Composite Fermions with Spin at $\nu = 1/2$.

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Summary. — The model of Composite Fermions for describing interacting electrons in two dimensions in the presence of a magnetic field is described. In this model, charged Fermions are combined with an even number of magnetic flux quanta in such a way that the external magnetic field is compensated on the average for half filling of Landau levels and the interaction is incorporated into an effective mass of the new composite particles. The fluctuations of the Chern-Simons gauge field, which describes formally the flux attachment, induce new interactions between the Composite Fermions. The effective interaction is investigated with particular emphasis on the role of the electron spin at filling factor $\nu = 1/2$. For a system with equal numbers of spin-up and spin-down electrons it is found that the dominant effective interaction is attractive in the spin-singlet channel. This can induce a ground state consisting of Cooper pairs of Composite Fermions that is separated from the excited states by a gap. The results are used to understand recent spin polarization measurements done in the region of the Fractional Quantum Hall Effect at different constant filling factors.

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1. – Introduction

Composite Fermions are quasi-particles constructed of interacting electrons confined to a plane and an even number of flux quanta attached to them [1]. They have been introduced in order to explain the features in the high-magnetic field magneto- and Hall-conductances in the region of the Fractional Quantum Hall Effect (FQHE) [2]. Basically, the attachment of fluxes is used to compensate on the average the external magnetic field at certain (even-denominator) values of the filling factor — the ratio between electron density and magnetic flux density — in such a way that the interaction between the electrons is incorporated into single-electron parameters as the effective mass which becomes dependent on the magnetic field. The system of CF behaves then as a Fermi liquid of non-interacting quasi-particles which can serve as a starting point for a perturbational treatment [3]. In this model, the rational filling factors with odd denominators at which the quantization features of the FQHE appear can be understood as the Integer Quantum Hall Effect (IQHE) [4] of the Composite Fermions [5]. A field theoretical approach to Composite Fermions has been successfully constructed by using the path integral formalism and by introducing a Chern-Simons gauge field [9, 10].

Direct experimental evidences for the existence of Composite Fermions have been found by investigating the magneto-transport in spatially modulated quantum Hall systems. The experimentally detected resistance oscillations have been interpreted by constructing certain trajectories of quasi particles that were commensurable with the spatial modulation of the two dimensional electronic system [11, 12, 13, 14].

Originally, the Composite Fermion description has been developed for fully spin-polarized quantum Hall states. There are, however, experiments in the region of low magnetic field which indicate that FQHE-ground states can be spin unpolarized [15, 16]. Recently, the spin polarization of FQHE-ground states have been measured by radiative recombination of electrons in the inversion layer of high-electron mobility GaAs/AlGaAs-heterostructures with holes bound to acceptors in the delta-doped region [17, 18, 19]. The spin polarization as a function of the magnetic field for several fixed filling factors has been investigated. Crossovers between FQHE-ground states with different spin polarizations have been found at certain values of the magnetic field B. When varying B, the polarization of a ground state remains constant within a large region until a certain crossover field $B_{\rm c}$ is reached. Then, the system is transferred to a new differently polarized ground state which remains again stable when B is changed.

The data have been found to be consistent with the model of non-interacting Composite Fermions but with an effective mass that scales $\propto \sqrt{B}$. The regions of constant spin polarization are then due to the occupation of a fixed number of spin split Composite Fermion Landau levels. The crossover occurs when intersections of the latter coincide with the Fermi Level. Most strikingly, near the crossover fields, plateaus with spin polarizations almost exactly intermediate between the fully spin polarized states appear for temperatures extrapolated to absolute zero. This indicates additional features beyond the non-interacting Composite Fermion model and could be signature of partially polarized collective states. In these experiments, the system has been tuned towards spin degeneracy by making use of the fact that the Zeeman splitting ΔE_Z and the cyclotron splitting $\hbar\omega_\mathrm{c}$ depend differently on B, $\Delta E_\mathrm{Z} \propto B$ and $\hbar\omega_\mathrm{c} \propto \sqrt{B}$, respectively, due to the $\sqrt{B}$-dependence of the effective mass of the Composite Fermions.

In NMR experiments [20], the spin polarization at filling factor 2/3 has also been investigated. A remarkably abrupt transition from a fully polarized state to a state with polarization 3/4 has been detected when decreasing the magnetic field. This has been interpreted as a first order quantum phase transition. For filling factors higher than 2/3, a strong depolarization has been observed that has been associated with two spin flips per additional flux quantum. In these measurements, the system has been tuned via tilting the magnetic field such that the Composite Fermions are subject to an in-plane magnetic field which only influences the Zeeman splitting.

The nature of the collective states under the conditions of the FQHE including the effect of the electron spin has been addressed in recent works. For instance, several spin polarization instabilities have been found by assuming the tilted field geometry [21, 22]. Related to the above described optical data, a non-translationally invariant charge density wave state of Composite Fermions has been proposed in the basis of of restricted Hartree Fock calculations [23]. From exact diagonalizations of a few interacting particles, a liquid of non-symmetric excitons has been suggested for explaining the experiments [24]. It appears that these suggestions do not exhaust the many possibilities of treating the effective interactions between the Composite Fermions and of explaining the spin polarization features. Therefore, it is worthwhile to attempt different approaches. To explain one of these is the main purpose of the present paper.

In the following chapter, we provide an introductory overview of the idea behind the model of Composite Fermions. We show that the main feature of the FQHE, namely the fractional quantization of the Hall conductance at fractional filling factors, can be understood for spin polarized systems once the fundamental idea of "attaching flux quanta" to a charged particle has been accepted. In the third chapter, we consider the specific case of half filling and explain how the formal theory of the Chern-Simons gauge transformation can be formulated for particles including their spin degree of freedom. In the fourth chapter, we calculate the propagators of the gauge field fluctuations that are needed in order to understand the effective interaction between the Composite Fermions. We find two contributions, symmetric and antisymmetric in the gauge fluctuations, that behave differently in the small-frequency and long-wavelength limit. In chapter five we calculate the propagators of the Composite Fermions and establish the self-consistent equations for the self-energies. In the sixth chapter the Dyson equation for the Composite Fermion propagator is solved and in the seventh chapter the self-consistent equation for the energy gap is derived.

2. – Introduction to Composite Fermion Theory

2˙1. *The Aharonov-Bohm Effect.* – A qualitative understanding of the idea behind the model of Composite Fermions can be gained by considering first the Aharonov-Bohm effect. Consider a particle with elementary charge e moving in the vector potential $\mathbf{a}(\mathbf{r})$

$$\mathbf{a}(\mathbf{r}) = \frac{\phi}{2\pi}\,\frac{\hat{\mathbf{z}}\times(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^2} \tag{1}$$

that corresponds to a localized Aharonov-Bohm magnetic field at position $\mathbf{r}_0$ pointing into the z-direction (unit vector $\hat{\mathbf{z}}$)

$$\mathbf{b}(\mathbf{r}) = \phi\,\delta(\mathbf{r}-\mathbf{r}_0)\,\hat{\mathbf{z}}\,. \tag{2}$$

The corresponding stationary Schrödinger equation is (light velocity $c = 1$)

$$\frac{1}{2m^*}\Big[i\hbar\nabla + e\mathbf{a}(\mathbf{r})\Big]^2\psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{3}$$

(m^* effective mass of the charge). With $\mathbf{r} = (x, y)$ we denote the coordinates of the particle in the (x, y)-plane. By replacing (for $|\mathbf{r}-\mathbf{r}_0| \neq 0$)

$$\psi(\mathbf{r}) = e^{-i(e\phi/h)\arg(\mathbf{r}-\mathbf{r}_0)}\,\psi'(\mathbf{r}) \equiv e^{-i\tilde{\phi}\arg(\mathbf{r}-\mathbf{r}_0)}\,\psi'(\mathbf{r}) \tag{4}$$

with $\tilde{\phi} \equiv \phi/\phi_0$ ($\phi_0 = h/e$ flux quantum), one notes that $\psi'(\mathbf{r})$ fulfills the Schrödinger equation of a free particle,

$$-\frac{\hbar^2\nabla^2}{2m^*}\,\psi'(\mathbf{r}) = E\psi'(\mathbf{r})\,. \tag{5}$$

In eq. (4), $\arg(\mathbf{r}-\mathbf{r}_0)$ is the angle of vector $\mathbf{r}-\mathbf{r}_0$ with the x-axis. Thus, eq. (4) appears to be similar to a gauge transformation. The Aharonov-Bohm vector potential eq. (1) is given by the gradient of the exponent in eq. (4). Despite the vector potential is absent in eq. (5), the wave function $\psi'(\mathbf{r})$ contains $\mathbf{a}$ in the boundary conditions. Indeed, assuming in eq. (4) a single-valued function for $\psi(\mathbf{r})$ implies necessarily a multi-valued gauge-transformed function $\psi'(\mathbf{r})$.

The above argument is completely independent of the spin of the charged particle since the starting Hamiltonian does not couple spin and charge and the gauge vector potential does not generate a magnetic field at the position of the charge, $\mathbf{r} \neq \mathbf{r}_0$. The argument is also not changed if an external magnetic field $\mathbf{B}$ is introduced. This leads to an additional vector potential $\mathbf{A}(\mathbf{r})$ and adds a spin dependent Zeeman term $\propto \sigma\cdot\mathbf{B}$ to the Hamiltonian in eq. (5) but does not influence the transformation of eq. (4).

2˙2. *Attaching Fluxes to Fermions*. – For a quantum system with N charges at positions $\mathbf{r}_1\ldots\mathbf{r}_j\ldots\mathbf{r}_N$, interacting via a $V(\mathbf{r}_i-\mathbf{r}_j)$-potential, the above consideration may be generalized as follows. Writing for the N-particle state

$$\psi_{\rm c}(\mathbf{r}_1\ldots\mathbf{r}_N) = \prod_{i\neq j} e^{-i\tilde{\phi}\arg(\mathbf{r}_i-\mathbf{r}_j)}\,\psi_{\rm e}(\mathbf{r}_1\ldots\mathbf{r}_N) \tag{6}$$

it is readily derived that if $\psi_{\rm e}$ fulfills the N-particle Schrödinger equation in the presence of an external vector potential $\mathbf{A}$

$$(7)\quad \left\{ \frac{1}{2m^*} \sum_j \left[i\hbar\nabla_j + e\mathbf{A}(\mathbf{r}_j) \right]^2 + \frac{1}{2} \sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j) \right\} \psi_{\rm e}(\mathbf{r}_1 \ldots \mathbf{r}_N) = E\psi_{\rm e}(\mathbf{r}_1 \ldots \mathbf{r}_N)$$

then $\psi_{\rm c}$ obeys the same Schrödinger equation, but with the effective vector potential

$$(8)\qquad \mathbf{A}_{\rm eff}(\mathbf{r}_j) \equiv \mathbf{A}(\mathbf{r}_j) - \mathbf{a}(\mathbf{r}_j)$$

where

$$(9)\qquad \mathbf{a}(\mathbf{r}) = \frac{\phi}{2\pi} \nabla \sum_i \arg(\mathbf{r} - \mathbf{r}_i) = \frac{\phi}{2\pi} \sum_i \frac{\hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^2}$$

is the generalization of eq. (1). Since $\nabla \times \mathbf{a}(\mathbf{r}) = 0$ for all $\mathbf{r} \neq \mathbf{r}_j$ $(j = 1 \ldots N)$ one may think of this as a gauge transformation. However, if $\mathbf{r} \to \mathbf{r}_i$ the non-single-valuedness of the phases implies singularities at $\mathbf{r}_j$ in the gauge field,

$$(10)\qquad \mathbf{b}(\mathbf{r}) \equiv \nabla \times \mathbf{a}(\mathbf{r}) = \phi \sum_j \delta(\mathbf{r} - \mathbf{r}_j)\hat{\mathbf{z}} \equiv \phi\rho(\mathbf{r})\hat{\mathbf{z}} \equiv \tilde{\phi}\phi_0\rho(\mathbf{r})\hat{\mathbf{z}}\,,$$

with the density of the particles

$$(11)\qquad \rho(\mathbf{r}) = \sum_j \delta(\mathbf{r} - \mathbf{r}_j)\,.$$

Since the above transformation changes only the phase of the N-particle wave function the probability density is not influenced

$$(12)\qquad |\psi_{\rm e}(\mathbf{r}_1 \ldots \mathbf{r}_N)|^2 = |\psi_{\rm c}(\mathbf{r}_1 \ldots \mathbf{r}_N)|^2\,.$$

There are several peculiarities associated with the above transformation which — similar as in the Aharonov-Bohm case — "attaches" $\tilde{\phi}$ flux quanta to each particle. If the number of flux quanta is an integer, the state in eq. (6) is a multivalued function of the angles of $\mathbf{r}_i - \mathbf{r}_j$ with respect to rotations about multiples of 2π. When $\tilde{\phi} = 2m$ $(m = 1, 2, 3, \ldots)$ the phase factors do not change the symmetry of the N-particle state $\psi_{\rm c}$. If $\psi_{\rm e}$ describes Fermions, so does $\psi_{\rm c}$, since when interchanging two particles, $\mathbf{r}_i \leftrightarrow \mathbf{r}_j$, the corresponding phase factor is even. On the other hand, when $\tilde{\phi} = 2m + 1$, the phase factor changes sign. In this case, the new state $\psi_{\rm c}$ is a Boson state. For non-integer $\tilde{\phi}$, particles with intermediate symmetries — anyons — can be generated [6, 7].

2˙3. *The Mean Field Approximation.* – By introducing the mean gauge field (ρ mean particle number density)

$$\overline{\mathbf{b}} = \tilde{\phi}\phi_0\rho\hat{\mathbf{z}} \tag{13}$$

the gauge field may be used to compensate on the average the external magnetic field

$$\mathbf{A}_{\text{eff}}(\mathbf{r}) \equiv \mathbf{A}(\mathbf{r}) - \overline{\mathbf{a}}(\mathbf{r}) + \overline{\mathbf{a}}(\mathbf{r}) - \mathbf{a}(\mathbf{r}) = \overline{\mathbf{a}}(\mathbf{r}) - \mathbf{a}(\mathbf{r}) \equiv -\delta\mathbf{a}(\mathbf{r})\,. \tag{14}$$

This can be achieved if $\mathbf{B}(\mathbf{r}) - \overline{\mathbf{b}}(\mathbf{r}) = 0$. Introducing the filling factor $\nu = \rho\phi_0/B$ and using the mean value eq. (13) $\overline{b}(\mathbf{r}) = \tilde{\phi}\phi_0\rho \equiv B$ implies

$$\tilde{\phi} = \frac{1}{\nu}\,. \tag{15}$$

In order to compensate on the average the external field at $\nu = 1/2$, two flux quanta have to be attached to each electron. We will consider this below in more detail.

Exactly at the filling factor where the gauge field is adjusted for compensating the external field, the interacting electron system in the external magnetic field becomes a system of Composite Fermions with the external field removed, but still in the presence of gauge field fluctuations. The Hamiltonian is now

$$\begin{aligned} H = {} & \frac{1}{2m^*}\sum_j \left[i\hbar\nabla_j - e\delta\mathbf{a}(\mathbf{r}_j)\right]^2 \\ & + \frac{1}{2\phi^2}\int \mathrm{d}\mathbf{r}\int \mathrm{d}\mathbf{r}'\,\nabla\times\delta\mathbf{a}(\mathbf{r})\,V(\mathbf{r}-\mathbf{r}')\,\nabla'\times\delta\mathbf{a}(\mathbf{r}')\,. \end{aligned} \tag{16}$$

We have used here the relation between the fluctuations of the gauge vector potential and the gauge magnetic field

$$\delta\mathbf{b} = \nabla\times\delta\mathbf{a}(\mathbf{r}) = \phi\rho(\mathbf{r})\hat{\mathbf{z}}\,, \tag{17}$$

in order to convert the interaction between the particles into a coupling between the gauge field fluctuations. The constant interaction terms stemming from the mean of the gauge field have been omitted. In the kinetic energy, a coupling between the particles and the gauge fluctuations occurs. If one assumes that via this effective particle-gauge field coupling the interaction between the particles can be incorporated completely into a modification of the effective mass, $m^* \to m_{\text{CF}}$, which will eventually depend on the magnetic field, we observe that the Hamiltonian is that of free, non-interacting particles. The corresponding Fermi momentum (assuming, for the moment, complete spin polarization) is

$$k_{\text{F}} = \sqrt{4\pi\rho} = \frac{1}{\ell_B\sqrt{m}}\,, \tag{18}$$

using eq. (15) with filling factor $\nu = 1/2m$ (m integer) and with the electron magnetic length $\ell_B \equiv \sqrt{\hbar/eB}$.

2·4. *The Fractional Quantum Hall Effect.* – If the external magnetic field is close to, but does not exactly coincide with, the one at $\nu = 1/2m$, one expects then that in analogy with the zero-external field limit the kinetic energy of the new composite particles is completely quenched, and the spectrum consists again of Landau levels at energies $E_n = \hbar\omega_{\rm CF}(n + 1/2)$ with $n > 0$. With $\overline{b} = 2m\rho\phi_0$ one obtains an effective mean field $B_{\rm eff} = B - \overline{b}$ with the Composite Fermion cyclotron frequency $\omega_{\rm CF} = e|B_{\rm eff}|/m_{\rm CF}(B)$. In mean field approximation, $m_{\rm CF}$ would be equal to m^*. However, taking into account the interaction between the particles in higher order, the CF mass is renormalized, thus introducing a dependence on the magnetic field [3]. If the effective filling factor p of the CF is

$$p = \frac{\rho\phi_0}{B_{\rm eff}}\,, \tag{19}$$

the filling factor ν of the original electrons corresponds to the integer filling of $|p|$ Composite Fermion Landau levels

$$\nu = \frac{p}{1 + 2mp}\,. \tag{20}$$

For $m = 1$ (compensation of the external magnetic field at half filling) one obtains for $p = 1, 2, 3\ldots$ the sequences $\nu = 1/3, 2/5, 3/7, \ldots$ and for $p = -1, -2, -3, \ldots$ the sequences $\nu = 1, 2/3, 3/5, \ldots$ which are consistent with the filling factors at which the FQHE is observed.

Using the gauge argument originally suggested by Laughlin for explaining the quantization of the Hall conductance in the IQHE [8] one can obtain also the fractional quantization of the Hall conductance. In this argument, the current is related to adiabatically changing the total electronic energy of a metallic loop via the change of a flux piercing the loop,

$$I = \frac{\Delta E}{\Delta\phi}\,. \tag{21}$$

This is obtained by considering the energy change corresponding to transferring between the edges of the loop the number of electrons that are associated with a flux change of ϕ_0. In the pth Composite Fermion Landau level, the total number of flux quanta associated with one electron is $(2mp + 1)/p = 2m + 1/p \equiv 1/n_p$; the $2m$ flux quanta are due to the Chern-Simons gauge transformation while $1/p$ is due to the occupation of the pth level. Thus, the energy change per flux quantum is $\Delta E = n_p e U_{\rm H}$ (Hall voltage $U_{\rm H}$). This gives for the Hall conductance at filling factor $\nu = p/(2mp + 1)$ the fractionally quantized values

$$G_{\rm H} = \frac{I}{U_{\rm H}} = \frac{p}{2mp + 1}\,\frac{e^2}{h} = \nu\,\frac{e^2}{h} \tag{22}$$

that are observed in the Fractional Quantum Hall Effect.

3. – The Chern-Simons Transformation with Spin

Formally, the transformation to Composite Fermions (CF) can be most straightforwardly introduced by starting from the Lagrangian of an interacting spin-degenerate two dimensional (2D) electron system with mean density $\rho = \rho_\uparrow + \rho_\downarrow$ in the presence of a magnetic field by introducing the statistical Chern-Simons gauge field (from now on we consider $\hbar = 1$) [9],

$$\mathcal{L}(\mathbf{r},t) = \mathcal{L}_{\rm F}(\mathbf{r},t) + \mathcal{L}_{\rm I}(\mathbf{r},t) + \mathcal{L}_{\rm CS}(\mathbf{r},t)\,. \tag{23}$$

The first term

$$\mathcal{L}_{\rm F}(\mathbf{r},t) = \sum_{s=\uparrow,\downarrow} \psi_s^\dagger(\mathbf{r},t)\Big\{ i\partial_t + \mu + e a_0^s(\mathbf{r},t) - \frac{1}{2m^*}\Big[i\nabla + e\Big(\mathbf{A}(\mathbf{r}) - \mathbf{a}^s(\mathbf{r},t)\Big)\Big]^2 \Big\}\psi_s(\mathbf{r},t)\,, \tag{24}$$

corresponds to the non-interacting Fermions, chemical potential μ, and the spins $s = \pm 1/2 = \uparrow,\downarrow$ in the presence of the vector potential of the homogeneous external magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$, and the gauge fields $(a_0^s, \mathbf{a}^s)$ with $\mathbf{a}^s = (a_x^s, a_y^s)$ corresponding to spin s. The Lagrangian of the interaction between the electrons

$$\mathcal{L}_{\rm I}(\mathbf{r},t) = -\frac{1}{2}\sum_{s,s'=\uparrow,\downarrow} \int {\rm d}^2 r' \rho_s(\mathbf{r},t) V(\mathbf{r}-\mathbf{r}') \rho_{s'}(\mathbf{r}',t)\,, \tag{25}$$

contains the densities of the Fermions with spin orientation s, $\rho_s(\mathbf{r},t) \equiv \psi_s^\dagger(\mathbf{r},t)\psi_s(\mathbf{r},t)$.

For the interaction, we assume a homogeneous isotropic potential $V(\mathbf{r}) = V(r) = V_\lambda/(r^2+d^2)^{\lambda/2}$ $(1 < \lambda < 2)$. For $\lambda = 1$, and $d \to 0$, this gives the pure Coulomb repulsion with $V_0 = e^2/\varepsilon$. The Fourier transform of this is $V(q) = 2\pi e^2/\varepsilon q$. For intermediate λ and $d \neq 0$, the potential decays as $r^{-\lambda}$ for large r, then for small q we have $V(q) \propto q^{\lambda-2}$. For $\lambda = \to 2$, $V(q \to 0) = \text{const}$.

The Chern-Simons Lagrangian of the gauge field

$$\mathcal{L}_{\rm CS}(\mathbf{r},t) = -\frac{e}{\tilde{\phi}\,\phi_0}\sum_{s=\uparrow\downarrow} a_0^s(\mathbf{r},t)\,\hat{\mathbf{z}}\cdot\nabla\times\mathbf{a}^s(\mathbf{r},t)\,, \tag{26}$$

is responsible for attaching $\tilde{\phi}$ flux quanta $\phi_0 \equiv hc/e = 2\pi/e$ to each Fermion ($\hat{\mathbf{z}}$ is the unit vector in the direction perpendicular to the plane),

$$\hat{\mathbf{z}}\cdot\nabla\times\mathbf{a}^s(\mathbf{r},t) \equiv b^s(\mathbf{r},t) = \tilde{\phi}\phi_0\rho_s(\mathbf{r},t). \tag{27}$$

This can be seen by minimizing the action with respect to a_0. We have assumed here that the gauge term does not couple the spins. This is equivalent to assuming that the orbital and the spin degrees of freedom are completely decoupled. The total wave function is then constructed from local individual spin-singlet pairs. An approach to the general case has been discussed in [25].

The above total Lagrangian eq. (23) can be shown to describe the same system of interacting electrons moving in a plane as without the Chern-Simons field.

With the Chern-Simons field, the effective magnetic field acting on an electron with the spin s is given by

$$B^s_{\rm eff}(\mathbf{r},t) = B - b^s(\mathbf{r},t)\,. \tag{28}$$

If the filling factors for the two spin directions, $\nu_s = \rho_s\phi_0/B$, are equal, it is possible to compensate the external magnetic field B on the average by the gauge field when adjusting $\nu_s \equiv \nu = \rho_s\phi_0/B = 1/\tilde{\phi}$. This implies $k_{\rm F} = \sqrt{2\pi\rho}$.

To be specific, we assume in the following that $\tilde{\phi} = 2$, such that the mean gauge magnetic field cancels the external one at half filling, $\nu = 1/2$. This is consistent with the above assumption of independent spin-up and spin-down gauge fields. In this case, the phase factor introduced by the gauge transformation into the many-electron wave function is even when interchanging particle indices. This means that the composite particles consisting of one electron and the two flux quanta are Fermions.

4. – The Propagator of the Gauge Field Fluctuations.

In deriving the propagator of the gauge fields, we use the transverse gauge, $\nabla\cdot\mathbf{a}^s = 0$. Then, the Bosonic variables associated with the gauge field fluctuations are the *transverse* components of their Fourier transforms, $a^s_1(\mathbf{q},\omega) \equiv \hat{\mathbf{z}}\cdot\hat{\mathbf{q}}\times[\mathbf{a}^s(\mathbf{q},\omega) - \langle\mathbf{a}^s(\mathbf{q},\omega)\rangle]$. By introducing the mean gauge field into $\mathcal{L}_{\rm F}$ the external field $\mathbf{A}$ is canceled.

The total action $S = \int d\mathbf{r}\,dt\,\mathcal{L}(\mathbf{r},t)$ can be written as ($\mu = 0,1$; $\nu = 0,1$)

$$\begin{aligned} S = {} & \sum_s \int \frac{d\mathbf{k}}{(2\pi)^2}\frac{d\omega}{2\pi}\,\psi^\dagger_s(\mathbf{k},\omega)\left[G^0_s(\mathbf{k},\omega)\right]^{-1}\psi_s(\mathbf{k},\omega) \\ & +\frac{1}{2}\sum_{\alpha\mu\nu}\int\frac{d\mathbf{q}}{(2\pi)^2}\frac{d\Omega}{2\pi}\,a^\alpha_\mu(\mathbf{q},\Omega)\left[D^{0\alpha}_{\mu\nu}(\mathbf{q},\Omega)\right]^{-1}a^{\alpha\dagger}_\nu(\mathbf{q},\Omega) \\ & +\sum_{s\mu}\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{d\omega}{2\pi}\frac{d\mathbf{q}}{(2\pi)^2}\frac{d\Omega}{2\pi}\,\psi^\dagger_s(\mathbf{k}+\mathbf{q},\omega+\Omega)\psi_s(\mathbf{k},\omega)a^s_\mu(\mathbf{q},\Omega)v^s_\mu(\mathbf{k},\mathbf{q}) \end{aligned} \tag{29}$$

Here, we have defined ($\alpha = \pm$)

$$a^\alpha_\mu = \frac{1}{2}(a^\uparrow_\mu + \alpha a^\downarrow_\mu)\,, \tag{30}$$

the free Fermion propagator (μ chemical potential)

$$\left[G^0_s(\mathbf{k},\omega)\right]^{-1} = \omega - \frac{k^2}{2m^*} + \mu + i\delta\,\mathrm{sgn}\,\omega\,, \tag{31}$$

and

$$\left[D^{0\alpha}_{\mu\nu}(\mathbf{q},\Omega)\right]^{-1} = \begin{pmatrix} 0 & \frac{2ieq}{\tilde{\phi}\phi_0} \\ -\frac{2ieq}{\tilde{\phi}\phi_0} & -\frac{e^2\rho}{m^*} - \frac{4q^2V(q)}{\tilde{\phi}^2\phi_0^2}\,\delta_{\alpha,+} \end{pmatrix}. \tag{32}$$

In eq. (29), the first term is due to the free electron part of $\mathcal{L}_{\rm F}$ and the second term contains the contribution of $\mathcal{L}_{\rm CS}$, $\mathcal{L}_{\rm I}$, with the electron density replaced by the constraint eq. (27) and the term $\propto \mathbf{a}^2$ in $\mathcal{L}_{\rm F}$ with the electron density replaced by the mean value. The third term represent the interactions between the gauge field fluctuations and the electrons due to terms $\propto \mathbf{a}$ in $\mathcal{L}_{\rm F}$. The term stemming from replacing the charge densities in the Coulomb interaction by the gauge fluctuations has been incorporated in the matrix element $[D^{0\alpha}_{11}]^{-1}$ in eq. (32). The vertex connecting two Fermions with one gauge field fluctuation operator $a^s_\mu(\mathbf{q},\omega)$ is

$$v^s_\mu(\mathbf{k},\mathbf{q}) = \begin{pmatrix} e \\ \frac{e}{m^*}\,\hat{\mathbf{z}}\cdot\frac{\mathbf{k}\times\mathbf{q}}{|\mathbf{q}|} \end{pmatrix}. \tag{33}$$

The above action is quadratic in the gauge field fluctuations. Thus, one could attempt to proceed by tracing out the gauge operators in order to derive the effective interaction between the Composite Fermions. The result would be exact, and of the form

$$D^0 v^2 \psi^\dagger\psi^\dagger\psi\psi,$$

independent of the frequency. This is due to the fact that the Chern-Simons field is purely topological and does not correspond to a Hamiltonian. A priori, the Chern-Simons gauge field does not have a dynamics. The latter is generated only via the coupling to the Fermions. For determining the dynamics of the *interacting* Composite Fermions one has to use further approximations. By starting from D^0, the lowest order approach does not lead to meaningful results. It is therefore important to generate the dynamics of the gauge field fluctuations, in order to derive the effective interaction between the Composite Fermions.

Although we are interested in the zero-temperature limit, it is more convenient to proceed by using the finite temperature formalism for the propagators [26]. The propagator of the gauge field fluctuations is defined by (T_τ time ordering operator)

$$\mathcal{D}^\alpha_{\mu\nu}(\mathbf{q},\tau) = -\langle T_\tau a^\alpha_\mu(\mathbf{q},\tau) a^{\dagger\alpha}_\nu(\mathbf{q},0)\rangle\,. \tag{34}$$

Formally, it can be obtained from a Dyson equation

$$\mathcal{D} = \mathcal{D}^0 + \mathcal{D}^0\,\Pi\,\mathcal{D}\,. \tag{35}$$

The polarization Π contains the interactions between the electrons and the gauge fluctuations. These are treated in lowest non-vanishing order, i. e. the last term in eq. (29)

enters in second order ($\Pi \approx \Pi^0$). In the limit $|\Omega_n| \ll v_F q \ll v_F k_F$ the result is

$$\Pi^0(\mathbf{q}, \Omega_n) = \begin{pmatrix} -\frac{e^2 m^*}{\pi}\left(1 - \frac{|\Omega_n|}{q v_F}\right) & 0 \\ 0 & \frac{e^2 q^2}{12\pi m^*} + \frac{2e^2|\Omega_n|\rho}{m^* q v_F} - \frac{e^2\rho}{m^*} \end{pmatrix} \tag{36}$$

This, together with $D^0 = \mathcal{D}^0$ (cf. eq. (32)), gives the propagator in random phase approximation ($\alpha = \pm 1$)

$$\mathcal{D}^\alpha(\mathbf{q}, \Omega_m) = \frac{1}{\zeta(q)[\gamma^+(q)\delta_{\alpha,+} + \gamma^-(q) - \eta|\Omega_m|/q] - \beta^2 q^2}$$

$$\times \begin{pmatrix} \gamma^+(q)\delta_{\alpha,+} + \gamma^-(q) - \eta\frac{|\Omega_m|}{q} & -i\beta q \\ i\beta q & \zeta(q) \end{pmatrix} \tag{37}$$

where $\zeta(q) = e^2 m^*(1 - |\Omega_n|/q v_F)/\pi$, $\beta = 2e/\tilde{\phi}\phi_0$, $\gamma^+(q) = -4q^2 V(q)/\tilde{\phi}^2\phi_0^2$, $\gamma^-(q) = -q^2 e^2/12\pi m^*$, $\eta = 2e^2\rho/m^* v_F$. Note that for zero Coulomb interaction $\gamma^+ = 0$ the symmetric and antisymmetric propagators are equal $\mathcal{D}^+ = \mathcal{D}^-$. For small q and Ω_n, the dominant matrix elements are

$$\mathcal{D}^+_{11}(\mathbf{q}, \Omega_n) \approx \frac{-q}{\alpha_+(q) q^2 + \alpha_- q^3 + \eta|\Omega_n|} \tag{38}$$

$$\mathcal{D}^-_{11}(\mathbf{q}, \Omega_n) \approx \frac{-q}{\alpha_- q^3 + \eta|\Omega_n|} \tag{39}$$

with $\alpha_+ = 4qV(q)/\tilde{\phi}^2\phi_0^2$ and $\alpha_- = (e^2/12\pi + 4\pi/\tilde{\phi}^2\phi_0^2)/m^*$. For Coulomb interaction ($\lambda = 1$), $V(q) \propto 1/q$ and $\alpha^+ \approx$ const. In this case, the matrix element $\mathcal{D}^-_{11}$ is much larger than $\mathcal{D}^+_{11}$ for $q \to 0$. On the other hand, when the interaction is screened ($\lambda = 2$), $V(q \to 0)=$ const, $\alpha^+ \propto q$, and $\mathcal{D}^-_{11}$ and $\mathcal{D}^+_{11}$ are of the same order. From now on we will focus on the unscreened Coulomb interaction ($\lambda = 1$), neglecting in eq. (38) the sub-leading term α^-.

5. – The Propagator of the Composite Fermions.

We calculate the Green function of the Composite Fermions by starting from the Nambu field

$$\mathbf{\Phi}(\mathbf{k}, \tau) = \begin{pmatrix} \psi_\uparrow(\mathbf{k}, \tau) \\ \psi^\dagger_\downarrow(-\mathbf{k}, \tau) \end{pmatrix} \equiv \begin{pmatrix} \Phi_1(\mathbf{k}, \tau) \\ \Phi^\dagger_2(\mathbf{k}, \tau) \end{pmatrix} . \tag{40}$$

Using the imaginary-time definition

$$\boldsymbol{\mathcal{G}}(\mathbf{k},\tau) = -\langle T_\tau \boldsymbol{\Phi}(\mathbf{k},\tau)\boldsymbol{\Phi}^\dagger(\mathbf{k},0)\rangle \tag{41}$$

gives the 2×2-matrix

$$\mathcal{G}_{ij}(\mathbf{k},\tau) = \begin{pmatrix} -\langle T_\tau\psi_\uparrow(\mathbf{k},\tau)\psi_\uparrow^\dagger(\mathbf{k},0)\rangle & -\langle T_\tau\psi_\uparrow(\mathbf{k},\tau)\psi_\downarrow(-\mathbf{k},0)\rangle \\ -\langle T_\tau\psi_\downarrow^\dagger(-\mathbf{k},\tau)\psi_\uparrow^\dagger(\mathbf{k},0)\rangle & -\langle T_\tau\psi_\downarrow^\dagger(-\mathbf{k},\tau)\psi_\downarrow(-\mathbf{k},0)\rangle \end{pmatrix}$$

The diagonal parts describe the propagation of Composite Fermions with spin up and down. The anomalous propagators in the off-diagonals, $\propto \langle\psi_\uparrow\psi_\downarrow\rangle$, describe the propagation of Fermion-Fermion pairs with opposite momenta and spins. We implicitly assume that they are different from zero. This has to be verified *a posteriori* at the end of the calculation. In terms of the self energy, the frequency-dependent Matsubara Green function is given by the Dyson equation

$$\boldsymbol{\mathcal{G}}^{-1}(\mathbf{k},\omega_n) = \boldsymbol{\mathcal{G}}_0^{-1}(\mathbf{k},\omega_n) - \boldsymbol{\Sigma}(\mathbf{k},\omega_n) \tag{42}$$

with the free Composite Fermion Green function

$$\boldsymbol{\mathcal{G}}^{\mathbf{0}}(\mathbf{k},\omega_n) = \begin{pmatrix} \frac{1}{i\omega_n-(k^2/2m^*-\mu)} & 0 \\ 0 & \frac{1}{i\omega_n+(k^2/2m^*-\mu)} \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{i\omega_n-\xi_k} & 0 \\ 0 & \frac{1}{i\omega_n+\xi_k} \end{pmatrix}. \tag{43}$$

In lowest non-vanishing order, the matrix elements of the self energy are determined by the interaction terms proportional to v_ν $(i=1,2; j=1,2)$

$$\begin{aligned} \Sigma_{ii}(\mathbf{k},\omega_n) = -\frac{1}{\beta}\sum_{\mu\nu}(-1)^\nu \int \frac{d\mathbf{q}}{(2\pi)^2}\sum_{\Omega_m}\left[\mathcal{D}^+_{\mu\nu}(\mathbf{q},\Omega_m)+\mathcal{D}^-_{\mu\nu}(\mathbf{q},\Omega_m)\right]\times \\ \times v_\mu(\mathbf{k},\mathbf{q})v_\nu(\mathbf{k},-\mathbf{q})\mathcal{G}_{11}(\mathbf{k}-\mathbf{q},\omega_n-\Omega_m) \end{aligned} \tag{44}$$

$$\begin{aligned} \Sigma_{i\neq j}(\mathbf{k},\omega_n) = \frac{1}{\beta}\sum_{\mu\nu}(-1)^\nu \int \frac{d\mathbf{q}}{(2\pi)^2}\sum_{\Omega_m}\left[\mathcal{D}^+_{\mu\nu}(\mathbf{q},\Omega_m)-\mathcal{D}^-_{\mu\nu}(\mathbf{q},\Omega_m)\right]\times \\ \times v_\mu(\mathbf{k},\mathbf{q})v_\nu(-\mathbf{k},-\mathbf{q})\mathcal{G}_{12}(\mathbf{k}-\mathbf{q},\omega_n-\Omega_m)\,. \end{aligned} \tag{45}$$

The self-energies can be chosen to satisfy the relations [27]

$$\Sigma_{21}(\mathbf{k},\omega_n) = \Sigma_{12}(\mathbf{k},\omega_n), \qquad \Sigma_{22}(\mathbf{k},\omega_n) = -\Sigma_{11}(-\mathbf{k},-\omega_n) \tag{46}$$

which can be obtained from the definitions eq. (44), and eq. (45) and from the definition of the Nambu Green function eq. (40). The Dyson equation eq. (42) together with the above eq. (44) and eq. (45) establish a self consistent set of equations for the Green functions.

6. – Solution of the Dyson Equation.

In order to solve the set of equations for the self energies it is useful to transform from the Matsubara propagators to the retarded propagators via analytic continuation to real frequencies [27]. One obtains for the self-energies

$$\begin{aligned}(47)\quad \Sigma^{\rm R}_{11}(\mathbf{k},\epsilon) = &-\frac{1}{2\pi^2}\sum_{\mu\nu}(-1)^\nu \int \frac{d\mathbf{q}}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega\, d\epsilon_1 \\ &\frac{{\rm Im}\big[D^{+,{\rm R}}_{\mu\nu}(\mathbf{k}-\mathbf{q},\omega) + D^{-,{\rm R}}_{\mu\nu}(\mathbf{k}-\mathbf{q},\omega)\big]}{\omega+\epsilon_1-\epsilon-i\delta} \\ &\times v_\mu(\mathbf{k},\mathbf{k}-\mathbf{q})\, v_\nu(\mathbf{k},\mathbf{q}-\mathbf{k})\, {\rm Im}G^{\rm R}_{11}(\mathbf{q},\epsilon_1)\left(\tanh\frac{\epsilon_1}{2T} + \coth\frac{\omega}{2T}\right)\end{aligned}$$

$$\begin{aligned}(48)\quad \Sigma^{\rm R}_{12}(\mathbf{k},\epsilon) = &\frac{1}{2\pi^2}\sum_{\mu\nu}(-1)^\nu \int \frac{d\mathbf{q}}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega\, d\epsilon_1 \\ &\frac{{\rm Im}\big[D^{+,{\rm R}}_{\mu\nu}(\mathbf{k}-\mathbf{q},\omega) - D^{-,{\rm R}}_{\mu\nu}(\mathbf{k}-\mathbf{q},\omega)\big]}{\omega+\epsilon_1-\epsilon-i\delta} \\ &\times v_\mu(\mathbf{k},\mathbf{k}-\mathbf{q})\, v_\nu(-\mathbf{k},\mathbf{q}-\mathbf{k})\, {\rm Im}G^{\rm R}_{12}(\mathbf{q},\epsilon_1)\left(\tanh\frac{\epsilon_1}{2T} + \coth\frac{\omega}{2T}\right)\,.\end{aligned}$$

The imaginary parts of G^R_{11} and G^R_{12} are obtained from the analytic continuation of $\mathcal{G}^R_{11}$ and $\mathcal{G}^R_{12}$ by observing that the Σ-functions depend only on the modulus of the momentum in an isotropic system

$$(49)\qquad G^R_{11}(\mathbf{q},\epsilon_1) = \frac{\epsilon_1+\xi_q+\Sigma^{R\,*}_{11}(q,-\epsilon_1)}{[\epsilon_1-\xi_q-\Sigma^R_{11}(q,\epsilon_1)][\epsilon_1+\xi_q+\Sigma^{R\,*}_{11}(q,-\epsilon_1)]-[\Sigma^R_{12}(q,\epsilon_1)]^2}$$

$$(50)\qquad G^R_{12}(\mathbf{q},\epsilon_1) = \frac{\Sigma^R_{12}(q,\epsilon_1)}{[\epsilon_1-\xi_q-\Sigma^R_{11}(q,\epsilon_1)][\epsilon_1+\xi_q+\Sigma^{R\,*}_{11}(q,-\epsilon_1)]-[\Sigma^R_{12}(q,\epsilon_1)]^2}$$

$$(51)\qquad \xi_q \equiv q^2/2m^* - \mu\,.$$

This can be rewritten in the form

$$(52)\qquad G^R_{11}(q,\epsilon_1) = \frac{\epsilon_1+\xi_q-\Sigma^R_{11}(q,\epsilon_1)}{[\epsilon_1-\Sigma^R_{11}(q,\epsilon_1)]^2-\xi_q^2-[\Sigma^R_{12}(q,\epsilon_1)]^2}$$

$$(53)\qquad G^R_{12}(q,\epsilon_1) = \frac{\Sigma^R_{12}(q,\epsilon_1)}{[\epsilon_1-\Sigma^R_{11}(q,\epsilon_1)]^2-\xi_q^2-[\Sigma^R_{12}(q,\epsilon_1)]^2}$$

due to the fact that ${\rm Im}\Sigma^R_{11}$ is an even function of ϵ_1. We are interested only in the odd part of ${\rm Re}\Sigma^R_{11}$ since the even part only gives a correction to the chemical potential that does not depend on the temperature [28].

We evaluate the imaginary parts of G_{11}^R and G_{12}^R for small imaginary parts of the self-energy, i.e. in the limit $\mathrm{Im}\Sigma_{11}^R, \mathrm{Im}\Sigma_{12}^R \to 0$. Since we are interested in the region of momenta next to the Fermi surface, we assume

$$\Sigma_{11}^R(k_{\mathrm{F}}, \epsilon_1) = \Sigma(\epsilon_1) - i\Gamma(\epsilon_1) \tag{54}$$

$$\Sigma_{12}^R(k_{\mathrm{F}}, \epsilon_1) = \phi(\epsilon_1) - i\Theta(\epsilon_1) \tag{55}$$

with $\Theta, \Gamma > 0$ because of the analytical properties of the retarded Green functions,

$$\mathrm{Im}G_{11}^R(q, \epsilon_1) = (A + \xi_q)\,\frac{-2\Gamma A}{B^2 + 4\Gamma^2 A^2} \tag{56}$$

$$\mathrm{Im}G_{12}^R(q, \epsilon_1) = \phi\,\frac{-2\Gamma A}{B^2 + 4\Gamma^2 A^2} \tag{57}$$

with the definitions $A \equiv \epsilon_1 - \Sigma(\epsilon_1)$ and $B = A^2 - \phi^2(\epsilon_1) - \xi_q^2$. For $\Gamma \to 0$ and $\Theta \to 0$ we get

$$\begin{aligned}\mathrm{Im}G_{11}^R(q, \epsilon_1) &= -\pi\,(A + \xi_q)\,\delta(B)\,\mathrm{sgn}A \\ &= -\pi\,\mathrm{sgn}[\epsilon_1 - \Sigma(\epsilon_1)]\,\frac{\epsilon_1 - \Sigma(\epsilon_1) + \xi_q}{2\Omega_1(\epsilon_1)} \\ &\qquad \times \{\delta[\xi_q - \Omega_1(\epsilon_1)] + \delta[\xi_q + \Omega_1(\epsilon_1)]\}\end{aligned} \tag{58}$$

$$\begin{aligned}\mathrm{Im}G_{12}^R(q, \epsilon_1) &= -\pi\,\phi\,\delta(B)\,\mathrm{sgn}A \\ &= -\pi\,\mathrm{sgn}[\epsilon_1 - \Sigma(\epsilon_1)]\,\frac{\phi(\epsilon_1)}{2\Omega_1(\epsilon_1)} \\ &\qquad \times \{\delta[\xi_q - \Omega_1(\epsilon_1)] + \delta[\xi_q + \Omega_1(\epsilon_1)]\}\end{aligned} \tag{59}$$

with

$$\Omega_1(\epsilon_1) = \sqrt{[\epsilon_1 - \Sigma(\epsilon_1)]^2 - \phi^2(\epsilon_1)}\,. \tag{60}$$

In order to perform the $\mathbf{q}$-integrations in eq. (47) and eq. (48), we consider the dominant contribution D_{11} and rewrite the expressions for the vertices with $p \equiv |\mathbf{k} - \mathbf{q}|$

$$v_1(\mathbf{k}, \mathbf{k} - \mathbf{q})\,v_1(\mathbf{k}, \mathbf{q} - \mathbf{k}) = -\frac{e^2}{m^{*2}}\,\frac{k^2 q^2}{p^2}\,\sin^2\theta \tag{61}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{q}$. Aligning the q_x axis parallel the $\hat{k}$-direction, the measure is changed to

$$\int_0^\infty q\mathrm{d}q \int_0^{2\pi} \mathrm{d}\theta = 2\int_0^\infty \mathrm{d}q \int_{|k-q|}^{k+q} \mathrm{d}p\,\frac{p}{k\sin\theta} \tag{62}$$

with

$$\sin\theta = \sqrt{1 - \left[\frac{k^2 + q^2 - p^2}{2kq}\right]^2}\,. \tag{63}$$

If we assume for the external momentum $k \approx k_{\rm F}$ and consider only the dominant contributions due to $q \sim k_{\rm F}$ we get for $\Sigma^{\rm R}_{11}(k_{\rm F}, \epsilon) \approx \Sigma(\epsilon)$ and $\Sigma^{\rm R}_{12}(k_{\rm F}, \epsilon) \approx \phi(\epsilon)$

$$\Sigma(\epsilon) = \frac{-1}{4\pi^4}\frac{k_{\rm F}^2 e^2}{m^{*2}} \int_0^\infty {\rm d}q \int_0^{2k_{\rm F}} {\rm d}p \sqrt{1 - \frac{p^2}{4k_{\rm F}^2}} \int {\rm d}\omega\, {\rm d}\epsilon_1 \\ \frac{{\rm Im}\left[D^{+,{\rm R}}_{11}(p,\omega) + D^{-,{\rm R}}_{11}(p,\omega)\right]}{\omega + \epsilon_1 - \epsilon - i\delta} {\rm Im}G^{\rm R}_{11}(q,\epsilon_1)\left(\tanh\frac{\epsilon_1}{2T} + \coth\frac{\omega}{2T}\right) \tag{64}$$

$$\phi(\epsilon) = \frac{-1}{4\pi^4}\frac{k_{\rm F}^2 e^2}{m^{*2}} \int_0^\infty {\rm d}q \int_0^{2k_{\rm F}} {\rm d}p \sqrt{1 - \frac{p^2}{4k_{\rm F}^2}} \int {\rm d}\omega\, {\rm d}\epsilon_1 \\ \frac{{\rm Im}\left[D^{+,{\rm R}}_{11}(p,\omega) - D^{-,{\rm R}}_{11}(p,\omega)\right]}{\omega + \epsilon_1 - \epsilon - i\delta} {\rm Im}G^{\rm R}_{12}(q,\epsilon_1)\left(\tanh\frac{\epsilon_1}{2T} + \coth\frac{\omega}{2T}\right) \tag{65}$$

since the $D^{\pm}_{11}$ depend only on the modulus of their argument.

The q-integral involves only ImG and yields, when linearizing $\xi_q \sim v_{\rm F}(q - k_{\rm F})$,

$$\int {\rm d}q\, {\rm Im}G^{\rm R}_{11}(q,\epsilon_1) = -\frac{\pi}{v_{\rm F}}{\rm sgn}[\epsilon_1 - \Sigma(\epsilon_1)]\,\frac{\epsilon_1 - \Sigma(\epsilon_1)}{\Omega_1(\epsilon_1)} \tag{66}$$

$$\int {\rm d}q\, {\rm Im}G^{\rm R}_{12}(q,\epsilon_1) = -\frac{\pi}{v_{\rm F}}{\rm sgn}[\epsilon_1 - \Sigma(\epsilon_1)]\,\frac{\phi(\epsilon_1)}{\Omega_1(\epsilon_1)} \tag{67}$$

In evaluating this integral, some assumptions have been made. First of all, Σ_{11} and Σ_{12} (cf. eqs. (54)), (55)), are assumed to have imaginary parts that are much smaller than the real parts. We have also neglected contributions to the self-energy that do not depend on the frequency. Although at this stage these assumption cannot really be justified they are *a posteriori* found to be consistent with the results. In any case, they are necessary in order to be consistent with the Fermi liquid picture for the Composite Fermions. The above equations are valid if Ω_1 is real, i.e. for $(\epsilon_1 - \Sigma)^2 - \phi^2 \geq 0$, otherwise, the integrals are zero due to the δ-functions.

By introducing the quantity

$$\epsilon z(\epsilon) \equiv \epsilon - \Sigma(\epsilon) \tag{68}$$

and defining the gap Δ,

$$\Delta(\epsilon) z(\epsilon) \equiv \phi(\epsilon)\,, \tag{69}$$

such that

$$\Omega_1(\epsilon_1) = |z(\epsilon_1)|\sqrt{\epsilon_1^2 - \Delta^2(\epsilon_1)}\,, \tag{70}$$

we also can write eq. (66) and eq. (67) in the form

$$\int \mathrm{d}q\, \mathrm{Im}G_{11}^{\mathrm{R}}(q,\epsilon_1) = -\frac{\pi}{v_{\mathrm{F}}}\frac{|\epsilon_1|}{\sqrt{\epsilon_1^2 - \Delta^2(\epsilon_1)}} \tag{71}$$

$$\int \mathrm{d}q\, \mathrm{Im}G_{12}^{\mathrm{R}}(q,\epsilon_1) = -\frac{\pi}{v_{\mathrm{F}}}\frac{\mathrm{sgn}\epsilon_1 \Delta(\epsilon_1)}{\sqrt{\epsilon_1^2 - \Delta^2(\epsilon_1)}} \tag{72}$$

We can now perform the p-integrations assuming $p \ll k_{\mathrm{F}}$, thus retaining only the first order in the square root. Using eq. (38) and eq. (39) and defining the integrals

$$P^+(\omega) = \int_0^\infty \mathrm{d}p\, \mathrm{Im}D_{11}^{+,R}(p,\omega) = -\frac{\pi}{4\alpha_+}\mathrm{sgn}\omega \tag{73}$$

$$P^-(\omega) = \int_0^\infty \mathrm{d}p\, \mathrm{Im}D_{11}^{-,R}(p,\omega) = -\frac{\pi}{3\sqrt{3}}\frac{1}{\alpha_-^{2/3}\eta^{1/3}}\omega^{-1/3} \tag{74}$$

we find in the limit $T \to 0$ where

$$\tanh\frac{\epsilon_1}{2T} \to \mathrm{sgn}\epsilon_1; \qquad \coth\frac{\omega}{2T} \to \mathrm{sgn}\omega \tag{75}$$

the expressions for the self-energies

$$\Sigma(\epsilon) = \frac{1}{4\pi^3}\frac{k_{\mathrm{F}}e^2}{m^*}\int \mathrm{d}\omega\, \mathrm{d}\epsilon_1\, \frac{\mathrm{sgn}\epsilon_1 + \mathrm{sgn}\omega}{\omega + \epsilon_1 - \epsilon - i\delta}\left[P^+(\omega) + P^-(\omega)\right] \times \mathrm{sgn}\epsilon_1 \frac{\epsilon_1}{\sqrt{\epsilon_1^2 - \Delta^2(\epsilon_1)}} \tag{76}$$

$$\phi(\epsilon) = \frac{1}{4\pi^3}\frac{k_{\mathrm{F}}e^2}{m^*}\int \mathrm{d}\omega\, \mathrm{d}\epsilon_1\, \frac{\mathrm{sgn}\epsilon_1 + \mathrm{sgn}\omega}{\omega + \epsilon_1 - \epsilon - i\delta}\left[P^+(\omega) - P^-(\omega)\right] \times \mathrm{sgn}\epsilon_1 \frac{\Delta(\epsilon_1)}{\sqrt{\epsilon_1^2 - \Delta^2(\epsilon_1)}}\,. \tag{77}$$

Now the energy integrations have to be performed. We begin by defining the integrals

$$F_+(\epsilon,\epsilon_1) = \int \mathrm{d}\omega\, \mathrm{sgn}\omega\, \frac{\mathrm{sgn}\epsilon_1 + \mathrm{sgn}\omega}{\omega + \epsilon_1 - \epsilon - i\delta} \tag{78}$$

$$F_-(\epsilon,\epsilon_1) = \int \mathrm{d}\omega\, \omega^{-1/3}\, \frac{\mathrm{sgn}\epsilon_1 + \mathrm{sgn}\omega}{\omega + \epsilon_1 - \epsilon - i\delta} \tag{79}$$

that must be evaluated as principal value integrals. For F_-,

$$\mathrm{Re}F_-(\epsilon,\epsilon_1) = -\frac{\pi}{\sqrt{3}}\frac{[1+3\,\mathrm{sgn}\epsilon_1\,\mathrm{sgn}(\epsilon_1-\epsilon)]}{(\epsilon-\epsilon_1)^{1/3}}\,, \tag{80}$$

$$\mathrm{Im}F_-(\epsilon,\epsilon_1) = \pi\,\frac{\mathrm{sgn}\epsilon_1 + \mathrm{sgn}(\epsilon-\epsilon_1)}{(\epsilon-\epsilon_1)^{1/3}}\,. \tag{81}$$

The integral F_+ must be calculated by introducing a cutoff Λ_c,

$$\mathrm{Re}F_+(\epsilon,\epsilon_1) = \int_{-\Lambda_\mathrm{c}}^{\Lambda_\mathrm{c}} \frac{\mathrm{d}\omega}{\omega+\epsilon_1-\epsilon} + \mathrm{sgn}\epsilon_1 \int_{-\Lambda_\mathrm{c}}^{\Lambda_\mathrm{c}} \mathrm{d}\omega\,\frac{\mathrm{sgn}\omega}{\omega+\epsilon_1-\epsilon}\,, \tag{82}$$

$$\mathrm{Im}F_+(\epsilon,\epsilon_1) = \pi(1-\mathrm{sgn}\,\epsilon_1\mathrm{sgn}(\epsilon_1-\epsilon))\,. \tag{83}$$

We finally find for the real part

$$\mathrm{Re}F_+(\epsilon,\epsilon_1) = \log\frac{|\Lambda_\mathrm{c}+\epsilon_1-\epsilon|}{|\Lambda_\mathrm{c}-\epsilon_1+\epsilon|} + \mathrm{sgn}\epsilon_1\log\frac{|\Lambda_\mathrm{c}^2-(\epsilon_1-\epsilon)^2|}{(\epsilon_1-\epsilon)^2} \tag{84}$$

The physically meaningful value of the cut-off can be estimated by considering with more detail the behavior of the integral over $\mathrm{Im}D_{11}^{\pm,R}(p,\omega)$,

$$\int_0^{2k_\mathrm{F}} \mathrm{d}p\,\mathrm{Im}D_{11}^{\pm,R}(p,\omega) = -\frac{1}{2\alpha_+}\left(\frac{\pi}{2} - \arctan\frac{\eta\omega}{4k_\mathrm{F}^2\alpha_+}\right) \tag{85}$$

This vanishes for $\omega \to \infty$. The scale for the vanishing of the integral can be obtained by considering the argument of the arctan-function

$$\frac{\eta\omega}{4k_\mathrm{F}^2\alpha_+} = \frac{\omega}{e^2/\epsilon l_B}\frac{1}{2k_\mathrm{F}l_B} \tag{86}$$

where $E_\mathrm{C} = e^2/\epsilon l_B$ is the energy scale of the Coulomb interaction and l_B is the magnetic length. Therefore, it is reasonable to choose as the cut-off $\Lambda_\mathrm{c} = \Lambda k_\mathrm{F} l_B E_\mathrm{C}$, where Λ represents the numerical value of the cut-off.

7. – The Energy Gap

In order to find the solutions of the above non-linear Eliashberg equations eq. (76) and eq. (77) it is convenient to define the constant

$$C = \frac{1}{4\pi^3}\frac{k_\mathrm{F}e^2}{m^*}\,, \tag{87}$$

and

$$M^+(\epsilon,\epsilon_1) = \int \mathrm{d}\omega\, P^+(\omega)\, \frac{\mathrm{sgn}\epsilon_1 + \mathrm{sgn}\omega}{\omega + \epsilon_1 - \epsilon - i\delta} = -\frac{\pi}{4\alpha_+} F^+(\epsilon,\epsilon_1)\,, \tag{88}$$

$$M^-(\epsilon,\epsilon_1) = \int \mathrm{d}\omega\, P^-(\omega)\, \frac{\mathrm{sgn}\epsilon_1 + \mathrm{sgn}\omega}{\omega + \epsilon_1 - \epsilon - i\delta} = -\frac{\pi}{3\sqrt{3}} \frac{1}{\alpha_-^{2/3}\eta^{1/3}} F^-(\epsilon,\epsilon_1)\,, \tag{89}$$

such that

$$\Sigma(\epsilon) = C \int \mathrm{d}\epsilon_1 [M^+((\epsilon,\epsilon_1) + M^-(\epsilon,\epsilon_1)]\mathrm{sgn}\epsilon_1\, \frac{\epsilon_1}{\sqrt{\epsilon_1^2 - \Delta^2(\epsilon_1)}} \tag{90}$$

$$\phi(\epsilon) = C \int \mathrm{d}\epsilon_1 [M^+((\epsilon,\epsilon_1) - M^-(\epsilon,\epsilon_1)]\mathrm{sgn}\epsilon_1\, \frac{\Delta(\epsilon_1)}{\sqrt{\epsilon_1^2 - \Delta^2(\epsilon_1)}} \tag{91}$$

which give after using the definitions eqs. (68), (69)

$$\begin{aligned}\Delta(\epsilon) = C \int & \frac{\mathrm{sgn}\epsilon_1 \mathrm{d}\epsilon_1}{\sqrt{\epsilon_1^2 - \Delta^2(\epsilon_1)}} \\ & \times \left\{ [M_+(\epsilon,\epsilon_1) + M_-(\epsilon,\epsilon_1)]\frac{\epsilon_1}{\epsilon}\Delta(\epsilon) + [M_+(\epsilon,\epsilon_1) - M_-(\epsilon,\epsilon_1)]\Delta(\epsilon_1) \right\}.\end{aligned} \tag{92}$$

If we assume that the gap is energy-independent, $\Delta(\epsilon) \approx \Delta$, this gives finally for the gap the condition

$$I_+ + I_- = 1\,, \tag{93}$$

with quantities $I_\pm$ that can be calculated by expanding with respect to ϵ_1 around $\epsilon = 0$. We first note that the imaginary parts of F_+, F_- do not give contributions to the ϵ_1-integral. Then, with $|\epsilon_1| > |\Delta|$, and assuming $\Delta > 0$, we get

$$\begin{aligned} I_- &= \frac{16\pi^2}{27}\eta^{-1/3}\alpha_-^{-2/3} \int_\Delta^\infty \frac{\mathrm{d}\epsilon_1 \epsilon_1^{-1/3}}{\sqrt{\epsilon_1^2 - \Delta^2}} \\ &= \frac{16\pi^{5/2}}{9}\eta^{-1/3}\alpha_-^{-2/3}\frac{\Gamma(7/6)}{\Gamma(2/3)}\Delta^{-1/3}\,, \end{aligned} \tag{94}$$

$$\begin{aligned} I_+ &= -\frac{\pi}{\alpha_+}\int_\Delta^{\Lambda_c} \mathrm{d}\epsilon_1 \frac{1}{\sqrt{\epsilon_1^2 - \Delta^2}} \left(\log\frac{\Lambda_c + \epsilon_1}{\epsilon_1} + \frac{\Lambda_c}{\Lambda_c + \epsilon_1} \right) \\ &\approx -\frac{\pi}{\alpha_+}\left[\frac{1}{2}\log^2\left(\frac{\Lambda_c}{\Delta}\right) + \frac{\Lambda_c}{\sqrt{\Lambda_c^2 - \Delta^2}}\log\left(\frac{\Lambda_c}{\Delta}\right) \right] \end{aligned} \tag{95}$$

in the limit $\Lambda_c/\Delta \gg 1$.

By replacing the expressions for η and $\alpha_\pm$ and Λ_c we find the final result

$$1 = C_- \left(\frac{E_F}{\Delta}\right)^{1/3} - C_+ \left[\log^2\left(\Lambda' \frac{E_F}{\Delta}\right) + \frac{\Lambda' E_F}{\sqrt{(\Lambda' E_F)^2 - \Delta^2}} \log\left(\Lambda' \frac{E_F}{\Delta}\right)\right] \tag{96}$$

with $\Lambda' \equiv 2\pi\Lambda/C_+$ and the constants

$$C_- \approx 1.4 \qquad C_+ = \frac{E_F}{2\pi e^2 k_F/\varepsilon} = \frac{E_F}{E_C}\frac{1}{2\pi k_F \ell_B}\,. \tag{97}$$

The first term in eq. (96) is completely independent of the interaction and describes the contribution due to D^-. The second term is due to D^+ and stems from the interaction between particles.

Independent of the value of the magnetic field there is *always* a solution $\Delta \neq 0$ to this equation. For E_F larger than E_C ($C_+ \gg 1$) Δ becomes vanishingly small. If E_C is much larger than E_F ($C_+ \ll 1$), the gap is nearly independent of the Coulomb interaction.

8. – Conclusion

The non-zero solution of eq. (96) indicates that in a *single* quantum Hall layer, when two Landau levels with opposite spins intersect at the Fermi energy in a perpendicular magnetic field, the system becomes unstable against formation of a spin-singlet state due to an effective attractive coupling of Composite Fermions via the gauge field fluctuations. The resulting condensate state is similar to the macroscopic state induced in a superconductor by the electron-phonon coupling. Is there any experimental indication that this indeed might be the case?

The existence of such a spin-singlet condensate state can contribute towards the understanding of the extra-plateaus in the spin polarization experiments of [19]. The splitting between the Landau levels of the Composite Fermions (CFLL) with spin up and spin down behaves as $\sqrt{B}$ for small B, and is proportional to B for large B due to the Zeeman splitting. Spin-up and spin-down components of different CFLL can intersect. As an example, we consider $\nu \equiv p/(2p+1) = 2/5$. This corresponds to two filled CFLL ($p = 2$). We adjust the Fermi level to the energy where the spin-down Zeeman level of the lowest CFLL becomes degenerate with the spin-up Zeeman level of the first CFLL. For magnetic fields smaller than the one corresponding to the point of degeneracy, B_c, only the Zeeman levels of the lowest CFLL are occupied at zero temperature. The spin polarization vanishes, $\gamma = (\rho_\downarrow - \rho_\uparrow)/(\rho_\downarrow + \rho_\uparrow) = 0$. Magnetic fields above B_c yield $\gamma = 1$. Exactly at B_c, two half-filled CFLL can be formed when defining the filling factor in terms of the ratio between the number of CF and the number of "effective" flux quanta crossing the sample. In analogy to the above, one could then perform a gauge transformation leading to "second generation" Composite Fermions with the corresponding gauge fluctuations mediating an effective attractive interaction. This would lead to the formation of a condensate.

The existence of the gap at the crossing point would imply that in a region of magnetic fields around this point, where the energy difference between the CFLL is less then Δ, the condensate remains stable. The formation of such a state of singlet CF-pairs was then responsible for the formation of a plateau exactly at half the distance between the neighboring plateaus in an interval of magnetic fields near the crossover point.

The possibility of generating long-range spin-pairing correlations in a single 2D Hall sample is similar to those discussed previously for QHE double layers [29, 30] by tuning the density and the magnetic field to induce the crossing between spin-up and spin-down Landau levels. It leads to interesting speculations. For instance, consider two QHE systems in the same plane, say at $\nu = 2/5$, separated by a tunnel junction. By tuning the two densities to the value of the point of degeneracy a "Josephson current" should flow. Such a current should vanish as soon as one of the two densities was detuned.

In conclusion, we have considered two Landau levels with opposite spins tuned to intersect at filling factor 1/2 at the Fermi level. By applying the Chern-Simons gauge transformation, we have derived an effective attractive interaction between the Composite Fermions. This yields an instability towards a spin-singlet condensate. We have discussed several experimental consequences. In order to observe the predicted spin-singlet state, a close-to-zero in-plane component of the magnetic field should be necessary as has been achieved in the spin-polarization experiments done in the region of the FQHE.

Our results suggest that different occupations of spin-up and spin-down Landau levels could account for instabilities at other fractional polarizations and that an in-plane component of the magnetic field could account for an anisotropic spin-singlet condensate.

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