Pricing Default Risk in Mortgage Backed Securities*

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Abstract

This paper values Mortgage Backed Securities (MBS) in an equilibrium framework that explicitly incorporates the default decisions of homeowners and essential contractual features of MBS. We first consider Collateralized Mortgage Obligations (CMOs), securities created by dividing a pool of mortgages into senior and residual tranches. We find that senior CMO bonds can be risk-free, low-risk or high-risk in equilibrium, depending on the relative size of the senior tranche. We extend the basic framework to value CMO-squared, securities created by pooling residual CMO bonds and dividing the pool into tranches. We find that senior CMO-squared bonds are riskier than senior CMO bonds of the same size, when CMO-squared are created using residual CMO bonds. Finally, we value Credit Default Swaps (CDSs), securities that provide insurance against default.

For house price data from the Case-Shiller index between 2006 and 2011, we find that senior CMO bond prices decline by 10% and residual bond prices decline by 60%. The price declines experienced by CMO-squared bonds are larger: senior bond prices drop 50% and residual bond prices drop 100%. The quantitative exercises suggest that default risk is an important factor for valuation of CDS written on residual CMO bonds, but not for CDS written on senior CMO bonds.

J.E.L Codes: G12, G21

Keywords: Mortgage default, Mortgage Backed Security, Collateralized Mortgage Obligations, Collateralized Debt Obligations, Credit Default Swaps.

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1 Introduction

Mortgage Backed Securities (MBS) were at the center of the financial crisis of 2007. Large unexpected drops in house prices resulted in mortgage defaults, which in turn impaired associated MBS. Large unprecedented losses on MBS, which were thought to be virtually immune to default risk and rated so, led analysts to conclude that these securities were "toxic" and difficult to value. Banks and other financial institutions that held large quantities of these assets were adversely affected. The crisis began.

The events leading up to the financial crisis highlight the need for a framework for MBS valuation that explicitly incorporates mortgage default. The goal of this paper is to provide such a framework. The analysis is conducted in two stages. First, we model the mortgage lending market as in John Krainer, Stephen F. LeRoy and Munpyung O (2009).[1] Housing services follow a geometric Brownian motion, implying that changes in housing services are exogenous and unforecastable. House prices equal the expected discounted value of services so they are also unforecastable. Borrowers buy houses using mortgages. They have the option to default on the mortgage, subject to a cost, which they exercise to maximize equity. Mortgage lenders make zero profits. We solve for equilibrium yield spreads in this setting. We characterize the equilibrium using boundary-crossing properties of geometric Brownian motion. To the best of our knowledge, this solution method is new to the mortgage valuation literature. The key advantage of this method is that it reduces the valuation of mortgages to a simple present value calculation.

Second, we model the MBS market. In practice, the term mortgage backed security is applied to a large class of mortgage bonds and their derivatives. We begin the analysis by focusing on Collateralized Mortgage Obligations (CMOs).[2] These securities are created by combining mortgages into a pool, dividing the pool into tranches, structuring the pool's cash flows so as to protect the senior tranche from default risk, and selling claims to cash flows on each tranche as bonds. We assume that CMOs are created either from pools that contain one type of mortgage only – homogeneous pools – or pools that contain two types of mortgages – heterogeneous pools.

For homogeneous pools, we find that senior bonds move from being risk-free to risky as the relative size of the senior tranche increases. Risk-free senior bonds do not experience principal or coupon shortfalls, whereas risky senior bonds may experience principal shortfalls. The equilibrium initial yield on risky senior bonds is a linearly decreasing function of the recovery rate on the tranche. As an application, we consider the valuation of mortgages with two liens. In practice, these mortgages usually consist of

[1] John Krainer, Stephen F. LeRoy and Munpyung O (2009) were the first to connect optimal mortgage-default to equilibrium yield spreads. They did so by adapting Robert C. Merton (1974) to the housing market and using the zero-profit condition to close Merton's model.

[2] In practice, CMOs can be broadly divided into "agency" CMOs, which are insured against default risk by Government Sponsored Enterprises, and "non-agency" CMOs, which are not insured by them. The analysis here focuses on default risk so it applies primarily to non-agency CMOs.

a first lien with an 80% loan-to-value ratio and a smaller second lien with a 10-15% loan-to-value ratio. We provide conditions under which the first-lien is risk-free in the presence of the second-lien, but not otherwise.

The heterogeneous pool contains mortgages that differ in terms of borrower default costs only; we refer to them as low default cost and high default cost mortgages. This pool experiences up to two default events. We find that, depending on the relative size of the senior tranche, senior bonds can be risk-free, low-risk, or high-risk in equilibrium. Risk-free senior bonds receive both principal and coupon payments even if both underlying mortgages default. Low-risk senior bonds receive all coupon payments even if both mortgages default, but do not recover their entire principal. High-risk senior bonds experience both principal and coupon shortfalls if the underlying mortgages default. The equilibrium initial yield on the senior tranche increases as the relative size of the tranche increases. We also find that an increase in the fraction of high-default cost mortgages has two opposing effects on equilibrium yields of senior bonds: it decreases the initial yield by reducing the likelihood of default and it increases the initial yield by reducing the net recovery on default. As a quantitative exercise, we calculate model implied bond prices when data from the Case-Shiller composite-20 index is fed through the model; the data is from July 2006 to July 2011. We find that senior bonds lose about 10% of their value and residual bonds lose about 60% of their value during this time period.

Next we turn to valuation of CMO-squared and other higher-order CMOs.[3] The standard models used by practitioners to value CMO-squared have been widely criticized because they do not connect valuation to mortgage default explicitly.[4] A contribution of our paper is to make this connection and provide an alternative to these models. A CMO-squared is created using a pool made from the residual tranche of the CMO, dividing this pool again into senior and residual tranches, and restructuring the pool's cash flows so as to protect the senior CMO-squared tranche from default risk. We find that senior CMO-squared bonds are riskier than senior CMO bonds when the relative sizes of the senior CMO and CMO-squared tranches are identical. The analysis in this section extends to higher-order CMOs. We present an example in which the senior tranche of a CMO-cubed suffers a hundred percent principal write down when low default cost mortgages are terminated. According to the quantitative exercise in this section, prices of senior CMO-squared bonds drop to 50% of their par values. Prices of residual CMO-squared bonds drop 100%, rendering these bonds worthless. Our quantitative findings are roughly consistent with the empirical work of Larry Cordell, Yilin Huang and Meredith Williams (2012). These authors report that, in the data, the average principal write-down on senior AAA-rated CMO-squared tranches was 67% in 2006 and 76% in 2007; the average write-down for other tranches was 93%.

[3]CMO-squared are often called Collateralized Debt Obligations in practice.

[4]Christopher L. Foote, Kristopher S. Gerardi and Paul S. Willen (2008) discuss how analysts of CMO-squared modeled correlation between mortgage defaults directly and estimated this correlation using historical data.

Finally, we consider the valuation of Credit Default Swaps (CDSs). A CDS is insurance against default. The quantitative exercise in this section suggests that default risk is a major factor in the pricing of CDS written on the residual CMO tranche, but not for CDS on the senior CMO tranche. Comparing these findings to the price declines observed in the ABX.HE CDS index, the model suggests that default risk by itself cannot account for the large price declines observed in the senior AAA-rated index.[5] It may, however, account for a large fraction of the price declines observed in the lower-rated indexes. These findings are consistent with recent research on the ABX.HE index: R. Stanton and N. Wallace (2011) show that the observed price declines in the ABX.AAA-HE index cannot be accounted for, by any reasonable expectation regarding defaults and recovery rates on the underlying mortgages. In contrast, Ingo Fender and Martin Scheicher (2009) conclude that default risk was important for the pricing of lower-rated ABX.HE indexes.

2 Mortgage Market

Setup.– Our setup of the mortgage market is as in John Krainer, Stephen F. LeRoy and Munpyung O (2009, KLO from here on). A house provides a stochastic flow of services. Housing services $x(t)$ are exogenous and follow a geometric Brownian motion:

$$dx(t) = \alpha x(t)dt + \sigma x(t)dw(t), \tag{1}$$

where α is the expected proportional growth rate, σ is the volatility parameter, and $w(t)$ is standard Brownian motion. We normalize initial housing services to one, $x(0) = 1$.

House prices are the expected discounted value of future services:

$$P(x(t)) \equiv \int_{z=t}^{\infty} e^{-r(z-t)} \mathbb{E}_t\big(x(z)\big)dz = \frac{x(t)}{r-\alpha}. \tag{2}$$

The operator $\mathbb{E}_t$ denotes the mathematical expectation conditional on information available at time t. The discount rate r is exogenous and constant. House prices can be represented by (2) if housing services follow geometric Brownian motion under the risk-neutral probability measure or if agents are risk-neutral. This specification of house prices rules out bubbles. By (1) and Ito's formula, house prices also follow a geometric Brownian motion, with α as the expected proportional growth rate and σ as the volatility

[5] The ABX.HE index was launched in January 2006 by Markit Group Ltd. in consortium with fifteen investment banks (these banks are usually CDS sellers). The ABX.HE index tracks the price of a single CDS written on a fixed basket of equally weighted 20 CMO pools. Every CMO comprising the index must meet certain criteria; see Markit (2008) for details. The CMOs are classified based on their ratings at the origination date of the index. For example, AAA rated bonds from all the 20 pools comprise the AAA ABX.HE index. These credit ratings are the ratings agencies' assessment of the CMOs at the date of index origination. A new series of the index was scheduled for release every six months. However, the decline in house prices significantly reduced the availibility of subprime CMOs, so no new series were released for vintages after 2007. The four vintages that were released according to the six month schedule are 2006-1, 2006-2, 2007-1, and 2007-2.

parameter. Since geometric Brownian motion is a Markov process, the current house price summarizes relevant past and future information. The best forecast for house prices is that they grow at the rate α.

Under the normalization $x(0) = 1$, the purchase price of the house is $P(1) = 1/(r - \alpha)$. An infinitely-lived borrower buys the house for its service flow $x(t)$. He does so using a mortgage. According to the mortgage contract, the lender supplies funds that are applied towards the purchase. In return the borrower pays the coupon c to the lender. The size of the mortgage equals the total amount of funds supplied by the lender. It is exogenous. Any difference between the purchase price and the size of the mortgage is provided for by the borrower's personal wealth, which we do not model. Once the mortgage-size is specified the loan-to-value (LTV) ratio follows. The borrower has the option to default on his mortgage, subject to a cost, at any time by paying the lender the current market value of the house. This assumption allows the market value of the house at the time of default to be less than the mortgage size. The assumption corresponds in reality to the ability of the borrower to turn over the keys and walk away from the house.

In practice, borrowers who choose to default have to bear relocation expenses, loss of future credit access, and loss of tax-benefits. We incorporate these costs into our model by assuming that the borrower faces positive default costs. The existence of negative equity mortgages in practice provides evidence in favor of positive borrower default costs.[6] Default costs, however, need not be positive for all borrowers. The popular press has reported instances of borrowers living rent-free in their houses after defaulting on their mortgage. Such cases are incorporated into the analysis by allowing borrower default costs to be negative. Borrower default costs are denoted k_β. These costs are exogenous and are constant over the lifetime of the mortgage.

In practice, default is also costly for lenders. Once borrowers default, lenders gain possession of the property. The cost of maintaining, repairing, and reselling the property is borne by lenders. Usually there is a lag, of a year or more, between the default date and the date at which lenders can repossess and sell the property. During this lag, lenders also lose income from coupon payments. We model these costs as lender default costs, denoted k_λ. These costs are exogenous and identical across mortgages. We also assume that default costs paid by borrowers and lenders are deadweight loss to the society.

We assume that the borrower cannot prepay his mortgage when its fair value exceeds its book value, even though he would like to do so. Thus credit risk is the only relevant risk faced by lenders. We also assume perfect competition in the mortgage market so lenders make zero profits.

Equilibrium.– The borrower chooses the threshold of housing services at which to default so as

[6] As noted in KLO, if borrower default costs were zero and lenders were to supply negative equity mortgages, then borrowers would default at the date of origination. Futhermore, the default behavior observed in the data is difficult to reconcile with the behavior implied by a model of costless default; see Yongheng Deng, John M. Quigley and Robert Van Order (2000).

to minimize his mortgage liability, or equivalently maximize his equity. We solve this problem using boundary-crossing properties of geometric Brownian motion. This method reduces the valuation of mortgages to a simple present value calculation. Alternatively, one could use dynamic programming; see KLO for this approach.

The mortgage liability minimization problem is

$$\min_{d} \quad \left\{ \mathbb{E}_0\left[\int_0^{\tau(d)} c\, e^{-rt} dt\right] + \mathbb{E}_0\left[e^{-r\tau(d)}\big(P(d) + k_\beta\big)\right]\right\}, \tag{3}$$

where d denotes a generic default threshold and $\tau(d)$ denotes the time at which housing services hit the default threshold. The time of default is random because housing services are random. All mathematical expectations in the minimization problem above are conditional on information available at origination. The random variable in all the expectations above is the default time $\tau(d)$. The term inside the first expectation is the total discounted present value of the coupons paid by the borrower until default. The term inside second expectation is the discounted present value of the borrower's payments on default: the market value of the house $P(d)$, and default costs k_β.

The objective function in (3) can be simplified by noting that c, $P(d)$, and k_β can be moved out of the expectation. After evaluating the integral within the first expectation, the problem in (3) becomes

$$\min_{d} \quad \left\{ \frac{c}{r}\left(1 - \mathbb{E}_0\left[e^{-r\tau(d)}\right]\right) + \big(P(d) + k_\beta\big)\mathbb{E}_0\left[e^{-r\tau(d)}\right] \right\}. \tag{4}$$

Calculating the mathematical expectation of $e^{-r\tau(d)}$ at origination is the key to solving the minimization problem. This expectation is the moment generating function of the random default time $\tau(d)$ evaluated at $-r$. It equals d^m, where $m > 0$ is the following constant

$$m = \frac{(\alpha - \sigma^2/2) + \sqrt{\big(\alpha - \sigma^2/2\big)^2 + 2r\sigma^2}}{\sigma^2}. \tag{5}$$

By the strong Markov property of geometric Brownian motion, the moment generating function of $\tau(d)$ conditional on information available at time t is $(d/x(t))^m$; see Ioannis Karatzas and Steven Shreve (1991) for a discussion.

After substituting for $\mathbb{E}_0[e^{-r\tau(d)}]$, the objective function is entirely in terms of the default threshold d. Standard optimization techniques apply. The optimal default threshold is

$$\delta = \frac{m(r-\alpha)}{m+1}\left(\frac{c}{r} - k_\beta\right). \tag{6}$$

It is strictly increasing in the mortgage coupon c, and strictly decreasing in borrower default costs k_β.

From here on we drop the word optimal and refer to δ as the default threshold. We also suppress the dependence of the optimal default time on δ and indicate the default time by τ.

Let $M(x(t))$ denote the value of the mortgage to the lender when housing services equal $x(t)$. The value of the mortgage at origination is $M(1)$. The lender makes zero profits so $M(1)$ equals the size of the mortgage. The zero-profit condition also implies that the equilibrium mortgage coupon c satisfies,

$$M(1) = \mathbb{E}_0 \left[\int_0^{\tau} c\, e^{-rt} dt \right] + \mathbb{E}_0 \left[e^{-r\tau} \left(P(\delta) - k_\lambda \right) \right]. \tag{7}$$

That is, c must be such that the value of the mortgage at origination $M(1)$ equals the total expected discounted value of mortgage coupons plus the expected discounted value of the lender's net recovery on default. After evaluating the mathematical expectations, we solve (7) for the coupon c to obtain

$$c = \frac{r\left[M(1) - (P(\delta) - k_\lambda)\delta^m\right]}{1 - \delta^m} \tag{8}$$

The equilibrium mortgage coupon c is strictly increasing in the size of the mortgage $M(1)$, and strictly decreasing in the lender's net recovery on default $P(\delta) - k_\lambda$.

The equilibrium values of δ and c are found by jointly solving equations (6) and (8). Under the normalization $x(0) = 1$, the default threshold δ is always less than one. Otherwise the borrower would default at the origination date; anticipating the borrower's behavior the lender would not supply the mortgage in the first place. Once the equilibrium coupon is calculated, the initial yield on the mortgage $c/M(1)$ follows. The equilibrium asset value of the mortgage at any time $t > 0$ can be calculated using an expression similar to (7),

$$M(x(t)) = \frac{c}{r} - \left(\frac{c}{r} + k_\lambda - P(\delta) \right) \left(\frac{\delta}{x(t)} \right)^m. \tag{9}$$

The first term on the right hand side of (9) is the value of the mortgage in the absence of default. The second term is the adjustment for default. On default, the lender loses all future coupon payments c/r, pays the default cost k_λ, and gains the market value of the house at the time of default $P(\delta)$. The equilibrium asset value of the mortgage is increasing in housing services $x(t)$ because default in the near future becomes less likely as $x(t)$ increases. As $x(t)$ approaches infinity, the value of the mortgage approaches its value in the absence of default. At the default threshold δ, the value of the mortgage equals the net recovery, $M(\delta) = P(\delta) - k_\lambda$. The recovery rate on the mortgage is $M(\delta)/M(1)$. Since $\delta < 1$, the recovery rate is always less than one.

3 Collateralized Mortgage Obligations

The analysis here focuses on the impact of default risk on valuation of MBS. So we consider a simple institutional structure in which the lender originates, restructures and services the MBS. The adopted simplification abstracts away from any informational asymmetries among the various other entities involved in the mortgage securitization process; see Adam B. Ashcraft and Til Schuermann (2008) for a discussion of possible asymmetries. Even though CMOs usually have many tranches, the analysis only considers pools that are divided into two tranches. This simplification captures the essential feature of CMOs: the disproportionate division of default risk among senior and subordinate tranches. None of our substantive conclusions, however, rely on the assumption of two tranches. The analysis proceeds in two stages. First we consider a CMO created from a pool that contains one type mortgage only – a homogeneous pool. Then we analyze CMOs created from a pool that contains two different types of mortgages – a heterogeneous pool. Pools with many mortgages can be studied by modifying the two-mortgage pool analysis appropriately. First we briefly discuss how MBS are created in reality.

The creation of MBS involves many different entities. The originator of the mortgage usually sells it to a servicer, a financial institution that is responsible for collecting coupon and recovery payments from the underlying mortgage. The servicer buys many different mortgages, combines them into a pool, and sells the pool to a trust.[7] The trust sells mortgage-backed bonds to investors. These bonds may simply represent pro-rata claims to the cash flows of the pool, such bonds are called mortgage pass-throughs. Alternatively the pool maybe divided into tranches and its cash flows restructured so as to divide default risk disproportionately among the tranches. The cash flows are structured such that each tranche is protected from default by its subordinate tranches. The resulting bonds are called Collateralized Mortgage Obligations (CMOs).

The disproportionate division of default risk is achieved by making the subordinate tranches absorb all losses first. So the senior-most bondholders do not lose principal or coupon payments until the losses are so large that the all subordinate bonds have been wiped-out. If any of the underlying mortgages default, the recovery from these mortgages are used to pay back the senior-most bondholders, while the losses on these mortgages are applied to the subordinate bonds. Similarly, any prepayments are applied to the senior-most bonds first. The bond administration is carried out by a trustee, who oversees the entire transaction on behalf of the investors and forwards all payments to them. The rating agencies rate the bonds. The senior-most bonds have the highest ratings because they have the lowest exposure to default risk; these bonds were usually rated AAA prior to the crisis. The ratings decline as the level of

[7] A trust is a "Special Purpose Vehicle" (SPV) that is legally separate from the servicer, even though it might be one of its subsidiaries. The legal separation ensures that the assets of the SPV do not belong to the servicer. So if the servicer declares bankruptcy, its creditors cannot claim the assets of the SPV; see Gary Gorton and Andrew Metrick (2012) for further discusion of SPVs.

subordination decreases.

3.1 Homogeneous pool

A homogeneous pool contains a unit measure of one type of mortgage only. The characteristics of the underlying mortgage and the pool are identical. For example the origination value, coupon, and recovery on the underlying mortgage and the pool are identical. Denote the value of the pool by $V_p(x(t))$, its coupon by c_p, and its recovery by R_p. The pool receives its coupon c_p until default and then it recovers R_p. The initial yield on the pool is $c_p/V_p(1)$ and recovery rate of the pool is $R_p/V_p(1)$.

Tranches.– The lender divides the pool by value into two tranches – a senior tranche and a residual tranche. Variables associated with the senior tranche are indexed by s and those associated with the residual tranche are indexed by j. The lender sells bonds that are pro-rata claims to cash flows on each tranche. The proceeds from the bond-sale finance the initial loan to borrowers. The cash flows to the pool are divided among the tranches so as to make the senior tranche relatively safe and the residual tranche relatively risky. The disproportionate division of default risk is obtained by giving the senior tranche first-claims to cash flows and making the residual tranche absorb all losses first.

The proportional value of the senior tranche at origination is exogenous. It is denoted by $0 \leq \theta \leq 1$. Let $V_s(x(t))$ denote the value of the senior tranche when housing services equal $x(t)$. The value of the senior tranche at origination is $V_s(1) = \theta V_p(1)$. The senior tranche receives its coupon c_s until the mortgage underlying the pool defaults. The recovery on the senior tranche is

$$R_s = \min \{V_s(1), R_p\}. \tag{10}$$

On default, the lender attempts to pay the senior tranche its entire principal $V_s(1)$. If he cannot do so, then he applies the entire recovery on the pool R_p to the senior tranche. The senior tranche has first-claims to the recovery on the pool. If the senior tranche recovers its entire principal then we call the recovery on the pool adequate, $R_s = V_s(1)$. Otherwise, we call the recovery inadequate. By the definition of $V_s(1)$, the recovery is adequate when the proportional value of the senior tranche at origination is less than the recovery rate of the pool, $\theta \leq R_p/V_p(1)$. Conversely, the recovery is inadequate when $\theta > R_p/V_p(1)$.

The recovery rate of the senior tranche is always greater than the recovery rate of the pool. When the recovery on the pool is adequate, the recovery rate on the tranche is one; recall that the recovery rate of the underlying mortgage and so that of the pool is always less than one. When the recovery on the pool is inadequate, the recovery on the tranche equals the recovery on the pool, $R_s = R_p$, and the value of the tranche at origination is less than the value of the pool, $V_s(1) \leq V_p(1)$. So the recovery rate of the tranche is greater than that of the pool, $R_s/V_s(1) \geq R_p/V_p(1)$. The recovery rate on the tranche and the pool equal each other when $\theta = 1$.

The value of the residual tranche is $V_j(x(t)) = V_p(x(t)) - V_s(x(t))$. Its coupon is $c_j = c - c_s$ and its recovery is $R_j = R_p - R_s$. The recovery on the residual tranche is strictly positive when the recovery on the pool is adequate and $R_p \neq V_s(1)$. It is zero when the recovery on the pool is inadequate or $R_p = V_s(1)$. The recovery rate of this tranche is less than the recovery rate of the pool. It is so because the recovery rate on the pool is the proportion-weighted average of the recovery rate of the tranches and the recovery rate of the senior tranche is greater than that of the pool.

The coupons c_s and c_j are endogenous. They reflect the default risk of the associated tranche. The recovery on the tranches admits the following interpretation: On default the bond manager first buys outstanding senior bonds at their market value. Any cash left over after the senior bond buyback is used to buy back residual bonds at their market value. This interpretation will be important for the analysis of heterogeneous pools below.

Equilibrium.– To find a CMO market equilibrium, we need to find the coupon at which the senior tranche is issued at par; the coupon on the residual tranche follows. So the equilibrium condition is

$$V_s(1) = \mathbb{E}_0 \left[\int_0^{\tau} e^{-rt} c_s dt \right] + \mathbb{E}_0 \left[e^{-r\tau} R_s \right]. \tag{11}$$

The left hand side of (11) is the value of the senior tranche at origination and the right hand side is the expected discounted value of the payments to this tranche. We use the moment generating function of τ and solve (11) for the senior coupon c_s. The implied initial yield on the senior tranche is

$$\frac{c_s}{V_s(1)} = \frac{r\left[1 - (R_s/V_s(1))\delta^m\right]}{1 - \delta^m}. \tag{12}$$

The initial yield on the senior tranche is a linear function of its recovery rate, $R_s/V_s(1)$. When the recovery on the pool is adequate, the recovery rate of the tranche is one and the initial yield equals r. The intuition behind this result is the following: When the recovery is adequate, the senior tranche does not face default risk. So the tranche must earn the risk-free rate r in equilibrium. When the recovery on the pool is inadequate, the initial yield on the senior tranche is greater than r. In this case, senior tranche-holders need to be compensated for default risk with an initial yield greater than the risk-free rate.

The equilibrium initial yields on the pool and the residual tranche are also given by an expression analogous to (12). The recovery rate of the senior tranche, the pool, and the residual tranche can be ordered as $R_s/V_s(1) \geq R_p/V_p(1) \geq R_j/V_j(1)$. So the initial yields on the tranches and the pool can be

Description	Symbol	Value
Expected return on mortgages	r	7%
Drift	α	3%
Volatility	σ	15%
Borrower default costs	k_β	0
Lender default costs	k_λ	2
Mortgage size	M(1)	20

Table 1: Benchmark parametrization

ordered as $c_s/V_s(1) \leq c/V_p(1) \leq c_j/V_j(1)$. The ordering of the recovery rates shows how the default risk is divided disproportionately among the tranches. The resulting ordering of the initial yields shows how the division of risk affects equilibrium yields.

Numerical examples.– We present a numerical example of senior and residual tranche yields implied by the model. Table 1 shows the benchmark parameterization. Our choice of parameters follows KLO. These authors calibrated the mortgage valuation model to data on California mortgages and found that model implied yield spreads at origination were close to the spreads observed in the data, for empirically plausible parameter values. The risk-free rate $r = 7\%$ and the drift parameter $\alpha = 3\%$ generate empirically realistic average real proportional gains of 7% on mortgages and home equity. These values also imply that the price-to-rent ratio in the model is 25. The price-to-rent ratios in data are closer to 10 or 15. This discrepancy between the model and the data is appropriate because the model abstracts from operating costs such as maintenence and utilities expenses. The chosen value of $\sigma = 15\%$ for the standard deviation of housing services is consistent with estimates of individual house price volatitility in the literature.[8] The mortgage size is chosen to obtain a LTV ratio of 80%, $M(1) = 20$. Borrower default costs are set to zero, $k_\beta = 0$. Lender default costs are set to ten percent of the mortgage size, $k_\lambda = 2$.

Under the normalization $x(0) = 1$, the implied purchase price of the house is $P(1) = 25$. The equilibrium default threshold δ and mortgage coupon c are obtained by numerically solving the system of nonlinear equations formed by (6) and (8). In equilibrium, borrowers terminate their mortgage when house prices drop to 67.57% of their original value. The equilibrium mortgage coupon is $c = 1.524$. The equilibrium initial yield on the mortgage is $c/M(1) = 7.62\%$. At the default date, the lender's net recovery is $M(\delta) = 14.89$, so the recovery rate on the mortgage is $M(\delta)/M(1) = 74.46\%$. The pool inherits all the mortgage characteristics. So the initial value of the pool is $V_p(1) = 20$, initial yield is $c/V_p(1) = 7.62\%$, and the recovery rate is $R_p/V_p(1) = 74.46\%$. The recovery on the pool is adequate provided $\theta \leq 0.7446$. In this case, the initial yield on the senior bond is $r = 7\%$.

When $\theta = 0.80$, the value of the senior tranche at origination is $V_s(1) = 16$. Since the recovery on the pool is inadequate, the recovery on the senior tranche is $R_s = R_p = 14.89$. The recovery rate of the

[8]For example, Flavin and Yamashita (2002) estimated the standard deviation of the real return on housing to be 14%. Similarly, Case and Shiller (1989) estimated the return on individual houses to be around 14-15%.

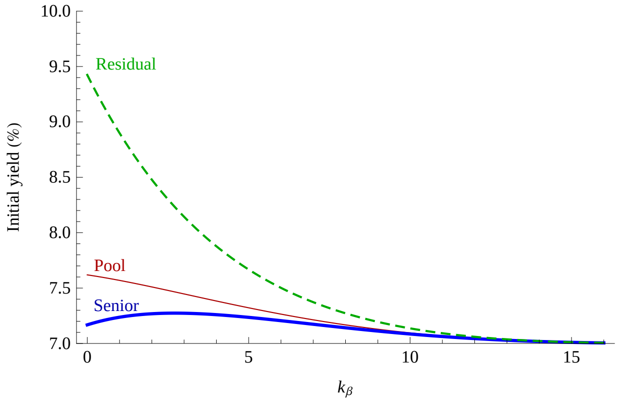


Figure 1: Initial yield as a function of borrower default costs k_β

senior tranche is $R_s/V_s(1) = 93.07\%$. The equilibrium senior coupon is $c_s = 1.147$ and the equilibrium initial yield on senior bonds is $c_s/V_s(1) = 7.17\%$. The equilibrium characteristics of the residual tranche follow from those of the senior tranche. The initial value of the residual tranche is $R_j(1) = 4$, its coupon is $c_j = 0.377$, and its initial yield is $c_j/R_j(1) = 9.42\%$. The recovery on this tranche is $R_j = 0$.

The disproportionate division of default risk is reflected in the recovery rates. The recovery rate of the senior tranche, pool, and residual tranche are 93.07%, 74.46% and 0%. The residual tranche is wiped out on default because the entire recovery on the pool is applied to the senior tranche. The initial yield spreads on the senior tranche, pool, and the residual tranche are 0.17%, 0.62%, and 2.42%. The yield spread on the residual tranche is higher because residual bondholders need to be compensated for the additional default risk borne by them. The example reiterates how the credit risk of the pool is divided disproportionately among the tranches to create a relatively safe senior tranche and a relatively risky residual tranche.

To gain further insights into the model solution, we analyze the initial yield on the pool, senior bonds, and residual bonds as a function of borrower default costs. An increase in borrower default costs k_β decreases the default threshold; see (6). The decrease in the default threshold has two opposing effects: It lowers the probability of default and it lowers the lender's net recovery on default. A lower default probability decreases the initial yield on the mortgage, but a lower net recovery increases the initial yield. The equilibrium initial yield on the pool, senior bonds, and residual bonds is the net of these two effects.

Figure 1 shows equilibrium initial yields as a function of borrower default costs; all other parameters equal their benchmark values. The solid curve in the middle represents the pool, the bold solid curve represents the senior tranche, and the dashed curve represents the residual tranche. The figure shows that the initial yield on the senior tranche is always less than that on the pool, and the initial yield on the

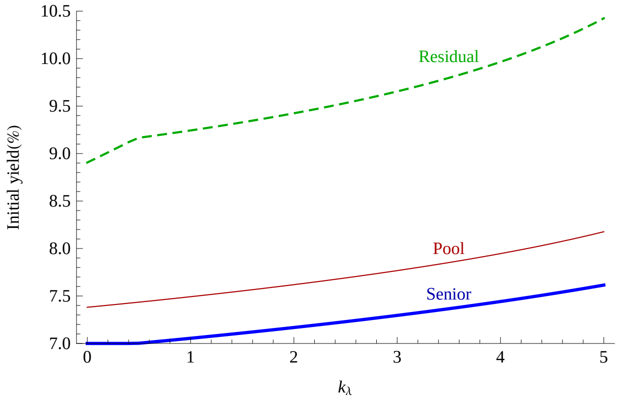


Figure 2: Initial yield as a function of lender default costs k_λ

residual tranche is always greater than that on the pool. The initial yield on the pool is monotonically decreasing in k_β, indicating that the probability effect dominates. The initial yield on the senior tranche first increases with k_β and then starts declining, indicating that the recovery effect dominates initially but is eventually taken over by the probability effect. Since the recovery on the residual tranche is zero when $k_\beta = 0$, further increase in k_β does not lower the net recovery; it only lowers the probability of default. So the initial yield on the residual tranche is decreasing in k_β. As k_β increases unboundedly, all three initial yields approach r. This finding is intuitive: if $k_\beta \to \infty$, then the exercise of the default option becomes extremely costly. So the underlying mortgage becomes risk-free and all initial yields approach the risk-free rate.

Figure 2 shows the equilibrium initial yield as a function of lender default costs k_λ. (All other parameters are equal to their benchmark values. In particular, $k_\beta = 0$.) Once again, the solid curve represents the pool, the bold solid curve represents the senior tranche, and the dashed curve represents the residual tranche. An increase in lender default costs k_λ reduces the net recovery on the pool and the tranches without changing the default probability; recall $R_p = P(\delta) - k_\lambda$. So all three initial yield curves increase in k_λ. The kinks on the yield curve for the tranches indicate the value of k_λ at which the recovery on the pool switches from being adequate to inadequate. The senior tranche is risk-free to the left of the kink and risky to the right of it. The recovery on the residual tranche is positive to the left of the kink and zero to the right of it.

In practice, for institutional reasons, mortgages with LTV ratios greater than 80% usually consist of two different liens. For example, a 92% mortgage usually consists of a first-lien with an LTV of 80%, and a second-lien of 12%. If the borrower defaults on the mortgage, then the first-lien has initial claims on the recovery. The recovery on the second-lien is positive only if the recovery on the underlying mortgage is greater than 80%. The disproportionate division of recovery among the two liens corresponds exactly to the division of recovery between the tranches in the model. The first-lien corresponds to the senior tranche and the second-lien corresponds to the residual tranche. So the analysis of CMOs created from homogeneous pools applies directly to the valuation of two-liened mortgages.

Consider the two-liened mortgage with a total LTV of 92%. (All other parameters equal their benchmark values.) The recovery rate on the mortgage is 87.17%. A pool containing this mortgage inherits all the characteristics of the mortgage. If the proportional value of the senior tranche at origination is 0.87, then the tranche corresponds to the first-lien of the mortgage. The proportional value of the tranche is less than the recovery rate of the pool, so the recovery on the pool is adequate and the senior tranche is risk-free. Correspondingly, the first-lien of the mortgage is also risk-free. On default, the entire loss of principal is borne by the residual tranche, which correponds to the second-lien of the mortgage. The recovery rate of the second-lien is only 1.67%. So the first-lien (senior tranche) is risk-free when the second-lien (residual tranche) is large enough to absorb the loss of principal on default.

3.2 Heterogenous pool

This section extends the analysis to heterogeneous pools, which contain two different types of mortgages. After appropriate modification, the analysis here also applies to pools with more than two types of mortgages. The two mortgages differ in borrower default costs. The default costs faced by the borrower of the first mortgage are lower. The other exogenous characteristics of the two mortgages are identical. In particular, one aggregate geometric Brownian motion governs the evolution of housing services for both the mortgaged properties. The cost of exercising the default option is lower for the first borrower, so his default threshold is higher. Therefore, contingent on default, the first mortgage is always terminated earlier. So we refer to the first and second mortgages as early-default and late-default mortgages. Variables associated with early- and late-default mortgages are indexed by e and l. Since the early-default mortgage has a higher default threshold, the value of the property when the mortgage is terminated is higher. So the lender's net recovery on this mortgage is higher; recall that lender default costs are identical across mortgages.

Consider a pool created by combining early- and late-default mortgages; normalize the total number of mortgages in the pool to one. The proportion of early-default mortgages in the pool is exogenous; it is denoted by $\eta \in (0, 1)$. The proportion of late-default mortgages is $1 - \eta$. Denote the value of the pool by $V_p(t, x(t))$. Time is a state variable when the pool is heterogeneous because we need to keep track of the composition of the pool. The value of the pool is the weighted average of the value of the underlying mortgages that have not defaulted. It equals the weighted average of both the mortgage values before the early default event; the value of the late-default mortgage weighted by $1 - \eta$ after the early-default event; zero after the late-default event.

$$V_p(t, x(t)) = \begin{cases} \eta M_e(x(t)) + (1-\eta)M_l(x(t)) & \text{if } 0 \le t < \tau_e \\ (1-\eta)M_l(x(t)) & \text{if } \tau_e \le t < \tau_l \\ 0 & \text{if } \tau_l \le t. \end{cases} \tag{13}$$

Denote the coupon on the pool by $c_p(t)$. It is the weighted average of the coupons on the underlying mortgages that have not defaulted. So it is $\eta c_e + (1-\eta)c_l$ before the early default event, $(1-\eta)c_l$ after the early default event, and zero after the late default event. The initial yield on the pool is $c_p(0)/V_p(0,1)$, and the yield at the time of early default is $c_p(\tau_e)/V_p(\tau_e, \delta_e)$. The yield on the pool can also be expressed as the weighted average of the underlying mortgage yields; the same holds for yields spreads on the pool. At the time of early default, the recovery on the pool is the recovery on early-default mortgages weighted by their proportion, $R_{pe} = \eta M_e(\delta_e)$; we refer to this as the early recovery on the pool. Similarly the late recovery on the pool is $R_{pl} = (1-\eta)M_l(\delta_l)$. We refer to the sum of the early and late recoveries on the pool, $R_p = R_{pe} + R_{pl}$, as the total recovery on the pool. The recovery rate of the pool is its total recovery divided by its value at origination, $R_p/V_p(0,1)$.

Bonds that represent pro-rata claims to the cash flows of the pool are labeled mortgage pass-throughs. The total value of pass-throughs at origination equals $V_p(0,1)$. These bonds receive the initial coupon $c_p(0)$ until the early default event. The lender uses the early recovery on the pool to buy back bonds at their market value. Bondholders are indifferent to selling their bonds at this value. The total value of the bonds that remain outstanding is $V_p(\tau_e, \delta_e)$. These bonds receive $c_p(\tau_e)$ until the late default event. The lender uses the late recovery on the pool to buy back the remaining bonds at their market value. The initial yield on pass-throughs is $c_p(0)/V_p(0,1)$ and the yield at the time of early default is $c_p(\tau_e)/V_p(\tau_e, \delta_e)$. The yield on pass-throughs is always greater than the risk-free rate r because these bonds carry default risk.

Tranches.– As in the case of a homogeneous pool, the lender divides the pool by value into a senior tranche and a residual tranche. He then sells bonds that are pro-rata claims to cash flows on each tranche. The senior tranche has first-claims to all cash flows on the pool, whereas the residual tranche is the first to bear all losses. This division of cash flows is done so as to make the senior tranche relatively safe and the residual tranche relatively risky.

The proportional value of the senior tranche at origination exogenous; it is denoted by $0 \le \theta \le 1$. Let $V_s(t, x(t))$ denote the value of the senior tranche. The initial value of the senior tranche is $V_s(0,1) = \theta V_p(0,1)$. Let $c_s(t)$ denote the coupon on the senior tranche. The cash flows to the tranche are as follows. The tranche receives its original coupon $c_s(0)$ until the early default event. The lender uses the early recovery on the pool to buy back senior bonds at their market value. The amount allocated towards the buyback is referred to as the early recovery on the senior tranche; it is denoted R_{se}. The fraction of senior bonds outstanding after the buyback is

$$q_s \equiv \frac{V_s(\tau_e, \delta_e)}{V_s(\tau_e, \delta_e) + R_{se}}, \tag{14}$$

where $V_s(\tau_e, \delta_e)$ is the market value of the senior tranche after the buyback. The lender attempts to pay the outstanding senior bonds their original coupon. If he cannot do so, then he forwards the entire coupon on the pool to the tranche. Therefore, the coupon on the senior tranche at the time of early default is

$$c_s(\tau_e) = \min\{q_s c_s(0), c_p(\tau_e)\}. \tag{15}$$

Senior bonds receive this coupon until the late default event. The lender uses the late recovery on the pool to buy back the remaining senior bonds at their market value. The amount allocated towards the buyback is termed the late recovery on the senior tranche; it is denoted R_{sl}.

The total recovery on the senior tranche is defined as the sum of the early and late recoveries; it is denoted R_s. The tranche has first-claims on the cash flows to the pool. So the entire early recovery on the pool is used to buy back senior bonds, unless the recovery exceeds the par value of the senior tranche. The early recovery on the tranche is $R_{se} = \min\{V_s(0,1),\ R_{pe}\}$. The entire late recovery on the pool is used to buy back senior bonds, unless the recovery exceeds the par value of the outstanding senior bonds. The late recovery on the tranche is $R_{sl} = \min\{V_s(0,1) - R_{se},\ R_{pl}\}$. In both cases, the minimum operator ensures that the total recovery on the tranche does not exceed its par value. The recovery rate of the senior tranche is its total recovery divided by the par value, $R_s/V_s(0,1)$.

As in the case of the homogeneous pool, we call the recovery on the pool adequate if it is large enough buy back the senior tranche at its par value, $R_s = V_s(0,1)$. Otherwise, the recovery is inadequate and $R_s = R_p$. The recovery rate of the tranche is one when the recovery on the pool is adequate and less than one when the recovery is inadequate. In both cases, the recovery rate of the tranche is greater than the recovery rate of the pool. We call the coupon on the pool adequate if it is large enough to continue paying the outstanding senior bonds their original coupon after the early buyback, $c_s(\tau_e) = q_s c_s(0)$. Otherwise, if $q_s c_s(0) > c_p(\tau_e)$ then the coupon is inadequate and $c_s(\tau_e) = c_p(\tau_e)$.

The value of the residual tranche is $V_j(t, x(t)) = V_p(t, x(t)) - V_s(t, x(t))$. Its coupon is $c_j(t) = c_p(t) - c_s(t)$. The coupon on the tranche after the early default event is positive when the coupon on the pool is adequate, and zero when the coupon is inadequate. The early recovery on the tranche is $R_{je} = R_{pe} - R_{se}$ and the late recovery is $R_{jl} = R_{pl} - R_{sl}$. The total recovery is $R_j = R_p - R_s$. The total recovery on the tranche is positive when the recovery on the pool is adequate and zero when the recovery is inadequate. The recovery rate of the tranche is less than the recovery rate of the pool.

Equilibrium.– We assume perfect competition in the CMO market, so the lender makes zero expected profits in equilibrium. The equilibrium coupon schedule on the senior tranche is such that this tranche is issued at par; the coupon schedule on the residual tranche follows. The equilibrium senior coupons $c_s(0)$ and $c_s(\tau_e)$ satisfy

$$V_s(0,1) = \mathbb{E}_0\left[\int_0^{\tau_e} e^{-rt}c_s(0)dt\right] + \mathbb{E}_0\left[e^{-r\tau_e}R_{se}\right] + \mathbb{E}_0\left[e^{-r\tau_e}V_s(\tau_e,\delta_e)\right], \tag{16}$$

$$V_s(\tau_e,\delta_e) = \mathbb{E}_{\tau_e}\left[\int_{\tau_e}^{\tau_l} e^{-r(t-\tau_e)}c_s(\tau_e)dt\right] + \mathbb{E}_{\tau_e}\left[e^{-r(\tau_l-\tau_e)}R_{sl}\right]. \tag{17}$$

where $c_s(\tau_e)$ is given by (15). According to (16) the market value of the senior tranche at origination is the expected discounted value of its initial coupon payments, its early recovery R_{se}, and its market value at the early default date $V_s(\tau_e,\delta_e)$. Similarly (17) states that the market value of the senior tranche at the time of early default is the expected discounted value of its remaining coupon payments and the late recovery payment R_{sl}.

The permutations of adequate coupon and adequate recovery on the pool suggest four types of equilibria: adequate coupon, adequate recovery; inadequate coupon, adequate recovery; adequate coupon, inadequate recovery; inadequate coupon, inadequate recovery. Next we discuss when each type of equilibrium arises, if at all, and characterize the coupon schedule of the senior tranche in that equilibrium.

First we focus on cases in which the recovery on the pool is adequate. Consider the case in which the senior tranche is so small that the entire tranche is bought back at its par value when early-default mortgages are terminated. The par value of this tranche must be less than the early recovery on the pool, $V_s(0,1) \leq R_{pe}$. Equivalently, let θ_1 denote the threshold at which the proportional value of the senior tranche is such that $V_s(0,1) = R_{pe}$; the threshold is $\theta_1 = R_{pe}/V_p(0,1)$. The entire senior tranche is bought back at the time of early default at its par value when $\theta \leq \theta_1$.

Now consider a senior tranche whose proportional value is slightly larger than θ_1. The par value of this tranche is slightly larger than R_{pe}. So the entire tranche is not bought-back at the time of early default. Instead the fraction $R_{pe}/V_s(0,1)$ is bought-back at the time of early default and the remaining fraction is bought-back at its par value at the time of late default. The early and late recoveries on the senior tranche are $R_{se} = R_{pe}$ and $R_{sl} = V_s(0,1) - R_{se}$. The tranche continues to be bought-back at its par value at the two default events as long as the late recovery on the pool is large enough to buyback outstanding senior bonds at their par value, $R_{pl} \geq V_s(0,1) - R_{pe}$. Rearranging terms in the inequality, we obtain that the recovery on the pool is adequate as long as the par value of the senior tranche is less than the total recovery on the pool. Let θ_2 denote the threshold at which the proportional value of the senior tranche is such that $V_s(0,1) = R_p$; the threshold is $\theta_2 = R_p/V_p(0,1)$. The recovery on the pool is adequate for all $\theta \leq \theta_2$.

Motivated by our finding for one-mortgage pools, we conjecture that the yield on senior bonds equals the risk free rate r when the recovery on the pool is adequate. The conjecture is correct; see the Appendix for the verification. The intuition is identical to the one-mortgage case: If the recovery is adequate, then senior bonds carry no default risk because the total recovery on these bonds equals their initial principal. So the equilibrium yield on senior bonds must equal the risk free rate r at all times. We label this equilibrium the risk-free equilibrium. Next we determine whether the coupon on the pool is adequate or inadequate in the risk-free equilibrium. Recall that the yield on the pool is always greater than r. The value of the senior tranche is always less than the value of the pool, $V_s(t, x(t)) \leq V_p(t, x(t))$. Together these two observations imply $c_p(t) \geq rV_p(t, x(t)) \geq rV_s(t, x(t))$. So the coupon on the pool is adequate in the risk-free equilibrium.

To establish the uniqueness of the risk-free equilibrium for $\theta \in [0, \theta_2]$, note that the definition of $c_s(\tau_e)$ in (15) implies that the coupon on the outstanding senior bonds cannot rise at the early default date. So the coupon either falls or remains constant at the early default date. Suppose that the coupon falls, then $c_s(\tau_e) = c_p(\tau_e)$ by definition. This case cannot be an equilibrium when $\theta \in [0, \theta_2]$ because it allows arbitrage: an investor can borrow $V_s(0, 1) - R_{se}$ at the risk free rate r, purchase the senior tranche, and earn a rate of return greater than r until the late default event (Recall that the yield on the pool is always greater than r and $V_s(t, x(t)) \leq V_p(t, x(t))$, implying that the yield on senior bonds is greater than r when $c_s(\tau_e) = c_p(\tau_e)$.) At the time of late default he will receive a recovery of $V_s(0, 1) - R_{se}$, which he can use to pay back his debt. Similarly, if the coupon on senior bonds is constant, then the yield must equal r to rule out arbitrage. The uniqueness of the risk-free equilibrium for $\theta in [0, \theta_2]$ implies that there are no equilibria for which the recovery on the pool is adequate while the coupon is inadequate.

If $\theta > \theta_2$, then the recovery on the pool is inadequate. To complete the classification of equilibria, we need to divide the interval $(\theta_2, 1]$ into two regions. One region in which the coupon on the pool is adequate and another region in which the coupon is inadequate. Consider a senior tranche whose proportional value is slightly larger than θ_2. The initial coupon on this tranche will be slightly larger than r. After the early default event, the coupon on the pool will be large enough to pay the outstanding senior bonds their initial coupon. So the coupon on the pool will be adequate. As the proportional size of the senior tranche increases, its recovery rate will decrease and so the required coupon will increase. The largest value of θ for which the coupon on the pool is adequate is found by solving $q_s c_s(0) = c_p(\tau_e)$. The resulting threshold, denoted θ_3, is

$$\theta_3 = 1 - \frac{(1 - \delta_e^m)\eta c_e / r}{M_e(1)} \left(1 - \frac{c_l / M_l(\delta_e)}{c_e / M_e(\delta_e)} \right). \tag{18}$$

The threshold (18) lies within the interval $(\theta_2, 1)$ provided $c_e/M_e(\delta_e) \geq c_l/M_l(\delta_e)$; see the Appendix for details. Since $M_e(1) = M_l(1)$, the required condition on the underlying mortgages holds when the initial yield on the early-default mortgage is greater than that on the late-default mortgage, $c_e/M_e(1) \geq c_l/M_l(1)$, and the net recovery on the early-default mortgage is less than the value of late-default mortgage at the same date, $M_e(\delta_e) \leq M_l(\delta_e)$. From here on, we assume that $c_e/M_e(\delta_e) \geq c_l/M_l(\delta_e)$. So when $\theta \in (\theta_2, \theta_3]$ the proportional value of the senior tranche at origination is small enough to leave the senior coupon unchanged after the early default event; the coupon on the pool is adequate. When $\theta \in (\theta_3, 1]$, however, the proportional value of the senior tranche at origination is so large that the entire coupon on the pool goes to this tranche after the early default event; the coupon on the pool is inadequate.

In summary, all thresholds lie within the interval $(0, 1)$ and satisfy $\theta_1 \leq \theta_2 < \theta_3$. They divide the unit interval into four equilibrium regions. When $\theta \in [0, \theta_1]$, the senior tranche is so small that all the principal on this tranche is repaid at the early default date. We refer to this region as risk-free region I. When $\theta \in (\theta_1, \theta_2]$, the senior tranche is still small enough that all the principal on this tranche is repaid on default. In this case, however, part of the principal remains outstanding after the early default date. The outstanding principal is repaid at the late default date. We refer to this region as risk-free region II. When $\theta \in (\theta_2, \theta_3]$, the senior tranche is so large that its entire principal cannot be repaid on default. It is, however, small enough that the senior bonds outstanding after the early default event continue to receive their initial coupon. We refer to this region as the low risk region. When $\theta \in (\theta_3, 1]$, the senior tranche is so large that its entire principal cannot be repaid on default, and the coupon on senior bonds outstanding after the early default event drops. We refer to this region as the high risk region.

Now that we have completed the classification of equilibria, we solve for the equilibrium coupons in the low risk and high risk equilibria; recall that the coupons for the risk-free equilibrium are such that the yield on the senior tranche equals r. In the low-risk equilibrium, the early and late recoveries on the senior tranche equals the early and late recoveries on the pool. The coupon on this tranche at the time of early default event is

$$c_s(\tau_e) = q_s c_s(0). \tag{19}$$

Equations (16), (17), and (19) form a system of nonlinear equations in $c_s(0)$, $c_s(\tau_e)$, and $V_s(\tau_e, \delta_e)$. We obtain the equilibrium senior coupon schedule by solving this system numerically.

In the high risk equilibrium also, the early and late recoveries on the senior tranche equals the early and late recoveries on the pool. However, after the early default event, the coupon on the pool is distributed pro-rata among the outstanding senior bonds, $c_s(\tau_e) = c_p(\tau_e)$. Since $c_s(\tau_e)$ is known, we find $V_s(\tau_e, \delta_e)$

using (17) and then solve (16) for $c_s(0)$ to obtain

$$c_s(0) = \frac{r}{1-\delta_e^m}\left[V_s(0,1) - \left(R_{pe} + \frac{c_p(\tau_e)}{r}\right)\delta_e^m - \left(R_{pl} - \frac{c_p(\tau_e)}{r}\right)\delta_l^m\right]. \tag{20}$$

The equilibrium characteristics of the residual tranche follow from those of the senior tranche. In risk-free region I, the early recovery on the residual tranche is strictly positive, except when $\theta = \theta_1$. After the early default event, the residual tranche mimics the pool: its coupon equals $c_p(\tau_e)$ and its late recovery equals R_{pl}. In risk-free region II, the early recovery on the residual tranche is zero. The late recovery is strictly positive, except when $\theta = \theta_2$. The coupon on this tranche is always strictly positive. It does, however, drop after the early default event. In the low risk region, the early and the late recoveries on the residual tranche are both zero. The coupon on this tranche is strictly positive, except when $\theta = \theta_3$. The coupon in this region also drops after the early default event. In the high risk region, the early and late recoveries on the residual tranche are zero, and its coupon drops to zero after the early default event.

Numerical Examples.– This subsection illustrates various characteristics of the model using numerical examples. It presents examples of yields on low-risk and high-risk senior bonds. It shows how the equilibrium thresholds θ_1, θ_2, and θ_3 change with the composition of the pool, η . Finally, it shows how model implied yields for low-risk senior bonds change with the securitization parameters θ and η. The parameter values are in the numerical example for the homogeneous pool; see Table 1. Borrower default costs for early- and late-default borrowers are set to zero and twenty percent of the mortgage size, $k_{\beta e} = 0$ and $k_{\beta l} = 4$. Lender default costs are set to ten percent of the mortgage size, $k_\lambda = 2$.

Table 2 summarizes the implied characteristics of early- and late-default mortgages. In equilibrium, borrowers with early-default mortgages terminate their mortgage when house prices drop to 67.57% of the purchase price. Late-default borrowers terminate when house prices drop to 53.06% of the purchase price. The recovery rates on early- and late-default mortgages are 74.46% and 56.32%; the equilibrium coupons are $c_e = 1.524$ and $c_l = 1.477$; the equilibrium initial yields are 7.62% and 7.38%. The initial yield on each mortgage reflects the compensation required by risk-neutral agents to bear the default risk associated with the mortgage. The higher initial yield on the early-default mortgage reflects the fact that the payoff on this mortgage is riskier than the payoff on the late-default mortgage.

As a benchmark, we set the measure of early- and late-default mortgages in the pool equal to each other, $\eta = 0.5$. Once we fix the composition of the pool, other characteristics of the pool are implied by those of the underlying mortgages. The implied origination value is $V_p(0,1) = 20$. The initial coupon is $c_p(0) = 1.500$. The initial yield on the pool is 7.50%. The value at the early default event is $V_p(\tau_e, \delta_e) = 8.42$. The coupon after the early default event is $c_p(\tau_e) = 0.738$, so the yield at the early default date is 8.77%. The early and late recoveries are $R_{pe} = 7.45$ and $R_{pl} = 5.63$. The total recovery

	Early-default mortgage ($k_{\beta e} = 0$)	Late-default mortgage ($k_{\beta l} = 4$)
LTV	80%	80%
Default threshold	67.57%	53.06%
Coupon	1.524	1.477
Initial yield	7.62%	7.38%
Recovery rate	74.46%	56.32%

Table 2: Benchmark mortgage characteristics

is $R_p = 13.08$ and the recovery rate is 65.39%.

Once the composition of the pool and the underlying mortgage parameters are chosen, the threshold for each type of equilibrium can be determined. The first threshold is $\theta_1 = 0.3723$. All senior bonds are bought back at the early default event if the initial value of the senior tranche, in proportion to that of the pool, is less than 37.23%. The second threshold is $\theta_2 = 0.6539$. So senior bonds are risk-free if the proportional value of the senior tranche at origination is less than 65.39%. When the proportional value is between 37.23% and 65.39% some risk-free senior bonds remain outstanding after the early default event; these bonds are bought back at par at the late default date. The third threshold is $\theta_3 = 0.9422$. If the proportional value of the senior tranche at origination is strictly larger than 65.39% but less than 94.22%, then senior bonds are low-risk in equilibrium. On the other hand, if this value is strictly larger than 94.22%, then senior bonds are high-risk in equilibrium.

We provide an example of each type of equilibrium; Table 3 summarizes these examples. The senior tranche is risk-free in equilibrium if its initial value is 40% of the pool's initial value. The implied initial value of the senior tranche is $V_s(0,1) = 8$. The coupons on the senior tranche are $c_s(0) = 0.560$ and $c_s(\tau_e) = 0.039$. Regardless of the evolution of house prices the yield on senior bonds is $r = 7\%$ until the late default event. The early recovery on the senior tranche equals the early recovery on the pool, $R_{se} = 7.45$. The late recovery on the senior tranche is $R_{sl} = 0.55$. The recovery rate on this tranche is 100%.

The residual tranche is 60% of the pool at origination, $V_j(0,1) = 12$. The implied initial coupon on residual bonds is $c_j(0) = 0.940$. In contrast to senior bond yields, residual bond yields depend on the evolution of house prices. Prior to the late-default event, residual bond yields decrease if house prices increase and vice-versa. This decrease in the residual bond yield reflects the decline in the default probability due to the increase in house prices. The initial yield on residual bonds is 7.84%. The initial yield spread on residual bonds is higher than the spread on the pool because all the default risk has been directed to the residual tranche. The coupon on the residual tranche at the early default event is $c_j(\tau_e) = 0.700$. The residual bond yield at the early default event increases to 8.89%. This increase in the residual bond yield reflects the increased likelihood of the late default event. Since the residual tranche absorbs all the losses due to default, the recovery rate on this tranche is only 42.32%.

The low-risk equilibrium occurs when $\theta = 0.80$. The implied initial value of the senior tranche is $V_s(0,1) = 16$. Unlike risk-free senior bonds, the yield on low-risk senior bonds depends on the evolution of house prices; the yield and house prices are inversely related. The yield on low-risk senior bonds approaches $r = 7\%$ only when house prices rise unboundedly. The initial coupon on the senior tranche is $c_s(0) = 1.158$. The initial yield on senior bonds is 7.24%; the yield spread of 24 basis points reflects the compensation to bondholders for default risk. The yield increases to 8.03% at the early default event. The early and late recoveries on the senior tranche and the pool are identical. The recovery rates, however, are not. The recovery rate on the pool is 65.39% and that on the senior tranche is 81.74%. The recovery rate on the senior tranche is higher because the residual tranche is the first to absorb losses.

Once we have the low-risk equilibrium characteristics of the senior tranche, those of the residual tranche follow. The initial value of the residual tranche is $V_j(0,1) = 4$. The initial yield on residual bonds is 8.55%, and the early default yield on these bonds is 12.34%. The recovery rate on the residual tranche is zero in the low-risk equilibrium.

The high-risk equilibrium occurs when $\theta = 0.95$. The initial value of the senior tranche is $V_s(0,1) = 19$. Since senior bonds are high-risk, the coupon on these bonds after the early default event equals the coupon on the pool, $c_s(\tau_e) = 0.738$. The value of the senior tranche at the early default event is $V_s(\tau_e, \delta_e) = 8.42$. The initial coupon on the senior tranche is $c_s(0) = 1.406$. The initial and early default yields on senior bonds are 7.40% and 8.77%. The early and late recoveries on the senior tranche and the pool are identical. The recovery rate on the pool is 65.39% and that on the senior tranche is 68.83%. The equilibrium characteristics of the senior tranche approach those of the pool as θ increases. When $\theta = 1$, the senior tranche and the pool are identical.

The initial value of the residual tranche in the high-risk equilibrium is $V_j(0,1) = 1$. The coupon on residual bonds is $c_j(0) = 0.094$ and the initial yield is 9.42%. In the high-risk equilibrium, the residual tranche stops receiving payments after the early default event.

Next we show how the model solution changes as the composition of the pool changes; the composition is determined by η. Figure 3 shows the equilibrium thresholds θ_1, θ_2, and θ_3 as functions of η. This figure partitions the unit square formed by admissible values of η and θ into four equilibrium regions: risk-free region I, risk-free region II, low-risk region, and high-risk region. To begin with, we restrict our attention to $\eta \in (0,1)$. The dashed line is the threshold $\theta_1 = R_{pe}/V_p(0,1)$, where $R_{pe} = \eta M_e(\delta_e)$. Since $M_e(1) = M_l(1)$, the initial value of the pool does not change with η, $V_p(0,1) = M_e(1)$. So θ_1 is linear in η with slope equal to the recovery rate on the early-default mortgage $M_e(\delta_e)/M_e(1)$. As η increases, the early recovery on the pool increases so more cash is available to buy back senior bonds at the time of early default. Therefore the size of the senior tranche that can be bought-back at par increases. Since

	Pool	Senior			Residual		
		Risk-free ($\theta = 0.40$)	Low-risk ($\theta = 0.80$)	High-risk ($\theta = 0.95$)	Risk-free ($\theta = 0.40$)	Low-risk ($\theta = 0.80$)	High-risk ($\theta = 0.95$)
Origination							
Value	20	8	16	19	12	4	1
Coupon	1.500	0.56	1.158	1.406	0.940	0.342	0.094
Yield	7.50%	7%	7.24%	7.40%	7.84%	8.55%	9.42%
Early default event							
Value	8.42	0.55	6.98	8.42	7.87	1.44	0
Coupon	0.738	0.039	0.560	0.738	0.700	0.178	0
Yield	8.77%	7%	8.03%	8.77%	8.89%	12.34%	-
Total recovery	13.08	8	13.08	13.08	5.08	0	0
Recovery rate	65.39%	100%	81.74%	68.83%	42.32%	0%	0%

Table 3: Equilibrium tranche characteristics for $\eta = 0.50$

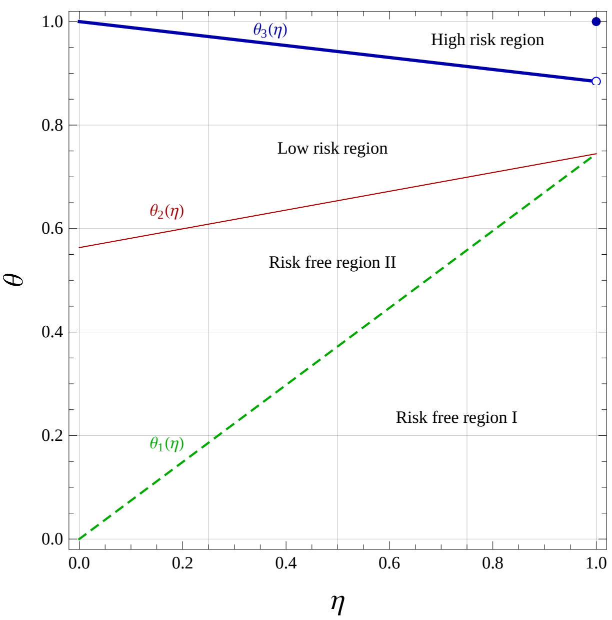


Figure 3: Equilibrium regions for admissible values of η and θ.

$V_p(0,1)$ does not change with η, an increase in the size of the senior tranche that can be bought-back implies an increase in the proportional size of the senior tranche that can be bought-back.

Similarly, $\theta_2 = R_p/V_p(0,1)$ is linear in η, and its slope equals the difference between the recovery rate on early- and late-default mortgages, $M_e(\delta_e)/M_e(1) - M_l(\delta_l)/M_l(1)$. This difference is positive because the default threshold, and so the recovery, of early-default mortgages is higher. As η increases, the weight of early-default mortgages in the pool increases so the total recovery on the pool increases. Therefore, the size of the senior tranche that can be bought back at its par value increases.

Equation (18) shows that the threshold θ_3 is linearly decreasing in η; recall that $c_e/M_e(\delta_e) \geq c_l/M_l(\delta_e)$ by assumption. An increase in η has two opposing effects – it increases the early recovery on the pool, R_{pe}, and it decreases the coupon on the pool after the early default event, $c_p(\tau_e)$. The increase in the early recovery implies that more senior bonds can be bought back at the time of early default, implying an increase in θ_3. The decrease in the $c_p(\tau_e)$, however, implies that the senior tranche should be smaller if outstanding senior bonds are to continue receiving their initial coupon after the early default event, implying that θ_3 decreases. The second effect dominates in the example considered.

So far the analysis has been restricted to $\eta \in (0,1)$. Now we extend it to include the endpoints of the interval. When η equals zero or one the pool contains one type of mortgage only, so the analysis in section 3.1 applies. The analysis of section 3.1 can be imbedded into the analysis here by setting the early and late default mortgage variables equal to each other. For example, $c_e = c_l$ and $M_e(\delta_e) = M_l(\delta_e)$ implies that $\theta_3 = 1$ at the end points. To maintain consistency with the two-mortgage pool framework, we assume that the pool experiences two default events when η equals zero or one, with one default event being inconsequential. If $\eta = 0$, then the pool contains late-default mortgages only. So the early default event is inconsequential: The coupon on the pool is unchanged, the early recovery on the pool is zero, and no senior bonds are bought back. All senior bonds continue to receive their initial coupon after the early default event. The coupon on the pool is adequate for all $\theta \in [0,1]$; see Figure 3. When $\eta = 1$ the pool contains early-default mortgages only. So the late-default event is inconsequential. The early recovery on the pool R_{pe} equals the total recovery R_p. Therefore the thresholds θ_1 and θ_2 coincide. The expression for θ_3 in (18) shows that $\theta_3 = 1$.

Figure 4 shows equilibrium initial yields on the senior tranche as a function of η, for various values of θ. The kink on an initial yield curve indicates the value of η at which the senior tranches switches equilibrium regions. For example, when $\theta = 0.60$, the senior tranche switches from the low-risk equilibrium to the risk-free equilibrium at the kink. When $\theta = 0.90$, the senior tranches switches from the low-risk equilibrium to the high-risk equilibrium at the kink. Depending on the value of θ, an increase in η can either increase or decrease the initial yield on senior bonds. To understand the effect of an increase in η on the yield, calculate the yield at the end points $\eta = 0$ and $\eta = 1$; the yield for $\eta \in (0,1)$ is a weighted

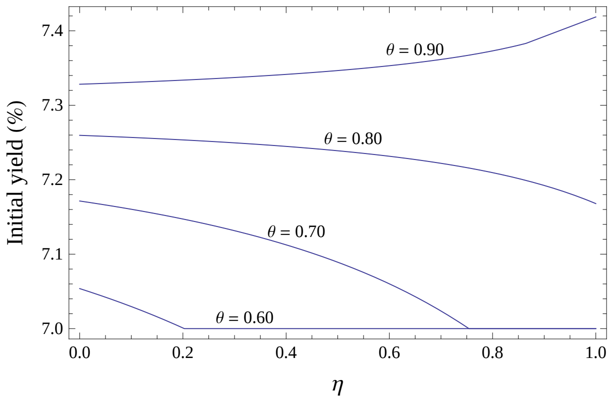


Figure 4: Equilibrium initial yields as a function of η.

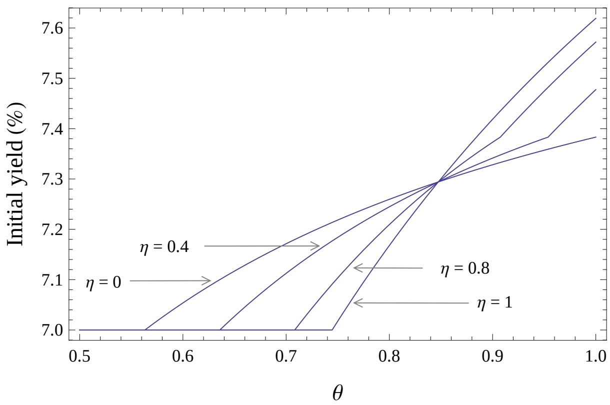


Figure 5: Equilibrium initial yields as a function of θ.

average of the yield at the end points. Our analysis of homogeneous pools applies at the endpoints. When $\eta = 0$, the pool consists of late-default mortgages only. When $\eta = 1$, the pool consists of early-default mortgages only. So an increase in η from zero to one is equivalent to a decrease in borrower default costs for a homogeneous pool; borrower default costs decrease from $k_{\beta l} = 4$ to $k_{\beta e} = 0$. As discussed earlier, a decrease in borrower default costs has two opposing effects: it increases the probability of default which raises the initial yield, and it increases the lender's net recovery which lowers the initial yield. The equilibrium initial yield at $\eta = 1$ maybe less or greater than the yield at $\eta = 0$, depending on which effect dominates. So the initial yield on the senior tranche is increasing in η for some θ, and decreasing in η for others.

Figure 5 shows equilibrium initial yields on senior bonds as a function of θ, for various η.[9] For a

[9]We only show the initial yield for $\theta \in [0.5, 1]$; this yield equals r for all $\theta \in [0, 0.5)$

given η, the initial yield on senior bonds is unambiguously increasing in θ. An increase in θ reduces the size of the residual tranche, so the senior tranche's buffer against default losses is decreased and its losses on default increase. The increase in initial yield with θ compensates senior bondholders for the possibility of larger losses. The kink on each initial yield curve shows the value of θ at which the senior tranche switches equilibrium regions. The first kink shows the switch from the risk-free region to the low-risk region and the second kink shows the switch from the low-risk region to the high-risk region. The high-risk equilibrium is ruled-out when there is only one type of mortgage in the pool. So the initial yield curves for $\eta = 0$ and $\eta = 1$ only have one kink.

This graph also provides a different perspective on how the interaction between the two securitization parameters θ and η affects the initial yield on senior bonds. Consider $\theta = 0.70$. In this case, an increase in η lowers the initial yield because the recovery effect dominates the probability effect; the senior tranche is risk-free when $\eta = 1$. Now consider $\theta = 0.90$. In this case, as increase in η increases the initial yield because the probability effect dominates the recovery effect. The initial yield curves cross at the value of θ at which the two opposing effects cancel each other out. The crossing point is found by equating the initial yield on senior bonds in the $\eta = 0$ and $\eta = 1$ cases, as given by (12), and solving for θ. In the benchmark, this crossing point is $\theta = 0.848$.[10]

4 Quantitative exercises

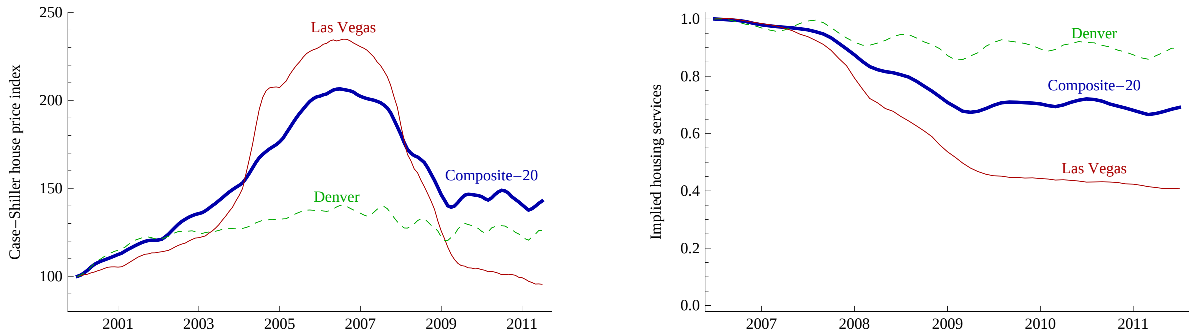
In this section we show the model implied security prices, yields, and net monthly returns for house prices observed in the data between July 2006 and July 2011. We conducted this exercise using the Case-Shiller house price index. We present our findings for three different indexes: the composite-20 index, the Las Vegas metropolitan area index, and the Denver metropolitan area index. We chose these indexes because the default experience of the benchmark pool is different for each index. Only early-default mortgages are terminated according to the composite-20 index. Both early- and late-default mortgages are terminated according to the Las Vegas index. None of the mortgages are terminated according to the Denver index.

Using a hand-collected dataset on subprime MBS, Park (2010) showed that the average LTV for non-agency securitizations during 2004-2007 was about 78%. Park (2010) also showed that the subordination for AAA-rated tranches during 2004-2007 ranged from 16.6% to 22.8%, with an average of 20.8%. Usually senior tranches of a CMO were rated AAA, so we assume that these tranches correspond to the senior tranche in the model. Motivated by the data, the low-risk equilibrium with LTV of 80% and $\theta = 0.80$ is

[10] The recovery on the early-default mortgage ($\eta = 1$ and $\theta = 1$) is greater than that on the late-default mortgage($\eta = 0$ and $\theta = 1$). So the initial yield curve for $\eta = 0$ exceeds $r = 7\%$ prior to curve for $\eta = 1$, implying that it is strictly above the initial yield curve for $\eta = 1$ for some θ. The initial yield curves for $\eta = 0$ and $\eta = 1$ cross provided the initial yield on the early-default mortgage is higher than the initial yield on the late-default mortgage. It is so in the example considered, the initial yield on the early-default mortgage is 7.62% and that on the late-default mortgage is 7.38%.

our preferred specification for the quantitative exercise.

Figure 6a shows the composite-20 index (bold solid line), the Las Vegas metropolitan area index (solid line), and the Denver metropolitan area index (dashed line) from January 2000 to July 2011; the composite-20 index aggregates house price information from twenty metropolitan areas. The composite-20 index displays rapid house price appreciation until July 2006. According to this index, house prices doubled between January 2000 and July 2006. House prices in the Las Vegas metropolitan area more than doubled during the same time period; house prices in July 2006 were approximately 2.4 times their January 2001 values. House price increases in the Denver metropolitan area were comparatively modest; they increased by about 30% in this time period. After reaching their peak in July 2006, house prices declined according to all three indexes. According to the composite-20 index, house prices declined at an average rate of 0.61% per month. By July 2011, the composite-20 index was 30.87% lower than its peak value. The Las Vegas index declined from its peak at an average rate of 1.47% per month, and was 59.25% lower than its peak value. The Denver index declined at a rate of 0.17% per month, and was only 10.19% lower than its peak.



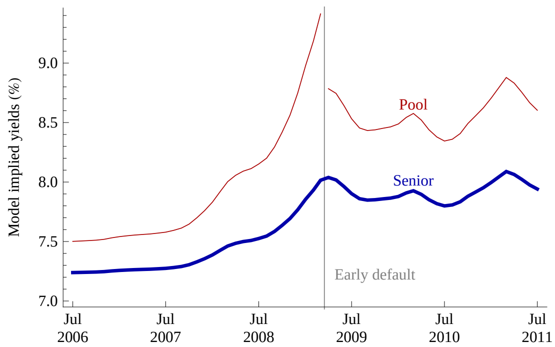
(a) Case-Shiller house price index

(b) Case-Shiller implied housing services

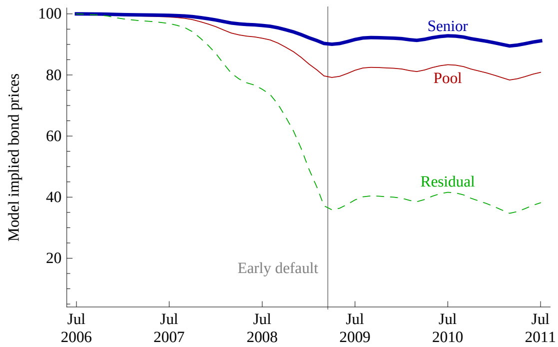
Figure 6: Case-Shiller house price index

In order to get the house price index data within the model framework, we calculated the housing service flow implied by the data; see Figure 6b. We normalize the flow of housing services to be one on July 2006, the date when housing services peak. This date will be the origination date for both mortgages in the pool.

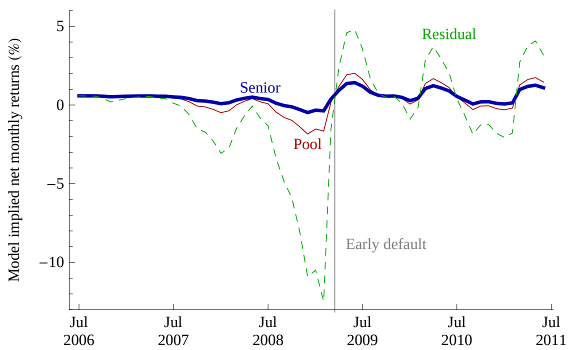
As noted earlier, the realization of the composite-20 index is such that only early-default mortgages are terminated. Figure 7a shows the realized yields on the pool (solid line) and the senior tranche (bold solid line), as implied by this index. The figure also shows the early-default date. As mentioned earlier, the bonds on the pool can be thought of as mortgage pass-through bonds. The yield on both mortgage pass-throughs and senior bonds rises as house prices fall because investors need to be compensated for the increased likelihood of default. The yield on pass-throughs drops discontinuously at the early default date because the coupon on these bonds drops at this date. Since senior bonds are low-risk, the yield on these bonds is unchanged at the early default date.



(a) Model implied bond yields



(b) Model implied bond prices

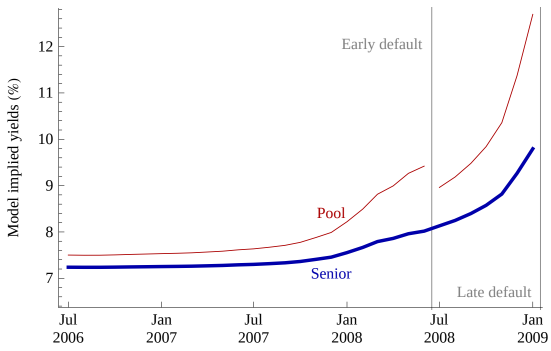


(c) Model implied net monthly returns

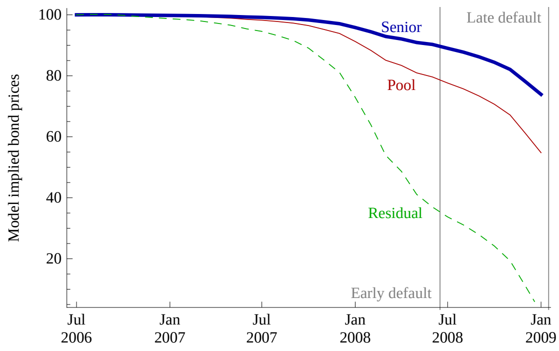
Figure 7: Composite-20 index

Figure 7b shows the market value of a mortgage pass-through, a senior bond, and a residual bond for the composite-20 index. The original bond value has been normalized to 100. Notice that the value of each bond is continuous. As expected, the value declines as house prices fall. The default risk of the pool is divided disproportionately among the tranches to create relatively safe senior bonds, and relatively risky residual bonds. The bond values reflects this division – senior bond values are always above the pass-through values, and residual bond values are always below. The model implied values of pass-throughs declined by 19.14% between July 2006 and July 2011. During the same time period, senior bond values declined by 8.84%, and residual bond values declined by 61.85%.

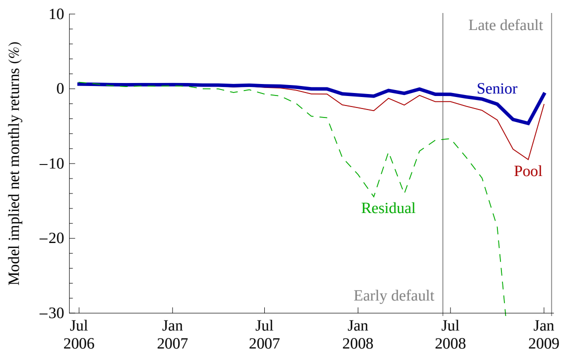
Figure 7c shows the net monthly return on all three bonds according to the composite-20 index. The net return was calculated as the sum of the monthly coupon payments and capital gains divided by the bond price last month. As the figure shows, the variability of monthly returns is highest for the residual tranche. The net monthly returns on this tranche range from about −12% right before the early default event, to around 5% after this default event. In contrast, the net monthly return on senior bonds stays around 1%.



(a) Model implied bond yields



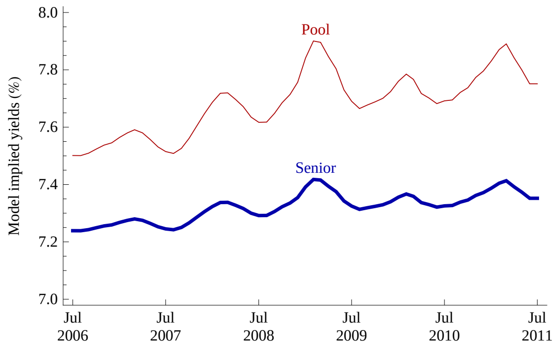
(b) Model implied bond prices



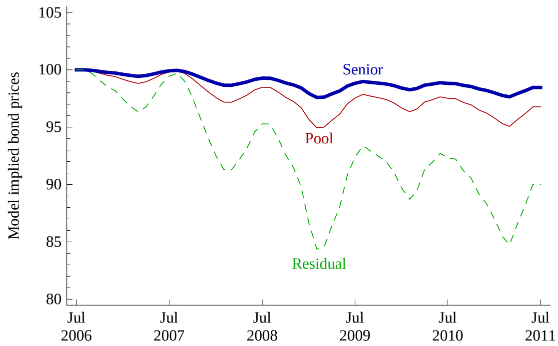
(c) Model implied net monthly returns

Figure 8: Las Vegas metropolitan area index

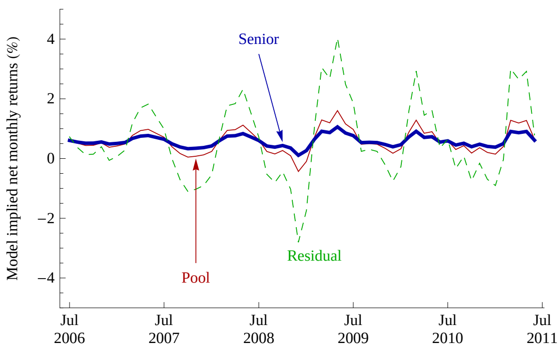
Figure 8 shows the model implied yields and prices for the housing services realized in the Las Vegas metropolitan area. The key difference between the Las Vegas index and the composite-20 index is that both early- and late-default mortgages are terminated according to the Las Vegas index. As expected, the realized pass-through and senior bond yields increase over time for this metropolitan area. The initial yield on pass-throughs is 7.50%, while the realized yield one month prior to the late default event is 12.69%. Similarly, the initial yield on senior bonds is 7.24%, and the yield prior to the late default event is 9.79%. The market value of all three bonds declines monotonically over time. A hundred dollar pass-through is worth 79.62 at the early default event, and 54.79 at the late default event. Hundred dollar senior and residual bonds are worth 90.29 and 36.98 at the early default date, respectively. The senior bond is worth 73.81 the month before late default date. Since the recovery on the residual bond is zero, it is only worth 2.37 a month before the late default date. According to the Las Vegas index, the net monthly returns on residual bonds are negative throughout the time period studied. In contrast, the net monthly return on senior bonds stays around 0% throughout, reaching its lowest value around −4% before the late-default event.



(a) Model implied bond yields



(b) Model implied bond prices



(c) Model implied net monthly returns

Figure 9: Denver metropolitan area index

Figure 9 shows the realized prices and yields for the Denver metropolitan area. Denver's index differs from the composite-20 and the Las Vegas index because neither the early-default nor the late-default mortgages are terminated according to this index. Realized bond yields for this region increase over time too. However, the increase much smaller. The yield on pass-throughs increases from 7.50% to 7.75% and the yield on senior bonds increases from 7.24% to 7.35%. Figure 9b shows that senior bond prices do not respond much to house price changes. In contrast, residual bond prices are very sensitive to house price changes. According to the Denver index, the net monthly returns fluctuate around 0.5% for all three bonds. As expected, the net return on senior bonds is close to 0.5%. However, the net return on residual bonds is more volatile; the lowest return is around -3% and the highest is around 4%.

5 CMO-squared

In practice, tranches from various CMOs are often combined together into a new pool. The pool is again divided into various tranches, and bonds on these tranches are sold in capital markets. Since the underlying assets of the pool are tranches of an existing CMO, the resulting CMO is called a CMO-squared. Prior to the crisis, CMO-squared were used extensively as collateral in the shadow banking system; the total notional amount of CMO-squared issued between 2005-2007 was about \$1.25 trillion.[11] CMO-squared were usually created from subordinate tranches of CMOs.[12] The basic principle behind creating CMO-squared was also to divide default risk disproportionately among the tranches; see Gary Gorton (2010) for an overview. In this section we study the valuation of CMO-squared. We also repeat the quantitative exercise of the previous section using the composite-20 index and calculate model implied CMO-squared yields, prices, and monthly returns.

Even though, in practice, tranches from different CMOs are combined to create the CMO-squared pool, we assume that CMO-squared are created either from the senior or from the residual tranche of a single CMO. This assumption allows us to analyze the interaction between the default risk of mortgages and CMO-squared in a simple setting. Throughout this section, we focus on the benchmark low-risk CMO. The analysis with risk-free and high-risk CMOs is similar. Consider a pool created from the low-risk senior tranche. The characteristics of the resulting pool are identical to the low-risk senior tranche: its initial value is 16, initial coupon is 1.158, early recovery is 7.45, value at the early default date is 6.98, coupon after early default is 0.560, and late recovery is 5.63; see Table 3.

The cash flows to the CMO-squared pool are re-structured so as to protect the senior tranche from default risk; the structure of cash flows is as in section 3.2. As earlier, the proportional size of the senior tranche of the CMO-squared created from the new pool is exogenous. In this case, the senior tranche of the CMO-squared can either be risk-free or low-risk in equilibrium. The high-risk equilibrium is ruled out because the coupon on the pool does not drop after the early default event. Consequently, all outstanding senior bonds also continue to receive their initial coupon after the early default event.

Following the analysis in section 3.2 we can calculate the thresholds for the equilibrium regions. The first and the second thresholds are 0.4656 and 0.8174. The senior tranche of the CMO-squared is risk-free when its proportional value at origination is less than 81.74%. When the proportional value is less than

[11] Securities Industry and Financial Markets Association, "Global CDO Issuance and Outstanding"(April 2013). <www.sifma.org/research/statistics.aspx>.

[12] For example, Sun Young Park (2013*b*) reports that only 1% of the value of tranches originally rated AAA was either placed-in CMO-squared issued during 2005-2007. In contrast, during the same time-period, this fraction was 47.03% for AA-rated, 68.38% for A-rated, 65.80% for BBB-rated tranches.

46.56%, the senior tranche is so small that all of it is bought back at the early default date. When the proportional value is greater than 46.56% and less than equal to 81.74% the senior tranche still recovers its entire principal. In this case, however, some senior bonds remain outstanding after the buy back at the time of early default. These senior bonds are bought back at their par value if late-default mortgages are terminated. If the proportional value is greater than 81.74%, then senior bonds are low-risk in equilibrium.

As a example, consider the case in which the proportional value is 0.80. The resulting senior tranche is risk-free in equilibrium. The initial value of this tranche is 12.8. Its coupons are 0.896 at origination, and 0.375 after the early default event. The early recovery is 7.45 and the late recovery is 5.35. The implied yield is $r = 7\%$ regardless of the evolution of housing services. So, in this case, re-tranching has created a senior CMO-squared tranche that is risk-free even though the CMO used to create it is risky. Note that the proportional value of the senior CMO-squared tranche is equal to that of its CMO counterpart.

Now we can calculate the implied characteristics of the residual tranche of the CMO-squared. The initial value of this tranche is 3.2. Its coupon is 0.262 at origination and 0.185 after the early default event. Its value at the early-default date is 1.63. Its early recovery is zero and its late recovery is 0.28. Its recovery rate is 8.75%. Its initial yield is 8.19% and its yield after the early default event is 11.35%. The yield spread at origination for the residual tranche of the CMO-squared is 1.19% whereas the spread for the residual tranche of the CMO is 1.55%. The lower spread on the CMO-squared tranche indicates that the residual tranche of the CMO-squared is less risky than its CMO counterpart.

Now consider a CMO-squared created from the residual tranche of the low-risk CMO. The characteristics of the resulting pool are identical to the residual tranche of the low-risk CMO: Its initial value is 4, initial coupon is 0.342, value at the early default date is 1.44, coupon after early default is 0.178, and total recovery is 0; see Table 3. Since the total recovery on the pool is zero, the resulting senior tranche cannot be risk-free in equilibrium, except in the trivial case in which the proportional value of this tranche is zero. Since the early recovery on the pool is zero, no senior bonds are bought back at the time of early default. So the senior tranche is low-risk in equilibrium as long as its coupon at origination is less than 0.178, the coupon on the pool after the early default event. The threshold at which the equilibrium switches from low-risk to high-risk is found by setting the coupon on the senior tranche equal to 0.178 in (16) and (17). The resulting value of the threshold is 0.565. So the senior tranche of the CMO-squared is low-risk in equilibrium when its proportional value is strictly greater than zero and less than 56.50%, and high-risk when its proportional value is strictly greater than 56.50%.

In particular, when the proportional value is 0.80, the resulting senior tranche is high-risk in equilibrium. The coupon on this tranche is 0.267 at origination and 0.178 after the early default event. The recovery rate on the senior CMO-squared tranche is zero, whereas the recovery rate of the senior CMO tranche is 81.74%. The implied initial yield is 8.34% and the yield at the early default date is 12.34%. The yield spread at origination on the senior tranche of the CMO-squared is 1.34%, while the yield spread at origination on the senior tranche of the CMO is 0.24%. The yield spreads show that the senior tranche of the CMO-squared is riskier than the senior tranche of the CMO. The resulting residual tranche is also riskier than the residual tranche on the CMO; the yield spread on the residual CMO-squared tranche is 2.42%, whereas the yield spread on its CMO counterpart is 1.55%.

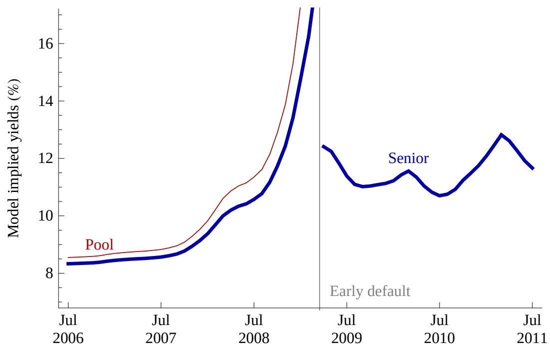
The analysis in this section highlights that exposure to default risk maybe very different for the tranches of a CMO and the tranches of a CMO-squared, even though the relative size of the tranches are identical. In the numerical examples presented, the senior tranche of the CMO-squared created from a low-risk CMO was risk-free or high-risk in equilibrium, depending on whether the senior or residual tranche of the CMO was used to create the CMO-squared. It is worth emphasizing that the differences in the riskiness of the CMO-squared were not due to differences in the characteristics of the underlying mortgages. Instead, the differences were solely due to securitization and the structure of cash flows at various levels of tranching.

We also calculated the yield, bond prices, and monthly returns on the CMO-squared created from the residual tranche of the benchmark CMO; we used the composite-20 index for this exercise. Figure 10 presents our findings. Since the resulting senior tranche is high-risk, its yield drops discontinuously at the early default date. The yield on the pool and the senior tranche are identical after early-default mortgages are terminated, so the solid line and the bold solid line overlap in Figure 10a. Figure 10b shows that the prices of all bonds decline monotonically. By the early default date, senior bonds are only worth half of their origination value and residual bonds are worthless.

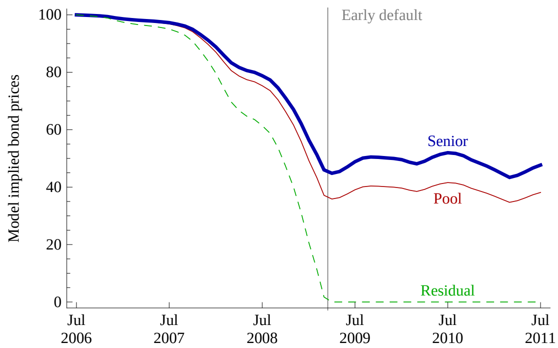
In practice, most CMO-squared are created from tranches of a CMO that have not been rated AAA; these tranches together correspond roughly to the residual tranche of the model CMO. For CMO-squared created from the residual tranche, the model implies that the senior tranches of CMO-squared are riskier than the senior tranches of CMOs. This prediction of the model seems to be consistent with average losses observed in the data. Larry Cordell, Yilin Huang and Meredith Williams (2012) report that the average principal write-down on publicly traded CMO-squared issued in 2006 and 2007 was above 93% for all tranches, except the Senior AAA tranche which suffered an average write-down of 67% in 2006 and 76% in 2007.[13] (For comparison, note that the model implied senior CMO-squared bond values dropped by 50% and residual bond values declined 100%.) In contrast, Sun Young Park (2013*a*) reports that the average write-down on AAA-rated tranches, for subprime CMOs issued during 2004-2007, was only 0.17%;

[13] A Senior AAA tranche or super senior tranche usually refers to tranches that have subordinate tranches which are AAA rated. By construction, the Senior AAA tranches had the lowest exposure to default risk.

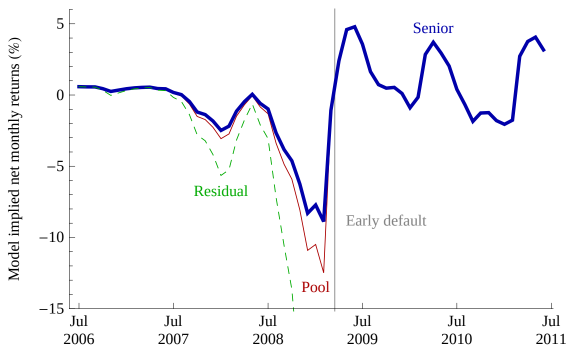
the average write-down on the lowest-rated BBB-rated tranches was 56.97%. Similarly, Christopher L. Foote, Kristopher S. Gerardi and Paul S. Willen (2008) report that only 10% of AAA-rated CMOs issued in 2006-2007 suffered losses, whereas 90% of CMO-squared issued during the same time period suffered losses.



(a) Model implied bond yields



(b) Model implied bond prices



(c) Model implied net monthly returns

Figure 10: Composite-20 index

Even though the analysis in this section involves considerable simplifications, it seems to capture how default risk of the underlying mortgages affects valuation of CMO-squared. Data support the prediction of the model that losses on senior tranches of CMO-squared created from residual tranches of CMOs maybe quite large, even though the losses on senior tranches of the same CMOs are small. The analysis so far has been limited to valuation of CMO-squared. However, it can easily be extended to incorporate valuation of higher-order CMOs. For example, the analysis implies that a CMO-cubed created from the residual tranche of the high-risk CMO-squared is such that the entire pool, and so the tranches, becomes worthless as soon as early-default mortgages are terminated.

6 Credit Default Swaps

A Credit Default Swap (CDS) is insurance against default. The CDS buyer pays insurance premiums to the CDS seller in exchange for payments contingent on some pre-specified credit events. CDS were a major asset class before the financial crisis. According to the annual market survey of the International Swaps and Derivatives Association (ISDA), the total amount of CDS outstanding in 2007 was $62.2 trillion. Beginning in 2005, CDS allowed market participants to take short positions on subprime MBS for the first time.[14] The launch of the ABX.HE index CDS aggregated and revealed the views of market participants on subprime MBS for the first time. Gary Gorton (2010) argues that this information regarding the subprime market, along with inadequate information regarding the location of subprime risk, began the financial crisis of 2007. This section extends the analysis to the valuation of CDS on mortgage bonds. It also shows the model implied CDS prices for the Case-Shiller house price index.

In practice, the ABX.HE index is traded as follows. The buyer pays a one time upfront fee and a fixed index-specific monthly premium to the seller in exchange for payments contingent on default. CDS prices are quoted as a percentage of par value. They equal the par value, normalized to 100 at origination, minus the upfront payment. For example, a price of 60 means that the upfront fee is 40. Since the insurance premium is fixed, it is the price that changes in response to market conditions so as to reflect the price of insurance against default. The CDS contract in the model looks similar. Consider a CDS written on senior bonds. These bonds are scheduled to pay $c_s(0)$ perpetually. The realized payments, however, depend on the realization of housing services. The seller of the CDS insures the buyer against any shortfall in scheduled payments. In return, the buyer pays the seller a one time upfront fee $I_s(t, x(t))$, and an insurance premium i_s; the premium is paid until the late default date.

The following thought experiment shows how to value a CDS contract on the senior bond. Suppose that the buyer of the CDS holds a senior bond, which he turns over to the seller at the time of the purchase along with the upfront fee; the buyer also pays the insurance premium until the late default date. In return, the seller pays the buyer $c_s(0)$ until the late default date. The CDS contract is terminated at this date with the seller giving the buyer an insurance payout of $c_s(0)/r$. The profits of a CDS seller from insuring one senior bond at some $t < \tau_l$ are

$$I_s(t, x(t)) + V_s(t, x(t)) + \mathbb{E}_t \left[\int_t^{\tau_l} e^{-r(\tau_l - z)} i_s dz \right] - \frac{c_s(0)}{r}, \tag{21}$$

where $I_s(t, x(t))$ is the upfront fee, $V_s(t, x(t))$ is the market value of the senior bond, i_s is the insurance premium, and $c_s(0)/r$ is the present value of the insurance payout. We assume that the insurance is

[14] The ISDA standardized its documentation, and successfully launched single-named asset-backed CDS contracts in 2005; see Ingo Fender and Martin Scheicher (2009).

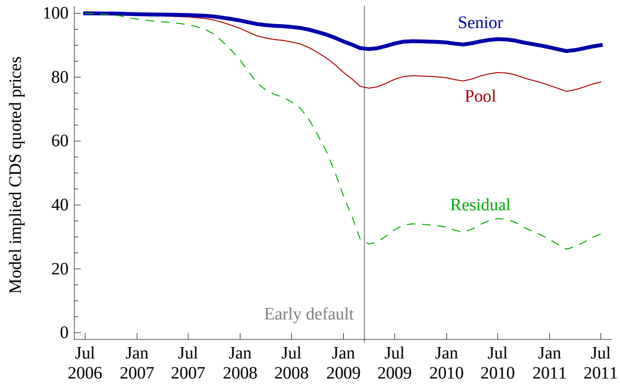


Figure 11: Composite-20 index

actuarially fair, so CDS sellers make zero profits. The insurance premium is such that the upfront fee is zero at origination. Once determined, the premium is fixed over lifetime of the CDS. So, as default probabilities change, the upfront fee fluctuates so as to keep the insurance fairly priced.

Even though the discussion so far has been restricted to the senior bonds, it carries over to CDS written on mortgage pass-throughs and residual bonds. As a numerical example, consider CDS written on each bond of the benchmark low-risk CMO separately. The insurance premium for the mortgage pass-through is 0.113, senior bonds is 0.043, and residual bonds is 0.070. As a percentage of the insured amount, the premium on pass-throughs is 0.56%, senior bonds is 0.26%, and residual bonds is 1.74%. Figure 11 shows the CDS prices on all three securities when the composite-20 index is fed through the model. The implied prices of all three CDS decrease as house prices decrease and default in the near future becomes more likely.

In practice, prior to the decline in house prices, senior bonds usually carried a AAA rating at origination. So model implied CDS prices for senior bonds correspond approximately to the ABX.HE-AAA index. The correspondence is not exact because the AAA tranches referenced by the corresponding ABX indices were not the senior most tranches in their CMOs. Figure 11 suggests that prices of CDS on senior bonds do not fall significantly below the par value; the lowest model implied price for this CDS is 88.18. In the data, however, the ABX.HE-AAA indexes were trading significantly below par; see Figure 1 in R. Stanton and N. Wallace (2011). For example, prices of both the 2007 vintages declined steadily and bottomed-out around 20, before recovering steadily to around 40 by July 2010. The quantitative exercise suggests that replicating the steep decline in the ABX.HE-AAA indices for reasonable parameter values might be difficult. This finding is in line with recent research on the ABX.AAA-HE index. For example, R. Stanton and N. Wallace (2011) conclude that no reasonable expectation regarding defaults and recovery rates on mortgages underlying the ABX.AAA-HE index can account for the observed decline in prices.[15]

CDS written on residual bonds in the model correspond approximately to ABX.HE index on bonds that were rated AA, A, BBB, BBB-. The index on these bonds experienced price declines that were larger than those experienced by the index on AAA-rated bonds. In fact, the ABX.HE index for some of the lowest-rated bonds experienced a 100% principal writedown, and was trading on an interest-only basis. Figure 11 suggests that mortgage default may account for a large fraction of the price decline in the ABX.HE indexes that were rated below AAA; the lowest implied price for CDS written on residual bonds is 26.14. This finding is consistent with the empirical work of Ingo Fender and Martin Scheicher (2009). Using regression analysis, these authors found that indicators of housing market activity were important for subordinate ABX.HE indexes, but not for AAA and AA-rated indexes.

7 Conclusions

This paper presented a structural model for pricing MBS in the presence of mortgage default risk. We modeled the default behavior of a borrower explicitly along with the essential contractual features of MBS. We began the analysis by valuing Collateralized Mortgage Obligations. For CMOs made from pools containing two types of mortgages, we found that senior bondholders may experience no principal or coupon shortfalls, principal shortfalls only, or both principal and coupon shortfalls; the type of equilibrium depends on the relative size of the senior tranche. The initial yield on senior bonds increased as the relative size of the senior tranche increased. In the quantitative exercise we found that senior bonds lose about 10% of their value and residual bonds lose about 60% of their value when housing services implied by the composite-20 Case-Shiller index, from July 2006 to July 2011, are fed through the model.

We extended the analysis to study CMO-squared. We found that a senior CMO-squared tranche, of the same relative size, is riskier than the senior tranche of the CMO when the CMO-squared is created from the residual tranche of the CMO. According to the quantitative exercise, senior CMO-squared bonds halved in value and residual bonds became worthless between July 2006 and July 2011. We also extended the benchmark model to price Credit Default Swaps on mortgage bonds. The model implied prices for CDS on residual bonds suggested that default risk was a major driver of the price declines for the ABX.HE indexes rated below AAA.

We understand that the quantitative predictions are not the outcome of a calibrated version of the

[15] R. Stanton and N. Wallace (2011) collected detailed data on the individual loans underlying the ABX.HE index, and calculated the default rates implied by the observed prices. They found that a prepayment rate of 25% and a recovery rate of 34% implied default rates of 100% at the observed prices for the ABX.HE-AAA index; the assumed prepayment rate is roughly consistent with historical prepayment rates on the underlying pools, and the recovery rate is below anything observed in U.S. mortgage markets. So an expected recovery rate greater than 34% implies that the observed prices are inconsistent with reasonable assumptions regarding default behavior.

model. Calibrating the model directly is challenging because of the presence of unobservable default cost parameters. We think that the estimation of these parameters from realized recovery rates along with a calibration of the model is a useful direction for future research. The calibrated version of the model can be used to analyze whether mortgage bonds were "mispriced" prior to, or during, the financial crisis. It would also serve as a benchmark with which to compare the rating agencys' assessments of MBS. Narratives of the crisis argue, with a considerable element of hindsight of course, that mispricing and inflated rating both exacerbated, if not caused, the financial crisis of 2007. The calibrated model can improve our understanding of the role, or the lack thereof, of these distortions in the financial crisis.

Why lenders choose a particular capital structure for MBS is another important area for future research. The Miller-Modigliani theorem applies to the environment laid out here, so lenders are indifferent between all capital structures for MBS. In practice, however, lenders were particular about the capital structure of the MBS; see Sun Young Park (2010) for some evidence that CMO pools with riskier collateral had higher subordination levels. The analysis also abstracted from informational frictions in the MBS market. Adam B. Ashcraft and Til Schuermann (2008) discuss seven key sources of informational frictions in the market for subprime MBS. We think that analyzing how these frictions affect MBS contractual features, yields, and equilibrium prices is also an important area for future work.

8 Omitted proofs

8.1 Risk-free equilibrium: Guess and verify

For $\theta \in [0, \theta_2]$, the senior tranche's early recovery equals late recovery is $R_{sl} = V_s(0,1) - R_{se}$. Calculate the senior tranche's value at the early default event, using (17), as implied by the guess for $c_s(\tau_e)$; denote the implied value by $V_s'(\tau_e, \delta_e)$.

$$
\begin{aligned}
V_s'(\tau_e, \delta_e) &= \mathbb{E}_{\tau_e}\left[\int_{\tau_e}^{\tau_l} e^{-r(t-\tau_e)} c_s(\tau_e) dt\right] + \mathbb{E}_{\tau_e}\left[e^{-r(\tau_l-\tau_e)} R_{sl}\right] \\
V_s'(\tau_e, \delta_e) &= (V_s(0,1) - R_{se})\left(1 - \left(\frac{\delta_l}{\delta_e}\right)^m\right) + (V_s(0,1) - R_{se})\left(\frac{\delta_l}{\delta_e}\right)^m \\
&= V_s(0,1) - R_{se}.
\end{aligned}
$$

Calculate the senior tranche's value at origination, using the right hand side of (16), as implied by the guess for $c_s(0)$ and $V_s'(\tau_e, \delta_e)$ calculated above. If the guessed coupon schedule is an equilibrium the implied origination value should equal $V_s(0,1)$, the actual value of the senior tranche at origination.

$$
\begin{aligned}
& \mathbb{E}_0\left[\int_0^{\tau_e} e^{-rt} c_s(0) dt\right] + \mathbb{E}_0\left[e^{-r\tau_e} R_{se}\right] + \mathbb{E}_0\left[e^{-r\tau_e} V_s'(\tau_e, \delta_e)\right] \\
= \; & V_s(0,1)(1 - \delta_e^m) + R_{se}\delta_e^m + (V_s(0,1) - R_{se})\delta_e^m \\
= \; & V_s(0,1).
\end{aligned}
$$

Thus the guessed coupon schedule is an equilibrium for $\theta \in [0, \theta_2]$. The uniqueness of this equilibrium was established in the body of the paper.

8.2 The threshold θ_3

In this subsection, we derive the expression for θ_3 along with the conditions necessary for it to lie in the interval $(\theta_2, 1)$.

Implicit differentiation of (16) and (17) shows that the senior coupon at origination $c_s(0)$ in increasing in θ. So there is a threshold θ_3 such that the coupon on the pool is inadequate for all $\theta > \theta_3$. The threshold is obtained by solving $q_s c_s(0) = c_p(\tau_e)$ for θ, with $c_s(0)$ given by (20). The calculation is outlined below.

$$
\begin{aligned}
q_s c_s(0) &= c_p(\tau_e) \\
V_s(0,1) &= R_{pe}\delta_e^m + R_{pl}\delta_l^m + \frac{c_p(\tau_e)}{r}(\delta_e^m - \delta_l^m) + \frac{c_p(\tau_e)(1 - \delta_e^m)}{rq_s}
\end{aligned}
$$

Add and subtract $(1 - \delta_e^m)c_p(0)/r$ to the right hand side and use the equilibrium valuation formula for the pool and the definition of q_s to obtain,

$$V_s(0,1) = V_p(0,1) - \frac{1-\delta_e^m}{r}\left(c_p(0) - c_p(\tau_e)\left(1 + \frac{R_{pe}}{V_p(\tau_e,\delta_e)}\right)\right)$$

Write $V_s(0,1) = \theta_3 V_p(0,1)$, divide both sides by $V_p(0,1)$, and rewrite all pool variables in terms of the underlying mortgage variables; recall that $V_p(0,1) = M_e(1)$ under the normalization $M_e(1) = M_l(1)$. After some algebra, we obtain

$$\theta_3 = 1 - \frac{(1-\delta_e^m)\eta c_e/r}{M_e(1)}\left(1 - \frac{c_l/M_l(\delta_e)}{c_e/M_e(\delta_e)}\right) \tag{22}$$

The threshold θ_3 is less than one when the condition $c_l/M_l(\delta_e) < c_e/M_e(\delta_e)$ holds. By (9) the leading fraction in the second term of (22) is less than one. So the threshold θ_3 is strictly positive when the required condition holds.

Next we verify that $\theta_3 > \theta_2$. By the definition of the thresholds θ_2 and θ_3 the inequality $\theta_3 > \theta_2$ can be written as follows; recall that $V_p(0,1) = M_e(1)$.

$$\begin{aligned}
1 - \frac{(1-\delta_e^m)\eta c_e/r}{M_e(1)}\left(1 - \frac{c_l/M_l(\delta_e)}{c_e/M_e(\delta_e)}\right) &> R_p/V_p(0,1) \\
V_p(0,1) - (1-\delta_e^m)\eta c_e/r\left(1 - \frac{c_l/M_l(\delta_e)}{c_e/M_e(\delta_e)}\right) &> R_p \\
\eta(M_e(1) - M_e(\delta_e)) + (1-\eta)(M_l(1) - M_l(\delta_l)) - (1-\delta_e^m)\eta c_e/r\left(1 - \frac{c_l/M_l(\delta_e)}{c_e/M_e(\delta_e)}\right) &> 0
\end{aligned}$$

Since $(1-\eta)(M_l(1) - M_l(\delta_l)) > 0$, we only need to show that

$$\begin{aligned}
M_e(1) - M_e(\delta_e) - (1-\delta_e^m)\frac{c_e}{r}\left(1 - \frac{c_l/M_l(\delta_e)}{c_e/M_e(\delta_e)}\right) &> 0 \\
\left(M_e(1) - (1-\delta_e^m)\frac{c_e}{r}\right) - M_e(\delta_e) + (1-\delta_e^m)\frac{c_e}{r}\frac{c_l/M_l(\delta_e)}{c_e/M_e(\delta_e)} &> 0 \\
M_e(\delta_e)(\delta_e^m - 1) + (1-\delta_e^m)\frac{c_l/M_l(\delta_e)}{r/M_e(\delta_e)} &> 0 \\
(1-\delta_e^m)\frac{c_l/M_l(\delta_e)}{r/M_e(\delta_e)} &> M_e(\delta_e)(1-\delta_e^m) \\
\frac{c_l}{r} &> M_l(\delta_e)
\end{aligned}$$

The last inequality holds by (9). Therefore $\theta_3 > \theta_2$.

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