FUNNEL CONTROL FOR SYSTEMS WITH RELATIVE DEGREE TWO*

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Abstract. Tracking of reference signals $y_{\text{ref}}(\cdot)$ by the output $y(\cdot)$ of linear (as well as a considerably large class of nonlinear) single-input, single-output systems is considered. The system is assumed to have strict relative degree two with (weakly) stable zero dynamics. The control objective is tracking of the error $e = y - y_{\text{ref}}$ and its derivative $\dot{e}$ within two prespecified performance funnels, respectively. This is achieved by the so-called funnel controller $u(t) = -k_0(t)^2 e(t) - k_1(t)\dot{e}(t)$, where the simple proportional error feedback has gain functions k_0 and k_1 designed in such a way to preclude contact of e and $\dot{e}$ with the funnel boundaries, respectively. The funnel controller also ensures boundedness of all signals. We also show that the same funnel controller (i) is applicable to relative degree one systems, (ii) allows for input constraints provided a feasibility condition (formulated in terms of the system data, the saturation bounds, the funnel data, bounds on the reference signal, and the initial state) holds, (iii) is robust in terms of the gap metric: if a system is sufficiently close to a system with relative degree two, stable zero dynamics, and positive high-frequency gain, but does not necessarily have these properties, then for small initial values the funnel controller also achieves the control objective. Finally, we illustrate the theoretical results by experimental results: the funnel controller is applied to a rotary mechanical system for position control.

1. Introduction. We study tracking of reference signals $y_{\text{ref}}(\cdot)$ by the output $y(\cdot)$ of a single-input, single-output system with (strict) relative degree two and (weakly) stable zero dynamics. For the purpose of illustration, we first explain our concept for the prototype of linear single-input, single-output systems

$$(1.1) \quad \begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \qquad x(0) = x^0, \\ y(t) &= cx(t), \end{aligned}$$

where $(A, b, c) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n \times \mathbb{R}^{1 \times n}$, $x^0 \in \mathbb{R}^n$, has *relative degree two and positive high-frequency gain*, i.e.,

$$(1.2) \quad cb = 0 \quad \text{and} \quad cAb > 0,$$

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and *asymptotically stable zero dynamics* (equivalently called *minimum phase*), i.e.,

$$(1.3) \qquad \forall\, s \in \mathbb{C} \quad \text{with} \quad \operatorname{Re} s \geq 0: \ \det \begin{bmatrix} sI_n - A & b \\ c & 0 \end{bmatrix} \neq 0.$$

1.1. Frequency domain: High-gain, zero dynamics, tracking. The zero dynamics of system (1.1) (and also of nonlinear systems) play an essential role in the design of a controller; see the nice textbooks [19, 20]. In the following we abuse notation and write $f(s)$ for the Laplace transform of $f(t)$. The transfer function of (1.1) may be written as $y(s) = c(sI - A)^{-1}b\, u(s) = \gamma \frac{q(s)}{d(s)}\, u(s)$ for coprime, monic $q, d \in \mathbb{R}[s]$ such that $\deg d > \deg q$, where the relative degree is $r := \deg d - \deg q$ and the high-frequency gain is $\gamma \neq 0$. By the Euclidean algorithm $d(s) = a(s)\, q(s) + l(s)$ for some $a, l \in \mathbb{R}[s]$ such that $\deg l < \deg q$, we may decompose system (1.1) as shown in the gray box of Figure 1(a).
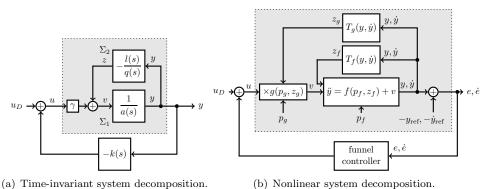

(a) Time-invariant system decomposition.   (b) Nonlinear system decomposition.

FIG. 1. *Linear and nonlinear system; for details on the latter see section* 3.1.

The subsystem $\Sigma_1 \colon v \mapsto y$ has relative degree r; the subsystem $\Sigma_2 \colon y \mapsto z$ is asymptotically stable if and only if $q(s)$ is Hurwitz. Recall that (provided (1.1) is stabilizable and detectable) (1.3) is equivalent to $q(s)$ being Hurwitz, [10, Prop. 2.1.2]; hence we see that Σ_2 captures the zero dynamics. Suppose we apply (time-invariant) derivative feedback of the form $u(s) = -k(s)y(s) + u_D(s)$, $k(s) \in \mathbb{R}[s]$ (see Figure 1(a)), where u_D is a disturbance or a new input to check stability of the closed-loop system; then the transfer function of the closed-loop system is

$$(1.4) \qquad y(s) = \frac{\gamma\, q(s)}{[a(s) + k(s)\gamma]\, q(s) + l(s)}\, u_D(s).$$

We stress the following observations: (i) If the zero dynamics are asymptotically stable, i.e., $q(s)$ is Hurwitz, then a Hurwitz polynomial $k(s)$ may be chosen independently of the special structure of the zero dynamics to yield an asymptotically stable system (1.4). (ii) If the systems entries are unknown and only the structural assumptions of minimum phase and sign of the high-frequency gain γ of (1.1) are assumed, then we may choose $k(s) = \kappa\, \widetilde{k}(s)$ such that $\widetilde{k}(s)$ is Hurwitz and has coefficients of the same sign as γ, and for sufficiently large κ system (1.4) becomes asymptotically stable.

We illustrate (ii) for relative degree two systems which are minimum phase and have high-frequency gain $\gamma > 0$. Let $a(s) = s^2 + a_1 s + a_0$, and choose $k(s) = k_1 s + k_0$ so that $k_0 = \gamma (k_1/2)^2$; then $a(s) + \gamma\, k(s)$ has zeros $s_{1;2} = -\frac{\gamma\, k_1 + a_1}{2} \pm$

$\frac{1}{2}\sqrt{2\gamma\,a_1 k_1 + a_1^2 - 4a_0}$, and for large k_1, approximately $s_{1;2} \approx -(\gamma k_1 \pm \sqrt{2\gamma a_1 k_1})/2$, and so the denominator in (1.4) becomes stable for sufficiently large k_1.

These properties, and generalizations thereof, will be exploited to design adaptive controllers in the time domain in the following.

Note also that if we want to track asymptotically some reference signals $y_{\mathrm{ref}}(\cdot)$, then the internal model principle [35, sect. 8.8] says, roughly speaking, that the feedback controller has to reduplicate the dynamics of the class of reference signal by an internal model. This can be circumvented and the controller can be kept simple by weakening the control objective slightly: asymptotic tracking is replaced by practical tracking; i.e., the tracking error ultimately gets smaller than a prespecified error bound.

1.2. Classical adaptive control. We now explain the classical concept of high-gain control where the gain is determined adaptively. To illustrate the idea, we consider relative degree one system (1.1) with positive high-frequency gain, i.e., $cb > 0$, and asymptotically stable zero dynamics, i.e., (1.3). It is well known that proportional output feedback $u(t) = -k\,y(t)$ applied to (1.1) yields a closed-loop system which is stable if $k > 0$ is sufficiently large; for a proof in time domain see, e.g., [10, Lem. 2.2.7].

This inherent high-gain property of the system class is used in adaptive control (see the pioneering contributions [3, 22, 23, 25, 34] and the survey [12]) as follows: Adaptive control means that k in the above feedback law becomes time-varying

$$(1.5) \qquad\qquad u(t) = -k(t)\,y(t),$$

and the gain is adapted by the output, e.g., $\dot{k}(t) = y(t)^2$, $k(0) = k^0$.

If (1.5) with $\dot{k} = y^2$, $k(0) = k^0$ is applied to (1.1), then, for any initial data $x^0 \in \mathbb{R}^n$, $k^0 \in \mathbb{R}$, the closed-loop system satisfies $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$. Intuitively speaking, this adaptive control strategy increases the gain $k(t)$ as long as $|y(t)|$ is large, until finally the gain is sufficiently large so that the closed-loop system becomes asymptotically stable. The drawbacks of $\dot{k} = y^2$ are that (i) the gain increases monotonically and, albeit bounded, may finally be very large and amplifies measurement noise; and (ii) no transient behavior is taken into account; an exception is [24] wherein the issue of prescribed transient behavior is successfully addressed.

1.3. The funnel controller for systems with relative degree one. The fundamentally different approach of *funnel control*, introduced by [14]—for systems of functional differential equations, including the class of systems of form (1.1) with relative degree one (i.e., satisfy $cb > 0$) which are minimum phase (i.e., satisfy (1.3)) resolves these drawbacks. To explain the concept, we stick to the relative degree one case and consider output stabilization, i.e., $y_{\mathrm{ref}} = 0$: the simplicity of (1.5) is preserved, but gain adaptation is replaced by

$$(1.6) \qquad\qquad k(t) = \frac{1}{\psi(t) - |y(t)|},$$

where $\psi \colon \mathbb{R}_{\geq 0} \to [\lambda, \infty)$ is, for some $\lambda > 0$, a bounded differentiable function representing the funnel boundary; see Figure 2. Now if (1.5), (1.6) is applied to (1.1), then, for any initial data $x^0 \in \mathbb{R}^n$ such that the initial output is in the funnel, i.e., $|cx^0| < \psi(0)$, the closed-loop system has a unique solution on $\mathbb{R}_{\geq 0}$, the gain $k(\cdot)$ is bounded, and the output evolves within the funnel, i.e., $|y(t)| < \psi(t)$ for all $t \geq 0$.

The intuition of funnel control is, roughly speaking, that the gain $k(t)$ is only "large" if $|y(t)|$ is "close" to the funnel boundary $\psi(t)$, and then the inherent high-gain

property of the system class precludes boundary contact. In contrast to the adaptive high-gain approach, the gain is no longer monotone, transient behavior within the funnel is guaranteed, and the gain is not dynamically generated and does not invoke any internal model. While in adaptive control the output (or the output error) tends to 0 as $t \to \infty$, in funnel control we may guarantee only that $\limsup_{t \to \infty} |y(t)| < \lambda$; however, $\lambda > 0$ is prespecified and may be arbitrarily small.

Funnel control has been successfully applied in experiments controlling the speed of electric devices [18]; see also the survey [12] and references therein.

As in adaptive control, funnel control becomes more difficult if one has to cope with the obstacle of higher relative degree. Clearly, if $u(t) = -k\,y(t)$ is applied to a relative degree two system, take, for example, $\ddot{y}(t) = u(t)$; then the closed-loop system is not asymptotically stable. In [15, 16] an extension to systems with higher relative degree is provided; however, the controller in [15] involves a filter, the feedback strategy is dynamic, and the gain occurs with $k(t)^6$; see [15, Rem. 4 (ii), (iii)].

1.4. Contributions of the present paper. We introduce a funnel controller with *derivative feedback* to achieve output tracking of relative degree two systems where a funnel for each output error and *its derivative* is prespecified to shape the transient behavior. The funnel controller is simply

$$(1.7) \qquad \boxed{u(t) = -k_0(t)^2\, e(t) - k_1(t)\, \dot{e}(t),}$$

where $k_0(\cdot), k_1(\cdot)$ are defined analogously as in (1.6) with funnel boundaries $\psi_0(\cdot)$ and $\psi_1(\cdot)$, resp., and $e(t) = y(t) - y_{\text{ref}}(t)$ is the error between the output and some desired reference signal. This simple controller applied to linear single-input, single-output systems of form (1.1) with stable zero dynamics (1.3) and relative degree two (1.2) ensures that the error and its derivative evolve within the funnels, and all internal variables remain bounded. We can enlarge the system class to encompass also nonlinear systems described by functional differential equations, and we are able to show that this controller also works for systems with relative degree one; i.e., we can apply this controller also in the case where only the upper bound two is known for the relative degree. Moreover, if input constraints are present, then the funnel controller is applicable provided the saturation is larger than a feasibility number. We also show that the funnel controller is robust in terms of the gap metric. Finally, our results are applied to position control with two stiff coupled machines; experimental results are shown.

The paper is structured as follows. In section 2, we introduce the funnel and state the main result for linear systems with relative degree two. Further results are presented in section 3: in section 3.1, nonlinear systems; in section 3.2, that funnel controller (1.7) also works for relative degree one systems; in section 3.3, control in the presence of input saturation; and in section 3.4, robustness. The application to a laboratory setup of two stiff coupled machines is described in section 4. To improve readability, proofs are given in Appendix A; however, sketches of the intuitions are discussed in the subsections. We finalize this introduction with the following nomenclature:

$$
\begin{aligned}
|x| &= \sqrt{x^\top x}, \text{ the Euclidean norm of } x \in \mathbb{R}^n \\
|M| &= \max\left\{\, |M\,x| \mid x \in \mathbb{R}^m,\ |x| = 1 \,\right\}, \text{ induced matrix norm of} \\
&\qquad M \in \mathbb{R}^{n\times m}
\end{aligned}
$$

$\mathcal{L}^\infty(I \to M)$: the space of essentially bounded functions $y \colon I \to M \subseteq \mathbb{R}^\ell$, $I \subseteq \mathbb{R}$ some interval, with norm

$\|y\|_\infty := \|y\|_{\mathcal{L}^\infty} = \operatorname{ess\,sup}_{t \geq 0} |y(t)|$

$\mathcal{L}^\infty_{\mathrm{loc}}(I \to M)$ the space of locally bounded functions $y \colon I \to M \subseteq \mathbb{R}^\ell$, with $\operatorname{ess\,sup}_{t \in K} |y(t)| < \infty$ for all compact $K \subseteq I$

$\mathcal{W}^{i,\infty}(I \to M)$: the Sobolev space of i-times weakly differentiable functions $y \colon I \to M \subseteq \mathbb{R}^\ell$ such that $y, \ldots, y^{(i)} \in \mathcal{L}^\infty(I \to \mathbb{R}^\ell)$ and norm

$\|y\|_{\mathcal{W}^{i,\infty}} = \sum_{j=0}^{i} \|y^{(j)}\|_{\mathcal{L}^\infty}$, $i \in \mathbb{N}$

$\mathcal{W}^{i,\infty}_{\mathrm{loc}}(I \to M)$: the space of i-times weakly differentiable functions $y \colon I \to M \subseteq \mathbb{R}^\ell$ such that $y, \ldots, y^{(i)} \in \mathcal{L}^\infty_{\mathrm{loc}}(I \to \mathbb{R}^\ell)$

$\|\cdot\|_{\mathcal{L}^\infty \times \mathcal{W}^{i,\infty}}$: some product norm on the product space $\mathcal{L}^\infty(I \to M) \times \mathcal{W}^{i,\infty}(I \to M)$

$\mathcal{C}^i(I \to M)$: the space of i-times continuously differentiable functions $y \colon I \to M \subseteq \mathbb{R}^\ell$

Note that $y \in \mathcal{W}^{i,\infty}_{(\mathrm{loc})}(I \to M)$ implies that $y^{(i-1)}$ is absolutely continuous. Furthermore, we consider solutions of differential equations in the sense of Carathéodory (see, e.g., [9, sect. 2.1.2]), and "a.a." stands for "almost all."

2. Funnel control for linear systems with relative degree two.

2.1. The performance funnels. The central ingredient of our approach is the concept of two performance funnels within which the tracking error $e = y - y_{\mathrm{ref}}$ and its derivative $\dot{e}$ are required to evolve; y_{ref} denotes a reference signal. A funnel

$$
\mathcal{F}_\varphi := \left\{\, (t,\eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid \varphi(t)|\eta| < 1 \,\right\}
$$

is determined by a function φ belonging to the class

$$
\mathcal{G}_1 := \left\{\, \varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \ \middle|\
\begin{array}{l}
\varphi \text{ is absolutely continuous } \forall\, t > 0 : \varphi(t) > 0 \text{ and} \\
\exists\, \lambda > 0\ \forall\, \varepsilon > 0 :\ 1/\varphi\big|_{[\varepsilon,\infty)} \in \mathcal{W}^{1,\infty}\big([\varepsilon,\infty) \to [\lambda,\infty)\big)
\end{array}
\right\}.
$$

Note that the funnel boundary is given by the *reciprocal* of φ. This formulation allows for $\varphi(0) = 0$ which, by $0 = \varphi(0)|e(0)| < 1$, puts no restriction on the initial value, and hence we are able to prove global results. In the presence of input saturations we cannot allow for arbitrary initial values, and hence we consider the class of *finite funnels*

$$
\mathcal{G}_1^{\mathrm{fin}} := \left\{\, \varphi \in \mathcal{G}_1 \mid 1/\varphi \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}) \,\right\}.
$$

An important property of the funnel class $\mathcal{G}_1$ is that each funnel $\mathcal{F}_\varphi$ with $\varphi \in \mathcal{G}_1$ is *bounded away from zero*; i.e., there exists λ (depending on φ) such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. This condition is equivalent to the assumption that φ is bounded, which should not be confused with the assumption that $1/\varphi$ is bounded corresponding to finite funnels in $\mathcal{G}_1^{\mathrm{fin}}$. Two typical funnels are illustrated in Figure 2.

As indicated in Figure 2, we do not assume that the funnel boundary decreases monotonically; while in most situations the control designer will choose a monotone funnel, there are situations where widening the funnel at some later time might be
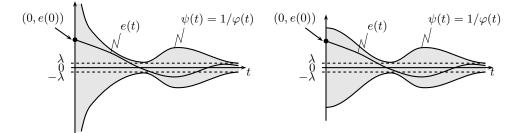
FIG. 2. *Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\psi(t) = 1/\varphi(t)$ for $t > 0$. Left: general funnel case $\varphi \in \mathcal{G}_1$. Right: a finite funnel $\varphi \in \mathcal{G}_1^{\mathrm{fin}}$.*

beneficial, e.g., when the reference signal changes strongly or the system is perturbed by some calibration so that a large error would enforce a large control action.

As mentioned above, we consider two funnels: one for the error and one for its derivative. The main control objective is to keep the error signal within prespecified error bounds, i.e., within some funnel. In order to achieve this control objective, we introduce a second funnel for the derivative of the error. This "derivative funnel" might originate in physical bounds on the derivative of the error or could be seen as a controller design parameter. If the error evolves within the funnel $\mathcal{F}_\varphi$ for some $\varphi \in \mathcal{G}_1$, then the derivative of the error eventually has to fulfill

$$\dot{e}(t) < \frac{\mathrm{d}}{\mathrm{d}t}(1/\varphi)(t) \quad \text{or} \quad \dot{e}(t) > -\frac{\mathrm{d}}{\mathrm{d}t}(1/\varphi)(t);$$

i.e., at some time the error must decrease faster than the upper funnel boundary grows smaller, or the error must increase faster than the lower funnel boundary grows larger. This implies that the derivative funnel must be *large enough* to allow the error to follow the funnel boundaries. Therefore, we consider the following family of tuples (φ_0, φ_1):

$$\mathcal{G}_2 := \left\{ (\varphi_0, \varphi_1) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{l} \varphi_0, \varphi_1 \in \mathcal{G}_1 \text{ and } \exists\, \delta \text{ such that} \\ \text{for a.a. } t > 0 : \; 1/\varphi_1(t) \geq \delta - \frac{\mathrm{d}}{\mathrm{d}t}(1/\varphi_0)(t) \end{array} \right\}$$

with corresponding funnel $\mathcal{F}_{\varphi_0}$ for the error and $\mathcal{F}_{\varphi_1}$ for the derivative of the error. The finite version $\mathcal{G}_2^{\mathrm{fin}}$ is defined in the same way as $\mathcal{G}_2$ by replacing $\mathcal{G}_1$ with $\mathcal{G}_1^{\mathrm{fin}}$ in the definition.
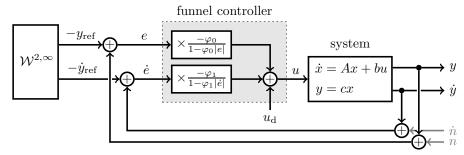

FIG. 3. *Closed-loop system* (1.1), (2.2) *subject to input disturbances u_d and measurement noise n; for the latter see Remark* 2.2.

2.2. Funnel control for linear systems with relative degree two. In this section we show funnel control for linear systems with relative degree two and stable

zero dynamics. This result is fundamental for various generalizations and aspects considered in section 3.

THEOREM 2.1 (funnel control for linear systems with relative degree two). *Consider linear systems* (1.1) *with relative degree two and positive high-frequency gain, i.e.,* (1.2), *and asymptotically stable zero dynamics, i.e.,* (1.3). *Let* $y_{\mathrm{ref}} \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *be a reference signal,* $u_{\mathrm{d}} \in \mathcal{L}^{\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *an input disturbance,* $(\mathcal{F}_{\varphi_0}, \mathcal{F}_{\varphi_1})$ *a pair of funnels for* $(\varphi_0, \varphi_1) \in \mathcal{G}_2$, *and* $x^0 \in \mathbb{R}^n$ *an initial value such that*

$$(2.1) \qquad \varphi_0(0)\,|y_{\mathrm{ref}}(0) - cx^0| < 1 \quad and \quad \varphi_1(0)\,|\dot{y}_{\mathrm{ref}}(0) - cAx^0| < 1.$$

Then the funnel controller

$$(2.2) \qquad \boxed{\begin{aligned} u(t) &= -k_0(t)^2\,e(t) - k_1(t)\,\dot{e}(t) + u_{\mathrm{d}}(t), \qquad e(t) = y(t) - y_{\mathrm{ref}}(t), \\ k_i(t) &= \frac{\varphi_i(t)}{1 - \varphi_i(t)|e^{(i)}(t)|}, \qquad\qquad\qquad\quad i = 0, 1, \end{aligned}}$$

applied to (1.1) *yields a closed-loop system, as shown in Figure* 3 *with the following properties:*

(i) *Precisely one maximal solution* $x : [0, \omega) \to \mathbb{R}^n$ *exists, and this solution is global (i.e.,* $\omega = \infty$); *in particular, the error and its derivative evolve within the corresponding funnels:*

$$\forall\, t \geq 0 : \ (t, e(t)) \in \mathcal{F}_{\varphi_0} \quad and \quad (t, \dot{e}(t)) \in \mathcal{F}_{\varphi_1}.$$

(ii) *The input* $u(\cdot)$ *and the gain functions* $k_0(\cdot)$, $k_1(\cdot)$ *are uniformly bounded.*

(iii) *The solution* $x(\cdot)$ *and its derivative are uniformly bounded; furthermore, the signals* $e(\cdot)$, $\dot{e}(\cdot)$ *are uniformly bounded away from the funnel boundaries:*

$$(2.3) \qquad \forall\, i \in \{0,1\}\ \exists\, \varepsilon_i > 0,\ \forall\, t > 0 : \ 1/\varphi_i(t) - |e^{(i)}(t)| \geq \varepsilon_i.$$

The proof is in Appendix A; however, we sketch its main ideas in the following.

First, assume without loss of generality, that the funnels are finite: $\varphi_0,\ \varphi_1 \in \mathcal{G}_1^{\mathrm{fin}}$; otherwise there will exist a local solution on $[0, \varepsilon)$, and we may consider the problem on the interval $[\varepsilon/2, \infty)$ instead of $[0, \infty)$. Therefore, $\psi_i := 1/\varphi_i$ denotes the finite funnel boundaries of $\mathcal{F}_{\varphi_i}$, $i = 0, 1$. To simplify the arguments, we assume that the derivatives of absolutely continuous functions are defined everywhere. Finally, we restrict our attention to positive errors $e(t)$; the negative case follows analogously.

In section 1.1 we have, although in a time-invariant setup, motivated the gains $k_0(t)^2$ for $e(t)$ (squared!) and $k_1(t)$ for $\dot{e}(t)$.

The standard theory of ordinary differential equations guarantees existence and uniqueness of a solution $x(\cdot)$ of (1.1) on $[0, \omega)$ for some maximal $\omega \in (0, \infty]$. Since e and $\dot{e}$ are bounded (they evolve within bounded funnels), the minimum phase condition (1.3) yields that z is bounded, so there exists a constant $M > 0$ such that

$$(2.4) \qquad \ddot{e}(t) \leq M + \gamma u(t) \quad \forall\, t \in [0, \omega).$$

In particular, if $u(t) \ll 0$, then $\ddot{e}(t) \ll 0$. If we knew that the product $k_0(\cdot)^2 e(\cdot)$ in the control law (2.2) is bounded, then it would follow from (2.4) that $\dot{e}$ remains bounded away from the boundaries of the funnel $\mathcal{F}_1$ because we were able to choose $\varepsilon_1 > 0$ in such a way that the following implications hold for all $t \in [0, \omega)$: $\dot{e}(t) = \psi_1(t) - \varepsilon_1 \implies \ddot{e}(t) < \dot{\psi}_1(t)$ and $\dot{e}(t) = -\psi_1(t) + \varepsilon_1 \implies \ddot{e}(t) > -\dot{\psi}_1(t)$. Hence, it suffices to prove
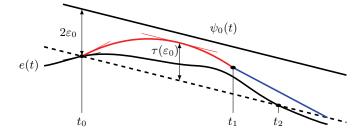
FIG. 4. *Illustration of the main idea of the proof of Theorem* 2.1 *showing the parabolic phase on* $[t_0, t_1)$ *and the linear phase on* $[t_1, t_2)$.

that k_0 is bounded or, equivalently, that e is uniformly bounded away from the funnel boundary; i.e., there must exist $\varepsilon_0 > 0$ such that $|e(t)| \leq \psi(t) - \varepsilon_0$ for all $t \in [0, \omega)$. This is the key step of the proof; it is illustrated in Figure 4 and goes as follows.

Consider, for some "small" ε_0, $t_0 \geq 0$ such that $e(t_0) = \psi_0(t_0) - 2\varepsilon_0$ and $e(t) < \psi_0(t) - 2\varepsilon_0$ for some $t < t_0$. Then we show that there exists $\tau(\varepsilon_0) > 0$ such that $e(t) \leq \psi_0(t) - 2\varepsilon_0 + \tau(\varepsilon_0)$ for $t > t_0$ and that $\tau(\varepsilon_0)/\varepsilon_0 \to 0$ as $\varepsilon_0 \to 0$. This implies that, for sufficiently small $\varepsilon_0 > 0$ and all $t \geq 0$, it follows that $e(t) \leq \psi_0(t) - \varepsilon_0$. We show that the following three properties hold:

- *Parabolic phase* on $[t_0, t_1)$: $\boxed{\ddot{e}(t) < -\overline{M}(\varepsilon_0)}$ for some $\overline{M}(\varepsilon_0) > 0$ with $\overline{M}(\varepsilon_0) \to \infty$ as $\varepsilon_0 \to 0$.
- *Linear phase* on $[t_1, t_2)$: $\boxed{\dot{e}(t) < \dot{\psi}_0(t)}$.
- Once in the linear phase, we remain in it until $e(t) < \psi_0(t) - 2\varepsilon_0$.

The parabolic phase is characterized by $\boxed{(\mathrm{P}): \ \dot{e}(t) \geq -\psi_1(t) + \delta/2}$, where $\delta > 0$ is given in the definition of $\mathcal{G}_2$. The linear phase is characterized by

$$\boxed{(\mathrm{L}1): \ e(t) \leq \psi_0(t) - 2\varepsilon_0 + \tau(\varepsilon_0)} \quad \text{and} \quad \boxed{(\mathrm{L}2): \ \dot{e}(t) \leq -\psi_1(t) + \delta/2};$$

additionally we may assume for both phases that $\boxed{(\mathrm{PL}): \ e(t) \geq \psi_0(t) - 2\varepsilon_0}$. Applying (PL) and (P) to the funnel controller (2.2) and for $2\varepsilon_0 \leq \lambda_0/2$, we obtain $u(t) < -\frac{1}{(2\varepsilon_0)^2}\frac{\lambda_0}{2} + \frac{1}{\delta/2}\|\psi_1\|_\infty + \|u_{\mathrm{d}}\|_\infty$, which, together with (2.4), yields the proposed property $\ddot{e}(t) < -\overline{M}(\varepsilon_0)$ of the parabolic phase, where $\overline{M}(\varepsilon_0) \to \infty$ as $\varepsilon_0 \to 0$. Hence the error is bounded by a *parabola* (recall $\dot{e}(t_0) \leq \|\psi_1\|_\infty$ and $e(t_0) \leq \|\psi_0\|_\infty$):

$$\forall\, t \in [t_0, t_1): \ e(t) < -\frac{\overline{M}(\varepsilon_0)}{2}(t - t_0)^2 + \dot{e}(t_0)(t - t_0) + e(t_0).$$

In particular, there exists a maximal "overshoot" $\tau(\varepsilon_0)$ of the error starting at $\psi_0(t_0) - 2\varepsilon_0$, and we can show that $\tau(\varepsilon_0)/\varepsilon_0 \to 0$ as $\varepsilon_0 \to 0$ (here we exploit the fact that the gain $k_0(\cdot)$ enters quadratically into the equation). The parabolic phase is active only as long as (P) holds; however, if (P) does not hold, then the property of $\mathcal{G}_2$ yields $\dot{e}(t) \leq -\psi_1(t) + \delta/2 < \dot{\psi}_0(t)$, which ensures that the distance between the error e and the funnel boundary ψ_0 increases. Finally, it can be shown that the parabolic phase is active for some time and the distance between the error and the funnel boundary gets bigger than $2\varepsilon_0$ either in this phase or in the subsequent linear phase. Altogether, by choosing ε_0 small such that $\tau(\varepsilon_0) \leq \varepsilon_0$, it follows that the error is uniformly bounded away from the funnel boundary with $e(t) \leq \psi_0(t) - \varepsilon_0$ for all $t \geq 0$.

REMARK 2.2 (measurement noise). *If system* (1.1) *is subject to measurement noise* $n(\cdot) \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$, *then the disturbed error signal is* $e = (y + n) - y_{\mathrm{ref}} = y - (y_{\mathrm{ref}} - n)$, *and the funnel controller tracks the disturbed reference signal* $y_{\mathrm{ref}} - n$. *Now Theorem* 2.1 *ensures that the disturbed error* e *and its derivative* $\dot{e}$ *remain within its funnels. Hence, the "real" error remains in the bigger funnel obtained by adding the corresponding bound of the noise to the funnel bounds used for the control.*

A remedy to suppress noise would be the introduction of a PI-extension as in [7, *Lem.* 3.4] *or* [18, *Figure* 2]. *To avoid technicalities, we omit it in the present paper.*

3. Nonlinear systems, systems of relative degree one or two, input saturations, and robustness. In this section, we show that the funnel controller (2.2) has far-reaching consequences. We show in section 3.1 that it is also applicable to a fairly large class of nonlinear strict relative degree two systems described by infinite-dimensional functional differential equations with weakly stable zero dynamics; in section 3.2 it is shown that the funnel controller is applicable whether or not the system is of relative degree one or two; in section 3.3 we show that the funnel controller copes with input saturations if a feasibility condition is satisfied; and in section 3.4 we show that the funnel controller is robust in terms of the gap metric.

3.1. Nonlinear and infinite-dimensional systems governed by functional differential equations. A careful inspection of the proof of Theorem 2.1 reveals that the essential property of system (1.1) is the existence of constants $M > 0$ and $\gamma > 0$ such that

$$(3.1) \qquad \forall t \geq 0: \; -M + \gamma u(t) \leq \ddot{e}(t) \leq M + \gamma u(t),$$

i.e., the property that a large u implies a large value for $\ddot{e}$ with the same sign. In the following (see also Figure 1(b)), we show that the funnel controller is therefore also applicable to a large class of nonlinear systems described by functional differential equations as long as (i) the system has strict relative degree two with positive high-frequency gain, (ii) it is in a certain Byrnes–Isidori form, (iii) the zero dynamics map bounded signals to bounded signals, and (iv) the operators involved are sufficiently smooth to guarantee local maximal existence of a solution of the close-loop system. We study the large class of infinite-dimensional nonlinear systems governed by functional differential equations with "memory" $h > 0$:

$$(3.2) \quad \ddot{y}(t) = f\big(p_f(t), T_f(y, \dot{y})(t)\big) + g\big(p_g(t), T_g(y, \dot{y})(t)\big)u(t),$$
$$y\big|_{[-h,0]} = y^0 \in \mathcal{W}^{1,\infty}([-h,0] \to \mathbb{R}),$$

where
- $p_f, p_g \in \mathcal{L}^{\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^P)$, $P \in \mathbb{N}$, are bounded disturbances;
- $f, g \in \mathcal{C}(\mathbb{R}^P \times \mathbb{R}^W \to \mathbb{R})$, $W \in \mathbb{N}$, such that for all $(p, w) \in \mathbb{R}^P \times \mathbb{R}^W$: $g(p, w) > 0$;
- $T_f, T_g : \mathcal{C}([-h, \infty) \to \mathbb{R}) \to \mathcal{L}^{\infty}_{\mathrm{loc}}\big([0, \infty) \to \mathbb{R}^W\big)$ are operators with the following properties, where $T = T_f$ and $T = T_g$, respectively:
  - T maps bounded trajectories to bounded trajectories; i.e., there exists a function $\alpha : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $y_0, y_1 \in \mathcal{L}^{\infty}([-h, \infty) \to \mathbb{R}) \cap \mathcal{C}([-h, \infty) \to \mathbb{R})$: $\|T(y_0, y_1)\|_{\infty} \leq \alpha(\|y_0\|_{\infty}, \|y_1\|_{\infty})$;
  - T is causal; i.e., for all $t \geq 0$, for all $\xi, \zeta \in \mathcal{C}([-h, \infty) \to \mathbb{R})^2$: $\xi\big|_{[-h,t)} = \zeta\big|_{[-h,t)} \implies T(\xi)\big|_{[0,t]} = T(\zeta)\big|_{[0,t]}$;

– T is "locally Lipschitz" continuous in the following sense: for all $t \geq 0$ $\exists \tau, \delta, c > 0$ such that for all $y_0, y_1, \Delta y_0, \Delta y_1 \in \mathcal{C}([-h, \infty) \to \mathbb{R})$ with $\Delta y_{0/1}\big|_{[-h,t]} \equiv 0$ and $\big\|(\Delta y_0, \Delta y_1)\big|_{[t,t+\tau]}\big\|_\infty < \delta$,

$$\big\|\big(T(y_0 + \Delta y_0, y_1 + \Delta y_1) - T(y_0, y_1)\big)\big|_{[t,t+\tau]}\big\|_\infty \leq c\,\big\|(\Delta y_0, \Delta y_1)\big|_{[t,t+\tau]}\big\|_\infty.$$

For relative degree one systems, the operators T_f, T_g and systems similar to (3.2) are well studied; see [28, 13, 14, 17], and see [16] for higher relative degree. In these references it is shown that system (3.2) encompasses linear system (1.1) with (1.2) and (1.3), and the generality of the operators T_f and T_g allows for infinite-dimensional linear systems, systems with hysteretic effects, systems with nonlinear delay elements, input-to-state stable (ISS) systems, and combinations thereof.

We state the nonlinear generalization of Theorem 2.1 for systems given by (3.2).

THEOREM 3.1 (funnel control for nonlinear functional differential equations with relative degree two). *Consider systems* (3.2). *Let* $y_{\mathrm{ref}} \in \mathcal{W}^{2,\infty}([0, \infty) \to \mathbb{R})$ *be a reference signal,* $u_{\mathrm{d}} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *an input disturbance,* $(\mathcal{F}_{\varphi_0}, \mathcal{F}_{\varphi_1})$ *a pair of funnels for* $(\varphi_0, \varphi_1) \in \mathcal{G}_2$, *and* $y^0 \in \mathcal{W}^{1,\infty}([-h, 0] \to \mathbb{R})$ *an initial trajectory such that*

(3.3) $$\varphi_0(0)\,|y_{\mathrm{ref}}(0) - y^0(0)| < 1 \quad and \quad \varphi_1(0)\,|\dot{y}_{\mathrm{ref}}(0) - \dot{y}^0(0)| < 1.$$

Then the funnel controller (2.2) *applied to* (3.2) *yields a closed-loop system which also satisfies properties* (i)–(iii) *of Theorem* 2.1.

The proof is in Appendix A.

Note that (3.2) may be written in block form as depicted in Figure 1(b).

Comparing the linear and the nonlinear case, i.e., Figure 1(a) and Figure 1(b), the zero dynamics captured by Σ_2 are now captured by T_f. In [19, sect. 4.1] it is shown that for nonlinear (as opposed to linear) systems of a relative degree two, the zero dynamics in the Byrnes–Isidori form are driven by y and $\dot{y}$ (not only by y). Now the weak condition that T_f is a bounded-input bounded-output operator allows the same design of the controller as in the linear case. The function g stands for the high-frequency gain (see γ in Figure 1(a)) and the assumptions on it ensures that it is uniformly bounded away from zero.

3.2. Linear systems with relative degree one. One may ask whether the funnel controller (2.2), which is designed for systems with relative degree two, also works for minimum-phase systems with relative degree one, i.e., (1.1) with (1.3) and $cb > 0$. The answer is affirmative.

THEOREM 3.2 (relative degree one case). *Consider linear system* (1.1) *with relative degree one and positive high-frequency gain, i.e.,* $cb > 0$, *and asymptotically stable zero dynamics, i.e.,* (1.3). *Let* $y_{\mathrm{ref}} \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *be a reference signal,* $u_{\mathrm{d}} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \cap \mathcal{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *an input disturbance,* $(\mathcal{F}_{\varphi_0}, \mathcal{F}_{\varphi_1})$ *a pair of funnels for* $(\varphi_0, \varphi_1) \in \mathcal{G}_2 \cap \mathcal{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R})^2$, $\varphi_1(0) = 0$, *and* $x^0 \in \mathbb{R}^n$ *an initial value such that* (2.1) *holds. Then the funnel controller* (2.2) *applied to* (1.1) *yields a closed-loop system which also satisfies properties* (i)–(iii) *of Theorem* 2.1.

The proof is in Appendix A.

The mathematical difficulty for application of the relative degree two funnel controller to a relative degree one system is as follows: Due to the derivative feedback, the resulting closed-loop system yields an *implicit* differential equation. To utilize the implicit function theorem to prove existence and uniqueness of solutions, we have to slightly restrict the class of allowed funnels and reference signals: φ_0, φ_1, and $\dot{y}_{\mathrm{ref}}$ are assumed to be continuously differentiable instead of just being absolutely continuous.

Additionally, we assume $\varphi_1(0) = 0$ for two reasons: (i) If $\varphi_1(0) > 0$, then $\dot{e}(0)$ has to fulfill $|\dot{e}(0)| < 1/\varphi_1(0)$, which might contradict the implicit differential equation. (ii) If $\varphi_1(0) = 0$, then $u(0)$ does not depend on $\dot{e}(0)$, and hence the implicit ordinary differential equation is explicit for $\dot{e}$ at $t = 0$, which yields existence and uniqueness of at least a local solution starting at $t = 0$.

3.3. Input saturation. In many practical applications, the input may be subject to certain bounds: say there is some maximal bound $\widehat{u} > 0$ such that $|u(t)| \leq \widehat{u}$ is required for all $t \geq 0$. In this case the funnel controller had to be replaced by

$$u(t) = \mathrm{sat}_{\widehat{u}} \big(- k_0(t)^2 e(t) - k_1(t)\dot{e}(t) + u_{\mathrm{d}}(t) \big)$$

with $e(\cdot)$, $k_0(\cdot)$, $k_1(\cdot)$ as in (2.2), and saturation function defined by $\mathrm{sat}_{\widehat{u}} \colon \mathbb{R} \to \{ w \in \mathbb{R} \mid |w| \leq \widehat{u} \}$, $v \mapsto \mathrm{sat}_{\widehat{u}}(v) := \mathrm{sgn}(v) \min\{|v|, \widehat{u}\}$. We will show that funnel control is also feasible in the presence of input constraints provided the saturation is larger than a certain feasibility number.

THEOREM 3.3 (funnel control with input saturation). *Suppose the linear system* (1.1) *has relative degree two with positive high-frequency gain* (1.2) *and asymptotically stable zero dynamics* (1.3), *and let* $Y_{\mathrm{ref}}, U_{\mathrm{d}}, \Psi_0, \Psi_1, \delta, \lambda_0, \lambda_1, X_0, \varepsilon_0^{x_0} \in \mathbb{R}_{>0}$. *Then there exists a feasibility number* $f_{\mathrm{feas}} > 0$ *such that, for any* $\widehat{u} \geq f_{\mathrm{feas}}$, *the saturated funnel controller*

$$(3.4) \qquad \boxed{\begin{aligned} u(t) &= \mathrm{sat}_{\widehat{u}} \big(- k_0(t)^2 e(t_0) - k_1(t)\dot{e}(t) + u_{\mathrm{d}}(t) \big), \qquad e(t) = y(t) - y_{\mathrm{ref}}(t), \\ k_i(t) &= \frac{\varphi_i(t)}{1 - \varphi_i(t)|e^{(i)}(t)|}, \qquad\qquad\qquad\qquad\qquad i = 0, 1, \end{aligned}}$$

applied to (1.1) *is applicable for all reference signals* $y_{\mathrm{ref}} \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *with* $\|y_{\mathrm{ref}}\|_{W^{2,\infty}} \leq Y_{\mathrm{ref}}$, *for all input disturbances* $u_{\mathrm{d}} \in \mathcal{L}^{\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *with* $\|u_{\mathrm{d}}\| \leq U_{\mathrm{d}}$, *for all pairs of finite funnels* $(\mathcal{F}_{\varphi_0}, \mathcal{F}_{\varphi_1})$ *with* $(\varphi_0, \varphi_1) \in \mathcal{G}_2^{\mathrm{fin}}$ *and* $\|1/\varphi_0\|_{W^{1,\infty}} \leq \Psi_0$, $\|1/\varphi_1\|_{W^{1,\infty}} \leq \Psi_1$, $\inf_{t\geq 0} 1/\varphi_0(t) \geq \lambda_0$, $\inf_{t\geq 0} 1/\varphi_1(t) \geq \lambda_1$, $\inf_{t\geq 0}(1/\varphi_0(t) + \frac{\mathrm{d}}{\mathrm{d}t}(1/\varphi_1)(t)) \geq \delta$, *and for all initial values* $x_0 \in \mathbb{R}^n$ *with* $\|x_0\| \leq X_0$ *and* $\psi(0) - |cx^0 - y_{\mathrm{ref}}(0)| \geq \varepsilon_0^{x_0}$; *i.e., the closed-loop system satisfies properties* (i)–(iii) *of Theorem* 2.1.

The proof is in Appendix A and uses arguments from the following.

As shown in Theorem 2.1, the input of the closed-loop system (1.1), (2.2) is bounded; however, in Theorem 3.3 we state that a saturated input yields the same result, provided the saturation bound is sufficiently large. In fact, we will show that the feasibility bound $f_{\mathrm{feas}} > 0$ depends on all parameters involved in the closed-loop system. In most cases the calculated f_{feas} may be very conservative; in applications of small dimension, it may be useful. However, for the position control problem considered in section 4, f_{feas} is already much larger than $\widehat{u}$ required in the experiments.

In the remainder of this section, we collect several bounds which in the end determine f_{feas}. This derivation has several consequences: (i) the bounds help us to understand the interplay between the two different "players" $k_0(\cdot)$ and $k_1(\cdot)$; (ii) if the entries of (1.1) are known, it may be possible to determine a sharper number f_{feas}; (iii) for simplicity we have considered only symmetric funnels, which is a rather hard assumption; this can be relaxed, and the feasibility bound becomes smaller; see [21] for a more detailed analysis in a comparable context.

In the following, we consider the closed-loop system (1.1), (3.4). Existence and uniqueness of a solution is treated in the proof of Theorem 3.3. Here we assume that a solution exists on the whole of $\mathbb{R}_{\geq 0}$, and we may also assume without loss of generality

that system (1.1) is in *Byrnes–Isidori form* as follows: see [19] and [16, Lem. 3.5] for an explicit derivation of the transformation which transforms (1.1) in Byrnes-Isidori form:

$$(3.5) \qquad \begin{aligned} \ddot{y}(t) &= r_0 y(t) + r_1 \dot{y}(t) + s^\top z(t) + \gamma u(t), \qquad \begin{pmatrix} y(0) \\ \dot{y}(0) \end{pmatrix} = \begin{pmatrix} cx^0 \\ cAx^0 \end{pmatrix}, \\ \dot{z}(t) &= p\,y + Q\,z, \qquad\qquad\qquad\qquad\qquad\qquad\quad z(0) = z^0, \end{aligned}$$

where $r_0, r_1 \in \mathbb{R}$, $s, p \in \mathbb{R}^{n-2}$, $Q \in \mathbb{R}^{(n-2)\times n(n-2)}$, $z^0 \in \mathbb{R}^{n-2}$. By (1.2), the high-frequency gain is $\gamma = cAb > 0$. For further analysis we need constants $Y_0^0, Y_0^1, Z_0 \in \mathbb{R}_{>0}$ such $|x_0| \leq X_0$ implies $|cx_0| \leq Y_0^0 \ \wedge\ |cAx_0| \leq Y_0^1 \ \wedge\ |z_0| \leq Z_0$.

3.3.1. A bound from the zero dynamics. Note that the minimum phase assumption (1.3) is equivalent to the matrix Q being *Hurwitz*, i.e.,

$$(3.6) \qquad\qquad \exists\, M_Q \geq 1 \ \exists\, \lambda_Q > 0 \ \forall\, t \geq 0: \ |\mathrm{e}^{Qt}| \leq M_Q \mathrm{e}^{-\lambda_Q t}.$$

Applying variations of constants to the second equation in (3.5) and taking norms yields, for all $t \geq 0$, $|z(t)| \leq M_Q \mathrm{e}^{-\lambda_Q t}|z^0| + \int_0^t M_Q \mathrm{e}^{-\lambda_Q(t-\tau)}|p|\,|y(\tau)|\,\mathrm{d}\tau \leq M_Q|z^0| + \frac{M_Q}{\lambda_Q}|p|\left[\|y_{\mathrm{ref}}\|_\infty + \|e|_{[0,t]}\|_\infty\right]$. Writing

$$(3.7) \qquad \begin{aligned} M_z &:= M_Q Z_0 + \frac{M_Q}{\lambda_Q}\,|p|\,[Y_{\mathrm{ref}} + \Psi_0], \\ M &:= |r_0|\Psi_0 + |r_1|\Psi_1 + |s^\top|M_z + \max\{|r_0|, |r_1|, 1\}Y_{\mathrm{ref}}, \end{aligned}$$

and observing that $\ddot{e}(t) = r_0 e(t) + r_1 \dot{e}(t) + s^\top z(t) + r_0 y_{\mathrm{ref}}(t) + r_1 \dot{y}_{\mathrm{ref}}(t) - \ddot{y}_{\mathrm{ref}}(t) + \gamma u(t)$, together with the fact that both e and $\dot{e}$ are bounded since they evolve within the bounded funnels, we conclude that the key inequality (3.1) holds.

3.3.2. Bounds from the parabolic phase. We consider the parabolic and linear phases as described in section 2.2 separately to determine a sufficiently large $\widehat{u}$. In the following we will consider only the case where the error e is positive; by symmetry the obtained bound will also be valid for negative errors. Choose $\varepsilon_0 > 0$ such that $2\varepsilon_0 \leq \frac{\lambda_0}{2}$, where $\lambda_0 := \inf_{t\geq 0} \psi_0(t) > 0$ and $2\varepsilon_0 \leq \psi(0) - |e(0)|$ (the latter is positive by (2.1)), and assume the parabolic phase is active on the interval $[t_0, t_1]$. Then, by (P) and (PL), $e(t) \geq \psi_0(t) - 2\varepsilon_0 \geq \lambda_0/2$, $\psi_1(t) > \dot{e}(t) \geq -\psi_1(t) + \delta/2$ for all $t \in [t_0, t_1)$, where $\delta > 0$ exists by definition of $\mathcal{G}_2$. Hence, for all $t \in [t_0, t_1)$,

$$(3.8) \qquad -k_0(t)^2 e(t) - k_1(t)\dot{e}(t) + u_d(t) < -\frac{\lambda_0}{8\varepsilon_0^2} + \frac{2\Psi_1}{\delta} + U_\mathrm{d} =: -U_{2\varepsilon_0},$$

and if $\widehat{u} \geq U_{2\varepsilon_0}$, we obtain by (3.1), which is proved in section 3.3.1, $e(t) < 1/2(M - \gamma U_{2\varepsilon_0})(t-t_0)^2 + \dot{e}(t_0)(t-t_0) + e(t_0)$. Since $e(t_0) = \psi(t_0) - 2\varepsilon_0$, we can easily obtain the following sufficient condition, which ensures that $e(t) \leq \psi_0(t) - \varepsilon_0$ for all $t \in [t_0, t_1)$:

$$(3.9) \qquad \frac{1}{2}(M - \gamma U_{2\varepsilon_0})(t - t_0)^2 + (\Psi_1 + \Psi_0)(t - t_0) - \varepsilon_0 \leq 0.$$

Under the assumption $M - \gamma U_{2\varepsilon_0} < 0$ or, equivalently, $\varepsilon_0 < \sqrt{\frac{\gamma \lambda_0}{M_0}}$, where

$$(3.10) \qquad\qquad M_0 := 8(M + 2\gamma\Psi_1/\delta + \gamma U_\mathrm{d}),$$

the parabola (3.9) obtains its maximum at $t_{\max} > t_0$, which is the solution of $(M - \gamma U_{2\varepsilon_0})(t_{\max} - t_0) + \Psi_1 + \Psi_0 = 0$. Basic calculations reveal that, with M_0 as in (3.10),

$$(3.11) \qquad 0 < \varepsilon_0 \leq \overline{\varepsilon}_0 := \frac{-2(\Psi_1 + \Psi_0)^2}{M_0} + \sqrt{\frac{\gamma\lambda_0}{M_0} + \frac{4(\Psi_1 + \Psi_0)^4}{M_0^2}} < \sqrt{\frac{\gamma\lambda_0}{M_0}},$$

together with $\widehat{u} \geq U_{2\varepsilon_0}$ ensures that $e(t) \leq \psi_0(t) - \varepsilon_0$ for all $t \in [t_0, t_1)$.

3.3.3. Bounds from the linear phase. It remains to consider the linear phase on $[t_1, t_2)$ characterized by $\psi_0(t) - 2\varepsilon_0 \leq e(t) \leq \psi_0(t) - \varepsilon_0$ and $\dot{e}(t) \leq -\psi_1(t) + \delta/2$ for all $t \in [t_1, t_2)$. Since $-\psi_1(t) + \delta/2 \leq \dot{\psi}_0(t) - \delta/2$ for almost all $t \geq 0$, the linear phase ensures $e(t) \leq \psi_0(t) - \varepsilon_0$ for all $t \in [t_1, t_2)$. Thus we have to find a sufficiently large $\widehat{u}$, which ensures that we remain in the linear phase until the distance between the error e and the funnel boundary ψ_0 is bigger than $2\varepsilon_0$. First observe that, for $2\varepsilon_0 \leq \lambda_0/2$, $\frac{\lambda_0}{8\varepsilon_0^2} - U_{\mathrm{d}} \leq k_0(t)^2 e(t) - u_{\mathrm{d}}(t) < \frac{\Psi_0}{\varepsilon_0^2} + U_{\mathrm{d}}$ for all $t \in [t_1, t_2)$, and thus the following implications hold for all $t \in [t_1, t_2)$ and all $\varepsilon_1 \in \big(0, \max\{\lambda_1/2, \delta/2\}\big]$, where $\lambda_1 := \inf_{t \geq 0} \psi_1(t)$:

$$\dot{e}(t) = -\psi_1(t) + \delta/2 \quad \Rightarrow \quad k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - u_{\mathrm{d}} > \frac{\lambda_0}{8\varepsilon_0^2} - U_{\mathrm{d}} - \frac{2\Psi_1}{\delta} = U_{2\varepsilon_0},$$

$$\dot{e}(t) = -\psi_1(t) + \varepsilon_1 \quad \Rightarrow \quad k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - u_{\mathrm{d}} < \frac{\Psi_0}{\varepsilon_0^2} + U_{\mathrm{d}} - \frac{\lambda_1/2}{\varepsilon_1} =: -U_{\varepsilon_0, \varepsilon_1}.$$

Clearly, by (3.1), the set $\{ (t, \dot{e}) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid -\psi_1(t) + \varepsilon_1 \leq \dot{e} \leq -\psi_1(t) + \delta/2 \} \subseteq \mathcal{F}_{\varphi_1}$ is positively invariant for small enough ε_0 and ε_1 (and corresponding large enough $\widehat{u} \geq \max\{U_{2\varepsilon_0}, U_{\varepsilon_0, \varepsilon_1}\}$); i.e., once in the linear phase, we remain there, and $\dot{e}$ is bounded away from the funnel boundary $-\psi_1$. In fact, with M_0 as in (3.10), and with M from (3.7), $\varepsilon_0 \leq \overline{\varepsilon_0}^* := \sqrt{\frac{\gamma\lambda_0}{M_0 + 8\Psi_1}}$ and $\varepsilon_1 \leq \overline{\varepsilon}_1(\varepsilon_0) := \frac{\gamma\lambda_1/2}{\Psi_1} + M + \frac{\gamma\Psi_0}{\varepsilon_0^2} + \gamma U_{\mathrm{d}}$, together with sufficiently large $\widehat{u}$ and (3.1), ensure that $\dot{e}(t) = -\psi_1(t) + \delta/2 \quad \Rightarrow \quad \ddot{e}(t) < -\|\dot{\psi}_1\|_\infty$, and $\dot{e}(t) = -\psi_1(t) + \varepsilon_1 \quad \Rightarrow \quad \ddot{e}(t) > \|\dot{\psi}_1\|_\infty$.

3.3.4. Feasibility number. In summary, if $\varepsilon_0^{\mathrm{m}} := \min\big\{\frac{\lambda_0}{4}, \frac{\varepsilon_0^{x_0}}{2}, \overline{\varepsilon}_0, \overline{\varepsilon}_0^*\big\}$, $\varepsilon_1^{\mathrm{m}} := \min\big\{\frac{\lambda_1}{2}, \frac{\delta}{2}, \overline{\varepsilon}_1(\varepsilon_0^{\mathrm{m}})\big\}$, and $f_{\mathrm{feas}} := \max\big\{\frac{\lambda_0}{8(\varepsilon_0^{\mathrm{m}})^2} - \frac{2\Psi_1}{\delta} - U_{\mathrm{d}}, \frac{\lambda_1}{2\varepsilon_1^{\mathrm{m}}} - \frac{\Psi_1}{(\varepsilon_0^{\mathrm{m}})^2} - U_{\mathrm{d}}\big\}$, then funnel control (3.4) with saturation is applicable if the saturation is larger than the feasibility number $\widehat{u} \geq f_{\mathrm{feas}}$.

3.4. Robustness in the sense of the gap metric. We now study robustness of the funnel controller (2.2) in terms of the gap metric [36]; see also [26] and the references therein.

Define the class of nominal systems of form (1.1) with asymptotically stable zero dynamics and relative degree two with positive high-frequency gain by

$$\mathcal{P} := \big\{ (A, b, c) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n \times \mathbb{R}^{1 \times n} \mid n \in \mathbb{N}, (A, b, c) \text{ satisfies (1.2) and (1.3)} \big\}.$$

Clearly, the funnel controller, as a universal controller, is already robust for disturbed systems within the class $\mathcal{P}$. Additionally, the aim of this section is to study robustness also for disturbances of a nominal plant $\theta = (A, b, c) \in \mathcal{P}$ which yield a disturbed plant $\widetilde{\theta} = (\widetilde{A}, \widetilde{b}, \widetilde{c}) \notin \mathcal{P}$. We will give sufficient conditions in terms of the gap metric for the funnel controller (2.2) to achieve the control objective if applied to a disturbed system belonging to the more general systems class

$$\widetilde{\mathcal{P}} := \left\{ (\widetilde{A}, \widetilde{b}, \widetilde{c}) \in \mathbb{R}^{q \times q} \times \mathbb{R}^q \times \mathbb{R}^{1 \times q} \;\middle|\; \begin{array}{l} q \in \mathbb{N}, \; (\widetilde{A}, \widetilde{b}, \widetilde{c}) \text{ is stabilizable} \\ \text{and detectable}, \; \widetilde{c}\widetilde{b} = 0 \end{array} \right\} \supsetneq \mathcal{P}.$$

In particular, the disturbance of the nominal plant can yield a plant which has a different state space dimension, has a higher relative degree than two, does not have a positive high-frequency gain, and/or is not minimum phase. Note that we do not consider disturbances which yield a relative degree one system; the reason for this is twofold: (i) due to the implicit nature of the resulting closed-loop system, we were not able to prove the general robustness result for $cb < 0$, and (ii) we have already shown in section 3.2 that the funnel controller works for any minimum-phase, relative degree one system with positive high-frequency gain.

In order to define the gap metric between plants in $\widetilde{\mathcal{P}}$, we first have to introduce the *plant operator* associated to $\theta = (A, b, c) \in \widetilde{\mathcal{P}}$ as follows:

$$(3.12) \qquad P_{\theta,x^0} \colon \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \to \mathcal{W}_{\mathrm{loc}}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}), \quad u \mapsto y,$$

where $x^0 \in \mathbb{R}^{\dim \theta}$, $\dim \theta$ is such that $A \in \mathbb{R}^{\dim \theta \times \dim \theta}$, and y is the unique output of the initial-value problem $\dot{x} = Ax + bu$, $x(0) = x^0$, $y = cx$. Since $cb = 0$, it is easy to see that P_{θ,x^0} is well defined and causal; i.e., for all $u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ it follows that the corresponding output fulfills $y \in \mathcal{W}_{\mathrm{loc}}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$, and $y\big|_{[0,\tau)}$ does not depend on $u\big|_{[\tau,\sup \mathrm{dom}\, u)}$ for all $\tau \in \mathrm{dom}\, u$. With abuse of notation, we write $P \in \widetilde{\mathcal{P}}$ if there exists $\theta \in \widetilde{\mathcal{P}}$ and $x^0 \in \mathbb{R}^{\dim \theta}$ such that $P = P_{\theta,x^0}$. For $P \in \widetilde{\mathcal{P}}$ define the *graph of* P as $\mathcal{G}_P := \big\{ (u, P(u)) \mid u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}),\ P(u) \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \big\}$.

We are now able to define the gap metric of two systems in $\widetilde{\mathcal{P}}$.

DEFINITION 3.4 (directed gap metric [6]). *For $P_1, P_2 \in \widetilde{\mathcal{P}}$ define the (possibly empty) set $\mathcal{O}_{P_1,P_2} := \big\{ \Phi : \mathcal{G}_{P_1} \to \mathcal{G}_{P_2} \mid \Phi \text{ is causal and surjective and } \Phi(0) = 0 \big\}$. The* directed gap *is given by*

$$\vec{\delta}(P_1, P_2) := \inf_{\Phi \in \mathcal{O}_{P_1,P_2}} \sup \left\{ \frac{\|(\Phi - I)(x)\|_{\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}}}{\|x\|_{\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}}} \ \middle| \ x \in \mathcal{G}_{P_1},\ \|x\|_{\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}} > 0 \right\},$$

with the convention that $\vec{\delta}(P_1, P_2) := \infty$ if $\mathcal{O}_{P_1,P_2} = \emptyset$.

Note that we here define the system graphs and the gap metric in the signal space setting of Theorem 2.1, i.e., $\mathcal{G}_P \subset \mathcal{L}^\infty \times \mathcal{W}^{2,\infty}$. It is also possible to define the system graphs and the gap metric, respectively, in different signal space settings. This may simplify the calculation of upper bounds for the gap metric. For an example of when systems are "close" in the gap metric, we refer the reader to [26, sect. 6.3.1]: the systems considered there may be extended to the case of a nominal system with relative degree two and a "disturbed system" with, e.g., relative degree three; see also Appendix B.

We are now ready to state the main robustness result. Note that we have to assume that the funnels are *not finite* at $t = 0$, the reason being that in the analysis we study the plant and controller as operators on certain signal spaces separately. In particular, the (bounded) signals can have arbitrarily big bounds, and if the funnels are finite at $t = 0$, we could in general not guarantee existence of a local solution for large "inputs" to the funnel controller operator because the initial values might not be contained within the funnels.

THEOREM 3.5 (robustness of the funnel controller). *Consider the funnel controller* (2.2) *with infinite funnels $\mathcal{F}_{\varphi_0}, \mathcal{F}_{\varphi_1}$, $(\varphi_0, \varphi_1) \in \mathcal{G}_2 \backslash \mathcal{G}_2^{\mathrm{fin}}$, input disturbance $u_\mathrm{d} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$, and reference signal $y_\mathrm{ref} \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Let $\theta \in \mathcal{P}$ be a nominal system with associated zero-initial-value plant operator $P_{\theta,0}$ given by* (3.12).

Then there exist functions $\eta : (0, \infty) \to (0, \infty)$ *and* $\alpha : \widetilde{\mathcal{P}} \to (0, \infty)$ *such that, for* $\widetilde{\theta} \in \widetilde{\mathcal{P}}$, $\widetilde{x}^0 \in \mathbb{R}^{\dim \widetilde{\theta}}$, *and* $r > 0$,

$$(3.13) \qquad \alpha(\widetilde{\theta})|\widetilde{x}^0| + \|(u_{\mathrm{d}}, y_{\mathrm{ref}})\|_{\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}} \leq r \quad \wedge \quad \vec{\delta}(P_{\theta,0}, P_{\widetilde{\theta},0}) \leq \eta(r)$$

implies that the closed loop of disturbed plant $P_{\widetilde{\theta},\widetilde{x}^0}$ *and funnel controller* (2.2) *works; that means properties* (i)–(iii) *of Theorem* 2.1 *hold.*

The proof is in Appendix A.

Theorem 3.5 also holds for $u_{\mathrm{d}} \in \mathcal{W}_0^{1,\infty}$, where $\mathcal{W}_0^{1,\infty}$ are all $\mathcal{W}^{1,\infty}$-functions, which are 0 at time 0, and $y_{\mathrm{ref}} \in \mathcal{W}^{2,\infty}$, which allows a simpler calculation of upper bounds for the gap metric; see Example B.1.

REMARK 3.6. *Assume we are given an input disturbance* u_{d} *and a reference signal* y_{ref} *with* $\|(u_{\mathrm{d}}, y_{\mathrm{ref}})\|_{\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}} \leq C$ *for some* $C > 0$, *and choose* $r > C$. *Then Theorem* 3.5 *ensures that for any disturbed plant* $\widetilde{\theta} \in \widetilde{\mathcal{P}}$ *which is "close enough" to the nominal plant* $\theta \in \mathcal{P}$ *in the sense that the directed gap metric is smaller than* $\eta(r)$, *the funnel controller will also work for the disturbed plant* $P_{\widetilde{\theta},\widetilde{x}^0} \in \widetilde{\mathcal{P}}$, *provided the initial value* $\widetilde{x}^0$ *of* $P_{\widetilde{\theta},\widetilde{x}^0}$ *is "small enough" in the sense that* $|\widetilde{x}^0| < (r - C)/\alpha(\widetilde{\theta})$. *In summary, if the directed gap between two plants is small enough and the initial value is small enough, then the funnel controller also works for the disturbed plant.*

4. Experimental results. In this section we consider a simple rotatory model for the standard position control problem and will apply the funnel controller to a laboratory setup of two stiff coupled machines; see Figure 5(a).
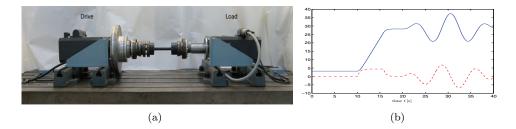

(a)   (b)

FIG. 5. (a) *Laboratory setup of rotatory system: stiff coupled machines (drive and load).* (b) *Reference signal for experiment:* —— $y_{\mathrm{ref}}(\cdot)\,[rad]$, - - $\dot{y}_{\mathrm{ref}}(\cdot)\,[rad/s]$.

4.1. Standard position control problem. The mathematical model of a rotatory system (a translational system is similar) with actuator for position control is given by

$$(4.1) \qquad \frac{\mathrm{d}}{\mathrm{d}t}x(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{\Theta} \end{pmatrix} \left(\mathrm{sat}_{\widehat{u}_A}(u(t) + u_A(t)) - u_L(t) - (Tx_2)(t) \right),$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t), \qquad\qquad\qquad x(0) = \left(\phi^0, \Omega^0 \right)^\top,$$

where the state variable $x(t) = (\phi(t), \Omega(t))^\top$ represents angle $\phi(t)$ and angular velocity $\Omega(t) = \dot{\phi}(t)$ at time $t \geq 0$ in $[rad]$ and $[rad/s]$, respectively.

In the "real world," the drive (or load) torque is generated by a saturated actuator comprising an inverter and machine (with current/torque control-loop) that is a nonlinear dynamical system. Since its dynamics are very fast, e.g., $u(t) \approx \mathrm{sat}_{\widehat{u}_A}(u(t) + u_A(t))$ for $|u(t) + u_A(t)| \leq \widehat{u}_A$ (see, e.g., [29, pp. 775–779]), we model the actuator by

the (small) disturbance $u_A \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) [Nm]$ and the saturation $\mathrm{sat}_{\widehat{u}_A}(\cdot)$ with $\widehat{u}_A > 0 [Nm]$. The input $u(\cdot) [Nm]$ represents the "desired" drive torque. It is additionally corrupted by an external load disturbance $u_L \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) [Nm]$ and by friction, which is modeled by an (unbounded) operator $T \colon \Omega(\cdot) \mapsto (T\Omega)(\cdot) [Nm]$. The friction operator $T \colon \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \to \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ maps angular velocity to the friction torque, covers viscous and dynamic friction effects, and allows for characterization of, e.g., the (nonlinear and dynamic) Lund–Grenoble friction model [4]. We omit further explanation of the friction model here; however, we refer the reader to [27, Chap. 3] and [18]; see also Appendix C.

The moment of inertia $\Theta > 0 [kg\,m^2]$ is a constant, the reciprocal of which is the high-frequency gain $\gamma := (1,0)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} 0 \\ 1/\Theta \end{pmatrix} = 1/\Theta > 0$. The influence of gears and elasticity in the shaft is neglected. Note that if a gear is applied and yields a negative high-frequency gain, then the gains $k_0(t)^2$ and $k_1(t)$ in funnel controller (2.2) have to be modified to $-k_0(t)^2$ and $-k_1(t)$, respectively, and the same results hold.

The output $y(\cdot) = \phi(\cdot)$ and its derivative $\dot{y}(\cdot) = \Omega(\cdot)$ are available for feedback and corrupted by measurement noise $n \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. The control objective is tracking of a reference signal $y_{\mathrm{ref}}(\cdot)$ and its derivative in the presence of input constraints; see Figure 5(b). Although in many applications derivative feedback is a problem, in the present setup of stiff coupled machines, or more generally in joint position control of robotics, it may be justified; see, e.g., [30, pp. 210–213 and 290–292].

Letting

$$f(p,w) := \frac{1}{\Theta}(p+w), \quad p(t) := -u_L(t) + u_A(t), \quad g(p,w) := \frac{1}{\Theta}, \quad T_f(y,\dot{y}) := -T(\dot{y}),$$

and $h := 0$, (4.1) without saturation reads as (3.2) and fulfills all its properties. Hence, for any funnels $\mathcal{F}_{\varphi_0}$ and $\mathcal{F}_{\varphi_1}$ with $(\varphi_0, \varphi_1) \in \mathcal{G}_2$, Theorem 3.1 ensures existence of a global solution $x \colon [0,\infty) \to \mathbb{R}^2$ of the closed-loop of the unsaturated system (4.1) and funnel controller (2.2); in particular, y and its derivative $\dot{y}$ evolve within the funnels $\mathcal{F}_{\varphi_0}$ and $\mathcal{F}_{\varphi_1}$ around the reference signal y_{ref} and its derivative $\dot{y}_{\mathrm{ref}}$, respectively.

Furthermore, if $(\varphi_0, \varphi_1) \in \mathcal{G}_2^{\mathrm{fin}}$, we can also show (3.1) for $\gamma = 1/\Theta$ and
(4.2)
$$M = \|\ddot{y}_{\mathrm{ref}}\|_\infty + \gamma\Big(\|u_L\|_\infty + \sup\{\,\|T(\Omega)\|_\infty \mid \Omega \in \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}) : \|\Omega\|_\infty \leq \|\psi_1\|_\infty \,\}\Big).$$

Note that Theorem 3.3 also holds for nonlinear systems with relative degree two if (1.1) is replaced by (3.2) (and the corresponding properties described in section 3.1) and (2.1) is replaced by (3.3). Therefore, properties (i)–(iii) of Theorem 2.1 hold in the presence of input saturations for (4.1).

4.2. Controller and funnel design. We are now ready to apply the saturated funnel controller (3.4) to the stiff coupled machines in the laboratory; see Figure 5(a). We have introduced a saturation with $\widehat{u} > 0$ to prevent destruction of the actuator and for safety reasons. The considered reference signal $y_{\mathrm{ref}} \colon [0,T] \to \mathbb{R}$ with $T = 40 [s]$ for the experiment is shown in Figure 5(b).

The functions $(\varphi_0, \varphi_1) \in \mathcal{G}_2^{\mathrm{fin}}$ determine the funnels $\mathcal{F}_{\varphi_0}$, $\mathcal{F}_{\varphi_1}$ and their reciprocals by
(4.3)
$$\psi_0(t) := (\Lambda_0 - \lambda_0)\exp\left(-t/T_E\right) + \lambda_0, \quad \psi_1(t) := -\dot{\psi}_0(t) + \lambda_1, \quad \Lambda_0 \geq \lambda_0 > 0,\ \lambda_1, T_E > 0,$$

respectively. Note that $\psi_0, \psi_1 \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{>0})$ with $\|\psi_0\|_\infty = \Lambda_0$, $\|\dot{\psi}_0\|_\infty =$

$(\Lambda_0 - \lambda_0)/T_E$, $\|\psi_1\|_\infty = (\Lambda_0 - \lambda_0)/T_E + \lambda_1$, $\|\dot{\psi}_1\|_\infty = (\Lambda_0 - \lambda_0)/T_E^2$, and, furthermore, $\inf_{t \geq 0} \psi_0(t) = \lambda_0$ and $\inf_{t \geq 0} \psi_1(t) = \lambda_1$.

To check the feasibility condition in Theorem 3.3, we collect the implementation, design, and system data in Table 4.1.

TABLE 4.1
Systems, implementation, and controller design data.

Description	Symbol(s) & Value(s)	Dimension(s)
Moment of inertia	$\Theta = 0.342$	$[kg\,m^2]$
Gain (assumed bounds)	$\gamma = 1/\Theta = 2.924$ ($\gamma_{min} = \gamma/2$, $\gamma_{max} = 3\gamma$)	$[rad/(Nms^2)]$
Initial values	$\phi^0 = 0$, $\Omega^0 = 0$	$[rad]$, $[rad/s]$
Initial reference values	$y_{\text{ref}}(0) = \pi$, $\dot{y}_{\text{ref}}(0) = 0$	$[rad]$, $[rad/s]$
Input saturation	$\widehat{u} = 7.0$ (chosen), $\widehat{u}_A = 22.0$ (specified)	$[Nm]$
Disturbance bounds	$\|u_A\|_\infty \leq 0.56$ (measured), $\|u_L\|_\infty \leq 4.0$	$[Nm]$
Measured noise bounds	$\|n\|_\infty \leq 5.8 \cdot 10^{-5}$, $\|\dot{n}\|_\infty \leq 0.024$	$[rad]$, $[rad/s]$
Reference bounds	$\|y_{\text{ref}}\|_\infty = 37.37$, $\|\dot{y}_{\text{ref}}\|_\infty = 6.81$, $\|\ddot{y}_{\text{ref}}\|_\infty = 6.05$	$[rad]$, $[\frac{rad}{s}]$, $[\frac{rad}{s^2}]$
Initial boundary values	$\psi_0(0) = \Lambda_0 = 2\pi$, $\psi_1(0) = 8.853$	$[rad]$, $[rad/s]$
Time constant	$T_E = 0.8189$	$[s]$
Asymptotic accuracies	$\lambda_0 = 0.2618$, $\lambda_1 = 1.5$	$[rad]$, $[rad/s]$
Sampling time (xPC)	$h = 1 \cdot 10^{-3}$	$[s]$

By Theorem 3.3 and neglecting (unknown) friction $T\Omega$ in (4.2), we conclude that $M = 41.14$, $M_0 = 473.72$, $\overline{\varepsilon}_0^* = 0.0265$, $\varepsilon_0^m = \overline{\varepsilon}_0 = 3.64 \cdot 10^{-4}$, $\overline{\varepsilon}_1(\varepsilon_0^m) = \varepsilon_1^m = 1.58 \cdot 10^{-8}$, where we used, based on worst case analysis, $\gamma = \gamma_{\max}$ for calculating M and $\gamma = \gamma_{\min}$ in (3.1) and hence in the rest of the calculation. Finally, the feasibility numbers are $\widehat{u}_A \geq 2.466 \cdot 10^5\,[Nm]$ and $\widehat{u} \geq 2.466 \cdot 10^5 + \|u_A\|_\infty \approx 2.467 \cdot 10^5\,[Nm]$. This computed lower bound of $\widehat{u}$ is very large and unrealistic compared to the actual required maximal torque of approximately $7.0\,[Nm]$ (see Figure 6(c)); it demonstrates how conservative the feasibility bound of Theorem 3.3 can be.

Finally, we illustrate the application of the funnel controller to the laboratory setup of two permanent magnetic synchronous machines, two power inverters, a remote host for monitoring, and a real-time xPC target rapid-prototyping system. Figure 5(a) depicts the coupled machines—drive and load. Both machines and inverters are identical in construction. Each machine is driven by its own power inverter. The actuators generate the torques $u(\cdot) + u_A(\cdot)$ and $u_L(\cdot)$, respectively. The built-in encoders of the machines provide position (and velocity) information. The motor drive accelerates or decelerates the inertia Θ, whereas the load drive emulates external loads u_L. The dynamics (faster than $1 \cdot 10^{-3}\,[s]$) of each actuator are negligible compared to those of the mechanical system (4.1) (see also the experiments in, e.g., [8, 18]).

Figure 6 depicts the measurements for the funnel controller (3.4) at the laboratory setup. The control error and its derivative remain within the prescribed funnel (see Figure 6(a),(b)). The control gains are adjusted "instantaneously" (see Figure 6(d)) so that boundary contact is excluded. The funnel controller is capable of tracking the time-varying reference with prescribed accuracy also when load torques $u_L(\cdot) \neq 0$ are induced (see Figure 6(c)). Noise amplification (see Figure 6(c)) and "oscillations" in speed, torque, and gains (see Figure 6(b),(c),(d)) are acceptable.

Appendix A. Proofs. To simplify the notation, we introduce for $(\varphi_0, \varphi_1) \in \mathcal{G}_2$ the funnel boundaries

$$(A.1) \qquad \psi_i \colon (0, \infty) \to (0, \infty), \quad t \mapsto 1/\varphi_i(t), \quad i = 0, 1.$$

(a) — $e(\cdot)\,[rad]$ and - - $\pm\psi_0(\cdot)\,[rad]$.

(b) — $\dot{e}(\cdot)\,[\frac{rad}{s}]$ and - - $\pm\psi_1(\cdot)\,[\frac{rad}{s}]$.

(c) — $u(\cdot)+u_A(\cdot)\,[Nm]$ and - - $-u_L(\cdot)\,[Nm]$.

(d) — $k_0(\cdot)^2\,[\frac{Nm}{rad}]$ and - - $k_1(\cdot)\,[\frac{Nms}{rad}]$.
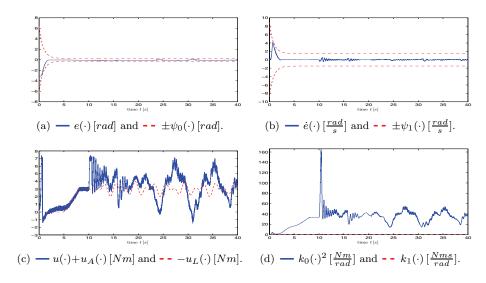
FIG. 6. *Experimental results of the controller design at the laboratory setup.*

A.1. Proof of Theorem 2.1: Funnel control for linear systems with relative degree two. Without loss of generality, we may assume that system (1.1) is in Byrnes–Isidori form (3.5). The main difficulties in proving Theorem 2.1 are, first, that the closed-loop initial-value problem (1.1), (2.2) has a potential singularity (a pole) on the right-hand side of the differential equation and, second, to show that the solution does not have a finite escape time, i.e., exists globally on $[0,\infty)$.

Step 1. We show existence and uniqueness of a maximal solution.

Define, for $(\varphi_0,\varphi_1) \in \mathcal{G}_2$,

$$(\text{A.2}) \quad \mathcal{D} := \big\{\, (t,\mu_0,\mu_1,\xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \;\big|\; (t,\mu_0) \in \mathcal{F}_{\varphi_0},\ (t,\mu_1) \in \mathcal{F}_{\varphi_1} \,\big\}$$

and $f \colon \mathcal{D} \to \mathbb{R}^n$ by

$$f(t,\mu_0,\mu_1,\xi) = \left(\begin{matrix} \begin{bmatrix} 0, & 1 \\ r_0, & r_1 \end{bmatrix} \begin{pmatrix} y_{\text{ref}}(t)+\mu_0 \\ \dot{y}_{\text{ref}}(t)+\mu_1 \end{pmatrix} - \begin{bmatrix} 0 \\ s^\top \end{bmatrix} \xi - \begin{pmatrix} \dot{y}_{\text{ref}}(t) \\ \ddot{y}_{\text{ref}}(t) \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ u_{\text{d}}(t) - \frac{\varphi_0(t)^2\mu_0}{(1-\varphi_0(t)|\mu_0|)^2} - \frac{\varphi_1(t)\mu_1}{1-\varphi_1(t)|\mu_1|} \end{pmatrix} \\ [p,0]\begin{pmatrix} y_{\text{ref}}(t)+\mu_0 \\ \dot{y}_{\text{ref}}(t)+\mu_1 \end{pmatrix} + Q\xi \end{matrix} \right).$$

The relative degree two condition (1.2) implies $\gamma = cAb > 0$, and the minimum-phase condition (1.3) is equivalent to Q being Hurwitz, i.e., (3.6). Then the initial-value problem (3.5), (2.2) may be written as

$$(\text{A.3}) \qquad \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} e(t) \\ \dot{e}(t) \\ z(t) \end{pmatrix} = f(t,e(t),\dot{e}(t),z(t)), \qquad \begin{pmatrix} e(0) \\ \dot{e}(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} cx^0 - y_{\text{ref}}(0) \\ cAx^0 - \dot{y}_{\text{ref}}(0) \\ z^0 \end{pmatrix}.$$

Clearly, f is locally Lipschitz in μ_0, μ_1, and ξ and measurable in t; hence the theory of ordinary differential equations (see, e.g., [33, Thm. III, section 10.XX]) ensures existence of a unique absolutely continuous solution $(e,\dot{e},z) \colon [0,\omega) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$, $0 < \omega \leq \infty$, which is maximally extended, i.e., the graph of the solution is not completely contained in any compact subset of $\mathcal{D}$.

In the following, let $(e,\dot{e},z) \colon [0,\omega) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ be the unique and maximally extended solution of the closed-loop initial-value problem (A.3).

Step 2. We show that there exists $M > 0$ such that

(A.4) $\qquad$ for a.a. $t \in [0, \infty):$ $\boxed{-M + \gamma u(t) \leq \ddot{e}(t) \leq M + \gamma u(t)}$.

By continuity of $e(\cdot)$, $\dot{e}(\cdot)$, $z(\cdot)$, and the corresponding $k_0(\cdot)$, $k_1(\cdot)$, there exists $\varepsilon \in (0, \omega)$ such that
(A.5)
$\forall i = 0, 1, \ \forall t \in [0, \varepsilon]: \ |e^{(i)}(t)| \leq |e^{(i)}(0)| + 1, \ \ |z(t)| \leq |z^0| + 1, \ \ k_i(t) \leq k_i(0) + 1.$

Hence it suffices to consider the interval $[\varepsilon, \omega)$, and we may adopt the notation (A.1). By definition of $\mathcal{G}_1$ we have that $\|\psi_i\|_{\varepsilon, \infty} := \left\|\psi_i\big|_{[\varepsilon, \infty)}\right\|_\infty$, $i = 0, 1$, and hence

(A.6) $\qquad \forall t \in [\varepsilon, \omega): \ |e(t)| < \psi_0(t) \leq \|\psi_0\|_{\varepsilon, \infty} \ \text{ and } \ |\dot{e}(t)| < \psi_1(t) \leq \|\psi_1\|_{\varepsilon, \infty}.$

Applying variation of constants to the third subsystem in (A.3) yields for all $t \in [\varepsilon, \omega): \ z(t) = e^{Q(t-\varepsilon)} z(\varepsilon) + \int_\varepsilon^t e^{Q(t-\tau)} p \left[y_{\mathrm{ref}}(\tau) - e(\tau)\right] \mathrm{d}\tau$, and thus, in view of (3.6), (A.5), and (A.6), it follows that

(A.7) $\qquad\qquad\qquad \forall t \in [0, \omega): \ \|z(t)\| \leq M_z,$

where, with $M_Q \geq 1$ and $\lambda_Q > 0$ as in (3.6), $M_z := M_Q[|z^0| + 1] + \frac{M_Q}{\lambda_Q} \|p\| \left[\|y_{\mathrm{ref}}\|_\infty + \|\psi_0\|_\infty\right]$. Since, for almost all $t \in [0, \omega)$, $\ddot{e}(t) = r_0[e(t) + y_{\mathrm{ref}}(t)] + r_1[\dot{e}(t) + \dot{y}_{\mathrm{ref}}(t)] + s^\top z(t) - \ddot{y}_{\mathrm{ref}}(t) + \gamma u(t)$, we obtain, by invoking (A.5) and (A.6), the claimed inequality (A.4) for

$$M := |r_0| \left[\max\{\|e(0)\| + 1, \|\psi_0\|_{\varepsilon, \infty}\} + \|y_{\mathrm{ref}}\|_\infty\right]$$
$$+ |r_1| \left[\max\{|\dot{e}(0)| + 1, \|\psi_1\|_{\varepsilon, \infty}\} + \|\dot{y}_{\mathrm{ref}}\|_\infty\right] + |s^\top| M_z + \|\ddot{y}_{\mathrm{ref}}\|_\infty.$$

Step 3. We show that $|e(\cdot)|$ is uniformly bounded away from funnel boundary $\psi_0(\cdot)$:

(A.8) $\qquad\qquad \exists \varepsilon_0 > 0 \ \forall t \in [\varepsilon, \omega): \ \boxed{\psi_0(t) - |e(t)| \geq \varepsilon_0}$.

Consider two phases: a *parabolic phase* and a *linear phase*. In the parabolic phase the distance of the error $e(\cdot)$ to the funnel boundary $\psi(\cdot)$ is bounded by a parabola as formalized by Step 3a below. Step 3b ensures that the "overshoot" of this parabola can be made sufficiently small. In the linear phase, the distance of the error and the funnel boundary grows linearly as formalized in Step 3c, and Step 3d ensures that the linear phase remains active as long as the error is close to the boundary.

Step 3a. We show that for $\varepsilon_0 \in (0, \lambda_0/2)$ the following implication holds on any interval $[t_0, t_1] \subseteq [\varepsilon, \omega)$:
(A.9)
$$\begin{bmatrix} \psi_0(t_0) - |e(t_0)| = 2\varepsilon_0 \ \wedge \ \text{for a.a. } t \in [t_0, t_1]: \\ \ddot{e}(t) \operatorname{sgn} e(t) \leq -(\|\psi_1\|_{\varepsilon, \infty} + \|\dot{\psi}_0\|_{\varepsilon, \infty})^2/(2\varepsilon_0) \end{bmatrix} \Rightarrow \forall t \in [t_0, t_1]: \ \psi_0(t) - |e(t)| \geq \varepsilon_0.$$

First note that there exists a countable family of pairwise disjoint intervals $T_i = [\underline{\tau}_i, \overline{\tau}_i]$, $i \in \mathcal{I}$, and $S_j = (\underline{\sigma}_i, \overline{\sigma}_i)$, $j \in \mathcal{J}$, with $[t_0, t_1] \subseteq \bigcup_{i \in \mathcal{I}} T_i \cup \bigcup_{j \in \mathcal{I}} S_j$ such that

$$\begin{aligned} \forall i \in \mathcal{I}: \ &\psi_0(\underline{\tau}_i) - |e(\underline{\tau}_i)| = 2\varepsilon_0 \ \wedge \quad \forall t \in T_i: \ \psi_0(t) - |e(t)| \leq 2\varepsilon_0, \\ \forall j \in \mathcal{J}: \ & \qquad\qquad\qquad\qquad\qquad\quad \forall t \in S_j: \ \psi_0(t) - |e(t)| > 2\varepsilon_0. \end{aligned}$$

On the intervals S_j, $j \in \mathcal{J}$, the conclusion of (A.9) is trivially true, and hence we have only to consider the intervals T_i, $i \in \mathcal{I}$, i.e., to show (A.9) under the additional assumption

$$(A.10) \qquad \forall\, t \in [t_0, t_1]: \; \psi_0(t) - |e(t)| \leq 2\varepsilon_0.$$

From $\lambda_0 > 2\varepsilon_0$ it follows that $\operatorname{sgn} e(\cdot)$ is constant on $[t_0, t_1]$. We consider only the case $\operatorname{sgn} e(\cdot) \equiv 1$; the case $\operatorname{sgn} e(\cdot) \equiv -1$ follows analogously. Integrating the inequality $\ddot{e}(\cdot) \leq -(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2/(2\varepsilon_0)$ twice over $[t_0, t]$ yields

$$\forall\, t \in [t_0, t_1]: \; e(t) \leq e(t_0) - \frac{(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2}{4\varepsilon_0}(t - t_0)^2 + \dot{e}(t_0)(t - t_0),$$

and in combination with $\dot{e}(t_0) \leq \|\psi_1\|_{\varepsilon,\infty}$ and $\psi_0(t) \geq \psi_0(t_0) - \|\dot{\psi}_0\|_{\varepsilon,\infty}(t - t_0)$ we conclude, for all $t \in [t_0, t_1]$,

$$\psi_0(t) - e(t) \geq \underbrace{\psi_0(t_0) - e(t_0)}_{=2\varepsilon_0}$$
$$- \left((\|\dot{\psi}_0\|_{\varepsilon,\infty} + \|\psi_1\|_{\varepsilon,\infty})(t - t_0) - \frac{(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2}{4\varepsilon_0}(t - t_0)^2 \right).$$

The parabola $t \mapsto (\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})(t - t_0) - \frac{(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2}{4\varepsilon_0}(t - t_0)^2$ attains its maximum at $t - t_0 = \frac{2\varepsilon_0}{\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty}}$ with the maximum value ε_0, and hence $\psi_0(t) - e(t) \geq \varepsilon_0$ for all $t \in [t_0, t_1]$. This proves Step 3a.

Step 3b. We show that there exists $\bar{\varepsilon}_0 \in (0, \lambda_0/4)$ such that the following implication holds on any interval $[t_0, t_1] \subseteq [\varepsilon, \omega)$ and for all $\varepsilon_0 \in (0, \bar{\varepsilon}_0]$:

$$(A.11) \quad \left[\forall\, t \in [t_0, t_1]: \; \dot{e}(t)\operatorname{sgn} e(t) \geq -\psi_1(t) + \delta/2 \; \wedge \; \psi_0(t) - |e(t)| \leq 2\varepsilon_0 \right]$$
$$\implies \quad \text{for a.a.}\, t \in [t_0, t_1]: \; \ddot{e}(t)\operatorname{sgn} e(t) \leq -(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2/(2\varepsilon_0).$$

The condition $2\varepsilon_0 \leq \lambda_0/2$, together with $\psi_0(t) - |e(t)| \leq 2\varepsilon_0$ on $[t_0, t_1]$, implies that $\operatorname{sgn} e(\cdot)$ is constant on $[t_0, t_1]$. We consider only the case $\operatorname{sgn} e(\cdot) \equiv 1$; $\operatorname{sgn} e(\cdot) \equiv -1$ follows analogously. The condition $\dot{e}(t) \geq \delta/2 - \psi_1(t)$ on $[t_0, t_1]$ implies that for all $t \in [t_0, t_1]: \; -k_1(t)\dot{e}(t) = \frac{-\dot{e}(t)}{\psi_1(t) - |\dot{e}(t)|} \leq \frac{2|\dot{e}(t)|}{\delta} < \frac{2\|\psi_1\|_{\varepsilon,\infty}}{\delta}$. From $\psi_0(t) - e(t) \leq 2\varepsilon_0$ and $2\varepsilon_0 \leq \lambda_0/2$, it follows that $e(t) \geq \lambda_0/2$ on $[t_0, t_1]$, and hence $-k_0(t)^2 e(t) \leq -\frac{\lambda_0}{8\varepsilon_0^2}$ for all $t \in [t_0, t_1]$. Inserting these inequalities into (A.4) and invoking (2.2) yields

$$(A.12) \qquad \text{for a.a.}\, t \in [t_0, t_1]: \; \ddot{e}(t) < M - \gamma\frac{\lambda_0}{8\varepsilon_0^2} + \gamma\frac{2\|\psi_1\|_{\varepsilon,\infty}}{\delta} + \gamma\|u_{\mathrm{d}}\|_\infty,$$

whence (A.11) for sufficiently small $\varepsilon_0 > 0$.

Step 3c. We show the following implication:

$$(A.13) \quad \left[\forall\, t \in [t_1, t_2]: \; \dot{e}(t)\operatorname{sgn} e(t) \leq -\psi_1(t) + \delta/2 \; \wedge \; \psi_0(t) - |e(t)| \leq 2\varepsilon_0 \right]$$
$$\implies \quad t \mapsto \psi_0(t) - |e(t)| \text{ is monotonically increasing on } [t_1, t_2].$$

Note that the presupposition in (A.13) precludes a sign change of $e(\cdot)$ on $[t_1, t_2]$. We consider only the case $\operatorname{sgn} e(\cdot) \equiv 1$; the other case follows analogously. Invoking the

definition of $\mathcal{G}_2$, for a.a. $t \in [t_1, t_2]:\ \dot{\psi}_0(t) - \dot{e}(t) \geq \dot{\psi}_0(t) + \psi_1(t) - \delta/2 \geq \delta - \delta/2 = \delta/2$ yields (A.13).

Step 3d. We show that there exists $\bar{\varepsilon}_0^* \in (0, \lambda_0/4]$ such that the following implication holds for any $[t_1, t_2] \subseteq [\varepsilon, \omega)$ and any $\varepsilon_0 \in (0, \bar{\varepsilon}_0^*]$:

$$(A.14) \quad \left[\dot{e}(t_1) \operatorname{sgn} e(t_1) = -\psi_1(t_1) + \delta/2 \quad \wedge \quad \forall t \in [t_1, t_2]:\ \psi_0(t) - |e(t)| \leq 2\varepsilon_0 \right]$$
$$\implies \quad \forall t \in [t_1, t_2]:\ \dot{e}(t) \operatorname{sgn} e(t) \leq -\psi_1(t) + \delta/2.$$

From $2\varepsilon_0 \leq \lambda_0/2$ and $\psi_0(t) - |e(t)| \leq 2\varepsilon_0$ it follows that $\operatorname{sgn} e(\cdot)$ is constant on $[t_1, t_2]$. We only consider $\operatorname{sgn} e(\cdot) \equiv 1$ here; the negative case follows analogously.

We show that the existence of $\hat{t} \in (t_1, t_2]$ with $\dot{e}(\hat{t}) > -\psi_1(\hat{t}) + \delta/2$ yields a contradiction to the assumptions of the implication (A.14). Therefore, choose $\hat{t}_1 \in [t_1, \hat{t})$ with $\dot{e}(\hat{t}_1) = -\psi_1(\hat{t}_1) + \delta/2$ and $\dot{e}(t) \geq -\psi_1(t) + \delta/2$ for all $t \in [\hat{t}_1, \hat{t}]$. Together with $\psi_0(t) - e(t) \leq 2\varepsilon_0$ we can conclude as in Step 3b that (A.12) holds for the interval $[\hat{t}_1, \hat{t}]$, and hence for small enough $\bar{\varepsilon}_0^*$ and all $\varepsilon_0 \in (0, \bar{\varepsilon}_0^*]$: $\ddot{e}(t) < -\|\dot{\psi}_1\|_{\varepsilon, \infty}$ for all $t \in [\hat{t}_1, \hat{t}]$. Now, $\delta/2 < \dot{e}(\hat{t}) + \psi_1(\hat{t}) = \dot{e}(\hat{t}_1) + \psi_1(\hat{t}_1) + \int_{\hat{t}_1}^{\hat{t}} \ddot{e}(\tau) + \dot{\psi}_1(\tau)\, d\tau < \dot{e}(\hat{t}_1) + \psi_1(\hat{t}_1) = \delta/2$, whence a contradiction to the choice of $\hat{t}$.

Step 3e. We show that for sufficiently small $\varepsilon_0 > 0$ the claim of Step 3 holds. Choose $\varepsilon_0 > 0$ such that (A.11), (A.14), and $\psi_0(\varepsilon) - |e(\varepsilon)| \geq 2\varepsilon_0$ hold. Seeking a contradiction, assume that there exists $t_2 \in (\varepsilon, \omega)$ such that $\psi_0(t_2) - |e(t_2)| < \varepsilon_0$. Choose $t_0 \in [\varepsilon, t_2)$ such that

$$(A.15) \qquad \psi_0(t_0) - |e(t_0)| = 2\varepsilon_0 \quad \text{and} \quad \forall t \in [t_0, t_2]:\ \psi_0(t) - |e(t)| \leq 2\varepsilon_0.$$

Since $2\varepsilon_0 < \lambda_0$, it follows that $e(\cdot)$ has a constant sign on $[t_0, t_2]$, we consider here only the positive case; the negative follows analogously. It follows from (A.15) that there exists $\nu > 0$ such that for a.a. $t \in (t_0, t_0 + \nu]:\ \dot{\psi}_0(t) - \dot{e}(t) \leq 0$, hence by the property of $\mathcal{G}_2$ we have for a.a. $t \in (t_0, t_0 + \nu]:\ \dot{e}(t) \geq -\psi_1(t) + \delta > -\psi_1(t) + \delta/2$, and by continuity of $\dot{e}$ and ψ_1 it follows that $\dot{e}(t_0) > -\psi_1(t_0) + \delta/2$, and hence there exists a maximal $t_1 \in (t_0, t_2]$ such that

$$(A.16) \qquad\qquad \forall t \in [t_0, t_1]:\ \dot{e}(t) \geq -\psi_1(t) + \delta/2.$$

Now implications (A.11) and (A.9) from Steps 3b and 3a, respectively, together with (A.15) and (A.16), show that $\psi_0(t) - e(t) \geq \varepsilon_0$ for all $t \in [t_0, t_1]$. Hence $t_1 < t_2$, which implies $\dot{e}(t_1) = -\psi_1(t_1) + \delta/2$. Combining this with (A.15) and implication (A.14) from Step 3d yields $\dot{e}(t) \leq -\psi_1(t) + \delta/2$ for all $t \in [t_1, t_2]$. Implication (A.13) from Step 3c now gives $\psi_0(t_2) - e(t_2) \geq \psi_0(t_1) - e(t_1) \geq \varepsilon_0$, which contradicts the choice of t_2. Hence Step 3 is shown.

Step 4. We show that $\dot{e}(\cdot)$ is uniformly bounded away from funnel boundary $\psi_1(\cdot)$:

$$(A.17) \qquad\qquad \exists\, \varepsilon_1 > 0 \ \forall t \in [\varepsilon, \omega):\ \boxed{\psi_1(t) - |\dot{e}(t)| \geq \varepsilon_1}.$$

We have, for $\varepsilon_0 > 0$ as in Step 3, that $k_0(t)^2 \leq 1/\varepsilon_0^2$ for all $t \in [\varepsilon, \omega)$ which, together with (A.6), yields for all $t \in [\varepsilon, \omega):\ k_0(t)^2 |e(t)| < \|\psi_0\|_{\varepsilon, \infty}/\varepsilon_0^2$. Assume $\varepsilon_1 \leq \min\{\lambda_1/2, \psi_1(\varepsilon) - |\dot{e}(\varepsilon)|\}$. Then, in view of (A.4) and (2.2), for almost all $t \in [\varepsilon, \omega):$ $\left[\psi_1(t) - |\dot{e}(t)| \leq \varepsilon_1 \implies \ddot{e}(t) \operatorname{sgn} \dot{e}(t) < M + \gamma \frac{\|\psi_0\|_{\varepsilon, \infty}}{\varepsilon_0^2} - \gamma \frac{\lambda_1/2}{\varepsilon_1} + \gamma \|u_d\|_\infty\right]$; hence for sufficiently small $\varepsilon_1 > 0$ and a.a. $t \in [\varepsilon, \infty):\ \psi_1(t) - |\dot{e}(t)| \leq \varepsilon_1 \implies \ddot{e}(t) \operatorname{sgn} \dot{e}(t) < -\|\dot{\psi}_1\|_{\varepsilon, \infty}$, which ensures that the set $\{\, (t, \xi) \in [\varepsilon, \omega) \times \mathbb{R} \mid \psi_1(t) - |\xi| \geq \varepsilon_1 \,\}$ is positively invariant for $\dot{e}(\cdot)$. Hence Step 4 is proved.

Step 5. We show assertions (i)–(iii).

Boundedness of $e(\cdot)$, $\dot{e}(\cdot)$, $z(\cdot)$, $k_0(\cdot)$, $k_1(\cdot)$ on $[0,\omega)$ follows from (A.5), (A.6), (A.7), (A.8), and (A.17). The inequality (2.3) holds on $[0,\omega)$ because $k_i(\cdot)$ is bounded, with $i=0,1$. Therefore, assertions (i)–(iii) hold if $\omega = \infty$. For ε_0 and ε_1 as in (2.3) and M_z as in (A.7), let

(A.18)
$$\mathfrak{C} := \left\{ \begin{array}{c} (t, e_0, e_1, z) \in [0,\omega] \\ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \end{array} \middle| \begin{array}{l} \forall i \in \{0,1\}: \ |e_i| \le |e^{(i)}(0)| + 1 \text{ if } t \in [0,\varepsilon], \\ \qquad\qquad |e_i| \le \psi_i(t) - \varepsilon_i \text{ otherwise,} \\ \|z\| \le M_z \end{array} \right\}.$$

Let $\mathcal{D}$ be as in Step 1. If $\omega < \infty$, then $\mathfrak{C} \subsetneq \mathcal{D}$ is a compact subset of $\mathcal{D}$ which contains the whole graph of the solution $t \mapsto (e(t), \dot{e}(t), z(t))$, which contradicts the maximality of the solution. Hence $\omega = \infty$. This completes the proof. $\square$

A.2. Proof of Theorem 3.1: Nonlinear systems described by functional differential equations. It suffices to show that there exists a maximal solution $y\colon [-h, \omega)$, and each solution fulfills the minor modification of (A.4):

(A.19) $\quad \exists M > 0 \ \exists \gamma > 0$ for a.a. $t \in [0,\omega): \ \ddot{e}(t) \operatorname{sgn} u(t) \ge -M \operatorname{sgn} u(t) + \gamma u(t)$.

Then Steps 3–5 of the proof of Theorem 2.1 can then be repeated identically to prove Theorem 3.1.

Step 1. We show existence of maximally extended solutions $y\colon [-h, \omega)$, $\omega \in (0, \infty]$. Define

$$F\colon [-h, \infty) \times \mathcal{D} \times \mathbb{R}^{2W} \to \mathbb{R}^3, \ \ (t, (\tau, e_0, e_1), (w_1, w_2)) \mapsto$$
$$\left(1, e_1, f(p_f(t), w_1) + g(p_g(t), w_2) \left(-\frac{\varphi_0(\tau)^2}{(1 - \varphi_0(\tau)|e_0|)^2} e_0 - \frac{\varphi_1(\tau)}{1 - \varphi_1(\tau)|e_1|} e_1 + u_{\mathrm{d}}(t) \right) - \ddot{y}_{\mathrm{ref}}(t) \right),$$

where $\mathcal{D} := \{ (\tau, e_0, e_1) \in [-h, \infty) \times \mathbb{R} \times \mathbb{R} \mid (|\tau|, e_0) \in \mathcal{F}_{\varphi_0}, \ (|\tau|, e_1) \in \mathcal{F}_{\varphi_1} \}$ and the operator $\widehat{T}\colon [-h, \infty) \times \mathcal{C}([-h, \infty) \to \mathbb{R})^2 \to \mathcal{L}_{\mathrm{loc}}^\infty \left([0, \infty) \to \mathbb{R}^W \right)^2$ with $\widehat{T}(\tau, e_0, e_1) :=$ $(T_f(e_0 + y_{\mathrm{ref}}, e_1 + \dot{y}_{\mathrm{ref}}), T_g(e_0 + y_{\mathrm{ref}}, e_1 + \dot{y}_{\mathrm{ref}}))$, where y_{ref} is extended to $[-h, 0)$ in such a way that $y_{\mathrm{ref}} \in \mathcal{W}^{2,\infty}([-h, \infty) \to \mathbb{R})$ and for all $t \in [-h, 0]: \ \varphi_0(|t|)\,|y^0(t) - y_{\mathrm{ref}}(t)| < 1$ and $\varphi_1(|t|)\,|\dot{y}^0(t) - \dot{y}_{\mathrm{ref}}(t)| < 1$. This is possible since (3.3) and $(\varphi_0, \varphi_1) \in \mathcal{G}_2$ hold.

Writing $\tau^0\colon [-h, 0] \to \mathbb{R}$, $t \mapsto t$, it follows that $x = (\tau, e, \dot{e})$ is a solution of $\dot{x} = F(t, x, \widehat{T}(x))$, $x\big|_{[-h,0]} = (\tau^0, y^0 - y_{\mathrm{ref}}|_{[-h,0]}, \dot{y}^0 - \dot{y}_{\mathrm{ref}}|_{[-h,0]})$ if and only if $y = y_{\mathrm{ref}} + e$ solves the closed-loop system (3.2), (2.2). Finally, [14, Thm. 5] ensures the existence of a maximally extended solution $y\colon [-h, \omega) \to \mathbb{R}$, $\omega \in (0, \infty]$.

Step 2. We show (A.19).

Consider a fixed solution $y : [-h, \omega) \to \mathbb{R}$ of (3.2), (2.2); i.e., for a.a. $t \in [0, \omega)$: $\ddot{e}(t) = f(p_f(t), T(y, \dot{y})(t)) + g(p_g(t), T_g(y, \dot{y})(t))u(t) - \ddot{y}_{\mathrm{ref}}(t)$. Choose $\varepsilon > 0$ such that $|y(t)| \le \|y^0\|_\infty + 1$ and $|y(t)| \le \|\dot{y}^0\|_\infty + 1$ for all $t \in [0, \varepsilon]$, and since

$$\forall t \in [\varepsilon, \omega): \ |y(t)| < \|y_{\mathrm{ref}}\|_\infty + \|1/\varphi_0\big|_{[\varepsilon,\infty)}\|_\infty \ \text{ and } \ |\dot{y}(t)| < \|\dot{y}_{\mathrm{ref}}\|_\infty + \|1/\varphi_1\big|_{[\varepsilon,\infty)}\|_\infty,$$

the trajectories y and $\dot{y}$ are bounded on $[0, \omega)$, and hence $T_f(y, \dot{y})\big|_{[0,\omega)}$ and $T_g(y, \dot{y})\big|_{[0,\omega)}$ are well defined and bounded, say by M_{T_f} and M_{T_g}. Let M_{p_f} and M_{p_g} be the corresponding bounds of $p_f(\cdot)$ and $p_g(\cdot)$; then by continuity of f and g,

$$\max_{|p| \le M_{p_f}, |w| \le M_{T_f}} |f(p, w)| =: M_f < \infty \ \text{ and } \ \min_{|p| \le M_{p_g}, |w| \le M_{T_g}} g(p, w) =: \gamma > 0,$$

and so (A.19) holds for $M := M_f + \|\ddot{y}_{\mathrm{ref}}\|_\infty$. $\square$

A.3. Proof of Theorem 3.2: Systems with relative degree one. The proof is based on the following existence and uniqueness of the solution of an implicit ordinary differential equation.

LEMMA A.1 (existence and uniqueness of the solution of an implicit ordinary differential equation). *Let $\mathcal{D} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$ be a nonempty and relatively open set, and let $(t_0, e_0^0, e_1^0, z^0) \in \mathcal{D}$. Let $F \in \mathcal{C}^1(\mathcal{D} \to \mathbb{R})$ be such that*

$$(A.20) \qquad\qquad F(t_0, e_0^0, e_1^0, z^0) = 0 \quad and$$

$$(A.21) \qquad \forall (t, e_0, e_1, z) \in \mathcal{D} : \ \frac{\partial F}{\partial e_1}(t, e_0, e_1, z) \neq 0.$$

Consider, for $p \in \mathbb{R}^{n-1}$, $Q \in \mathbb{R}^{(n-1) \times (n-1)}$, and $g \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$, the implicit initial-value problem

$$(A.22) \qquad \begin{aligned} 0 &= F(t, e, \dot{e}, z), & e(t_0) &= e^0, \\ \dot{z} &= pe + Qz + g(t), & z(t_0) &= z^0. \end{aligned}$$

Then there exists a unique maximal solution $(e, z) \colon [t_0, \omega) \to \mathbb{R} \times \mathbb{R}^{n-1}$, $\omega \in (t_0 \,\infty]$, of (A.22) such that $\mathrm{graph}(e, \dot{e}, z) = \{ \, (t, e(t), \dot{e}(t), z(t)) \mid t \in [0, \omega) \, \} \subseteq \mathcal{D}$, and maximality implies that $\mathrm{graph}(e, \dot{e}, z)$ is not completely contained in any compact subset of $\mathcal{D}$.

Proof.

Step 1. We show existence and uniqueness of a local solution of the initial-value problem (A.22).

Differentiability of $F(\cdot)$, together with (A.20) and (A.21), allows us to apply the implicit function theorem (see, for example, [1, Thm. VII.8.2]) to conclude the following: there exist a relatively open neighborhood $\mathcal{U} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^{n-1}$ of (t_0, e_0^0, z^0), an open neighborhood $\mathcal{V} \subseteq \mathbb{R}$ of e_1^0, a unique function $f \in \mathcal{C}^1(\mathcal{U} \to \mathcal{V})$ such that $f(t_0, e_0^0, z^0) = e_1^0$, and $F(t, e_0, f(t, e_0, z), z) = 0$ for all $(t, e_0, z) \in \mathcal{U}$; moreover,

$$(A.23) \qquad \forall (t, e_0, z) \in \mathcal{U} : \ [F(t, e_0, e_1, z) = 0 \ \wedge \ e_1 \in \mathcal{V}] \iff e_1 = f(t, e_0, z).$$

Consider next the initial-value problem

$$(A.24) \qquad \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} e_0 \\ z \end{pmatrix} = \begin{pmatrix} f(t, e_0, z) \\ pe_0 + Qz + g(t) \end{pmatrix}, \quad \begin{pmatrix} e_0(t_0) \\ z(t_0) \end{pmatrix} = \begin{pmatrix} e_0^0 \\ z^0 \end{pmatrix}.$$

The right-hand side of (A.24) is continuous on the relatively open set $\mathcal{U}$ and locally Lipschitz in e_0 and z; hence the standard theory of ordinary differential equations (see, e.g., [33, Thm. III, sect. 11.III]) yields existence of a unique solution $(e, z) \colon [t_0, \omega) \to \mathbb{R} \times \mathbb{R}^{n-1}$, $\omega \in (t_0, \infty]$, of the initial-value problem (A.24). From (A.23) it follows that this solution is also a unique (local) solution of (A.22).

Step 2. We show that every solution of (A.22) can be maximally extended.

Let $(e, z) \colon [t_0, \omega) \to \mathbb{R} \times \mathbb{R}^{n-1}$, $\omega \in (t_0, \infty]$, be a solution of (A.22). If $\omega = \infty$, nothing is shown; hence assume $\omega < \infty$. Define

$$\mathcal{A}_{(\omega, e, z)} := \left\{ (\sigma, \xi(\cdot)) \ \middle| \ \begin{array}{l} \sigma \in [\omega, \infty], \ \xi \in \mathcal{C}^1([t_0, \sigma) \to \mathbb{R}^n), \ \xi\big|_{[t_0, \omega)} = (e, z) \\ (\xi_1, \dot{\xi}_1, \xi_2, \ldots, \xi_n) \text{ solves (A.22) on } [t_0, \sigma) \end{array} \right\},$$

that is, the set comprising the solution (e, z) and all proper right extensions of (e, z) that are also solutions. Define on this nonempty set a partial order $\preceq$ by $(\sigma_1, \xi_1(\cdot)) \preceq$

$(\sigma_2, \xi_2(\cdot))$:$\Longleftrightarrow$ $\sigma_1 \leq \sigma_2$ and $\xi_1(\cdot) = \xi_2|_{[t_0,\sigma_1]}$. Let $\mathcal{A}_1$ be a totally ordered subset of $\mathcal{A}_{(\omega,e,z)}$. Set $\sigma^* := \sup\{\, \sigma \in [\omega,\infty] \mid \exists\, (\sigma,\xi(\cdot)) \in \mathcal{A}_1 \,\}$ and let $\xi^* : [t_0,\sigma^*) \to \mathbb{R}^n$ be defined by the property that, for every $(\sigma,\xi) \in \mathcal{A}_1$, $\xi^*|_{[t_0,\sigma)} = \xi(\cdot)$. Then $(\sigma^*;\xi^*) \in \mathcal{A}_{(\omega,e,z)}$ and it is an upper bound for $\mathcal{A}_1$. By Zorn's lemma (see, e.g., [33, sect. 7.XIII]), it follows that $\mathcal{A}_{(\omega,e,z)}$ contains at least one maximal element. Hence there exists a maximal solution $(e,z): [t_0,\omega^*) \to \mathbb{R} \times \mathbb{R}^{n-1}$, $\omega^* \in (t_0,\infty]$ of the initial-value problem (A.22).

Step 3. We show uniqueness of the solution of the initial-value problem (A.22).

Let $(e,z): [t_0,\omega) \to \mathbb{R}^n$, $\omega \in (t_0,\infty]$, and $(\widetilde{e},\widetilde{z}): [t_0,\widetilde{\omega}) \to \mathbb{R}^n$, $\widetilde{\omega} \in (t_0,\infty]$, be two solutions of the initial-value problem (A.22). Seeking a contradiction, suppose that there exists a first time $t_1 \in [t_0,\infty)$ where the two solutions separate; more precisely, $t_1 := \max\{\, t \in [t_0,\min\{\omega,\widetilde{\omega}\}) \mid (e,z)\big|_{[t_0,t]} = (\widetilde{e},\widetilde{z})\big|_{[t_0,t]} \,\} \in \mathbb{R}$. According to Step 1, the corresponding initial-value problem (A.22) at t_1 with initial value $(e^1,z^1) := (e(t_1),z(t_1)) = (\widetilde{e}(t_1),\widetilde{z}(t_1))$ has a unique local solution on $[t_1,t_1+\delta) \subseteq [t_0,\min\{\omega,\widetilde{\omega}\})$ for some $\delta > 0$; hence $(e,z)\big|_{[t_1,t_1+\delta)} = (\widetilde{e},\widetilde{z})\big|_{[t_1,t_1+\delta)}$. This contradicts the definition of t_1 and proves the claim of Step 3.

Step 4. We show that the graph of the maximal solution $(e,\dot{e},z)$ is not contained in any compact subset of $\mathcal{D}$.

Let $(e,\dot{e},z): [t_0,\omega) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$, $\omega \in (t_0,\infty]$, be the unique maximal solution of (A.22). An equivalent formulation of the claim of Step 4 is that the closure of $\mathrm{graph}(e,\dot{e},z)$ is not a compact subset of $\mathcal{D}$. Denote the closure of $\mathrm{graph}(e,\dot{e},z)$ by $\mathfrak{C} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^{n-1}$ and, seeking a contradiction, assume that $\mathfrak{C}$ is a compact subset of $\mathcal{D}$. Then, first of all, $\omega < \infty$. By continuity of F and by construction of $\mathfrak{C}$ we have

$$\forall\, (t,e_0,e_1,z) \in \mathfrak{C}: \ F(t,e_0,e_1,z) = 0.$$

Hence the implicit function theorem ensures, for each $(t^*,e_0^*,e_1^*,z^*) \in \mathfrak{C}$, existence of (relatively) open neighborhoods $\mathcal{U}_{(t^*,e_0^*,z^*)}$ of (t^*,e_0^*,z^*) and $\mathcal{V}_{e_1^*}$ of e_1^*, as well as a function $f_{(t^*,e_0^*,e_1^*,z^*)} \in \mathcal{C}^1(\mathcal{U}_{(t^*,e_0^*,z^*)} \to \mathcal{V}_{e_1^*})$ such that (A.23) holds. Let $\mathcal{W}_{(t^*,e_0^*,e_1^*,z^*)} := \{\, (t,e_0,e_1,z) \mid (t,e_0,z) \in \mathcal{U}_{(t^*,e_0^*,z^*)}, \ e_1 \in \mathcal{V}_{e_1^*} \,\}$ which is a (relatively) open neighborhood of (t^*,e_0^*,e_1^*,z^*), and $\bigcup_{(t^*,e_0^*,e_1^*,z^*) \in \mathfrak{C}} \mathcal{W}_{(t^*,e_0^*,e_1^*,z^*)}$ is an open covering of $\mathfrak{C}$. By compactness of $\mathfrak{C}$ we may choose a finite subcovering of $\mathfrak{C}$; in particular, there exist $\varepsilon > 0$ and $(t^\omega,e_0^\omega,e_1^\omega,z^\omega) \in \mathfrak{C}$ such that $\mathrm{graph}\big((e,\dot{e},z)\big|_{[\omega-\varepsilon,\omega)}\big) \subseteq \mathcal{W}_{(t^\omega,e_0^\omega,e_1^\omega,z^\omega)}$. Hence, by (A.23), $\dot{e}(t) = f_{(t^\omega,e_0^\omega,e_1^\omega,z^\omega)}(t,e(t),z(t))$ on $[\omega-\varepsilon,\omega)$; i.e., $(e,z)\big|_{[\omega-\varepsilon,\omega)}$ is a solution of an (explicit) ordinary differential equation whose graph is contained in the compact set $\mathfrak{C}$. Now an application of [33, Lem. II, sect. 6.VI] ensures that this solution can be extended to the closed interval $[\omega-\varepsilon,\omega]$; in particular, $(e(\omega),\dot{e}(\omega),z(\omega)) = \lim_{t \nearrow \omega}(e(t),\dot{e}(t),z(t)) \in \mathfrak{C} \subseteq \mathcal{D}$ is well defined and $F(e(\omega),\dot{e}(\omega),z(\omega)) = 0$. Hence, by Step 1 with initial time ω and corresponding initial value, the solution can be extended locally to the interval $[\omega,\omega^*)$ for some $\omega^* > \omega$ which contradicts maximality of the solution. This shows the assertion of Step 4, and the proof of the lemma is complete. $\square$

Proof of Theorem 3.2.

Step 1. We show existence of a maximal solution.

Without loss of generality, we may assume that system (1.1) is in Byrnes–Isidori form, i.e., $\dot{y} = ry + s^\top z + \gamma u(t)$, $y(0) = cx^0$, $\dot{z} = py + Qz$, $z(0) = z^0$, where $r \in \mathbb{R}$, $s,p \in \mathbb{R}^{n-1}$, $Q \in \mathbb{R}^{(n-1 \times n-1)}$ is Hurwitz by (1.3), $z^0 \in \mathbb{R}^{n-1}$, and $\gamma := cb > 0$. The closed-loop system (1.1), (2.2) may be written as the following implicit differential

equation:

$$\dot{e}(t) = r(e(t) + y_{\text{ref}}(t)) + s^\top z(t) - \dot{y}_{\text{ref}}(t)$$

(A.25)
$$+ \gamma \left(u_d(t) - \frac{\varphi_0(t)^2 e(t)}{(1 - \varphi_0(t)|e(t)|)^2} - \frac{\varphi_1(t)\dot{e}(t)}{1 - \varphi_1(t)|\dot{e}(t)|} \right),$$

$$\dot{z}(t) = p(e(t) + y_{\text{ref}}(t)) + Qz(t),$$

with initial values $e(0) = e^0 := cx^0 - y_{\text{ref}}(0)$ and $z(0) = z^0$. For $\mathcal{D} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$ defined analogously as in (A.2), the implicit ordinary differential equation (A.25) can be written as $0 = F(t, e(t), \dot{e}(t), z(t))$, $\dot{z}(t) = pe(t) + Qz(t) + py_{\text{ref}}(t)$, with corresponding $F : \mathcal{D} \to \mathbb{R}$. Simple observations, taking also into account the absolute value function $|\cdot|$, reveal that F is differentiable with $\frac{\partial F}{\partial e_1}(t, e_0, e_1, z) = 1 + \gamma \frac{\varphi_1(t)}{(1 - \varphi_0(t)|e_1|)^2} \geq 1$ for all $(t, e_0, e_1, z) \in \mathcal{D}$. Since $\varphi_1(0) = 0$, it follows that (A.25) is *explicit* in $\dot{e}$ at $t = 0$; hence

$$F(0, e_0^0, e_1^0, z^0) = 0 \iff e_1^0 = r(e_0^0 + y_{\text{ref}}(0)) + s^\top z^0 - \dot{y}_{\text{ref}}(0) - \gamma \frac{\varphi_0(0)e_0^0}{(1 - \varphi_0(0)e_0^0)^2} + \gamma u_d(0).$$

Lemma A.1 now yields that there exists a unique and maximally extended solution $(e, z) : [0, \omega) \to \mathbb{R} \times \mathbb{R}^{n-1}$ of (A.25) with $(t, e(t), \dot{e}(t), z(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$.

Step 2. We show existence of $M > 0$ such that

(A.26)
$$\forall t \in [0, \omega) : \quad -M + \gamma u(t) \leq \dot{e}(t) \leq M + \gamma u(t).$$

This follows analogously as in Step 2 of the proof of Theorem 2.1.

Step 3. We show $\exists \varepsilon_0 \in (0, \lambda_0/2)$ for all $t \in (0, \omega) : 1/\varphi_0(t) - |e(t)| \geq \varepsilon_0$.

Adopting the notation (A.1) and choosing $\varepsilon > 0$ as in (A.5), it suffices to show that the set $\{ (t, e_0) \in [\varepsilon, \omega) \times \mathbb{R} \mid \psi_0(t) - |e_0| \geq \varepsilon_0 \}$ is positively invariant for sufficiently small $\varepsilon_0 > 0$ and $\psi_0(\varepsilon) - |e(\varepsilon)| \geq \varepsilon_0$. The former clearly follows if the following implication holds for all $t \in [\varepsilon, \omega)$:

$$\psi_0(t) - |e(t)| = \varepsilon_0 \implies \dot{e}(t) \operatorname{sgn} e(t) \leq -\psi_1(t) + \delta/2,$$

because, by definition of $\mathcal{G}_2$, $-\psi_1(t) + \delta/2 \leq \dot{\psi}_0(t) - \delta/2$. Seeking a contradiction, we assume there exists $t \in [\varepsilon, \omega)$ with $\psi_0(t) - |e(t)| = \varepsilon_0$ and $\dot{e}(t) \operatorname{sgn} e(t) > -\psi_1(t) + \delta/2$. From $\varepsilon_0 < \lambda_0/2$ together with (2.2) and (A.26) it then follows that

$$\dot{e}(t) \operatorname{sgn} e(t) < M - \gamma \frac{\lambda_0/2}{\varepsilon_0^2} + \gamma \frac{\|\psi_1|_{[\varepsilon, \infty)}\|_\infty}{\delta/2} + \gamma \|u_d\|_\infty,$$

and hence, for sufficiently small ε_0, we have a contradiction to the assumption.

Step 4. We show $\exists \varepsilon_1 \in (0, \lambda_1/2]$ for all $t \in (0, \omega) : 1/\varphi_1(t) - |\dot{e}(t)| \geq \varepsilon_1$.

Adopting the notation (A.1) and choosing $\varepsilon > 0$ as in (A.5), it suffices to show that $|\dot{e}(t)| \leq \psi_1(t) - \varepsilon_1$ for all $t \in [\varepsilon, \omega)$ and for sufficiently small ε_1. Seeking a contradiction, assume $|\dot{e}(t)| > \psi_1(t) - \varepsilon_1$ for some $t \in [\varepsilon, \omega)$ and arbitrary small $\varepsilon_1 > 0$. We consider only the case $\dot{e}(t) > 0$; the other case follows analogously. Choose $\varepsilon_0 > 0$ accordingly to Step 3. From $\varepsilon_1 \leq \lambda_1/2$ together with (2.2) and (A.26) it follows that

$$\lambda_1/2 \leq \psi_1(t) - \varepsilon_1 < \dot{e}(t) < M + \gamma \frac{\|\psi_0|_{[\varepsilon, \omega)}\|_\infty}{\varepsilon_0^2} - \gamma \frac{\lambda_1/2}{\varepsilon_1} + \gamma \|u_d\|_\infty,$$

which is a contradiction for sufficiently small ε_1.

Step 5. We show that the maximal solution is global.

Assume $\omega < \infty$; then $\mathfrak{C} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$ defined as in (A.18) is a compact subset containing graph$(e, \dot{e}, z)$, which according to Lemma A.1 contradicts maximality of the solution; hence $\omega = \infty$, and the proof of Theorem 3.2 is complete. $\quad\square$

A.4. Proof of Theorem 3.3: Input saturations. Existence and uniqueness of a maximal solution $(e, \dot{e}, z) : [0, \omega) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ follows similarly to Step 1 in the proof of Theorem 2.1. Now all the inequalities derived in sections 3.3.1–3.3.4 hold on $[0, \omega)$ instead of $\mathbb{R}_{\geq 0}$, and with minor modifications the steps of the proof of Theorem 2.1 can be repeated to prove Theorem 3.3. We omit the details. $\quad\square$

A.5. Proof of Theorem 3.5: Robustness in the gap metric.

A.5.1. Prerequisites. To match the notation of the gap metric (see, e.g., [26]), we rename the signals from Theorem 3.5 as $u_0 := u_{\mathrm{d}}$, $u_1 := u$, $u_2 := u_0 - u_1 = k_0^2 e + k_1 \dot{e}$, $y_0 := y_{\mathrm{ref}}$, $y_1 := y$, $y_2 := y_0 - y_1 = -e$. Corresponding to this notation, we consider the plant operator and the operator representing the funnel controller $P_{\theta, x^0} : u_1 \mapsto y_1$, $C_{\varphi_0, \varphi_0} : y_2 \mapsto u_2$, respectively. Due to possible finite escape time, we introduce the *ambient signal spaces* $\mathcal{L}_a^\infty$ and $\mathcal{W}_a^{2,\infty}$ (see [26, sect. 6.1] for more details), so that the plant and the controller can be considered as the maps $P_{\theta, x^0} : \mathcal{L}_a^\infty \to \mathcal{W}_a^{2,\infty}$, $C_{\varphi_0, \varphi_1} : \mathcal{W}_a^{2,\infty} \to \mathcal{L}_a^\infty$. Finally, let the closed-loop equations be given by

$$[P_{\theta, x^0}, C_{\varphi_0, \varphi_1}] : y_1 = P_{\theta, x^0}(u_1), \ u_2 = C_{\varphi_0, \varphi_1}(y_2), \ u_0 = u_1 + u_2, \ y_0 = y_1 + y_2.$$

Theorem 2.1 ensures that for all $(u_0, y_0) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ there exists unique $(u_1, y_1), (u_2, y_2) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ which solves the closed loop $[P_{\theta, x^0}, C_{\varphi_0, \varphi_1}]$. This implies, in the terminology of [26], that the closed loop $[P_{\theta, x^0}, C_{\varphi_0, \varphi_1}]$ is globally well posed and $\big(\mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})\big)$ stable.

We now study the closed loop $[\widetilde{P}, C_{\varphi_0, \varphi_1}]$ of the disturbed plant $\widetilde{P} \in \widetilde{\mathcal{P}}$ and the (unchanged) funnel controller C_{φ_0, φ_1}. In general, this closed loop will not generate globally defined solutions; however, we can show the following properties.

LEMMA A.2. *Let* $(\varphi_0, \varphi_1) \in \mathcal{G}_2 \setminus \mathcal{G}_2^{\mathrm{fin}}$, $\widetilde{P} \in \widetilde{\mathcal{P}}$, *and* $(u_0, y_0) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. *Then the closed loop* $[\widetilde{P}, C_{\varphi_0, \varphi_1}]$ *has the following properties:*

(i) *There exist unique, maximally extended solutions* $(u_1, y_1), (u_2, y_2) : [0, \omega) \to \mathbb{R}^2$, *for some* $\omega \in (0, \infty]$.

(ii) *If* $(u_2, y_2) \in \mathcal{L}^\infty([0, \omega) \to \mathbb{R}^m) \times \mathcal{W}^{2,\infty}([0, \omega) \to \mathbb{R}^m)$, *then* $\omega = \infty$; y_2 *and* $\dot{y}_2$ *are uniformly bounded away from the funnel boundaries* $\varphi_i(\cdot)^{-1}$, $i = 0, 1$, *respectively.*

(iii) $[\widetilde{P}, C_{\varphi_0, \varphi_1}]$ *is regularly well posed* [26]; *i.e., it is locally well posed and*

$$\omega < \infty \implies \big\| (u_2, y_2) \big|_{[0, \tau)} \big\|_{\mathcal{L}^\infty \times \mathcal{W}^2} \to \infty \ \ as \ \ \tau \nearrow \omega.$$

Proof. (i) Let $\widetilde{\theta} = (\widetilde{A}, \widetilde{b}, \widetilde{c}) \in \widetilde{\mathcal{P}}$ and $\widetilde{x}^0 \in \mathbb{R}^{\dim \widetilde{\theta}}$ be such that $\widetilde{P} = P_{\widetilde{\theta}, \widetilde{x}^0}$. The closed loop can then be rewritten as $\dot{x} = f(t, x)$, $x(0) = \widetilde{x}^0$, where

$$f : \mathcal{D} \to \mathbb{R}^n, \quad (t, x) \mapsto \widetilde{A}x + \widetilde{b}u_0(t)$$

$$+ \widetilde{b}\frac{\varphi_0(t)^2}{(1 - \varphi_0(t)|y_0(t) - cx|)^2}(y_0(t) - \widetilde{c}x) + \widetilde{b}\frac{\varphi_1(t)}{1 - \varphi_1(t)|\dot{y}_0(t) - \widetilde{c}\widetilde{A}x|}(\dot{y}_0(t) - \widetilde{c}\widetilde{A}x)$$

for $\mathcal{D} := \{\ (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \ | \ (t, y_0(t) - cx) \in \mathcal{F}_{\varphi_0},\ (t, \dot{y}_0(t) - cAx) \in \mathcal{F}_{\varphi_1}\ \}$, and

$$y_1 = \widetilde{c}x, \ \dot{y}_1 = \widetilde{c}\widetilde{A}x, \qquad\qquad\qquad y_2 = y_0 - y_1, \ \dot{y}_2 = \dot{y}_0 - \dot{y}_1,$$

$$u_2 = -\left(\frac{\varphi_0}{1 - \varphi_0|y_2|}\right)^2 y_2 - \frac{\varphi_1}{1 - \varphi_0|\dot{y}_2|}\dot{y}_2, \qquad u_1 = u_0 - u_2.$$

Now, as in the proof of Theorem 2.1, the theory of ordinary differential equations [33, Thm. III, sect. 10.XX] ensures existence and uniqueness of a maximally extended solution.

(ii) For $t \in [0, \omega)$, let $k_i(t) = \frac{\varphi_i(t)}{1 - \varphi_i(t)|y_2^{(i)}(t)|}$, $i = 0, 1$.

By construction $\varphi_i(t)|y_2^{(i)}(t)| < 1$ for all $t \in [0, \omega)$. We may choose $\varepsilon \in (0, \omega)$ such that

$$\boxed{\forall\, t \in [0, \varepsilon]\ \forall\, i \in \{0, 1\}:\ |y_2^{(i)}(t)| \leq |y_2^{(i)}(0)| + 1\ \wedge\ k_i(t) \leq k_i(0) + 1}.$$

In the following we adopt the notation (A.1), i.e., $\psi_{0/1}(\cdot)$ denote the funnel boundaries. We will show that boundedness of u_2 implies boundedness of $k_0(\cdot)$ and $k_1(\cdot)$ on the interval $[0, \omega)$. Then the same line of argument as in the proof of Theorem 2.1 shows that $\omega = \infty$ and that $y_2, \dot{y}_2$ are uniformly bounded away from their corresponding funnel boundaries.

Seeking a contradiction, assume (a) k_0 is unbounded and k_1 is bounded, (b) k_0 is bounded and k_1 is unbounded, or (c) both k_0 and k_1 are unbounded. Cases (a) and (b) can be treated analogously; therefore we consider only case (a). Boundedness of u_2 implies that the product $k_0^2 y_2$ is bounded; hence unboundedness of k_0 implies that we may choose a sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n \nearrow \omega$ and $k_0(t_n) \to \infty$ and $y_2(t_n) \to 0$. This is a contradiction because $|y_2(t_n)| < \lambda_0/2$ implies $\psi_0(t_0) - |y_2(t_n)| > \lambda_0/2$; hence $k_0(t_n) < 2/\lambda_0$.

It remains to consider (c). Assume that k_0 and k_1 are both unbounded. Since the (weak) derivative of ψ_i, $i = 0, 1$, is essentially bounded on $[\varepsilon, \omega)$ and the (weak) derivative of $y_2^{(i)}$, $i = 0, 1$, is essentially bounded on $[0, \omega)$ by assumption, it follows that for all $i \in \{0, 1\}$ and all $s, t \in [\varepsilon, \omega)$ with $t > s$,

$$\psi_i(t) - |y_2^{(i)}(t)| \leq \underbrace{\psi_i(s) - |y_2^{(i)}(s)|}_{=1/k_i(s)} + \underbrace{\big(\|\dot{\psi}_i\|_{\varepsilon,\infty} + \|y_2^{(i+1)}\|_\infty\big)}_{=:M_i}(t - s).$$

Hence, by choosing s such that $0 < t - s \leq \omega - s$ is small enough and $k_i(s)$ is big enough it holds that

$$(A.27) \quad \forall\, M > 0\ \forall\, i \in \{0, 1\}\ \exists\, s_i \in [\varepsilon, \omega)\ \forall\, t \in [s, \omega):$$
$$k_i(t) = 1/(\psi_i(t) - |y_2^{(i)}|) \geq \frac{1}{1/k_i(s_i) + M_i(\omega - s_i)} \geq M.$$

This implies that $k_i(t) \to \infty$ as $t \nearrow \omega$, and therefore by positivity and continuity of ψ_i we have $\lim_{t \nearrow \omega} |y_2^{(i)}(t)| \to \psi_i(\omega)$, and close to ω no sign change occurs for $y_2^{(i)}$, $i = 0, 1$. First, assume that $\dot{y}_2$ is positive near ω; then choose $t^* \in [\varepsilon, \omega)$ such that, in view of (A.27) and the properties of $\mathcal{G}_2$, for a.a. $t \in [t^*, \omega)$: $\dot{y}_2(t) \geq \psi_1(t) - \delta > -\dot{\psi}_0(t)$. Hence $t \mapsto \psi_0(t) + y_2(t)$ is strictly increasing on $[t^*, \omega)$ which, in view of $\lim_{t \nearrow \omega} \psi_0(t) - |y_2(t)| = 0$, is possible only if $y_2(t)$ is positive on $[t^*, \omega)$. Second, the analogue argument shows that a negative sign of $\dot{y}_2$ near ω implies a negative

sign of y_2 near ω. Altogether this shows that y_2 and $\dot{y}_2$ have the same sign near ω. In particular, boundedness of u_2 implies that both products $k_0^2 y_2$ and $k_1 \dot{y}_2$ must be bounded, which yields a contradiction in the same way as in cases (a) and (b).

(iii) This follows directly from (i) and (ii). $\quad\square$

A.5.2. Proof of Theorem 3.5. Since the perturbed closed loop $[P_{\widetilde{\theta},\widetilde{x}^0}, C_{\varphi_0,\varphi_1}]$ is, according to Lemma A.2, regularly well posed, we can repeat the proofs of [11, Props. 4.3, 4.4] (see also [26, Thms. 6.5.3, 6.5.4] for signal spaces in the present setting) to show existence of functions η and α such that (3.13) implies that the closed loop $[P_{\widetilde{\theta},\widetilde{x}^0}, C_{\varphi_0,\varphi_1}]$ maps $(u_0, y_0) = (u_{\mathrm{d}}, y_{\mathrm{ref}}) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ to $(u_1, y_1), (y_2, u_2) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. In particular, there exists a unique global and uniformly bounded solution. As shown in the proof of Lemma A.2(ii), boundedness of (u_2, y_2) implies that the gain functions k_0 and k_1 of the funnel controller are bounded, which in turn shows that the error and its derivative, i.e., y_2 and $\dot{y}_2$, are uniformly bounded away from the funnel boundaries.

It remains to show that the state variable x of the linear system corresponding to $\widetilde{\theta} = (\widetilde{A}, \widetilde{b}, \widetilde{c})$ and its derivative are bounded. Detectability of $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ yields the existence of $F \in \mathbb{R}^q$, $q := \dim \widetilde{\theta}$, such that $\mathrm{spec}\,(\widetilde{A} + F\widetilde{c}) \subseteq \mathbb{C}_-$. Setting $g := -\left[F - k_0^2\,\widetilde{b}\right]y_2 + k_1\,\widetilde{b}\dot{y}_2 + \widetilde{b}\,u_0$ gives

$$(A.28) \qquad \dot{x} = \left[\widetilde{A} - k_0^2\,\widetilde{b}\widetilde{c}\right]x - k_1\,\widetilde{b}\widetilde{c}\,\dot{x} + \widetilde{b}\,u_0 + k_0^2\,\widetilde{b}\,y_0 + k_1\,\widetilde{b}\,\dot{y}_0 = \left[\widetilde{A} + F\widetilde{c}\right]x + g.$$

Since $y_2 \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and $k_i \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$, $i \in \{0,1\}$, and since $w_0 = (u_0, y_0) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ it follows that $g \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. Hence, by (A.28) and variation of constants we obtain $x \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ and, by (A.28), also $\dot{x} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. $\quad\square$

Appendix B. "Close" systems in terms of the gap metric.

EXAMPLE B.1. *Consider the linear system $P \in \mathcal{P}$, given by $\dot{x} = \begin{bmatrix} 0 & 1 \\ -a^2 & 2a \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $y = [1,0]x$, where $a > 0$, and the "disturbed system" $\widetilde{P} \in \widetilde{\mathcal{P}}$ given by*

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2a^2M^2 & 4aM^2 - 3a^2M & 6aM - 2M^2 - a^2 & 2a - 3M \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2M \end{bmatrix} u, \quad y = \begin{bmatrix} -M, 1, 0, 0 \end{bmatrix} x,$$

where $M > 0$. Their transfer functions are given by $g(s) = \frac{1}{(s-a)^2}$ and $\widetilde{g}(s) = \frac{-2M(s-M)}{(s-a)^2(s+2M)(s+M)}$, respectively; hence $\widetilde{P}$ is a system with relative degree three, with negative high-frequency gain $-2M$, and with a zero M in the right-half plane: the system is not minimum phase. Both systems arise from the example in [26, sect. 6.3.1] by multiplication with $\frac{1}{s-a}$. Note that the line of argument in [26, sect. 6.3.1] is incomplete and we were not able to prove the following estimation for the gap metric defined in $\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}$. However, if we replace $\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}$ by $\mathcal{W}_0^{1,\infty} \times \mathcal{W}^{2,\infty}$ in the definition of graphs and gap metric, one can adopt the idea from [26, sect. 6.3.1] to show that $\lim_{M\to\infty} \vec{\delta}(P, \widetilde{P}) = 0$; i.e., in an arbitrary small neighborhood of the nominal plant $P \in \mathcal{P}$, we find a plant $\widetilde{P}$ having relative degree three, negative high-frequency gain, and being nonminimum phase.

Appendix C. Friction model. Friction counteracts the acceleration of a body in motion. The popular (nonlinear and dynamic) Lund–Grenoble friction model introduced in [4] cannot reproduce hysteretic behavior with nonlocal memory (see [32]) and nonphysical drift phenomena may occur for small vibrational forces (see [5]). However, it is adequate for the position control problem since most of the friction effects

observed in "reality" are covered, e.g., sticking, break-away (and varying break-away forces), presliding displacement, and frictional lag; moreover, stick-slip and hunting for controllers with integrational part can be reproduced (see, e.g., [4, 27]) and the Lund–Grenoble friction model can be rendered passive [2].

To explain the Lund–Grenoble friction model, we first introduce, following [27], the Stribeck function. For Coulomb friction torque u_C and static friction (stiction) torque u_S such that $0 < u_C \leq u_S$, Stribeck velocity $\Omega_S > 0$, stiffness $\sigma_0 > 0$ of the bristles, and $\delta_S \in [1/2, 2]$, let the Stribeck function be given by

$$(C.1) \qquad \beta : \mathbb{R} \to [u_C/\sigma_0, u_S/\sigma_0], \quad \Omega \mapsto \sigma_0^{-1} \left(u_C + (u_S - u_C) \exp^{-(|\Omega|/\Omega_S)^{\delta_S}} \right).$$

The function $\beta(\cdot)$ covers the Stribeck effect (Stribeck curve): a "rapid" decrease in friction for increasing but very low speeds close to standstill [31].

Next, the dynamics of the average bristle deflection $\vartheta(\cdot)$ of the asperity junctions is modeled, for some angular velocity $\Omega \in \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and initial average bristle deflection $\vartheta^0 \in \mathbb{R}$, by

$$(C.2) \qquad \dot{\vartheta}(t) = \Omega(t) - \frac{|\Omega(t)|}{\beta(\Omega(t))} \, \vartheta(t), \qquad \vartheta(0) = \vartheta^0 \,.$$

The damping (of the deflection rate $\dot{\vartheta}(\cdot)$) and the viscous friction are modeled, for $\sigma_1, \sigma_2, \Omega_D > 0$ and $\delta_D, \delta_V \geq 1$, by

$$\sigma_D : \mathbb{R} \to [0, \sigma_1], \quad \Omega \mapsto \sigma_1 \exp^{-(|\Omega|/\Omega_D)^{\delta_D}} \quad \text{and} \quad \sigma_V : \mathbb{R} \to \mathbb{R}, \quad \Omega \mapsto \sigma_2 |\Omega|^{\delta_V} \operatorname{sgn}(\Omega).$$

We are now ready to define the friction operator mapping the angular velocity to the friction torque and which is parameterized by ϑ^0:

$$(C.3) \quad \begin{aligned} &T : \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \to \mathcal{L}_{\text{loc}}^{\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}), \\ &\Omega(\cdot) \mapsto \sigma_0 \vartheta_{\Omega(\cdot)} + \sigma_D(\Omega) \left(\Omega - \frac{|\Omega|}{\beta(\Omega)} \vartheta_{\Omega(\cdot)} \right) + \sigma_V(\Omega), \text{ with } \vartheta_{\Omega(\cdot)} \text{ solves (C.2).} \end{aligned}$$

Some care must be exercised to show that (C.3) is well defined. We first show that the initial-value problem (C.2) has a unique solution for each $\Omega \in \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R})$:

$$\vartheta_{\Omega(\cdot)} : \mathbb{R}_{\geq 0} \to \left[- \max \left\{ u_S/\sigma_0, |\vartheta^0| \right\}, \max \left\{ u_S/\sigma_0, |\vartheta^0| \right\} \right].$$

Existence, uniqueness, and extension on $\mathbb{R}_{\geq 0}$ follow from the standard theory of linear, time-varying, differential equations; furthermore, it is easy to see that if $|\vartheta_{\Omega(\cdot)}(t)| \geq u_S/\sigma_0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \big(\vartheta_{\Omega(\cdot)}(t)^2 \big) = -2 |\vartheta_{\Omega(\cdot)}(t) \Omega(t)| \left(\underbrace{- \operatorname{sgn} \big(\vartheta_{\Omega(\cdot)}(t) \Omega(t) \big)}_{\in \{-1, 0, 1\}} + \frac{|\vartheta_{\Omega(\cdot)}(t)|}{\beta(\Omega(t))} \right) \leq 0,$$

and hence $|\vartheta_{\Omega(\cdot)}(t)| \leq \max\{u_S/\sigma_0, |\vartheta^0|\}$ for all $t \geq 0$. Therefore, we have, for all $\Omega \in \mathcal{L}^{\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$,
(C.4)

$$\|T(\Omega)\|_{\infty} \leq \sigma_0 \max \left\{ \frac{u_S}{\sigma_0}, |\vartheta^0| \right\} + \sigma_1 \|\Omega\|_{\infty} \left(1 + \frac{\sigma_0}{u_C} \max \left\{ \frac{u_S}{\sigma_0}, |\vartheta^0| \right\} \right) + \sigma_2 \|\Omega\|_{\infty}^{\delta_V},$$

and so T is well defined.

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