# Generic stabilizability for time-delayed feedback control

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Time delayed feedback control is one of the most successful methods to discover dynamically unstable features of a dynamical system in an experiment. This approach feeds back only terms that depend on the difference between the current output and the output from a fixed time T ago. Thus, any periodic orbit of period T in the feedback controlled system is also a periodic orbit of the uncontrolled system, independent of any modelling assumptions.

It has been an open problem whether this approach can be successful in general, that is, under genericity conditions similar to those in linear control theory (controllability), or if there are fundamental restrictions to time-delayed feedback control. We show that there are no restrictions in principle. This paper proves the following: for every periodic orbit satisfying a genericity condition slightly stronger than classical linear controllability, one can find control gains that stabilise this orbit with extended time-delayed feedback control.

While the paper's techniques are based on linear stability analysis, they exploit the specific properties of linearisations near autonomous periodic orbits in nonlinear systems, and are, thus, mostly relevant for the analysis of nonlinear experiments.

## 1. Introduction

Time-delayed feedback control was originally proposed by Pyragas in 1992 as a tool for discovery of unstable periodic orbits (one frequent building block in nonlinear systems with chaotic dynamics or multiple attractors) in experimental nonlinear dynamical systems [1]. Pyragas proposed that one take the output $x(t) \in \mathbb{R}^n$ of a dynamical system and feed back in real time the difference between this output and the output time $T$ ago into an input $u(t) \in \mathbb{R}^{n_u}$ of the system (multiplied by some *control gains* $K^T \in \mathbb{R}^{n_u \times n}$):

$$u(t) = K^T[x(t-T) - x(t)]. \tag{1}$$

In a first experimental demonstration, Pyragas and Tamaševičius successfully identified and stabilised an unstable periodic orbit in a chaotic electrical circuit [2]. Socolar *et al* in 1994 [3]

introduced a generalisation of time-delayed feedback (which is often used in place of (1) and is implemented as shown in Figure 1 as a block diagram):

$$u(t) = K^T[\tilde{x}(t) - x(t)], \quad \text{where}$$
$$\tilde{x}(t) = (1 - \varepsilon)\tilde{x}(t - T) + \varepsilon x(t - T), \tag{2}$$

and $\varepsilon \in (0, 1]$, called *extended time-delayed feedback*. If $\varepsilon = 1$, feedback law (2) reduces to time-delayed feedback (1), if $\varepsilon = 0$ feedback law (2) degenerates to classical linear feedback with a fixed T-periodic reference signal $\tilde{x}(t)$ (see below (3) for a discussion). Note that, for example in [3], the variable $\tilde{x}(t)$ was eliminated in the mathematical discussion by writing

$$u(t) = K^T \left( \varepsilon \left[ \sum_{j=1}^{\infty} (1 - \varepsilon)^{j-1} x(t - jT) \right] - x(t) \right).$$

While this would suggest that knowledge of all history of $x$ is required to initialise the system, in the experiment the feedback control was implemented as shown in the block diagram in Figure 1, which is equivalent to (2). By construction of the feedback laws (1) and (2), for
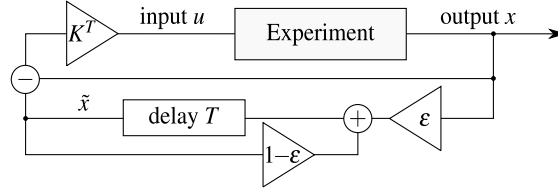


**Figure 1:** Block diagram for extended time-delayed feedback (2), as applied to an experiment, for example, in [3]. We prove generic stabilisablity for the case when the output $x$ is the whole internal state of the dynamical system and the input $u$ is scalar. Triangle block symbols are multiplications of the signal by the factor in the block.

$\varepsilon > 0$ every periodic orbit of period T of the dynamical system with feedback control is also a periodic orbit of the uncontrolled system ($u = 0$).[1] However, the stability of the periodic orbit may change from unstable without control to asymptotically stable with control for appropriately chosen gains K.

The delayed terms $x(t - T)$ and $\tilde{x}(t - T)$ make extended time-delayed feedback control different from the classical linear feedback control, which has the form

$$u(t) = K^T[x_*(t) - x(t)], \tag{3}$$

where $x_*(t)$ is, for example, a known unstable periodic orbit of the dynamical system governing $x$. While the goal of (3) is to stabilise a known *reference output* (in this case a periodic orbit), time-delayed feedback is able to stabilise and, thus, *find* a-priori unknown periodic orbits. For this reason time-delayed feedback originated, and has found most attention, in the physics and science community, rather than in the control engineering community. It can be used to discover features of nonlinear dynamical systems inaccessible in conventional experiments, such as unstable equilibria, periodic orbits and their bifurcations, non-invasively. A few examples where time-delayed feedback (or its extended version) have been successfully used are: control

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[1]For $\varepsilon \neq 0$, (2) with T-periodic $\tilde{x}$ implies that $\tilde{x} = x$ for all t.

of chemical turbulence [4], all-optical control of unstable steady states and self-pulsations in semiconductor lasers [5, 6, 7], control of neural synchrony [8, 9, 10], control of the Taylor-Couette flow [11], atomic force microscopy [12] and (with further modifications) systematic bifurcation analysis in mechanical experiments in mechanical engineering [13, 14, 15].

One difficulty for time-delayed feedback is that there are until now no general statements guaranteeing the existence of stabilising control gains K under some genericity condition on the dynamical system governing $x$ and its input $u$, such as controllability. This is in contrast to the situation for classical linear feedback control (3), where the following is known [16]: if the periodic orbit $x_*$ is linearly controllable by input $u$ in $p$ periods (this is a genericity condition) then one can assign its period-$pT$ monodromy matrix to any matrix with positive determinant by $pT$-periodic feedback gains $K(t)^T \in \mathbb{R}^{n_u \times n}$.

The greater level of difficulty for (extended) time-delayed feedback is unsurprising since the feedback-controlled system acquires memory. Let us assume that the measured quantity $x$ is governed by an ordinary differential equation (ODE) $\dot{x}(t) = f(x(t), u(t))$ (which is autonomous without control ($u = 0$) and non-autonomous with classical feedback control (3)). Then $x$ and $\tilde{x}$ will be governed by a delay differential equation (DDE) if $u$ is given by time-delayed feedback (1), or by an ODE coupled to a difference equation if $u$ is given by extended time-delayed feedback (2) with $\varepsilon \in (0, 1)$ (we will refer to both cases simply as DDEs). This means that the initial value for both, $x$ and $\tilde{x}$, is a history segment, a function on $[-T, 0]$ with values in $\mathbb{R}^N$. In DDEs periodic orbits have infinitely many Floquet multipliers.[2] Section 2 will review the development of analysis for the time-delayed feedback laws (1) and (2). This paper proves a first simple generic stabilisability result for extended time-delayed feedback control (2) with time-periodic gains $K(t)$ (similar to results for classical linear feedback control).

**Main result** The following theorem states that the classical approach to periodic feedback gain design by Brunovsky [17] can be applied to make (2) stable in the limit of small $\varepsilon > 0$ in the simplest and most common case of a scalar input $u$ (thus, $n_u = 1$) and linear controllability of the periodic orbit by an input at a single time instant.

**Theorem 1.1 (Generic stabilisability with extended time-delayed feedback)**

Assume that the dynamical system

$$\dot{x}(t) = f(x(t), u(t)) \quad (f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n \text{ smooth}) \tag{4}$$

with $u = 0$ has a periodic orbit $x_*(t)$ of period $T > 0$, and assume that the monodromy matrix[3] $P_0$ of $x_*$ from time 0 to $T$ is controllable with $b_0 = \partial_u f(x_*(0), 0)$ (that is, $\det[b_0, P_0 b_0, \ldots, P_0^{n-1} b_0] \neq 0$).

Then there exist gains $K_0 \in \mathbb{R}^n$ such that $x_*$ as a periodic orbit of the feedback controlled system (4) with (see below for the definition of the function $\Delta_\delta$)

$$u(t) = \Delta_\delta(t) K_0^T [\tilde{x}(t) - x(t)], \qquad \tilde{x}(t) = (1 - \varepsilon)\tilde{x}(t - T) + \varepsilon x(t - T) \tag{5}$$

has one simple Floquet multiplier at 1 and all other Floquet multipliers inside the unit circle for all sufficiently small $\varepsilon$ and $\delta$.

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[2]Floquet multipliers are the eigenvalues of the linearisation of the time-$T$ map along the periodic orbit.

The function $\Delta_\delta$ is zero except for a short interval of length $\delta$ every period $T$ such that the feedback $u$ has the form of a short but large near-impulse:[4]

$$\Delta_\delta(t) = \begin{cases} 1/\delta & \text{if } t|_{\text{mod}[0,T)} \in [0,\delta] \\ 0 & \text{if } t|_{\text{mod}[0,T)} \notin [0,\delta]. \end{cases} \tag{6}$$
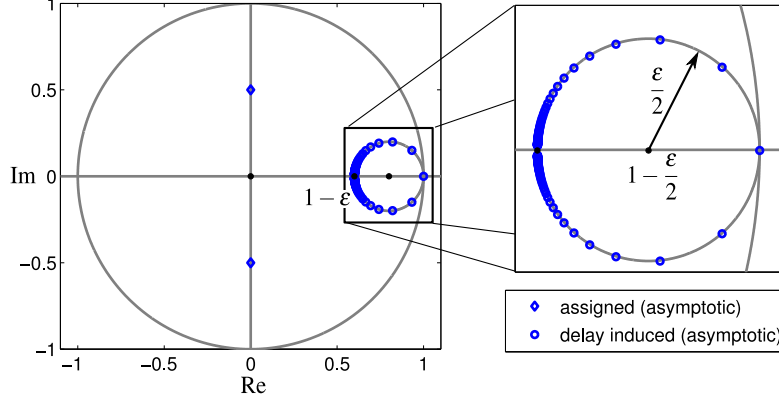


**Figure 2:** Illustration of Floquet multiplier spectrum for extended time-delayed feedback with single input. Using the appropriate control gains $K_0$, $n$ Floquet multipliers can be freely assigned up to determinant restrictions ($n = 2$ in the illustrated case). The other Floquet multipliers lie on a circle of radius $\varepsilon/2$ around $1 - \varepsilon/2$, accumulating at $1 - \varepsilon$. This spectrum is achieved asymptotically for sufficiently short and strong impulses ($\delta \ll 1$) and small $\varepsilon$. A simple trivial multiplier at 1 is always present.

**Remarks — Constant gains** The gains as constructed are periodic. This is to be expected since there are no general results for constant gains $K \in \mathbb{R}^n$ for the classical linear feedback case (3), either. Furthermore, simple examples show that the above statement can definitely not be made when we restrict ourselves to constant gains in (5): $u(t) = K^T[\tilde{x}(t) - x(t)]$ with $K \in \mathbb{R}^n$. See Section 5 for an example).

**Properties of the spectrum of the linearisation** (see also Figure 2 for an illustration) The claim of Theorem 1.1 is about linear stability of the periodic orbit $(x(t), \tilde{x}(t)) = (x_*(t), x_*(t))$ of (4)–(5). Thus, we have to consider the problem (4)–(5), linearised in $(x(t), \tilde{x}(t)) = (x_*(t), x_*(t))$:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + b(t)\Delta_\delta(t)K_0^T[\tilde{x}(t) - x(t)], \\ \tilde{x}(t) &= (1-\varepsilon)\tilde{x}(t-T) + \varepsilon x(t-T), \end{aligned} \tag{7}$$

where $A(t) = \partial_x f(x_*(t), 0)$ and $b(t) = \partial_u f(x_*(t), 0)$.

The gains $K_0$ are identical to those chosen by Brunovsky [17] for the classical feedback spectrum assignment problem (note that Brunovsky made weaker assumptions on $A(t)$ and $b(t)$ than Theorem 1.1). One can choose the gains $K_0$ to place the $n$ Floquet multipliers $\lambda_k$ ($k = 1, \ldots, n$) of

$$\dot{x} = A(t)x - b(t)\Delta_\delta(t)K_0^T x,$$

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[4]The notation $t|_{\text{mod}[0,T)}$ refers to the number $\tau \in [0,T)$ such that $(t - \tau)/T$ is an integer.

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anywhere inside the unit circle subject to the restriction that they have to be eigenvalues of a real matrix with positive determinant.

However, DDEs such as (7) may have infinitely many Floquet multipliers. Theorem 1.1 rests on a perturbation argument for small $\varepsilon > 0$ for the other, delay-induced, Floquet multipliers.[5] At $\varepsilon = 0$ the difference equation for $\tilde{x}$ in (7) simplifies to $\tilde{x}(t) = \tilde{x}(t-T)$. Thus, an arbitrary initial history $\tilde{x}$ with period $T$ will not change under the time-$T$ map of (7). This results for $\varepsilon = 0$ in a spectrum of (7) consisting of

- the finitely many assigned Floquet multipliers $\lambda_k$ ($k = 1, \ldots, n$) as determined by the gains $K_0$, and (assuming all $\lambda_k \neq 1$)

- the spectral point $\lambda_\infty = 1$ with an infinite-dimensional eigenspace. Specifically, if we choose the space of continuous functions $C([-T, 0]; \mathbb{R}^n \times \mathbb{R}^n)$ as phase space for (7) then, for $\varepsilon = 0$, the eigenspace for $\lambda_\infty = 1$ is

$$\Big\{ (x, \tilde{x}) \in C([-T, 0]; \mathbb{R}^n \times \mathbb{R}^n) : \quad x(0) = x(-T), \quad \tilde{x}(0) = \tilde{x}(-T),$$
$$\dot{x} = A(t)x + b(t)\Delta_\delta(t)K_0^\mathsf{T} [\tilde{x}(t) - x(t)] \Big\}.$$

Note that, since $\lambda_k \neq 1$ for $k = 1, \ldots, n$, the ODE $\dot{x} = A(t)x + b(t)\Delta_\delta(t)K_0^\mathsf{T} [\tilde{x}(t) - x(t)]$ has a unique periodic solution $x$ for all periodic functions $\tilde{x}$. This means that for every $T$-periodic $\tilde{x}$ there is an eigenvector for $\lambda_\infty = 1$ with this $\tilde{x}$-component.

The general theory for DDEs [18] ensures that for positive (small) $\varepsilon$ the Floquet multipliers $\lambda_k$ ($k = 1, \ldots, n$) are only slightly perturbed, and that the infinitely many Floquet multipliers emerging from $\lambda_\infty$ accumulate to the spectrum of the *essential part*, the difference equation in (7) with the $\tilde{x}$ terms only: $\tilde{x}(t) = (1 - \varepsilon)\tilde{x}(t - T)$. Specifically, the only accumulation point of the spectrum of (7) for $\varepsilon \in (0, 1)$ is at $1 - \varepsilon$ and the stability of (7) is determined by the location of the Floquet mulitpliers emerging from the perturbation of $\lambda_\infty$ (of which at most finitely many can lie outside the unit circle). The detailed analysis in Section 3 will show that for small $\varepsilon > 0$ the Floquet multipliers emerging from $\lambda_\infty$ lie close to a circle of radius $\varepsilon/2$ around $1 - \varepsilon/2$, inside the unit circle (except for the unit Floquet multiplier), as shown in Figure 2 for $n = 2$.

**Trivial multiplier**   The eigenvector to the trivial multiplier 1 is $\dot{x}_*(0)$, corresponding to the linearised phase shift (for every $s \in \mathbb{R}$, $t \mapsto x_*(t + s)$ is also a solution of the system with extended time-delayed feedback (4), (5)). Section 4 gives a modification of Theorem 1.1 with a function $\Delta_\delta$ depending on $x(t)$ instead of $t$ switching the gains on and off. Then the feedback controlled system becomes autonomous. In this modified system with autonomous (but nonlinear) extended time-delayed feedback the periodic orbit $x_*$ is asymptotically stable in the classical sense.

**Timing of impulse**   In (6) we chose the timing of the impulse (the part of the period $[0, T)$ where $\Delta_\delta$ is non-zero) as $[0, \delta]$ without loss of generality. The genericity condition in its most general form requires that there must be a time $t \in [0, T]$ such that the monodromy matrix from $t$ to $t + T$ and $\partial_u f(x_*(t), 0)$ are controllable. As the uncontrolled system is autonomous, we can shift the phase of the periodic orbit $x_*$, considering $x_*(t + \cdot)$ instead of $x_*$.

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[5]The perturbation is *not* a small-delay perturbation since the delay $T$ and one coefficient in front of the delay, $1 - \varepsilon$, are not small.

**Practical considerations**  The result gives precise control over the Floquet multipliers in the limit of small δ and ε. For small δ the feedback control corresponds to a sharp kick once per period, which is not practical for strongly unstable periodic orbits. However, the gains found with the help of Theorem 1.1 provide a feasible starting point for optimisation-based spectrum assignment methods (*continuous pole placement*) as constructed by Michiels *et al* [19, 20] and adapted to time-delayed feedback (1) [21, 22, 23]. In the context of continuation one can combine the gains provided by Theorem 1.1 as starting points, continuous pole placement, and the automatic adjustment of the time delay T demonstrated in [24, 25] to create a feedback control that non-invasively tracks a family of periodic orbits in a system parameter.

## 2. Review: analysis of (extended) time-delayed feedback

The initial proposals of time-delayed feedback (1) and its extended version (2) were accompanied with demonstrations in simulations and experiments, showing that this type of feedback control can be successful [1, 2, 3], but not with general necessary or sufficient conditions for applicability or with constructive ways to design the feedback gains.

However, it was quickly recognised that time-delayed feedback can be applied to periodic orbits that are weakly unstable due to a period doubling bifurcation or torus bifurcation [26, 27]. Hence, time-delayed feedback is often associated with control of chaos, because it can be used to suppress period doubling cascades. However, general sufficient criteria were rather restrictive [28], requiring full access to the state (x governed by $\dot{x}(t) = f(x(t)) + u(t)$ with $u \in \mathbb{R}^n$). A first general result was negative, the so-called *odd number limitation* for periodically forced systems [29], showing that extended time-delayed feedback cannot stabilise periodic orbits in periodically forced systems with an odd number of Floquet multipliers λ with Re λ > 1 (and no Floquet multiplier at 1). This theoretical limitation is not a severe restriction in practice since one can extend the uncontrolled system with an artificial unstable degree of freedom before applying time-delayed feedback [30]. Fiedler *et al* showed that this limitation does not apply to autonomous periodic orbits [31, 32]. Since then general results have been proven for weakly unstable periodic orbits with a Floquet multiplier close to 1 (but larger than 1, [33]), or near subcritical Hopf bifurcations [34, 35]. A review of developments up to 2010 is given in [36].

An extension of the odd number limitation to autonomous periodic orbits (with trivial Floquet multiplier) was given by Hooton & Amann [37, 38] for both, time-delayed feedback (1) and its extension (2). However, these limitations merely impose restrictions on the gains K. They do not rule out feedback stabilisability a priori (which is in contrast to the statements about periodic orbits in forced systems).

## 3. Spectrum of linearisation for extended time-delayed feedback-controlled system

Let us consider a feedback controlled dynamical system with extended time-delayed feedback control and arbitrary time-dependent gains $K(t) \in \mathbb{R}^n$:

$$\dot{x}(t) = f(x(t), u(t)), \tag{8}$$

$$u(t) = K(t)^{\mathsf{T}}[\tilde{x}(t) - x(t)], \tag{9}$$

$$\tilde{x}(t) = (1 - \varepsilon)\tilde{x}(t - T) + \varepsilon x(t - T). \tag{10}$$

This system is governed by an ordinary differential equation (ODE) without control ($u = 0$) and a delay differential equation (DDE) with control. We assume that the uncontrolled system $\dot{x}(t) = f(x(t), 0)$ has a periodic orbit $x_*$ of period $T$. This periodic orbit $x_*$ is also a periodic orbit of (8)–(10) if $\varepsilon > 0$: $x(t) = \tilde{x}(t) = x_*(t)$. System (8)–(10) is a DDE with the phase space

$$\{(x, \tilde{x}) \in C([-T, 0]; \mathbb{R}^n \times \mathbb{R}^n) : \tilde{x}(0) = (1 - \varepsilon)\tilde{x}(-T) + \varepsilon x(-T)\}.$$

Hale & Verduyn-Lunel [18] treated DDEs of the type of system (8)–(10) (which contains difference equations) as part of their discussion of neutral DDEs. The essential part of the semiflow generated by (8)–(10) is governed by the part of (10) containing $\tilde{x}$: $\tilde{x}(t) = (1 - \varepsilon)\tilde{x}(t - T)$, which is linear and has spectral radius $1 - \varepsilon$. Thus, it fits into the scope of the theory as described in the textbook by Hale & Verduyn-Lunel [18]. Specifically, the asymptotic stability of the periodic orbit given by $x(t) = \tilde{x}(t) = x_*(t)$ is determined by the point spectrum of the linearisation of (8)–(10). Hence, the periodic orbit $x_*$ is stable if all Floquet multipliers of the linearisation along $x_*$ except the trivial multiplier 1 are inside the unit circle (and the trivial Floquet multiplier 1 is simple). We denote the monodromy matrix[6] of

$$\dot{x} = [A(t) - \mu b(t) K(t)^\mathsf{T}] x(t), \quad \text{where } A(t) = \partial_x f(x_*(t), 0), \quad b(t) = \partial_u f(x_*(t), 0) \tag{11}$$

for $\mu \in \mathbb{C}$ by $P(\mu)$. Thus, the monodromy matrix of the uncontrolled system $\dot{x}(t) = A(t)x(t)$ equals $P(0)$, which we denote by

$$P_0 = P(0). \tag{12}$$

With this definition of $P(\mu)$, Floquet multipliers of the linearisation of (8)–(10) in $x_*$ different from $1 - \varepsilon$ are given as roots of

$$h(\lambda; \varepsilon) := \det\left[\lambda I - P\left(1 - \frac{\varepsilon}{\lambda - (1 - \varepsilon)}\right)\right]$$

(I is the identity matrix; see Section A.1 for detailed proof). The following lemma states that the gains $K(t)$ can only stabilise a periodic orbit $x_*$ with extended time-delayed feedback and small $\varepsilon$, if they are stabilising with classical linear feedback (that is, when replacing the recursively determined signal $\tilde{x}$ by the target orbit $x_*$: $u(t) = K(t)[x_*(t) - x(t)]$). (Recall that $A(t) = \partial_x f(x_*(t), 0)$, $b(t) = \partial_u f(x_*(t), 0)$.)

**Lemma 3.1 (Extended time-delayed feedback stabilisation implies classical stabilisation)**

> If the linear system
>
> $$\dot{x}(t) = [A(t) - b(t)K(t)^\mathsf{T}]x(t) \tag{13}$$
>
> has at least one Floquet multiplier outside the unit circle, then there exists a $\varepsilon_{\max} \in (0, 1)$ such that the periodic orbit $x_*$ is unstable for the extended time-delayed feedback (8)–(10) for all $\varepsilon \in (0, \varepsilon_{\max})$.

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[6]Thus, $P(\mu)$ is defined as the solution $y$ at time $T$ of the linear differential equation $\dot{y}(t) = [A(t) - \mu b(t)K(t)^\mathsf{T}]y(t)$ with initial value $y(0) = I$ (I is the identity matrix).

**Proof** The Floquet multipliers of (13) are given as roots of $h(\lambda; 0) = \det(\lambda I - P(1))$. We denote the root with modulus greater than 1 by $\lambda_0$ such that $h(\lambda_0; 0) = 0$. Consequently, for all $\lambda$ in the ball $B_r(\lambda_0)$, where $r = (|\lambda_0| - 1)/2$, the difference $h(\lambda; \epsilon) - h(\lambda; 0)$ is uniformly bounded and analytic for all $\epsilon \in (0, 1)$ and all $\lambda$ in $B_r(\lambda_0)$. Since $\lambda_0$ must have finite multiplicity as a root of $h(\cdot; 0)$, $h(\lambda; \epsilon)$ must have a root in $B_r(\lambda_0)$ for sufficiently small $\epsilon > 0$ (say, $\epsilon \in (0, \epsilon_{max})$), too. By choice of $r$ this root lies outside of the unit circle. (*This ends the proof of Lemma 3.1.*) □

Lemma 3.1 shows that gains $K(t)$ that stabilise with extended time-delayed feedback with small $\epsilon$ also have to feedback-stabilise in the classical sense. Since $K$ is an arbitrary periodic function there are many ways to construct gains for the classical linear feedback control $u(t) = K(t)[x_*(t) - x(t)]$ for periodic orbits $x_*$ [16]. We choose the approach comprehensively treated by Brunovsky [17], which is particularly amenable to analysis in the extended time-delayed feedback case and for which one can then prove the converse of Lemma 3.1:

> *extended time-delayed feedback is stabilising for the same gains for which Brunovksy's approach is stabilising the classical feedback control.*

**Near-impulse feedback and its parametrised monodromy matrix** We pick state feedback control in the form of a single large but short impulse. That is, we consider a short time $\delta \in (0, T)$ and define the linear feedback control

$$u_\delta(t; y) = \Delta_\delta(t) K_0^\mathsf{T} y, \text{ where } \qquad \Delta_\delta(t) = \begin{cases} 1/\delta & \text{if } t|_{\mathrm{mod}[0, T)} \in [0, \delta] \\ 0 & \text{if } t|_{\mathrm{mod}[0, T)} \notin [0, \delta]. \end{cases} \qquad (14)$$

where $t|_{\mathrm{mod}[0, T)}$ is the number $\tau \in [0, T)$ such that $(t - \tau)/T$ is an integer, and $K_0 \in \mathbb{R}^n$ is a vector of constant control gains. Let us first look at classical feedback $u(t) = \Delta_\delta(t) K_0^\mathsf{T}[x_*(t) - x(t)]$ (where we assume that we know the periodic orbit $x_*$). Using feedback law (14) the feedback controlled system reads

$$\dot{x}(t) = f\left(x(t), \Delta_\delta(t) K_0^\mathsf{T}[x_*(t) - x(t)]\right). \qquad (15)$$

We define the nonlinear time-$T$ map $X(x; \delta, K_0)$ as the solution at time $T$ (the period of the periodic orbit $x_*$) of (15) when starting from $x$ at time 0 (including the dependence on parameters $\delta$ and $K_0$ as additional arguments of $X$). Then, for small deviations $y_0$ from $x_*(0)$, the map $X(\cdot; \delta, K_0)$ has the form $X(x_*(0) + y_0; \delta, K_0) = y(T) + O(\|y_0\|^2)$, where $y$ satisfies the linear differential equation (recall that $A(t) = \partial_x f(x_*(t), 0)$, $b(t) = \partial_u f(x_*(t), 0)$)

$$\dot{y}(t) = \left[A(t) - b(t) \Delta_\delta(t) K_0^\mathsf{T}\right] y(t), \qquad y(0) = y_0, \qquad (16)$$

and the term $O(\|y_0\|^2)$ is uniformly small (including its derivatives) for all $\delta$. Let us introduce a complex parameter $\mu$ into (16), which will become useful later in our consideration of extended time-delayed feedback: define for a general complex $\mu$ with $|\mu| \leqslant C$ (with an arbitrary fixed $C > 0$) the linear ODE

$$\dot{y}(t) = \left[A(t) - \mu b(t) \Delta_\delta(t) K_0^\mathsf{T}\right] y(t), \qquad y(0) = y_0. \qquad (17)$$

Denote the monodromy matrix of (17) from $t = 0$ to $t = T$ by $P(\mu; \delta, K_0)$ to keep track of its dependence on the parameters $\delta \in (0, T)$ and $K_0 \in \mathbb{R}^n$. Thus, $P(\mu; \delta, K_0)$ refers to the

same monodromy matrix as $P(\mu)$, defined by (11), for the special case $K(t) = \Delta_\delta(t)K_0$. Then $P(\mu; \delta, K_0)$ satisfies

$$P(\mu; \delta, K_0) = P_0 \exp(-b(0)K_0^\mathsf{T}\mu) + O(\delta), \tag{18}$$

where the error term $O(\delta)$ is uniform for $|\mu| \leqslant C$ and bounded $\|K_0\|$, including its derivatives with respect to all arguments. Hence, we can extend the definition of $P(\mu; \delta, K_0)$ to $\delta = 0$:

$$P(\mu; 0, K_0) = \lim_{\delta \to 0} P(\mu; \delta, K_0) = P_0 \exp(-b(0)K_0^\mathsf{T}\mu),$$

$$= P_0[I - \sigma(\mu)b(0)K_0^\mathsf{T}] \quad \text{where} \tag{19}$$

$$\sigma(\mu) = \begin{cases} \dfrac{\exp\left(\mu K_0^\mathsf{T} b(0)\right) - 1}{K_0^\mathsf{T} b(0)} & \text{if } K_0^\mathsf{T} b(0) \neq 0 \\ \mu & \text{if } K_0^\mathsf{T} b(0) = 0. \end{cases} \tag{20}$$

The limit is uniform for all $\mu$ with modulus less than $C$. For $\mu = 0$, $P$ is the monodromy matrix $P_0$ of the uncontrolled system, and, thus, independent of $\delta$ and $K_0$.

**Approximate spectrum assignment for finitely many Floquet multipliers** The control (14) is a simplification of the general case of finitely many (at most $n$) short impulses treated in [17]. Feedback of type (14) permits us to assign arbitrary spectrum approximately under the assumption that the pair $(P_0, b(0))$ is controllable (recall that, according to the definition of $P_0$ in (12), $P_0$ is the monodromy matrix of the uncontrolled system $\dot{x}(t) = f(x(t), 0)$ along the periodic orbit $x_*$). This is a stronger assumption than the assumption made in [17], but it is still a genericity assumption.

**Lemma 3.2 (Approximate spectrum assignment for classical state feedback control, simplified from [17])**

> Let $r > 0$ be arbitrary. If the pair $(P_0, b(0))$ is controllable (that is, the $n \times n$ controllability matrix $\left[b(0), P_0 b(0), \ldots, P_0^{n-1} b(0)\right]$ is regular), then there exist a $\delta_{max} > 0$ and a vector of control gains $K_0 \in \mathbb{R}^n$ in (14) such that all Floquet multipliers of $x_*$ for the differential equation (15) have modulus less than $r$ for all $\delta \in (0, \delta_{max})$, where $\Delta_\delta$ is as defined in (14).

Note that the vector $K_0$ can be chosen independent of the $\delta \in (0, \delta_{max})$, but it may depend on the radius $r$ into which one wants to assign the spectrum. This result follows from classical linear feedback control theory ([17] proves a more general result). In short, linear feedback control theory [17] makes the following argument (thus, proving Lemma 3.2): the linearisation of $X$ with respect to its initial condition can be expanded in $\delta$ as

$$\partial_x X(x_*(0); \delta, K_0) = P(1; \delta, K_0) = P_0 \exp(-b(0)K_0^\mathsf{T}) + O(\delta)$$

(where $P(\cdot; \delta, K_0)$ was the generalised monodromy matrix defined for (17)). Since $\det P_0$ is positive we can for every matrix $R$ with positive determinant find a vector $K_0$ such that $\operatorname{spec} R = \operatorname{spec}(P_0 \exp(-b(0)K_0^\mathsf{T}))$ (using the assumption of controllability; see auxiliary Lemma A.1, which is a special case from the more general treatment in [17], and [39] for a Matlab implementation). Hence, if we choose the spectrum of $R$ inside a circle $B_{r/2}(0)$ of radius $r/2$ around $0$, then the spectrum of $\partial_x X(x_*(0); \delta, K_0)$ is also inside $B_r(0)$ for sufficiently small $\delta > 0$.

**Approximate spectrum for extended time-delayed feedback** We fix the control gains $K_0$ such that $\lim_{\delta \to 0} P(1; \delta, K_0) = P_0 \exp(-b(0)K_0^\mathsf{T})$ has all eigenvalues inside $B_r(0)$ for some $r \in (0, 1)$. Consider now again the extended time-delayed feedback control (8)–(10) with the particular choice of short impulse linear feedback law (14):

$$\dot{x}(t) = f(x(t), \Delta_\delta(t)K_0^\mathsf{T}[\tilde{x}(t) - x(t)]) \tag{21}$$

$$\tilde{x}(t) = (1 - \varepsilon)\tilde{x}(t - T) + \varepsilon x(t - T), \tag{22}$$

where $\varepsilon \in (0, 1)$.

**Lemma 3.3 (Floquet multipliers of extended time-delayed feedback)**

Assume that the matrix $P_0 \exp(-b(0)K_0^\mathsf{T})$ has all eigenvalues inside the ball $B_r(0)$ with $r < 1$. Then, for all sufficiently small $\varepsilon$ and $\delta$, the periodic orbit $x(t) = \tilde{x}(t) = x_*(t)$ of system (21), (22) has a simple Floquet multiplier $\lambda = 1$ and all its other Floquet multipliers are inside the unit circle.

**Outline of proof (details are given in Section A.2)** Eigenvalues $\lambda$ of the linearisation of (21)–(22) are roots of the function

$$h(\lambda; \varepsilon, \delta) = \det\left[\lambda I - P\left(1 - \frac{\varepsilon}{\lambda - (1 - \varepsilon)}; \delta, K_0\right)\right]. \tag{23}$$

Roots of $h$ with a non-small distance from $1 - \varepsilon$ are close to the roots of $\det(\lambda I - P(1; \delta, K_0))$, which are inside the unit circle by assumption. Roots $\lambda$ of $h$ close to $1 - \varepsilon$ with modulus greater than $1 - \varepsilon/2$ have the form $\lambda = 1 - \varepsilon + \varepsilon/\kappa$ where $|\kappa|$ is bounded away from $0$ and infinity. The roots $\kappa$ of $h(1 - \varepsilon + \varepsilon/\kappa; \delta, \varepsilon)$ are small perturbations of the roots $\kappa_{\ell,0}$ of $\det(I - P_0 - P_0 b(0)K_0^\mathsf{T}\sigma(\kappa - 1))$, where $\sigma$ is as defined in (20). These roots $\kappa_{\ell,0}$ have the form

$$\kappa_{\ell,0} = 1 + \frac{2\pi i\ell}{K_0^\mathsf{T} b(0)} \tag{24}$$

(if $K_0^\mathsf{T} b(0) \neq 0$, otherwise, only a single root $\kappa_{0,0} = 1$ exists). The roots $\kappa_{\ell,0}$ have all modulus greather than unity (except for $\ell = 0$, which corresponds to the trivial eigenvalue $\lambda = 1$) such that the corresponding roots $\lambda_\ell$ of $h$ have modulus smaller than unity. (*This ends the proof of Lemma 3.3.*) □

**Remark — two types of Floquet multipliers** The proof of Lemma 3.3 shows that there are two distinct types of roots: those approximating the spectrum assigned by the choice of control gains $K_0$, and those close to $1 - \varepsilon$ (called $\lambda_\ell$ above). The roots $\lambda_\ell$ lie close to the circle of radius $\varepsilon/2$ around the center $1 - \varepsilon/2$ in the complex plane and have the form

$$\lambda_\ell \approx 1 - \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\left[\frac{K_0^\mathsf{T} b(0) - 2\pi i\ell}{K_0^\mathsf{T} b(0) + 2\pi i\ell}\right] \qquad\qquad (\ell \in \mathbb{Z}).$$

For $\ell = 0$, the expression is exact (giving the simple root at unity), for the others the approximation is sufficiently accurate for small $\delta$ and $\varepsilon$ to ensure that they stay inside the unit circle. The illustration in Figure 2 shows the two distinct groups for the Hopf normal form example discussed in Section 5.

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**Importance of scalar input and trivial Floquet multiplier**  The proof of Lemma 3.3 hinges on one argument that depends on the presence of a trivial Floquet multiplier: we need to find the roots $s_j$ of $s \mapsto \det(I - P_0 - P_0 b(0) K_0^{\mathsf{T}} s)$ and then find solutions $\kappa$ of $\sigma(\kappa - 1) = s_j$ for all these roots $s_j$. Since $b(0) K_0^{\mathsf{T}}$ has rank one we know that $s \mapsto \det(I - P_0 - P_0 b(0) K_0^{\mathsf{T}} s)$ is a first-order polynomial (see Section A.2 for details). The presence of a trivial Floquet multiplier then ensures that this first-order polynomial has the root 0. Hence, 0 is its only root, restricting the possible location for the $\kappa_{\ell,0}$ to the list in (24). This simple argument would not apply for cases where the uncontrolled periodic orbit $x_*$ has no trivial Floquet multiplier, or for control with non-scalar inputs $u$, or for control with more than one kick per period.

## 4. Autonomous feedback control

The feedback control constructed in Lemma 3.3 introduces an explicit time dependence into the system. The controlled system has the form

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), \Delta_\delta(t) K_0^{\mathsf{T}} [\tilde{x}(t) - x(t)]) \\
\tilde{x}(t) &= (1 - \varepsilon)\tilde{x}(t - T) + \varepsilon x(t - T),
\end{aligned}
\tag{25}
$$

where $\Delta_\delta$ is time-periodic with period $T$, but the system still has a Floquet multiplier $\lambda = 1$. The neutrally stable direction corresponding to this Floquet multiplier is a phase shift: if $(x(t), \tilde{x}(t)) = (x_*(t), x_*(t))$ is a periodic orbit of (25) then so is $(x(t), \tilde{x}(t)) = (x_*(t+s), x_*(t+s))$ for any $s \in \mathbb{R}$. Hence, the controlled system with the gains $K(t) = \Delta_\delta(t) K_0$ is susceptible to arbitrarily small time-dependendent perturbations (say, experimental disturbances): the phase $s$ of the stabilised solution may drift until $\Delta_\delta$ is non-zero at a time $s$ where the gains $K_0$ are no longer stabilising. This problem does not occur if, instead of applying the feedback $K_0^{\mathsf{T}} [\tilde{x}(t) - x(t)]$ at a fixed time per period, we apply it in a strip in $\mathbb{R}^n$ close to a Poincaré section at $x_*(0)$ (as illustrated in Figure 3), putting a factor depending on $x(t)$ in front of $K_0^{\mathsf{T}} [\tilde{x}(t) - x(t)]$. Specifically, we let the function $\Delta_\delta$ not depend explicitly on time $t$ but on a function $\tilde{t} : \mathbb{R}^n \mapsto \mathbb{R}$, where the argument of $\tilde{t}$ is $x(t)$. Then the common notion of asymptotic stability of periodic orbits in autonomous dynamical systems applies. One would then always
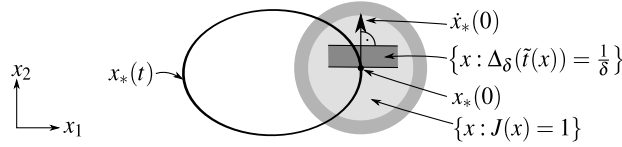


**Figure 3:** Illustration of choice for strip and Poincaré section where gains should be non-zero.

apply control near $x_*(0)$ despite phase drift. A possible explicit expression for $u$ is

$$
u(t) = \Delta_{\rho,\delta}(x(t)) K_0^{\mathsf{T}} [\tilde{x}(t) - x(t)], \qquad \text{where} \quad \Delta_{\rho,\delta}(x) = J_\rho(x) \Delta_\delta(\tilde{t}(x)),
\tag{26}
$$

$$
J_\rho(x) = \begin{cases} 1 & \text{if } |x - x_*(0)| < \rho, \\ 0 & \text{if } |x - x_*(0)| > 2\rho, \end{cases}
\qquad\qquad
\tilde{t}(x) = \frac{\dot{x}_*(0)^{\mathsf{T}}}{\dot{x}_*(0)^{\mathsf{T}} \dot{x}_*(0)} [x - x_*(0)],
\tag{27}
$$

$\rho > 0$ is a small radius and $J_\rho$ is smooth. In (27), $\tilde{t}(x_*(t)) = t + O(t^2)$ for $|t| \ll 1$, and $J_\rho$ restricts control to the neighborhood of radius $\rho$ around $x_*(0)$. With $u$ as defined in (26), the right-hand side of the now autonomous system

$$
\dot{x}(t) = f(x(t), \Delta_{\delta,\rho}(x(t)) K_0^{\mathsf{T}} [\tilde{x}(t) - x(t)]), \qquad \tilde{x}(t) = (1 - \varepsilon)\tilde{x}(t - T) + \varepsilon x(t - T)
\tag{28}
$$

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has a right-hand side that depends discontinuously on $x(t)$ (because $\Delta_\delta$ is discontinuous in its argument. Since the general mathematical theory for DDEs coupled to difference equations is not well developed, one may replace the discontinuous $\Delta_\delta$ in (26) with a smooth approximation of $\Delta_\delta$. This does not affect the final result, which we can state as a lemma (see Appendices A.3 and A.4 for the details of the choice for $\rho$ and the smoothing of $\Delta_{\delta,\rho}$):

**Lemma 4.1 (Autonomous stabilisability of periodic orbits with extended time-delayed feedback)**

> Assume that the matrix $P_0 \exp(-b(0)K_0^\mathsf{T})$, as used in Lemma 3.3, has all eigenvalues inside the ball $B_r(0)$ with $r < 1$. Then, for all sufficiently small $\rho$, there exist $\varepsilon_{max} > 0$ and $\delta_{max} > 0$ such that the periodic orbit $x(t) = \tilde{x}(t) = x_*(t)$ of system (28) is asymptotically exponentially stable for all $\varepsilon \in (0, \varepsilon_{max})$ and $\delta \in (0, \delta_{max})$.

**Remark: other arguments for $\Delta_{\delta,\rho}$** In (26) we can replace the argument $x(t)$ of $\Delta_{\delta,\rho}$ with $\tilde{x}(t)$, $x(t-T)$ or $\tilde{x}(t-T)$ without changing the linearisation in $x(t) = \tilde{x}(t) = x_*(t)$. Thus, (28) successfully stabilises the periodic orbit $x_*$ also with these modifications.

**Robustness** We assumed perfect knowledge of the periodic orbit $x_*$ and the right-hand side $f$ in the construction of $K_0$ and $\Delta_{\delta,\rho}$. However, we know that stable periodic orbits persist under small perturbations. Thus, for gains near $K_0$ and functions close to $\Delta_{\delta,\rho}$ the periodic orbit of the controlled system persists. Due to the non-invasive nature of extended time-delayed feedback, the periodic orbit of the system with perturbed $K_0$ and $\Delta_{\delta,\rho}$ is still identical to $x_*$.

## 5. Illustrative example: Hopf normal form

The construction of gains as described in Section 4 has been implemented as a Matlab function (publically available at [39], depending on DDE-Biftool [40, 41, 42]). The supplementary material demonstrates how one can find stabilising gains for two examples:

1. a family of period-two unstable oscillations around the hanging-down position of the parametrically excited pendulum, and

2. the unstable periodic orbits in the subcritical Hopf normal form.

We discuss example 2 in more detail in this section, because for this example we can prove that stabilisation with ETDF is not possible with constant gains and small $\varepsilon$. The subcritical Hopf bifurcation has also been used commonly in the literature as a benchmark example. Here we choose the Hopf normal form with constant speed of rotation (such that in polar coordinates the angle $\theta$ satisfies $\dot{\theta} = 1$ and all periodic orbits have period $2\pi$). Note that the control constructed by Fiedler *et al* [31] depended on changing rotation and was stabilising only in a small neighborhood of the bifurcation. Flunkert & Schöll [32] analysed time-delayed feedback control (with $\varepsilon = 1$) of the subcritical Hopf bifurcation completely, but also excluded the case of constant rotation and restricted themselves to a small neighbourhood of the bifurcation. Thus, even though example 2 is seemingly simple, it shows that the method proposed in the paper is able to stabilise periodic orbits that are beyond the approaches previously suggested in the literature. Without loss of generality we choose a linear control input $b = [1, 1]^\mathsf{T}$ such

that the system with control has the form:

$$\dot{x}_1 = px_1 - x_2 + x_1[x_1^2 + x_2^2] + u,$$
$$\dot{x}_2 = x_1 + px_2 + x_2[x_1^2 + x_2^2] + u,$$

(29)

where $p < 0$. This system has for $u = 0$ an unstable periodic orbit of the form $x_*(t) = [r\sin t, -r\cos t]^T$ with radius $r = \sqrt{-p}$ and period $T = 2\pi$. The monodromy matrix $P_0$ for the uncontrolled system along the periodic orbit $x_*$ equals

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & \exp(-4\pi p) \end{bmatrix}.$$

Since the derivative of the right-hand side with respect to the control input equals $b(t) = b = [1, 1]^T$, the periodic orbit is controllable in time $T$. In fact, the pair $(P_0, b)$ is controllable as required for the applicability of Lemma 3.3. Extended time-delayed feedback control, applied to a two-dimensional system has the form

$$u(t) = K_1(t)[\tilde{x}_1(t) - x_1(t)] + K_2(t)[\tilde{x}_2(t) - x_2(t)]$$
$$\tilde{x}_j(t) = (1 - \varepsilon)\tilde{x}_j(t - T) + \varepsilon x_j(t - T) \quad (j = 1, 2).$$

(30)

We can state two simple corollaries from our general considerations. First, it is impossible to stabilise the periodic orbit $x_*$ with extended time-delayed feedback using time-independent gains $K_1$ and $K_2$ for small $\varepsilon$:

**Lemma 5.1 (Lack of stabilisability for constant control gains)**

Let $p < 0$ and let $K_1(t)$ and $K_2(t)$ be arbitrary constants (also calling them $K_1$ and $K_2$). Then there exists an $\varepsilon_{max} > 0$ such that the periodic orbit $x_*$ is unstable with the extended time-delayed feedback control (30) for all $\varepsilon \in (0, \varepsilon_{max})$.

**Proof**  Amann & Hooton [38] proved a general topological restriction on the gains $K(t)$ for extended time-delayed feedback control: let $K(t) \in \mathbb{R}^n$ be arbitrary (continuous), $\theta \in [0, 1]$ be arbitrary, and let $u$ be of the form

$$u = \theta K(t)^T[\tilde{x}(t) - x(t)].$$

The scalar $\theta$ provides a homotopy from the uncontrolled system ($\theta = 0$) to the controlled system ($\theta = 1$). Assume that the trivial Floquet multiplier $\lambda_1 = 1$ of $x_*$ is isolated for $u = 0$ (which is the case for example (29) with $p < 0$). Then the Floquet multiplier $\lambda_1$ depends smoothly on $\theta$ at least for small $\theta$ and will be real: $\lambda_1(\theta) \in \mathbb{R}$ for $0 < \theta \ll 1$. A necessary condition for extended time-delayed feedback with gains $K(t)$ to be stabilising for $x_*$ and arbitrary $\varepsilon \in (0, 1)$ is that $\lambda_1'(0) \geqslant 0$ if the number of Floquet multipliers in $\{z \in \mathbb{C} : \text{Re}\, z > 1\}$ is odd for $\theta = 0$. If we denote an adjoint eigenvector for the trivial Floquet multiplier by $\bar{x}_*(t)$ (the right eigenvector is $\dot{x}_*(t)$), this criterion can be simplified to

$$\frac{\int_0^T \bar{x}_*(t)^T b(t) K(t)^T \dot{x}_*(t)\, dt}{\int_0^T \bar{x}_*(t)^T \dot{x}_*(t)\, dt} \leqslant 0,$$

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where $b(t) = \partial_u f(x_*(t), 0)$ (this simplifying criterion was formulated in general in [33]). For our particular example, we have

$$\dot{x}_*(t) = \bar{x}_*(t) = \begin{bmatrix} r\cos t \\ r\sin t \end{bmatrix}, \qquad b(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad T = 2\pi$$

and constant gains $K_1$ and $K_2$ such that the necessary condition of [33, 38] is

$$K_1 + K_2 \leqslant 0. \tag{31}$$

On the other hand, if $K_1 + K_2 \leqslant 0$ the Jacobian of (29) with classical linear feedback control

$$u = K_1(x_{1,*}(t) - x_1) + K_2(x_{2,*}(t) - x_2) = K_1(r\sin t - x_1) + K_2(-r\cos t - x_2) \tag{32}$$

along $x = x_*(t)$ has the trace

$$\text{tr}\, \partial_x f(x(t), K(x_*(t) - x(t)))|_{x(t) = x_*(t)} = 2p + 4r^2 - K_1 - K_2 = -2p - K_1 - K_2.$$

Since $p < 0$, this trace is positive if $K_1 + K_2 \leqslant 0$ for all $t \in [0, 2\pi]$ such that the classical linear feedback control (32) cannot be stabilising for the periodic orbit $x_* = [r\sin t, -r\cos t]^\top$. Thus, Lemma 3.1 implies that extended time-delayed feedback cannot be stabilising either, for sufficiently small $\varepsilon > 0$.
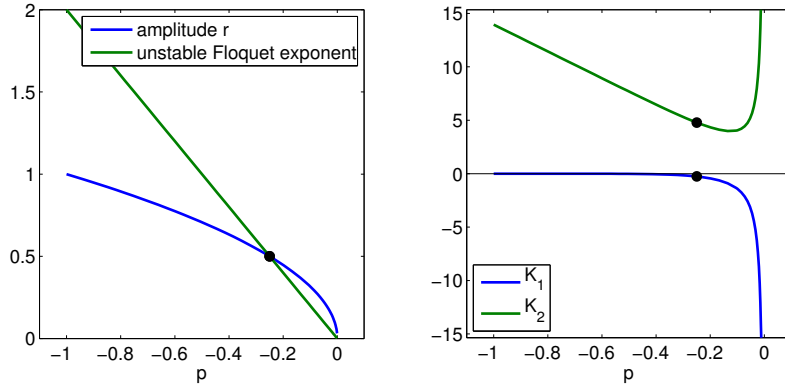
*(This ends the proof of Lemma 5.1.)* $\square$



**Figure 4:** Left: Amplitude $r = \sqrt{-p}$ and unstable Floquet exponent (equals $-2p$) along family of periodic orbits. Right: gains $K_1$ (note that $K_1 < 0$ always) and $K_2$ along family of periodic orbits in Hopf normal form (29) with feedback law (33).

**Construction of gains** For the periodic control gains $\Delta_\delta(t)K_0^\top$ (or the autonomous nonlinear gains $\Delta_{\delta,\rho}(x(t))K_0^\top$) the gains $K_0$ are constructed such the matrix $P_0 \exp(-b(0)K_0^\top)$ has all eigenvalues inside the unit circle (for our illustration we choose the target location at $\pm i/2$). Figure 4 shows the amplitude and unstable Floquet exponent of the periodic orbits and the gains obtained in this manner (called $K_1$ and $K_2$ in Figure 4). Since the pair $(P_0, b(0))$ is not controllable at the Hopf point, the gains diverge to infinity for $p \to 0$. In particular, $P_0 = I$ for $p = 0$ such that it cannot be linearly controllable with a single input.
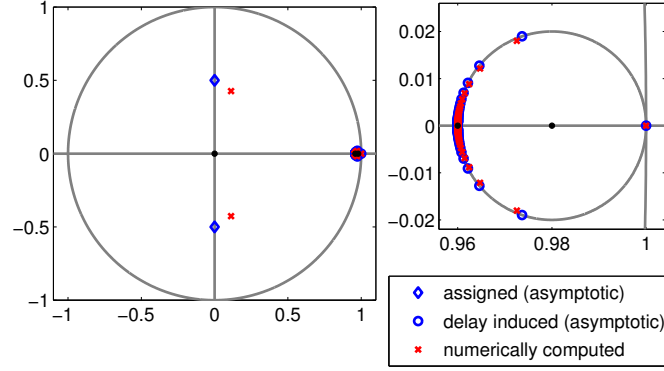
**Figure 5:** Numerically computed versus asymptotic spectrum for $p = -0.25$ ($\varepsilon = 0.04$, $\delta = T/500$, $\rho = 0.3$, $K_1 = -0.258$, $K_2 = 4.786$).Left: unit circle in the complex plane. Right: zoom into the circle around $1 - \varepsilon$ or radius $\varepsilon/2$. Computed with DDE-Biftool [40, 41, 42], see supplementary material and [39] for the code.

**Illustration of asymptotics**    Figure 5 shows how the true Floquet multipliers approximate their asymptotic values when using the autonomous time-delayed feedback control (44) with gains depending on $x$:

$$u(t) = \Delta_{\delta,\rho}(x(t)) \left[ K_1[\tilde{x}_1(t) - x_1(t)] + K_2[\tilde{x}_2(t) - x_2(t)] \right]$$
$$\tilde{x}_j(t) = (1 - \varepsilon)\tilde{x}_j(t - T) + \varepsilon x_j(t - T) \quad (j = 1, 2). \tag{33}$$

For the construction of $\Delta_{\delta,\rho}$ we used the construction of $\tilde{t}$ proposed in (26)–(27):

$$\tilde{t}(x) = \frac{y_0^T}{y_0^T y_0}[x - x_0] \qquad\qquad \text{for } x \in B_\rho(x_0), \tag{34}$$

where $y_0 = [r, 0]^T$, $x_0 = [0, -r]^T$ and $r = \sqrt{-p}$. At the particular parameter value $p = -0.25$ shown in Figure 5 the uncontrolled periodic orbit is already strongly unstable: the unstable Floquet multiplier equals $\exp(-4\pi p) \approx 23.141$. The gains $K_1$ and $K_2$, designed to assign the Floquet mulitpliers $\pm i/2$, are also large after division by $\delta$. Hence, the range of $\delta$ and $\varepsilon$, for which stabilisation is successful is small. The values for $\delta$ and $\varepsilon$ used in the illustration are chosen such that deviations from the asymptotic limit are visible but small.

## 6. Conclusion and Outlook

The paper proves conclusively that there are no restrictions inherent in extended time-delayed (state) feedback control if one accepts time-periodic gains, while for constant gains general positive results are unlikely (as they are absent for classical feedback control of linear time-periodic systems). The particulars of the gain construction presented here, following the approach of Brunovksy, are merely for the purpose of proving their existence analytically. While the result raises the possibility that more general assignment is feasible (since there is a lot of freedom in the choice of general time-periodic $K(t)$) the techniques for proving this may have to be different from those in the paper. The central argument of the paper rests on the rank-one nature of the control input, making it easy to locate all roots $\kappa$ of the transcendental function $\kappa \mapsto \det[I - P_0 \exp(b(0)K_0^T(\kappa - 1))]$ (the matrix $b(0)K_0^T$ has rank one

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in our case). The argument also exploits the presence of the trivial Floquet multiplier, thus, making the result (even if it is based entirely on linear theory) mostly relevant to the analysis of nonlinear systems.

## A. Auxiliary Lemmas and detailed arguments of proofs

### Lemma A.1

> Let $(A, b)$ with $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ be controllable, let $\det A > 0$, and let $(\lambda_1, \ldots \lambda_n) \in \mathbb{C}^n$ be the spectrum of a real matrix with positive determinant. Then one can find a vector $K \in \mathbb{R}^n$ such that $\mathrm{spec}(A \exp(bK^\mathsf{T})) = (\lambda_1, \ldots, \lambda_n)$.

The proof is given in [17]. A Matlab implementation of the explicit construction is `SpecExpAssign.m` in [39].

### A.1. Floquet multipliers for extended time-delayed feedback

### Lemma A.2 (Characteristic equation for Floquet multipliers)

> Let $A(t) \in \mathbb{R}^{n \times n}$, $b(t)$, $K(t) \in \mathbb{R}^n$ be T-periodic and let $\varepsilon$ be positive. Then the Floquet multipliers different from $1 - \varepsilon$ of the linear system
>
> $$\dot{x}(t) = A(t)x(t) + b(t)K(t)^\mathsf{T}[\tilde{x}(t) - x(t)] \tag{35}$$
> $$\tilde{x}(t) = (1 - \varepsilon)\tilde{x}(t - T) + \varepsilon x(t - T) \tag{36}$$
>
> are roots of the function
>
> $$h : \lambda \mapsto \det\left[\lambda I - P\left(1 - \frac{\varepsilon}{\lambda - (1 - \varepsilon)}\right)\right].$$

For $\mu \in \mathbb{C}$ the matrix $P(\mu)$ was defined as the solution $x$ at time $T$ of $\dot{x} = [A(t) - \mu b(t)K(t)^\mathsf{T}]x$ with $x(0) = I$ (that is, $P(\mu)$ is the monodromy matrix of $\dot{x} = [A(t) - \mu b(t)K(t)^\mathsf{T}]x$).

**Proof**  Let $\lambda \in \mathbb{C}$ be an eigenvalue of the time-T map $M$ of (35)–(36). Let $x_0$, $\tilde{x}_0$ (both $[-T, 0] \mapsto \mathbb{R}^n$) be the components of an eigenvector corresponding to $\lambda$, and let $x_1$, $\tilde{x}_1$ (also both $[-T, 0] \mapsto \mathbb{R}^n$) be the corresponding components of $M[x, \tilde{x}]$. Then $\tilde{x}_1(t) = \lambda \tilde{x}_0(t)$ and, by definition of the time-T map $M$, $\tilde{x}_1(t) = (1 - \varepsilon)\tilde{x}_0(t) + \varepsilon x_0(t)$. Hence, $\lambda \tilde{x}_0(t) = (1 - \varepsilon)\tilde{x}_0(t) + \varepsilon x_0(t)$, which implies (since $\lambda \neq 1 - \varepsilon$)

$$\tilde{x}_0(t) = \frac{\varepsilon}{\lambda - (1 - \varepsilon)} x_0(t)$$

for $t \in [-T, 0]$. Using this relation, we can solve (35) on the interval $[-T, 0]$ as

$$\dot{x}_0(t) = A(t)x_0(t) + b(t)K(t)^\mathsf{T}\left[\frac{\varepsilon}{\lambda - (1 - \varepsilon)} - 1\right]x_0(t)$$

with boundary condition $x_0(0) = \lambda x_0(-T)$. By definition of the monodromy matrix $P$ this is equivalent to

$$P\left(1 - \frac{\varepsilon}{\lambda - (1 - \varepsilon)}\right)x_0(-T) = \lambda x_0(-T),$$

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which has a non-trivial solution $x_0(-T)$ if and only if the function $h$ in Lemma A.2 is non-zero. (*This ends the proof of Lemma A.2.*)  □

## A.2. Proof of Lemma 3.3

The characteristic function $h(\lambda; \varepsilon, \delta) = \det[\lambda I - P(1 - \varepsilon/(\lambda - (1 - \varepsilon)); \delta, K_0)]$, defined in (23), has a root $\lambda = 1$ for all small $\delta$ and all $\varepsilon \in (0, 1)$: a nullvector of $I - P(0; \delta, K_0)$ is $\dot{x}_*(0)$ (corresponding to a linearised phase shift).

For $\lambda$ with modulus larger than $1 - \varepsilon/2$ the term $\varepsilon/(\lambda - (1 - \varepsilon))$ has modulus less or equal than 2. Let us pick $\delta_1 > 0$ and a $C_1 \in (0, 1)$ (both small) such that the polynomial

$$\lambda \mapsto \det(\lambda I - P(1 - \mu; \delta, K_0))$$

has all roots inside the ball $B_{(r+1)/2}(0) \subset B_1(0)$ for all $\delta \in [0, \delta_1]$ and $\mu$ with $|\mu| \leqslant C_1$. This is possible since $\lambda \mapsto \det(\lambda I - P(1; 0, K_0))$ has all roots inside the ball $B_r(0)$ by assumption of the Lemma and the limit of $P(1 - \mu; \delta, K_0)$ for $\delta \to 0$ was uniform for bounded $\mu$ (recall $\lim_{\delta \to 0} P(1 - \mu; \delta, K_0) = P_0 \exp((\mu - 1)b(0)K_0^\mathsf{T}))$. Thus, $h(\lambda; \varepsilon, \delta)$ cannot have roots $\lambda$ on or outside the unit circle for which

$$\left| \frac{\varepsilon}{\lambda - (1 - \varepsilon)} \right| \leqslant C_1$$

holds. Hence, for all $\delta \in [0, \delta_1]$ all roots of $h(\cdot; \varepsilon, \delta)$ on or outside of the unit circle must satisfy

$$C_1 \leqslant \left| \frac{\varepsilon}{\lambda - (1 - \varepsilon)} \right| \leqslant 2. \tag{37}$$

We introduce the new variable $\kappa \in \mathbb{C}$ defined via

$$\lambda = 1 - \varepsilon + \varepsilon/\kappa. \tag{38}$$

Restriction (37) for $\lambda$ is equivalent to the restriction $C_1 \leqslant |\kappa| \leqslant 2$ for $\kappa$. Hence, for all $\delta \in [0, \delta_1]$ and $\varepsilon \in (0, 1)$, every root $\lambda$ of $h(\cdot; \delta, \varepsilon)$ on or outside of the unit circle corresponds to a root $\kappa$ of

$$\begin{aligned} g(\kappa; \delta, \varepsilon) &= h(1 - \varepsilon + \varepsilon/\kappa; \delta, \varepsilon) \\ &= \det\left[ \left(1 - \varepsilon + \frac{\varepsilon}{\kappa}\right) I - P(1 - \kappa; \delta, K_0) \right] \end{aligned}$$

with $C_1 \leqslant |\kappa| \leqslant 2$. This one-to-one correspondence of roots of $h$ and $g$ is given via relation (38) and includes multiplicity of the roots. Relation (38) also implies that $|\kappa| \leqslant 1$, because, otherwise, $|\lambda| < 1$. The function $g$ has a limit

$$g(\kappa; \delta, \varepsilon) \to g(\kappa; 0, 0) \qquad\qquad \text{for } (\delta, \varepsilon) \to 0$$

uniformly for $\kappa$ with $C_1 \leqslant |\kappa| \leqslant 2$. Hence, the set of roots $\kappa$ of $g(\cdot; \delta, \varepsilon)$ with $C_1 \leqslant |\kappa| \leqslant 2$ is a small perturbation of the set of roots of $g(\cdot; 0, 0)$:

$$\begin{aligned} g(\kappa; 0, 0) &= \det\left[ I - P_0 \exp\left( b(0)K_0^\mathsf{T} [\kappa - 1]\right) \right] \\ &= \det\left[ I - P_0 - P_0 b(0)K_0^\mathsf{T} \sigma(\kappa - 1) \right], \text{ where} \end{aligned}$$

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$$\sigma(\kappa - 1) = \begin{cases} \dfrac{\exp\left(K_0^\mathsf{T} b(0)(\kappa - 1)\right) - 1}{K_0^\mathsf{T} b(0)} & \text{if } K_0^\mathsf{T} b(0) \neq 0 \\[2ex] \kappa - 1 & \text{if } K_0^\mathsf{T} b(0) = 0. \end{cases}$$

The generalised eigenvalue problem for the matrix pair $(I - P_0, P_0 b(0) K_0^\mathsf{T})$ with characteristic polynomial $\sigma \mapsto \det(I - P_0 - P_0 b(0) K_0^\mathsf{T} \sigma)$ is regular because $\det(I - P_0) = 0$ but $\det(I - P_0 - P_0 b(0) K_0^\mathsf{T} \sigma(-1))$ is regular (since $P_0 \exp(-b(0) K_0^\mathsf{T})$ has all eigenvalues inside the unit circle, $I - P_0 \exp(-b(0) K_0^\mathsf{T})$ is regular). As $P_0 b(0) K_0^\mathsf{T}$ has rank 1 the characteristic polynomial corresponding to $\sigma \mapsto \det(I - P_0 - P_0 b(0) K_0^\mathsf{T} \sigma)$ has degree 1. Moreover, its only root equals 0 (which must be simple due to the regularity of $\det(I - P_0 - P_0 b(0) K_0^\mathsf{T} \sigma)$). Hence, we know that $g(\kappa; 0, 0) = 0$ if and only if $\sigma(\kappa - 1) = 0$.

**Case $K_0^\mathsf{T} b(0) = 0$** If $K_0^\mathsf{T} b(0) = 0$, this implies that the only root $\kappa$ of $g(\cdot; 0, 0)$ with $C_1 \leqslant |\kappa| \leqslant 2$ equals unity. Hence, also for sufficiently small $\varepsilon$ and $\delta$, the only root $\kappa$ with $C_1 \leqslant |\kappa| \leqslant 2$ of $g(\cdot; \delta; \varepsilon)$ equals unity (since $g(1; \delta; \varepsilon) = 0$).

**Case $K_0^\mathsf{T} b(0) \neq 0$** If $K_0^\mathsf{T} b(0) \neq 0$, we have that $g(\kappa; 0, 0) = 0$ if and only if $\exp(K_0^\mathsf{T} b(0)(\kappa - 1)) = 1$ such that the roots are

$$\kappa_{\ell,0} = 1 + \frac{2\pi i \ell}{K_0^\mathsf{T} b(0)} \qquad\qquad (\ell \in \mathbb{Z})$$

The first part of the subscript, $\ell$, numbers the roots, the second part of the subscript, 0, indicates that $\delta = \varepsilon = 0$. Thus, the roots $\kappa_{\ell,0}$ of $g(\cdot; 0, 0)$ have a modulus

$$|\kappa_{\ell,0}| = \sqrt{1 + \frac{4\pi^2 \ell^2}{(K_0^\mathsf{T} b(0))^2}}$$

such that only the roots $\kappa_{\ell,0}$ with index

$$-\ell_{\max} \leqslant \ell \leqslant \ell_{\max}, \qquad\qquad \text{where } \ell_{\max} = \frac{\sqrt{3}}{2\pi} |K_0^\mathsf{T} b(0)|,$$

are in the admissible range with $C_1 \leqslant |\kappa_{\ell,0}| \leqslant 2$. (Hence, both cases, $K_0^\mathsf{T} b(0) = 0$ and $K_0^\mathsf{T} b(0) \neq 0$ can be treated equally.) The admissible roots $\kappa_{\ell,0}$ of $g(\cdot; 0, 0)$ are all simple. Hence, for sufficiently small $\delta$ and $\varepsilon$, $g(\cdot; \delta, \varepsilon)$ will have roots $\kappa_\ell$ for $|\ell| \leqslant \ell_{\max}$ that are small perturbations of $\kappa_{\ell,0}$, and these roots $\kappa_\ell$ are the only roots of $g(\cdot; \delta, \varepsilon)$ with modulus in $[C_1, 2]$. Since we know that $g(1; \delta, \varepsilon) = 0$, we know that $\kappa_0 = 1$ (hence, for $\ell = 0$ the perturbation is zero). Furthermore, for non-zero $\ell$ with $|\ell| \leqslant \ell_{\max}$, the modulus of $\kappa_{\ell,0}$ is greater than 1. Hence, the perturbed roots $\kappa_\ell$ also have modulus greater then 1 for sufficiently small $\varepsilon$ and $\delta$, and non-zero $|\ell| \leqslant \ell_{\max}$.

Consequently, by relation (38), the only roots of $h(\cdot; \delta, \varepsilon)$ that could be on or outside the unit circle are

$$\lambda_\ell = 1 - \varepsilon + \frac{\varepsilon}{\kappa_\ell} \qquad\qquad \text{where } |\ell| \leqslant \ell_{\max}.$$

However, these roots $\lambda_\ell$ are simple and satisfy $\lambda_0 = 1$ and $|\lambda_\ell| < 1$ for non-zero $\ell$, since $|\kappa_\ell| > 1$ for non-zero $\ell$. *(This ends the proof of Lemma 3.3.)* □

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### A.3. Details of construction for autonomous feedback gains — Regularisation of the short impulse $\Delta_\delta(t)$

To avoid discontinous dependence of the right-hand side on the solution, we first regularise the discontinuity of the time-dependent gain $K(t)$. Define for $\delta \in (0, \sqrt{T + 1/16} - 1/4)$ (such that $2\delta^2 + \delta < T$) the regularised version of $\Delta_\delta$:

$$\Delta_\delta(t) = \begin{cases} 1/\delta & \text{if } t|_{\text{mod}[0,T)} \in [0, \delta], \\ 0 & \text{if } t|_{\text{mod}[0,T)} \in [\delta + \delta^2, T - \delta^2], \\ \dfrac{1}{\delta} \, m\left(\dfrac{\delta + \delta^2 - t|_{\text{mod}[0,T)}}{\delta^2}\right) & \text{if } t|_{\text{mod}[0,T)} \in (\delta, \delta + \delta^2) \\ \dfrac{1}{\delta} \, m\left(\dfrac{t|_{\text{mod}[0,T)} - T + \delta^2}{\delta^2}\right) & \text{if } t|_{\text{mod}[0,T)} \in (T - \delta^2, T), \end{cases} \tag{39}$$

where $m : \mathbb{R} \mapsto [0, 1]$ is an arbitrary smooth monotone increasing function with $m(s) = 0$ for $s \leqslant 0$ and $m(s) = 1$ for $s \geqslant 1$. When using $\Delta_\delta$ as defined in (39) instead of (6) to define the linear (now approximately) short-impulse feedback law

$$u_\delta(t; y) = \Delta_\delta(t) K_0^\mathsf{T} y \tag{40}$$

the nonlinear time-$T$ map is still linearisable. Denoting the monodromy matrix of the linear system (recall that $A(t) = \partial_x f(x_*(t), 0)$, $b(t) = \partial_u f(x_*(t), 0)$ and using definition (39) for $\Delta_\delta$)

$$\dot{y}(t) = \left[A(t) - \mu b(t)\Delta_\delta(t)K_0^\mathsf{T}\right] y(t), \qquad\qquad y(0) = y_0$$

again by $P(\mu; \delta, K_0)$ then the monodromy matrix of the smoothed system still satisfies (identical to (18))

$$P(\mu; \delta, K_0) = P_0 \exp(-b(0)K_0^\mathsf{T}\mu) + O(\delta), \tag{41}$$

where the error term $O(\delta)$ is uniform for bounded $\mu \in \mathbb{C}$ and $K_0 \in \mathbb{R}^n$. Hence, we can replace the discontinuous definition (6) for $\Delta_\delta(t)$ by (39) in (21), and Lemma 3.3 still applies to the modified (regularised) system.

### A.4. State-dependent gains $K(x(t))$

Consider a sufficiently small radius $\rho > 0$ such that the equation $\dot{x}_*(0)^\mathsf{T}[x - x_*(t)] = 0$ has a unique solution $t \in \mathbb{R}$ close to 0 for all $x \in B_{2\rho}(x_*(0)) \subset \mathbb{R}^n$, thus defining implicitly a smooth function $t_\rho$:

$$t_\rho : \mathbb{R}^n \mapsto \mathbb{R}, \qquad t_\rho(x) = \begin{cases} \text{root } t \text{ of } \dot{x}_*(0)^\mathsf{T}[x - x_*(t)] = 0 & \text{if } x \in B_{2\rho}(x_*(0)), \\ \text{arbitrary such that } t_\rho \text{ is smooth} & \text{otherwise.} \end{cases} \tag{42}$$

The function $\tilde{t}$, defined in (27), is approximately equal to $t_\rho$ along the periodic orbit $x_*$ and near $x_*(0)$: $t_\rho(x_*(t)) - \tilde{t}(x_*(t)) = t - \tilde{t}(x_*(t)) = O(t^2)$. We consider also a regularised indicator function for the neighbourhood of $x_*(0)$

$$J_\rho : \mathbb{R}^n \mapsto \mathbb{R}, \qquad J_\rho = \begin{cases} 1 & \text{if } x \in B_\rho(x_*(0)), \\ 0 & \text{if } x \notin B_{2\rho}(x_*(0)), \\ \text{arbitrary such that } J_\rho \text{ is smooth} & \text{if } x \in B_{2\rho}(x_*(0)) \setminus B_\rho(x_*(0)). \end{cases}$$

and combine $t_\rho$ and $J_\rho$ with $\tilde{\Delta}_\delta$ as defined in (39) to the smooth globally defined function

$$\tilde{\Delta}_{\delta,\rho} : \mathbb{R}^n \mapsto \mathbb{R}, \qquad\qquad \tilde{\Delta}_{\delta,\rho}(x) = J_\rho(x)\tilde{\Delta}_\delta(t_\rho(x))$$

When applying $\tilde{\Delta}_{\delta,\rho}$ to $x_*(t)$ the result is identical to the timed impulse $\tilde{\Delta}_\delta$ for small $\delta$: if $\|x_*(t) - x_*(0)\| < \rho$ for all $t \in [-\delta^2, \delta + \delta^2]$ then

$$\tilde{\Delta}_{\delta,\rho}(x_*(t)) = \tilde{\Delta}_\delta(t)$$

for all $t \in \mathbb{R}$. Consequently, the system with extended time-delayed feedback and state-dependent gains

$$\dot{x}(t) = f(x(t), \tilde{\Delta}_{\delta,\rho}(x(t))K_0^\mathsf{T}[\tilde{x}(t) - x(t)]) \tag{43}$$

$$\tilde{x}(t) = (1-\varepsilon)\tilde{x}(t-T) + \varepsilon x(t-T), \tag{44}$$

which is now autonomous with a smooth right-hand side, has for sufficiently small $\rho$ and $\delta + \delta^2 < \rho$ exactly the same linearisation along the periodic orbit $x(t) = \tilde{x}(t) = x_*(t)$ as system (21),(22). The derivative of $\tilde{\Delta}_{\delta,\rho}$ with respect to its argument is multiplied by 0 if $\tilde{x}(t) = x(t) = x_*(t)$ for all times $t$ in the term $\tilde{\Delta}_{\delta,\rho}(x(t))K_0^\mathsf{T}[\tilde{x}(t) - x(t)]$ in (43).

The time reconstruction function $t_\rho$ as defined in (42) satisfies $t_\rho(x_*(t)) = t$ as long as $x_*(t) \in B_{2\rho}(x_*(0))$. The definition of $t_\rho$ as proposed in (27) in the main text is a $O(t^2)$ perturbation of (42) for $t$ of order $\delta$. Thus, continuity of the Flqouet multipliers implies that the perturbed version of $t_\rho$ preserves the stability of the periodic orbit $x_*$.

**Declaration**

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