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,n)=1}} 1. \tag{4.10}$$

By invoking Lemma 3.3 we establish equation (4.9).

For (4.8) we write

$$\begin{aligned}
\sum_{D \in \mathcal{H}_{2g+1,q}} \left(\frac{D}{n}\right) &= \sum_{2\alpha+\beta=2g+1} \sum_{\deg(B)=\beta} \sum_{\deg(A)=\alpha} \mu(A) \left(\frac{A^2 B}{n}\right) \\
&= \sum_{0 \le \alpha \le g} \sum_{\deg(A)=\alpha} \mu(A) \left(\frac{A^2}{n}\right) \sum_{\deg(B)=2g+1-2\alpha} \left(\frac{B}{n}\right) \\
&\le \sum_{0 \le \alpha \le g} \sum_{\deg(A)=\alpha} \sum_{\deg(B)=2g+1-2\alpha} \left(\frac{B}{n}\right). \qquad (4.11)
\end{aligned}$$

If $n \neq \square$ then $\sum_{\deg(B)=2g+1-2\alpha} \left(\frac{B}{n}\right)$ is a character sum to a non–principal character modulo n. So using Lemma 3.2 we have that

$$\sum_{\deg(B)=2g+1-2\alpha} \left(\frac{B}{n}\right) \ll |n|^{1/2}. \tag{4.12}$$

Further we can estimate trivially the non–principal character sum by

$$\sum_{\deg(B)=2g+1-2\alpha} \left(\frac{B}{n}\right) \ll \frac{|D|}{|A|^2} = q^{2g+1-2\alpha}. \tag{4.13}$$

Thus, if $n \neq \square$, we obtain that

$$\begin{aligned}
\sum_{D \in \mathcal{H}_{2g+1,q}} \left(\frac{D}{n}\right) &\ll \sum_{0 \le \alpha \le g} \sum_{\deg(A)=\alpha} \min\left(|n|^{1/2}, \frac{|D|}{|A|^2}\right) \\
&\ll |D|^{\frac{1}{2}} |n|^{\frac{1}{4}}, \qquad (4.14)
\end{aligned}$$

upon using the first bound (4.12) for $\alpha \le g - \frac{\deg(n)}{4}$ and the second bound (4.13) for larger α. And this concludes the proof of the lemma. $\square$

Using Lemma 4.1 in (4.7) we obtain that

$$S_2 = \sum_{\substack{n_1,\dots,n_k \\ \deg(n_j)\leq x \\ j=1,\dots,k \\ n_1\dots n_k=\square}} \frac{1}{\sqrt{|n_1|\dots|n_k|}} \left(\frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n_1\dots n_k}} \left(\frac{|P|}{|P|+1} \right) \right)$$
$$+ \sum_{\substack{n_1,\dots,n_k \\ \deg(n_j)\leq x \\ j=1,\dots,k \\ n_1\dots n_k=\square}} \frac{1}{\sqrt{|n_1|\dots|n_k|}} O\left(|D|^{\frac{1}{2}} |n_1\dots n_k|^{\varepsilon} \right)$$
$$+ \sum_{\substack{n_1,\dots,n_k \\ \deg(n_j)\leq x \\ j=1,\dots,k \\ n_1\dots n_k\neq\square}} \frac{1}{\sqrt{|n_1|\dots|n_k|}} O\left(|D|^{\frac{1}{2}} |n_1\dots n_k|^{\frac{1}{4}} \right). \tag{4.15}$$

After some arithmetic manipulations with the O–terms we get that

$$S_2 = \frac{|D|}{\zeta_A(2)} \sum_{\substack{n_1,\dots,n_k \\ \deg(n_j)\leq x \\ j=1,\dots,k \\ n_1\dots n_k=\square}} \frac{1}{\sqrt{|n_1|\dots|n_k|}} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n_1\dots n_k}} \left(\frac{|P|}{|P|+1} \right)$$
$$+ O\left(|D|^{\frac{1}{2}} q^{\left(\frac{3}{4}+\varepsilon\right)x} \right). \tag{4.16}$$

Since $x = \frac{2(2g)}{15k}$, the error term above is $\ll |D|^{\frac{3}{5}}$. So,

$$S_2 = \frac{|D|}{\zeta_A(2)} \sum_{\substack{n_1,\dots,n_k \\ \deg(n_j)\leq x \\ j=1,\dots,k \\ n_1\dots n_k=\square}} \frac{1}{\sqrt{|n_1|\dots|n_k|}} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n_1\dots n_k}} \left(\frac{|P|}{|P|+1} \right) + O\left(|D|^{\frac{3}{5}} \right). \tag{4.17}$$

Writing $n_1 \dots n_k = m^2$ we see that

$$\begin{aligned}
&\sum_{\substack{m^2 \text{ monic} \\ \deg(m^2)\le x}} \frac{d_k(m^2)}{|m|} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(\frac{|P|}{|P|+1}\right) \\
&\le \sum_{\substack{n_1,\ldots,n_k \\ \deg(n_j)\le x \\ j=1,\ldots,k \\ n_1\ldots n_k=\square=m^2}} \frac{1}{\sqrt{|n_1|\ldots|n_k|}} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n_1\ldots n_k}} \left(\frac{|P|}{|P|+1}\right) \\
&\le \sum_{\substack{m^2 \text{ monic} \\ \deg(m^2)\le kx}} \frac{d_k(m^2)}{|m|} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(\frac{|P|}{|P|+1}\right),
\end{aligned} \tag{4.18}$$

where $d_k(m)$ represents the number of ways to write the monic polynomial m as a product of k factors.

We need to obtain an estimate for

$$\sum_{\substack{m \text{ monic} \\ \deg(m)=x}} d_k(m^2) a_m, \tag{4.19}$$

where $a_m = \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(\frac{|P|}{|P|+1}\right)$.

To obtain the desired estimate we consider the corresponding Dirichlet series

$$\zeta_f(s) = \sum_{m \text{ monic}} \frac{d_k(m^2)a_m}{|m|^s} = \sum_{n=0}^{\infty} \sum_{\deg(m)=x} d_k(m^2) a_m u^x = Z_f(u), \tag{4.20}$$

with $u = q^{-s}$. Writing the above as an Euler product

$$\zeta_f(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 + \frac{d_k(P^2)a_P}{|P|^s} + \frac{d_k(P^4)a_{P^2}}{|P|^{2s}} + \cdots\right), \tag{4.21}$$

we can identify the poles of $\zeta_f(s)$. Similar calculations carried out in the classical case by Soundararajan and Rudnick [20, page 9] and Selberg [22, Theorem 2], and for function fields by Andrade and Keating [2, Section 4.3] shows us that $\zeta_f(s)$ has a pole at $s = 1$ of order $\frac{k(k+1)}{2}$. Therefore we can write

$$\zeta_f(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|^s}\right)^{-\frac{k(k+1)}{2}}$$

$$\times \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 + \left(\frac{|P|}{|P|+1} \sum_{j=1}^{\infty} \frac{d_k(P^{2j})}{|P|^{js}}\right)\right) \left(1 - \frac{1}{|P|^s}\right)^{\frac{k(k+1)}{2}}, \tag{4.22}$$

where the first product has a pole at $s = 1$ of order $\frac{k(k+1)}{2}$ and the second product above (4.22) is convergent for $\mathrm{Re}(s) > 1$ and holomorphic in $\{s \in B \mid \mathrm{Re}(s) = 1\}$ with

$$B = \left\{ s \in \mathbb{C} \mid -\frac{\pi i}{\log(q)} \leq \Im(s) < \frac{\pi i}{\log(q)} \right\}. \tag{4.23}$$

Thus we can use Theorem 17.4 from [18] to obtain the desired estimate. But we sketch below how this can be done. A standard contour integration (Cauchy's theorem)

$$\frac{1}{2\pi i} \oint_{C_\varepsilon + C} \frac{Z_f(u)}{u^{x+1}} du = \sum \mathrm{Res}(Z_f(u) u^{-x-1}), \tag{4.24}$$

where C is the boundary of the disc $\{u \in \mathbb{C} \mid |u| \leq q^{-\delta}\}$ for some $\delta < 1$ and C_ε a small circle about $s = 0$ oriented clockwise. There is only one pole in the integration region $C_\varepsilon + C$ and it is located at $u = q^{-1}$ as can be seen from (4.22). To find the residue there, we expand both $Z_f(u)$ and u^{-x-1} in Laurent series about $u = q^{-1}$, multiply the results together, and pick out the coefficient of $(u - q^{-1})^{-1}$. After this residue calculation we obtain that

$$\sum_{\substack{m \text{ monic} \\ \deg(m) = x}} d_k(m^2) a_m \sim C(k) q^x x^{\frac{k(k+1)}{2} - 1}, \tag{4.25}$$

for a positive constant $C(k)$ explicitly given by

$$C(k) = \frac{\log(q)^{\frac{k(k+1)}{2}}}{\left(\frac{k(k+1)}{2} - 1\right)!} \alpha, \tag{4.26}$$

with

$$\alpha = \lim_{s \to 1} \left[(s-1)^{\frac{k(k+1)}{2}} \zeta_f(s) \right]. \tag{4.27}$$

In the end we obtain that

$$\sum_{\substack{m \text{ monic} \\ \deg(m)\leq z}} \frac{d_k(m^2)}{|m|} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(\frac{|P|}{|P|+1}\right) \sim C(k)(z)^{k(k+1)/2}. \tag{4.28}$$

Therefore we can conclude that

$$S_2 \asymp |D|(\log_q |D|)^{k(k+1)/2}. \tag{4.29}$$

4.2. **Estimating S_1.** It remains to evaluate S_1 and for that we need an "approximate" functional equation for $L(\frac{1}{2}, \chi_D)$. Using Lemma 3.1 with $s = \frac{1}{2}$ we have that

$$\begin{aligned}
S_1 = &\sum_{D\in\mathcal{H}_{2g+1,q}} \left(\sum_{\deg(f_1)\leq g} \frac{\chi_D(f_1)}{|f_1|^{1/2}} + \sum_{\deg(f_2)\leq g-1} \frac{\chi_D(f_2)}{|f_2|^{1/2}} \right) \\
&\times \left(\sum_{\substack{n_1,\ldots,n_{k-1} \\ \deg(n_j)\leq x \\ j=1,\ldots,k-1}} \frac{\chi_D(n_1\ldots n_{k-1})}{\sqrt{|n_1|\ldots|n_{k-1}|}} \right) \\
= &\sum_{\deg(f_1)\leq g} \frac{1}{\sqrt{|f_1|}} \sum_{\substack{n_1,\ldots,n_{k-1} \\ \deg(n_j)\leq x \\ j=1,\ldots,k-1}} \frac{1}{\sqrt{|n_1|\ldots|n_{k-1}|}} \sum_{D\in\mathcal{H}_{2g+1,q}} \left(\frac{D}{f_1 n_1\ldots n_{k-1}}\right) \\
&+ \sum_{\deg(f_2)\leq g-1} \frac{1}{\sqrt{|f_2|}} \sum_{\substack{n_1,\ldots,n_{k-1} \\ \deg(n_j)\leq x \\ j=1,\ldots,k-1}} \frac{1}{\sqrt{|n_1|\ldots|n_{k-1}|}} \sum_{D\in\mathcal{H}_{2g+1,q}} \left(\frac{D}{f_2 n_1\ldots n_{k-1}}\right).
\end{aligned} \tag{4.30}$$

In the last equality in equation (4.30) the sums over f_1 and f_2 are exactly the same, with the only difference being the size of the sums, i.e., $\deg(f_1) \leq g$ and $\deg(f_2) \leq g-1$. We estimate only the f_1 sum in the last equality and the result being the same for the f_2 sum just replacing g by $g-1$.

If $f_1 n_1 \ldots n_{k-1}$ is not a square then an application of Lemma 4.1 gives us that

$$\sum_{\deg(f_1)\le g}\frac{1}{\sqrt{|f_1|}}\sum_{\substack{n_1,\ldots,n_{k-1}\\ \deg(n_j)\le x\\ j=1,\ldots,k-1}}\frac{1}{\sqrt{|n_1|\ldots|n_{k-1}|}}\sum_{D\in\mathcal{H}_{2g+1,q}}\left(\frac{D}{f_1n_1\ldots n_{k-1}}\right)$$

$$\ll \sum_{\deg(f_1)\le g}\frac{1}{\sqrt{|f_1|}}\sum_{\substack{n_1,\ldots,n_{k-1}\\ \deg(n_j)\le x\\ j=1,\ldots,k-1}}\frac{1}{\sqrt{|n_1|\ldots|n_{k-1}|}}|D|^{\frac{1}{2}}|f_1n_1\ldots n_{k-1}|^{\frac{1}{4}}$$

$$= |D|^{\frac{1}{2}}\sum_{\deg(f_1)\le g}|f_1|^{-\frac{1}{4}}\sum_{\deg(n_1)\le x}|n_1|^{-\frac{1}{4}}\cdots\sum_{\deg(n_{k-1})\le x}|n_{k-1}|^{-\frac{1}{4}}$$

$$\ll |D|^{\frac{1}{2}}q^{\frac{3}{4}g}(q^x)^{(k-1)\frac{3}{4}}. \tag{4.31}$$

With our choice of x, we have that for $f_1n_1\ldots n_{k-1}$ not a square

$$\sum_{\deg(f_1)\le g}\frac{1}{\sqrt{|f_1|}}\sum_{\substack{n_1,\ldots,n_{k-1}\\ \deg(n_j)\le x\\ j=1,\ldots,k-1}}\frac{1}{\sqrt{|n_1|\ldots|n_{k-1}|}}\sum_{D\in\mathcal{H}_{2g+1,q}}\left(\frac{D}{f_1n_1\ldots n_{k-1}}\right)$$

$$\ll |D|^{\frac{39}{40}}. \tag{4.32}$$

For $f_2n_1\ldots n_{k-1}$ not equal to a perfect square, the same reasoning gives

$$\sum_{\deg(f_2)\le g-1}\frac{1}{\sqrt{|f_2|}}\sum_{\substack{n_1,\ldots,n_{k-1}\\ \deg(n_j)\le x\\ j=1,\ldots,k-1}}\frac{1}{\sqrt{|n_1|\ldots|n_{k-1}|}}\sum_{D\in\mathcal{H}_{2g+1,q}}\left(\frac{D}{f_2n_1\ldots n_{k-1}}\right)$$

$$\ll |D|^{\frac{39}{40}}. \tag{4.33}$$

It remains to estimate the main–term in S_1. If $f_1n_1\ldots n_{k-1}$ is a perfect square then

$$\sum_{D\in\mathcal{H}_{2g+1,q}}\sum_{\substack{f_1,n_1,\ldots,n_{k-1}\\ \deg(f_1)\le g\\ \deg(n_j)\le x\\ j=1,\ldots,k-1}}\frac{1}{\sqrt{|f_1||n_1|\ldots|n_{k-1}|}}\chi_D(f_1n_1\ldots n_{k-1}) \tag{4.34}$$

$$= \sum_{\substack{f_1,n_1,\ldots,n_{k-1}\\ \deg(f_1)\le g\\ \deg(n_j)\le x\\ j=1,\ldots,k-1}}\frac{1}{\sqrt{|f_1||n_1|\ldots|n_{k-1}|}}\sum_{D\in\mathcal{H}_{2g+1,q}}\chi_D(f_1n_1\ldots n_{k-1}).$$

By Lemma 3.3 we have that (4.34) becomes

$$\sum_{D\in\mathcal{H}_{2g+1,q}} \sum_{\substack{f_1,n_1,\dots,n_{k-1}\\ \deg(f_1)\le g\\ \deg(n_j)\le x\\ j=1,\dots,k-1}} \frac{1}{\sqrt{|f_1||n_1|\dots|n_{k-1}|}}\chi_D(f_1n_1\dots n_{k-1})$$

$$= \sum_{\substack{f_1,n_1,\dots,n_{k-1}\\ \deg(f_1)\le g\\ \deg(n_j)\le x\\ j=1,\dots,k-1}} \frac{1}{\sqrt{|f_1||n_1|\dots|n_{k-1}|}}\frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic}\\ \text{irreducible}\\ P|f_1n_1\dots n_{k-1}}} \left(\frac{|P|}{|P|+1}\right)$$

$$+O\left(\sum_{\substack{f_1,n_1,\dots,n_{k-1}\\ \deg(f_1)\le g\\ \deg(n_j)\le x\\ j=1,\dots,k-1}} \frac{1}{\sqrt{|f_1||n_1|\dots|n_{k-1}|}}|D|^{\frac{1}{2}}|f_1n_1\dots n_{k-1}|^{\varepsilon}\right) \tag{4.35}$$

If we call $a_f = \prod_{P|f}\left(\frac{|P|}{|P|+1}\right)$, then we have

$$\sum_{D\in\mathcal{H}_{2g+1,q}} \sum_{\substack{f_1,n_1,\dots,n_{k-1}\\ \deg(f_1)\le g\\ \deg(n_j)\le x\\ j=1,\dots,k-1}} \frac{1}{\sqrt{|f_1||n_1|\dots|n_{k-1}|}}\chi_D(f_1n_1\dots n_{k-1})$$

$$= \frac{|D|}{\zeta_A(2)} \sum_{\deg(m)\le \frac{g+(k-1)x}{2}} \frac{a_{m^2}d_k(m^2)}{|m|} + O\left(|D|^{\frac{1}{2}}q^{g(\varepsilon-\frac{1}{2})+g}(q^{x(\varepsilon-\frac{1}{2})+x})^{k-1}\right). \tag{4.36}$$

With our choice of x we have that the O–term above is $\ll |D|^{39/40}$.

The last step is to estimate the main term contribution

$$\frac{|D|}{\zeta_A(2)} \sum_{\deg(m)\le \frac{g+(k-1)x}{2}} \frac{a_{m^2}d_k(m^2)}{|m|}. \tag{4.37}$$

By employing the same reasoning of Rudnick and Soundararajan [20, page 10] we write $n_1\cdots n_{k-1} = rh^2$ where r and h are monic polynomials and r is square–free. Then f_1 is of the form rl^2. With this notation the main term contribution is

$$\frac{|D|}{\zeta_A(2)} \sum_{\substack{n_1,\dots,n_{k-1}\\ n_1\cdots n_{k-1}=rh^2\\ \deg(n_j)\le x\\ j=1,\dots,k-1}} \frac{1}{|rh|} \sum_{\substack{l \text{ monic}\\ \deg(l)\le \frac{g-\deg(r)}{2}}} \frac{1}{|l|}a_{rhl}. \tag{4.38}$$

Note that $\deg(r) \leq (k-1)x$ and an easy calculation as those used in [20, page 10] and [1, Lemma 5.7 and pages 2812–2813] gives that the sum over l above is

$$\sum_{\substack{l \text{ monic} \\ \deg(l) \leq \frac{g-\deg(r)}{2}}} \frac{1}{|l|} a_{rhl} \sim C(r,h) a_{rh} g, \tag{4.39}$$

for some positive constant $C(r,h)$.

Therefore follows that the main term contribution to (4.36) is

$$\begin{aligned}
&\gg |D|(\log_q |D|) \sum_{\substack{n_1,\ldots,n_{k-1} \\ n_1 \cdots n_{k-1} = rh^2 \\ \deg(n_j) \leq x \\ j=1,\ldots,k-1}} \frac{1}{|rh|} a_{rh} \\
&\gg |D|(\log_q |D|) \sum_{\substack{r,h \\ \deg(rh^2) \leq x}} \frac{d_{k-1}(rh^2)}{|rh|} a_{rh} \\
&\gg |D|(\log_q |D|)^{k(k+1)/2}, \qquad (4.40)
\end{aligned}$$

where the last bound follows by activating the same estimate as proved in past section, replacing k by $k-1$. The same argument applies to the second sum in (4.29) replacing g by $g-1$. Therefore we can conclude that

$$S_1 \gg |D|(\log_q |D|)^{k(k+1)/2}. \tag{4.41}$$

Combining (4.29) and (4.41) finishes the prove of Theorem 2.1.

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