How Fast is Too Fast?
Rate-induced Bifurcations in Multiple Time-scale Systems

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Abstract

This thesis studies the phenomenon of rate-induced bifurcations. Externally forced systems may have a critical rate above which they undergo some sort of destabilisation, and move away suddenly to a new state. Mathematically, the phenomenon is a non-autonomous instability.

We present a framework in which rate-induced bifurcations can be studied. This is based on geometric singular perturbation theory which is derived from Fenichel’s Theorem. In particular we make use of folded singularities and canard trajectories, which are modern concepts from geometric singular perturbation theory.

We concentrate on systems with multiple time-scales where the mechanism for a rate-induced bifurcation is not obvious. So much so, that once a multiple time-scale system has undergone a rate-induced bifurcation, the instability threshold which separates initial states that destabilise from those that adiabatically follow a changing stable state is described as non-obvious. We study in detail the complicated non-obvious instability threshold that arises near a folded saddle-node (type I) singularity. In particular, we show how the complicated threshold structure depends on two parameters – the ratio of time-scales and the folded singularity bifurcation parameter. In contrast, we also show single time-scale systems where the rate-induced bifurcation is caused by a large perturbation in the boundary of the basin of attraction for the stable state.
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1. Introduction

This work is motivated by the phenomena of rate-induced bifurcations (see also, rate-induced tipping points) in multiple time-scale systems. This phenomenon was only recently identified, first in a simplified model for soil carbon decomposition [47, 74], then in models for type III neurons [49, 72]. The theory of rate-induced bifurcations is under development, see [74, 2, 3]. The body of the thesis focuses on multiple time-scale systems, however some new results for simple, single time-scale systems are included in the Chapter 4.

1.1. Motivation

In science, “tipping points” or “critical transitions” are sudden, often unexpected, changes in the state of a complex system, triggered by slowly-varying external conditions.

These changes are usually associated with a critical level of external conditions at which the stable state (an attractor) disappears or destabilises in a bifurcation, causing the system to move to another state [66, 40]. For example, in Fig. 1.1(c) the system tracks the changing stable state until the external conditions reach a critical level at which the stable state disappears in a saddle-node bifurcation. It is intuitively clear that past the critical level there is no (nearby) stable state to track, and all trajectories move away to another part of the state space, directly as a result of the bifurcation.

Such bifurcation-induced tipping points have been studied quite extensively [66, 40, 2]. This is because the slow passage through a saddle-node bifurcation [Fig. 1.1(c)] became a paradigm of a tipping point after it was identified in idealised models as a mechanism responsible for the collapse of thermohaline circulation at a critical level of fresh-water influx into the North Atlantic [18, Ch.16], loss of submerged vegetation in shallow turbid lakes at a critical level of nutrient concentration [57, Ch.7], forest-to-desert transitions at a critical level of precipitation [57, Ch.11], power outage blackouts at a critical level of energy consumption [20], and in many other critical transitions. However, recent mathematical work showed that tipping phenomena are not just bifurcation-induced [74, 19, 2]. There are other independent tipping mechanisms, and these mechanisms may be associated with different critical factors.

It turns out that some systems simply do not have any critical levels of external conditions, but they may have critical rates of change. These systems fail to track the changing stable state only if external conditions change too fast. For example, in Fig. 1.1(a)-(b) the stable state never bifurcates — it exists continuously for all fixed values of external conditions.
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Intuitively, one might expect that the system tracks the continuously changing stable state and never moves away. Indeed, this happens if external conditions change slowly enough. However, tracking is not always the case. The system may fail to track the continuously changing stable state, and move away, if external conditions change above some critical rate [58, 74]. This phenomenon is a rate-induced bifurcation, or a rate-induced tipping point [2].

Rate-induced bifurcations encompass various phenomena from the natural world that are being explored by scientists. For example, in the current and future climate, the critical factor may be the rate of global warming rather than the temperature itself. This is evidenced by examples of a rapid decrease in the ability of many animals and plants to migrate or adapt with an increasing rate of climate change [43], critical dependence of thermohaline circulation on the rate of North-Atlantic fresh-water influx [46], and the sudden release of soil carbon from peatlands into the atmosphere above some critical rate of global warming [47, 74]. In neuroscience, nerves can accommodate [31] slow changes in an externally applied voltage but an excitation occurs when the voltage increases too fast. For example, type III excitable neurons fire action potentials [33, 72, 49] and electrical excitation waves start to propagate in heart tissue [8] only above some critical rate of voltage increase. In ecology, there is a critical-rate hypothesis in the context of regime shifts for ecosystems which are subject to too rapid change in external conditions such as wet El Niño Southern Oscillation years, droughts, or disease outbreaks [58]. In quantum mechanics, the adiabatic theorem states that the final state of a quantum system with continuously changing Hamiltonian depends, critically, on the rate at which the Hamiltonian changes [36, 4].

Nonetheless, scientists often find rate-induced bifurcations counter-intuitive because the ensuing critical transitions cannot be associated with any obvious loss of stability. Secondly, there might not even be any obvious intuitive threshold, such as a nearby unstable state or basin boundary, that would indicate moving away from the changing stable state [74, 50]. Finally, the critical rate may be slower than the slowest time-scale of the static system [2].

Figure 1.1. The conceptual difference between (a)–(b) a rate-induced bifurcation and (c) a bifurcation-induced tipping point in systems with a time-varying external input. The “stable state” is an asymptotic stable state when the external input is fixed in time. In (a)–(b), in response to a varying external input either a trajectory tracks the moving stable state (blue), or moves (rapidly) away and the initial state destabilises (red).
1. Introduction

In mathematics, finding rate-induced bifurcations is an interesting problem because it involves some kind of non-autonomous instability that cannot, in general, be described by traditional bifurcation theory or by an asymptotic approach \[9, 74\]. For example, the rate dependence and threshold in the compost-bomb instability \[74\] and type III neuron excitability \[72\] have been explained only recently, using canards and folded-saddle singularities, which are concepts from modern geometric singular perturbation theory.

Here, we use an approach to study critical rates of change and their corresponding instability thresholds based on geometric singular perturbation theory, which derives from Fenichel’s Theorem \[26\] (detailed in Appendix A.1). It allows us to study rate-induced bifurcations for systems with multiple time-scales in terms of slow manifolds and canard trajectories in suitably extended systems. A similar approach can also be used to study rate-induced bifurcations in single time-scale systems, where trajectories track some perturbed stable state, and the critical rate is identified with initial states lying on a perturbed basin boundary.

1.2. Rate-induced bifurcations in forced systems

Let us consider an arbitrary high-dimensional complex system where we have identified a low-dimensional subsystem that we wish to check for a rate-induced bifurcation. If the influence of the subsystem’s environment can be approximated by a time-varying external forcing function \(\lambda(\epsilon t)\), then the subsystem can be modelled by a non-autonomous ordinary differential equation

\[
\frac{dx}{dt} = f(x, \mu, \lambda(\epsilon t)).
\]  

(1.1)

Here, \(x \in \mathbb{R}^n\) is the subsystem state vector, \(\mu \in \mathbb{R}^k\) represents subsystem parameters that do not vary in time, \(t \in \mathbb{R}\) is time, and \(\lambda \in \mathbb{R}^l\) represents external conditions that vary on a time-scale \(\epsilon t\). The functions \(f\) and \(\lambda\) are sufficiently smooth.

In line with other theoretical work on rate-induced bifurcations \[74, 2, 3\], we make the following assumption:

**Assumption 1.1.** For every fixed value of external input \(\lambda\), that is when \(\epsilon = 0\), system (1.1) has a stable steady state, and this stable steady state depends smoothly on \(\lambda\), such that we can write \(\tilde{x}(\lambda)\). Where a system also has unstable steady states, we will denote the stable steady states \(\tilde{x}^s(\lambda)\) and the unstable steady states \(\tilde{x}^u(\lambda)\).

We want to know, when the external input \(\lambda(\epsilon t)\) varies in time at a rate \(\epsilon > 0\), whether the system can keep pace with the continuously changing steady state \(\tilde{x}(\lambda(\epsilon t))\).

**Definition 1.1.** When the external input \(\lambda(\epsilon t)\) varies in time, that is when \(\epsilon > 0\), we refer to \(\tilde{x}(\lambda(\epsilon t))\) as the moving steady state. Note, \(\tilde{x}(\lambda(\epsilon t))\) is typically not a steady state of system (1.1). In the existing literature \(\tilde{x}(\lambda(\epsilon t))\) has been referred to as the quasi-static equilibrium (see \[74, 2\]).
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When the external input $\lambda(\epsilon t)$ changes slowly, that is at small $\epsilon$, trajectories started at initial states $x_0 = x(t_0)$ near the moving steady state $\tilde{x}(\lambda(\epsilon t))$ will track $\tilde{x}(\lambda(\epsilon t))$ [see Fig. 1.1(a)] – they may not converge to $\tilde{x}(\lambda(\epsilon t))$, but remain nearby, with the same qualitative behaviour. However, when the external input changes too fast, that is at larger $\epsilon$, the initial states $x_0$ near the moving steady state $\tilde{x}(\lambda(\epsilon t))$ may destabilise – the trajectory started at $x_0$ undergoes a qualitative change in behaviour from its small $\epsilon$ case, and moves away from $\tilde{x}(\lambda(\epsilon t))$ to another, often distant, state [see Fig. 1.1(b)]. The initial state $x_0$ may have a well-defined critical rate $\epsilon_c(x_0)$ at which it first destabilises.

What is more, the whole system may have a critical rate $\epsilon_c$, when there is a qualitative change in the phase portrait, for example there may be a specified set of initial states which destabilise. In such cases, the system undergoes a rate-induced bifurcation.

Exactly what “destabilising” and “a qualitative change in the phase portrait” look like, depends on the system being studied. For example, it may be that trajectories move rapidly away from $\tilde{x}(\lambda(\epsilon t))$, but then return to track $\tilde{x}(\lambda(\epsilon t))$ – whether this transient destabilisation is of importance depends on the application (see neuronal excitability [49]). The mechanisms for rate-induced bifurcation phenomena are different in different systems, so we have an approach to identify rate-induced bifurcations in multiple time-scale systems (see Chapter 2), and a different approach for single time-scale systems (see Chapter 4).

Lastly, the concept of “destabilising” is closely related to the concept of “threshold”. For example, one speaks of “excitation thresholds” in neuronal excitability [31]. The idea is to fix $\epsilon$, and identify the separatrix in the $x$ state space separating those initial states $x_0$ that destabilise, from those that do not. This threshold can be very intricate, as is shown in Chapter 2.

1.3. A brief overview of different approaches to rate-induced bifurcations

A rate-induced bifurcation is a non-autonomous instability in which the system loses its ability to track all or part of the moving steady state. The phenomenon does not involve any loss of stability of the fixed stable state and cannot in general be described by traditional bifurcation theory. In recent years, various approaches have been proposed to address stability in non-autonomous dynamical systems

$$\frac{dx}{dt} = f(x, \mu, t),$$  \hspace{1cm} (1.2)

but can these approaches describe rate-induced bifurcations?

A typical approach is to define non-autonomous pullback attractors or bounded complete solutions in the $(x, t)$-phase space, and study non-autonomous bifurcations, where a pullback attractor becomes topologically different [1, 53] or a bounded complete solution folds or branches [52]. However, non-autonomous bifurcations of pullback attractors may not contain any information about the system behaviour relative to the moving steady state,
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so they may not capture rate-induced bifurcations. More promising is the concept of finite-time bifurcations [54], but the theory has not been been fully developed yet.

Another approach is to prescribe a safe region about the moving steady state in the x-space, and give criteria for the forcing function ensuring the system remains within the safe region [9, 2]. For example, Bishnani and MacKay [9] estimated the size $\epsilon$ of a suitably shaped set of forcing functions for which the solution of the forced system $dx/dt = f(x, \mu, t)$ remains within some prescribed $\eta$-neighbourhood, or safe region, of a uniformly hyperbolic reference solution of the unforced system $dx/dt = f_0(x, \mu, t)$. More recently, Ashwin et al [2] simplified forced system (1.1), by assuming a fixed-in-time stable linear operator $M$:

$$\frac{dx}{dt} = M(x - \tilde{x}(\lambda(\epsilon t))),$$

and a spherical safe region with a fixed “tipping radius”. They obtained general criteria in terms of a maximum rate of change of the moving steady state, that is sup$(d\tilde{x}/dt)$, to ensure the system remained in the sphere at time $t$.

In general, the tipping radius approach does not necessarily capture whether there has been a qualitative change in the phase portraits at some rate of forcing $\epsilon$, that is, a rate-induced bifurcation. This is illustrated by examples in Section 1.5. However, the tipping radius approach gives a clear definition of which trajectories are safe, that is near $\tilde{x}(\lambda(\epsilon t))$, and which are not, and is suited to many applications, an obvious example being models that lose validity at the safe region boundary.

1.4. Limitations of the tipping radius approach

The examples of rate-induced bifurcations we consider in this thesis are drawn from Ashwin et al [2].

We revisit the general criteria for staying within the tipping radius for non-linear $\lambda(\epsilon t)$ from [2]. This material corrects and extends the general criteria in [2, Sec.2(b)] and has, in part, been published as an erratum [51]:


We define a maximum rate of change $r_{\text{max}}$ for any non-linear forcing function $\lambda(\epsilon t)$:

$$r_{\text{max}}(t) = \sup_{t_0 < s < t} \left| \frac{d}{dt} \tilde{x}(\lambda(\epsilon s)) \right|.$$  \hfill (1.4)

The simplified, linearised system (1.3) gives the instantaneous deviation from the moving
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steady state $\tilde{x}(\lambda(\epsilon t))$ (using the integrating factor $e^{-Ms}$):

$$x(t) - \tilde{x}(\lambda(\epsilon t)) = e^{M(t-t_0)} (x(t_0) - \tilde{x}(\lambda(\epsilon t_0))) - \int_{t_0}^{t} e^{M(t-s)} \frac{d}{dt} \tilde{x} (\lambda(\epsilon s)) \, ds.$$  \hspace{1cm} (1.5)

We assume that the dependence on initial values in (1.5) decays, meaning either $x(t_0) = \tilde{x}(\lambda(\epsilon t_0))$ or $(t-t_0) \to \infty$, and use (1.4) to obtain:

$$|x(t) - \tilde{x}(\lambda(\epsilon t))| \leq r_{\max}(t) \int_{0}^{t-t_0} \|e^{Mu}\| \, du.$$  \hspace{1cm} (1.6)

If $M$ is stable, then [32, Lemma 3.3.19]:

$$\|e^{Mu}\| \leq ce^{-\beta u}$$  \hspace{1cm} (1.7)

for some $c, \beta > 0$, so

$$|x(t) - \tilde{x}(\lambda(\epsilon t))| \leq r_{\max}(t) \frac{c}{\beta} \left(1 - e^{-\beta(t-t_0)}\right).$$

So the trajectory stays within the tipping radius $R$ up to to time $t$, that is:

$$|x(t) - \tilde{x}(\lambda(\epsilon t))| < R,$$

if,

$$r_{\max}(t) \frac{c}{\beta} \left(1 - e^{-\beta(t-t_0)}\right) < R.$$  \hspace{1cm} (1.8)

If $M$ is a scalar, we have

$$c = 1, \quad \beta = -M,$$

and (1.8) reduces to [2, Eq. (2.9)] in the limit $(t-t_0) \to \infty$. However, if $M$ is a matrix, Eq. (2.9) in [2] is no longer valid. Rather, we need a good choice of $c$ and $\beta$ in (1.7) so condition (1.8) is optimal, see [32, Sec. 5.5].

If $M$ is diagonalisable by an invertible matrix $D$ whose columns are eigenvectors of $M$, and $M$ has eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n < 0$, then

$$c = \|D\| \|D^{-1}\| \geq 1, \quad \beta = |\lambda_n|,$$

which is known as the eigenvector bound. This bound may be far from optimal, because it simply uses the eigenvectors of $M$. Better bounds can be obtained for any matrix $M$ with negative real-part eigenvalues, by using a self-adjoint matrix $H$ that solves the (standard) Lyapunov equation:

$$M^* H + H M = -2I.$$  \hspace{1cm} (1.9)

For example, an elegant estimate in terms of $H$ was given by Godunov [68, Eq. (13)]:

$$c = \sqrt{\|H\| \|H^{-1}\|} \geq 1, \quad \beta = \|H^{-1}\|.$$
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We remark that Eqs. (1.4)–(1.5) do not easily give sufficient conditions for

$$|x(t) - \tilde{x}(\lambda(t))| \geq R$$

in terms of $r_{\text{max}}(t)$. That is, Eq. (2.10) in [2] is no longer valid, except when $r_{\text{max}}$ does not vary in time.

The tipping radius approach in Ashwin et al [2] can be used to study rate-induced bifurcations. Indeed, the examples in [2, Sec. 3] all exhibit rate-induced bifurcations. However, choosing a suitable tipping radius $R$ such that crossing the tipping radius coincides with a rate-induced bifurcation is a challenge, even for simple non-linear systems with linear external forcing. In [2], where feasible, an effective tipping radius $R_c$ is given so crossing $R_c$ coincides with the rate-induced bifurcation. The simplified linear theory in Ashwin et al [2] does not take into account the importance of non-linear terms in (1.1), or the more general geometry of the tipping threshold which may change with time and may be far from being shaped like a sphere.

We present frameworks to study rate-induced bifurcations in multiple and single time-scale systems that take all these complications into account. Our work is based on geometric singular perturbation theory [26, 35], including the modern concepts of folded singularities and canard trajectories [5, 24, 62]. We do not need to estimate an effective tipping radius $R_c$, instead we compute the (perturbed) moving steady states, discover at what value of $\epsilon$ a qualitative change in the dynamics occurs, and declare the system to have had a rate-induced bifurcation.

1.5. Example of leaving a safe region and of a rate-induced bifurcation

In general, the safe region approach works for models that do not produce a rate-induced bifurcation phenomenon as well as for models that do produce a rate-induced bifurcation phenomenon. This is illustrated by:

**Example 1.5.1.** Consider an initial value problem for a non-autonomous differential equation

$$\frac{dx}{dt} = -(x - \lambda(\epsilon t)), \quad x(0) = x_0, \quad (1.10)$$

with steady drift $\lambda(\epsilon t) = -\epsilon t$. The system has a unique moving stable steady state $\tilde{x}(\lambda) = \lambda$.

Integrating (1.10) gives a continuous deviation, in $x_0$ and $\epsilon$, from the moving steady state:

$$x(t) - \tilde{x}(\lambda(\epsilon t)) = e^{-t}x_0 + \epsilon(1 - e^{-t}).$$

Notice, the deviation from the moving steady state at large time $t$ increases linearly with $\epsilon$ [Fig. 1.2(a)], thus there is no qualitative difference in the phase portrait as $\epsilon$ increases. Although there is no rate-induced bifurcation for (1.10), one can prescribe a safe region
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Figure 1.2. A deviation of the trajectory \( x(t, x_0, \epsilon) \) from the moving stable steady state \( \tilde{x}(\lambda(\epsilon t)) \) as a function of time \( t \) and \( \epsilon \). At a fixed time \( t_c \): (a) the deviation is a linear function of \( \epsilon \) in system (1.10), so \( \epsilon_c \) can only be defined with reference to a tipping radius \( R \) (given by grey plane); whereas in (b) the deviation increases ‘abruptly’ with \( \epsilon \) in system (1.11), so \( \epsilon_c(x_0, t_c) \) is at the point of abrupt change. In (a)-(b), \( x_0 = 0.5, t_c = 8 \), and the deviation is delineated at \( \epsilon = 0.8 \) and 1.2.

about the moving steady state [see Fig. 1.2(a)] and say that the system undergoes rate-induced tipping if the trajectory \( x(t, x_0) \) leaves the safe region. For example, we follow [2] and say the system “tips” when \( x(t) - \tilde{x}(\lambda(\epsilon t)) = R \).

The approach developed in this thesis differs from the safe region approach in three aspects. Firstly, it is designed for systems that produce a rate-induced bifurcation – that is a qualitative change in the phase portrait. Secondly, it avoids specifying any ad hoc tipping threshold/safe region. Instead, our approach exploits an intrinsic threshold – some discontinuity in the deviation from the moving steady state, which arises from strong system non-linearities. Thirdly, critical rates and instability thresholds give conditions for the system to undergo a rate-induced bifurcation, rather than to avoid a rate-induced bifurcation.

This is illustrated by:

**Example 1.5.2.** Consider an initial value problem for a non-autonomous differential equation

\[
\frac{dx}{dt} = (x - \lambda(\epsilon t))(x - \lambda(\epsilon t) - 2\mu), \quad x(0) = x_0, \quad (1.11)
\]

with steady drift \( \lambda(\epsilon t) = -\epsilon t \). This system has a stable moving steady state \( \tilde{x}^a(\lambda) = \lambda \), and an unstable moving steady state \( \tilde{x}^r(\lambda) = \lambda + 2\mu \). One could impose a safe region about \( \tilde{x}^a(\lambda(\epsilon t)) \) that gives a lower bound of the critical rate [9, 2], but it does not seem to easily give the critical rate itself. Instead, we exploit an intrinsic instability due to \( \tilde{x}^r(\lambda) \). At \( \epsilon_c = \mu^2 \), the phase portrait changes qualitatively, see Fig. 1.2(b). More precisely, qualitative analysis of

\[
\frac{d(x - \tilde{x}^a(\lambda(\epsilon t)))}{dt} = (x - \tilde{x}^a(\lambda(\epsilon t)))(x - \tilde{x}^a(\lambda(\epsilon t)) - 2\mu) + \epsilon,
\]

reveals that long-term deviation from the moving stable steady state has a discontinuity in
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$x_0$ and $\epsilon$:

$$\lim_{t \to \infty} (x(t) - \tilde{x}(\lambda(\epsilon t))) = \begin{cases} 
\mu - \sqrt{\mu^2 - \epsilon} & \text{if } \epsilon < \mu^2 \text{ and } x_0 < x_{\text{thr}} = \mu + \sqrt{\mu^2 - \epsilon}, \\
\infty & \text{if } \epsilon < \mu^2 \text{ and } x_0 > x_{\text{thr}} = \mu + \sqrt{\mu^2 - \epsilon}, \\
\infty & \text{if } \epsilon > \mu^2 \text{ and } x_0 \in \mathbb{R}.
\end{cases}$$

This means that for $\epsilon < \mu^2$ there is an $\epsilon$-dependent instability threshold $x_{\text{thr}} = \mu + \sqrt{\mu^2 - \epsilon}$.

Furthermore, system (1.11), initialised at $x_0$ below $x_{\text{thr}}$, destabilises if, and only if, the drift rate $\epsilon$ exceeds the critical value $\epsilon_c = \mu^2$.

In general, the computation of critical rates and instability thresholds is far more complicated than in Example 1.5.2 (see Chapter 4 for further examples in single time-scale systems).

1.6. Outline of thesis

The thesis is structured into three chapters, the first two of which each constitute a paper. As such, each chapter can be read in isolation, and important concepts from geometric singular perturbation theory are reintroduced in each. Chapter 2 presents the non-obvious banded instability thresholds that form following a rate-induced bifurcation in multiple time-scale system. Chapter 3 reveals how the canard trajectories that comprise the complicated banded instability threshold change as the ratio of time-scales and rate of external forcing change. Finally, Chapter 4 presents two methods for computing critical rates for rate-induced bifurcations in single time-scale systems.

Material in Chapter 2 is based on [50]:


Material in Chapter 3 is based on a paper that is being finally redrafted before submission:

C. Perryman and S. Wieczorek. Bifurcations of canard trajectories near a type I folded saddle-node singularity. For submission to *Nonlinearity*. 

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2. Adapting to a Changing Environment: Non-Obvious Thresholds in Multi-Scale Systems

Many natural and technological systems fail to adapt to changing external conditions and move to a new state if the conditions vary too fast. Such “non-adiabatic” processes are ubiquitous, but little understood. We identify these processes with a new non-linear phenomenon – an intricate threshold where a forced system fails to adiabatically follow a changing stable state. In systems with multiple time-scales, we derive existence conditions that show such thresholds to be generic, but non-obvious, meaning they cannot be captured by traditional stability theory. Rather, the phenomenon is organised by concepts from modern singular perturbation theory: folded singularities and canard trajectories, including composite canards. Thus, non-obvious thresholds should explain the failure to adapt to a changing environment in a wide range of multi-scale systems including: tipping points in the climate system, regime shifts in ecosystems, excitability in nerve cells, adaptation failure in regulatory genes, and adiabatic switching in technology.

2.1. Introduction

The time evolution of real-world systems often takes place on multiple time-scales, and is paced by aperiodically changing external conditions. Of particular interest are situations where, if the external conditions change too fast, the system fails to adapt and moves to a new state. In climate science and ecology one speaks of “rate-induced tipping points” [74, 44, 60, 43], the “critical rate hypothesis” [58], and “adaptation failure” [10] to describe the sudden transitions caused by too rapid changes in external conditions (e.g. dry and hot climate anomalies or wet periods due to El Niño-Southern Oscillation). In neuroscience, type III excitable nerves [34, Ch. 7] accommodate slow changes in an externally applied voltage, but an excitation requires a rapid enough increase in the voltage [31, 8]. However, such rate-induced transitions cannot be explained by classical stability theory, and require an alternative approach.

This chapter conceptualises the failure to adapt to a changing environment as a rate-induced bifurcation [74, 2] – a non-autonomous instability characterised by critical rates of external forcing [74, 2] and instability thresholds [74, 49]. Rate-induced bifurcations can be counter-intuitive because they occur in systems where a stable state exists continuously for all fixed values of the external input [Fig. 1.1(a)–(b)]. When the external input varies
in time, the position of the stable state changes and the system tries to keep pace with the changes. The forced system *adiabatically follows or tracks* the continuously changing stable state if the external input varies slowly enough [Fig. 1.1(a)]. However, many systems fail to track the changing stable state if the external input varies too fast. These systems have initial states that destabilise – trajectories move away to a new, distant state – above some critical rate of forcing [Fig. 1.1(b)]. This happens even though there is no obvious loss of stability. Moreover, in systems with multiple time-scales there may be no obvious threshold separating the adiabatic and non-adiabatic responses in Fig. 1.1(b). This is in contrast to dynamic bifurcations [6], which can be explained by classical bifurcations of the stable state at some critical level of external input [Fig. 1.1(c)]. In this case, the forced system destabilises predictably around the critical level, independently of the initial state and of the rate of change.

In the absence of an obvious threshold, scientists are often puzzled by the actual boundary separating initial states that adapt to changing external conditions from those that fail to adapt. The first non-obvious threshold was identified only recently, in the context of a rate-induced climate tipping point termed the “compost-bomb instability”, as a *folded saddle canard* [74]. This finding explained a sudden release of soil carbon from peat lands into the atmosphere above some critical rate of warming, which puzzled climate carbon-cycle scientists [47, 74]. Subsequently, similar non-obvious “firing thresholds” explained the spiking behaviour of type III neurons [49, 72].

Here, we reveal a non-obvious threshold with an intricate band structure. The uncovered threshold is generic, and should explain the failure to adapt to a changing environment in a wide range of non-linear multi-scale systems. Specifically, the intricate band structure is shown to arise from a combination of the complicated dynamics due to a folded node singularity [62] and the simple threshold behaviour due to a folded saddle singularity [74] near a type I folded saddle-node singularity [39, 28, 70]. What is more, the threshold is identified with special *composite canards* – trajectories that follow canard segments of different folded singularities. More generally, we derive existence results for critical rates and non-obvious thresholds, and discuss our contribution in the context of canard theory and its applications.

### 2.2. A general framework and existence results for non-obvious thresholds

Our general framework is based on geometric singular perturbation theory [25, 35]. It builds on the ideas developed in [74], and extends the analysis to a general forcing type. Specifically, we consider multi-scale dynamical systems akin to simple climate, neuron,
and electrical circuit models [47, 74, 56, 12, 49, 72, 67]:

\[
\begin{align*}
\delta \frac{dx}{dt} &= f(x, y, \lambda(\epsilon t), \delta), \\
\frac{dy}{dt} &= g(x, y, \lambda(\epsilon t), \delta),
\end{align*}
\]

with a fast variable \(x\), slow variable \(y\), and sufficiently smooth functions \(f\) and \(g\). The small parameter \(0 < \delta \ll 1\) quantifies the ratio of the \(x\) and \(y\) time-scales. The time-varying external input \(\lambda(\epsilon t)\) is bounded between \(\lambda_{\text{min}}\) and \(\lambda_{\text{max}}\), and evolves on a slow time-scale \(\tau = \epsilon t\), where \(\tau \in (\tau_{\text{min}}, \tau_{\text{max}})\).

The system has two small parameters: \(\delta\) and \(\epsilon\). While the analysis of rate-induced bifurcations is greatly facilitated by the singular limit \(\delta = 0\), it requires non-zero \(\epsilon\). The limit \(\epsilon = 0\) gives the conceptual starting point for the analysis.

When \(\lambda\) does not vary in time, i.e. when \(\epsilon = 0\), Eqs. (2.1)–(2.2) define a dynamical system with one fast and one slow variable, and a parameter \(\lambda\). In the singular limit \(\delta = 0\), the slow subsystem \(\frac{dy}{dt} = g(x, y, \lambda, 0)\) evolves on the one-dimensional critical manifold \(S(\lambda)\), defined by \(f(x, y, \lambda, 0) = 0\). Alternatively, \(S(\lambda)\) consists of steady states of the fast subsystem \(\frac{dx}{dT} = f(x, y, \lambda, 0)\), where \(T = t/\delta\) is the fast time-scale, and \(y\) acts as a second parameter. The critical manifold can have an attracting part \(S^a(\lambda)\) and a repelling part \(S^r(\lambda)\), which are separated by a fold point \(F(\lambda)\) tangent to the fast \(x\)-direction (Fig. 2.1).

To give precise statements about non-obvious thresholds we assume for every fixed \(\lambda\) between \(\lambda_{\text{min}}\) and \(\lambda_{\text{max}}\):

**Assumption 2.1** (The system has a quadratic non-linearity). The critical manifold \(S(\lambda)\) is locally a graph over \(x\) with a single fold \(F(\lambda)\) tangent to the fast \(x\)-direction, defined by

\[
\frac{\partial f}{\partial x} \bigg|_S = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} \bigg|_S \neq 0.
\]

**Assumption 2.2** (The system has a stable state for all fixed external conditions). Near \(F(\lambda)\), \(S^a(\lambda)\) contains just one steady state \(\tilde{x}(\lambda)\) which is asymptotically stable and varies continuously with \(\lambda\).

The geometric structure of the phase space in the singular limit \(\delta = 0\) gives insight into the dynamics for \(\delta\) small, but non-zero. Specifically, where steady states of the fast subsystem are hyperbolic (i.e. on \(S^a(\lambda)\) and \(S^r(\lambda)\) but not on \(F\)), system (2.1)–(2.2) with \(0 < \delta \ll 1\) has a slow attracting manifold \(S^a_\delta(\lambda)\) and a slow repelling manifold \(S^r_\delta(\lambda)\). Both \(S^a_\delta(\lambda)\) and \(S^r_\delta(\lambda)\) are locally invariant, lie close to, and have the same stability type as \(S^a(\lambda)\) and \(S^r(\lambda)\), respectively. This follows from Fenichel’s Theorem [25, 35].

When \(\lambda\) varies smoothly in time such that \(0 < \epsilon \lesssim 1\) and \(0 < \delta \ll 1\), Eqs. (2.1)–(2.2)
define a dynamical system with one fast and two slow variables:

\[\begin{align*}
\delta \epsilon \frac{dx}{d\tau} &= f(x, y, \lambda(\tau), \delta), \\
\epsilon \frac{dy}{d\tau} &= g(x, y, \lambda(\tau), \delta), \\
\frac{d\tau}{d\tau} &= 1.
\end{align*}\] (2.4) (2.5) (2.6)

Now the critical manifolds \(S^a\) and \(S^r\), as well as the slow manifolds \(S^a_\delta\) and \(S^r_\delta\) are two-dimensional, and \(\tilde{x}\) and \(F\) form curves (Fig. 2.1). When \(\lambda(\tau)\) varies slowly enough, the forced system (2.1)–(2.2) tracks the continuously changing stable state \(\tilde{x}(\lambda(\tau))\). However, the system may fail to track, and destabilise. To be more precise, we define:

**Definition 2.1.** For a given initial state in \(S^a_\delta\), we say that system (2.1)–(2.2) destabilises if the trajectory leaves \(S^a_\delta\) and moves away along the fast \(x\)-direction. Otherwise, we say that system (2.1)–(2.2) tracks the moving stable state \(\tilde{x}(\lambda(\tau))\).

**Definition 2.2.** The critical rate \(\epsilon_c\) is the largest \(\epsilon\) below which there are no initial states in \(S^a_\delta\) that destabilise.

**Definition 2.3.** The instability threshold is the boundary within \(S^a_\delta\) separating initial states that track \(\tilde{x}(\lambda(\tau))\) from those that destabilise.

Figure 2.1(a)–(b) shows two trajectories of Eqs. (2.1)–(2.2) for different initial states on \(S^a\). Below the critical rate, all trajectories track, and eventually converge to \(\tilde{x}(\lambda(\tau))\) [Fig. 2.1(a)]. However, above the critical rate there are initial states near \(\tilde{x}\) that fail to track \(\tilde{x}(\lambda(\tau))\), and destabilise [red in Fig. 2.1(b)]. Interestingly, some trajectories leave \(S^a_\delta\) but, instead of destabilising along the fast \(x\)-direction, return to \(S^a_\delta\) and converge to \(\tilde{x}\) [blue in Fig. 2.1(b)]. The two qualitatively different behaviours in Fig. 2.1(b) show that there is an instability threshold within \(S^a_\delta\). What is more, the threshold can be simple [Fig. 2.1(c)] as reported in [74, 49], or can have an intriguing band structure [Fig. 2.1(d)] that has not been reported to date. In both cases, it is not immediately obvious what determines the threshold.

The analysis of the dynamical mechanism for non-obvious thresholds is greatly facilitated by the singular limit, where the fold and slow manifolds are unique and known exactly. System (2.4)–(2.6) is reduced to the slow dynamics on \(S\) by setting \(\delta = 0\), and then projected onto the \((x, \tau)\)-plane by differentiating Eq. (2.4) with respect to slow time \(\tau\):

\[\begin{align*}
\frac{dx}{d\tau} &= -\frac{g}{\epsilon} \frac{\partial f}{\partial x} \bigg|_S, \\
\frac{d\tau}{d\tau} &= 1.
\end{align*}\] (2.7) (2.8)

It now becomes clear that if a trajectory deviates too much from \(\tilde{x}\) and approaches a typical point on \(F\) then, according to fold condition (2.3), \(\partial f / \partial x\) in Eq. (2.7) approaches
2. Non-Obvious Thresholds in Multi-Scale Systems

Figure 2.1. (a)–(b) Trajectories starting at two different initial states (dots) on $S^a$, near the changing stable state $\tilde{x}$, (a) below and (b) above the critical rate. (c)–(d) Above the critical rate, the initial states on $S^a$ that (red) destabilise or (blue) track $\tilde{x}(\lambda)$ highlight different threshold types. We used Eqs. (2.1)–(2.2), (2.17), and [(a), (b), (d)] Eq. (2.18) with (a) $\epsilon = 0.06$ and [(b), (d)] $\epsilon = 0.216$; and (c) Eq. (2.23) with $\epsilon = 1$. Other parameters were $\delta = 0.01$, $\lambda_{\text{max}} = 2.5$. For (a)–(d) the critical manifold $S(\lambda)$ is given by $y = -\lambda - x(x - 1)$, has a fold $F(\lambda)$ at $(x, y) = (1/2, -\lambda + 1/4)$ and a unique stable steady state $\tilde{x}(\lambda)$ at $(x, y) = (0, -\lambda)$. For clarity, the plots are shown in the co-moving coordinate system $(x, y + \lambda, \lambda)$. The $\lambda$ axis can be transformed into a slow time axis using [(a), (b), (d)] Eq. (2.18) or (c) Eq. (2.23).

zero, and $x$ diverges off to infinity in finite slow time $\tau$. However, there may be special points on $F$ where

$$\left[ g \frac{\partial f}{\partial y} + \epsilon \frac{\partial f}{\partial \lambda} \right] = 0,$$

so $dx/d\tau$ remains finite. Such special points are referred to as folded singularities [64, 62]. The corresponding trajectories, that cross from $S^a$ along the eigendirections of a folded singularity onto $S^r$, are referred to as singular canards [62]. The distinction between systems that have a critical rate and those that do not appears to be whether all trajectories started on $F$ flow onto $S^a$, or whether there are trajectories started on $S^a$ that reach $F$ away from a folded singularity. (We do not explicitly consider trajectories that reach $F$ at a folded singularity as it is sufficient to identify either of the above two behaviours.) Furthermore, canard trajectories, being solutions that separate these two behaviours, are candidates for non-obvious thresholds.

An obstacle to the analysis of critical rates and instability thresholds is that the flow on $F$, specifically the right hand side of Eq. (2.7), is not well defined. This obstacle can be overcome by a special time rescaling [24]:

$$d\tau = -ds \epsilon \frac{\partial f}{\partial x},$$
where the new time $ds$ is infinitely faster on $F$, and reverses direction on $S^r$. This gives the desingularised system

$$
\frac{dx}{ds} = \left[ g \frac{\partial f}{\partial y} + \epsilon \frac{\partial f}{\partial \lambda} \frac{d\lambda}{d\tau} \right]_{F},
$$

(2.10)

$$
\frac{d\tau}{ds} = -\epsilon \frac{\partial f}{\partial x} \bigg|_{S},
$$

(2.11)

where trajectories remain the same as in Eqs. (2.7)–(2.8), the vector field on $F$ becomes well defined, folded singularities become regular steady states, and singular canards become trajectories tangent to an eigenspace of a steady state. One speaks of “folded nodes”, “folded saddles” and “folded foci” for Eqs. (2.7)–(2.8) if a steady state for Eqs. (2.10)–(2.11) has real eigenvalues with the same sign, real eigenvalues with opposite signs, and complex eigenvalues with non-zero real parts, respectively. Most importantly, the difference between tracking and destabilising can easily be analysed using Eqs. (2.10)–(2.11).

Specifically, we derive conditions for the existence of critical rates and non-obvious thresholds:

**Theorem 2.1** (Existence of critical rates: a dissipative Adiabatic Theorem). *Suppose the forced system (2.1)–(2.2) with assumptions 2.1–2.2 satisfies the folded singularity condition (2.9) for some $\tau \in (\tau_{\text{min}}, \tau_{\text{max}})$ and $\epsilon > 0$. Then, system (2.1)–(2.2) has a critical rate $\epsilon_c$. The critical rate is approximately the largest $\epsilon$ below which (2.9) is never satisfied within $(\tau_{\text{min}}, \tau_{\text{max}})$:

$$
\epsilon_c = \inf \left\{ \epsilon > 0 : \left[ g \frac{\partial f}{\partial y} + \epsilon \frac{\partial f}{\partial \lambda} \frac{d\lambda}{d\tau} \right]_{F} = 0 \right\} + O(\delta).
$$

**Theorem 2.2** (Existence of non-obvious thresholds). *The forced system (2.1)–(2.2) with assumptions 2.1–2.2 is guaranteed to have an instability threshold if a folded saddle is the only folded singularity within $(\tau_{\text{min}}, \tau_{\text{max}})$. Then, the threshold is given by the folded saddle maximal canard.

Moreover, if $\tau \in (\tau_{\text{min}}, \infty)$ and

$$
\lim_{\tau \to \infty} \frac{d\lambda}{d\tau} = 0,
$$

(2.12)

then the system has an instability threshold if, and only if, there is a folded saddle singularity.

**Note.** Often in real-life applications the changing external conditions $\lambda$ are expressed as a prescribed function of time $t$, but not $\epsilon$ or $\tau$. Specifying $\epsilon$ is not necessary. If one replaces $\tau$ with $\epsilon t$ in Eqs. (2.4)–(2.11) the dependence on $\epsilon$ disappears. However, $\epsilon$ and $\tau$ are useful for defining critical rates of change, and facilitate the derivation of the statements in Theorems 2.1 and 2.2.

For the proof of Theorems 2.1 and 2.2, we consider system (2.1)–(2.2) with assumptions 2.1–2.2, and restrict the discussion to $\tau \in (\tau_{\text{min}}, \tau_{\text{max}})$. The proofs are based on two steps. In the first step, a qualitative analysis of Eqs. (2.10)–(2.11) identifies a folded singularity
with a critical rate, and certain singular canards as candidates for an instability threshold. In the second step, recent results from canard theory [62, 71, 70] are used that state the singular canards due to a folded saddle, a folded node, and a folded saddle-node type I, perturb to maximal canards in (2.4)–(2.6) with 0 < δ ≪ 1. Maximal canards are those trajectories crossing from $S^a_\delta$ onto $S^r_\delta$, which remain on $S^r_\delta$ for the longest time, equivalently they are robust intersections of $S^a_\delta$ and $S^r_\delta$. In this chapter, we numerically compute maximal canards $\gamma_\delta$, shown in Fig. 2.4, and their approximations by singular canards $\gamma$, shown in Figs. 2.3 and 2.5.

2.2.1. Proof of Theorem 2.1

Let $p$ be a point on the fold $F$ in the desingularised system (2.10)–(2.11). By assumption 2.1 in the desingularised system, the vector field at $p$ only has a component in the $x$-direction. When $\epsilon = 0$, by assumption 2.2 in the desingularised system, the vector field points towards the attracting critical manifold $S^a$ at every $p \in F$. This means all trajectories starting on $F$ flow onto $S^a$, and no trajectories starting on $S^a$ reach $F$. When $\epsilon > 0$, there may be trajectories that reach $F$ from $S^a$. This happens if, and only if, the vector field changes sign at some $p \in F$ as $\epsilon$ is varied:

$$\frac{dx}{ds}\bigg|_p = \left[ g \frac{\partial f}{\partial y} + \epsilon \frac{\partial f}{\partial \lambda} \frac{d\lambda}{d\tau} \right]_p = 0,$$

(2.13)

$$\frac{d}{d\epsilon} \frac{dx}{ds}\bigg|_p = \left[ \frac{\partial f}{\partial \lambda} \frac{d\lambda}{d\tau} \right]_p \neq 0.$$

(2.14)

Furthermore, by assumption 2.1 $S$ can be expressed as a graph over $y$ meaning $(\partial f/\partial y)|_p \neq 0$, and by assumption 2.2 there are no steady states on $F$ in the full system meaning $g|_p \neq 0$, so (2.13) already implies (2.14).

By [35, Th. 1], if system (2.10)–(2.11) has no trajectories started on $S^a$ that reach $F$, then system (2.4)–(2.6) has no trajectories that leave $S^a_\delta$ for $0 < \delta \ll 1$. Furthermore, by [63, Th. 1], if system (2.4)–(2.6) has trajectories starting on $S^a$ that reach $F$ away from a folded singularity, then system (2.4)–(2.6) has trajectories that leave $S^a_\delta$ and move away along the fast $x$-direction for $0 < \delta \ll 1$. Hence, the folded singularity condition (2.13) implies a critical rate for system (2.4)–(2.6), and for the original system (2.1)–(2.2).

By definition 2.2, the critical rate is approximately the largest $\epsilon$ below which (2.13) is never satisfied within $(\tau_{\text{min}}, \tau_{\text{max}})$:

$$\epsilon_c \approx \inf \left\{ \epsilon > 0 : \left[ g \frac{\partial f}{\partial y} + \epsilon \frac{\partial f}{\partial \lambda} \frac{d\lambda}{d\tau} \right]_F = 0 \right\}.$$

The smaller the value of $\delta$, the better the approximation [62].
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Figure 2.2. Sketches of selected phase portraits for system (2.7)-(2.8), containing folded saddles (FS), folded nodes (FN), and folded saddle-nodes (FSN). Singular canards are shown in bold. On $S^a$, there are trajectories that (white) approach $F$ away from a folded singularity, (blue) leave $S^a$ via a folded singularity, and (grey) never reach $F$.

2.2.2. Proof of Theorem 2.2

Consider a fixed value of $\epsilon > \epsilon_c$. We are interested in phase portraits of system (2.7)-(2.8) which have two types of trajectories starting on $S^a$: those that reach $F$ away from a folded singularity, and those that never reach $F$ and remain on $S^a$. We refer to the separatrix dividing these two types of trajectories as the singular threshold. Phase portraits of system (2.7)-(2.8) that may contain a singular threshold are identified as follows. We keep in mind that $d\tau/dt > 0$, construct possible phase portraits of the desingularised system (2.10)-(2.11), reverse the flow on $S^r$, and keep those portraits that allow a singular threshold.

The proof consists of three parts. Firstly, we analyse an arbitrary external input $\lambda(\tau)$ to show that an isolated folded saddle guarantees a singular threshold. Secondly, we analyse an asymptotic external input satisfying condition (2.12) to show that there is a singular threshold if, and only if, there is a folded saddle. Lastly, we use recent results from canard theory to show that singular thresholds persist as instability thresholds for $\delta$ small, but
Part 1

Firstly, assume condition (2.9) is satisfied, meaning there is a folded singularity \( p \). Without loss of generality, suppose \( p \) is at the origin. According to [62, Prop. 2.1], under assumption 2.1 and condition (2.9), there is a smooth change of coordinates that projects the fold curve \( F \) orthogonally onto the \( \tau \)-axis and, in the neighbourhood of \( p \), brings the desingularised system (2.10)–(2.11) to the normal form

\[
\begin{align*}
\frac{d\hat{x}}{d\hat{s}} &= b\hat{\tau} + c\hat{x} + O(\hat{x}^2, \hat{x}\hat{\tau}, \hat{\tau}^2), \\
\frac{d\hat{\tau}}{d\hat{s}} &= -2c\hat{x} + O(\hat{x}^2, \hat{x}\hat{\tau})
\end{align*}
\] (2.15) (2.16)

where \( \hat{x} \) and \( \hat{\tau} \) are the new coordinates, the fold \( F \) is defined by \( \hat{x} = 0 \), and the attracting critical manifold \( S^a \) is defined by \( \hat{x} < 0 \). The eigenvalues of \( p \):

\[
\xi_{1,2} = \frac{c \pm \sqrt{c^2 - 8cb}}{2},
\]

determine the type of the folded singularity in system (2.7)–(2.8). In particular, \( p \) is a folded saddle if \( b < 0 \), a folded saddle-node if \( b = 0 \), and a folded node, focus or centre if \( b > 0 \). The key observation for our purposes is that \( b \neq 0 \) determines the direction of the flow on \( F \), where \( d\hat{\tau}/d\hat{s} = 0 \) and \( d\hat{x}/d\hat{s} = b\hat{\tau} + O(\hat{\tau}^2) \).

In the case of a folded saddle (\( b < 0 \)), trajectories starting on \( S^a \) and near \( F \) reach \( F \) when \( \hat{\tau} < 0 \) and sufficiently small, or flow away from \( F \) onto \( S^a \) when \( \hat{\tau} > 0 \) and sufficiently small [Fig. 2.2(a)]. If a folded saddle is the only folded singularity, then there are no additional changes in the direction of the flow on \( F \). The local behaviour for \( 0 < \hat{\tau} \ll 1 \) extends to \( 0 < \hat{\tau} < \hat{\tau}_{\text{max}} \), meaning no trajectories started on \( S^a \) for \( \hat{\tau} > 0 \) ever reach \( F \). Hence, an isolated folded saddle implies a singular threshold. What is more, the threshold is given by the singular folded saddle canard. This can be seen by noting that, in the desingularised system (2.10)–(2.11), the separatrix between trajectories starting on \( S^a \) that reach \( F \) and those that never reach \( F \) is the stable manifold of the saddle equilibrium. This stable manifold becomes the singular folded saddle canard \( \gamma_S \) in system (2.7)–(2.8) [Fig. 2.2(a)].

If, in addition to a folded saddle, there are other folded singularities, a singular threshold can no longer be guaranteed [e.g. Fig. 2.2(c)], nor excluded [e.g. Fig. 2.2(b)]. To obtain the threshold, one needs to study the behaviour of trajectories started on \( S^a \); see the analysis of case 1 in Section 3.

In the special case of a folded saddle-node (\( b = 0 \)), the flow on \( F \) in system (2.15)–(2.16) is determined by \( d\hat{x}/d\hat{s} = O(\hat{\tau}^2) \). This means there is no change in the sign of the flow at \( p \) [e.g. Fig. 2.2(f)]. A folded saddle-node is structurally unstable. Under arbitrarily small variation of system parameters, it unfolds into a folded saddle at positive \( \hat{\tau} \) and a folded node at negative \( \hat{\tau} \) (multiple singularities discussed in the paragraph above), or vanish.
In the case of a folded node, focus or centre \((b > 0)\), trajectories starting on \(S^a\) and sufficiently close to \(F\) flow away from \(F\) onto \(S^a\) when \(-1 \ll \hat{\tau} < 0\), or reach \(F\) when \(0 < \hat{\tau} \ll 1\); see an example of an unstable folded node in Fig. 2.2(d).

For \(b \geq 0\), a singular threshold cannot be guaranteed [e.g. Fig. 2.2(e) and (f)], nor excluded [e.g. Fig. 2.2(d) and (g)].

Secondly, assume there are no folded singularities. If the flow on \(F\) in system (2.15)–(2.16) points towards \(S^a\), a singular threshold can be excluded. If the flow on \(F\) points towards \(S^r\), a singular threshold cannot be guaranteed, nor excluded [for example, restricting the \((\hat{\tau}_{\text{min}}, \hat{\tau}_{\text{max}})\) interval to the lower part of the phase portrait below the folded node in Fig. 2.2(d) gives a singular threshold without a folded singularity.]

Finally, if \(\hat{\tau}_{\text{max}}\) is positive and finite, there may be ‘spurious’ singular thresholds in phase portraits with a folded singularity and \(b \geq 0\), or with no folded singularities, where all trajectories starting on \(S^a\) and near \(F\) for \(\hat{\tau} > 0\) flow towards \(F\). Because \(\hat{\tau}_{\text{max}}\) is finite, some of these trajectories will simply fail to reach \(F\) by \(\hat{\tau}_{\text{max}}\).

It turns out that many examples of a singular threshold described above, including the ‘spurious’ singular threshold, can be eliminated with a sensible assumption about \(\lambda(\tau)\).

**Part 2**

A more definitive statement about instability thresholds can be made when \(\tau \in (\tau_{\text{min}}, \infty)\), and the external input is asymptotic, i.e. \(\lambda(\tau)\) satisfies condition (2.12).

Assume there is a singular threshold. On the one hand, it follows from assumption 2.2 and from condition (2.12) that, for sufficiently large \(\tau\), trajectories started on \(S^a\) and near \(F\) must flow onto \(S^a\) and approach \(\tilde{x}\). On the other hand, a singular threshold requires trajectories that start on \(S^a\) and reach \(F\). Hence, the flow on \(F\) in the desingularised system (2.10)–(2.11) must point towards \(S^a\) for large values of \(\tau\), and towards \(S^r\) for lower values of \(\tau\). Such a change in the direction of the flow on \(F\) requires a folded singularity with \(b < 0\) in (2.15)–(2.16). Hence, a folded saddle is necessary for a singular threshold.

Assume there is a folded saddle singularity. There are two possible situations. First, a folded saddle is the only folded singularity. Second, a folded saddle is one of many folded singularities. In the second situation, assumption 2.1 and condition (2.12) require that, typically, the folded singularity with the largest \(\tau\)-component is a folded saddle. “Typically” excludes a folded saddle-node which is not structurally stable. In both situations, there is a singular threshold by the argument used for an isolated folded saddle in part 1 of this proof. Hence, a folded saddle is sufficient for a singular threshold.

**Part 3**

In the last step of the proof we use theorems from canard theory stating that the singular canards due to a folded saddle [62, Th. 4.1], a folded node [62, Th. 4.1][71, Prop. 4.1], and a
2. Non-Obvious Thresholds in Multi-Scale Systems

folded saddle-node type I [70, Ths. 4.1 and 4.4], perturb to maximal canards in (2.4)–(2.6) with $0 < \delta \ll 1$. Maximal canards are transverse, robust intersections of two-dimensional attracting $S_a^\delta$ and repelling $S_r^\delta$ slow manifolds [62, 71]. Such intersections are possible in system (2.4)–(2.6) because the slow manifolds $S_a^\delta$ and $S_r^\delta$ can be extended across the fold [17]. Starting on $S_a^\delta$ and near the fold, trajectories jump off $S_a^\delta$ in the fast $x$-direction on one side of such intersections, and flow onto $S_a^\delta$ on the other side [62, Fig. 13]. Thus, a singular threshold in system (2.7)–(2.8) implies an instability threshold in system (2.4)–(2.6), and in the original system (2.1)–(2.2).

2.3. Two cases of a non-obvious threshold

Guided by the proof of Theorem 2.2, specifically the analysis of the phase portraits containing a folded saddle [Fig. 2.2(a)–(b)], we distinguish two cases of a non-obvious threshold. Furthermore, we identify one case with the complicated threshold shown in Fig. 2.1(d), and uncover the underlying dynamical mechanism.

We illustrate the two cases using an example of (2.1)–(2.2) with

$$f = x(x - 1) + y + \lambda(\tau) \quad \text{and} \quad g = -x, \quad (2.17)$$

and two different aperiodic forcing functions $\lambda(\tau)$ satisfying (2.12).

2.3.1. Case 1: Complicated threshold due to a type I folded saddle-node singularity

Consider example (2.17) subject to logistic growth at a rate $\epsilon$:

$$\lambda(\tau) = \lambda_{\max} \tanh(\tau), \quad (2.18)$$

where $\tau \in (-\infty, \infty)$ and $\lambda \in (-\lambda_{\max}, \lambda_{\max})$. The desingularised system (2.10)–(2.11) becomes

$$\frac{dx}{ds} = -x + \epsilon \frac{\lambda_{\max}^2}{\lambda_{\max}^2 - \lambda^2(\tau)}, \quad (2.19)$$

$$\frac{d\tau}{ds} = \epsilon (1 - 2x). \quad (2.20)$$

Steady states of (2.19)–(2.20) lie on the fold $x = 1/2$, at $\lambda(\tau)$ satisfying the folded singularity condition (2.9):

$$\lambda^2(\tau) - \lambda_{\max}^2 \left( \lambda_{\max} - \frac{1}{2\epsilon} \right) = 0, \quad (2.21)$$
and their eigenvalues $\xi$ are found from the characteristic polynomial

$$
\xi^2 + \xi - 4\epsilon^2 \lambda(\tau) \left[ 1 - \left( \frac{\lambda(\tau)}{\lambda_{\max}} \right)^2 \right] = 0.
$$

(2.22)

The folded singularity condition (2.21) has no real roots when $\epsilon < (2\lambda_{\max})^{-1}$. When $\epsilon = (2\lambda_{\max})^{-1}$, there is a double root within $(\tau_{\min}, \tau_{\max})$, corresponding to a folded saddle-node of type I [39] at $(x, \lambda) = (1/2, 0)$. When $\epsilon > (2\lambda_{\max})^{-1}$, there are two distinct roots within $(\tau_{\min}, \tau_{\max})$, corresponding to a stable folded node (focus) $FN(FF)$ at

$$(x, \lambda) = (1/2, -\sqrt{\lambda_{\max}(\lambda_{\max} - (2\epsilon)^{-1}))}$$

and a folded saddle $FS$ at

$$(x, \lambda) = (1/2, \sqrt{\lambda_{\max}(\lambda_{\max} - (2\epsilon)^{-1}))}.$$ 

This means that, upon increasing $\epsilon$, there is a generic saddle node bifurcation of folded singularities at $\epsilon_{SN} = (2\lambda_{\max})^{-1}$, which by Theorem 2.1 is approximately the critical rate $\epsilon_c$ for $0 < \delta \ll 1$. According to Theorem 2.2, condition (2.12) and the presence of a folded saddle guarantee an instability threshold. However, unlike the case of an isolated folded saddle, it is not clear what the threshold is.

The instability threshold is defined on the attracting slow manifold $S^a_\delta$, which is difficult to compute near the fold $F$. To facilitate numerical computations, we consider initial states on the critical manifold $S_{a,c}$, which is known exactly. The results are shown in Fig. 2.3, where the white regions indicate destabilising, and the grey regions indicate tracking. Away from $F$, the critical manifold $S^a$ closely approximates the slow manifold $S^a_\delta$. Here, the instability threshold is well approximated by the boundaries between the white and grey regions. However, caution is required near $F$, especially around $FN$, where $S^a_\delta$ twists in a complicated manner [17, Fig. 6], and the chosen surface of initial conditions, $S^a$, intersects these twists. There, the boundaries between the white and grey regions deviate from the instability threshold due to the choice of initial states. We also show what happens to initial states on $S^r$ just to the right of $F$, as some are mapped along the fast flow onto $S^a_\delta$ and converge to $\tilde{x}$. This is why a “reflection” of the band structure from $S^a$ can be seen on $S^r$.

Shortly past the saddle-node bifurcation, there are three bands of initial states on $S^a_\delta$ [Fig. 2.3(a)]. The threshold separating these bands is formed by two canard trajectories: the folded saddle maximal canard $\gamma^S_\delta$, and the strong folded node maximal canard $\gamma^N_\delta$. On $S^a$, trajectories from the white band enclosed by $\gamma^S_\delta$ and $\gamma^N_\delta$ move directly towards the fold, then leave the attracting slow manifold $S^a_\delta$ and destabilise along the fast $x$-direction. Trajectories from the grey band below $\gamma^S_\delta$ approach the faux saddle maximal canard $\gamma^C_\delta$ straight away, thereby staying on the attracting slow manifold $S^a_\delta$ and tracking $\tilde{x}$. This is in contrast to trajectories from the other grey band on $S^a$, the one above $\gamma^N_\delta$. These trajectories initially approach and twist around the weak folded node maximal canard $\gamma^C_\delta$, and leave $S^a_\delta$. However, rather than destabilising, they are fed back along $\gamma^C_\delta$, onto $S^a_\delta$. 

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Figure 2.3. Initial states on the critical manifold $S$ that (white) destabilise or (grey) track $\tilde{x}(\lambda(\tau))$ in Eqs. (2.1)–(2.2) and (2.17)–(2.18) with $\delta = 0.01$, $\lambda_{\text{max}} = 2.5$, and $\epsilon = (a) 0.201$, (b) 0.212, (c) 0.216, and (d) 0.270, shown projected onto the $(x, \lambda)$ plane. Away from $F$, the instability threshold in $S^a$ is well approximated by the white-grey boundary in $S^a$. Points $FN$, $FS$, and $FF$ are folded node, folded saddle, and folded focus singularities, respectively; the strong folded node singular canard $\gamma_N$ and the folded saddle singular canard $\gamma_S$ approximate projections of the maximal canards $\gamma_N^\delta$ and $\gamma_S^\delta$, respectively, onto $S$. The projection of the maximal canard $\gamma_C^\delta$ onto $S$ is approximated by the weak folded node/faux saddle singular canard $\gamma_C^\delta$ when $\lambda > -1$, but lies below $\gamma_C^\delta$ for $-2.5 < \lambda < -1$, e.g. within the wide grey band around $\lambda = -2$ in (c). Although it is difficult to see, $\gamma_C^\delta$ terminates on $F$ just above $FF$. Compare (c) with Fig. 2.1(d).

and eventually remain on $S^a$ [Fig. 2.1(b), blue trajectory]. Finally, grey and white initial states on $S^r$ are mapped along the fast flow onto the grey and white, respectively, bands of $S^a$.

As $\epsilon$ increases, the threshold becomes more complicated due to the presence of the stable folded node $FN$. Additional threshold curves appear successively above $\gamma_N^\delta$, giving up to five white bands of initial states above $\gamma_N^\delta$ that destabilise [Fig. 2.3(b)]. Trajectories started within these additional white bands twist around $\gamma_C^\delta$ before destabilising, [Fig. 2.1(b), red trajectory]. These white bands are separated by narrow grey bands which are difficult to see in Fig. 2.3; see the narrow grey band in the inset of Fig. 2.4, or narrow blue bands in Fig. 2.1(d). Trajectories started within these narrow grey bands leave $S^a$, and follow a maximal canard on $S^a$ for some time, but then return to $S^a$ into the region below $\gamma_S^\delta$, and remain on $S^a$ and converge to $\tilde{x}$. The white bands expand with $\epsilon$ and approach the weak folded node maximal canard $\gamma_C^\delta$ on both sides [Fig. 2.3(c)]. When the folded node $FN$ turns into a folded focus $FF$ at $\epsilon = (2 + \sqrt{4 + \lambda_{\text{max}}^2})/8\lambda_{\text{max}}$, its canards disappear [62] and so does the band structure [Fig. 2.3(d)]. We are left with a simple threshold, given

\[ \lambda_{\text{max}} = 2.5, \quad \epsilon = \frac{2 + \sqrt{4 + \lambda_{\text{max}}^2}}{8\lambda_{\text{max}}} \]
2. Non-Obvious Thresholds in Multi-Scale Systems

Figure 2.4. (a) Initial states on the critical manifold $S$ that (white) destabilise or (grey) track $\tilde{x}(\lambda(\tau))$ in Eqs. (2.1)–(2.2) and (2.17)–(2.18) with $\delta = 0.01$ and $\epsilon = 0.204$. Inset shows grey band between c and d; a similar band exists between e and f. Labels b–g at $\lambda = -0.7$, or $\tau = -\tanh^{-1}(0.28)$, denote different threshold components including: (b) the folded saddle maximal canard $\gamma_\delta^S$, (c) the strong folded node maximal canard $\gamma_\delta^N$, (d) a composite canard that follows $\gamma_\delta^N$ and $\gamma_\delta^S$, (e) a secondary folded node maximal canard, (f) a composite canard that follows a secondary maximal canard and $\gamma_\delta^S$, (g) a secondary folded node maximal canard.

The key mechanism for complicated thresholds is the phenomenon whereby trajectories leave $S^a_\delta$ through the folded node region and then, rather than destabilising, are fed back to $S^a_\delta$ through the folded saddle region. This phenomenon has two consequences. Firstly, not all initial states on $S^a_\delta$ and above $\gamma_\delta^N$ destabilise. Secondly, the initial states on $S^a_\delta$ that destabilise or track $\tilde{x}$ form alternating bands, and these bands have not been identified before. More specifically, the alternating bands are related to the known rotational sectors of a folded node. However, whilst rotational sectors are separated by a single canard trajectory [17, 71], our white bands are separated by a narrow grey band bounded by two different canard trajectories.

Figure 2.4 identifies the different components of the complicated threshold. They consist of known maximal canards such as (b) $\gamma_\delta^S$, (c) $\gamma_\delta^N$, and [(e) and (g)] secondary folded node maximal canards that bifurcate off $\gamma_\delta^C$ [71]. These canards form the lower boundaries of the narrow grey bands. Most interestingly, we uncover composite canards that follow canard segments of different folded singularities. These canards form the upper boundaries of the narrow grey bands. Figure 2.4 shows composite canards which initially (d) follow $\gamma_\delta^N$, or (f) follow the first secondary folded node maximal canard, and then [(d) and (f)] follow...
Figure 2.5. Initial states on the critical manifold $S$ that (white) destabilise or (grey) track $\tilde{x}(\lambda(\tau))$ for Eqs. (2.1)–(2.2), (2.17), and (2.23) with $\delta = 0.01$, $\lambda_{\text{max}} = 2.5$, and (a) $\epsilon = 0.25$, (b) $\epsilon = 1$, shown projected onto the $(x, \lambda)$ plane. Away from $F$ the instability threshold in $S^a$ is well approximated by the white-grey boundary in $S^a$. Compare (b) with Fig. 2.1(c). For labels see Fig. 2.3.

$\gamma^S_\delta$. This explains the intriguing band structure with intermingled regions of white and grey in Figs. 2.3(b)-(c) and 2.4(a), or red and blue in Fig. 2.1(d). It is interesting to note, the composite canards in Fig. 2.4(d) and (f) are reminiscent of trajectories that switch between different primary and secondary canards of the same folded node in a stellate cell model [73].

2.3.2. Case 2: Simple threshold due to an isolated folded saddle singularity

Consider example (2.17) subject to an exponential approach at a rate $\epsilon$:

$$\lambda(\tau) = \lambda_{\text{max}} \left(1 - e^{-\tau}\right),$$

where $\tau \in (0, \infty)$ and $\lambda \in (0, \lambda_{\text{max}})$. The desingularised system (2.10)–(2.11) becomes

$$\frac{dx}{ds} = -x + \epsilon \left(\lambda_{\text{max}} - \lambda(\tau)\right), \quad (2.24)$$

$$\frac{d\tau}{ds} = \epsilon(1 - 2x). \quad (2.25)$$

The steady state of (2.24)-(2.25) lies on the fold $x = 1/2$, at $\lambda(\tau)$ satisfying the folded singularity condition (2.9):

$$\lambda(\tau) = \lambda_{\text{max}} - \frac{1}{2\epsilon}, \quad (2.26)$$

and its eigenvalues $\xi$ are found from the characteristic polynomial

$$\xi^2 + \xi + 2\epsilon^2 (\lambda(\tau) - \lambda_{\text{max}}) = 0. \quad (2.27)$$

The main difference from case 1 is that the different forcing $\lambda(\tau)$ gives a folded singularity condition (2.26) with just a single root, corresponding to an isolated folded saddle $FS$ at $(x, \lambda) = (1/2, \lambda_{\text{max}} - (2\epsilon)^{-1})$. Upon increasing $\epsilon$, the folded saddle enters $(\tau_{\text{min}}, \tau_{\text{max}})$ via its lower boundary when $\epsilon = (2\lambda_{\text{max}})^{-1}$, which by Theorem 2.1 is approximately the
critical rate $\epsilon_c$ for $0 < \delta \ll 1$. According to Theorem 2.2, there is an instability threshold given by the folded saddle maximal canard $\gamma^S_\delta$, as in the compost-bomb and the type III neuron examples [74, 49]. Numerical computations in Fig. 2.5 confirm that for $\delta = 0.01$, and away from $F$, the threshold is well approximated by the singular canard $\gamma^S$. It is interesting to note, the threshold in Fig. 2.5 is very similar to that in Fig. 2.3(d) because there are no canard trajectories generated by the folded focus.

*Note on simple thresholds.* When the external input $\lambda(\tau)$ does not satisfy (2.12), there can be an instability threshold that is not associated with a folded saddle [Fig. 2.2(d)]. Moreover, it follows from the proof of Theorem 2.2 that such a threshold is simple, like in the case of an isolated folded saddle.

### 2.4. Conclusions

In summary, we analysed multiple time-scale systems subject to a changing environment, identified non-linear mechanisms for the failure to adapt, and derived conditions for the existence of these mechanisms. Specifically, we described instability thresholds where a system fails to adiabatically follow a continuously changing stable state. Despite their cross-disciplinary nature, these thresholds are largely unexplored because they are “non-obvious”, meaning they cannot, in general, be revealed by traditional stability theory. Thus, they require an alternative approach. We presented a framework, based on geometric singular perturbation theory, that led us to a novel threshold type with an intriguing band structure. The threshold has alternating bands, where trajectories track the moving stable state, or destabilise. We showed that this structure is organised by a type I folded saddle-node singularity. Intuitively, it arises from an interplay of the complicated dynamics of twisting canard trajectories due to a folded node singularity, and the simple threshold behaviour illustrated for a folded saddle singularity. Most importantly, trajectories which leave the attracting slow manifold through the folded node region can be fed back to the attracting slow manifold through the folded saddle region. In more technical terms, the band structure is related to the rotational sectors of a folded node, but also differs from them in one key aspect. Whereas the rotational sectors are separated by a single canard trajectory, namely the maximal canard [17, 71], the corresponding wide bands are separated by a narrow band. These separating narrow bands are bounded by two different canard trajectories. One of them is a known maximal canard, and the other is a composite canard that follows maximal-canard segments of different folded singularities.

Whilst non-obvious thresholds can be complicated, they are generic, and should explain counter-intuitive responses to a changing environment in a wide range of multi-scale systems. We highlighted their importance by examples of climate and ecosystems failing to adapt to a rapidly changing environment [74, 2, 47], and type III excitable cells “firing” only if the voltage stimulus rises fast enough [49, 31]. More generally, our results give new insight into non-adiabatic processes in multi-scale dissipative systems, and should stimulate further work in canard theory.
3. Bifurcations of Canard Trajectories Near a Type I Folded Saddle-Node Singularity

This chapter considers bifurcation of canard trajectories for a (structurally unstable) folded saddle-node (type I) singularity. The new results for this system are: the doubling up of the $S^r_\delta$ manifold; a two dimensional bifurcation diagram in $\delta$ (time-scale separation), and $\epsilon$ (the folded saddle-node bifurcation parameter); identifying new canard trajectories which limit to canard trajectories with infinite integration time; new scaling laws in $\delta$ and $\epsilon$ for canard bifurcations; and a dense area of canard bifurcations in $\delta-\epsilon$ space, which we term a sprite.

3.1. Introduction

Consider a non-autonomous, multiple time-scale system with (aperiodic) external forcing:

\begin{align*}
\delta \frac{dx}{dt} &= f(x, y, \lambda(\epsilon t)), \\
\frac{dy}{dt} &= g(x, y, \lambda(\epsilon t)),
\end{align*}

where $x \in \mathbb{R}$ is a fast variable, $y \in \mathbb{R}$ is a slow variable, and $0 < \delta << 1$ is the ratio of the $x$ and $y$ time-scales. The function $\lambda(\epsilon t)$ describes the external forcing, which varies in time at a rate $0 < \epsilon < 1$. The functions $f, g$ and $\lambda$ are sufficiently smooth.

Because of the multiple time-scales, system (3.1)–(3.2) exhibits both slow motion, and abrupt, often large amplitude, fast motion, which allows phenomena such as rate induced bifurcations, excitable behaviour, and mixed mode oscillations [74, 72, 17]. In applications, the transition between slow and large amplitude fast motion has been termed “jump”, “tipping” and “excitation”, and can be dependent on the initial states, the form of the external forcing function, and rate of change of the external forcing [74, 49, 72, 50].

To analyse the dynamics, system (3.1)–(3.2) is studied in the singular limit $\delta = 0$, and the dynamic behaviour is extended to the $0 < \delta << 1$ case using geometric singular perturbation theory [25, 35, 62]. This is done by extending system (3.1)–(3.2) by $\tau = \epsilon t$.
3. Bifurcations of Canard Trajectories Near a Type I Folded Saddle-Node Singularity

to an autonomous, one fast-two slow variables system:

\[
\begin{align*}
\delta \frac{dx}{dt} &= f(x, y, \lambda(\tau)), \\
\frac{dy}{dt} &= g(x, y, \lambda(\tau)), \\
\frac{d\tau}{dt} &= \epsilon.
\end{align*}
\]  

(3.3) (3.4) (3.5)

In the singular limit \( \delta = 0 \), the slow subsystem

\[
\frac{dy}{dt} = g(x, y, \lambda(\tau)), \quad \frac{d\tau}{dt} = \epsilon
\]

(3.6)

evolves on the two-dimensional critical manifold \( S \), defined by:

\[
S := \{ (x, y, \tau) : f(x, y, \lambda(\tau)) = 0 \}.
\]

(3.7)

Equivalently, \( S \) consists of steady states of the fast subsystem \( dx/dT = f(x, y, \lambda(\tau)) \), where \( T = t/\delta \) is the fast time-scale, and \( y \) and \( \tau \) are assumed to stand still.

In multiple time-scale systems, interesting phenomena such as rate induced bifurcations, excitability and mixed mode oscillations occur when switching between fast and slow dynamics. This happens if there is a folded critical manifold.

**Assumption 3.1.** Suppose the critical manifold \( S \) for system (3.1)–(3.2) has a fold \( F \) tangent to the fast \( x \)-direction, defined by:

\[
\left. \frac{\partial f}{\partial x} \right|_S = 0 \quad \text{and} \quad \left. \frac{\partial^2 f}{\partial x^2} \right|_S \neq 0.
\]

(3.8)

The fold \( F \) separates the critical manifold into an attracting part \( S^a \), and repelling part \( S^r \), relative to the fast \( x \) direction [Fig. 3.1 (a)].

To study the dynamics near \( F \), suppose \( S \) can be expressed as a graph over \( x \) and \( \tau \), i.e. \( (\partial f/\partial y)|_S \neq 0 \), and project the dynamics within \( S \) onto the \( (x, \tau) \)-plane by differentiating (3.7) with respect to slow time \( t \):

\[
\begin{align*}
\frac{dx}{dt} &= -g \left. \frac{\partial f}{\partial y} + \epsilon \frac{\partial f}{\partial \lambda} \right|_S \\
\frac{d\tau}{dt} &= \epsilon.
\end{align*}
\]

(3.9) (3.10)

For trajectories which approach a typical point on \( F \), \( \partial f/\partial x \to 0 \), and \( x(t) \) diverges off to infinity in finite slow time \( t \) (this corresponds to a fast jump, tipping, excitation, or a large relaxation oscillation in the applications). However, at special points on \( F \) called folded singularities

\[
\left. \left[ g \frac{\partial f}{\partial y} + \epsilon \frac{\partial f}{\partial \lambda} \right] \right|_F = 0,
\]

(3.11)

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3. Bifurcations of Canard Trajectories Near a Type I Folded Saddle-Node Singularity

so trajectories can cross $F$ with finite speed, and, surprisingly, follow the repelling manifold $S^r$ [Fig. 3.1 (a)]. The folded singularities are classified according to the type of steady state they are, after a time rescaling

$$dt = -ds \left. \frac{\partial f}{\partial x} \right|_S,$$

in the desingularised system

$$\frac{dx}{ds} = \left[ g \frac{\partial f}{\partial y} + \epsilon \frac{\partial f}{\partial \lambda} \frac{d\lambda}{d\tau} \right]_S,$$

$$\frac{d\tau}{ds} = -\epsilon \left. \frac{\partial f}{\partial x} \right|_S.$$

(3.12)

(3.13)

A folded singularity for Eqs. (3.9)–(3.10) is classified as a “folded saddle”, “folded node” or “folded focus”, if the steady state for Eqs. (3.12)–(3.13) has non-zero real eigenvalues with opposite signs, non-zero real eigenvalues with the same sign, or complex eigenvalues with non-zero real parts, respectively [62]. There are also structurally unstable folded singularities, such as the “folded saddle-node” which has one zero eigenvalue and one non-zero real eigenvalue.

Situations with structurally stable, isolated folded singularities have been much studied [62, 71, 28, 16, 15, 73]. This chapter focuses on the interaction between a folded saddle and a folded node, following a folded saddle-node (type I) bifurcation of folded singularities. By comparison, a folded saddle-node (type II) bifurcation corresponds to a steady state crossing a folded singularity [39].

Assumption 3.2. Suppose for the extended system (3.1)–(3.2) when $\delta = 0$, at some value of $\epsilon_{FSN}$ there is a folded saddle-node (type I) bifurcation, and following the bifurcation, the folded node is attracting. That is, the folded singularity condition (3.11) for

- $\epsilon < \epsilon_{FSN}$ has no real solutions
- $\epsilon = \epsilon_{FSN}$ has one real solution,
- $\epsilon > \epsilon_{FSN}$ has two real solutions, one corresponds to a folded saddle, and the other to a stable folded node.

From here on, we are only interested in stable folded nodes, and just refer to them as folded nodes.

Recall, folded singularities are points where the slow flow crosses the fold $F$ in finite slow time. The trajectories that cross through a folded singularity from $S^a$ onto $S^r$ are called singular canards $\gamma^S$ and the trajectories that cross from $S^r$ onto $S^a$ are called singular faux canards [62]. These singular canard and faux canard trajectories follow the eigendirections for the folded singularity in the desingularised system (3.12)–(3.13). Thus, different types of folded singularity have different arrangements of singular canard and faux canard trajectories. A folded saddle has a unique singular canard $\gamma^S$ and a unique singular faux canard [Fig. 3.1 (a)]. In contrast, a folded node has an entire section of
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![Image of a fast-slow system following a folded saddle-node (type I) bifurcation](image)

Figure 3.1. Sketch of a fast-slow system following a folded saddle-node (type I) bifurcation (a) in the singular limit $\delta = 0$, (b) for $0 < \delta << 1$. In (a) the attracting and repelling parts of the critical manifold $S^a$ (red) and $S^r$ (blue) are separated by the fold $F$. The folded node $FN$ (dot) feeds into the folded saddle $FS$ (dot) as a folded node weak singular canard trajectory $\gamma^C$ is connected to the unique folded saddle faux canard trajectory. Shown also are the unique folded node strong singular canard trajectory $\gamma^N$ and folded saddle singular canard trajectory $\gamma^S$. Note, all trajectories started in the section of $S^a$ between by $\gamma^N$ and $F$ which contains $\gamma^C$ go through the folded node. In (b) the attracting and repelling slow manifolds $S^a_\delta$ (red) and $S^r_\delta$ (blue) persist, as do some canard trajectories, including the strong folded node canard $\gamma^N_\delta$, and folded saddle canard $\gamma^S_\delta$.

singular canards bounded by $F$ and the unique singular canard tangent to the strong eigendirection $\gamma^N$. The section contains a (non-unique) singular canard tangent to the weak eigendirection $\gamma^C$ [Fig. 3.1 (a)].

There are two cases of a folded saddle-node (type I) bifurcation that give rise to a stable folded node [62]. This chapter focuses on the case which gives rise to a complicated instability threshold [see Fig. 2.2(b)], as trajectories started on $S^a$ may do one of three things: not approach $F$ and remain on $S^a$, approach $F$ and leave $S^a$ with $x(t)$ becoming infinite, or cross $F$ at the folded node, then cross back and return to $S^a$.

**Assumption 3.3.** Suppose for the extended system (3.1)–(3.2) when $\delta = 0$ and $\epsilon = \epsilon_{FSN}$, the folded saddle-node centre manifold lies on $S^a$. Equivalently, when $\epsilon > \epsilon_{FSN}$, the trajectory that connects the folded node and folded saddle, flows from the folded node to the folded saddle.

Note, it follows from assumption 3.3 that the folded node singular canard trajectory tangent to the weak eigendirection $\gamma^C$ is also the folded saddle singular faux canard trajectory [Fig. 3.1 (a)].

3.1.1. Motivation

Folded singularities and their canard trajectories were originally studied as a mathematical curiosity [5, 62]. However, more recently, their importance has been identified in applications [74, 49, 73, 15, 17, 72, 38]. Specifically, the folded saddle-node (type I) singularity arises in a diverse range of applications [69, 65, 70].
As described in the previous chapter, a rate-induced bifurcation is the transition in an externally forced multi-scale system (3.1)–(3.2) from tracking a time-varying equilibrium (when the forcing $\lambda(\epsilon t)$ varies slowly), to diverging to a different state (when the forcing varies too fast) [2, 74]. The rate-induced bifurcation corresponds to the appearance of a folded singularity, and its canards form the threshold, which separates trajectories that track from those that diverge to a different state [50]. Rate-induced bifurcations occur in neuron excitation, and environmental systems, including the sudden release of carbon from organic soils termed the compost-bomb [47, 74]. A folded saddle-node singularity arises for generic external forcing, for example, a shift from a low to high state [50]. What is more, the threshold formed near a folded saddle-node has an intricate band structure [50], which depends on $\epsilon$ (the rate of forcing) and $\delta$ (the separation of time-scales).

Neurons are cells which can be electrically excited causing them to fire (have a rapid change in the cell membrane voltage). In fast-slow models of neurons, above a given rate of external electrical forcing, the canards form firing thresholds that separate areas of initial conditions at which the neuron does and does not fire [49]. Specifically, a folded saddle-node (type I) singularity arises in a hybrid of the classic Morris-Lecar and FitzHugh-Nagumo models [72], and in pituitary gland models [65, 69].

Systems like (3.1)–(3.2) can exhibit another phenomenon – mixed mode oscillations. Here trajectories have periodic cycles consisting of a number of small amplitude oscillations and a number of large amplitude oscillations. Canard trajectories separate initial states with different oscillation patterns [17]. Mixed mode oscillations arise in the externally forced Van der Pol system (a classic oscillator used in biology, physics and economics) following a folded saddle-node (type I) bifurcation [67, 29].

Mathematically, the dynamics following a folded saddle-node (type I) bifurcation are interesting, but, as yet, have not been studied much. The first studies show that the interacting folded singularities produce rich and complex behaviour [28, 50, 70]. For example, results in the previous chapter show the intricate threshold that arises [50], and numerical work by Guckenheimer [28] uses flow maps to show that the dynamics are more complicated than for isolated folded singularities. (Whereas Guckenheimer considered a folded saddle feeding into a folded node, here we consider a folded node feeding into a folded saddle [Fig. 3.1(a)].) Lastly, the first analytical study specifically of a folded saddle-node (type I) has recently been done by Vo et al [70].

Our results give new insight to the complicated dynamics near a folded saddle-node (type I) singularity by computing bifurcations of canard trajectories as the parameters $\delta$ and $\epsilon$ vary. Local bifurcations classically describe changes in the number and stability of compact invariant sets such as steady states and limit cycles. We extend this concept to study the appearance and disappearance of canard trajectories, which are non-compact sets. (Note, we focus on the existence of canard trajectories and do not define their stability.) Previous studies of canard trajectory bifurcations in two slow-one fast systems were limited to systems with structurally stable isolated folded singularities [71, 16].

The outline of this chapter is as follows. Section 3.2 describes the current state-of-the-art
understanding of dynamics near folded singularities, and Section 3.3 introduces the example equations used in this chapter. By compactifying the system with respect to infinite time we are able to identify new canard trajectories which limit to canard trajectories with infinite integration time. There are two different viewpoints for studying canards in fast-slow systems – using flow maps [28] and using a boundary value problem [15]. In Section 3.4, the $0 < \delta << 1$ dynamics near a folded saddle-node for different $\epsilon$ are illustrated using flow maps. These maps have apparent discontinuities which correspond to canard trajectories. In Section 3.5, the canards are computed as solutions to a boundary value problem using boundary solver routines in AUTO [21]. By the numerical continuation of canard trajectories in $\epsilon$, we reveal the three types of canard bifurcations near a folded saddle-node (type I) (Section 3.5.2). In Section 3.5.3 the full bifurcation diagram in $\epsilon$ and $\delta$ is constructed, with the global and local structures of the bifurcation diagram explained in Sections 3.5.3.c–3.5.3.f. We conclude in Section 3.6 with a comparison to existing theoretical results, highlighting new contributions that have not been captured by the existing theory [70].

3.2. The state-of-the-art for dynamics near folded singularities

Section 3.1 introduced analytical results for system (3.1)–(3.2) in the singular limit $\delta = 0$. What is exciting is the analytic results for $\delta = 0$ from Eqs. (3.9)–(3.10), can inform us about the real world dynamics when $0 < \delta << 1$. This is done by geometric singular perturbation theory [25, 35], which, heuristically, glues slow dynamics and fast dynamics together. However, there are subtleties for the region near the fold $F$ where the slow dynamics become tangential to the fast dynamics.

When $0 < \delta << 1$, and $O(\delta^{1/4})$ away from $F$, the manifolds $S^a$ and $S^r$ persist as nearby, invariant, attracting and repelling slow manifolds $S^a_\delta$ and $S^r_\delta$ [Fig. 3.1(b)] [25, 70]. So, away from $F$, the dynamical behaviour in the limit $\delta = 0$ is a good approximation to the behaviour when $0 < \delta << 1$. However, what happens to the trajectories that approach $F$?

When $0 < \delta << 1$, the slow manifolds $S^a_\delta$ and $S^r_\delta$ can be extended beyond $F$ by integrating trajectories started on these manifolds forwards or backwards in time [15]. Typically at $F$, $S^a_\delta$ and $S^r_\delta$ separate, coming above or below the other manifold [Fig. 3.1(b)]. From here on, where “above” and “below” are used, it is assumed the manifolds $S^a_\delta$ and $S^r_\delta$ are positioned as in Fig. 3.1(b). The behaviour of the trajectories (near $S^a_\delta$) that approach $F$ depends on the position of the slow manifolds. (Because $S^a_\delta$ is strongly attracting, it suffices to describe the dynamics of trajectories started on $S^a_\delta$.) Heuristically, if $S^a_\delta$ goes above $S^r_\delta$, the trajectories will move away from $S^r_\delta$, along the fast $x$ direction, to another part of the state space, whereas if $S^a_\delta$ dives below $S^r_\delta$, when the trajectories move away from $S^r_\delta$ they go back, along the fast $x$ direction, to $S^a_\delta$ [Fig. 3.1(b)]. Near $F$, the manifolds $S^a_\delta$ and $S^r_\delta$ may twist together, and intersect in a complex manner [17, Fig. 6]. Thus, to
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understand the dynamics for system (3.1)–(3.2) when \(0 < \delta \ll 1\), further results are
needed which describe the intersections of \(S^a_\delta\) and \(S^r_\delta\).

The singular canard trajectories \(\gamma^c\) defined for \(\delta = 0\), may persist as nearby canard trajectories \(\gamma_\delta\) when \(0 < \delta \ll 1\). The canard trajectories cross from \(S^a_\delta\) to the repelling manifold \(S^r_\delta\), then follow \(S^r_\delta\) for \(O(1)\) slow time \(t\) [62]. Geometrically, they are the transverse intersections of \(S^a_\delta\) and \(S^r_\delta\) [62, 15]. (Whereas in one fast-one slow systems canard trajectories exist only for an exponentially small parameter range, in one fast-two slow systems they are robust, generic trajectories for a range of parameters.) The faux canard trajectories do not involve intersections of \(S^a_\delta\) and \(S^r_\delta\), and are not interesting because most trajectories move away from \(S^r_\delta\) and follow \(S^a_\delta\).

The persistence of singular canard trajectories depends on the type of folded singularity the system had in the limit \(\delta = 0\). Note, when \(0 < \delta \ll 1\), the folded singularities themselves no longer exist [Fig. 3.1(b)]. The dynamics generated by the isolated folded saddle and the isolated folded node when \(0 < \delta \ll 1\) have been researched thoroughly [62, 11, 71, 16, 15, 17, 72]. Because we are interested in a system where a folded node interacts with a nearby folded saddle, we first review the dynamics of these isolated singularities, before summarising the current, partial, understanding of the dynamics following a folded saddle-node (type I) bifurcation. The folded saddle and folded node produced by the bifurcation have some of the canards present in the isolated cases, and most interestingly, have new canards that cannot be inferred from the individual folded singularities, but arise from their interaction.

3.2.1. Dynamics near an isolated folded saddle

Recall, when \(\delta = 0\), a folded saddle has a unique singular canard \(\gamma^S\) and a unique faux canard [Fig. 3.1(a)]. When \(0 < \delta \ll 1\), the dynamics in the folded saddle region are simple: there is a unique canard trajectory \(\gamma^S_\delta\) [Fig. 3.1(b)] [62].

Near a folded saddle, the canard \(\gamma^S_\delta\) acts as a threshold: on one side of \(\gamma^S_\delta\), trajectories approach \(F\), go above \(S^r_\delta\), and move away to another part of the state space, whereas on the other side of \(\gamma^S_\delta\), trajectories do not approach \(F\) and stay near \(S^a_\delta\) for all time.

3.2.2. Dynamics near an isolated folded node

Recall, when \(\delta = 0\), a folded node has a unique strong singular canard trajectory \(\gamma^N\) tangent to the strong eigendirection, a weak singular canard trajectory \(\gamma^C\) tangent to the weak eigendirection, and an entire section (funnel) of singular canard trajectories bounded by \(F\) and \(\gamma^N\) [Fig. 3.1 (a)]. When \(0 < \delta \ll 1\), the dynamics in the folded node region are complicated [17, Fig. 6]: the number of canard trajectories depends on the ratio of the weak and strong eigenvalues \(\mu < 1\) [62, 71]. For all \(\mu\), the folded node strong canard \(\gamma^N_\delta\) persists [Fig. 3.1 (b)]. When \(\mu^{-1} \not\in \mathbb{N}\), the folded node weak canard \(\gamma^C_\delta\) persists. For \(\mu^{-1} \not\in \mathbb{N}\), where \(2k + 1 < \mu^{-1} < 2k + 3\), there are \(k\) other canard trajectories from the
funnel that persist. These are called secondary canard trajectories $\gamma^i_\delta$, $i = 1 \ldots k$, and have the following properties [62, 71, 11, 16]:

- $\gamma^i_\delta$ is $O(\delta^{(1-\mu)/2})$ close to $\gamma^i_N$ at $O(1)$ distance from $F$,
- $\gamma^i_\delta$ is between $\gamma^C_\delta$ and $\gamma^N_\delta$,
- $\gamma^i_\delta$, $i = 1 \ldots k$ are in order, with $\gamma^k_\delta$ closest to $\gamma^C_\delta$, and
- each $\gamma^i_\delta$ rotates $i$ times about $\gamma^C_\delta$. (By “rotate” we consider the trajectory to have turned 360°.)

On one side of $\gamma^N_\delta$, trajectories go above $S^r_\delta$, and move away immediately, whereas on the other side, trajectories rotate around $\gamma^C_\delta$ before moving away. If the trajectory lies between $\gamma^{i-1}_\delta$ and $\gamma^i_\delta$ it has $i$ rotations. Thus, the canards separate areas of $S^a_\delta$, known as rotational sectors, where the trajectories have different numbers of small amplitude rotations [71, 17, 11].

The increase in the number of canard trajectories as $\mu \to 0$ has been described by bifurcations of canard trajectories [71, 16]. Specifically, at every odd $\mu^{-1} \in \mathbb{N}$ there is a transcritical bifurcation with $\gamma^C_\delta$ where the secondary canard trajectory $\gamma^i_\delta, \mu^{-1} = 2i + 1$, is formed. At every even $\mu^{-1} \in \mathbb{N}$ there are pitchfork bifurcations where new branches bifurcate off $\gamma^C_\delta$, however these branches only exist for a negligible parameter interval so are usually ignored. The canard trajectories are transverse intersections of the slow manifolds $S^a_\delta$ and $S^r_\delta$, and the canard bifurcations correspond to $S^a_\delta$ and $S^r_\delta$ being tangent [71, 16].

In real-world systems with a folded node, where $\delta$ is not sufficiently small, or some normal form assumptions are violated, the dynamics are even more complex [15, 73]. In a self-coupled FitzHugh-Nagumo system, as $\delta$ is increased, there can be more than one canard trajectory with $i$ rotations [15, Fig. 3]. In a Hodgkin-Huxley model, there are two canard trajectories with the $i$ rotations, the secondary canard $\gamma^i_\delta$, and a canard that has a relaxation oscillation then follows $\gamma^{i-1}_\delta$ [15, Fig. 5, 6]. These canards with an initial relaxation oscillation are a consequence of the folded node interacting with the dynamics of a steady state, as is also seen in a stellate cell model [73].

### 3.2.3. Dynamics near a folded saddle-node (type I)

Consider $\epsilon > \epsilon_{FSN}$. Recall, when $\delta = 0$, following the folded saddle-node (type I) bifurcation, the folded node feeds into the folded saddle. That is, the folded node weak singular canard $\gamma^C$ connects to the folded saddle singular faux canard [Fig. 3.1 (a)].

When $0 < \delta << 1$, following the folded saddle-node (type I) bifurcation, the dynamics are only partially understood. In [70] the authors analytically identify two distinct regions of behaviour termed “near-field” and “far-field” which correspond to $\epsilon$ near $\epsilon_{FSN}$, and $\epsilon$ far from $\epsilon_{FSN}$. Geometrically when $\epsilon$ is near $\epsilon_{FSN}$, the folded singularities are close, the folded node weak eigenvalue is close to zero, and the ratio of folded node eigenvalues $\mu$ is close to zero. Whereas when $\epsilon$ is far from $\epsilon_{FSN}$, the folded singularities are far apart, and
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Figure 3.2. Example of different canard trajectories. The critical manifold $S$ is shown for reference. (a) The secondary canard $\gamma^1_\delta$ (green) starts between the (folded node) strong canard $\gamma^S_\delta$ (black) and centre trajectory $\eta^C_\delta$ (red), and rotates once around $\eta^C_\delta$. (b) The composite canard $\tilde{\gamma}^N_\delta$ (orange) follows the strong canard $\gamma^N_\delta$ (black) a fast section, then the unique folded saddle canard $\gamma^S_\delta$ (black). (c) The tertiary canard $\zeta^4_\delta$ (cyan) starts away from the strong canard $\gamma^N_\delta$ (black), between the centre trajectory $\eta^C_\delta$ (red) and $\tau \rightarrow -\infty$, and rotates four times around $\eta^C_\delta$. The centre trajectory $\eta^C_\delta$ is approximated by the trajectory with the smallest amplitude rotations.

$\mu$ may be close to one. We make use of this “near” and “far” terminology in our study of the dynamics following the folded saddle-node (type I) bifurcation.

When $0 < \delta << 1$, many of the canard trajectories that are present for the isolated folded singularities exist, however, a canard may form part of a concatenated canard-faux canard trajectory [70]. The concatenated canard-faux canard trajectories initially behave as folded node canards and cross from $S^a_\delta$ to $S^r_\delta$ through the folded node region, however, they then behave as folded saddle faux canards and cross back from $S^r_\delta$ to $S^a_\delta$ through the folded saddle region [70, 50]. In this work, we do not refer to concatenated trajectories as canard trajectories, as they cannot be computed as intersections of the manifolds $S^a_\delta$ and $S^r_\delta$ when extended from a distance $O(1)$ away from $F$. (Note, closer than $O(\delta^{1/4})$ to the fold, the $S^r_\delta$ manifold is only defined by an extension starting further away. Thus, it is not clear whether concatenated trajectories that cross $F$ and stay closer than $O(\delta^{1/4})$ to $F$ should be described as canard trajectories.)

For any $\epsilon > \epsilon_{FSN}$, there is a (folded saddle) canard trajectory $\gamma^S_\delta$ and (folded node) strong canard trajectory $\gamma^N_\delta$ [Fig. 3.1 (b)] [70]. There is a special concatenated canard-faux canard trajectory called the centre trajectory $\eta^C_\delta$ [Fig. 3.2(a)], which in the singular limit $\delta \rightarrow 0$ becomes the folded node weak singular canard $\gamma^C_\delta$ connected to the folded saddle singular faux canard [Fig. 3.1(a)] [70]. (This is instead of the weak canard trajectory created by an isolated folded node.) For $\epsilon$ near $\epsilon_{FSN}$, many trajectories follow $\eta^C_\delta$ back through the folded saddle and so are concatenated canard-faux canard trajectories. There is a transition regime between $\epsilon$ near and $\epsilon$ far from $\epsilon_{FSN}$ where the maximal distance a trajectory follows $\eta^C_\delta$ is not as far as to the folded saddle region [70, Lemma 4.11]. Finally, when $\epsilon$ is far from $\epsilon_{FSN}$ the maximal distance a trajectory follows $\eta^C_\delta$ is not even halfway from the folded node region to the folded saddle region [70, Lemma 4.11].

Note, when $0 < \delta << 1$, the weak attractivity of $\eta^C_\delta$ makes it difficult to compute. However, the neighbouring trajectories rotate around $\eta^C_\delta$, and it is well approximated by the neighbouring trajectory with the smallest amplitude rotations. Where we wish to approximate $\eta^C_\delta$ for the purpose of illustration, we interpolate initial conditions and select the trajectory with the smallest amplitude rotations [Fig. 3.2(a),(c)].
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Folded singularities involved | Canard trajectory name | Notation | Number of rotations |
--- | --- | --- | --- |
FS | folded saddle canard | \( \gamma_S^N \) | 0 |
FN | (folded node) strong canard | \( \gamma_N^N \) | 0 |
FN and FS | composite strong canard | \( \tilde{\gamma}_N^N \) | 1 |
FN | (folded node) secondary canard | \( \gamma_1^1 \), \( \gamma_i^i \) | 1, \( i \) |
FN and FS | composite secondary canard | \( \tilde{\gamma}_1^1 \), \( \tilde{\gamma}_i^i \) | 2, \( i + 1 \) |
FN | tertiary canard | \( \zeta_N^N \), \( \zeta_1^1 \), \( \zeta_i^i \) | 0, 1, \( i \) |
FN and FS | composite tertiary canard | \( \tilde{\zeta}_N^N \), \( \tilde{\zeta}_1^1 \), \( \tilde{\zeta}_i^i \) | 1, 2, \( i + 1 \) |

Table 3.1. The different canard trajectories following a folded saddle-node bifurcation (type I). The canard trajectory may go through the folded saddle region (FS), the folded node region (FN), or both. Tertiary canard trajectories are first identified in results presented in this chapter.

There are a number of (folded node) secondary canard trajectories \( \gamma_i^i \), \( i = 1...k \), where \( k \) depends on \( \delta \) and \( \epsilon \). When \( \epsilon \) is near to \( \epsilon_{FSN} \), \( k \) is small, because most trajectories are concatenated canard-faux canard trajectories [50, 70]. When \( \epsilon \) is far from \( \epsilon_{FSN} \), the singularities are far apart, and are assumed to act as isolated folded singularities [70]. So when \( \epsilon \) is far from \( \epsilon_{FSN} \), and \( 2k + 1 < \mu^{-1} < 2k + 3 \), it is assumed that there are \( k \) secondary canard trajectories \( \gamma_i^i \), \( i = 1...k \), which lie between \( \eta_C^C \) and \( \gamma_N^N \), and each \( \gamma_i^i \) rotates \( i \) times around the centre trajectory \( \eta_C^C \) [Fig. 3.2(a)] [70].

We find that there are further, novel, canard trajectories that do not exist for the isolated folded singularities. Firstly, composite canard trajectories \( \tilde{\gamma} \) follow a canard trajectory generated by the folded node \( \gamma_N \), a fast section, then the unique (folded saddle) canard trajectory \( \gamma_S^S \) [Fig. 3.2(b)] [50].

For a completeness, we describe the second novel type of canard trajectory, which was first identified in the results presented in this chapter. Tertiary canard trajectories \( \zeta \) go through the folded node region, and lie between \( \eta_C^C \) and \( F \), with \( \zeta^N \) having no rotations and each \( \zeta_i^i \), \( i = 1...k \), having \( i \) rotations about \( \eta_C^C \) [Fig. 3.2(c)]. Note, they are distinct from secondary canards \( \gamma_i^i \), which lie between \( \gamma_N^N \) and \( \eta_C^C \) [Fig. 3.2(a)]. Lastly, there are also composite tertiary canard trajectories \( \tilde{\zeta} \) that follow \( \zeta \), a fast section, then \( \gamma_S^S \). The different types of canard trajectories following a folded saddle-node (type I) bifurcation are given in Table 3.1.

Following a folded saddle-node (type I) bifurcation, the threshold behaviour of the folded saddle is combined with the rotational behaviour of a folded node, and creates a threshold with an intricate band structure [50]. Specifically, following the bifurcation, there are three
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distinct dynamical zones [Fig. 3.1(b)] [70, 50]:

Zone 1. Between $\gamma_S^\delta$ and $\tau \to \infty$, trajectories do not approach $F$ and stay near $S^a_\delta$ for all time.

Zone 2. Between $\gamma_N^\delta$ and $\gamma_S^\delta$, trajectories approach $F$, go above $S^r_\delta$, and immediately move away to another part of the state space.

Zone 3. Between $\gamma_N^\delta$ and $\tau \to -\infty$, trajectories initially converge to, and rotate around $\eta_C^\delta$. These trajectories approach $F$ and may either go back to $S^a_\delta$, or move away to another part of the state space.

3.3. The model equations and compactification

Specifically, in this research we consider a slow-fast system of the form (3.1)–(3.2) satisfying assumptions 3.1–3.3:

$$\frac{\delta}{dt} \frac{dx}{dt} = x(x-1) + y + \lambda(\epsilon t), \tag{3.14}$$
$$\frac{dy}{dt} = -x, \tag{3.15}$$

and external forcing $\lambda(\epsilon t)$ as a non-linear shift from $-A$ to $A$:

$$\lambda(\epsilon t) = A \tanh(\epsilon t). \tag{3.16}$$

Moreover, for this forcing, $d\lambda/dt$ can be expressed as a function of $\lambda$

$$\frac{d\lambda}{dt} = \frac{\epsilon}{A}(A^2 - \lambda^2(\epsilon t)), \tag{3.17}$$

so we can extend system (3.14)–(3.16) using $\lambda$ instead of $\tau$:

$$\frac{\delta}{dt} \frac{dx}{dt} = x(x-1) + y + \lambda, \tag{3.18}$$
$$\frac{dy}{dt} = -x, \tag{3.19}$$
$$\frac{d\lambda}{dt} = \frac{\epsilon}{A}(A^2 - \lambda^2). \tag{3.20}$$

Figure 3.3 compares system (3.14)–(3.16) extended by $\tau$ and extended by $\lambda$. Notice, the time variable $\tau \in (-\infty, \infty)$, whereas the external forcing variable $\lambda(\tau) \in [-A,A]$, which is compact. Hence we refer to (3.18)–(3.20) as the compactified system.

The advantages of having a compactified system (3.18)–(3.20) are, firstly, we are able to compute the asymptotic dynamics as $\tau \to \pm \infty$ using finite $\lambda(\tau) \to \pm A$. Secondly, when $0 < \delta << 1$, in the $\tau$-extended system there are no invariant sets, whereas in the compactified system some invariant sets are preserved. Specifically, for all $\delta \geq 0$ and $\epsilon$, system (3.18)–(3.20) has invariant planes at $\{(x,y,\lambda) : \lambda = \pm A\}$. What is more, as system (3.14) has equilibria for fixed $\lambda$, system (3.18)–(3.20) has a saddle steady state $p$ at $(x,y,\lambda) = (0,A,-A)$, and a node steady state $q$ at $(x,y,\lambda) = (0,-A,A)$ [Fig. 3.3(b)]. These properties allow us to compute the entire $\lambda(\tau)$ range of the slow manifolds $S^a_\delta$ and
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![Figure 3.3. System (3.14)–(3.16) for $A = 2.5$ and $\log(\epsilon - \epsilon_{FSN}) = -1.8$, in the limit $\delta = 0$. The critical manifolds $S^\alpha$ and $S^\rho$ with fold $F$, folded singularities $FN$ and $FS$, and associated singular canards and faux canard $\gamma^N$, $\gamma^S$ and $\gamma^C$ are illustrated (a) in $(x, y, \tau)$ co-ordinates and (b) in $(x, y + \lambda, \lambda)$ co-ordinates. In (b) there are invariant sets at $\lambda = \pm A$, a saddle steady state at $p$, and a node steady state at $q$. $S^\rho_{\delta}$, and canard trajectories with infinite integration time that connect to the saddle steady state $p$.

Recall, by assumption 3.3, the folded node weak canard $\gamma^C$ is also the (unique) folded saddle singular faux canard. For the compactified system, the unstable manifold of the saddle steady state $p$ converges to the folded node tangent to the weak eigendirection, and is an obvious candidate for $\gamma^C$ [Fig. 3.3(b)]. Note, however, the results show that the concatenated canard-faux canard $\eta^C_\delta$ typically detaches from $p$ when $0 < \delta << 1$.

The numerical computations in the rest of the chapter are performed for the compactified system (3.18)–(3.20). Results can easily be mapped back to $(x, y, \tau)$ space by $\tau = \tanh^{-1}(\lambda/A)$. The equations from the analysis in Section 3.1 can simply be changed by $d\lambda/d\tau = \epsilon^{-1}d\lambda/dt$ and replacing $d\tau/du$ with $d\lambda/du = d\lambda/d\tau \cdot d\tau/du$.

System (3.18)–(3.20) has a parabolic critical manifold with slope $y + \lambda$. Note, in figures the vertical axis is $y + \lambda$ to remove the slope on the critical manifold. The fold $F$ is the line $\{(x, y, \lambda) : x = 1/2\}$ and has two folded singularities at

$$\lambda_{+,-}(\tau) = \pm A \sqrt{1 - \frac{1}{2A\epsilon}},$$

with eigenvalues given by

$$\xi_{1,2} = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{8\epsilon}{A} \lambda_{+,-}^{-1}(\tau)}.$$ 

Following a folded saddle-node bifurcation (type I) at $\epsilon_{FSN} = (2A)^{-1}$, there is a folded saddle at $\lambda_-$ and a folded node at $\lambda_+$. At $\epsilon_{FF} = (2 + \sqrt{A^2 + 4})/(8A)$ the folded node degenerates into a folded stable focus. Note, $\epsilon$ controls both the separation of the folded singularities, and the ratio of folded node eigenvalues $\mu$.

We study system (3.14)–(3.16) for fixed $A = 2.5$, $0 < \delta << 1$ and $0.2 < \epsilon < 0.2601$ (i.e. ...
$\epsilon_{FSN} < \epsilon < \epsilon_{FF}$). As the most varied behaviour for system (3.14)–(3.16) occurs as $\delta \to 0$ and $\epsilon \to \epsilon_{FSN}$, parameter values are expressed as $\log_{10}(\delta)$ and $\log_{10}(\epsilon - \epsilon_{FSN})$. Note, for $A = 2.5 \log_{10}(\epsilon_{FF} - \epsilon_{FSN}) \approx -1.2213$.

### 3.4. Dynamics near a folded saddle-node (type I) from a viewpoint of flow maps

A flow map $\phi : L_{in} \to \Sigma_{out}$ illustrates how trajectories map between two suitably chosen “in” and “out” cross-sections, $L_{in}$ and $\Sigma_{out}$.

Consider a line of initial states $L_{in}$ on $S^a$, parallel to the fold $F$, and sufficiently distant $O(1)$ from the (complicated) dynamics near $F$ [Fig. 3.4(a)]. (Because the nearby slow manifold $S^a_\delta$ is invariant and strongly attracting it is sufficient to consider a line of initial states on $S^a_\delta$ because all trajectories near $S^a_\delta$, away from $F$, collapse onto $S^a_\delta$ [28]). Set

$$L_{in} := \{(x, y, \lambda) = (-0.5, -0.75 - \lambda_{in}, \lambda_{in}) : \lambda_{in} \in [-A, A]\}. \quad (3.23)$$

Note, $L_{in}$ is one-dimensional so it is parameterised by $\lambda_{in}$. Terminate the trajectories started at $L_{in}$ when they reach a surface $\Sigma_{out}$ on the repelling side of $F$, and $O(1)$ distant from $F$ [Fig. 3.4(a)]. Set

$$\Sigma_{out} := \{(x, y, \lambda) = (1.5, y_{out}, \lambda_{out})\}. \quad (3.24)$$

As $L_{in}$ is transverse to the trajectories in system (3.18)–(3.20) (which move parallel to the folded node strong eigendirection), the map $\phi : L_{in} \to \Sigma_{out}$ describes the dynamics of all generic trajectories in $L_{in}$. This is the advantage of a flow map method, whereas later results in Section 3.5 describe only canard trajectories.

Previous studies of flow maps near a folded singularity have not demonstrated how the flow map changes with a varying parameter [28]. From the flow map, we show that the dynamics in zones 1 and 2 for the folded saddle-node (type I) remain qualitatively the same for $\epsilon_{FSN} < \epsilon < \epsilon_{FF}$. Moreover, we expose the complicated, changing dynamics of trajectories in zone 3 [Fig. 3.1(b), the lower $\lambda$ and $\tau$ side of $\gamma^N_\delta$]. We show that these changes in zone 3 are qualitatively different when $\epsilon$ is near, and when $\epsilon$ is far from $\epsilon_{FSN}$, which corroborates the existing theory [70]. What is more, when $\epsilon$ is far from $\epsilon_{FSN}$, we infer the existence of the novel tertiary canard $\zeta_\delta^i$. This new tertiary canard does not exist in systems with an isolated folded singularity. Thus, our result contradicts the prevailing wisdom that, following a folded saddle-node (type I) bifurcation, when $\epsilon$ is far from $\epsilon_{FSN}$, the folded saddle and folded node act as isolated folded singularities [70].

Figure 3.5 shows the flow map $\phi$ for system (3.18)–(3.20) for $\log(\delta) = -2$ at six different values of $\epsilon$. Panels (a)-(p) show $\epsilon_{FSN} < \epsilon < \epsilon_{FF}$, and the final panels (q)-(r) show $\epsilon > \epsilon_{FF}$. The panels on the left (a), (c), (e), (g), (i), (k), (m), (o), and (q) show the image on $\Sigma_{out}$, and the panels on the right (b), (d), (f), (h), (j), (l), (n), (p), and (r) show the $\lambda$ component of the flow map, $\lambda_{in}$ and $\lambda_{out}$. The trajectories are coloured in order along
3. Bifurcations of Canard Trajectories Near a Type I Folded Saddle-Node Singularity

Figure 3.4. Illustration of the set-up for the flow map \( \phi : L_{\text{in}} \rightarrow \Sigma_{\text{out}} \). Trajectories started on the line \( L_{\text{in}} \) on the attracting side of the slow manifold \( S \) are terminated at the surface \( \Sigma_{\text{out}} \). Trajectories are shown for \( \log(\delta) = -2 \), \( \log(\epsilon - \epsilon_{\text{FSN}}) = -2.5 \) for initial states \( \lambda_{\text{in}} = -1.3, -1, -0.7, -0.4 \). Note, not all trajectories from \( L_{\text{in}} \) map to \( \Sigma_{\text{out}} \) (\( \lambda_{\text{in}} = -0.4 \) (black)), and the map is non-monotonic (trajectory from \( \lambda_{\text{in}} = -1.3 \) (orange) rotates under trajectory from \( \lambda_{\text{in}} = -1 \) (green)).

Recall, there are trajectories that stay near \( S^\delta_a \) for all time [Fig. 3.4(a), black trajectory]. These trajectories do not map to \( \Sigma_{\text{out}} \). Thus, for open intervals on \( L_{\text{in}} \), the flow map \( \phi \) is undefined. These intervals are indicated by grey bands in the \( \lambda \) component of the flow map (Fig. 3.5).

Recall, canard trajectories lie in the invariant slow manifolds \( S^\delta_a \) and \( S^\delta_r \). However, because \( S^\delta_a \) is strongly attracting in the fast \( x \) direction near \( L_{\text{in}} \), each canard trajectory is closely followed by a trajectory started on \( L_{\text{in}} \). The canard and approximate canard trajectories reach \( \Sigma_{\text{out}} \) at \( y_{\text{out}} + \lambda_{\text{out}} = -0.75 + O(\delta) \). However, when on \( S^\delta_r \), the canard trajectories become repelling in the fast \( x \) direction, so they are difficult to follow [Fig. 3.1(b)]. Thus, in the flow map \( \phi : L_{\text{in}} \rightarrow \Sigma_{\text{out}} \), the approach to (approximate) canard trajectories is seen as increasingly sparse end points at low \( y_{\text{out}} + \lambda_{\text{out}} \) on \( \Sigma_{\text{out}} \), and vertical asymptotes at apparent discontinuities in the \( \lambda \) component of the map (Fig. 3.5). The discontinuities are “apparent”, because, in fact, the canard trajectories bound (narrow) intervals of \( L_{\text{in}} \) where the map is undefined [Fig. 3.5, narrow grey bands]. (This is different to the “apparent” discontinuity encountered for an isolated folded node flow map, where the
Figure 3.5. Illustration of dynamics of system (3.18)–(3.20) for $A = 2.5$, $\log(\delta) = -2$ and $\log(\epsilon - \epsilon_{FSN}) = (a)-(b) -3$, (c)-(d) -2.5, (e)-(f) -1.8, (g)-(h) -1.79, (i)-(j) -1.75, (k)-(l) -1.73, (m)-(n) -1.7, (o)-(p) -1.65, and (q)-(r) -1.15. Panels on left (a), (c), (e), (g), (i) (k), (m), (o), and (q) show end points of trajectories on $\Sigma_{out}$. Panels on right (b), (d), (f), (h), (j), (l), (n), (p), and (r) show the $\lambda$ component of $\phi : L_{in} \rightarrow \Sigma_{out}$. A trajectory end point is coloured the same in both panels. Grey bands show initial states $\lambda_{in}$ which do not map to $\Sigma_{out}$. The distance marked on the panel is $\delta(1-\mu)/2$. In panels (q)-(r), $\epsilon > \epsilon_{FF}$, so $\mu$ is not defined.

map is continuous, but sharp near a canard trajectory (28, 17)).

Specifically, for any canard trajectory $\gamma_{\delta}$, there is an approximate canard trajectory and vertical asymptote at $\lambda_{in} \approx \gamma_{\delta} \cap \{x = -0.5\}$. Thus, the canard trajectories separate $L_{in}$ into intervals where the flow map $\phi$ is defined or undefined, as seen in the $\lambda$ component [Fig. 3.5(d)]. Moreover, the image of each separate interval for which $\phi$ is defined, forms a distinct “branch” on $\Sigma_{out}$ [Fig. 3.5(c)].

3.4.1. A summary of the dynamics for fixed $\epsilon$

Following a folded saddle-node bifurcation, when $\epsilon > \epsilon_{FSN}$, the dynamics are simple if $\epsilon$ is still very close to $\epsilon_{FSN}$ [Fig. 3.5(a)-(b)]. (Note, before the folded saddle-node bifurcation, no trajectory started on $L_{in}$ approaches $F$, and thus nothing maps to $\Sigma_{out}$.)
For system (3.18)–(3.20), when \( \log(\delta) - 2 \) and \( \log(\epsilon - \epsilon_{\text{FSN}}) = -3 \), trajectories from a small interval of \( L_{\text{in}} \) map to \( \Sigma_{\text{out}} \) [Fig. 3.5(a)-(b)]. The image on \( \Sigma_{\text{out}} \) forms a parabolic curve. The parabolic curve gets sparse as \( y_{\text{out}} + \lambda_{\text{out}} \to -0.75 \) (not shown in figure), and the trajectories approach canard trajectories. Specifically, the ends of the parabolic curve correspond to the two canard trajectories that exist for all of \( \epsilon_{\text{FSN}} < \epsilon < \epsilon_{\text{FF}} \) which are \( \gamma_{S}^{\delta} \) (at higher \( \lambda_{\text{in}}, \lambda_{\text{out}} \)) and \( \gamma_{N}^{\delta} \) (at lower \( \lambda_{\text{in}}, \lambda_{\text{out}} \)).

For \( \log(\epsilon - \epsilon_{\text{FSN}}) = -3 \), \( L_{\text{in}} \) is clearly separated into the three zones of dynamics that occur near the folded saddle-node (type I) [Fig. 3.1(b), Fig. 3.5(b)]:

**Zone 1. Right of \( \gamma_{S}^{\delta} \) \((-0.82 < \lambda_{\text{in}} < A)\).** Trajectories do not approach \( F \), so stay near \( S_{a}^{\delta} \) and do not map to \( \Sigma_{\text{out}} \) [e.g. Fig. 3.4(a), black trajectory]. This forms the right large grey band in the \( \lambda \) component of \( \phi \), Fig. 3.5(b).

**Zone 2. Between \( \gamma_{S}^{\delta} \) and \( \gamma_{N}^{\delta} \) \((-1.16 < \lambda_{\text{in}} < -0.82)\).** Trajectories approach \( F \), and immediately move away from \( S_{a}^{\delta} \) to \( \Sigma_{\text{out}} \) [e.g. Fig. 3.4, blue and green trajectories]. In the \( \lambda \) component of \( \phi \), this is characterised by trajectories starting at lower \( \lambda_{\text{in}} \) mapping to lower \( \lambda_{\text{out}} \) [Fig. 3.5(b)].

**Zone 3. Left of \( \gamma_{N}^{\delta} \) \((-A < \lambda_{\text{in}} < -1.16)\).** Trajectories approach \( F \), and rotate around \( \eta_{C}^{\delta} \). For this value of \( \epsilon \), all the trajectories in zone 3 go back to \( S_{a}^{\delta} \) through the folded saddle region and stay near \( S_{a}^{\delta} \) so do not map to \( \Sigma_{\text{out}} \). This forms the left large grey band in the \( \lambda \) component of \( \phi \), Fig. 3.5(b).

### 3.4.2. Changes in dynamics as \( \epsilon \) varies

First we discuss in general terms how the flow map \( \phi \) near the folded saddle-node changes as \( \epsilon \) increases.

For any \( \epsilon > \epsilon_{\text{FSN}} \), the flow map \( \phi \) is undefined for the interval near \( \lambda_{\text{in}} = A \) as trajectories started in zone 1 do approach \( F \) and thus do not map to \( \Sigma_{\text{out}} \) (Fig. 3.5). Adjacent to this undefined interval, is a growing interval of \( L_{\text{in}} \) where \( \phi \) is defined, which has \( O(1) \) width and maps to an outer parabolic curve on \( \Sigma_{\text{out}} \) [Fig. 3.5(c)-(f)]. This interval correspond to zone 2. The outer parabolic curve bounds all the other branches in the image on \( \Sigma_{\text{out}} \) [e.g. Fig. 3.5(c)]. Recall, zone 2 is bounded by \( \gamma_{S}^{\delta} \) and \( \gamma_{N}^{\delta} \). As \( \epsilon \) increases, \( \gamma_{S}^{\delta} \) and \( \gamma_{N}^{\delta} \) move further apart, so the outer parabola becomes wider. In this way, the flow map \( \phi \) illustrates the qualitatively unchanging dynamics of zones 1 and 2.

As \( \epsilon \) increases, \( \phi \) is defined for an increasing number of small intervals in zone 3, that is, at low \( \lambda_{\text{in}} \). These intervals map to branches on \( \Sigma_{\text{out}} \) that are inside the outer parabola [Fig. 3.5 (c)-(j)]. Consequently, as \( \epsilon \) increases, the wide undefined interval in zone 3 near \( \lambda_{\text{in}} = -A \), shrinks [Fig. 3.5 (c)-(f)] and, at some transition value of \( \epsilon \), disappears [Fig. 3.5(g)-(h)]. Until, when \( \epsilon \) is far from \( \epsilon_{\text{FSN}} \), there are only narrow undefined intervals [Fig. 3.5(k)-(p)]. Note, these intervals map to branches on \( \Sigma_{\text{out}} \) that no longer fill the outer parabola. What is more, as \( \epsilon \) approaches \( \epsilon_{\text{FF}} \), adjacent defined intervals merge [Fig. 3.5(k)-(p)]. So when \( \epsilon \geq \epsilon_{\text{FF}} \), there is again only one, now large, interval of \( L_{\text{in}} \) where the flow map \( \phi \) is defined [Fig. 3.5(q)-(r)].
Thus, the flow map illustrates that the dynamics in zone 3 change dramatically as \( \epsilon \) increases. Recall, trajectories started in zone 3 rotate around the centre trajectory \( \eta_C^\delta \) and may either go back to \( S_a^\delta \), or move away \( S_r^\delta \) and map to \( \Sigma_{\text{out}} \). So, in zone 3, there may be few, or many intervals where \( \phi \) is undefined, or defined. What is more, these intervals are separated by apparent discontinuities which correspond to canard trajectories in the compactified system (3.18)–(3.20).

We are interested in changes in the number of intervals that \( \phi \) is defined for, and the position of the corresponding branches on \( \Sigma_{\text{out}} \). Specifically, near a folded saddle-node singularity, we find there are seven qualitatively different features. These features are detailed below in the order in which they first appear with increasing \( \epsilon \). For each feature, we identify the types of canard trajectory that bound the intervals, and the qualitative dynamics of trajectories in the intervals.

### 3.4.2.a. For \( \epsilon \) near \( \epsilon_{\text{FSN}} \)

As \( \epsilon \) increases from \( \epsilon_{\text{FSN}} \), and the folded node and folded saddle separate, the outer parabola on \( \Sigma_{\text{out}} \) bounded by \( \gamma_N^\delta \) and \( \gamma_S^\delta \) widens.

**Feature 1.** Extra branches appear on \( \Sigma_{\text{out}} \) inside the outer parabola [Fig. 3.5(c),(e)]. The extra inner branches trace the \( \gamma_S^\delta \) side of the outer parabola, then go to low \( y_{\text{out}} + \lambda_{\text{out}} \). Equivalently, the flow map \( \phi \) is defined for an increasing number of small intervals, as shown in the \( \lambda \) component of \( \phi \) [Fig. 3.5(d),(f)].

This feature of the flow map is similar to that of an isolated folded node. (However, for an isolated folded node, the image on \( \Sigma_{\text{out}} \) is unbounded on one side [28].) Thus, we infer properties of \( \phi \) from the flow map for a folded node [28, 71, 17]. For any given branch on \( \Sigma_{\text{out}} \), all the trajectories that map to that branch have the same number of rotations about \( \eta_C^\delta \). Specifically, the trajectories which map to the inner branch at lowest \( \lambda_{\text{out}} \) (closest to \( \gamma_N^\delta \)) have one rotation, the trajectories that map to the next inner branch have two rotations, and so on. This happens because the trajectories with more rotations follow \( \eta_C^\delta \) for longer, and consequently map to higher \( \lambda_{\text{out}} \). This also makes a distinct pattern in the \( \lambda \) component of the flow map as branches have monotonically increasing minima [Fig. 3.5(d),(f)] [28].

As \( \lambda_{\text{in}} \) decreases, each branch moves towards \( y_{\text{out}} + \lambda_{\text{out}} = -0.75 \), where there is a secondary canard trajectory \( \gamma_i^\delta \), \( i = 1 \ldots m \), with \( i \) rotations. However, unlike for an isolated folded node, the branches are bounded on the right side by the composite secondary canard \( \tilde{\gamma}_i^{\delta -1} \), rather than the (folded node) secondary canard \( \gamma_i^{\delta -1} \). The bounding by \( \tilde{\gamma}_i^{\delta -1} \) is evidenced on \( \Sigma_{\text{out}} \) by successive end points tracing the outer parabola towards \( \gamma_S^\delta \cap \Sigma_{\text{out}} \), which is, equivalently, the end point for any composite canard [Fig. 3.5(c),(e), Fig. 3.2(b)].

For example, in the first inner branch, bounded by \( \gamma_1^\delta \) and \( \tilde{\gamma}_i^{\delta -1} \), trajectories rotate once around \( \eta_C^\delta \), then move away from \( S_a^\delta \) [Fig. 3.4, \( \lambda_{\text{in}} = -1.3 \) (orange)].

Recall, each branch corresponds to a small interval where \( \phi \) is defined. Moreover, each in-
interval corresponds to a folded node rotational sector, with trajectories in the \( i \)-th rotational sector having \( i \) rotations. The scaling is the same following a folded saddle-node (type I) bifurcation as for an isolated folded node, with each rotation sector being \( O(\delta^{(1-\mu)/2}) \) wide (Fig. 3.5) \([11, 17]\).

Finally, the flow map \( \phi \) clearly shows, following a folded saddle-node (type I) bifurcation, the maximum number of inner branches \( m \), depends on the space between \( \gamma^S_\delta \) and \( \gamma^N_\delta \). Equivalently, this is the maximum number of secondary canard trajectories, and rotational sectors. From our results, one would expect \( m \) to be the given by the distance in \( \lambda \) between \( \gamma^S_\delta \) and \( \gamma^N_\delta \) divided by \( O(\delta^{(1-\mu)/2}) \). This adds insight to the theory, which simply establishes for \( \epsilon \) near \( \epsilon_{FSN} \), \( m \) is small \([70]\). Moreover, this result may provide extra structure for future analytical work on the folded saddle-node (type I).

**Feature 2.** There are narrow intervals of \( L_{in} \) where the map \( \phi \) is undefined \([Fig. 3.5(d),(f) narrow grey bands]\). These narrow undefined intervals separate the intervals described in feature 1. Each undefined interval is bounded on the left by the secondary canard trajectory \( \gamma^i_\delta \) and on the right by the associated composite secondary canard trajectory \( \tilde{\gamma}^i_\delta \). Trajectories in this interval closely follow \( \gamma^i_\delta \), a long way down \( S^r_\delta \), before moving back to \( S^a_\delta \) and the zone 1 region \([50]\). They do not approach \( F \) again, and so remain near \( S^a_\delta \). Thus, these trajectories do not map to \( \Sigma_{out} \), and the flow map \( \phi \) is undefined.

For \( \epsilon \) near \( \epsilon_{FSN} \), there still remains an \( O(1) \) interval of \( L_{in} \), near \( \lambda_{in} = -A \), where the flow map \( \phi \) is undefined. Trajectories in this interval rotate around \( \eta^C_\delta \), however there is insufficient distance between \( \gamma^S_\delta \) and \( \gamma^N_\delta \) for them to complete more than \( m \) rotations, so they fed back through the folded saddle region to \( S^a_\delta \) \([70, 50]\). Notice, on \( \Sigma_{out} \) there is insufficient space for another inner branch \([Fig. 3.5(c),(e)]\).

### 3.4.2.b. Transition

We characterise the transition regime as the value of \( \epsilon \) where trajectories started near \( \lambda_{in} = -A \) no longer follow the centre trajectory \( \eta^C_\delta \) through the folded saddle region back to \( S^a_\delta \) \([Fig. 3.5(g)-(h)]\). Notice, there is still an \( O(1) \) wide undefined interval separated from \( \lambda_{in} = -A \) where trajectories follow \( \eta^C_\delta \) back to \( S^a_\delta \) \([Fig. 3.5(g)-(j)]\).

This transition regime relates to the analytical theory for a folded saddle-node, which gives an intermediate interval of \( \epsilon \) where the maximal distance a trajectory follows \( \eta^C_\delta \) is not as far as to the folded saddle region \([70, Lemma 4.11]\). Whereas, in contrast, when \( \epsilon \) is far from \( \epsilon_{FSN} \), the maximal distance a trajectory follows \( \eta^C_\delta \) is not even halfway from the folded node region to the folded saddle region \([70, Lemma 4.11]\). Thereby, we will see for \( \epsilon \) far from \( \epsilon_{FSN} \), no trajectory follows \( \eta^C_\delta \) back to \( S^a_\delta \).

The transition can happen in different ways, one of which is shown in Fig. 3.5(g)-(j). In the transition regime there are new features, the first of these is important in applications.
Feature 3. Trajectories started near $\lambda_{in} = -A$ no longer converge to $S^a_0$. That is, the flow map $\phi$ is defined for the interval of $L_{in}$ bounded by $\lambda_{in} = -A$ [Fig. 3.5(h)]. On $\Sigma_{out}$, the image of this interval forms a branch that is not connected to the outer parabola [Fig. 3.5(g)]. This branch has an end that does not correspond to a canard trajectory, but to the limit $\lambda_{in} \to -A$, and so terminates at $y_{out} + \lambda_{out} \approx 0.22$ [Fig. 3.5(g)]. The other end of the branch terminates at a canard trajectory which is described in more detail below.

Note, the trajectory started at $\lambda_{in} \to -A$ is strongly attracting, because it is the unstable manifold of the saddle steady state $p$ [Fig. 3.3(b)]. So, all trajectories started very near $\lambda_{in} = -A$ converge to a point on $\Sigma_{out}$ [Fig. 3.5(g)], and map to the same value of $\lambda_{out}$ [Fig. 3.5(h)].

There is still an $O(1)$ undefined interval, separated from $\lambda_{in} = -A$. For example, when $\log(\delta) = -2$, $\log(\epsilon - \epsilon_{FSN}) = -1.79$, and $-2.28 < \lambda_{in} < -2.14$ [Fig. 3.5(h), grey band].

This undefined interval is important, because from it we can infer the relative position of the centre trajectory $\eta^C_0$. Trajectories started in this $O(1)$ undefined interval are closest to $\eta^C_0$, as evidenced by their following $\eta^C_0$ back through the folded saddle region to $S^a_0$, and having the smallest amplitude rotations. (Note, when $\delta = 0$, the centre trajectory $\eta^C_0$ is asymptotic to the unstable manifold of the saddle point $p$ at $\lambda_{in} = -A$ [Fig. 3.3(b)], and this result shows when $\delta > 0$, the centre trajectory $\eta^C_0$ no longer connects to $p$.)

We can now say more about the canard trajectory that bounds the defined interval near $\lambda_{in} = -A$, and equivalently, bounds the new branch on $\Sigma_{out}$. This canard trajectory is of a different nature to the secondary canards $\gamma^+_0$ identified in feature 1. Firstly, notice as $\lambda_{in}$ decreases, the end points on $\Sigma_{out}$ trace a branch that moves away from, rather than towards, $y_{out} + \lambda_{out} = -0.75$ [Fig. 3.5(g)]. That is, the trajectories go orange to red as $y_{out} + \lambda_{out}$ increases. Secondly, from the location of the $O(1)$ undefined interval, we know this new canard trajectory is positioned between $\eta^C_0$ and $\lambda_{in} = -A$, rather than between $\eta^C_0$ and $\gamma^+_0$ [Fig. 3.5(h)]. Thus, this canard is different to the secondary canards $\gamma^+_0$ that appear for an isolated folded node, and are positioned between $\eta^C_0$ and $\gamma^+_0$ [71]. This is a new type of canard trajectory, which we call a tertiary canard $\zeta^+_i$, where $i$ is the number of rotations the canard has about $\eta^C_0$ [Fig. 3.2(c)].

Lastly, we want to know how the trajectories near $\lambda_{in} = -A$ change when they map to $\Sigma_{out}$, instead of returning to $S^a_0$. These trajectories have large amplitude rotations around $\eta^C_0$ in the folded saddle region. At small values of $\epsilon$, they are fed back through the folded saddle region to $S^a_0$. However, as $\epsilon$ increased, their rotations became larger. Until, in the transition regime, the trajectories try to do too large a rotation, go above $S^a_0$, and move away along the fast $x$ direction to another part of the state space, thereby mapping to $\Sigma_{out}$.

Feature 3 is important in applications which have a shift in the external forcing $\lambda(\tau)$ from $-A$ to $A$. This is because if the system has been at $\lambda(\tau) = -A$ for a long time, typical initial states will be near $\lambda_{in} = -A$. Feature 3 gives the smallest $\epsilon$ at which these trajectories started near $\lambda_{in} = -A$ no longer converge to $S^a_0$, and instead move away along the fast $x$ direction to another part of the state space.
Feature 4. A new type of branch appears on $\Sigma_{\text{out}}$ that starts at $y_{\text{out}} + \lambda_{\text{out}} \to -0.75$ and connects to the outer parabola [Fig. 3.5(i)]. Equivalently, the flow map $\phi$ is defined for another interval at low $\lambda_{\text{in}}$, which is bounded by a new type of canard trajectory.

Consider the canard trajectory that bounds the end of the branch which is not by the outer parabola [Fig 3.5(i)]. As in feature 3, as $\lambda_{\text{in}}$ decreases, the end points on $\Sigma_{\text{out}}$ trace a branch that moves away from $y_{\text{out}} + \lambda_{\text{out}} \to -0.75$, [Fig 3.5(i)]. That is, the trajectories go light to dark orange as $y_{\text{out}} + \lambda_{\text{out}}$ increases. As described in feature 3, this corresponds to a tertiary canard trajectory $\zeta_{i}^{\delta}$, with $i$ rotations.

Consider the canard trajectory that bounds the end of the branch which traces the outer parabola [Fig 3.5(i)]. As in feature 1, the end points on $\Sigma_{\text{out}}$ tracing the outer parabola correspond to a composite canard trajectory. However, the branch traces the outer parabola in the opposite direction, as shown by the colour change, to the branches identified in feature 1. Thus, this corresponds to a further new type of canard trajectory, a composite tertiary canard trajectory $\tilde{\zeta}_{i}^{\delta-1}$, with $i$ rotations. Like the composite secondary canard trajectory, the composite tertiary canard trajectory $\tilde{\zeta}_{i}^{\delta-1}$ is formed from the tertiary canard trajectory $S_{\delta}^{\delta-1}$ followed by the saddle canard trajectory $S_{\delta}^{S}$. Again, we assume all trajectories that map to this branch have the same number of rotations $i$. Thus, the branch is bounded at one end by $\zeta_{i}^{\delta}$, and at the other end by the composite tertiary canard trajectory with $i$ rotations, $\tilde{\zeta}_{i}^{\delta-1}$.

The new type of branch occurs following the flow map $\phi$ being temporarily undefined at $\lambda_{\text{in}} \to -A$, at $\epsilon = -1.783$. Consequently, the defined interval near $\lambda_{\text{in}} = -A$ disconnects from $\lambda_{\text{in}} = -A$, and is able to move away. Almost immediately, $\phi$ is defined again at $\lambda_{\text{in}} \to -A$, and another interval is formed bounded by $\lambda_{\text{in}} = -A$ (see feature 3). It appears as if the defined interval near $\lambda_{\text{in}} = -A$ widens and splits into two intervals, with the interval near $\lambda_{\text{in}} = -A$ mapping to lower $\lambda_{\text{out}}$ [Fig. 3.5(h),(j)].

Dynamically, this change corresponds to trajectories near $\lambda_{\text{in}} = -A$ moving away from $S_{\delta}^{\delta}$ after one fewer large rotation about $\eta_{\delta}^{C}$ in the folded saddle region. This is seen in the $\lambda$ component of the flow map – the trajectories that move away after one fewer rotation, leave $S_{\delta}^{\delta}$ sooner, so map to a lower value of $\lambda_{\text{out}}$ [Fig. 3.5(i)-(j)]. Note, this means the $\lambda$ component of the flow map no longer has branches with monotonically increasing minima [Fig. 3.5(j)], which makes it unlike the flow map for an isolated folded node [28, 17].

Feature 5. At low $\lambda_{\text{in}}$, there is a narrow interval for which the flow map $\phi$ is undefined [Fig. 3.5(j), narrow grey band]. This is similar to feature 2. The undefined interval is bounded by $\zeta_{i}^{\delta}$ and the associated composite canard $\tilde{\zeta}_{i}^{\delta}$. The trajectories in this interval closely follow $\zeta_{i}^{\delta}$ a long way down $S_{\delta}^{\delta}$, before moving back to $S_{\delta}^{\delta}$ and the zone 1 region. They do not approach $F$ again, and so remain on $S_{\delta}^{\delta}$. Thus, they do not map to $\Sigma_{\text{out}}$. 

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3.4.2.c. For $\epsilon$ far from $\epsilon_{FSN}$

In this regime, the $O(1)$ interval of $L_{in}$ near $\eta_{G}^{C}$ where the map $\phi$ is undefined disappears [Fig. 3.5(j),(l)]. Note, the centre trajectory $\eta_{G}^{C}$ still exists, but for $\epsilon$ far from $\epsilon_{FSN}$, trajectories rarely follow it [70, Lemma 4.11].

Feature 6. There is a branch on $\Sigma_{out}$ with both ends connected to the outer parabola [Fig. 3.5(k),(o)]. This shows the branch is bounded on both sides by composite canard trajectories. Moreover, from the direction the branch traces along the outer parabola for decreasing $\lambda_{in}$, we see the right side of the interval is bounded by $\tilde{\gamma}_{G}^{i}$ and the left side is bounded by $\tilde{\zeta}_{G}^{i}$. (This can also be inferred from continuity as $\epsilon$ changes from the canards bounding the intervals in Fig. 3.5(i)-(j).)

The first time this feature occurs for increasing $\epsilon$ is as we are coming out of the transition regime [Fig. 3.5(k)-(l)]. This case is atypical for feature 6, as the branch is formed from the merging of two defined intervals [Fig. 3.5(j),(l)]. However, there are some interesting aspects about coming out of the transition regime. Firstly, the $O(1)$ undefined interval disappears [Fig. 3.5(l)], because no trajectory started on $L_{in}$ follows $\eta_{G}^{C}$ back to $S_{a}^{G}$. (In fact, the trajectory from $L_{in}$ that follows $\eta_{G}^{C}$ most closely, that is, has the smallest rotations, appears to be the trajectory which also maps to the smallest value of $y_{out} + \lambda_{out} -$ at the fold in the branch on $\Sigma_{out}$.) Secondly, when the $O(1)$ undefined interval disappears, the two canard trajectories that bound it merge and disappear. For this to occur, those canard trajectories must have been qualitatively the same, with the same number of rotations $i$.

3.4.2.d. For $\epsilon$ approaching $\epsilon_{FF}$

As $\epsilon$ approaches $\epsilon_{FF}$, the number of intervals $\phi$ is defined for reduces, until for $\epsilon \geq \epsilon_{FF}$, there is only one defined interval [Fig. 3.5(q)-(r)]. Equivalently, the number of canard trajectories reduces, and for $\epsilon \geq \epsilon_{FF}$ there is only one canard trajectory. The reduction is a repeated two stage process, described by features 7 and 6, the first iteration is shown in Fig. 3.5(k)-(n). The images on $\Sigma_{out}$ for further iterations are similar, but after each iteration there is one fewer of the feature 1 branches, connecting from the outer parabola to $y_{out} + \lambda_{out} = -0.75$ [Fig. 3.5(o)-(p)].

Feature 7. The interval of $\phi$ near $\lambda_{in} = -A$ maps to a branch on $\Sigma_{out}$ that is connected to the outer parabola [Fig. 3.5(m)-(n)]. This interval of $\phi$ is now bounded by $\lambda_{in} = -A$ and a secondary composite canard $\tilde{\gamma}_{G}^{i}$, for some value of $i$.

For feature 7 to form as $\epsilon$ increases, the branch with both ends connected to the outer parabola (feature 6) shrinks to nothing. Simultaneously, the abutting narrow undefined intervals join and disappear. Thereby, the branch from $\lambda_{in} = -A$ merges with the branch below, and that branch connects to the outer parabola [Fig. 3.5(k)-(n)]. Thus, there are now two fewer defined intervals for the flow map $\phi$, which have been replaced by one large interval [Fig. 3.5(l),(n)]. Equivalently, four canard trajectories have disappeared.
Note, in the $\lambda$ component of the flow map, the branches have monotonically increasing minima [Fig. 3.5(m)-(n)]. This is as similar as $\phi$ gets to the flow map for an isolated folded node [28, 17]. In particular, there are no tertiary canard trajectories. However, there are subtle differences – neither narrow undefined intervals, nor composite canard trajectories arise in the flow map for an isolated folded node.

Feature 6 occurs repeatedly as $\epsilon$ approaches $\epsilon_{FF}$ [Fig. 3.5(o)-(p)]. There is a branch on $\Sigma_{out}$ with both ends connected to the outer parabola. This arises when the defined interval near $\lambda_{in} = -A$ splits into two defined intervals, with the interval near $\lambda_{in} = -A$ mapping to lower $\lambda_{out}$ [Fig. 3.5(n),(p)]. As in feature 4, this corresponds to trajectories near $\lambda_{in} = -A$ having one fewer rotation about $\eta_{C}$ . Again, the two defined intervals are separated by a narrow undefined interval. As $\epsilon$ approaches $\epsilon_{FF}$, feature 6 corresponds to an increase in the number of intervals $\phi$ is defined for, and equivalently to two more canard trajectories.

As $\epsilon$ approaches $\epsilon_{FF}$, the number of defined intervals initially decreases by two (feature 7) then increases by one (feature 6). Until, at $\epsilon = \epsilon_{FF}$, on $\Sigma_{out}$ only the outer parabola remains, that is, the flow map $\phi$ has only one defined interval, bounded by $\lambda_{in} = -A$ and the (folded saddle) canard trajectory $\gamma_{S}^{C}$ [Fig. 3.5(q)-(r)]. This is because when $\epsilon = \epsilon_{FF}$, the folded node degenerates into a folded focus. A folded focus has no canard trajectories, so all the canards that go through the folded node have disappeared. The dynamics for $\epsilon \geq \epsilon_{FF}$ can simply be described by zone 1 and zone 2, which now extends all the way to $\lambda_{in} = -A$, as zone 3 no longer exists.

It has been generally considered that for $\epsilon$ far from $\epsilon_{FSN}$, the folded node and folded saddle act as isolated folded singularities [70]. There is not an equivalent study of changing flow maps with a varying parameter for an isolated folded node. However, the following features in the flow map for the folded saddle-node would not be reproduced for an isolated folded node: the composite canards, the extra tertiary canards $\zeta_{6}$ positioned between $\eta_{6}^{C}$ and $F$, the narrow undefined intervals, and the non-monotonic branches in the $\lambda$ component of the flow map. This suggests treating the folded node and folded saddle as isolated folded singularities when $\epsilon$ is far from $\epsilon_{FSN}$ is an over simplification.

The maps in Fig. 3.5 show the importance of knowing what canard trajectories are present near a folded saddle-node, as they determine the dynamics of generic trajectories. Using the flow map $\phi$, we have identified changes in the number and the type of intervals the flow map $\phi$ is defined for (Fig. 3.5). What is more, we know these changes correspond to the appearance and disappearance of different types of canard trajectories. These changes can be studied as bifurcations of canard trajectories, both for varying $\epsilon$ and $\delta$.
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3.5. Dynamics near a folded saddle node (type I) from a viewpoint of bifurcating canard trajectories

In Section 3.4, the (approximate) canard trajectories were difficult to follow when they become repelling in the fast \( x \) direction. Whilst canard trajectories cannot easily be computed by forward time integration, we can compute the canard trajectories as solutions to a boundary value problem “\( L_{\text{in}} \) to \( L_{\text{out}} \)” [16, 15]:

Consider system (3.18)–(3.20), with variable integration time \( T \), and four boundary conditions:

\[
(x(0), y(0), \lambda(0)) \in L_{\text{in}} \quad \text{and} \quad (x(T/\delta), y(T/\delta), \lambda(T/\delta)) \in L_{\text{out}},
\]

where \( L_{\text{in}} \) and \( L_{\text{out}} \) are lines on \( S_a^0 \) and \( S_r^0 \), respectively, at a distance \( O(1) \) from \( F \).

The solutions to the boundary value problem “\( L_{\text{in}} \) to \( L_{\text{out}} \)” are canard trajectories. Note, system (3.18)–(3.20) has time rescaled to \( \delta dt \) for computations, so the boundary conditions are dependent on \( \delta \).

As described in Section 3.4, we can closely approximate canard trajectories using \( L_{\text{in}} \) on \( S^a \), and \( L_{\text{out}} \) on \( S^r \) (Fig. 3.6). Following Section 3.4, we choose \( L_{\text{in}} \) given by Eq. (3.23) and \( L_{\text{out}} = \Sigma_{\text{out}} \cap S^r \) given by Eq. (3.24), so the boundary conditions are:

\[
x(0) = -0.5, \quad y(0) + \lambda(0) = -0.75, \quad x(T/\delta) = 1.5, \quad y(T/\delta) + \lambda(T/\delta) = -0.75.
\]

The boundary value problem “\( L_{\text{in}} \) to \( L_{\text{out}} \)” can be solved using boundary value solver routines in AUTO [21]. (In AUTO, a boundary value problem needs an initial solution. Section 3.5.1 describes how this is computed.) The accuracy of the approximate canard trajectories and their bifurcations depends on the boundary conditions. The results shown in this chapter are verified as the same (that is, the difference is smaller than the accuracy of the boundary solver routine, which is set by a choice of convergence criteria) if the boundaries \( L_{\text{in}} \) and \( L_{\text{out}} \) are twice as far from \( F \), at \( x(0) = -1.5 \) and \( x(T/\delta) = 2.5 \).

We can use the numerical continuation routines in AUTO to find approximate canard trajectories for different values of \( \delta > 0 \) and \( \epsilon > 0 \). Note, for any fixed \( \delta \) and \( \epsilon \), there may be multiple canard trajectories, all of which are solutions to the boundary value problem “\( L_{\text{in}} \) to \( L_{\text{out}} \)”. However, because each canard trajectory is typically isolated and difficult to follow, the boundary value problem is well posed when continuing solutions in \( \delta \) and \( \epsilon \).

At the end of this section, we fully describe the local and global features of the dynamics near a folded saddle-node (type I) using a bifurcation diagram for the varying parameters \( \delta \) and \( \epsilon \) (Section 3.5.3). There are three stages in computing this bifurcation diagram. First, canard trajectories are computed for fixed \( \delta \) and \( \epsilon \) (Section 3.5.1). These are used as initial solutions for the full boundary value problem, “\( L_{\text{in}} \) to \( L_{\text{out}} \)”. Second, the canard trajectories are continued in \( \epsilon \) and we identify co-dimension one bifurcations (Section 3.5.2). Finally, the co-dimension one bifurcations are continued in both \( \delta \) and \( \epsilon \) to form the two parameter bifurcation diagram (Section 3.5.3).
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Our results verify that changes in the number of defined intervals for the flow map $\phi$ correspond to bifurcations of canard trajectories. We show, for the first time, which canard trajectory bifurcations occur in systems with a folded saddle-node (type I) singularity. By using numerical computation we find canard trajectory bifurcations at large $\delta$, which is beyond the scope of the analytical theory [70]. Thus, we get a complete picture of the relationship between the bifurcations when $\epsilon$ is near, and when $\epsilon$ is far from $\epsilon_{FSN}$. Lastly, we reveal a novel dense area of turning-point canard trajectory bifurcations, which we term a sprite. Our results corroborate existing theoretical results, and uncover new dynamical features in systems with folded saddle-node (type I) singularities.

3.5.1. Computing canard trajectories as intersections of $S^a_\delta$ and $S^r_\delta$

We use the fact that canard trajectories are intersections of $S^a_\delta$ and $S^r_\delta$ to compute the canard trajectories for fixed parameters, $\log(\delta) = -2$ and $\log(\epsilon - \epsilon_{FSN}) = -1.8$. This is a multiple stage process. (See code in Appendix B.4.1)

Stage 1. The approximate slow manifolds $S^a_\delta$ and $S^r_\delta$ are computed up to the surface $\Sigma_{FN}$, perpendicular to $F$ at the folded node (Fig. 3.6):

$$\Sigma_{FN} := \{(x, y, \lambda) : \lambda = \lambda_- \}.$$

With the exception of the folded saddle canard trajectory $\gamma^S_{\delta}$, all canard trajectories near a folded saddle-node (type I) pass through the folded node region, so the cross-section of $S^a_\delta$ and $S^r_\delta$ at $\Sigma_{FN}$ is very informative. Note, because system (3.18)–(3.20) is compactified, we are able to compute $S^a_\delta$ and $S^r_\delta$ for all $\lambda_{in} \in [-A, \lambda_-]$ and $\lambda_{out} \in [\lambda_-, A]$.

Stage 2. The curves $S^a_\delta \cap \Sigma_{FN}$ and $S^r_\delta \cap \Sigma_{FN}$ intersect (Fig. 3.7), and these points correspond to (approximate) canard trajectories [28]. The trajectories from Stage 1 that map to these points are concatenated to form (approximate) canard trajectories (Fig. 3.6, red and blue.
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We now describe Stage 1 in more detail. This stage is set up as two boundary value problems, one for trajectories on the attracting slow manifold $S^a_\delta$, in forwards time going from $L_{in}$ to $\Sigma_{FN}$; and the other for the trajectories on the repelling slow manifold $S^r_\delta$, in backwards time going from $L_{out}$ to $\Sigma_{FN}$ (Fig. 3.6). The boundary value problems need initial solutions. These are computed via the homotopy method detailed in [16]. In brief, the folded node in $\Sigma_{FN}$ is taken as the initial solution, grown to have non-zero integration time $T$ by moving the start boundary along $F$, then the start boundary is moved down $S^a$ (respectively $S^r$) to $L_{in}$ (respectively $L_{out}$).

In detail, for the boundary value problem $L_{in}$ to $\Sigma_{FN}$, take general boundary conditions that the trajectory starts on $S^a$ and ends in $\Sigma_{FN}$:

\[ y(0) + \lambda(0) = -x(0)(x(0) - 1), \quad \text{and} \quad \lambda(T/\delta) = \lambda_. \]

and $x(0)$, $\lambda(0)$ and $T$ may be used as continuation parameters or free variables.

Stage 1a. The boundary value problem has the extra condition $x(0) = 0.5$ (the trajectory starts on $F$), and is initialised with the folded node and $T = 0$. The trajectory is grown by varying $y(0)$, $\lambda(0)$ and $T$ until $\lambda(0) = -1$.

Stage 1b. The boundary value problem has variable $x(0)$, and the extra condition $\lambda(0) = -1$. The trajectory is grown by varying $x(0)$, $y(0)$ and $T$ until $x(0) = -0.5$. Now $(x(0), y(0), \lambda(0)) \in L_{in}$, and we have an initial solution to the boundary value problem $L_{in}$ to $\Sigma_{FN}$ as required.

Note, by choosing $L_{in}$ below the saddle steady state $p = (0.5, A, -A)$ [Fig. 3.3(b)], we avoid having any “duplicate” start points that are on the same trajectory. This problem was encountered in computations for the rescaled isolated folded node, and created “turning-point” bifurcations [71, 16].

Figure 3.7 shows the intersection of approximate slow manifolds $S^a_\delta$ and $S^r_\delta$ and the surface $\Sigma_{FN}$. The intersection of $S^a_\delta$ and $S^r_\delta$ with a surface through a folded singularity has been computed many times for isolated folded nodes [71, 28, 16, 15, 17], but this is the first time it has been computed near a folded saddle-node. At first glance, the results are like those for an isolated folded node; $S^a_\delta \cap \Sigma_{FN}$ and $S^r_\delta \cap \Sigma_{FN}$ spiral in together, creating secondary canard trajectories $\gamma^S_i$ with an increasing number of rotations $i$ (Fig. 3.7). What is new, and distinctly different near a folded saddle-node, is the repelling manifold $S^r_\delta$ folds back on itself and spirals out [Fig. 3.7(c), blue]. Consequently, the intersection points are repeated. These correspond to composite secondary canard trajectories $\tilde{\gamma}^S_i$ that are closely positioned by $\gamma^S_i$ and perform one extra rotation by following the folded saddle canard $\gamma^S_0$ [Fig. 3.2(b)]. (Note, the folded back section of $S^r_\delta \cap \Sigma_{FN}$ is the image of a narrow interval of $L_{out}$, close to the folded saddle canard $\gamma^S_0$.) What is more, by compactifying the system, we unusually are able to compute $S^a_\delta \cap \Sigma_{FN}$ all the way to the centre tip, which is the accumulation point of trajectories started at $\lambda_{in} \to -A$ [Fig. 3.7(c), red]. For the system without compactification (3.14)–(3.16), this corresponds to trajectories started at
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Figure 3.7. Intersections of the attracting and repelling slow manifolds $S_a^\delta$, $S_r^\delta$ with $\Sigma_{FN}$ (red, blue) for $\log(\delta) = -2$, $\log(\epsilon - \epsilon_{FSN}) = -1.8$. Panels (b)-(c) show successive zooms. Curves intersect at canard trajectories (diamonds). Note, a canard $\gamma_\delta$ and its corresponding composite canard $\tilde{\gamma}_\delta$ are coincident. (The flow map $\phi$ for the same parameters was shown in Fig. 3.5(e)-(f).)

$\tau \to -\infty$.

Each pair of trajectories that meet on $\Sigma_{FN}$ at the intersection point of $S_a^\delta \cap \Sigma_{FN}$ and $S_r^\delta \cap \Sigma_{FN}$ is concatenated to form an (approximate) canard trajectory (Fig. 3.6). This can now be used as an initial solution to the boundary value problem “$L_{in}$ to $L_{out}$” for continuation in $\epsilon$.

From here on the following colours will correspond to the number of rotations a canard trajectory has: black: 0 (or more than 5), green: 1 (or 7), red: 2, blue: 3, cyan: 4, magenta: 5. Note, wherever there are both ordinary and composite canard trajectories, the composite canard trajectory will be plotted on top.

### 3.5.2. Canard bifurcations in one parameter

The canard trajectories are continued for fixed $\log(\delta) = -2$ and varying $\log(\epsilon - \epsilon_{FSN})$. This reveals the two types of co-dimension one canard trajectory bifurcation that occur near the folded saddle-node singularity in system (3.18)–(3.20): turning-point bifurcations, and (unprecedented) infinite-time bifurcations.

Turning-point bifurcations are also known as a saddle-node or fold bifurcations. We use this terminology to prevent confusing the canard trajectory bifurcation with the folded saddle-node singularity bifurcation in system (3.18)–(3.20). (Note, this is not the turning-point bifurcation described for an isolated folded node in [71], where the line of initial states comes tangent to a turn in the trajectories, resulting in two trajectories either side of the “turning-point” mapping to the same image.) In our results we distinguish between structurally different left and right turning-point bifurcations. A turning-point bifurcation at $\log(\epsilon - \epsilon_{FSN}) = C$ is “left” if there are no solutions when $\log(\epsilon - \epsilon_{FSN}) < C$, or “right”
Figure 3.8. Bifurcation diagram for canard trajectories for fixed $\log(\delta) = -2$ and varying $\log(\epsilon - \epsilon_{FSN})$, against the start point of the trajectory $\lambda_{in}$. Each curve consists of canard trajectories with the same number of rotations $i$ (black: 0, green: 1, red: 2, blue: 3, cyan: 4). There are left and right turning-point bifurcations (filled dots), and at $\lambda_{in} = -A$ there are infinite-time bifurcations (empty dots). Note, where a canard trajectory and a composite canard trajectory coincide, the composite canard trajectory branch is shown on top. Canard trajectories were initially computed for $\log(\epsilon - \epsilon_{FSN}) = -1.8$ (dashed line) (Fig. 3.7).

if there are no solutions when $\log(\epsilon - \epsilon_{FSN}) > C$.

The infinite-time bifurcation is unprecedented because no previous studies of folded singularities reveal any similar bifurcation [5, 71, 16, 15, 17, 28, 62]. That we are able to compute the infinite-time bifurcation is as a direct result of compactifying system (3.14)–(3.16), as the bifurcation corresponds to the formation of a canard trajectory with infinite integration time $T$.

Figure 3.8 shows the bifurcation diagram for fixed $\log(\delta) = -2$ and varying $\log(\epsilon - \epsilon_{FSN})$, plotted against the start point $\lambda_{in}$ of each canard trajectory. Canard trajectories are created and destroyed in left and right turning-point bifurcations (filled dots), and at $\lambda_{in} = -A$ in infinite-time bifurcations (empty dots). Notice, there is a structural difference between the left and the right turning-point bifurcations – the left turning-points are all isolated, whereas the right turning-points occur nearly simultaneously with another right turning-point.

Note, for different values of $\log(\epsilon - \epsilon_{FSN})$ we can compare the flow map $\phi$ (Fig. 3.5) with the one parameter bifurcation diagram (Fig. 3.8), and see the number and position of the apparent discontinuities in $\phi$ matches the number and position of the canard trajectories.

The one parameter bifurcation diagram in Fig. 3.8 has five distinct curves. Neighbouring curves coincide, and both curves continue to $\lambda_{in} = -A$. Each curve consists of canard trajectories with some number of rotations $i$, and is coloured accordingly. Notice, the curves for $i \neq 0$ are self-similar.

We can identify different parts of each curve with a specific type of canard trajectory (see Table 3.1 for a list of possible types). Each curve is divided into parts by the bifurcation points. These parts of the curve are called branches. Which canard trajectory is identified with each branch is noted here, and justified in Subsections 3.5.2.a–3.5.2.c by the detailed
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description of the dynamics near each canard bifurcation. We make the assumption that, if a branch has the same type of canard trajectory at the bifurcation point at either end, it has that same type of canard trajectory at any point along the branch.

The $i = 0$ (black) curve is divided into three branches. The branch started at the left turning-point bifurcation and continuing for $\epsilon > \epsilon_{FF}$ corresponds to the folded saddle canard trajectory $\gamma_{\delta}^S$. The branch bounded by the left and right turning-point bifurcations corresponds to the folded node canard trajectory $\gamma_{\delta}^N$. Finally, the branch bounded by the right turning-point bifurcation and the infinite-time bifurcation corresponds to the only other canard with no rotations, the tertiary canard trajectory $\zeta_{\delta}^N$.

Any $i \neq 0$ curve is divided into four branches. Two of these branches are bounded by the left turning-point bifurcation and a right turning-point bifurcation. The one bounded by the right turning-point bifurcation at larger $\epsilon$ corresponds to the composite secondary canard trajectory $\tilde{\gamma}_{\delta}^{i-1}$ (or $\tilde{\gamma}_{\delta}^N$ when $i = 1$). The other one, bounded by the right turning-point bifurcation at smaller $\epsilon$, corresponds to the secondary canard trajectory $\gamma_{\delta}^i$. The remaining two branches are bounded by right turning-point bifurcations and infinite-time bifurcations. The one bounded by the right turning-point bifurcation and infinite-time bifurcation at larger $\epsilon$ corresponds to the composite tertiary canard trajectory $\tilde{\zeta}_{\delta}^{i-1}$ (or $\tilde{\zeta}_{\delta}^N$ when $i = 1$). The other branch, bounded by the other right turning-point and infinite-time bifurcations at smaller $\epsilon$, corresponds to the tertiary canard trajectory $\zeta_{\delta}^i$.

Recall, the composite secondary canard trajectory $\tilde{\gamma}_{\delta}^{i-1}$ initially follows $\gamma_{\delta}^{i-1}$, then performs one more rotation [Fig. 3.2(b)]. Likewise, the composite tertiary canard trajectory $\tilde{\zeta}_{\delta}^{i-1}$ follows $\zeta_{\delta}^{i-1}$, then performs one more rotation. Thus, in Fig. 3.8, the $\tilde{\gamma}_{\delta}^{i-1}$ and the $\tilde{\zeta}_{\delta}^{i-1}$ branches of the $i$ curve coincide with the $i - 1$ curve, and are plotted on top.

We describe the canard bifurcations identified in Fig. 3.8 in detail, for a typical example of each type. The chosen examples are for canard trajectories with the minimal number of rotations as they are simplest to illustrate. The bifurcations are illustrated both in terms of the individual canard trajectories (plotted in three dimensions with the critical manifold $S$ as reference), and in terms of the change in the intersections of the slow manifolds $S_a^\delta$ and $S_r^\delta$ (plotted as one dimensional cross-sections with the surface $\Sigma_{FN}$) (Figs. 3.9–3.11). The three dimensional plots are a novel way of illustrating the canard trajectory bifurcations, which gives some intuition as to how the canard trajectory bifurcations relate to classic bifurcations of invariant sets. Typically, canard bifurcations are illustrated the second way – by cross-sections of $S_a^\delta$ and $S_r^\delta$, to highlight how the bifurcations correspond to changes in the intersection of $S_a^\delta$ and $S_r^\delta$ [71, 28, 17]. We compare these canard bifurcations near the folded saddle-node with the canard bifurcations near an isolated folded node [62, 71, 16].

3.5.2.a. Left turning-point bifurcation

Left turning-point bifurcations correspond to the smallest value of $\epsilon$ for which there are canard trajectories with some number $i$ rotations [Fig. 3.8, (left) filled dots]. We illustrate
Figure 3.9(a) is a zoom-in of the bifurcation diagram in Fig. 3.8 showing the values of log(\(\epsilon - \epsilon_{FSN}\)) used to illustrate the dynamics before, at, and after the bifurcation (dashed lines). Also highlighted are the start points \(\lambda_{in}\) of the canard trajectories \(\gamma^N_\delta\) (black diamond), \(\gamma^1_\delta\) (green filled dot or diamond), and \(\tilde{\gamma}^1_\delta\) (orange diamond). The start point of the canard trajectory \(\gamma^S_\delta\) is not shown because it is at a much larger value of \(\lambda_{in}\) (see Fig. 3.8).

First, we consider the bifurcation in terms of the canard trajectories, shown in Fig. 3.9(b)-(d). The critical manifold \(S\), and the two canard trajectories with zero rotations \(\gamma^S_\delta\) and \(\gamma^N_\delta\) (thin black lines) are shown for reference. Before the bifurcation, there are no canard trajectories with one rotation [Fig. 3.9(b)]. At the bifurcation, the first canard trajectory \(\gamma^1_\delta\) with one rotation appears [Fig. 3.9(c)]. Note, the canard trajectory \(\gamma^1_\delta\) begins at fractionally lower \(\lambda\) than \(\gamma^N_\delta\), and ends at lower \(\lambda\) than \(\gamma^S_\delta\) [Fig. 3.9(c)]. Then, after the bifurcation, this canard splits into two canards with \(i = 1\) rotations, one of which moves away from \(\gamma^N_\delta\) and \(\gamma^S_\delta\) with a shrinking rotation (forming the secondary canard \(\gamma^N_\delta\)), and the other converges to \(\gamma^N_\delta\) and \(\gamma^S_\delta\) with a growing rotation (forming the composite canard \(\tilde{\gamma}^N_\delta\)) [Fig. 3.9(d), green and orange trajectories, respectively].

Second, we consider the bifurcation in terms of changes in the shape of the slow manifolds \(S^a_\delta\) and \(S^r_\delta\), shown in Fig. 3.9(e)-(g). Notice, near a folded saddle-node singularity, as \(\epsilon\) increases, the repelling slow manifold curve \(S^r_\delta \cap \Sigma_{FN}\) spirals (anti-clockwise) further in, before folding back on itself and spiralling out [Fig. 3.9(e)-(g), blue curve]. Before the bifurcation, the curves \(S^a_\delta \cap \Sigma_{FN}\) and \(S^r_\delta \cap \Sigma_{FN}\) only intersect once, at the canard trajectory.
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trajectory $\gamma^N_\delta$ [Fig. 3.9(c)]. At the bifurcation, $S^r_\delta \cap \Sigma_{FN}$ comes tangent to $S^a_\delta \cap \Sigma_{FN}$, creating another canard trajectory $\gamma^1_\delta$ [Fig. 3.9(f)]. After the bifurcation, $S^r_\delta \cap \Sigma_{FN}$ crosses $S^a_\delta \cap \Sigma_{FN}$ creating two intersection points, one of which is $\gamma^1_\delta$, the other of which, where $S^r_\delta \cap \Sigma_{FN}$ spirals out, is the composite canard trajectory $\tilde{\gamma}^N_\delta$ [Fig. 3.9(g)].

This bifurcation gives further insight into the composite canard trajectories $\tilde{\gamma}^i_\delta$. Recall, $\tilde{\gamma}^i_\delta$ is a composition of $\gamma^i_\delta$ and $\gamma^S_\delta$ [Fig. 3.2(b)]. However, the composite canard trajectory $\tilde{\gamma}^i_\delta$ does not exist for the same parameter region as the corresponding ordinary canard trajectory $\gamma^i_\delta$. Rather, $\tilde{\gamma}^i_\delta$ exists for the same parameter region as $\gamma^{i+1}_\delta$ (Fig. 3.9).

The existing folded saddle-node singularity theory shows as $\epsilon$ near $\epsilon_{FSN}$ increases, the number of canard trajectories increases [70]. However, the theory gives no indication as to how the new canard trajectories are formed. Our results show that new (approximate) canard trajectories are formed in the left turning-point bifurcation (Fig. 3.8). Moreover, from studying the trajectories near the left turning-point bifurcation, it appears an $i$-th left turning-point bifurcation can be characterised as the trajectories that rotate around the centre trajectory $\eta^C_\delta$ having the $(i+1)$-th rotation become “big enough”. Where “big enough” is when this rotation follows the repelling manifold $S^r_\delta$ down to $L_{out}$ (Fig. 3.6). It may seem that “big enough” would be dependent on the boundary $L_{out}$, however, even when $L_{out}$ is twice as far from $F$, the left turning-point bifurcations occur at the same values of log($\epsilon - \epsilon_{FSN}$).

Secondly, in the existing theory, Vo et al ignore composite canard trajectories [70]. However, our results show that composite canard trajectories should not be ignored, as they form one of the branches in the left turning-point bifurcation (Fig. 3.9).

3.5.2.b. Right turning-point bifurcation

Each curve of canard trajectories with $i$ rotations has two right turning-point bifurcations, one where the ordinary canard trajectories disappear, and the other, at larger $\epsilon$, where the composite canard trajectories disappear (Fig. 3.8). Notice, the right turning-point bifurcation for the composite canard trajectories with $i$ rotations is nearly concurrent with the right turning-point bifurcation of the ordinary canard trajectories with $i-1$ rotations (Fig. 3.8). In fact, the composite canard trajectory bifurcation occurs at fractionally smaller $\epsilon$, but for most values of $\delta$ and $i$ these bifurcations are too close to be distinguished.

Figure 3.10 shows the simplest typical right turning-point bifurcation, when $i = 0$. This is the final bifurcation as $\epsilon$ increases, at $\epsilon \approx \epsilon_{FF}$, where the folded node degenerates to a folded focus and only the folded saddle canard $\gamma^S_\delta$ remains. The bifurcation is described as $\epsilon$ decreases, to make it easier to compare with both the infinite-time bifurcation (Fig. 3.11) and canard bifurcations near an isolated folded node [71, 16]. Note, the nearly simultaneous bifurcation of the composite canard trajectories $\tilde{\gamma}^N_\delta$ and $\tilde{\zeta}^N_\delta$ is not included in the illustration.

Figure 3.10(a) is a zoom-in of the bifurcation diagram in Fig. 3.8 (without the $i = 1$ curve of composite canard trajectories), showing the values of log($\epsilon - \epsilon_{FSN}$) used to illustrate
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Figure 3.10. The right turning-point bifurcation of canards with zero rotations, for \( \lambda = -2 \). Panel (a) is a zoom-in of the bifurcation diagram in Fig. 3.8 (without the \( i = 1 \) curve) showing the location of panels (b)-(g) before, at, and after the bifurcation, at \( \log(\epsilon - \epsilon_{FSN}) = -1.21, -1.219, -1.23 \), respectively. Panels (b)-(d) show the bifurcating canard trajectories \( \gamma_\delta^N \) and \( \zeta_\delta^N \), also shown for reference is the critical manifold \( S \). Panels (e)-(g) show the bifurcation in terms of intersection points at \( \gamma_\delta^N \) and \( \zeta_\delta^N \) of the attracting and repelling slow manifolds \( S_\delta^a \) and \( S_\delta^r \) at the surface \( \Sigma_{FN} \). Note, the saddle canard trajectory (not shown) persists.

The dynamics before, at, and after the bifurcation (dashed lines). Also highlighted are the start points \( \lambda_{in} \) of the canard trajectories \( \gamma_\delta^N \) (black filled dot or diamond) and \( \zeta_\delta^N \) (orange diamond). The saddle canard trajectory \( \gamma_\delta^S \) exists for all values of \( \epsilon \) in Fig. 3.10, however it is not in the illustrated range of \( \lambda \).

First, we consider the bifurcation in terms of the canard trajectories, shown in Fig. 3.10(b)-(d). The saddle canard trajectory \( \gamma_\delta^S \) exists for all \( \epsilon > \epsilon_{FSN} \), so is not of interest. Rather, we consider canard trajectories in the folded node region. Before the bifurcation, there are no canards with zero rotations – in fact, there are no canards at all in the folded node region [Fig. 3.10(b)]. At the bifurcation, there is one canard with zero rotations \( \gamma_\delta^N \) [Fig. 3.10(c)]. After the bifurcation, this canard splits into two canard trajectories with zero rotations, \( \gamma_\delta^N \) and \( \zeta_\delta^N \) [Fig. 3.10(d)].

Second, we consider the bifurcation in terms of changes in the shape of the slow manifolds \( S_\delta^a \) and \( S_\delta^r \), shown in Fig. 3.10(e)-(g). As before, the repelling manifold \( S_\delta^r \) folds back on itself at the tip, however the fold is too tight to see in Fig. 3.10(e)-(g). As \( \epsilon \) decreases, \( S_\delta^a \cap \Sigma_{FN} \) and \( S_\delta^r \cap \Sigma_{FN} \) move closer together. Before the bifurcation, \( S_\delta^a \cap \Sigma_{FN} \) and \( S_\delta^r \cap \Sigma_{FN} \) are nearby, but do not intersect, so there are no canard trajectories near the folded node [Fig. 3.10(e)]. At the bifurcation, \( S_\delta^a \cap \Sigma_{FN} \) becomes tangent to \( S_\delta^r \cap \Sigma_{FN} \) at the canard trajectory \( \gamma_\delta^N \) [Fig. 3.10(f)]. After the bifurcation, the curves \( S_\delta^a \cap \Sigma_{FN} \) and \( S_\delta^r \cap \Sigma_{FN} \) cross, and intersect at canard trajectories \( \gamma_\delta^N \) and \( \zeta_\delta^N \) [Fig. 3.10(g)].

There are composite canard trajectories \( \tilde{\gamma}_\delta^N \) and \( \tilde{\zeta}_\delta^N \) (not shown) that are nearly concurrent with \( \gamma_\delta^N \) and \( \zeta_\delta^N \). The right turning-point bifurcation for composite canard trajectories \( \tilde{\gamma}_\delta^N \) and \( \tilde{\zeta}_\delta^N \) is as in Fig. 3.10, but the canard trajectories go back to \( S_\delta^a \) then follow the
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For canard trajectories with \( i \neq 0 \) rotations, the right turning-point bifurcation is like that for \( i = 0 \) rotations. However, before the bifurcation, \( S^a_\delta \cap \Sigma_{FN} \) and \( S^r_\delta \cap \Sigma_{FN} \) intersect [e.g. Fig. 3.11(g)], and the central tips of \( S^a_\delta \cap \Sigma_{FN} \) and \( S^r_\delta \cap \Sigma_{FN} \) curl inward, making them locally similar to Fig. 3.10(e).

When \( \epsilon \) is far from \( \epsilon_{FSN} \), the folded node and folded saddle are considered to behave as isolated folded singularities [70]. Therefore, it is interesting to compare the right turning-point bifurcation with the transcritical bifurcation for an isolated folded node. We ignore the composite canards, because they only exist for a folded saddle-node system. In both cases, the bifurcations correspond to a central tangency of \( S^a_\delta \cap \Sigma_{FN} \) and \( S^r_\delta \cap \Sigma_{FN} \) (compare Fig. 3.10(e)-(f) with [71, Fig. 13] and [16, Fig. 4]). However, there are two fundamental differences. Firstly, in the case of an isolated folded node, there is a weak canard trajectory which the secondary canard trajectories \( \gamma_i \) bifurcate off. Whereas, in the case of a folded node following a folded saddle-node bifurcation, there is a centre trajectory \( \eta_C \). The centre trajectory \( \eta_C \) remains near \( S^a_\delta \), thus the secondary canard trajectories \( \gamma_i \) cannot bifurcate off \( \eta_C \), as these trajectories are not topologically equivalent. Nonetheless, \( \eta_C \) continues to act as an organising centre, and lies between the bifurcating canard trajectories \( \gamma_i \) and \( \zeta_i \).

Secondly, for an isolated folded node the bifurcation is transcritical, rather than a turning-point bifurcation. In particular, this means that before the bifurcation, the intersections of \( S^a_\delta \cap \Sigma_{FN} \) and \( S^r_\delta \cap \Sigma_{FN} \) are very different. In the case of an isolated folded node, \( S^a_\delta \cap \Sigma_{FN} \) and \( S^r_\delta \cap \Sigma_{FN} \) intersect at the weak canard trajectory and, just before the bifurcation, at another, sporadic, canard trajectory. Whereas, in the case of a folded saddle-node, before the bifurcation, \( S^a_\delta \cap \Sigma_{FN} \) and \( S^r_\delta \cap \Sigma_{FN} \) have no (local) intersections [Fig. 3.10(e)].

3.5.2.c. Infinite-time bifurcation

Each curve of canard trajectories with \( i \) rotations has two infinite-time bifurcations, one where the tertiary canard trajectory \( \zeta_i \) appears, and the other, at larger \( \epsilon \), where the composite tertiary canard trajectory \( \tilde{\zeta}_{i-1} \) appears (Fig. 3.8). The infinite-time bifurcation for the composite canard with \( i \) rotations is nearly concurrent with the infinite-time bifurcation of the ordinary canard with \( i - 1 \) rotations (Fig. 3.8). Again, the composite canard trajectory bifurcation occurs at fractionally smaller \( \epsilon \), but the bifurcations are often too close to be distinguished.

Studying the infinite-time bifurcations provides insight into the newly identified tertiary canards \( \zeta_i \). Recall, these canard trajectories are not seen for an isolated folded node [71, 16], nor included in the existing theory for folded saddle-node (type I) [70]. It may be they have been overlooked in previous studies, as they occur away from the principal canard trajectories, which are the folded node strong and weak canard trajectories.

The infinite-time bifurcations are best be studied in a compactified system, of which this is the first such study. This is because by compactifying the system, the asymptotic behaviour as \( \tau \to \pm \infty \) is captured by \( \lambda \to \pm A \) (Fig. 3.3). Specifically, this gives the
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Figure 3.11. The infinite-time bifurcation of canards with zero rotations, for log(δ) = −2. Panel (a) is a zoom-in of the bifurcation diagram in Fig. 3.8 (without the i = 1 curve) showing the location of panels (b)-(g) before, at, and after the bifurcation, at log(ε − ε_{FSN}) = −1.24, −1.23, −1.22, respectively. Panels (b)-(d) show the bifurcating canard trajectory ζ_δ^N, also shown for reference is the critical manifold S and saddle steady state p. Panels (e)-(g) show the bifurcation in terms of the intersection point at ζ_δ^N of the attracting and repelling slow manifolds S_δ^r and S_δ^a at the surface Σ_{FN}. Note, the tip of S_δ^a ∩ Σ_{FN} is an accumulation point of trajectories starting near the saddle steady state p.

compactified system the following topological features: there is a persistent saddle steady state p at (x, y, λ) = (0, −A, −A); and there is a finite range of λ_{in}, so S_δ^a ∩ Σ_{FN} can be computed all the way to the centre tip.

Figure 3.11 shows the simplest infinite-time bifurcation, when i = 0. The bifurcation is described as ε decreases, and follows from the right turning-point bifurcation shown in Fig. 3.10. There is nearly simultaneous bifurcation of the composite canard trajectory ζ_δ^N, however it is not included in the illustration.

Figure 3.11(a) is a zoom-in of the bifurcation diagram in Fig. 3.8 (without the i = 1 curve of composite canard trajectories), showing the values of log(ε − ε_{FSN}) used to illustrate the dynamics before, at, and after the bifurcation (dashed lines). Also highlighted are the start points λ_{in} of the canard trajectories γ_δ^N (black diamond) and ζ_δ^N (orange filled dot or diamond). The saddle canard trajectory γ_δ^S exists for all values of ε in Fig. 3.11, however it is not in the illustrated range of λ.

First, we consider the bifurcation in terms of the canard trajectories, shown in Fig. 3.11(b)-(d). The saddle canard trajectory γ_δ^S exists for all ε > ε_{FSN}, so is not of interest. Rather, we consider canard trajectories in the folded node region. Before the bifurcation, there are two canard trajectories with zero rotations γ_δ^N and ζ_δ^N [Fig. 3.11(b)]. At the bifurcation, the canard trajectory ζ_δ^N connects to the saddle steady state p and disappears [Fig. 3.11(c)]. Notice, because p is a saddle, when the canard trajectory connects to p it has infinite integration time T. After the bifurcation, there is just γ_δ^N and no tertiary canards [Fig. 3.11(d)].
Second, we consider the bifurcation in terms of changes in the shape of the slow manifolds $S^a_\delta$ and $S^r_\delta$, shown in Fig. 3.11(e)-(g). As before, the repelling manifold $S^r_\delta$ folds back on itself at the tip, however the fold is too tight to see in Fig. 3.11(e)-(g). As $\epsilon$ decreases, $S^a_\delta \cap \Sigma_{FN}$ and $S^r_\delta \cap \Sigma_{FN}$ move apart [Fig. 3.11(e)-(g)]. Before the bifurcation, $S^a_\delta \cap \Sigma_{FN}$ and $S^r_\delta \cap \Sigma_{FN}$ cross, and intersect at canard trajectories $\gamma^N_\delta$ and $\zeta^N_\delta$ [Fig. 3.11(e)]. At the bifurcation, the tip of $S^a_\delta \cap \Sigma_{FN}$ only just meets $S^r_\delta \cap \Sigma_{FN}$ at the canard trajectory $\zeta^N_\delta$ [Fig. 3.11(f)]. After the bifurcation, the curves $S^a_\delta \cap \Sigma_{FN}$ and $S^r_\delta \cap \Sigma_{FN}$ have separated at the tip, so the canard trajectory $\zeta^N_\delta$ disappears [Fig. 3.11(e)].

There are composite canard trajectories $\tilde{\gamma}^N_\delta$ and $\tilde{\zeta}^N_\delta$ (not shown) that are nearly concurrent with $\gamma^N_\delta$ and $\zeta^N_\delta$. More precisely, the composite canard trajectories are on the inner side of $S^r_\delta \cap \Sigma_{FN}$ [Fig. 3.11(e)-(g)]. The infinite-time bifurcation for the composite canard trajectory $\tilde{\zeta}^N_\delta$ is as in Fig. 3.11, but the canard trajectories go back to $S^a_\delta$ and follow the folded saddle canard $\gamma^S_\delta$.

Notice, the infinite-time bifurcation is topologically necessary for the tips of $S^a_\delta \cap \Sigma_{FN}$ and $S^r_\delta \cap \Sigma_{FN}$ to be free to curl inwards before the next right turning-point bifurcation [Fig. 3.11(g), Fig. 3.10(e)].

Because $p$ is a saddle steady state, when $\xi^i_\delta$ attaches to $p$ at the bifurcation, it has infinite integration time, hence the name of the bifurcation. Moreover, this can be used to numerically continue these bifurcations in $\delta$ and $\epsilon$, by finding canard trajectories with a fixed integration time $T/\delta = 10^6$. By comparison, the secondary canard trajectories $\gamma^i_\delta$ at most had integration time $T/\delta = O(10^4)$.

Next, we continue the co-dimension one bifurcations in both parameters $\delta$ and $\epsilon$.

### 3.5.3. Canard bifurcations in two parameters

In this section, we present the two parameter, $\delta$ and $\epsilon$, bifurcation diagram. From the bifurcation diagram, we discover the number of canard trajectories for any $\delta$ and $\epsilon$; scaling laws for co-dimension one bifurcations; the sprite feature; and co-dimension two bifurcations. We compute the two parameter bifurcation diagram for all canard trajectories with $i \leq 7$ rotations, for $\log(\delta) \in [-3.6, 0]$ and $\log(\epsilon - \epsilon_{FSN}) \in [-4, 0]$. It is difficult to compute the canard trajectories for smaller $\delta$, especially those with many rotations [16]. As the curves are self-similar, results can be predicted for $\delta$ and $\epsilon$ outside this range.

To construct the two parameter bifurcation diagram, the co-dimension one bifurcations computed for fixed $\log(\delta) = -2$ (Fig. 3.8) are continued in both parameters $\delta$ and $\epsilon$. There are boundary value solver routines in AUTO for the continuation of turning-point bifurcations in two parameters [21]. The infinite-time bifurcations are continued as solutions to the boundary value problem “$L_{in}$ to $L_{out}$” with specified integration time $T/\delta = 10^6$. However, when $\log(\delta)$ and $\log(\epsilon - \epsilon_{FSN})$ are small, AUTO no longer accurately detects the right turning-point bifurcations, nor can canard trajectories with integration time $T/\delta = 10^6$ be continued. In which case, we identify bifurcations for fixed values of $\log(\delta)$ by computing the continuation of canard trajectories $\epsilon$ (as done in Section 3.5.2) and,
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where required, for fixed values of $\log(\epsilon - \epsilon_{FSN})$ and continuing in $\delta$. In this case, the turning-point bifurcations are identified by eye (the one parameter bifurcation diagrams can be zoomed in to sufficiently high resolution) and the infinite-time bifurcations are identified with canard trajectories starting at $\lambda_{in} = -A + 0.001$.

To ensure completeness of the bifurcation diagram, we checked the number of intersections for $S^c_0 \cap \Sigma_{FN}$ and $S^c_\delta \cap \Sigma_{FN}$ agreed with the anticipated number of canard trajectories at different values of $\delta$ and $\epsilon$. To ensure there was no dependence on the boundary conditions, parts of the bifurcation diagram were computed with $L_{in}$ and $L_{out}$ twice as far away from $F$, and the results were qualitatively the same.

In the two parameter bifurcation diagram, as before, the colour of the curves corresponds to the number of rotations $i$ the canard trajectories have (black: 0 or more than 5, green: 1 or 7, red: 2, blue: 3, cyan: 4, and magenta: 5). The solid curves are turning-point bifurcations, and the dashed curves are infinite-time bifurcations. The bifurcation curves divide the $\log(\delta) - \log(\epsilon - \epsilon_{FSN})$ plane into different regions, and the number in each region corresponds to the number of canard trajectories there. As expected from studying the co-dimension one bifurcations, at turning-point bifurcation curves (solid) the number of canard trajectories changes by two, and at infinite-time bifurcation curves (dashed) the number of canard trajectories changes by one. Note, where the curves become nearly concurrent (because the bifurcations of composite and ordinary canard trajectories occur nearly simultaneously), the number of canard trajectories changes by four and two, respectively. Finally, the black dots show co-dimension two bifurcation points.

In this section, we first show part of the two parameter bifurcation diagram to demonstrate how it fits with one parameter, “cross-section”, fixed $\delta$ bifurcation diagrams. Then we show the complete two parameter bifurcation diagram, with successive figures zooming in to show the detail.

### 3.5.3.3.a. Partial bifurcation diagram

Figure 3.12(a) shows the two parameter bifurcation diagram for large $\delta$, where it is simplest. Panels (b) and (c) show “cross-section” one parameter bifurcation diagrams for fixed $\delta$. For this range of $\delta$ there are only canard trajectories with $i = 0$ rotations and with $i = 1$ rotation (black and green curves, respectively). The turning-point bifurcations in panels (b)-(c) (filled dots) correspond to crossing solid curves in panel (a). Likewise, the infinite-time bifurcations in panels (b)-(c) (empty dots) correspond to crossing dashed curves in panel (a).

In Fig. 3.12(a), the numbers show how many canard trajectories there are in each region. Panels (b)-(c) can be used to verify these numbers (note, $\gamma^S_\delta$ is not in the illustrated range of $\lambda_{in}$). For example, when $\log(\delta) = -0.65$ and $\log(\epsilon - \epsilon_{FSN}) = -1.6$, there are only two canard trajectories, and no canard trajectories with $i = 1$ rotation [Fig. 3.12(a), (b)]. In fact, the only canard trajectories that exist in this region are $\gamma^S_\delta$ and $\gamma^N_\delta$. Whereas, when $\log(\delta) = -0.8$ and $\log(\epsilon - \epsilon_{FSN}) = -1.6$, there are two canard trajectories with $i = 0$
Figure 3.12. Panel (a) shows the two parameter, δ and ε, bifurcation diagram for large δ. There are curves of turning-point bifurcations (solid) and infinite-time bifurcations (dashed) for canard trajectories with zero rotations (black) and one rotation (green). Numbers show how many canard trajectories exist in each region. There are co-dimension two bifurcations (black dots), discussed in Section 3.5.3.d. Dotted lines show the position of “cross-section” one parameter bifurcation diagrams (b) and (c). Panels (b)-(c) show one parameter bifurcation diagrams for fixed log(δ) = -0.65, -0.8. There are turning-point bifurcations (filled dots) and infinite-time bifurcations (empty dots). The branches of canard trajectories with i = 1 rotations are labelled with which canard trajectory they correspond to.

3.5.3.b. Full bifurcation diagram

We now reveal the complete two parameter, δ and ε, bifurcation diagram, Fig. 3.13. The linear behaviour for ε near ε_{FSN} continues to at least log(ε - ε_{FSN}) = -4 (not shown in figure). The set of curves for each i ≠ 0 are approximately self-similar, being like the i = 1 bifurcation curves shown in Fig. 3.12. The curves are ordered, and the bifurcation curves for i are nearly concurrent with i - 1 and i + 1 curves. This is a consequence of the nearly concurrent bifurcations for composite and ordinary canard trajectories. The numbers give the number of canard trajectories in each region (for smaller regions see the zoomed in bifurcation diagram, Fig. 3.14). The bifurcation diagram is not computed for canard trajectories with i > 7 rotations, but we anticipate that there are more, self-similar bifurcation curves for canard trajectories with i > 7 rotations, resulting in more that 16 canard trajectories (Fig. 3.13).
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Figure 3.13. Complete \(\delta\) and \(\epsilon\) bifurcation diagram for canard trajectories with less than seven rotations. The curve colour corresponds to the number of rotations \(i\) (black: 0 or more than 5, green: 1 or 7, red: 2, blue: 3, cyan: 4, and magenta: 5). Solid curves are turning-point bifurcations, and the dashed curves are infinite-time bifurcations, numbers show how many canards there are in a region, black dots mark co-dimension two canard bifurcations.

Figure 3.14. Successive zoom ins of the two parameter bifurcation diagram (see Fig. 3.13 for labels). Boxes in panel (b) show locations of later zoom ins [Figs. 3.18(a), 3.19(a)].
Specifically, the \( i = 0 \) curve (black) for \( \epsilon \) near \( \epsilon_{FSN} \) corresponds to the first appearance of canard trajectories following the folded saddle-node bifurcation. As expected from the theory, as \( \delta \to 0 \), this bifurcation curve tends to \( \epsilon = \epsilon_{FSN} \). The \( i = 0 \) curve (black) for large \( \epsilon \) corresponds to the disappearance of all canard trajectories apart from \( \gamma^S_\delta \). As expected from the theory, as \( \delta \to 0 \), this bifurcation curve tends to \( \epsilon = \epsilon_{FF} \).

First, we discuss the global scaling laws that can be inferred for canard bifurcations near a folded saddle-node singularity from the two parameter bifurcation diagram (Fig. 3.13). Then we look in more detail at the local behaviour at the co-dimension two bifurcations (black dots), and the sprite feature.

### 3.5.3.c. Scaling laws near a folded saddle-node singularity

**Scaling law 1.** For system (3.18)–(3.20), for small \( \delta, \) and \( \epsilon \) near \( \epsilon_{FSN} \), the left turning-point bifurcations follow a linear law [Fig. 3.13, for \( \log(\epsilon - \epsilon_{FSN}) < -2 \)]:

\[
\log(\delta) = \log(\epsilon - \epsilon_{FSN}) + c, \tag{3.25}
\]

for some constant \( c \), given in Table 3.2. What is more, for fixed \( \epsilon \), the constant \( c \) depends on the number of rotations \( i \) the canard trajectories have, and follows an approximate power law (Fig. 3.15):

\[
c = -1.5 \log(i) + 0.9. \tag{3.26}
\]

The constant \( c \) is computed in two ways, first, it is assumed the curves have unit gradient, and \( c \) is given by the point on the curve at \( \log(\epsilon - \epsilon_{FSN}) = -3.5 \). Second, \( c \) is given by the equation for a straight line segment connecting two points on the curve at \( \log(\epsilon - \epsilon_{FSN}) = -3.5 \) and \( -2.5 \). The second method gives an approximation of the gradient, which is very close to one. However, as the linear parts of the bifurcation curves span different intervals of \( \log(\epsilon - \epsilon_{FSN}) \) and are slightly convex, using the same interval of \( \log(\epsilon - \epsilon_{FSN}) = -3.5 \) and \( -2.5 \) gives a decreasing gradient for increasing \( i \) (Table 3.2).

Equivalently, in \( \delta-\epsilon \) space the left turning-point bifurcations of canard trajectories with \( i \) rotations form straight lines through the origin with different gradient \( C = 10^c \), depending on \( i \):

\[
\delta = C(\epsilon - \epsilon_{FSN}), \tag{3.27}
\]

with

\[
C = 10^{0.9i^{-1.5}}. \tag{3.28}
\]

There is no analytical justification made for the coefficients in Eqs. (3.25)–(3.26), (3.27)–(3.28). It may be that the -1.5 exponent follows from a re-scaling to blow-up the folded singularity, as is done in [70].

Canard bifurcations are not studied in the existing theoretical results [70]. However, in the system in [70], for \( \epsilon \to \epsilon_{FSN} \), the maximum number of canard rotations is \( O(\delta^{-1/4}) \), which suggests, equivalently, that the canard with \( i \) rotations first appears at \( \delta = O(i^{-4}) \).
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</table>

Table 3.2. The coefficient $c$ in Eq. (3.25) for the linear law of left turning-point bifurcations of canard trajectories at small $\delta$ and $\epsilon$ near $\epsilon_{\text{FSN}}$. Computed using $\log(\epsilon - \epsilon_{\text{FSN}}) = -3.5$ (one point) and $\log(\epsilon - \epsilon_{\text{FSN}}) = -3.5$ and $-2.5$ (two points).

Figure 3.15. Comparison of constant $c$ (computed from one point) (diamonds) for canard trajectories with $i$ rotations, with the power law $\log(\delta) = -1.5 \log(i) + 0.9$, Eq. (3.25).

A linear law relating $\epsilon$ and $\delta$ is a novel, unprecedented result for canard bifurcations near a folded singularity. What is more, the coefficients in Eq. (3.27), and the power law for $C$, Eq. (3.28), are an interesting area for future research.

Scaling law 2. For system (3.18)–(3.20), the first infinite-time bifurcations for increasing $\delta$ and $\epsilon$ can be approximated by the line:

$$\log(\delta) = 2.5 \log(\epsilon - \epsilon_{\text{FSN}}) + 2.5,$$

(3.29)
as shown in Fig. 3.16 (dashed line). Notice, $A = 2.5$, so Eq. (3.29) may depend on $A$, but this has not been verified.

This line marks a transition between the linear behaviour when $\epsilon$ is near $\epsilon_{\text{FSN}}$ (scaling law 1), to vertical asymptotes when $\epsilon$ is far from $\epsilon_{\text{FSN}}$ (see scaling law 3), via a region of co-
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Figure 3.16. The two parameter bifurcation diagram compared to scaling law results. Dashed line shows linear approximation to the first infinite-time bifurcations (see scaling law 2). Solid lines show values of \( \log(\epsilon - \epsilon_{FSN}) \) at odd \( \mu^{-1} \) (see scaling law 3). “Sprite” shows asymptote of dense turning-point bifurcations. Note, \( \mu^{-1} = 7 \) at \( \log(\epsilon - \epsilon_{FSN}) = -1.85442 \) and the sprite is at \( \log(\epsilon - \epsilon_{FSN}) = -1.85831 \).

dimension two bifurcations (Fig. 3.13). Moreover, this line is important for applications as it corresponds to the smallest value of \( \delta \) and \( \epsilon \) at which trajectories started near \( \lambda_{in} = -A \) no longer converge to \( S^\delta_{\delta} \) (see Section 3.4, feature 3 for the flow map).

As infinite-time bifurcations are first identified in system (3.18)–(3.20), this scaling law, following a folded saddle-node bifurcation is a novel and unprecedented result.

**Scaling law 3.** For system (3.18)–(3.20), for \( \epsilon \) far from \( \epsilon_{FSN} \), the number of secondary canard trajectories \( \gamma^0_\delta \) and composite secondary canard trajectories \( \tilde{\gamma}^1_\delta \) depends on the ratio of the folded node eigenvalues \( \mu \).

Specifically, for \( 2k + 1 < \mu^{-1} < 2k + 3 \) there are \( i = 1 \ldots k \) canard trajectories \( \gamma^i_\delta \) and \( \tilde{\gamma}^i_\delta \). Also, the strong canard trajectory and composite strong canard trajectory, \( \gamma^N_\delta \) and \( \tilde{\gamma}^N_\delta \), only exist for \( \mu^{-1} > 1 \). Notice, for the secondary canard trajectories \( \gamma^i_\delta \) and the strong canard trajectory \( \gamma^N_\delta \), this is the same scaling law as in the case of an isolated folded node [71].

Recall, \( \mu \) depends on \( \epsilon \), and for any \( i \), \( \gamma^i_\delta \) and \( \tilde{\gamma}^{i-1}_\delta \) disappear at right turning point bifurcations at smaller and larger \( \epsilon \), respectively (see Section 3.5.2.b, and Fig. 3.12 for \( i = 1 \)). In the two parameter bifurcation diagram, the right turning-point bifurcations correspond to solid curves at larger \( \epsilon \) (Fig. 3.13). For \( \epsilon \) far from \( \epsilon_{FSN} \), these right turning-point bifurcation curves are vertically asymptotic (Figs. 3.13, 3.16). This shows the bifurcations are occurring at (approximately) the same values of \( \epsilon \) for all \( 0 < \delta << 1 \).

What is more, for \( i = 0, 1, 2, 3, 4 \), the right turning-point bifurcation curves asymptote to values of \( \epsilon \) where \( \mu^{-1} = 1, 3, 5, 7 \), respectively (Fig. 3.16, black, green, red, blue and cyan curves). The values of \( \log(\epsilon - \epsilon_{FSN}) \) at odd \( \mu^{-1} \) for system (3.18)–(3.20) are given in
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Table 3.3. The values of $\log(\epsilon - \epsilon_{FSN})$ at odd $\mu^{-1} \in \mathbb{N}$ for system (3.18)–(3.20). Also given are which secondary canard trajectory $\gamma^i_\delta$ is formed at odd $\mu^{-1}$ in the case of an isolated folded node [71].

<table>
<thead>
<tr>
<th>$\log(\epsilon - \epsilon_{FSN})$</th>
<th>$\mu^{-1}$</th>
<th>Canard trajectory formed if isolated folded node</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.22</td>
<td>1</td>
<td>$\gamma^N_\delta$</td>
</tr>
<tr>
<td>-1.43</td>
<td>3</td>
<td>$\gamma^1_\delta$</td>
</tr>
<tr>
<td>-1.66</td>
<td>5</td>
<td>$\gamma^2_\delta$</td>
</tr>
<tr>
<td>-1.85</td>
<td>7</td>
<td>$\gamma^3_\delta$</td>
</tr>
<tr>
<td>-2.01</td>
<td>9</td>
<td>$\gamma^4_\delta$</td>
</tr>
<tr>
<td>-2.15</td>
<td>11</td>
<td>$\gamma^5_\delta$</td>
</tr>
<tr>
<td>-2.27</td>
<td>13</td>
<td>$\gamma^6_\delta$</td>
</tr>
</tbody>
</table>

Finally, it is not apparent if the right turning-point bifurcation curves for $i = 5, 6, 7$ asymptote to $\mu^{-1} = 9, 11, 13$ (Fig. 3.16, magenta, black and green curves). It may be that they do not, and that this is as a consequence of $\epsilon$ not being sufficiently far from $\epsilon_{FSN}$ [70].

In the existing theoretical results, it is assumed that, for $\epsilon$ far from $\epsilon_{FSN}$, following a folded saddle-node bifurcation, the folded node behaves like an isolated folded node [70]. It is thereby implied that the canard bifurcations follow a scaling law similar to our scaling law 3, however it is not proven.

For $\epsilon$ far from $\epsilon_{FSN}$, the infinite-time bifurcations also appear to asymptote to specific values of $\epsilon$, which are not related to $\mu^{-1}$ (Fig. 3.16, dashed curves). This is another aspect of the new, infinite-time bifurcations which requires further research.

3.5.3.d. Co-dimension two bifurcations

For system (3.18)–(3.20), following the folded saddle-node bifurcation, there are two different types of co-dimension two canard bifurcation for varying $\delta$ and $\epsilon$ (Fig. 3.17).

Firstly, there are *cusp bifurcations* (C) where two turning-point bifurcation curves converge (Fig. 3.17). Specifically, a curve of left turning-point bifurcations converges with a curve of right turning-point bifurcations.

Secondly, there are *infinite turning-point bifurcations* (I) where a turning-point bifurcation curve tangentially meets an infinite-time bifurcation curve (Fig. 3.17). At an infinite turning-point bifurcation, there are local changes in the number of canard trajectories. Effectively, the “+2 canard trajectories” turning-point bifurcation curve (solid) is split into two “+1 canard trajectory” infinite-time bifurcation curves (dashed) [Fig. 3.17(b)].

In the state space, at the infinite turning-point bifurcation, there are two canard trajecto-
Figure 3.17. The two parameter bifurcation diagram near co-dimension two bifurcations of canard trajectories with (a) $i = 1$ rotations and (b) $i = 2$ rotations. The curves of turning-point bifurcations (solid) and infinite-time bifurcations (dashed) join at co-dimension two cusp bifurcations (C) and infinite turning-point bifurcations (I). The inset in panel (a) shows two cusp bifurcations $O(10^{-4})$ close. The number of canard trajectories in each region is given.

For system (3.18)–(3.20), the co-dimension two bifurcations are either near the line given by Eq. 3.25, or near the sprite (Fig. 3.16). At co-dimension two bifurcations near the line, for each $i \neq 0$, two infinite-time bifurcation curves and one turning-point bifurcation curve meet, and disappear (Figs. 3.13, 3.14). When these bifurcation curves meet, there is a similar transition in the one parameter “cross-section” bifurcation diagrams at fixed $\delta$ [see Fig. 3.12(b)-(c) showing transition for $i = 1$]. For any $i \neq 0$, at larger $\delta$, the curve of canard trajectories forms a closed loop and has only one left and one right turning-point bifurcation [Fig. 3.12(b)]. Whereas, at smaller $\delta$, this curve has grown a tail so has one left and two right turning-point bifurcations, and two infinite-time bifurcations [Fig. 3.12(c)].

Whilst the transition when the bifurcation curves meet is similar for all $i \neq 0$, locally there can be different configurations of the co-dimension two bifurcations [Fig. 3.17(a)-(b)]. For $i = 1$, the local configuration is distinct, and comes close to a co-dimension three swallowtail bifurcation, with two cusp bifurcations $O(10^{-4})$ close in $\log(\delta) - \log(\epsilon - \epsilon_{FSN})$ space [Fig. 3.17(a)]. Note, for $i = 1$ the turning-point bifurcation curve is below the infinite-time bifurcation. More typically, for curves with $1 < i < 6$, the turning-point bifurcation curve is above the infinite-time bifurcation curve, and there is one cusp bifurcation [Fig. 3.17(b)]. For $i = 7$, the infinite-time bifurcation curve is met by the turning-point bifurcation at larger, rather than smaller $\epsilon$, and this creates a third configuration [Fig. 3.14(b)]. We assume this third configuration of co-dimension two bifurcations is typical for curves with $i > 7$ rotations.
3.5.3.e. The bifurcation sprite

The two parameter bifurcation diagram reveals that system (3.18)–(3.20) has a totally novel, unprecedented, dense area of turning-point bifurcations – the sprite (Figs. 3.16, 3.18). Here the bifurcation curves come vertically asymptotic to \( \log(\epsilon - \epsilon_{FSN}) = -1.85831 \), and adjacent left and right turning-point bifurcation curves meet in cusp bifurcations. We describe this feature as a sprite, because the bifurcation curves have an upward branching structure with increasing \( \delta \).

The sprite is a very distinctive feature, the like of which has neither been observed for isolated folded singularities [71, 17, 28], nor been anticipated in the existing theory for folded saddle-node (type I) bifurcations [70]. From a global perspective, the sprite separates regions of different dynamical behaviour for \( \epsilon \) near, and far from \( \epsilon_{FSN} \), and organises the bifurcation curves (Figs. 3.13, 3.14). Locally, the sprite creates regions with many canard trajectories (Fig. 3.18).

In the two parameter bifurcation diagram, the sprite separates two regions of different behaviour (Figs. 3.13, 3.14). There are two particular behaviours that are different on either side of the sprite. First, for each \( i \), consider the behaviour of the turning-point bifurcation curve at largest \( \epsilon \). If that turning-point bifurcation curve is left of the sprite, it disappears in a co-dimension two bifurcation [Fig. 3.14(a), green curves]. Whereas, if that turning-point bifurcation curve is right of the sprite, it exists for all smaller values of \( \epsilon \) (Figs. 3.13, 3.14). Second, consider the behaviour of the infinite-time bifurcation curves as \( \delta \) decreases. To the left of the sprite, the infinite-time bifurcation curves converge to turning-point bifurcation curves (Fig. 3.13). Whereas, to the right of the sprite, the infinite-time bifurcation curves and turning-point bifurcation curves for each \( i \) have distinct vertical asymptotes (Figs. 3.13, 3.16, and scaling law 3). Thus, it may be that the sprite forms the theoretically expected boundary between different dynamics for \( \epsilon \) near, and far from \( \epsilon_{FSN} \) [70].

In the two parameter bifurcation diagram, the sprite organises the surrounding bifurca-
3. Bifurcations of Canard Trajectories Near a Type I Folded Saddle-Node Singularity

tion curves (Figs. 3.13, 3.14). More specifically, for increasing $\delta$ the sprite has an upward branching structure, with turning-point bifurcation curves branching off the sprite. Looking at successive, decreasing values of $\delta$, we see turning-point bifurcation curves branching off the sprite, first for $i = 6$, then for $i = 5$ [Fig. 3.14(a)-(b), black and magenta curves].

The $i = 4$ turning-point bifurcation curves (cyan) also branch off the sprite at smaller $\delta$ (Fig. 3.13). Whilst the $i = 4$ turning-point bifurcation curve (to the right of the sprite) is not seen to converge for the range of $\delta$ in Fig. 3.13, it is connected to the sprite when $\log(\delta) = -3.5$, in the same manner as the $i = 5$ curve was connected when $\log(\delta) = -2.383$ [Fig. 3.19(e)].

Because the turning-point bifurcation curves branch off the sprite, it could be considered that the sprite generates the $i = 4, 5, 6$ curves. It may be that the turning-point bifurcation curves for canard trajectories with $i < 4$ rotations are also generated by the sprite at smaller $\delta$, however we are not able to continue the sprite to sufficiently small $\delta$ to verify this. (Note, were this the case, scaling law 3 would not hold.) The sprite does not generate all the turning-point bifurcation curves; notice the $i = 7$ curves are bounded away from the sprite (Fig. 3.13, green curves at small $\epsilon$). It may be that there is a sprite at smaller $\epsilon$ that organises the curves for canard trajectories with $i \geq 7$ rotations.

We now describe what the sprite corresponds to in “cross-section” one parameter bifurcation diagrams for fixed $\delta$. At the sprite, adjacent curves of canard trajectories with $i$ and $i - 1$ rotations spiral together, creating the dense region of turning-point bifurcations [Fig. 3.19(e)]. Unexpectedly, the number of canard trajectory rotations changes continuously from $i$ to $i - 1$ in the sprite region. This is discussed further in Section 3.5.3.f. The sprite gets more dense as $\delta$ decreases, with many turning-point bifurcation curves close together (Figs. 3.14, 3.18). In the one parameter bifurcation diagram this corresponds to the loop formed by the $i$ and $i - 1$ curves of canard trajectories [Fig. 3.19(e)] spiralling in further, and having a smaller diameter.

Note, the one-parameter bifurcation diagram in Fig. 3.19(e) shows the sprite between the $i = 6$ and $i = 5$ curves. However, successive transcritical bifurcations, as described in Section 3.5.3.f, move the sprite to lie between curves of canard trajectories with $i = 5$ and 4 rotations when $\log(\delta) \in (-3, -2.49)$, and $i = 4$ and 3 rotations when $\log(\delta) \in (-3.6, -3)$.

Lastly, we describe what the sprite corresponds to in the state space. From the cross-section of $S^\alpha_\delta$ and $S^\delta_\delta$ at the surface $\Sigma_{FN}$, we see the sprite corresponds to the repelling slow manifold $S^\alpha_\delta$ having many tight rotations. This results in many intersection points with the attracting slow manifold $S^\alpha_\delta$, or equivalently, many canard trajectories. Moreover, the large number of rotations in $S^\delta_\delta$ implies some of the canard trajectories have a similarly large number of rotations $i$. However, $i$ cannot be exactly determined. This is because the rotations are small, and the exact position of the axis of rotation, formed by the centre trajectory $\eta^C_\delta$, is not known. Because $i$ is unknown, all the bifurcation curves for the sprite were coloured black to denote canard trajectories with $i > 6$ rotations.

In summary, the sprite of canard bifurcation curves is a novel, important aspect of system (3.18)–(3.20) that requires further research. It may be that it forms the theoretically
expected boundary between different dynamics for $\epsilon$ near, and far from $\epsilon_{FSN}$. The sprite is asymptotic to $\log(\epsilon - \epsilon_{FSN}) = -1.85831$ as $\delta$ decreases, however the reason for this value is not clear, and may be dependent on the boundary conditions $L_{in}$ and $L_{out}$, and the forcing amplitude $A$. Likewise, it is not clear whether the sprite is a generic feature for systems with a folded saddle-node (type I) singularity, or whether it belongs to a subclass of systems.

### 3.5.3.f. Co-dimension one bifurcations along special parameter paths

For special parameter paths in $\log(\delta) - \log(\epsilon - \epsilon_{FSN})$ space, it is possible to see further co-dimension one bifurcations.

As is usual, near the cusp bifurcations [Figs. 3.13, 3.17], taking a parameter path tangent to the turning-point bifurcation curves gives a transcritical bifurcation, and taking a parameter path through the cusp bifurcation gives a pitchfork bifurcation [61, Sec. 3.6].

Recall, there are turning-point bifurcation curves that branch off the left of the sprite. Notice, for system (3.18)–(3.20) these turning-point bifurcation curves are atypical, because when they are crossed in the direction of decreasing $\delta$, the number of canard trajectories decreases (Fig. 3.14). Taking a parameter path through the maxima of one these turning-point bifurcation curves gives a transcritical bifurcation (Fig. 3.19). What is more, the behaviour near these transcritical bifurcations is surprising, and so it is described below in more detail.

We consider three choices of parameter paths for fixed $\delta$. These parameter paths are above, at, and below the maxima of a turning-point bifurcation curve to the left of the sprite [Fig. 3.19(a), dotted lines]. We compare the local “cross-section” one parameter bifurcation diagrams along these parameter paths [Fig. 3.19(c)-(e)]. Note, there is no global change in the one parameter bifurcation diagrams for our three parameter paths [Fig. 3.19(b)].

For the parameter path above the maxima, there is a curve of canard trajectories with $i$ rotations, and a curve of canard trajectories with $i - 1$ rotations [Fig. 3.19(c), $i = 6$ (black) and $i = 5$ (magenta)]. For the parameter path at the maxima, the $i$ and $i - 1$ curves meet in a transcritical bifurcation [Fig. 3.19(d), filled square]. For the parameter path below the maxima, the transcritical bifurcation unfolds into two turning-point bifurcations [Fig. 3.19(e)]. Moreover, the curve of canard trajectories with $i$ rotations, and the curve of canard trajectories with $i - 1$ rotations have merged [Fig. 3.19(e), black-magenta curve].

The merging of the $i$ and $i - 1$ curves is worth describing in a little more detail, as it has not been observed before in continuations of canard trajectories (to the author’s best knowledge) [16, 15]. These curves merge as $\epsilon$ varies, so there is a canard trajectory where the number of rotations changes continuously between $i$ and $i - 1$. Although this change is topologically straightforward, it has not been observed before. Note, when this occurs in system (3.18)–(3.20), the exact number of canard trajectory rotations cannot be
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Figure 3.19. The transcritical bifurcation at the maxima of a turning-point bifurcation curve along parameter paths with fixed $\delta$. Panel (a) is a zoom in of the two parameter bifurcation diagram near the sprite [see Fig. 3.13 for labels, and Fig. 3.14(b) for location]. Dotted lines in panel (a) show the parameter paths before, at and after the transcritical bifurcation, at fixed $\log(\delta) = -2.382$, (d) $-2.3827$ and (e) $-2.383$. Panels (b)-(e) show the one parameter bifurcation diagrams along these parameter paths (see Fig. 3.8 for labels). Panel (b) is the global one parameter bifurcation diagram for all three parameter paths, with a box showing the location of local, zoomed in panels (c)–(e). The transcritical bifurcation (filled square) is shown in panel (d). In panels (c)–(e) $i = 6$ (black) and $i = 5$ (magenta) curves are slightly offset to allow both to be seen. In panel (e) the $i = 6$ and $i = 5$ curves merge, as shown by colour gradient.

The complete two parameter, $\delta$ and $\epsilon$ bifurcation diagram in Fig. 3.13 reveals the self-similar structure of the bifurcations near a folded saddle-node (type I). We inferred scaling laws for the co-dimension one canard bifurcations, some of which were unprecedented, and others corroborate theoretical results by Vo et al [70]. Moreover, we described a novel sprite feature, from both a global and local perspective. Lastly, we con-
considered special paths of co-dimension one bifurcations, which highlight special behaviour near the sprite.

3.6. Conclusions

From our study of a system following a folded saddle-node (type I) bifurcation, we build a complete picture of the dynamics of a multiple time-scale system (3.1)–(3.2) with a shift in external forcing, and pose questions for future research about more general systems. Moreover, our results can be used to predict the instability threshold and numbers of small amplitude oscillations for a range of real world systems with a folded saddle-node [2, 47, 49, 69, 65, 29]. Note, some of the results, in particular those pertaining to infinite-time bifurcations, may only be relevant to systems which can be compactified. An externally forced system can be compactified if the external forcing is strictly increasing, biasymptotic and may be expressed as the solution of a differential equation.

We find that, following a folded saddle-node (type I) bifurcation, there are five different types of canard trajectories. Three of these exist for the isolated folded singularities – the folded saddle canard trajectory \( \gamma^S \), and the folded node strong and secondary canard trajectories \( \gamma^N \) and \( \gamma^i \). The remaining two are new, novel canard trajectories that have first been observed in this work, and in our paper [50] – the tertiary canard trajectory \( \zeta^i \), and the composite canard trajectories \( ˜\gamma^i \) and \( ˜\zeta^i \). These novel canard trajectories may be special to systems where a folded node interacts with a folded saddle, however this is an area that requires further analytical study.

This chapter presents the first study of bifurcations of canard trajectories near a folded saddle-node (type I), and reveals that there are three different types of co-dimension one bifurcation, distinguishing between left and right turning-point bifurcations. The left turning-point bifurcation corresponds to the creation of canard trajectories shortly after the folded saddle-node bifurcation – giving new insight to the behaviour of the system near a folded saddle-node (type I). The infinite-time bifurcation is a new type of canard trajectory bifurcation, which we were able to identify because of compactifying infinite time to finite \( \lambda \). Lastly, the right turning-point bifurcation occurs when the folded saddle and folded node are far apart. This canard trajectory bifurcation is akin to the transcritical bifurcation for an isolated folded node, as it follows the same scaling law dependent on the ratio of folded node eigenvalues.

Whilst the common wisdom was that, following a folded saddle-node (type I) bifurcation, when the folded node and folded saddle are far apart they behave as isolated singularities, our results show this is an over-simplification. Treating the folded singularities as isolated folded singularities does not capture important dynamics behaviour, such as the tertiary canard trajectories \( \zeta^i \) and the composite canard trajectories \( ˜\gamma^i \) and \( ˜\zeta^i \).

By continuing the co-dimension one canard trajectory bifurcations in both small parameters \( \delta \) and \( \epsilon \), we get a complete two parameter bifurcation diagram for systems near a folded saddle-node (type I). This illustrates how the internal separation of the system time-
scales $\delta$ and the rate of the external forcing $\epsilon$ interact, which has important implications for phenomena like rate-induced tipping.

Using a continuation method based on boundary value solvers, we are able to get reliable results for large values of $\delta$, beyond the scope of an analytical approach. Our results thereby show how the curves of co-dimension one bifurcations present shortly after the folded saddle-node (type I) bifurcation attach to curves of co-dimension one bifurcations present when the folded saddle and folded node are far apart.

Specifically, for systems like (3.18)–(3.20), we show the two parameter bifurcation diagram is approximately self-similar, and uncover new scaling laws that pose interesting questions for further research. The first novel scaling law identifies a linear relationship near the folded saddle-node singularity between $\epsilon$ and $\delta$, with the formation of canard trajectories with $i$ rotations scaling like $\delta^{-3/2}$. The second novel scaling law shows, as $\epsilon$ increases, the value at which trajectories near $\lambda \to -A$ destabilise follows an exponential relationship between $\epsilon$ and $\delta$.

Excitingly, the two parameter bifurcation diagram revealed a further unprecedented result for systems like (3.18)–(3.20) - a dense region of turning-point bifurcations, we term the sprite. Mathematically, this feature is of interest as it may account for the analytically identified, qualitatively different dynamics shortly after a folded saddle-node (type I) bifurcation, compared to when the folded saddle and folded node have moved further apart. In applications, this feature is important as it creates regions of $\delta$ and $\epsilon$ parameter space with very many canard trajectories, which could give rise to dense bands for an instability threshold, or a rich area of mixed mode oscillations.

To summarise, this first comprehensive study of the complicated dynamics following a folded saddle-node (type I) bifurcation delineates new dynamical features, and highlights the need for further developments to be made in the theoretical description of the folded saddle-node (type I).
4. Rate-induced Bifurcations in Single Time-Scale Systems

Rate-induced bifurcation in single time-scale systems are different to rate-induced bifurcations in multiple time-scale systems. In single time-scale systems, rate-induced bifurcations typically arise in systems with a stable state and a nearby unstable state [2, 3]. In such systems, external forcing causes the basin boundary for the stable state to perturb. Initial states that previously tracked the stable state, then destabilise. The difficulty lies in accurately computing the perturbed basin boundary for non-linear systems [2].

4.1. Introduction

Consider non-autonomous, single time-scale system:

\[
\frac{dx}{dt} = f(x, \lambda(\epsilon t)),
\]

4.1

\[x \in \mathbb{R}^n\] with external forcing \(\lambda(\epsilon t)\), which varies in time at a rate \(\epsilon > 0\). The functions \(f\) and \(\lambda\) are sufficiently smooth.

Assumption 4.1. For every fixed \(\lambda\), system (4.1) has a stable state \(\tilde{x}^a(\lambda)\) and a nearby unstable state \(\tilde{x}^r(\lambda)\), which vary continuously with \(\lambda\).

In the simplest case for every fixed \(\lambda\), \(\tilde{x}^a(\lambda)\) and \(\tilde{x}^r(\lambda)\) are nodes, but they may be more general invariant manifolds, such as periodic orbits.

When \(\epsilon > 0\), \(\tilde{x}^a(\lambda(\epsilon t))\) and \(\tilde{x}^r(\lambda(\epsilon t))\) are typically not steady states, or even trajectories, of system (4.1). We refer to \(\tilde{x}^a(\lambda(\epsilon t))\) and \(\tilde{x}^r(\lambda(\epsilon t))\) as moving steady states.

The precise definition of destabilising for a rate-induced bifurcation depends on the system and external forcing function to be studied. For example, see [3] for precise definitions for one-dimensional systems externally forced by some parameter shift.

In this chapter we consider external forcing \(\lambda(\epsilon t)\) that satisfies the following assumptions.

Assumption 4.2. The external forcing function \(\lambda(s)\) is

(a) strictly increasing:

\[
\frac{d\lambda}{ds} > 0,
\]
4. Rate-induced Bifurcations in Single Time-Scale Systems

(b) and asymptotic: when $s \to +\infty$,

$$\lambda(s) \to \lambda_{\text{max}} \text{ and } \frac{d\lambda}{ds} \to 0,$$

(c) or even biasymptotic: also, when $s \to -\infty$,

$$\lambda(s) \to \lambda_{\text{min}} \text{ and } \frac{d\lambda}{ds} \to 0.$$

(d) Moreover, $\lambda(s)$ can be expressed as the solution to a differential equation:

$$\frac{d\lambda}{ds} = g(\lambda).$$

For asymptotic external forcing we consider system (4.1) only for $t$ greater than some start time $t_{\text{min}}$, and let $\lambda_{\text{min}} = \lambda(\epsilon t_{\text{min}})$. For biasymptotic external forcing we allow $t \in (-\infty, \infty)$, and boundaries $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ are as defined in assumption 4.2.

Now we make our definition of “destabilising” more precise:

**Definition 4.1.** For system (4.1) with external forcing $\lambda(\epsilon t)$ that satisfies assumption 4.2:

An initial state $x_0 = x(t_0)$,

- tracks $\tilde{x}^a(\lambda(\epsilon t))$ if the trajectory started at $x_0$ converges to $\tilde{x}^a(\lambda_{\text{max}})$, or
- destabilises if the trajectory started at $x_0$ moves away to another part of the state space and does not converge to $\tilde{x}^a(\lambda_{\text{max}})$, and the
- critical rate $\epsilon_c(x_0)$ is the largest value such that for all $0 < \epsilon < \epsilon_c(x_0)$, the initial state tracks $\tilde{x}^a(\lambda(\epsilon t))$.

For any $\epsilon$, the instability threshold separates the initial states that track from those that destabilise.

Let $U$ be some small neighbourhood of $\tilde{x}^a(\lambda_{\text{min}})$, such that all initial states $x_0 \in U$ track $\tilde{x}^a(\lambda(\epsilon t))$.

- The critical rate of the system $\epsilon_c$ is the largest value such that for all $0 < \epsilon < \epsilon_c$, there exists such a $U$.
- At the critical rate, the system undergoes a rate-induced bifurcation.
4.1.1. Towards a general framework to study rate-induced bifurcations in single time-scale systems

Following assumption 4.1, we can apply Fenichel’s Theorem to the curve of stable states $\tilde{x}^a(\lambda)$ and the curve of unstable states $\tilde{x}^r(\lambda)$ for $\epsilon = 0$ (see Appendix A.1) [26, 25, 35].

For sufficiently small $0 < \epsilon < \epsilon_*$, system (4.1) has two special trajectories:

- a perturbed stable state $\tilde{x}^a_\epsilon(\lambda)$,
- and a perturbed unstable state $\tilde{x}^r_\epsilon(\lambda)$.

Both $\tilde{x}^a_\epsilon(\lambda)$ and $\tilde{x}^r_\epsilon(\lambda)$ have the same stability type as $\tilde{x}^a(\lambda)$ and $\tilde{x}^r(\lambda)$, respectively. They are also $O(\epsilon)$ close and diffeomorphic to $\tilde{x}^a(\lambda)$ and $\tilde{x}^r(\lambda)$, respectively. For ease of reading, we refer to $\tilde{x}^a(\lambda(\epsilon t))$, $\tilde{x}^r(\lambda(\epsilon t))$, $\tilde{x}^a_\epsilon(\lambda(\epsilon t))$ and $\tilde{x}^r_\epsilon(\lambda(\epsilon t))$ as $\tilde{x}^a$, $\tilde{x}^r$, $\tilde{x}^a_\epsilon$ and $\tilde{x}^r_\epsilon$, respectively.

The perturbed steady state trajectories $\tilde{x}^a_\epsilon$ and $\tilde{x}^r_\epsilon$ can be used to infer results about system (4.1) for $0 < \epsilon < \epsilon_*$. First, the perturbed stable state $\tilde{x}^a_\epsilon$ is attracting and $O(\epsilon)$ close to $\tilde{x}^a$. Thus, if a trajectory converges to $\tilde{x}^a_\epsilon$, it can be considered to track $\tilde{x}^a$. Second, the perturbed unstable state $\tilde{x}^r_\epsilon$ typically forms the basin boundary for $\tilde{x}^a_\epsilon$. If this is the case, for a given $0 < \epsilon < \epsilon_*$ the instability threshold corresponds to the perturbed unstable state $\tilde{x}^r_\epsilon$, and for a given initial state $x_0$ the critical rate of the initial state $\epsilon_c(x_0)$ corresponds to $x_0$ lying on $\tilde{x}^r_\epsilon$. For example, we use these ideas in [3] for systems with one variable $x \in \mathbb{R}$ and biasymptotic external forcing $\lambda(\epsilon t)$ to show that there exists some $\epsilon_* > 0$ such that for all $\epsilon < \epsilon_*$ there is at least one trajectory which tracks, that is, remains sufficiently close to, $\tilde{x}^a$ for all time.

The critical rate of the system $\epsilon_c$ is bounded below by the maximum value of $\epsilon_*$ such that Fenichel’s Theorem is satisfied. However, the perturbed steady state $\tilde{x}^a_\epsilon$ in Fenichel’s Theorem satisfies the slightly stricter condition that it is also diffeomorphic to $\tilde{x}^a$, which we do not require, thus $\epsilon_c \geq \epsilon_*$ (see Appendix A.1). The critical rate $\epsilon_c$ can be considered to give an (upper) validity boundary of Fenichel’s Theorem.

In conclusion, to understand rate-induced bifurcations in single time-scale systems, we compute the perturbed steady states $\tilde{x}^a_\epsilon$ and $\tilde{x}^r_\epsilon$. This can be done with a variety of methods.

Firstly, for a generic single time-scale systems (4.1), we can take advantage of $\tilde{x}^a_\epsilon$ and $\tilde{x}^r_\epsilon$ corresponding to slow manifolds, and numerically compute the slowest attracting and repelling trajectory. This is shown in Section 4.2. Secondly, for systems with external forcing which satisfies assumption 4.2, the ends of $\tilde{x}^a$ and $\tilde{x}^r$ persist as saddle steady states for $\epsilon > 0$. In this case, $\tilde{x}^a_\epsilon$ and $\tilde{x}^r_\epsilon$ correspond to manifolds of saddle steady states, so are closely approximated by many trajectories. What is more, we show when both $\tilde{x}^a_\epsilon$ and $\tilde{x}^r_\epsilon$ correspond to manifolds of saddle steady states, the system has a critical rate $\epsilon_c$, which is given by a heteroclinic connection. This is shown in Section 4.3.

We consider a variety of simple systems: one dimensional with $x \in \mathbb{R}$, and two dimensional with $x \in \mathbb{R}^2$ and an unstable periodic orbit, that may have shear or be asymmetric; with
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the two external forcing functions $\lambda(\epsilon t)$ used in Chapter 2. For these examples we compute $\tilde{x}_a^\epsilon$ and $\tilde{x}_r^\epsilon$, illustrate the system before, at, and after a rate-induced bifurcation, and show the relationship between the critical rate $\epsilon_c$ and the system parameters.

These systems closely follow the examples given in Ashwin et al [2]. We make a comparison in Section 4.4 between our results for the critical rate found from perturbed manifolds $\tilde{x}_a^\epsilon$ and $\tilde{x}_r^\epsilon$, and the results in [2] which identifies when trajectories cross a “tipping radius” which approximates the basin of attraction for $\tilde{x}^a$.

4.2. Method 1: Numerical computation of slow manifolds

This method exploits the fact that the perturbed steady states $\tilde{x}_a^\epsilon$ and $\tilde{x}_r^\epsilon$ are slow manifolds. Thus, we term it the slow manifold method. Following the technique given in [42], we use numerical continuation software AUTO [21] to identify the slowest attracting and repelling trajectory in a region.

The slow manifold method is suitable for finite intervals of a generic single time-scale system (4.1). So we can assess the accuracy of this method, we consider a system with external forcing given by an exponential approach,

$$\lambda(\epsilon t) = \lambda_{\text{max}}(1 - e^{-\epsilon t}) \quad \text{for } t \in [t_{\text{min}}, \infty),$$

(4.2)

where $\epsilon, \lambda_{\text{max}} > 0$. This is an asymptotic external forcing function so $\lambda_{\text{min}}$ is defined by $t_{\text{min}}$. Suppose $t_{\text{min}} = 0$, then $\lambda_{\text{min}} = 0$, and the system has a rate-induced bifurcation if initial states near $\tilde{x}_a^a(0)$ destabilise. The external forcing $\lambda(\epsilon t)$ satisfies assumption 4.2, parts (a), (b) and (d), and is expressed as the solution of the differential equation

$$\frac{d\lambda}{dt} = \epsilon(\lambda_{\text{max}} - \lambda).$$

(4.3)

This means for this choice of external forcing, the non-autonomous, single time-scale system (4.1) can be extended by $\lambda$, and the perturbed unstable state $\tilde{x}_r^\epsilon$ is the stable manifold of a persistent saddle steady state at

$$q_{\text{max}} := \{(x, \lambda) = (\tilde{x}_r^\epsilon(\lambda_{\text{max}}), \lambda_{\text{max}})\}.$$ 

As such, $\tilde{x}_r^\epsilon$ can be computed accurately using an ordinary differential equation solver, and the results compared with our more general method which identifies the slowest repelling trajectory.

4.2.1. AUTO boundary value problem to find slow manifolds

Finding the slow manifolds can be posed as a boundary value problem and solved in AUTO [42].
Consider a section \( \Sigma \) transverse to \( \tilde{x}^a \) (Fig. 4.1). For the extended system (4.1) with (4.3) take

\[
\Sigma := \{ \lambda = \lambda_{\text{max}} - \upsilon \}
\]

for \( \upsilon > 0 \).

Candidate trajectories for \( \epsilon \) with some initial state \((x_0, \lambda_0)\) are terminated at \((x_1, \lambda_1) \in \Sigma\) a distance \(d\) from \(\tilde{x}^a\) (Fig. 4.1).

Consider trajectories of a fixed arc length \(L^*\) ending at a point \((x_1, \lambda_1)\) on \(\Sigma\). The slow manifold, equivalently \(\tilde{x}^a\), is the trajectory that takes the longest time \(T\) to reach \(\Sigma\).

Consider a section \(\Sigma\) transverse to \(\tilde{x}^r\), again the slow manifold, equivalently \(\tilde{x}^r\), is the trajectory that takes the longest time \(T\) to reach \(\Sigma\).

For \(0 < \epsilon < \epsilon^*\) and small we find there exist well defined perturbed steady states \(\tilde{x}^a_\epsilon\) and \(\tilde{x}^r_\epsilon\). To be well defined, the different candidates for \(\tilde{x}^a_\epsilon\) and \(\tilde{x}^r_\epsilon\) should be exponentially close as \(L^*\) is varied.

The boundary conditions are

\[
x(0) = x_0, \quad \lambda(0) = \lambda_0, \quad x(T) = x_1, \quad \lambda(T) = \lambda_1,
\]

where \(x_0\) and \(\lambda_0\) are not specified, and \(x_1\) and \(\lambda_1\) are given by (Fig. 4.1):

\[
x_1 - \tilde{x}^a_\epsilon(\lambda_1) - d = 0
\]

\[
\lambda_1 - \lambda_{\text{max}} + \upsilon = 0.
\]

The integral condition is:

\[
\int_0^T \sqrt{x(t) \cdot x(t)} - \frac{L}{T} \, dt = 0,
\]
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Figure 4.2. System (4.8) forced by an exponential approach (4.2) with \( t_{\text{min}} = 0 \), \( \lambda_{\text{max}} = 3 \), and \( \mu = 1 \), (a) before, (b) at, and (c) after the rate-induced bifurcation, when \( \epsilon = 0.5, 1.767, \) and 2.5, respectively. Stable and unstable moving steady states \( \tilde{x}^a \) and \( \tilde{x}^r \) (dashed lines) end at node and saddle steady states \( p_{\text{max}} \) and \( q_{\text{max}} \) (dots), respectively. Arbitrary initial states either track \( \tilde{x}^a \) and converge to \( p_{\text{max}} \) (grey), or destabilise (white). The instability threshold \( \tilde{x}^r \) (blue) is computed as the stable manifold of \( q_{\text{max}} \) using an ordinary differential equation solver in MATLAB. Also shown is the trajectory from the special initial state \((x_0, \lambda_0) = (\tilde{x}^a(\lambda_{\text{min}}), \lambda_{\text{min}}) = (-1, 0)\) (empty dot).

where \( L \) is the length of the trajectory, and \( T \) is the integration time.

Step 1. The system is initialised with a point solution \((x_0, \lambda_0) = (x_1, \lambda_1) \in \Sigma\) with \( d = 0 \). Note \( T = 0 \).

Step 2. Do not use the integral condition (4.7) as \( T = 0 \), instead use an extra boundary condition \( \lambda_0 = l \), for some small \( l > 0 \). Using \( l \) as the continuation parameter, increase the trajectory to have non-zero length.

Step 3. Now use the integral condition (4.7) not the extra boundary condition, and extend the trajectory further to some length \( L = L_* \).

Step 4. Fix \( L = L_* \) and vary \((x_1, \lambda_1)\) by changing \( d \) and compare integration time \( T \). The slowest trajectory for a fixed arc length is \( \tilde{x}^a_\epsilon \) or \( \tilde{x}^r_\epsilon \). AUTO detects these local maxima as fold points. (Note, equivalently, for fixed \( T \) the folds can be found in \( L - d \) space.)

Step 5. We continue the folds in \((L, d, T)\) parameter space to compare results for different choices of arc length \( L_* \).

First solve the boundary value problem for small \( \epsilon \), then increase \( \epsilon \) to identify the critical rate \( \epsilon_c(x_0) \) for \( \lambda_0 \).

4.2.2. Example 1: Stable node and unstable node

Consider a one dimensional, two node system forced by an exponential approach:

\[
\frac{dx}{dt} = (x + \lambda)^2 - \mu, \tag{4.8}
\]

\[
\frac{d\lambda}{dt} = \epsilon(\lambda_{\text{max}} - \lambda), \tag{4.9}
\]
with \( \mu, \epsilon, \lambda_{\text{max}} > 0, \lambda \in [0, \lambda_{\text{max}}] \). For fixed \( \lambda \), system (4.8)–(4.9) has a stable state at

\[
\tilde{x}^a(\lambda) := -\lambda - \sqrt{\mu}
\]

and an unstable state at

\[
\tilde{x}^r(\lambda) := -\lambda + \sqrt{\mu}.
\]

Moreover, for time-varying \( \lambda(\epsilon t) \), that is \( \epsilon > 0 \), system (4.8)–(4.9) has a persistent saddle steady state

\[
q_{\text{max}} := \{(x, \lambda) = (-\lambda_{\text{max}} + \sqrt{\mu}, \lambda_{\text{max}})\}
\]

and node steady state

\[
p_{\text{max}} := \{(x, \lambda) = (-\lambda_{\text{max}} - \sqrt{\mu}, \lambda_{\text{max}})\}.
\]

The parameters \( \mu \) and \( \lambda_{\text{max}} \) control the distance between \( \tilde{x}^a \) and \( \tilde{x}^r \).

Figure 4.2 shows system (4.8)–(4.9) before, at, and after the rate-induced bifurcation when initial states in the neighbourhood of \( \tilde{x}^a(0) \) destabilise. In Fig. 4.2, the instability threshold \( \tilde{x}^r_\epsilon \) is found by computing the stable manifold of the saddle steady state \( q_{\text{max}} \) using MATLAB. Equivalently, \( \tilde{x}^r_\epsilon \) can be considered to be a perturbation of the unstable state \( \tilde{x}^r \). Notice, for smaller \( \epsilon \), the perturbed unstable state \( \tilde{x}^r_\epsilon \) is closer to the (unperturbed) unstable state \( \tilde{x}^r \) [Fig. 4.2(a)].

We now illustrate computing \( \tilde{x}^r_\epsilon \) and the perturbed stable state \( \tilde{x}^a_\epsilon \) using our slow manifold method. We take \( \Sigma := \{\lambda_{\text{max}} - 0.5\} \), so we are a distance \( \nu = 0.5 \) from the node \( p_{\text{max}} \) and saddle \( q_{\text{max}} \) and the dynamics of these have less effect. Because \( \tilde{x}^a \) and \( \tilde{x}^r \) are parallel, the same section \( \Sigma \) can be used to simultaneously find \( \tilde{x}^a_\epsilon \) and \( \tilde{x}^r_\epsilon \) (Fig. 4.1).

Note, system (4.8)–(4.9) has the property that:

**Property 4.1.** The slow-fast separation of \( x \) and \( \lambda \) time-scales breaks down for \( \lambda << \lambda_{\text{max}} \).

For example for \( \mu = 1 \), at \( x = -\lambda \) midway between \( \tilde{x}^a \) and \( \tilde{x}^r \), the rates of change are

\[
\frac{dx}{dt} = -1 \quad \text{and} \quad \frac{d\lambda}{dt} = \epsilon(\lambda_{\text{max}} - \lambda).
\]

Thus, the time-scale separation breaks down for \( \lambda \approx \lambda_{\text{max}} - 1/\epsilon \). Property 4.1 means system (4.8)–(4.9) may not have well defined perturbed steady states \( \tilde{x}^a_\epsilon \) and \( \tilde{x}^r_\epsilon \) for \( \lambda << \lambda_{\text{max}} \), as shown later.

We compute \( \tilde{x}^a_\epsilon \) and \( \tilde{x}^r_\epsilon \) for \( \epsilon = 0.2 \) and \( \epsilon = 0.5 \) (Fig. 4.3). The slow manifold method works best for small \( \epsilon \), as then there is a larger separation of time-scales. We initially chose a fixed arc length \( L_\epsilon = 5 \). We show that other values of \( L_\epsilon \) give less good results because of the breakdown in time-scale separation for \( \lambda << \lambda_{\text{max}} \) [Figs. 4.3, 4.4].

Recall, \( d \) is the distance of the trajectory end point \((x_1, \lambda_1)\) in \( \Sigma \) from \( \tilde{x}^a \) (Fig. 4.1). We expect there to be a locally slowest trajectory at \( d \approx 0 \) corresponding to \( \tilde{x}^a_\epsilon \), and a locally slowest trajectory at \( d \approx 2\sqrt{\mu} \) corresponding to \( \tilde{x}^r_\epsilon \).

Figure 4.3 shows there is a local slowest trajectory at \( d \approx 0 \) which corresponds to \( \tilde{x}^a_\epsilon \) (label
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Figure 4.3. For system (4.8)–(4.9) with \( \lambda_{\text{max}} = 3, \mu = 1 \) and \( \epsilon = 0.2 \) and 0.5. Trajectories with fixed arc length \( L_\ast \) from different end points in \( \Sigma \) (green line) a distance \( d \) from \( \tilde{x}^a \) have different integration times \( T \). In (a)-(b) \( \epsilon = 0.2 \) and \( L_\ast = 5 \), in (c)-(d) \( \epsilon = 0.5 \) and \( L_\ast = 5 \) (solid) or \( L_\ast = 7 \) (dashed). Panels (a) and (c) show how \( T \) varies with \( d \), dots 1-7 correspond to the trajectories shown in panels (b) and (d), respectively. Key points are: 2 (red) the maximum corresponding to \( \tilde{x}^a_\epsilon \); 6 (blue) the maximum corresponding to \( \tilde{x}^r_\epsilon \); and 4 (purple) the minimum which corresponds to the fastest trajectory between \( \tilde{x}^a_\epsilon \) and \( \tilde{x}^r_\epsilon \).

For smaller \( \epsilon \) (further from the rate-induced bifurcation) the slowest manifolds are more distinct. That is, the local maximum \( T \) for varying \( d \) is narrower [Fig. 4.3(a)], and the trajectories for different \( d \) clearly converge [Fig. 4.3(b)]. This is because for smaller \( \epsilon \) there is a larger separation of time-scales. Note, for \( \epsilon = 0.2 \) the maximum near \( d = 0 \) is sharp [Fig. 4.3(a)], showing all the trajectories that converge to \( \tilde{x}^a_\epsilon \) are very close. Thus, the accuracy of \( \tilde{x}^a_\epsilon \) is sensitive to changes in \( d \), as seen by comparing trajectories 1, 2 and 3 in Fig. 4.3(b).

The ability to detect the perturbed steady states \( \tilde{x}^a_\epsilon \) and \( \tilde{x}^r_\epsilon \) depends on a suitable choice of arc length \( L_\ast \). When \( \epsilon = 0.5 \), the perturbed stable state \( \tilde{x}^a_\epsilon \) is more difficult to detect, as shown in Fig. 4.3(c). In fact, for \( L_\ast = 7 \) the maximum near \( d = 0 \) disappears, and there is no suitable choice of trajectory for \( \tilde{x}^a_\epsilon \) [Fig. 4.3(d)]. Note, at this value of \( \epsilon \) the slow manifold method can no longer be used to compute \( \tilde{x}^a_\epsilon \), although \( \epsilon \) is less than \( \epsilon_c = 1.767 \). Possibly \( \epsilon \approx 0.5 \) corresponds to the validity boundary of Fenichel’s Theorem with \( \tilde{x}^a_\epsilon \) no longer being \( O(\epsilon) \) close to \( \tilde{x}^a \). Essentially, the slow manifold method no longer
works because the separation of time-scales required for Fenichel’s Theorem breaks down for \( \lambda \ll \lambda_{\text{max}} \), see Property 4.1.

To assess how robust the results are for different choices of fixed arc length \( L^* \), we continue the fold points with varying \( L \) (also \( d \) and \( T \)) to identify the slowest trajectories for different choices of \( L^* \). This is shown in Fig. 4.4 for \( \epsilon = 0.2 \).

For the candidate trajectory for \( \tilde{x}^a_\epsilon \), as \( L \) is varied, the position of the end point \( d \) varies on a scale \( O(10^{-5}) \) [Fig. 4.4(a)]. Moreover, this difference in \( (x_1, \lambda_1) \) corresponds to distinct trajectories [Fig. 4.4(b)]. As expected, because of the breakdown in the separation of time-scales, \( \tilde{x}^a_\epsilon \) is only well defined for \( \lambda \) near \( \lambda_{\text{max}} \). That is, for all choices of \( L^* \), the candidate trajectory for \( \tilde{x}^a_\epsilon \) are only in agreement for \( \lambda \in [0.5, 2.5] \), and not up to \( \lambda_{\text{min}} = 0 \). Note, because of the separation of time-scales, if the arc length is too long, \( L > 13 \), there is a rapid increase in \( d \) [Fig. 4.4(a)] and the slowest trajectory initially lies close to \( \tilde{x}^r_\epsilon \) [Fig. 4.4(b), trajectory 4]. For \( L^* > 13 \) we can no longer distinguish a solution for \( \tilde{x}^a_\epsilon \).

In contrast, the candidate trajectory for \( \tilde{x}^r_\epsilon \) is very robust under changes in arc length \( L \). As \( L \) is varied, there is little change in the position of the end point \( d \) which varies on a scale \( O(10^{-7}) \) [Fig. 4.4(c)]. The unique corresponding trajectory is shown in Fig. 4.4(d).

Why is the candidate trajectory for \( \tilde{x}^r_\epsilon \) robust, while the candidate trajectory for \( \tilde{x}^a_\epsilon \) is not.
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Figure 4.5. The perturbed unstable steady state $\tilde{x}_r^\epsilon$ computed as a slow manifold using AUTO (green), and computed as the stable manifold of the saddle $q_{\text{max}}$ using MATLAB (blue) give exactly the same result. Figure shown for $\epsilon = 0.5, \lambda_{\text{max}} = 3$ and $\mu = 1$, see Fig. 4.3 for labels.

robust? As discussed, for our chosen system (4.8)–(4.9), $\tilde{x}_r^\epsilon$ is unique and is the stable manifold of the saddle $q_{\text{max}}$. This makes it less sensitive to changes in $d$, as all trajectories in backwards time converge to it. Whereas, $\tilde{x}_a^\epsilon$ is not unique, and is a stable manifold of the node $p_{\text{max}}$. Unless $\tilde{x}_a^\epsilon$ coincides with a (very) weak eigenvector for $p_{\text{max}}$, trajectories in backwards time diverge from it, making it highly sensitive to changes in $d$.

For $\epsilon = 0.5$, we can see from Fig. 4.3 that if $L_\ast = 7$ then $\tilde{x}_a^\epsilon$ is not well defined. In fact, a continuation of the fold points for $\epsilon = 0.5$ shows the maximum corresponding to $\tilde{x}_a^\epsilon$ disappears in a fold bifurcation with the local minimum when $L \approx 6.8$, thus there is is no well-defined $\tilde{x}_a^\epsilon$ for $L > 6.8$. This is because there is less separation of time-scales when $\epsilon$ is larger.

Ultimately, for system (4.8)–(4.9) results are not good if a large arc length $L_\ast$ is chosen, because then candidate trajectories for $\tilde{x}_a^\epsilon$ and $\tilde{x}_r^\epsilon$ extend to small $\lambda$ where there is a breakdown in the separation of time-scales (property 4.1).

We show our slow manifold method to compute $\tilde{x}_r^\epsilon$ with AUTO is as good as computing $\tilde{x}_a^\epsilon$ as the stable manifold of the saddle $q_{\text{max}}$ with MATLAB. Figure 4.5 shows that for system (4.8)–(4.9) with $\lambda_{\text{max}} = 3, \mu = 1$ and $\epsilon = 0.5$, the two methods give exactly the same result for $\tilde{x}_r^\epsilon$. Note, the breakdown in $x$ and $\lambda$ time-scale separation is reflected by how much $\tilde{x}_r^\epsilon$ deviates from $\tilde{x}_r^\epsilon$ for $\lambda < -1$.

In conclusion, the slow manifold method gives a lot of information about the system. For small $0 < \epsilon < \epsilon_\ast$, we can compute the perturbed manifolds $\tilde{x}_a^\epsilon$ and $\tilde{x}_r^\epsilon$. For system (4.8)–(4.9), the repelling manifold $\tilde{x}_r^\epsilon$ computed by the slow manifold method remains well defined across the range of $\lambda$ and $\epsilon$. Therefore, for a given initial state $x_0$, the critical rate $\epsilon_c(x_0)$ can be identified by $x_0$ lying on $\tilde{x}_r^\epsilon$. We also have a well defined instability threshold, given by $\tilde{x}_a^\epsilon$, separating the initial states that track $\tilde{x}_a^\epsilon$ from those that destabilise.

The slow manifold method can be evaluated with different $L_\ast$ to show that the perturbed stable state $\tilde{x}_a^\epsilon$ for system (4.8)–(4.9) is no longer well defined if $\lambda_\ast < \lambda_{\text{max}} - 0.5$, even when $\epsilon < \epsilon_c$. That is, the initial states that track $\tilde{x}_a^\epsilon$ and converge to $p_{\text{max}}$ no longer
converge to a unique trajectory $\tilde{x}_a^\epsilon$. As discussed above, this is because of the breakdown in the separation of $x$ and $\lambda$ time-scales, and, in some sense, corresponds to the validity boundary of Fenichel’s Theorem. For system (4.8)–(4.9), the maximum value of $\epsilon_a$ such that Fenichel’s Theorem holds is not well defined.

For system (4.8)–(4.9), the critical rate of the system $\epsilon_c = 1.767$ is determined by the (arbitrary) choice of $t_{\text{min}} = 0$. We are not able to use the slow manifold method to compute $\tilde{x}_{a}^\epsilon$ as $\epsilon$ approaches $\epsilon_c$. However, for system (4.8)–(4.9), the critical rate $\epsilon_c$ can be given by the critical rate of the initial condition $x_0 = \tilde{x}_a^\epsilon(0)$, which can be computed from well defined $\tilde{x}_r^\epsilon$.

The slow manifold method is general and can be used to compute $\tilde{x}_{a}^\epsilon$ and $\tilde{x}_{r}^\epsilon$ for $\lambda$ extended systems for finite (compact) intervals of $\lambda$, where there is a distinction in $x$ and $\lambda$ time-scales. This method could also be generalised to work with a $\tau = \epsilon t$ extended system for finite intervals of time. It may be possible to derive the critical rate of the system $\epsilon_c$ from $\tilde{x}_a^\epsilon$ or $\tilde{x}_r^\epsilon$.

4.3. Method 2: Heteroclinic connections in an extended system

Consider a single time-scale system (4.1) that satisfies assumptions 4.1–4.2 and is biasymptotic. This system can be extended to:

\[
\begin{align*}
\frac{dx}{dt} & = f(x, \lambda), \\
\frac{d\lambda}{dt} & = \epsilon g(\lambda),
\end{align*}
\]

where $\lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}]$.

System (4.10)–(4.11) has moving steady states $\tilde{x}_a^\epsilon(\lambda)$ and $\tilde{x}_r^\epsilon(\lambda)$, the ends of which, at $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$, persist as steady states for $\epsilon > 0$. Therefore, for $\epsilon > 0$ the end points of the perturbed steady states $\tilde{x}_a^\epsilon(\lambda)$ and $\tilde{x}_r^\epsilon(\lambda)$ persist as steady states given by $\tilde{x}_a^\epsilon(\lambda_{\text{min}})$, $\tilde{x}_a^\epsilon(\lambda_{\text{max}})$, $\tilde{x}_r^\epsilon(\lambda_{\text{min}})$ and $\tilde{x}_r^\epsilon(\lambda_{\text{max}})$. What is more, by assumption 4.2 the external forcing is strictly increasing, so the steady states at $\tilde{x}_a^\epsilon(\lambda_{\text{min}})$ and $\tilde{x}_r^\epsilon(\lambda_{\text{max}})$ are of saddle type in $x-\lambda$ space. Thus, the perturbed steady states $\tilde{x}_a^\epsilon$ and $\tilde{x}_r^\epsilon$ correspond to unstable manifolds of saddle steady states. Note, the unstable manifolds of the saddle steady states $\tilde{x}_a^\epsilon(\lambda_{\text{min}})$ and $\tilde{x}_r^\epsilon(\lambda_{\text{max}})$ are well defined for all $\epsilon > 0$, but after a rate-induced bifurcation these manifolds are no longer $O(\epsilon)$ close to $\tilde{x}_a^\epsilon(\lambda)$ and $\tilde{x}_r^\epsilon(\lambda)$. In this Section, for $\epsilon > \epsilon_c$ we still denote the unstable manifolds of the saddle steady states by $\tilde{x}_a^\epsilon$ and $\tilde{x}_r^\epsilon$, although it is an abuse of notation because they no longer satisfy Fenichel’s Theorem. In the simplest case these steady states are fixed points, but they may be more general invariant manifolds, such as periodic orbits.

Proposition 4.1. For system (4.1), satisfying assumptions 4.1 and 4.2(a)-(d), the critical rate $\epsilon_c$ corresponds to a heteroclinic connection between $\tilde{x}_a^\epsilon(\lambda_{\text{min}})$ and $\tilde{x}_r^\epsilon(\lambda_{\text{max}})$.
Proposition 4.1 follows from a suggestion by Jan Sieber in [2, Section 3bii]. Our results in Subsections 4.3.2–4.3.5 demonstrate that Proposition 4.1 is valid for a range of examples for $x \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}^1$. We present a simple argument in Subsection 4.3.2 to show why Proposition 4.1 is true, however it may require refining for higher dimensions.

Being able to compute the critical rate $\epsilon_c$ from the heteroclinic connection gives a quicker, more reliable way to compute $\epsilon_c$ than repeatedly testing for increasing $0 < \epsilon < \epsilon^*$ if $\tilde{x}_a^\epsilon$ and $\tilde{x}_r^\epsilon$ exist, and if so, where they are positioned relative to some initial state $x_0$ (as done in Method 1).

We compute the critical rate $\epsilon_c$ from the heteroclinic connection for four example systems in Subsections 4.3.2–4.3.5. These four examples are chosen as because they form a progression of simple canonical models for systems that may exhibit rate-induced bifurcations. The simplest system is the one-dimensional stable and unstable node system (Subsection 4.3.2), the next is a simple two dimensional system which introduces oscillatory behaviour (Subsection 4.3.3), the third is a two dimensional oscillatory system with shear - an important property in laser systems [75] (Subsection 4.3.4), and the final system is a two dimensional oscillatory system which is assymmetric and thus has added directional dependence on the external forcing (Subsection 4.3.5). We compute how the critical rate $\epsilon_c$ depends on the system parameters and illustrate the systems before, at, and after the rate-induced bifurcation. For these systems we use the same external forcing function, a logistic growth from $\lambda_{\text{min}} = 0$ to $\lambda_{\text{max}}$:

$$\lambda(\epsilon t) = \frac{\lambda_{\text{max}}}{2} \left( \tanh(\epsilon t) + 1 \right)$$

expressed by the differential equation

$$\frac{d\lambda}{dt} = \frac{2\epsilon}{\lambda_{\text{max}}} \lambda(\lambda_{\text{max}} - \lambda).$$

We have a numerical method to compute the heteroclinic connection between two saddle steady states based on Lin’s method [45], which we implement using numerical continuation software AUTO [37, 21].

**4.3.1. Computing heteroclinic connections using Lin’s method**

Computing the heteroclinic connection between two saddle steady states using Lin’s method can be implemented in AUTO (see [37] and [21, Section 15.13, demo pc1]). We give an overview of the method below, see Appendices A.2 and B.5.1 for detailed equations and code for AUTO, respectively.

Using Lin’s method to find the heteroclinic connection is more robust than homotopy or shooting methods if the saddle steady states are in a high dimension system, because the manifold is grown from both saddle steady states, and there is a well defined test function to decrease until a connection is found [37].
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Figure 4.6. Set-up to use Lin’s method to compute the heteroclinic connection between two saddle steady states $p_{\text{min}}$ and $q_{\text{max}}$, where $u_-$ is the trajectory from $p_{\text{min}}$, $u_+$ is the trajectory from $q_{\text{max}}$, $\Sigma$ is the Lin section and $\eta$ is the distance in $\Sigma$ between $u_-$ and $u_+$.

For ease of exposition, we first consider the simplest case where $x \in \mathbb{R}$, $\lambda \in \mathbb{R}$, and $\tilde{x}_e^p(\lambda_{\text{min}})$ and $\tilde{x}_e^q(\lambda_{\text{max}})$ are both saddle points, which we denote $p_{\text{min}}$ and $q_{\text{max}}$, respectively.

We want to find a heteroclinic connection between two saddle points $p_{\text{min}}$ and $q_{\text{max}}$. See Fig. 4.6 for the problem setup. Consider a Lin’s section $\Sigma$ halfway between $p_{\text{min}}$ and $q_{\text{max}}$:

$$\Sigma := \left\{ (x, \lambda) : \lambda = \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{2} \right\}.$$ 

Denote the unstable manifold of $p_{\text{min}}$ by

$$u_-(t) = \{x_-(t), \lambda_-(t)\}$$

and the stable manifold of $q_{\text{max}}$ by

$$u_+(t) = \{x_+(t), \lambda_+(t)\}.$$ 

Step 1. Set $\epsilon > 0$ to an estimated value of the critical rate for system (4.10)–(4.11).

Step 2. Compute the unstable manifold of $p_{\text{min}}$ to $\Sigma$. Numerically, this is an initial value problem, and $u_-(0)$ starts a small distance $\varepsilon_p > 0$ along the unstable eigenvector $e_p$ of $p_{\text{min}}$. Increase the integration time $T_-$ from 0 until $u_-(T_-) \in \Sigma$.

Step 3. Compute the stable manifold of $q_{\text{max}}$ from $\Sigma$. Numerically, this is an initial value problem, and $u_+(T_+)$ ends a small distance $\varepsilon_q > 0$ along the stable eigenvector $e_q$ of $q_{\text{max}}$. Increase the integration time $T_+$ from 0 until $u_+(0) \in \Sigma$.

Step 4. Compute the distance $\eta$ between the end points of these manifolds $u_-(T_-)$ and $u_+(0)$ in $\Sigma$.

Step 5. Reduce the distance $\eta$ to zero by varying $\epsilon$, and the distances from $p_{\text{min}}$ and $q_{\text{max}}$, $\varepsilon_p$ and $\varepsilon_q$, and integration times $T_-$ and $T_+$. (This is a boundary value problem.) When $\eta = 0$ we have a heteroclinic connection and $\epsilon$ is the critical rate.

Step 6. The heteroclinic connection can be continued in parameter space to find the dependence of $\epsilon$ on the system parameters.
If we have a saddle periodic orbit $\Gamma_{\text{max}}(t)$ instead of the saddle fixed point $q_{\text{max}}$, step three is more complicated. Firstly, the periodic orbit $\Gamma_{\text{max}}(t)$ needs to be defined; this may be done by growing $\Gamma_{\text{max}}$ from a fixed point, or by reading in a list of all the points in $\Gamma_{\text{max}}$. Secondly, the stable eigenfunction $e_\gamma(t)$ for $\Gamma_{\text{max}}(t)$ needs to be defined. Using AUTO, we can compute the stable Floquet multiplier and hence find the eigenfunction $e_\gamma(t)$ [23]. (In Appendix A.3, the Floquet multiplier is computed by hand for a basic periodic orbit, corresponding to $\Gamma_{\text{max}}$ in Subsection 4.3.3). By varying the starting distance $\varepsilon$ from a point on the periodic orbit $\Gamma_{\text{max}}(0)$ along the eigenfunction $e_\gamma(0)$, all trajectories in the stable manifold of $\Gamma_{\text{max}}(t)$ are traced out.

In Subsections 4.3.2–4.3.5, we use this method to compute the heteroclinic connection for a range of different example systems, starting with the simplest system where $\tilde{x}_a(\lambda_{\text{min}})$ and $\tilde{x}_r(\lambda_{\text{max}})$ are both saddle points, then where $\tilde{x}_r(\lambda_{\text{max}})$ is a periodic orbit.

4.3.2. Example 1: Stable node and unstable node

Consider a one dimensional, two node system forced by a logistic growth:

$$\frac{dx}{dt} = (x + \lambda)^2 - \mu, \quad (4.14)$$

$$\frac{d\lambda}{dt} = \frac{2\varepsilon}{\lambda_{\text{max}}} \lambda(\lambda_{\text{max}} - \lambda), \quad (4.15)$$

where constants $\mu, \varepsilon, \lambda_{\text{max}} > 0$. The constant $2\sqrt{\mu}$ is the separation in $x$ between the stable and unstable steady states.

For fixed $\lambda$, system (4.14)–(4.15) has a stable state at

$$\tilde{x}_a(\lambda) := -\lambda - \sqrt{\mu}$$

and an unstable state at

$$\tilde{x}_r(\lambda) := -\lambda + \sqrt{\mu}.$$  

Moreover, for time-varying $\lambda(\varepsilon t)$, that is $\varepsilon > 0$, system (4.14)–(4.15) has invariant sets

$$\{(x, \lambda) : \lambda = 0\} \text{ and } \{(x, \lambda) : \lambda = \lambda_{\text{max}}\},$$

so has persistent steady states

$$p_{\text{min}} := \tilde{x}_a(0) = \{(x, \lambda) = (-\sqrt{\mu}, 0)\},$$

$$p_{\text{max}} := \tilde{x}_a(\lambda_{\text{max}}) = \{(x, \lambda) = (-\lambda_{\text{max}} - \sqrt{\mu}, \lambda_{\text{max}})\},$$

$$q_{\text{min}} := \tilde{x}_r(0) = \{(x, \lambda) = (\sqrt{\mu}, 0)\},$$

$$q_{\text{max}} := \tilde{x}_r(\lambda_{\text{max}}) = \{(x, \lambda) = (-\lambda_{\text{max}} + \sqrt{\mu}, \lambda_{\text{max}})\},$$

where $p_{\text{min}}$ and $q_{\text{max}}$ are saddles and $q_{\text{min}}$ and $p_{\text{max}}$ are nodes. The eigenvalues and eigenvectors at each steady state are:
Figure 4.7. The two node system forced by a logistic growth (4.14)–(4.15) has invariant manifolds at \( \{ \lambda = 0 \} \) and \( \{ \lambda = \lambda_{\text{max}} \} \) (dashed lines), four steady states (dots). There are two saddle steady states \( p_{\text{min}} \) and \( q_{\text{max}} \), these have the same eigenvector \( v \). Is there a heteroclinic connection between \( p_{\text{min}} \) and \( q_{\text{max}} \)?

At \( p_{\text{min}} \): \(-2\sqrt{\mu} \) along \((1,0)\); and \( \epsilon \lambda_{\text{max}} \) along \( v \).

At \( p_{\text{max}} \): \(-2\sqrt{\mu} \) along \((1,0)\); and \(-\epsilon \lambda_{\text{max}} \) along \((-2\sqrt{\mu}, 2\sqrt{\mu} - \epsilon \lambda_{\text{max}})\).

At \( q_{\text{min}} \): \(2\sqrt{\mu} \) along \((1,0)\); and \( \epsilon \lambda_{\text{max}} \) along \((2\sqrt{\mu}, 2\sqrt{\mu} - \epsilon \lambda_{\text{max}})\).

At \( q_{\text{max}} \): \(2\sqrt{\mu} \) along \((1,0)\); and \(-\epsilon \lambda_{\text{max}} \) along \( v \).

Where \( v = (-2\sqrt{\mu}, 2\sqrt{\mu} + \epsilon \lambda_{\text{max}}) \) is the eigenvector for the saddle steady states. The dynamics of system (4.14)–(4.15) are sketched in Fig. 4.7.

When \( \epsilon > 0 \) is sufficiently small, trajectories from initial states near \( \tilde{x}^a \), track \( \tilde{x}^a \) and converge to \( p_{\text{max}} \). System (4.14)–(4.15) has a rate-induced bifurcation if a trajectory started in the neighbourhood of \( p_{\text{min}} \) destabilises. We want to find the heteroclinic connection between the saddle steady states \( p_{\text{min}} \) and \( q_{\text{max}} \) which corresponds to the rate-induced bifurcation for system (4.14)–(4.15).

Why should a heteroclinic connection correspond to a rate-induced bifurcation? This is what we proposed in Proposition 4.1 and we now discuss for this simple one dimensional example why that should be the case. Consider Figure 4.7. If the unstable manifold of \( p_{\text{min}} \) goes to the \( \{ \lambda = \lambda_{\text{max}} \} \) plane below \( q_{\text{max}} \) then all trajectories started in a neighbourhood of \( p_{\text{min}} \) will track and converge to the stable node \( p_{\text{max}} \). However, if the unstable manifold of \( p_{\text{min}} \) goes above \( q_{\text{max}} \), then all these trajectories will destabilise and escape to \((+\infty, \lambda_{\text{max}})\). The heteroclinic connection is the boundary between these two possibilities.

Figure 4.8 shows system (4.14)–(4.15) before, at, and after the rate-induced bifurcation. Generally, the perturbed steady states \( \tilde{x}_\epsilon^a \) and \( \tilde{x}_\epsilon^r \) correspond to the unstable manifold of \( p_{\text{min}} \) and the stable manifold of \( q_{\text{max}} \), respectively. Note, even after the rate-induced bifurcation, these manifolds are still well defined, but no longer correspond to \( O(\epsilon) \) perturbations of the moving steady states \( \tilde{x}^a \) and \( \tilde{x}^r \) [Fig. 4.8(d)]. It is therefore an abuse of notation to refer to them as \( \tilde{x}_\epsilon^a \) and \( \tilde{x}_\epsilon^r \); however, we do so for ease. The manifold \( \tilde{x}_\epsilon^r \) forms the instability threshold, separating initial states for trajectories that track \( \tilde{x}^a \) and converge to \( p_{\text{max}} \) (shaded grey), from those initial states which destabilise.
Figure 4.8. Two node system forced by a logistic growth (4.14)–(4.15) for $\lambda_{\text{max}} = 3$. (a) Bifurcation diagram showing how critical rate $\epsilon_c$ varies with $\mu$. This has a vertical asymptote at $\frac{\lambda^2}{4}$ (dashed line). (b)-(d) Phase portraits (b) before, (c) at, and (d) after the rate-induced bifurcation when $\mu = 1$ and $\epsilon = 1.05$, $\epsilon = \epsilon_c = 2$, and $\epsilon = 3$, respectively. The system has invariant manifolds at $\{\lambda = 0\}$ and $\{\lambda = \lambda_{\text{max}}\}$, moving steady states $\tilde{x}^a$ and $\tilde{x}^r$ (dashed lines), ending in persistent steady states $p_{\text{min}}$, $p_{\text{max}}$, $q_{\text{min}}$, and $q_{\text{max}}$ (dots). Before the rate-induced bifurcation there is a perturbed stable state $\tilde{x}^a_\epsilon$ (red line), and perturbed unstable state $\tilde{x}^r_\epsilon$ (blue line). After the bifurcation, the unstable manifolds of $p_{\text{min}}$ and the stable manifolds of $q_{\text{max}}$ still persist, and we label them $\tilde{x}^a_\epsilon$ and $\tilde{x}^r_\epsilon$, respectively, although this is an abuse of notation. The initial states for trajectories that track $\tilde{x}^a_\epsilon$ are shaded grey, and those that destabilise are shaded white.

Trajectories started near $p_{\text{min}}$ converge to $\tilde{x}^a_\epsilon$. Before the rate-induced bifurcation, this corresponds to tracking the moving stable state $\tilde{x}^a$ [Fig. 4.8(b)]. Notice, for $\epsilon = 1.05$, already $\tilde{x}^a_\epsilon$ does not satisfy Fenichel’s Theorem as it is not tangent to, thus is not diffeomorphic to, $\tilde{x}^a$ at $\lambda = \lambda_{\text{max}}$ [Fig. 4.8(b)]. In this way the validity boundary of Fenichel’s Theorem $\epsilon_*$ is a lower bound for the critical rate $\epsilon_c$, as whether the perturbed steady state $\tilde{x}^a_\epsilon$ is diffeomorphic to the unperturbed steady state $\tilde{x}^a$ is not important for rate-induced bifurcations.

At the rate-induced bifurcation, there no longer exists any open neighbourhood of $p_{\text{min}}$ from which all trajectories track. This corresponds to a heteroclinic connection between $p_{\text{min}}$ and $q_{\text{max}}$ [Fig. 4.8(c)]. What is more, the heteroclinic connection is in some sense, “neutrally stable”. Geometrically, this can be seen by trajectories converging and diverging from the heteroclinic connection in equal measure. More formally, the stability can be
assessed by Lyapunov exponents, see Appendix A.4. This change in the stability of the perturbed steady state $\tilde{x}_a$ when it becomes a heteroclinic connection may be useful to assess, as $\epsilon$ increases, when a system is approaching a rate-induced bifurcation.

The heteroclinic connection is computed using Lin’s method. What is more, using numerical continuation software AUTO, the heteroclinic connection can be computed for varying parameters $\mu$ and $\lambda_{\text{max}}$. This gives the dependence of the critical rate $\epsilon_c$ on the parameters $\mu$ and $\lambda_{\text{max}}$ [Fig. 4.8(a) shows dependence on $\mu$]. For system (4.14)–(4.15), the heteroclinic connection can also be computed by hand, which gives the explicit dependence of $\epsilon_c$ on $\mu$ and $\lambda_{\text{max}}$.

**Analytical equation for the critical rate**

The heteroclinic connection between $p_{\text{min}}$ and $q_{\text{max}}$ for system (4.14)–(4.15) shown in Fig. 4.8 appears to be a straight line. We assume it is the straight line that goes through $p_{\text{min}}$ and $q_{\text{max}}$:

$$x = \frac{2\sqrt{\mu} - \lambda_{\text{max}}}{\lambda_{\text{max}}} \lambda - \sqrt{\mu}. \quad (4.16)$$

We verify that the line (4.16) gives a heteroclinic connection and, moreover, show that the heteroclinic connection exists if

$$\epsilon_c = \frac{2\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}}. \quad (4.17)$$

Equation (4.17) gives the critical rate $\epsilon_c$ for system (4.14)–(4.15).

The line (4.16) passes through $p_{\text{min}}$ and $q_{\text{max}}$, it remains to show that it is a trajectory for system (4.14)–(4.15) for some value of $\epsilon$ by showing it is tangent to the flow. Consider the derivative of the line equation (4.16) and substitute in the equations for $dx/dt$ and $d\lambda/dt$ in system (4.14)–(4.15):

$$\frac{dx}{dt} = \frac{2\sqrt{\mu} - \lambda_{\text{max}}}{\lambda_{\text{max}}} \frac{d\lambda}{dt}. \quad (4.18)$$

Find the right hand side of (4.18) by substituting (4.16) into (4.14),

$$\frac{dx}{dt} = \left(\frac{2\sqrt{\mu}}{\lambda_{\text{max}}} \lambda - \sqrt{\mu}\right)^2 - \mu, \quad (4.19)$$

$$= \frac{4\mu \lambda}{\lambda_{\text{max}}^2} (\lambda - \lambda_{\text{max}}). \quad (4.20)$$

Find the right hand side of (4.18) by using $d\lambda/dt$ from (4.15),

$$\frac{2\sqrt{\mu} - \lambda_{\text{max}}}{\lambda_{\text{max}}} \frac{d\lambda}{dt} = \frac{2\epsilon \lambda}{\lambda_{\text{max}}^2} (\lambda_{\text{max}} - 2\sqrt{\mu})(\lambda - \lambda_{\text{max}}). \quad (4.21)$$

Equate (4.20) and (4.21) and rearrange to solve for $\epsilon$ to get equation (4.17).

Equation (4.17) gives the explicit dependence of the critical rate $\epsilon_c$ on the system parameters. Specifically, that system (4.14)–(4.15) may destabilise if the distance $2\sqrt{\mu}$ between the stable and unstable node is reduced; the magnitude of the external forcing $\lambda_{\text{max}}$ is
increased; or if the rate of change of the external forcing $\epsilon$ is increased.

Notice, for equation (4.17) to be solvable, that is for (4.14)–(4.15) to have a critical rate:

$$\lambda_{\text{max}} > 2\sqrt{\mu}.$$  

In terms of the geometry of the system, this implies $q_{\text{max}}$ must be lower than $p_{\text{min}}$ (Fig. 4.7). Otherwise, even if the external forcing $\lambda(\epsilon t)$ is so fast that there is no change in $x(t)$, at $\lambda_{\text{max}}$ we still be in the basin of attraction for $p_{\text{max}}$. This motivated Theorem 3.2 in Ashwin et al [3].

### 4.3.3. Example 2: Stable node and unstable periodic orbit

Consider a two dimensional node and periodic orbit system forced by a logistic growth:

$$\begin{align*}
\frac{dz}{dt} &= F(z - \lambda), \\
\frac{d\lambda}{dt} &= \frac{2\epsilon}{\lambda_{\text{max}}} \lambda(\lambda_{\text{max}} - \lambda)
\end{align*}$$

(4.22) \hspace{1cm} (4.23)

where $z = x + iy \in \mathbb{C}$, $\lambda \in \mathbb{R}$, and

$$F(z) = (a + i\omega)z + |z|^2z.$$  

(4.24)

Constants $a, \omega, \epsilon, \lambda_{\text{max}} \in \mathbb{R}$ with $a, \epsilon, \lambda_{\text{max}} > 0$. The constant $a$ is the radius squared of the unstable periodic orbit, and $\omega$ describes the rotation.

For fixed $\lambda$, system (4.22)–(4.24) has a stable state at

$$\tilde{z}^a(\lambda) := \lambda$$

equivalently, $\{(x, y) = (\lambda, 0)\}$

and an unstable periodic orbit at

$$\tilde{z}^r(\lambda) := \{(z, \lambda) : |z - \lambda|^2 = a\}$$

equivalently, $\{(x, y) : (x - \lambda)^2 + y^2 = a\}$.

For time-varying $\lambda(\epsilon t)$ system (4.22)–(4.24) has invariant sets

$$\{(z, \lambda) : \lambda = 0\} \text{ and } \{(z, \lambda) : \lambda = \lambda_{\text{max}}\},$$

so has persistent steady state fixed points $p_{\text{min}}$ and $p_{\text{max}}$, and periodic orbits $\Gamma_{\text{min}}$ and $\Gamma_{\text{max}}$. The fixed point $p_{\text{min}}$ and periodic orbit $\Gamma_{\text{max}}$ are of saddle type.

When $\epsilon > 0$ is sufficiently small, trajectories from initial states near $\tilde{z}^a$, track $\tilde{z}^a$ and converge to $p_{\text{max}}$. System (4.22)–(4.24) has a rate-induced bifurcation if a trajectory started in the neighbourhood of $p_{\text{min}}$ destabilises. We want to find the heteroclinic connection between the saddle steady states $p_{\text{min}}$ and $\Gamma_{\text{max}}$, which corresponds to the rate-induced bifurcation for system (4.22)–(4.24).
Figure 4.9. Node and periodic orbit system forced by a logistic growth (4.22)–(4.24) for $\lambda_{\text{max}} = 2$ and $r = 1$. (a) Bifurcation diagram showing how critical rate $\epsilon_c$ varies with $\omega$. (b)-(d) Phase portrait (b) before, (c) at, and (d) after the rate-induced bifurcation when $\omega = 2$ and $\epsilon = 1.3$, $\epsilon = \epsilon_c = 1.579$, and $\epsilon = 2.0$, respectively. The system has moving node stable state $\tilde{z}^a$ (dashed line), ending at persistent steady states $p_{\text{min}}$ and $p_{\text{max}}$ (dots), and moving periodic orbit unstable state (not marked) ending at persistent steady states $\Gamma_{\text{min}}$ and $\Gamma_{\text{max}}$. Before the rate-induced bifurcation there is a perturbed stable state $\tilde{z}_a^\epsilon$ (red line), and perturbed unstable state $\tilde{z}_r^\epsilon$ (blue tube). After the bifurcation the unstable manifolds of $p_{\text{min}}$ and the stable manifolds of $\Gamma_{\text{max}}$ still persist, and we label them as $\tilde{z}_a^\epsilon$ and $\tilde{z}_r^\epsilon$, respectively, although this is an abuse of notation. Only trajectories started at initial states within $\tilde{z}_r^\epsilon$ track $\tilde{z}^a$.

Figure 4.9 shows system (4.22)–(4.24) before, at, and after the rate-induced bifurcation. Generally the perturbed steady state manifolds $\tilde{z}_a^\epsilon$ and $\tilde{z}_r^\epsilon$ correspond to the unstable manifold of $p_{\text{min}}$ and the stable manifolds of $\Gamma_{\text{max}}$, respectively. Note, even after the rate-induced bifurcation, these manifolds are still well defined, but no longer correspond to $O(\epsilon)$ perturbations of the moving steady states $\tilde{z}^a$ and $\tilde{z}_r^\epsilon$ [Fig. 4.9(d)]. It is therefore an abuse of notation to refer to them as $\tilde{z}_a^\epsilon$ and $\tilde{z}_r^\epsilon$, however, we do so for ease.

For all values of $\epsilon$, trajectories started near $p_{\text{min}}$ converge to $\tilde{z}_a^\epsilon$. Before the rate-induced bifurcation, this corresponds to tracking the moving stable state $\tilde{z}^a$ [Fig. 4.9(b)]. For any $\epsilon$, the manifold $\tilde{z}_r^\epsilon$ forms the instability threshold, separating initial states for trajectories that track $\tilde{z}^a$ and converge to $p_{\text{max}}$, from those initial states which destabilise.

Notice, at the rate-induced bifurcation there is a heteroclinic connection between $p_{\text{min}}$ and $\Gamma_{\text{max}}$ [Fig. 4.9(c)]. The heteroclinic connection is computed using Lin’s method. What
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is more, using numerical continuation software AUTO, the heteroclinic connection can be computed for varying parameters $a$, $\omega$ and $\lambda_{\text{max}}$. This gives the dependence of the critical rate $\epsilon_c$ on the system parameters. Note, the stable Floquet multiplier for $\Gamma_{\text{max}}$ becomes very large as $\omega \to 0$ (see Appendix A.3) so the heteroclinic connection for $|\omega| < 0.5$ is instead computed by a shooting method in MATLAB, computing $\tilde{x}_\epsilon$ for varying $\epsilon$ until the end point lies within a tolerance of the periodic orbit $\Gamma_{\text{max}}$ (code given in Appendix B.3.1).

Figure 4.9(a) shows the dependence of $\epsilon_c$ on the rotation $\omega$, in particular, we see having rotation stabilises the system, as was seen in [2].

Figure 4.10 more fully shows the dependence of $\epsilon_c$ on the parameters $\omega$ and $\lambda_{\text{max}}$. The surface is drawn by interpolating between continuation curves for fixed $\lambda_{\text{max}} = 1.5, 2, 2.5, 3, 3.5, 4$, $\omega = 0, \pm 0.5, \pm 1, \pm 1.5, \pm 2, \pm 2.5, \pm 3$, and $\rho = 2\epsilon/\lambda_{\text{max}} = 0.5, 1, 1.5, 2, 2.5, 3$. We see system (4.22)–(4.24) only has a rate-induced bifurcation if $\lambda_{\text{max}} > \sqrt{a}$. Similar to the stable and unstable node example, this corresponds to the shift $\lambda_{\text{max}}$ being sufficiently big that $\Gamma_{\text{max}}$ is lower than $p_{\text{min}}$.

4.3.4. Example 3: Stable node and unstable periodic orbit with shear

We consider the effects of shear $\alpha$ on the rate-induced bifurcation for a periodic orbit. Shear is an important component of oscillatory behaviour in the dynamics of laser systems [75].
Consider a two dimensional node and periodic orbit system with shear $\alpha$ forced by a logistic growth:

$$\frac{dz}{dt} = F(z - \lambda), \quad (4.25)$$
$$\frac{d\lambda}{dt} = \frac{2\epsilon}{\lambda_{\text{max}}} \lambda(\lambda_{\text{max}} - \lambda) \quad (4.26)$$

where $z = x + iy \in \mathbb{C}$, $\lambda \in \mathbb{R}$, and

$$F(z) = a + i(\omega + \alpha(|z|^2 - a))z + |z|^2z. \quad (4.27)$$

Constants $a, \omega, \alpha, \epsilon, \lambda_{\text{max}} \in \mathbb{R}$ and $a, \epsilon, \lambda_{\text{max}} > 0$. The constant $a$ is the radius squared of the unstable periodic orbit, $\omega$ describes the rotation, and shear $\alpha$ makes the rotation of the flow non-uniform. System (4.25)–(4.27) has a reflective $x-y$ symmetry when $\alpha \to -\alpha$ and $\omega \to -\omega$, so it is sufficient to consider $\alpha \geq 0$.

System (4.25)–(4.27) is the same as system (4.25)–(4.27), but with the addition of shear $\alpha$. It therefore has the same steady states $\tilde{z}^u$ and $\tilde{z}^r$, invariant sets, and persistent steady states $p_{\text{min}}, p_{\text{max}}, \Gamma_{\text{min}}$ and $\Gamma_{\text{max}}$.

The shear in system (4.25)–(4.27) has no effect on the rotation around the periodic orbit $|z - \lambda|^2 = a$. If $\omega > 0$, the shear opposes the rotation inside, and supports the rotation outside the periodic orbit [Fig. 4.11]. On the other hand, if $\omega < 0$, the shear supports the rotation inside, and opposes the rotation outside. Moreover, for $\omega > 0$, there is some $\alpha$ at which the rotation in the periodic orbit is reversed [Fig. 4.11]. Reversing happens when $\alpha > \omega$, as shown by a sign change in the stable eigenvalues of (4.25)–(4.27) at $(0, 0, 0)$:

$$\epsilon_{1,2} = -a \pm i(\alpha - \omega).$$

System (4.25)–(4.27) before, at, and after the rate-induced bifurcation looks very similar to system (4.22)–(4.24) (shown in Fig. 4.9). Again, by using AUTO to continue the heteroclinic connection, we get the dependence of the critical rate $\epsilon_c$ on the system parameters.
Figure 4.12 shows the dependence of $\epsilon_c$ on the rotation $\omega$ for different shear $\alpha$.

Figure 4.12. Critical rate $\epsilon_c$ for the node and periodic orbit system with shear forced by logistic growth (4.25)–(4.27) with $\lambda_{\text{max}} = 2$ and $r = 1$. (a) Bifurcation diagram showing how critical rate $\epsilon_c$ varies with $\omega$ for shear $\alpha = 0, 1, 2, 3$. (b) Elliptical relationship between $\alpha$ and $\omega$ with $\epsilon_c$ as colour, contours are shown at $\epsilon_c = 1, 2, 3$.

When there is no shear $\alpha = 0$ and the results in Subsection 4.3.3 are recovered (compare Fig. 4.9(a) and Fig. 4.12(a)). Introducing shear $\alpha > 0$, breaks the symmetry of the system with respect to $\omega$ [Fig. 4.12(a)]. The bifurcation diagram for $\epsilon_c$ for varying $\omega$ and $\alpha > 0$ appears to be shifted left. Recall, rotation has a stabilising effect and, for $\omega > 0$, the shear opposes the rotation near $\tilde{z}^a$ (see Fig. 4.11), so system (4.25)–(4.27) destabilises at lower $\epsilon_c$. Whereas, for $\omega < 0$, the shear supports the rotation near $\tilde{z}^a$, so system (4.25)–(4.27) destabilises at higher $\epsilon_c$.

For each value of $\alpha$, there is a local minimum which corresponds to the value $\omega$ where the system has the lowest critical rate $\epsilon_c$. One would expect this to be at $\omega = \alpha$ as then the rotation cancels near $\tilde{z}^a$ (see Fig. 4.11), however it happens at slightly lower $\omega$ [Fig. 4.12(a)]. The reason for this is not clear.

In fact, for $\alpha \neq 0$ the $\omega-\epsilon$ profile is slightly asymmetric, showing the impact of shear opposing $\omega$ is different to the impact of shear supporting $\omega$.

4.3.5. Example 4: Stable node and unstable asymmetric periodic orbit

We now consider a system with an asymmetric unstable periodic orbit to demonstrate the effect of path dependence on the direction of the external forcing $\lambda(\epsilon t)$ for rate-induced bifurcations.

Consider a Van der Pol system which has a stable node, surrounded by an asymmetric unstable periodic orbit for all $\mu > 0$, $\omega > 0$:

$$\frac{dx}{dt} = y,$$  (4.28)

$$\frac{dy}{dt} = \mu (1 - x^2)y - \omega^2 x.$$  (4.29)
Apply external forcing $\lambda(\epsilon t)$ along a path at an angle $\theta$ from the $x$-axis:

$$\{ \lambda(\epsilon t) \cos \theta, \lambda(\epsilon t) \sin \theta \}$$

where the magnitude of the forcing $\lambda(\epsilon t)$ is a logistic growth, as before.

So the full system is:

\[
\begin{align*}
\frac{dx}{dt} &= v, \\
\frac{dy}{dt} &= \mu (1 - u^2) v - \omega^2 u \\
\frac{d\lambda}{dt} &= 2\epsilon \lambda_{\text{max}} \lambda(\lambda_{\text{max}} - \lambda),
\end{align*}
\]  

(4.30) (4.31) (4.32)

where

$$u = x - \lambda \cos \theta, \quad v = y - \lambda \sin \theta,$$

(4.33)

and constants $\mu, \omega, \epsilon, \lambda_{\text{max}} > 0$, and $\theta \in [0, \pi]$.

As before, for every fixed $\lambda$, system (4.30)–(4.33) has a stable state

$$\tilde{x}^a := \{(x, y) = (\lambda \sin \theta, \lambda \cos \theta)\}$$

surrounded by a periodic orbit unstable state $\tilde{x}^r$.

As before, for time-varying $\lambda(\epsilon t)$ system (4.30)–(4.33) has a saddle steady state $p_{\text{min}} := \tilde{x}^a(0)$ and saddle periodic orbit $\Gamma_{\text{max}} := \tilde{x}^r(\lambda_{\text{max}})$.

We want to find the critical rate $\epsilon_c$ for system (4.30)–(4.33). Again, we use Proposition 4.1, that is the critical rate $\epsilon_c$ corresponds to a heteroclinic connection between $p_{\text{min}}$ and $\Gamma_{\text{max}}$, and use Lin’s method to find the heteroclinic connection. Again, the system before,
Figure 4.14. (a) The radius of the asymmetric periodic orbit for system (4.30)–(4.33) varies with angle $\theta$, consequently (b) the critical rate $\epsilon_c$ varies depending on the direction of forcing $\theta$. This effect is more marked when the maximum value of the forcing function $\lambda_{\text{max}}$ is smaller than the radius.
after, at the rate-induced bifurcation is not illustrated, as it looks very similar to Fig. 4.9 for system (4.22)–(4.24). System (4.30)–(4.33) has a squarer shaped unstable manifold $\tilde{x}_{r}$ than in Fig. 4.9, as given by the shape of the asymmetric periodic orbit (Fig. 4.13).

We continue the heteroclinic connection for varying $\theta$ to find the dependence of the critical rate $\epsilon_{c}$ on the choice of direction of the external forcing. This is shown in Fig. 4.14(b), the radius of the unstable periodic orbit is shown in Fig. 4.14(a) for comparison. We see the critical rate $\epsilon_{c}$ at which system (4.30)–(4.33) destabilises depends on the choice of $\theta$. Moreover, if the radius at the chosen angle of $\theta$ is greater than the maximum value of the forcing function $\lambda_{\text{max}}$ then the system may not have a rate-induced bifurcation for that forcing function $\lambda(\epsilon t)$.

Since the radius of the periodic orbit changes with $\mu$ and $\omega$ it is difficult to isolate the effect of changing these parameters on $\epsilon_{c}$, as $\lambda_{\text{max}}$ needs to be changed as well to negate the effect of the change in the shape of the periodic orbit. The radius does not change much along $\theta = 0$, and is always $\approx 2$. We see if we fix $\theta = 0$ and vary $\omega$ the increased rotation stabilises the system (as was seen for systems (4.22)–(4.24) and (4.25)–(4.27)). If we change $\mu$ this introduces different $x$ and $y$ time-scales to system (4.30)–(4.33) - movement is slow about the $x$-axis and fast away from it. Roughly speaking, increasing $\mu$ elongates the orbit in the $y$ direction and stabilises system, as trajectories escape the basin of attraction for $\tilde{x}^{a}$ through the $y$-direction which is further away, and faster, so more difficult to overcome. This would be an interesting system to research further, using the ideas about multiple time-scale systems from Chapter 2.

### 4.4. Tipping radius approach revisited

In this section we compare our results with results in Ashwin et al. [2], where the critical rate is computed using a tipping radius approach (see Section 1.4).

Ashwin et al. [2] use the example systems we labelled 1 and 2, which were a stable and an unstable node, and a stable node and an unstable periodic orbit, respectively. We compare critical rates for examples 1 and 2 when they are forced by an exponential approach or logistic growth function, with the critical rate when forced by a steady drift

$$\lambda(\tau) = r \tau$$

as given in [2]. Furthermore, we consider whether example 1 avoids a rate-induced bifurcation when the maximum rate of change of the external forcing $r_{\text{max}}(t)$ (Eq. (1.4)) satisfies the avoidance condition (Eq. (1.8)). We use the avoidance condition with a tipping radius $R$ given by the distance between $\tilde{x}^{a}$ and $\tilde{x}^{r}$, and also with the effective tipping radius $R_{c}$ given in [2].

Consider example 1, the stable and unstable node system, either:

(a) forced by an exponential approach (4.8)–(4.9), starting at $\lambda(\epsilon t_{\text{min}}) = \lambda_{\text{min}}$, so $r_{\text{max}} =$
4. Rate-induced Bifurcations in Single Time-Scale Systems

\[ \epsilon(\lambda_{\text{max}} - \lambda_{\text{min}}), \]

or

(b) forced by a logistic growth (4.14)–(4.15) so \( r_{\text{max}} = \epsilon \lambda_{\text{max}}/2 \).

The phase portraits for these system (Figs. 4.2, 4.8) can be compared with Fig. 3 in Ashwin et al, both show a similar transition.

When example 1 is forced by a logistic growth, we have a formula for the critical rate \( \epsilon_c \), Eq. (4.17). This can be compared to the equivalent formula when forced by a steady drift \( r_c = \mu [2] \). This is done by setting \( r = \epsilon \lambda_{\text{max}}/2 \) so both forcing functions have the same gradient. Then for a steady drift we have

\[ \epsilon_c = \frac{2\mu}{\lambda_{\text{max}}} \]

and for logistic growth forcing (4.17)

\[ \epsilon_c = \frac{2\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}}. \]

The steady drift forcing is akin to logistic growth forcing as \( \lambda_{\text{max}} \to \infty \), and we then see that the critical rates converge to each other.

When example 1 is forced by an exponential approach, we do not have a formula for the critical rate \( \epsilon_c \). However, we can still compare the values for the \( \epsilon_c \) with the avoidance condition for example 1 [2, Sec.3(a)]:

\[ \frac{1}{2\sqrt{\mu}} r_{\text{max}} < R \quad (\text{or } R_c), \]  

(4.34)

where the tipping radius \( R = 2\sqrt{\mu} \) or \( R_c = \sqrt{\mu}/2 \). Note, \( R_c \) is given in [2] such that for steady drift forcing there is a rate-induced bifurcation exactly when there is equality in (4.34).

Our results for example 1 forced by an exponential approach were given in Section 4.2, for \( \mu = 1, \lambda_{\text{max}} = 3 \). Recall, the critical rate \( \epsilon_c \) is dependent on the start time \( t_{\text{min}} \). For \( t_{\text{min}} = 0, \epsilon_c = 1.767 \). More generally, we can find \( t_{\text{min}} \) such that \( \lambda_{\text{min}} = \lambda(\epsilon t_{\text{min}}) \), and for \( \lambda_{\text{min}} = -2, \epsilon_c = 0.5 \), or for \( \lambda_{\text{min}} = -6, \epsilon_c = 0.2 \). The avoidance condition (4.34) with \( R \) does not hold for the last two results, but, unexpectedly, with \( R_c \) it holds for all these results.

When forced by a logistic growth, the formula for the critical rate \( \epsilon_c \), Eq. (4.17), satisfies the avoidance condition (4.34) with \( R_c \), but not with \( R \). This is not surprising, as the critical rate with logistic growth forcing is bounded above by the critical rate for steady drift forcing.

Consider example 2, the stable node and unstable periodic orbit system forced by a logistic growth (4.22)–(4.24), and \( r_{\text{max}} = \epsilon \lambda_{\text{max}}/2 \). We can compare the two parameter bifurcation diagram for \( a = 1 \) and varying \( \omega \) and \( \epsilon_c \) (Fig. 4.9) with Fig. 4 in Ashwin et al. By rescaling \( r = \epsilon \lambda_{\text{max}}/2 \) we see these bifurcation diagrams converge as \( \lambda_{\text{max}} \) increases. This may mean
the formula for the critical rate with steady drift forcing in [2], can be rescaled by some $h(\lambda_{\text{max}}, a, \omega)$ to get a formula for the critical rate with logistic growth. It remains to be seen whether for logistic growth forcing, as for steady drift forcing, destabilisation for small $\omega$ is via a saddle-node bifurcation, and for large $\omega$ is via a Hopf bifurcation.

4.5. Conclusions

In this chapter we introduce a general framework for rate-induced bifurcations for single time-scale systems based on Fenichel’s Theorem. Is this the correct framework to use for rate-induced bifurcations? Generally, it is a powerful method that gives much information about the system – for example it shows whether a perturbed stable state $\tilde{x}^a_\epsilon$ exists, and illustrating the perturbed steady states provides geometrical insight into the dynamics of the system. Moreover, it may be that other ideas from Fenichel’s Theorem such as the normal form or generalised Lyapunov type numbers are of use for assessing the stability of trajectories or to give bounds on critical rates for generic forcing functions. The limitations of the framework are that it can only be applied for to a finite (compact) interval of $\lambda$ or $\tau$, and that it assumes there is a separation of time-scales between the internal system dynamics and the external forcing. Note, it has been generally assumed for rate-induced bifurcations that the external forcing is slower (see [2]).

In this chapter we give two methods to compute the critical rate, and demonstrate these methods for a range of examples. Our first method identifies slow manifolds $\tilde{x}^a_\epsilon$ and $\tilde{x}^r_\epsilon$ which correspond to the perturbed attracting state that the trajectories track, and the instability threshold. For some systems it may not be possible to compute $\tilde{x}^a_\epsilon$ or $\tilde{x}^r_\epsilon$ for large $\epsilon < \epsilon_c$. However, there may still be enough information to derive the critical rate. Our second method identifies the critical rate with a heteroclinic connection between persistent saddle steady states for systems forced by a logistic growth function. It provides a neat short cut to find the critical rate, rather than repeatedly evaluating the system at different $\epsilon$. Also for all $\epsilon > 0$ it is possible to compute the unstable manifolds of the saddle steady states, which, while $\epsilon < \epsilon_*$ and Fenichel’s Theorem is valid, are $\tilde{x}^u_\epsilon$ and $\tilde{x}^r_\epsilon$. Moreover, through numerical continuation, the second method gives the dependence of the critical rate on the system parameters. Currently the second method is only applied to systems forced by a logistic growth function, however it may be more widely applicable to generic systems with persistent saddle steady states, for example, systems forced by an external forcing functions which has a local maximum, or is a periodic oscillation.

We apply our methods to four different example systems. It may be that these example systems can be used as normal forms for qualitatively similar systems. This would be especially pertinent for the stable and unstable node system forced by a logistic growth, as, if the re-scaling were known, that gives a formula for the critical rate.

A key aim of this chapter was to use the examples to give insight into the rate-induced bifurcation phenomenon. Indeed, the examples in this chapter have informed the development of definitions and theorems in [3], which studies rate-induced bifurcation for single
time-scale, \( x \in \mathbb{R} \), and external forcing \( \lambda \in \mathbb{R} \) given by a generic shift function. Specifically, our results show that a rate-induced bifurcation requires that \( \tilde{x}^a(\lambda_{\text{min}}) \) lies in the basin of attraction of \( \tilde{x}^a(\lambda_{\text{max}}) \), (i.e. that \( p_{\text{min}} \) must be at higher \( \lambda \) than \( q_{\text{max}} \)). This result formed the basis of Theorem 3.2 in Ashwin et al [3].

We compared our results for examples one and two forced by a logistic growth with equivalent results in [2], and showed that our results converge to their linear equivalent as \( \lambda_{\text{max}} \to \infty \).

Finally, example four highlights that if \( x \in \mathbb{R}^2 \), then the critical rate may be path dependent. This is an aspect that needs to be kept in mind for the future development of both definitions and numerical methods for rate-induced bifurcations.
5. Discussion

This thesis presents new results and insight into the phenomenon of rate-induced bifurcations. An externally forced system can have a moving steady state which is tracked, or adiabatically followed, by trajectories started from nearby initial states. However, the system may have a rate-induced bifurcation, in which case, above a critical rate of external forcing, nearby trajectories no longer track the moving steady state and instead move away to a different part of the state space. This can be described as a destabilisation, or failure to adapt to changing external conditions.

For a given system and external forcing function we ask:

- Whether the system has a rate-induced bifurcation?
- If so, is the critical rate well-defined (either for a given initial state, or for the whole system)?
- If so, how can the critical rate be computed?
- If not all the initial states destabilise, is there a well defined instability threshold separating those that do from those that do not?
- If so, how can the instability threshold be computed?

These questions can be addressed numerically or analytically, for systems with multiple time-scales or with one time-scale.

In this thesis, rate-induced bifurcations are studied using a framework that we base on geometric single perturbation theory, and more generally, on Fenichel’s Theorem [26, 25, 35, 24, 62]. We prove two theorems for the existence of critical rates and instability thresholds in systems with two time-scales. Moreover, we extend the existing body of knowledge on geometric singular perturbation theory in order to be able to compute the complicated instability threshold that arises for two time-scale systems with an external forcing given by a shift. We demonstrate in single time-scale systems that the rate-induced bifurcation depends on the position of the perturbed basin boundary. We use two different numerical methods to compute the perturbed basin boundary and are thereby able to compute critical rates and instability thresholds.

Our framework for multiple time-scale systems in Chapter 2 extends and formalises preliminary ideas in Wieczorek et al [74], culminating in the proof of existence results. Our framework complements existing work on rate-induced bifurcations in single time-scale systems – Bishnani and MacKay [9] use uniform hyperbolicity, and Ashwin et al [2] use
5. Discussion

linear estimates of the system to compute the critical rate at which trajectories leave a prescribed safe region. Two other proposed frameworks for studying rate-induced bifurcations are to develop non-autonomous bifurcation theory using pullback attractors [53] or to develop finite-time bifurcation theory [54]. Our results for single time-scale systems in Chapter 4 have been influential in forming definitions and theorems for rate-induced bifurcations in non-autonomous bifurcation theory [3]. Similarly, the two multiple time-scale systems presented in Chapter 2 are taken as good examples of finite-time bifurcations.

This thesis principally studies rate-induced bifurcations in multiple time-scale systems in Chapter 2, with Chapter 3 investigating folded-saddle node (type I) singularities in order to explain the complexities of the banded instability threshold. Chapter 4 complements these results with motivating examples for rate-induced bifurcations in single time-scale systems.

With rate-induced bifurcations we have an instance of existing, esoteric mathematical theory – specifically canard trajectories in modern geometric singular perturbation theory – becoming important for real-world systems [5, 24, 62, 74, 72]. Moreover, the application of rate-induced bifurcations is driving the need to advance canard theory. For example, work in Chapter 2 and by the neuroscience community [49, 69, 65] has stimulated further study of folded saddle-node (type I) singularities in [70] and Chapter 3.

Our results in Chapter 2 gave sufficient conditions for multiple time-scale systems to have critical rates and instability thresholds. We showed that a rate-induced bifurcation corresponds to the appearance of a folded singularity, and which configurations of folded singularities give rise to instability thresholds. In particular, we revealed that near a folded saddle-node (type I) singularity, the instability threshold has an intricate band structure – which surprises experts. There are wide bands of initial states that destabilise, which are akin to the rotational sectors near a folded node, but distinctly different, as they are separated by narrow bands of initial states that track. This complicated banded instability threshold could arise in a range of different applications, as it occurs in two time-scale systems with a shift in external conditions from $\lambda_{\min}$ to $\lambda_{\max}$. What is more, we uncover new composite canard trajectories, formed from canard trajectories from two different folded singularities, which create part of the banded instability threshold.

There were insufficient results in the existing theory of canard trajectories to explain the complicated banded threshold we uncovered in Chapter 2, therefore we used numerical methods to explore the system further. Our findings were presented in Chapter 3.

Chapter 3 focused on the canard trajectories of a generic two time-scale system following the bifurcation of a folded saddle-node (type I) singularity into a folded saddle singularity and a folded node singularity. Analytical results for the existence and properties of canard trajectories near a folded saddle-node (type I) have recently been developed by Vo et al [70]. However, our results in Chapter 3 reveal further properties that are not captured by [70].

In Chapter 3 we publish a series of flow maps for a system with a folded node feeding into a folded saddle, different distances apart. The features of this flow map point to the
existence of another new canard trajectory we term a *tertiary canard trajectory*, which may be unique to systems near a folded saddle-node (type I) singularity.

We compute cross-sections of the attracting and repelling slow manifolds, showing how they twist around near a folded saddle-node (type I) singularity. Specifically, we reveal that the repelling slow manifold folds back on itself, which is not the case with other folded singularities [71, 16, 28, 15]. The folding back of the slow manifold creates the composite canard trajectories identified in Chapter 2.

In total there are five types of canard trajectory near a folded saddle-node (type I) singularity, two of which, the tertiary and composite canard trajectories, are novel, and not included in the existing theory [70]. These canard trajectories are found by numerical continuation in one parameter, having been predicted by the flow maps, and verified by cross-sections of the attracting and repelling slow manifolds.

This is the first study of canard trajectory bifurcations near a folded saddle-node (type I), and it reveals key ways in which the dynamics change with the two parameters: the internal time separation in the system $\delta$, and folded saddle-node bifurcation parameter $\epsilon$. We construct a two parameter bifurcation diagram which captures the behaviour when both parameters vary simultaneously. There are no other published two parameter bifurcation diagrams for any systems with folded singularities – Desroches et al [15] consider varying two parameters, $\mu$ and $\delta$, but do not show the effect of varying both at once.

Our two parameter bifurcation diagram reveals that there are three scaling laws for the canard trajectory bifurcations, two of which were unexpected, and the third corroborates an idea in [70] that, for $\epsilon$ away from $\epsilon_{FSN}$, the canard trajectory bifurcations are like those near an isolated folded node. Note, in general our study highlights that, for $\epsilon$ away from $\epsilon_{FSN}$, it is an over-simplification to treat the folded node as an isolated folded node as has been done in previous studies [70, 71].

Finally, our two parameter bifurcation diagram uncovers an exciting sprite feature – a dense region of canard trajectory bifurcations in $\delta-\epsilon$ space with an upward branching structure in $\delta$. Here there are many canard trajectories, generated by tightly twisted attracting and repelling slow manifolds. Moreover, the dynamics in this region are unusual, with some of the canard trajectories having a changing number of rotations as they are continued in $\delta$ or $\epsilon$. Globally, the sprite feature generates many of the bifurcation curves in $\delta-\epsilon$ space, and separates two regions of different behaviour – which possibly correspond to the near field and far field identified in [70].

Chapter 3 highlights that there is both further analytical and numerical work to be done to understand the folded saddle-node (type I) singularity.

Our study of the canard bifurcations can be used to inform future analytical theory for a folded saddle-node (type I) singularity. The existing theory does not capture canard bifurcations, nor describe the continual transition in dynamics as $\epsilon$ moves away from $\epsilon_{FSN}$, identifying only distinct near and far field behaviour [70]. Specifically, there are several features we identified in our system which require further study to establish whether they
are universal.

Firstly, a study incorporating composite and tertiary canard trajectories may reveal that they are only present in compactified systems, or that, as a tertiary canard does not interact with a folded saddle, it can arise in systems with only an isolated folded node. Secondly, we wish to determine whether any system following a type I folded saddle-node bifurcation, and satisfying the assumptions in Chapter 3, has exactly the three types of canard bifurcation we found in Chapter 3. Thirdly, the three scaling laws we identified require analytical proof. Finally, the sprite feature, which is both important for applications and mathematically intriguing, merits further research.

There are particular numerical challenges that should be addressed to deepen our understanding of folded saddle-node singularities, and of fast-slow systems in general. The results in Chapters 2 and 3 could potentially be derived within the framework developed in [38] for the study of three time-scale systems (with fast $x$, slow $y$ and super-slow $\lambda$). In particular, by re-writing and rescaling the system to an integrable form, it may be possible to find analytical formulations for the intersections of $S^a_\delta$ and $S^r_\delta$ with a cross-section lying along the fold, thus allowing for the analytical treatment of secondary, tertiary and composite canard trajectories following a folded saddle-node (type I) bifurcation. Can we construct the attracting and repelling slow manifolds $S^a_\delta$ and $S^r_\delta$ near a folded saddle-node as full surfaces (like in [16]) to be able to better understand their intersections? Possibly this can be done using a boundary value problem similar to the one in Chapter 3. More ambitiously, can we compute the concatenated canard-faux canard centre trajectory $\eta^C_\delta$, and other faux canard trajectories? Our results highlight the need to better understand $\eta^C_\delta$, as it appears to acts as an organising centre for the canard trajectories. This need has also been recognised in [70] amongst other places. Lastly, further development of numerical methods may overcome the computational limitation for continuing canard trajectories with a large number of rotations as $\delta \to 0$, and thereby extend results for the two parameter bifurcation diagram.

In Chapter 4 we turn our attention to single time-scale systems. We demonstrate that the instability threshold, the critical rate, and the rate-induced bifurcation, all depend on the perturbed basin boundary of the moving steady state. We compute the perturbed basin boundary for simple examples with external forcing given by an exponential approach or a logistic growth. The intention is that these motivating examples become canonical examples for rate-induced bifurcations in single time-scale systems and provide inspiration for proofs of general results. For example, [3] builds on the findings in Chapter 4 to develop a rigorous theory for rate-induced bifurcations for systems with one time-scale, $x \in \mathbb{R}$, and $\lambda \in \mathbb{R}$ as a generic parameter shift (that is, a bounded, biasymptotic function).

The first method we use to compute the perturbed basin boundary is to identify the slowest manifold. This directly uses our framework based on Fenichel’s Theorem, which assumes the external forcing is slower than the internal system dynamics so a moving steady state perturbs to a slow manifold.

We use the second method for systems with biasymptotic forcing, specifically a logistic
5. Discussion

growth forcing. We identify the perturbed basin boundary with the stable manifold of a saddle steady state. Moreover, we demonstrate that the critical rate corresponds to a heteroclinic connection between two saddle steady states, which we compute using Lin’s method. This enables us to immediately find the critical rate for systems with logistic growth forcing. Furthermore, using numerical continuation methods, we can find the dependence of the critical rate on the system parameters. Proof is required that this method works in general for systems with biasymptotic external forcing. This is particularly important for $x \in \mathbb{R}^n, n > 1$, as it is not obvious what conditions are required on the contraction rates in different directions.

In general, for single time-scale systems we propose a framework for studying rate-induced bifurcations that is based on Fenichel’s Theorem. That is, below the critical rate, the trajectories track a perturbed attracting steady state, which potentially has a basin boundary given by a perturbed repelling steady state. As the rate of forcing increases, the perturbed attracting steady state continues to perturb as long as it remains normally hyperbolic. In many cases, a rate-induced bifurcation corresponds to the loss of a perturbed attracting steady state—that is, when it stops being normally hyperbolic. As such, a rate-induced bifurcation could be regarded as the validity boundary of Fenichel’s Theorem. This is an appealing idea, as the normal hyperbolicity of a manifold can be found numerically, and potentially be used to for early warning signals for rate-induced bifurcations. Specifically, numerical methods could be written to compute generalised Lyapunov type numbers along a trajectory which give the normal hyperbolicity [26, 76]. However, there are some limitations—Fenichel’s Theorem is only applicable to finite (compact) intervals, and a loss of normal hyperbolicity may instead correspond to a loss of diffeomorphism of the perturbed attracting steady state.

Next, we consider combining the multiple time-scale results presented in Chapters 2 and 3, with the single time-scale results presented in Chapter 4. For example, we can consider the slow manifold of the multiple time-scale system as a single time-scale subsystem. So, when $\delta = 0$ and $\epsilon > 0$, within the slow manifold of the multiple time-scale system, the moving steady state $\tilde{x}(\lambda)$ perturbs to an attracting, slowest trajectory $\tilde{x}_\epsilon(\lambda)$ which nearby trajectories track. For a system with a folded saddle-node singularity, $\tilde{x}_\epsilon(\lambda)$ closely approximates the special centre trajectory $\eta^C$ introduced in Chapter 3. When $\delta > 0$, the trajectory $\tilde{x}_\epsilon(\lambda)$ is further perturbed by $\delta$. A combined framework may give further insight into the perturbation and stability of tracking trajectories in multiple time-scale systems, with respect to both small parameters $\delta$ and $\epsilon$ (see [38] for a framework for three time-scale systems). A combined framework could be tested on more geometrically complicated systems, for example with unstable steady states on the slow manifold, with bifurcating steady states, or with fold curves that do not span the full range of $\lambda$.

We now discuss ideas to further extend results presented in the whole of this thesis to study rate-induced bifurcations more effectively. In general, we want to study rate-induced bifurcations in one and multiple time-scale systems, for $x \in \mathbb{R}^n$ and various classes of external forcing functions $\lambda(\epsilon t) : \mathbb{R} \to \mathbb{R}^l$.

Much of the work presented in this thesis assumes a finite interval of time or external
forcing \( \lambda \). This is because, it is not clear what determines if a trajectory has destabilised if trajectories do not converge as \( t \to \infty \). Therefore, it is difficult to define rate-induced bifurcations for systems over infinite time.

There are “transient” rate-induced bifurcations where, above a critical rate, there is a sudden, large deviation from the moving steady state. Trajectories then return to the moving steady state, and may destabilise again. This occurs in excitable type III neurons [49] and the compost-bomb [47]. The framework presented in this thesis is not immediately applicable to these cases.

In many real world applications, following a rate-induced bifurcation, a trajectory may move to an alternative steady state not “too far” away from the moving steady state. In general, all theoretical frameworks for rate-induced bifurcations struggle to incorporate these “safe” rate-induced bifurcations, as the notion of “too far” is very application dependent [9, 74, 2, 3].

In this thesis we give examples of rate-induced bifurcations in a catalogue of simple dynamical systems. Further informative numerics could be done for these systems with different external forcing functions. For example, there are other types biasymptotic forcing functions that are common in applications – a pulse, a periodic oscillation, or shift with a local minimum or maximum – and many other forcing functions that are not biasymptotic.

We now consider what future expansions are needed for any framework for rate-induced bifurcations to be able to address a wider class of real world problems.

What would a suitable framework be for systems of partial differential equations? This extension would allow rate-induced bifurcations to be identified in spatially extended systems, for example in dramatically changing vegetation patterns [27], or a spatially extended compost-bomb model [47].

Can we compute rate-induced bifurcations in systems with noise? This could be by using stochastic analysis or a probabilistic approach [55]. Recently this has been researched for bifurcations induced-tipping points [7, 40, 41].

Can we identify early warning mechanisms for rate-induced bifurcations [55]? Early warning mechanisms for tipping points has been a policy relevant, hot topic of research over the last few years. As discussed in [2], tipping points may be caused by dynamic bifurcations or rate-induced bifurcations. Much attention has been given to identifying early warning mechanisms for dynamic bifurcations [13, 19, 66] – identifying a change in the stability of an underlying (deterministic) trajectory as the bifurcation is approached from changes in the real world noisy trajectory. Therefore, if a rate-induced bifurcation is associated with a loss of normal hyperbolicity, we have an equivalent change in stability of the underlying attracting perturbed stable state. For multiple time-scale systems with a folded slow manifold (see Chapter 2), a deeper understanding of canard trajectory stability, and the effect of noise is required. For example, neither Chapter 3, nor other studies of folded singularities [39, 71, 16, 70] discuss canard trajectory stability at all.
Lastly, can we find more examples of rate-induced bifurcations in real world systems? The results presented in this thesis characterise the types of system which we expect to see rate-induced bifurcations in. For example, the two time-scale system in Chapter 2 is, in essence, a system with fast (exponential) positive feedback, and slow negative feedback, inspired by the compost-bomb model [47]. A similar sort of feedback system could be useful in economic models for bubble bursts. We expect that systems with rate-induced bifurcations are all around us, however, because the phenomenon is counter-intuitive, it has proved difficult to identify. To this end, we need to communicate results about rate-induced bifurcations to the applied science community – particularly to the climate, ecological, and neuroscience communities where “tipping” is already of interest, but is principally understood as a bifurcation-induced, as that literature is more developed and that phenomenon is less counter intuitive.
Appendices
A. Supporting Theory

A.1. Fenichel’s Theorem

Our approach is based on Fenichel’s Theorem \[26, 25, 35\]. The key insight is, when the rate of change of the forcing is small, there exists an attracting solution near to the moving stable state \(\tilde{x}^a\). When the rate of change is too large, the attracting solution may cease to exist near \(\tilde{x}^a\). Therefore, the trajectory started at some initial state \(x_0\), may diverge from \(\tilde{x}^a\), and destabilise. Note, an initial state \(x_0\) will also destabilise if, above some rate of change, \(x_0\) is outside the basin of attraction for the attracting solution.

Fenichel’s Theorem addresses the existence of the attracting solution and the repelling solution which may form the basin boundary of the attracting solution. In taking this approach we directly compare the behaviour a trajectory to the moving stable state \(\tilde{x}^a\); and identify an intrinsic instability in the forced system, without imposing an ad hoc safe region.

Fenichel’s Theorem \[26, 25, 35\] is used to analyse the singular perturbation of systems with multiple time-scales. The fast \(x\) dynamics evolve on a time-scale \(t\), and slow \(y\) dynamics evolve on a time-scale \(\epsilon t\), where \(0 < \epsilon \ll 1\). Specifically, Fenichel’s Theorem shows that manifolds of hyperbolic equilibria for \(\epsilon = 0\) persist as nearby manifolds, with the same stability, for \(\epsilon > 0\) and sufficiently small. The basic system is

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y, \epsilon), \quad (A.1) \\
\frac{dy}{dt} &= \epsilon g(x, y, \epsilon), \quad (A.2)
\end{align*}
\]

where \(x \in \mathbb{R}^n\), \(y \in \mathbb{R}^m\), and functions \(f\) and \(g\) are \(C^r\), \(r \geq 1\). In the limit \(\epsilon = 0\), system (A.1)–(A.2) shows the behaviour of the fast \(x\) dynamics when the slow \(y\) dynamics are assumed to stand still. System (A.1)–(A.2) can be written with respect to the slow time-scale \(\tau = \epsilon t\),

\[
\begin{align*}
\frac{dx}{d\tau} &= f(x, y, \epsilon), \quad (A.3) \\
\frac{dy}{d\tau} &= g(x, y, \epsilon). \quad (A.4)
\end{align*}
\]

In system (A.3)–(A.4), in the limit \(\epsilon = 0\), the \(x\) dynamics are assumed to act infinitely fast. So the slow motion of \(y\) is constrained to the manifold \(\{f(x, y, 0) = 0\}\), which is the set of equilibria for Eq. (A.1) when \(\epsilon = 0\).
**Definition: Critical Manifold** [35]. A critical manifold $M$ is a compact subset of the hyperbolic equilibria of the fast variable $x$, that is a set of points where $f(x, y, 0) = 0$ and the eigenvalues of the Jacobian $\frac{df}{dx}(x, y, 0)$ have non-zero real part.

The stable and unstable manifolds of all the points in $M$ are denoted $W^s(M)$ and $W^u(M)$, respectively.

Fenichel’s Theorem states for sufficiently small $\epsilon > 0$, the critical manifold $M$ persists as a nearby slow manifold $M_\epsilon$ which has fast motion transverse to it, akin to (A.1)–(A.2) when $\epsilon = 0$, and slow flow along it, akin to (A.3)–(A.4) when $\epsilon = 0$.

**Theorem: Fenichel’s Theorem** [26, 25]. Suppose $M \subset \{ f(x, y, 0) = 0 \}$ is compact (possibly with a boundary) and composed of hyperbolic equilibria. Then there exists $\epsilon_\ast > 0$, such that for all $\epsilon \in [0, \epsilon_\ast)$, there exists a manifold $M_\epsilon$ that has the following properties:

(F1) $M_\epsilon$ is *locally invariant* under the flow of (A.1)–(A.2), i.e. solutions can only leave $M_\epsilon$ via a possible boundary inherited from $M$ [35, Def. 10].

(F2) $M_\epsilon$ is $O(\epsilon)$ close and diffeomorphic to $M$.

(F3) The stable and unstable manifolds of $M$ persist as stable and unstable manifolds of $M_\epsilon$, denoted $W^s(M_\epsilon)$ and $W^u(M_\epsilon)$. These are locally invariant under the flow of (A.1)–(A.2), and $O(\epsilon)$ close and diffeomorphic to $W^s(M)$ and $W^u(M)$, respectively.

(F4) $M_\epsilon$ retains the *same attractivity* as $M$. That is, solutions started on $W^s(M_\epsilon)$ converge exponentially fast to $M_\epsilon$, and solutions started on $W^u(M_\epsilon)$ diverge exponentially fast from $M_\epsilon$, for as long as they remain near $M_\epsilon$.

(F5) $M_\epsilon$ is $C^{r-1}$, with respect to $x$, $y$, and $\epsilon$.

Commonly, in applications of Fenichel’s Theorem, $\epsilon$ is said to be sufficiently small, but no maximal value for $\epsilon_\ast$ is given. This may be because no maximum exists, or because it is difficult to determine. In systems where rate-induced bifurcations occur, there typically is a maximal value of $\epsilon_\ast$, and it can be identified. We term the maximal value of $\epsilon_\ast$ the *validity boundary* of Fenichel’s Theorem.

The validity boundary of Fenichel’s Theorem and the critical rate for the rate-induced bifurcation are closely related. Consider system (A.1)–(A.2) with a critical manifold $M$ consisting of stable hyperbolic equilibria. Note, for all $\epsilon$ any solution to (A.1)–(A.2) is $C^r$ and locally invariant, and, if $M$ is stable, then $W^s(M)$ is locally all of $\mathbb{R}^n$, so (F1), (F3), and (F5) hold. Past the validity boundary at least one of properties (F2) and (F4) in Fenichel’s Theorem do not hold. That is, system (A.1)–(A.2) no longer has a solution near $M$ that is diffeomorphic to $M$, and attractive. Moreover, assume in the area $O(\epsilon)$ close to $M$ all the solutions to (A.1)–(A.2) are diffeomorphic to $M$, then, there is *no attractive solution near to $M$*. If the solutions to (A.1)–(A.2) cannot converge to some attracting solution near to $M$, they cannot track $M$ – we have exceeded the critical rate.
A. Supporting Theory

More generally Fenichel’s Theorem is expressed for a normally hyperbolic manifold [26, 76, 35]. Normally hyperbolic manifolds are more general than curves of hyperbolic equilibria – they are invariant manifolds with the rates of contraction and expansion normal to the manifolds dominating the rates of contraction and expansion within the manifold.
A. Supporting Theory

A.2. Detailed equations to compute heteroclinic connection using Lin’s method

The equations below are required to use Lin’s method to compute the heteroclinic con-
nection between two points \( p \) and \( s \), or a point \( p \) and periodic orbit \( \Gamma \). These are a
simplification of the general equations given in [37]. The AUTO code for the heteroclinic
connection between a point and periodic orbit is given in Appendix B.5.1.

Let \( u \in \mathbb{R}^m \) denote the variables \( x \in \mathbb{R}^n, \lambda \in \mathbb{R}^l \), \( m = n + l \), and \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m \) denote the
governing dynamics, with some parameter \( \epsilon \):

\[
\dot{u} = f(u, \epsilon). \tag{A.5}
\]

Let \( \Sigma \) denote the Lin’s Section which is an \( m - 1 \) dimensional plane splitting the \( u \) phase
space, such that \( p \) and \( s \), or \( p \) and \( \Gamma \), are in different portions. The section \( \Sigma \) contains
point \( p_\Sigma \), has unit normal \( n_\Sigma \), and has \( i = 1...m - 1 \) orthonormal Lin vectors \( z_i \) that lie
in \( \Sigma \).

The unstable manifold from the critical point \( p \) to the section \( \Sigma \) is denoted \( u^{-}(t) \).
The stable manifold from the section \( \Sigma \) to the critical point \( s \) or periodic orbit \( \Gamma \) is denoted
\( u^{+}(t) \).

Let \( \eta_i \) denote the difference along \( z_i \) between \( u^{-}(t) \) and \( u^{+}(t) \) in \( \Sigma \).

The equations below formulate Lin’s method as a boundary value problem. Once a solution
has been found, continue it in AUTO, varying \( \epsilon \) until \( \eta_i = 0 \) for all \( i = 1...m - 1 \). In this
case, \( u^{-}(t) \) and \( u^{+}(t) \) meet in \( \Sigma \), and we have a heteroclinic connection between \( p \) and \( s \),
or \( p \) and \( \Gamma \).

A.2.1. Point to point

For \( u \in \mathbb{R}^2 \) and critical points \( p \) and \( s \).

Equation to find the critical point \( p \):

\[
f(p, \epsilon) = 0. \tag{A.6}
\]

Equation to find the critical point \( s \neq p \):

\[
f(s, \epsilon) = 0. \tag{A.7}
\]

Assume the eigenvectors for the system at \( s \) and \( p \) are known and are given by \( e_s \) and \( e_p \).
Equations for the unstable trajectory $u^-(t)$ from the critical point $p$ to the section $\Sigma$:

\[
\dot{u}^-(t) = T^- f(u^-(t), \epsilon),
\]

\[
u^-(0) = p + \varepsilon_pe_p,
\]

\[
\langle u^-(1) - p\Sigma, n\Sigma \rangle = 0.
\]  

(A.8)  
(A.9)  
(A.10)

The problem is written as a boundary value problem with the time rescaled from $T^-$ to 1. The start point of $u^-(t)$ is a small distance $\epsilon_p$ from $p$ along the unstable eigenvector $e_p$. The end point of $u^-(t)$ is in $\Sigma$, so $(u^-(1) - p\Sigma)$ is perpendicular to unit normal $n\Sigma$.

Likewise, equations for the stable trajectory $u^+(t)$ from the section $\Sigma$ to the critical point $s$:

\[
\dot{u}^+(t) = T^+ f(u^+(t), \epsilon),
\]

\[
u^+(1) = s + \varepsilon_se_s,
\]

\[
(u^+(0) - u^-(1)) = \eta z.
\]  

(A.11)  
(A.12)  
(A.13)

The problem is written as a boundary value problem with the time rescaled from $T^+$ to 1. The end point of $u^+(t)$ is a small distance $\epsilon_s$ from $s$ along the stable eigenvector $e_s$. The start point of $u^+(t)$ is in $\Sigma$ a distance $\eta$ away from $u^-(1)$ along the Lin vector $z$.

**A.2.2. Point to periodic orbit**

For $u \in \mathbb{R}^3$, critical point $p$ and periodic orbit $\Gamma$.

Equation to find the critical point $p$:

\[
f(p, \epsilon) = 0.
\]  

(A.14)

Equations to find the periodic orbit, $\Gamma$, described by the trajectory $u_\gamma$:

\[
\dot{u}_\gamma(t) = T_\gamma f(u_\gamma(t), \epsilon),
\]

\[
u_\gamma(0) = u_\gamma(1),
\]

\[
\int_0^1 \langle \dot{u}_\gamma(\tau), u_\gamma(\tau) \rangle d\tau = 0.
\]  

(A.15)  
(A.16)  
(A.17)

Finding a periodic orbit is formulated as a boundary value problem with the period adjusted from $T_\gamma$ to 1, a periodic boundary condition, and phase condition [22].

Equations to determine the eigenfunctions $e_\gamma(t)$ for the periodic orbit:

\[
\dot{e}_\gamma(t) = T_\gamma D_u f(u_\gamma(t), \epsilon)e_\gamma(t) - \mu e_\gamma(t),
\]

\[
e_\gamma(0) = e_\gamma(1),
\]

\[
\langle e_\gamma(0), e_\gamma(0) \rangle = 1.
\]  

(A.18)  
(A.19)  
(A.20)
Again this is expressed as a boundary value problem with the orbit period adjusted from $T_\gamma$ to 1. The variable $\mu$ is the stable Floquet exponent. Note, it is standard notation to use $\mu$ for the Floquet multiplier, but here we use it for the exponent. Equation (A.19) is a periodic boundary condition for the eigenfunction, and Eq. (A.20) normalises the eigenfunction. The equations are equivalent to the alternative equations for the boundary value problem given in [23].

Equations for the unstable manifold $u^-(t)$ from the critical point $p$ to the section $\Sigma$:

\[ \dot{u}^-(t) = T^- f(u^-(t), \epsilon), \quad (A.21) \]
\[ u^-(0) = p + \epsilon p e_p, \quad (A.22) \]
\[ \langle u^-(1) - p_\Sigma, n_\Sigma \rangle = 0. \quad (A.23) \]

The problem is written as a boundary value problem with the time rescaled from $T^-$ to 1. The start point of $u^-(t)$ is a small distance $\epsilon_p$ from $p$ along the unstable eigenvector $e_p$. The end point of $u^-(t)$ is in $\Sigma$ so $(u^-(1) - p_\Sigma)$ is perpendicular to unit normal $n_\Sigma$.

Equations for a trajectory $u^+$ in the stable manifold of the periodic orbit $u_\gamma(t)$ are formulated as a boundary value problem starting at the section $\Sigma$ and ending near a a point $u_\gamma(0) \in \Gamma$:

\[ \dot{u}^+(t) = T^+ f(u^+(t), \epsilon), \quad (A.24) \]
\[ u^+(1) = u_\gamma(0) + \epsilon_\gamma e_\gamma(0), \quad (A.25) \]
\[ (u^+(0) - u^-(1)) = \eta_1 z_1 + \eta_2 z_2. \quad (A.26) \]

The problem is written as a boundary value problem with the time rescaled from $T^+$ to 1. The end point of $u^+(t)$ is a small distance $\epsilon_\gamma$ from $u_\gamma(0)$ along the stable eigenfunction $e_\gamma(0)$. The start point of $u^-(t)$ is in $\Sigma$, so distances $\eta_i$ away from $u^-(1)$ along the Lin vector $z_i, i = 1, 2$. 
A.3. Analytically calculating the unstable Floquet multiplier

By using polar co-ordinates and a Poincare map we can analytically calculate the Floquet multiplier for the periodic orbit $\Gamma_{\text{max}}$ in system (4.22)–(4.23) in the $\{\lambda = \lambda_{\text{max}}\}$ plane. This follows the example in Guckenheimer [30, pg 23-25].

Since the dynamics collapse onto the $\{\lambda = \lambda_{\text{max}}\}$ plane we will just consider $\text{Re}(z), \text{Im}(z)$ behaviour at $\lambda = \lambda_{\text{max}}$. Translate the system so $(\lambda_{\text{max}}, 0)$ is at the origin. Let $\bar{z} = z - \lambda_{\text{max}}$:

$$\dot{\bar{z}} = -a\bar{z} + i\omega\bar{z} + |\bar{z}|^2\bar{z}. \quad (A.27)$$

By expressing $\bar{z} = re^{i\theta}$ and equating real and imaginary parts of $\dot{\bar{z}}$ we get the system in polar coordinates,

$$\dot{r} = -ar + r^3, \quad (A.28)$$
$$\dot{\theta} = \omega. \quad (A.29)$$

When $a = r^2$ we have a periodic orbit as expected. By integrating (A.28) we obtain an equation for $r(t)$. Note,

$$\frac{1}{-ar + r^3} = \frac{1}{r(r - \sqrt{a})(r + \sqrt{a})} = \frac{1}{2a} \left( -\frac{1}{2r} + \frac{1}{r - \sqrt{a}} + \frac{1}{r + \sqrt{a}} \right).$$

So,

$$\int_{r(0)}^{r(t)} \frac{1}{r} dr = \left. \frac{1}{2a} \left( \log \left( \frac{r^2 - a}{r^2} \right) \right) \right|_{r(0)}^{r(t)} = \left. \frac{1}{2a} \left( \log \left( 1 - \frac{a}{r^2} \right) \right) \right|_{r(0)}^{r(t)} = t. \quad (A.30)$$

Giving,

$$\frac{a}{r^2(t)} = 1 - e^{2at} \left( 1 - \frac{a}{r^2(0)} \right). \quad (A.33)$$

Thus,

$$r(t) = \sqrt{a} \left[ 1 - e^{2at} \left( 1 - \frac{a}{r^2(0)} \right) \right]^{-\frac{1}{2}}, \quad (A.34)$$

where $r(0) = r_0$.

Thus the trajectory, $\phi_t$, for (A.28)–(A.29) given starting conditions $r(0) = r_0, \theta(0) = \theta_0$ is,

$$\phi_t(r_0, \theta_0) = \left( \sqrt{a} \left[ 1 - e^{2at} \left( 1 - \frac{a}{r^2(0)} \right) \right]^{-\frac{1}{2}}, \omega t + \theta_0 \right). \quad (A.35)$$

We want to find the Poincare map for (A.28)–(A.29). Consider the section $\Sigma := \{\theta = 0\}$. 

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Table A.1. Value for unstable Floquet multiplier for periodic orbit $\Gamma_{\text{max}}$ for system (4.22)–(4.23) with $\lambda_{\text{max}} = 1$, $\omega = 0.5$ for different $a$ returned by AUTO, and correct values from analytical result computed in MATLAB.

<table>
<thead>
<tr>
<th>$a$</th>
<th>AUTO</th>
<th>Correct Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000e-04</td>
<td>1.00252</td>
<td>1.0025</td>
</tr>
<tr>
<td>2.48264e-02</td>
<td>1.86630</td>
<td>1.8663</td>
</tr>
<tr>
<td>6.01186e-01</td>
<td>3.64710e+06</td>
<td>3.6471e+06</td>
</tr>
<tr>
<td>9.52785e-01</td>
<td>2.51000e+10</td>
<td>2.5099e+10</td>
</tr>
<tr>
<td>1.15758</td>
<td>4.31566e+12</td>
<td>4.3152e+12</td>
</tr>
<tr>
<td>1.50584</td>
<td>9.75335e+15</td>
<td>2.7306e+16</td>
</tr>
<tr>
<td>2.00000</td>
<td>4.74450e+17</td>
<td>6.7612e+21</td>
</tr>
</tbody>
</table>

The Poincare map $P: \Sigma \rightarrow \Sigma$ for $q \in \Sigma$ is $P(q) = \phi_{\tau}(q)$ where $\tau$ is the first return time. The time of flight for the orbit is $\frac{2\pi}{\omega}$, thus

$$P(r_0) = \sqrt{a} \left[ 1 - e^{\frac{4\pi a}{\omega}} \left( 1 - \frac{a}{r_0^2} \right) \right]^{-\frac{1}{2}}.$$ (A.36)

There is a fixed point, i.e. $P(r_0) = r_0$, when $r_0 = \sqrt{a}$ as expected, which corresponds to our periodic orbit. $P$ is a one-dimensional map. We find the stability of the fixed point, i.e. our periodic orbit, by considering the derivative of the map evaluated at $r_0 = \sqrt{a}$.

$$DP = \frac{dP}{dr_0} = -\sqrt{a} \frac{1}{2} \left[ 1 - e^{\frac{4\pi a}{\omega}} \left( 1 - \frac{a}{r_0^2} \right) \right]^{-\frac{3}{2}} \left( -2ae^{\frac{4\pi a}{\omega}} \frac{r_0^3}{r_0^3} \right).$$ (A.37)

$$DP(\sqrt{a}) = \sqrt{a} [1 - 0]^{-\frac{3}{2}} \left( \frac{ae^{\frac{4\pi a}{\omega}}}{a^2} \right)$$ (A.38)

$$= e^{\frac{4\pi a}{\omega}}.$$ (A.39)

Thus $r_0 = \sqrt{a}$ is a fixed point with Floquet multiplier $e^{\frac{4\pi a}{\omega}}$, so the fixed point is unstable if $\frac{4\pi a}{\omega} > 0$, i.e. for all $a > 0$ and $\omega > 0$. Moreover, the unstable Floquet multiplier grows exponentially as $a$ increases.

A.3.1. Comparing with the numerics

The values in Table A.1 were returned by AUTO for $\Gamma_{\text{max}}$ in system (4.22)–(4.23) for the unstable Floquet multiplier at $\lambda_{\text{max}} = 1$ for $\omega = 0.5$, and varying $a \in (0.0, 2.0)$. The step size was variable with DSMIN=0.001 and DSMAX=0.5, and the step size increased with increasing $a$ from 0.015 to 0.11. Also shown in Table A.1 is the value of $e^{\frac{4\pi a}{\omega}}$

for increasing $a$ calculated in MATLAB.
Figure A.1. Continuing the heteroclinic connection in system (4.22)–(4.23) for $a = 1$, $\lambda_{\text{max}} = 1.5, 2, 2.5$ and varying $\omega$ and $\epsilon$. When $\omega$ is near 0 the unstable Floquet exponent (green curve) is too big for AUTO to compute accurately (see Table A.1). AUTO continuation is therefore only done for $|\omega| > 0.5$ (dashed lines).

As can be seen from Table A.1, the Floquet multiplier rapidly increases with $a$. AUTO behaves well for $\omega = 0.5$, until $a > 1$ when Floquet multipliers become very big and AUTO struggles to approximate them well [48]. Thus, computations should be restricted to smaller values of $a$, and larger values of $\omega$.

Figure A.1 shows results for continuing the heteroclinic connection in system (4.22)–(4.23) for $a = 1$ against $\omega$. Note, as $\omega$ approaches 0, the Floquet exponent $4\pi/\omega$ becomes infinite, so the AUTO routine cannot be used to find the heteroclinic connection for small $\omega$. Instead, for $\omega \in (-0.5, 0.5)$ the heteroclinic connection is found by a shooting method in MATLAB (see code in Appendix B.3.1).
A.4. Lyapunov exponents to show loss of attractivity

We compute the Lyapunov exponents along the heteroclinic connection between the two saddle steady states in system (4.14)–(4.15). This shows the loss of attractivity of the perturbed steady state $\tilde{x}_a^\epsilon$ when it becomes a heteroclinic connection. More generally, there are the related generalised Lyapunov type numbers [26, 76], which would show the loss of normal hyperbolicity of a perturbed steady state $\tilde{x}_a^\epsilon$.

Suppose
\[
\dot{x} = f(x),
\] (A.40)

where $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $x^*(t)$ be a trajectory of system (A.40). We want to know the stability of this trajectory. Consider what happens to a nearby trajectory – a small perturbation $\xi(t_0)$ is made at time $t_0$, the trajectory $x(t)$ is started from $x^*(t_0) + \xi(t_0)$.

Let $\xi(t)$ denote the evolved difference in position, $x(t) - x^*(t)$.

Using Taylor’s expansion:
\[
\begin{align*}
\dot{\xi} &= f(x(t)) - f(x^*(t)) \\
&= f(x^*(t) + \xi(t)) - f(x^*(t)) \\
&= f(x^*(t)) + Df(x^*(t))\xi(t) + ... - f(x^*(t)) \\
&= Df(x^*(t))\xi(t) + ... \\
\end{align*}
\] (A.44)

Assuming $\xi(t)$ is small,
\[
\dot{\xi} = Df(x^*(t))\xi(t).
\] (A.45)

The initial perturbation $\xi(t_0)$ could be in any of $n$ directions, to solve for these simultaneously consider an $n \times n$ matrix $M(t)$,

\[
\dot{M}(t) = Df(x^*(t))M(t),
\] (A.46)

with $M(t_0) = I$. Note, generally equation (A.46) cannot be solved explicitly. To recover the equation for any $\xi(t)$, solve

\[
\xi(t) = M(t)\xi(0).
\] (A.47)

The eigenvalues $m_i(t)$ of $M(t)$ give the dominant directions of motion at time $t$. Conventional ordering is $|m_1(t)| \geq |m_2(t)| \geq ... |m_n(t)|$.

The Lyapunov exponents $l_i$ are real numbers quantifying the average exponential rate of contraction or expansion about $x^*(t)$ as $t \to \infty$:
\[
l_i = \lim_{t \to \infty} \frac{1}{t - t_0} \log |m_i(t)|.
\] (A.48)

Since the Lyapunov exponents give the average rate of contraction or expansion, $l_i < 0$ indicates stable behaviour, and $l_i > 0$ unstable behaviour. For example, if $x^*(t)$ is a critical point then the Lyapunov exponents are the eigenvalues.
A. Supporting Theory

Important properties of the Lyapunov exponents are:

- Along any trajectory the Lyapunov exponents are constant, this is because given any \( t_0 \), as \( t \to \infty \) the same behaviour is recovered.

- For any \( x^\ast(t) \) that is not a fixed point there is at least one zero Lyapunov exponent corresponding to the direction of motion along the trajectory.

- A Lyapunov exponent may have an associated eigenvector, recovered from the eigenvectors of \( M(t) \). If it exists, the eigenvectors show the directions of exponential contraction and expansion for the perturbation \( \xi(t) \) after infinite time.

The following lemmas can help simplify calculations of \( M(t) \).

**Lemma 1.** If \( A = VDV^{-1} \) then \( e^A = Ve^DV^{-1} \).

**Lemma 2.** If \( D \) is diagonalisable then \( e^{D_{ij}} = e^{D_{ij}} \).

**Lemma 3.** If \( Df(x^\ast(t)) \) is of the form \( g(t) \cdot J \) where \( g \) is a continuous, bounded scalar function and \( J \) is an \( n \times n \) matrix, then

\[
M(t) = M(t_0) \exp \left[ \int_{t_0}^{t} g(t) dt \cdot J \right].
\]

**Proof:** Assume there is some function \( \Omega(t,t_0) \) such that \( M(t) = M(t_0) \exp \left[ \Omega(t,t_0) \right] \). Differentiate, then \( \dot{M}(t) = \frac{d}{dt} \Omega(t,t_0)M(t) \). So \( \frac{d}{dt} \Omega(t,t_0) = g(t) \cdot J \) and under conditions in the lemma, \( \int_{t_0}^{t} g(t) dt \cdot J \) is well defined, so take \( \Omega(t,t_0) = \int_{t_0}^{t} g(t) dt \cdot J \).

### A.4.1. For the stable and unstable node system with logarithmic growth

We restate system (4.14)–(4.15) here for completeness:

\[
\begin{align*}
\dot{x} &= (x + \lambda)^2 - \mu, \\
\dot{\lambda} &= \rho \lambda (\lambda_{\text{max}} - \lambda),
\end{align*}
\]

where \( \rho = 2\epsilon/\lambda_{\text{max}} \) for brevity.

At the critical rate \( \epsilon_c \) the perturbed steady state becomes a heteroclinic connection. We denote the heteroclinic connection \( x^\ast = \tilde{x}_{\epsilon_c}^a \) (Eq. (4.16)):

\[
x^\ast(\lambda(t)) = \left(-1 + \frac{2\sqrt{\rho}}{\lambda_{\text{max}}} \right) \lambda(t) - \sqrt{\mu}.
\]

The critical rate (Eq. (4.17)) can be written in terms of \( \rho \):

\[
\rho = \frac{4\mu}{\lambda_{\text{max}}(\lambda_{\text{max}} - 2\sqrt{\rho})}.
\]

We want to find the stability of the heteroclinic connection. Unusually the matrix perturbation equation, (A.46), can be solved explicitly in this case using properties given by
Lemmas 1–3. Linearise (A.49)–(A.50) about $x^*(\lambda(t))$,

$$
Df(x^*(t)) = \begin{bmatrix}
2(x + \lambda) & 2(x + \lambda) \\
0 & \frac{4\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}} \left(1 - \frac{2}{\lambda_{\text{max}}} \lambda \right)
\end{bmatrix}_{x=x^*(\lambda(t))} \quad (A.53)
$$

$$
= \left(1 - \frac{2}{\lambda_{\text{max}}} \lambda(t) \right) \begin{bmatrix}
-2\sqrt{\mu} & -2\sqrt{\mu} \\
0 & \frac{4\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}}
\end{bmatrix} \quad (A.54)
$$

$$
= 2\sqrt{\mu} \left(1 - \frac{2}{\lambda_{\text{max}}} \lambda(t) \right) \begin{bmatrix}
-1 & -1 \\
0 & \frac{1}{\lambda_{\text{max}} - 2\sqrt{\mu}}
\end{bmatrix} \quad (A.55)
$$

The matrix $Df(x^*(t))$ has time varying eigenvalues, $-2\sqrt{\mu}(1 - \frac{2}{\lambda_{\text{max}}} \lambda(t))$, $\frac{4\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}}(1 - \frac{2}{\lambda_{\text{max}}} \lambda(t))$, and can be diagonalised using the time-invariant eigenvectors $(1, 0)$, $(\lambda_{\text{max}} - 2\sqrt{\mu}, \lambda_{\text{max}})$. Thus, the eigenvalues of $M(t)$ are

$$
m_1 = \exp \left[-2\sqrt{\mu} \int_{t_0}^{t} \left(1 - \frac{2}{\lambda_{\text{max}}} \lambda(s) \right) ds \right],
$$

$$
m_2 = \exp \left[\frac{4\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}} \int_{t_0}^{t} \left(1 - \frac{2}{\lambda_{\text{max}}} \lambda(s) \right) ds \right].
$$

with eigenvectors $(1, 0)$ and $(\lambda_{\text{max}} - 2\sqrt{\mu}, \lambda_{\text{max}})$. The direction $(\lambda_{\text{max}} - 2\sqrt{\mu}, \lambda_{\text{max}})$ is the direction of motion along the heteroclinic connection trajectory, (see Eq. (A.51)). The eigenvalue $m_1$ with eigendirection $(1, 0)$ describes the behaviour normal to the heteroclinic connection. Hence, $m_1$ characterises the stability of the heteroclinic connection.

Since

$$
\lambda(s) = \frac{\lambda_{\text{max}}}{2} \tanh \left(\frac{2\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}} s \right) + \frac{\lambda_{\text{max}}}{2}, \quad \text{where } \lambda(0) = \lambda_{\text{max}}/2,
$$

$$
\int_{t_0}^{t} \left(1 - \frac{2}{\lambda_{\text{max}}} \lambda(s) \right) ds = -\int_{t_0}^{t} \tanh \left(\frac{2\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}} s \right) ds \quad (A.56)
$$

$$
= -\frac{\lambda_{\text{max}} - 2\sqrt{\mu}}{2\mu} \log \cosh \left(\frac{2\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}} s \right) \bigg|_{t_0}^{t} \quad (A.57)
$$

$$
= -\frac{\lambda_{\text{max}} - 2\sqrt{\mu}}{2\mu} \log \left(\frac{\cosh \left(\frac{2\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}} t \right)}{\cosh \left(\frac{2\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}} t_0 \right)} \right). \quad (A.58)
$$

Let

$$
c(s) = \cosh \left(\frac{2\mu}{\lambda_{\text{max}} - 2\sqrt{\mu}} s \right).
$$

Therefore, the Lyapunov exponents are given by

$$
l_1 = \lim_{t \to \infty} \frac{2\sqrt{\mu} - \lambda_{\text{max}}}{t - t_0} \log \left(\frac{c(t)}{c(t_0)} \right),
$$
\[ l_2 = \lim_{t \to \infty} \frac{2}{t - t_0} \log \left( \frac{c(t)}{c(t_0)} \right). \]

As \( t \to \infty \), \( c(t) > c(t_0) \). So, for any start time \( t_0 \), \( l_2 \to 0 \), as is expected for the Lyapunov exponent along the direction of motion of the trajectory.

Consider \( l_1 \) which gives the transverse stability of the heteroclinic connection. If we suppose the heteroclinic connection is pinned at \( \lambda(0) = \lambda_{\text{max}}/2 \), and take balanced limits with \( t_0 = -r \) and \( t = r \), then \( l_1 = 0 \). As we take this balanced limit \( r \to \infty \) we describe the transverse stability of the whole heteroclinic connection \( x^* \). This shows that the heteroclinic connection \( x^* \) has some loss of globally averaged transverse stability. It may be possible to extend this idea to show that as the perturbed steady state \( \tilde{x}_\epsilon^a \) approaches the heteroclinic connection there is a gradual loss of attractivity of \( \tilde{x}_\epsilon^a \).
B. Computer Programs

This Appendix contains a selection of the computer codes which I wrote to find the numerical results detailed in this thesis.

B.1. MATLAB codes for Chapter 2

B.1.1. Functions for equations and Jacobian for fast-slow system

These functions support the codes given Subsections B.1.2 and B.2.1.

System (2.17)–(2.18), equivalently system (3.18)–(3.20), can be written as

\[
\begin{align*}
\delta \frac{dx}{dt} &= x(x-1) + y + \lambda, \quad (B.1) \\
\frac{dy}{dt} &= -x, \quad (B.2) \\
\frac{d\lambda}{dt} &= \varepsilon \bar{\lambda}(\Delta - \bar{\lambda}). \quad (B.3)
\end{align*}
\]

where \( \lambda_{\text{max}} = A, \varepsilon = \epsilon/A, \bar{\lambda} = \lambda + \sqrt{A}, \Delta = 2\sqrt{A} \).

This code use a MATLAB ordinary differential equation solver \texttt{ode15s} with a specified Jacobian, which is suitable for stiff systems, see [59, Sec. 3] for detail.

Equations (B.1)–(B.3) in forward time – fs_fullsys.m

```matlab
function fp = fs_fullsys_bwd(t, traj, epsilon, Delta, delta)
    x = traj(1);
    y = traj(2);
    lambda = traj(3);
    % shift in lambda
    xp = 1/delta*(y + lambda + x*(x-1));
    yp = -x;
    lambdap = epsilon*lambda*(Delta - lambda);
    fp = [xp; yp; lambdap];
    return
```
Equations (B.1)–(B.3) in backward time – fs_fullsys_bwd.m

```matlab
function fp = fs_fullsys_bwd(t, traj, epsilon, Delta, delta)
    x = traj(1);
y = traj(2);
lambda = traj(3);

%shift in lambda
xp = -1/delta * (y + lambda + x*(x-1));
yp = x;
lambdap = -epsilon*lambda*(Delta - lambda);
fp = [xp; yp; lambdap];
return
```

Jacobian in forward time – fs_fullsys_jac.m

```matlab
function J = fs_fullsys_jac(t, traj, epsilon, Delta, delta)
    J = [1/delta*(2*traj(1)-1) 1/delta 1/delta
          -1 0 0
          0 0 0 epsilon*(Delta-2*traj(3))];
```

Jacobian in backward time – fs_fullsys_bwd_jac.m

```matlab
function J = fs_fullsys_bwd_jac(t, traj, epsilon, Delta, delta)
    J = -1*[1/delta*(2*traj(1)-1) 1/delta 1/delta
           -1 0 0
           0 0 0 epsilon*(Delta-2*traj(3))];
```

Termination event – trajevent.m

```matlab
function [value, isterminal, direction] = trajevent(t, traj)
    value = traj(1) - 1.5; % when intersects x=1.5
    isterminal = 1; % stop the integration
    direction = 0;
```

B.1.2. Which initial states on the critical manifold destabilise – does_it_tip_for_ic.m

This code produces the results shown in Fig. 2.3. Where the instability threshold is below computational resolution, it is inferred as the boundary between regions with trajectories with different numbers of oscillations.
If running at typical resolution, recommend splitting into 4 sections in parallel:

**SECTIONS**

**SECTION 1:** x0 from -0.1 to 0.3, lambda0 0 to 2.5

**SECTION 2:** x0 from 0.302 to 0.7, lambda0 0 to 2.5

**SECTION 3:** x0 from -0.1 to 0.3, lambda0 2.502 to 5

**SECTION 4:** x0 from 0.302 to 0.7, lambda0 2.502 to 5

format long

turn off annoying warnings
warning('off', 'MATLAB:ode15s:IntegrationTolNotMet')
warning('off', 'signal:findpeaks:noPeaks')

eps_new = 0.216

% output file
create folder delta01/eps_new216 before running code
filename = ['delta01/eps_new', num2str(eps_new*1000,'%03d'), '/doesittip ']

% choice of params (usually fixed)
delta = 1e-2;
Delta = 5;
eps = eps_new * (2/Delta)
error_tol = 1e-10

x0_vector = [0, 0.5];
% x0_vector = -0.1:0.002:0.7; % all (this takes around 100 hours on 1 core)
% x0_vector = -0.1:0.002:0.3;
% x0_vector = 0.302:0.002:0.7;

lambda0 = 1.1:0.01:1.6;
% % lambda0 = 0:0.002:5; % all
% lambda0 = 0:0.002:0.3; % 2.5;
% lambda0 = 2.502:0.002:5;

time = 0:0.01:10;

options = odeset('Jacobian', @(t, traj) f_sys.fullysys_jac(t, traj, eps, Delta, delta), 'RelTol', error_tol, 'AbsTol', error_tol);

ll = length(lambda0);
lx = length(x0_vector);

% matrix of 1s or 0s, 1 if tips.
clear doesittip

doesittip = uint8(zeros(ll, lx)); % assume nothing tips and all zero

% colors if plotting traj
mycolor = ['c', 'm', 'g', 'k', 'r', 'b'];
chcol = ceil(ll/6 + 1) % when to change color

mytrajfig = figure % put here if want all plots on same figure
hold on

% initialize other data matrices
failpoint = 10; % smallest x coord of trajectories with bad data (int time too short)
mintip = 9999; % min x coord of trajectories which tip
data = {}; % max x coord of trajectories which don't tip
nopeaks = uint8(99+ones(ll, lx)); % number of peaks

troughearly = uint8(NaN(ll, lx));
heightpeak1 = single(NaN(ll, lx));
heightpeak2 = single(NaN(ll, lx));
heightpeak3 = single(NaN(ll, lx));
heightpeak4 = single(NaN(ll, lx));
depthtrough1 = single(NaN(ll, lx));
depthtrough2 = single(NaN(ll, lx));
depthtrough3 = single(NaN(ll, lx));
lambdapeak1 = single(NaN(ll, lx));
lambdapeak2 = single(NaN(ll, lx));
lambdapeak3 = single(NaN(ll, lx));
lambdapeak4 = single(NaN(ll, lx));
lambdatrough1 = single(NaN(ll, lx));
lambdatrough2 = single(NaN(ll, lx));
lambdatrough3 = single(NaN(ll, lx));

for xi = 1:length(x0_vector)
    x0 = x0_vector(xi)
    % code to run
end
for li = 1:11
y0 = lambda0 - x0*(x0-1);
clear traj peaks locp lp troughs loc t t
[t, traj] = ode15s(@(t, traj) fs_fullsys(t, traj, eps, Delta, delta), time, [x0, y0(li), lambda0(li)], options);

% Flag up dubious points which don't integrate all the way and stop near fold
% Occurs for trajectories started near lambda0=Delta/2 and small epsilon
% Need to be re-run with longer integration time.
if (length(traj) < length(time)) % guaranteed to break down when tips as too stiff
    [t, traj] = ode45(@(t, traj) fullsys(t, traj, eps, Delta, delta), time, [x0, y0(li), lambda0(li)], options);
end

% Use built-in Matlab function to get x and lambda location of peaks
[peaks, locp] = findpeaks(traj(:,1));
lp = length(peaks);

% Use built-in Matlab function to get x and lambda location of troughs
[troughs, loc t t] = findpeaks(-traj(:,1));
l t = length(troughs);

% Clean data
if (lp > 1) % Ignore oscillations more than 2 apart in lambda
    for i = 2:lp
        if (locp(i) == locp(i-1)+2)
            locp(i) = 0;
            loc t t = loc t t (loc t t (i-1)+1) = 0;
        end
    end
end

if (lt > 1)
    for i = 2:lt
        if (loc t t (i) == loc t t (i-1)+2)
            loc t t (i) = 0;
            loc p = loc p (loc p (i-1)+1) = 0;
        end
    end
end

peaks(locp == 0) = []; % Remove 0 data
troughs(loc t t == 0) = [];
locp(locp == 0) = [];
loc t t (loc t t == 0) = [];
lp = length(peaks);
l t = length(troughs);

% Check if early trough
l t e a r l y = 0;
if (lt > 0)
    if (lp == 0 || loc t t (1) < locp(1)) % i.e. if a trough happens first, then near origin
        l t e a r l y = 1;
    end
end

% Populate data matrices for this (li, xi)
npeaks(li, xi) = lp;
if (lp > 0)
    heightpeak1(li, xi) = peaks(1);
    lambdapeak1(li, xi) = traj(locp(1), 3);
    lambdapeak1(li, xi) = traj(locp(1), 3);
    depthtrough1(li, xi) = troughs(1);
    lambdatrough1(li, xi) = traj(loc t t (1), 3);
end

if (lp > 1)
    heightpeak2(li, xi) = peaks(2);
    lambdapeak2(li, xi) = traj(locp(2), 3);
    lambdapeak2(li, xi) = traj(locp(2), 3);
end
B. Computer Programs

B.2. MATLAB codes for Chapter 3

B.2.1. Computing the flow map – does_it_tip_map.m

This code produces the results shown in Fig. 3.5. It uses the equations files given in Section B.1, and MATLAB ordinary differential equation solver ode15s with a specified Jacobian, which is suitable for stiff systems, see [59, Sec. 3] for detail.

%does_it_tip_map

%traj event is x=1.5 (see trajevent.m)
%terminates trajectory as at end point on L_out

format long

warning('off', 'MATLAB:ode15s:IntegrationTolNotMet')

delta=1e-10;
lepsilond=-1.65 %log_10(epsilon - 0.2)
neweps=10^(lepsilond)+0.2
% Rescale to use old system of equations
eps=neweps/2.5

Delta=5;
options=odeset('Jacobian',@(t,traj)fullsys_jac(t,traj,eps,Delta,delta),'RelTol',1e-8,'AbsTol',1e-8,'Events',@trajevent);

% Initial state on L_in
% Y = y + lambda
x0=-0.5
y0=-0.75

% lambda_in between -2.5 and 0 corresponds to lambda0 being 0 to 2.5.
% At high resolution (25,000 points)
%lambda0=0:0.0001:2.5;
%lambda0=0:0.01:2.5;
y0=y0-lambda0;

% Maximum integration time
time=0:0.01:50;

% Variables to store end points of each trajectory
x1=zeros(size(lambda0));
y1=zeros(size(lambda0));
lambdal=zeros(size(lambda0));

% Compute trajectory for every initial state
for li=1:length(lambda0)
    % For monitoring purposes: print every 10th value of lambda
    if(mod(li,10)==0)
        lambda0(li)
    end
    [t,traj]=ode15s(@(t,traj)fullsys(t,traj,eps,Delta,delta),time,[x0,y0(li),lambda0(li)],options);
    % Store end points of trajectory
    x1(li)=traj(end,1);
y1(li)=traj(end,2);
lambdal(li)=traj(end,3);
end

% Copy initial state lambda0 and end point lambdal and y1
% for "good" trajectories that map to Sigma_out
% and for "bad" trajectories that do not map to Sigma_out.
lambda0gd=lambda0(x1>1.5);
lambdalgd=lambdal(x1>1.5);
y1gd=y1(x1>1.5);
lambda0bd=lambda0(x1<1.5);

% lambda component of flow map

figure; hold on;

% Plot end points
for i=1:length(lambdalgd)
    plot(lambda0gd(i),lambdalgd(i),'.');
end

% Image on Sigma_out of flow map

figure; hold on;

% Plot end points
for i=1:length(lambdalgd)
    plot(y1gd(i)+lambdalgd(i),lambdalgd(i),'.');
end
B.3. MATLAB codes for Chapter 4

B.3.1. Computing heteroclinic connection from a point to periodic orbit for $\omega = 0$

This code produces the results for system (4.22)–(4.24) when $\omega = 0$ in Figs. 4.9(a) and 4.10.

System (4.22)–(4.24) can equivalently be expressed as

$$\frac{dx}{dt} = -\alpha (x - \lambda) - \omega y + ((x - \lambda)^2 + y^2)(x - \lambda), \quad (B.4)$$
$$\frac{dy}{dt} = -\alpha y + \omega (x - \lambda) + ((x - \lambda)^2 + y^2)y, \quad (B.5)$$
$$\frac{d\lambda}{dt} = \rho \lambda (\Delta - \lambda), \quad (B.6)$$

where $z = x + iy$, $\alpha = a$, $\rho = \epsilon/\lambda_{\text{max}}$, $\Delta = 2\lambda_{\text{max}}$.

This uses the MATLAB ordinary differential equation solver `ode45` based on a fourth order Runge-Kutta method.

```
function ntop = hopf_shift_fwd_vary_rDo(t, traj, rho, Delta, omega)
alpha = 1.0;
x = traj(1);
y = traj(2);
lambda = traj(3);
xp = -alpha*(x-lambda) - omega*y + ([x-lambda]*[x-lambda] + y*y)*(x - lambda);
yp = omega*(x-lambda) - alpha*y + ([x-lambda]*[x-lambda] + y*y)*y;
lambdap = rho*lambda*(Delta - lambda);
ntop=[xp; yp; lambdap];
end
```

```
% check finding_critical_param and hopf_shift_fwd_vary_rDo have correct
% alpha = 1.0
folder='around_omega0_data/
% turn off annoying warnings
warning('off', 'MATLAB:ode45:IntegrationTolNotMet')

rho = 0.5:0.5:3.0;
omega = -0.5:0.1:0.5;
for i=1:length(rho)
delta_critical=zeros(length(omega),1);
for j=1:length(omega)
delta_critical(j)=finding_critical_param('Delta', rho(i), -9999, omega(j));
end
filename=strcat(folder, 'rho', num2str(rho(i)+10, '%2d'), '_-0.dat');
save(char(filename), '-ascii', 'delta_critical');
end
```

B. Computer Programs

Delta = 1.5:0.5:4.0;
omega = -0.5:0.1:0.5;
for i = 1:length(Delta)
    rho_critical=zeros(length(omega),1);
    for j = 1:length(omega)
        rho_critical(j)=finding_critical_param('rho',-9999,Delta(i),omega(j));
    end
    filename=strcat(folder,'delta',num2str(Delta(i)*10,'%02d'),'.0.dat')
    save(char(filename),'ascii','rho_critical')
end

function param_c = finding_critical_param(vary_param_name,rho,Delta,omega)
% Adapted to find critical parameter "vary_param_name", for other two
% specified parameter values.
% Need to give an original input value for critical parameter but will be
% overwritten (use -9999).
% e.g. to find the omega=0 curve in rho-Delta when omega=0 using omega0
% code, fixing omega=0 and for each Delta in some array in omega0 code
% call finding_critical_param('rho',-9999,<Delta_value>,0.0)

disp(['Finding critical ', vary_param_name])

alpha = 1.0; % NOT read in to ode function so can't vary, check set correctly in ..._vary_rDO.
eps=1e-7; %displacement in lambda from p.o.
t_fwd = [0:0.01:50];
tol = 0.0001; %how accurate answer wanted
param_min=0; %approx suitable values to look between
param_max=5;

%now set up correct starting parameters. The critical parameter is still assigned a value.
y_max=sqrt(alpha)+0.5; %what we use to check - i.e. y should not get bigger than this -
x_min=-(sqrt(alpha)+0.5); %correct even allowing for bend in manifold
x_max=sqrt(alpha)+Delta+0.5; %x moves between zero and delta should not get bigger than delta
    plus circle radius
if (vary_param_name == 'rho')
    rho = param;
elseif (vary_param_name == 'Delta')
    Delta = param;
    x_max=sqrt(alpha)+param+0.5;
elseif (vary_param_name == 'omega')
    omega = param;
else
    disp('Invalid parameter name')
    return
end
ev=zeros(1,3);
while (param_max-param_min > tol)
    param=param_min + 0.5*(param_max - param_min);
    if (vary_param_name == 'rho ')
        rho = param;
    elseif (vary_param_name == 'Delta ')
        Delta = param;
    elseif (vary_param_name == 'omega ')
        omega = param;
    else
        disp('Invalid parameter name')
        return
    end
    ev(1) = alpha * (alpha + rho * Delta) + omega*omega;
    ev(2) = - rho * Delta + omega;
    ev(3) = omega+omega + (alpha + rho+Delta)* (alpha + rho+Delta);
    nev = sqrt(ev'*ev);
    ev(1:3) = ev(1:3) / nev;

    %calculate the stable manifold (i.e. trajectory from the equilibrium
    [t, traj_stab]= ode45(@(hopfshift_fwd,vary_rDO, t_fwd, eps,ev, [], rho, Delta, omega);
    if ((max(abs(traj_stab(:,2))) > y_max) || (max(abs(traj_stab(:,1))) > x_max) ... || (min(abs(traj_stab(:,1))) < x_min))
        param_max=param;
    end

end

**B.4. AUTO codes for Chapter 3**

Numerical continuation software AUTO [21] requires a .f90 equations file, and optionally a .c. constants file. Programs can be automated by writing a .auto script.

**B.4.1. Computing canard trajectories and continuing for varying $\epsilon$**

This code is used for computation in Section 3.5 using the system of equations (3.18) – (3.20), with $A$ called $\lambda_{\text{max}}$; $\delta$ parameterised by $\log_{10}(\delta)$ called $\logdelta$; and $\epsilon$ parameterised by $\log_{10}(\epsilon - 0.2)$ called $\logepsilon$. Different initial values of $\log_{10}(\delta)$ and $\log_{10}(\epsilon - 0.2)$ are set in attr.f90 and rep.f90.

This code is run in multiple stages, so there are three different .f90 equations files, and two different .auto command files. The code follows AUTO demo fnc developed for [14].

The first stage is to compute all the candidate trajectories that go from $L_{\text{in}}$ to $\Sigma_{\text{FN}}$, and $\Sigma_{\text{FN}}$ to $L_{\text{out}}$ (attrrep.auto). This is set up as two boundary value problems, one for trajectories on the attracting slow manifold $S^a_{\delta}$, in forwards time going from $L_{\text{in}}$ to $\Sigma_{\text{FN}}$ (attr.f90 and c.attr); and the other for the trajectories on the repelling slow manifold $S^r_{\delta}$, in backwards time going from $L_{\text{out}}$ to $\Sigma_{\text{FN}}$ (rep.f90 and c.rep). For a boundary value problem in AUTO, an initial solution is required that satisfies the boundary value problem. We first compute the initial solution by solving a homotopy method with $(x(T), y(T), \lambda(T)) \in \Sigma_{\text{FN}}$, increasing the integration time $T$ from 0 (attrrep.auto).

The second stage is to identify by eye where pairs of candidate trajectories meet in $\Sigma_{\text{FN}}$ – these are canard trajectories. We specify user breakpoints in the attrrep.auto command file to be able to save these trajectories, ready for stage three.

The third stage is concatenate the pairs of trajectories identified in stage two to form canard trajectories (fsn_vareps.auto), which we use as initial solutions to the full boundary problem from $L_{\text{in}}$ to $L_{\text{out}}$ (fsn.f90).

The fourth stage is to continue the canard trajectories with varying parameters. Below we show the code for varying $\epsilon$, fsn_vareps.auto using constants files c.fsn_vareps and c.fsn_vareps.deltaplus (not shown, has $DS = 0.01$). Recall, canard trajectories are...
solutions to the full boundary problem from $L_{in}$ to $L_{out}$ (fsn.f90). Further continuations of fold points can also readily be done.

**attr.f90**

```fortran
SUBROUTINE FUNC(NDIM, U, ICP, PAR, IJAC, F, DFDU, DDFP)
IMPLICIT NONE
INTEGER, INTENT(IN) :: NDIM, ICP(*) , IJAC
DOUBLE PRECISION, INTENT(IN) :: U(NDIM), PAR(*)
DOUBLE PRECISION, INTENT(OUT) :: F(NDIM)
DOUBLE PRECISION, INTENT(INOUT) :: DFDU(NDIM,NDIM), DDFP(NDIM,*)
DOUBLE PRECISION x, y, lambda, mu, lambda_max, logepsilon, logdelta, T

! Define the state variables
x = U(1)
y = U(2)
lambda = U(3)

! Define the system parameters
mu = PAR(1)
lambda_max = PAR(2)
logepsilon = PAR(3)
logdelta = PAR(4)

! Define the integration time as a parameter
T = PAR(11)

! Define the right-hand sides
F(1) = T * (y + lambda + x * (x - 1))
F(2) = T * ((10**logdelta) * -1 * x)
F(3) = T * ((10**logepsilon + 0.2)/lambda_max * ( lambda_max**2 - lambda**2))

END SUBROUTINE FUNC
```

```fortran
SUBROUTINE STPNT(NDIM, U, PAR, T)
IMPLICIT NONE
INTEGER, INTENT(IN) :: NDIM
DOUBLE PRECISION, INTENT(INOUT) :: U(NDIM), PAR(*)
DOUBLE PRECISION, INTENT(IN) :: T

DOUBLE PRECISION x, lambda, lambda_max, logepsilon

PAR(1) = 1.0 !mu
PAR(2) = 2.5 !lambda_max
PAR(3) = -3 !-1.796 * epsilon = 10^-1.796+0.2 = 0.216
PAR(4) = -3.6 !-2 * delta = 0.01
PAR(11) = 0

! position of FN for epsilon=0
lambda_max = PAR(2)
logepsilon = PAR(3)

U(1) = 0.5 !x
U(3) = -SQRT(lambda_max * ( lambda_max - 1/(2*(10**logepsilon + 0.2)))) !lambda
x = U(1)
```
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```
lambda = U(3)
U(2) = -x*(x-1) - lambda ! y

! start point at the same point
PAR(5) = U(1) ! x(0)
PAR(6) = U(3) ! lambda(1)
PAR(7) = U(3) ! lambda(0)
PAR(8) = U(2) ! y(0)

END SUBROUTINE STPNT

SUBROUTINE BCND(NDIM,PAR, ICP ,NBC, U0 , U1 ,FB, IJAC ,DBC)

IMPLICIT NONE
INTEGER, INTENT(IN) :: NDIM, ICP(*), NBC, IJAC
DOUBLE PRECISION, INTENT(IN) :: PAR(*), U0(NDIM), U1(NDIM)
DOUBLE PRECISION, INTENT(OUT) :: FB(NBC)
DOUBLE PRECISION, INTENT(INOUT) :: DBC(NBC,*)

! Define boundary conditions */
FB(1) = U0(2)+U0(3)+U0(1) ! initial point stays on S
FB(2) = U0(1) - PAR(5) ! Initial point is on L_in, the intersection between S and the plane {x=-0.5}
FB(3) = U0(2) - PAR(8) ! and the plane {x=-0.5}
FB(4) = U0(3) - PAR(7)
FB(5) = U1(3) - PAR(6) ! End point is in a cross-section containing the folded node: Sigma:=\{lambda=const\}

IF (NBC==5) RETURN

! FB(6) gives that eqn of the fold F x=0.5
FB(6) = 0.5 - U0(1)

END SUBROUTINE BCND

SUBROUTINE PVLS(NDIM,U,PAR)

IMPLICIT NONE
INTEGER, INTENT(IN) :: NDIM
DOUBLE PRECISION, INTENT(IN) :: U(NDIM)
DOUBLE PRECISION, EXTERNAL :: GETP

! Define external parameter which monitors the x- and the y-coordinate of the end point in section Sigma_fn
PAR(9) = GETP("BV1", 1, U)
PAR(10) = GETP("BV1", 2, U)

END SUBROUTINE PVLS

SUBROUTINE ICND
END SUBROUTINE ICND

SUBROUTINE FOPT
END SUBROUTINE FOPT

rep.f90

! Slow manifolds computation for the fast slow shift system
! Comp. of the repelling slow manifold
! Homotopy step 1: "away from the folded node along the fold curve"
! followed by
! Homotopy step 2: "away from the fold curve on the critical manifold"
! followed by
! Actual computation of the repelling slow manifold
```
B. Computer Programs

Based on AUTO demo fnc

SUBROUTINE FUNC(NDIM, U, ICP, PAR, IJAC, F, DFDU, DFPD)

IMPLICIT NONE

INTEGER, INTENT(IN) :: NDIM, ICP(*), IJAC
DOUBLE PRECISION, INTENT(IN) :: U(NDIM), PAR(*)
DOUBLE PRECISION, INTENT(OUT) :: F(NDIM)
DOUBLE PRECISION, INTENT(INOUT) :: DFDU(NDIM, NDIM), DFPD(NDIM, *)

DOUBLE PRECISION x, y, lambda, mu, lambda_max, logepsilon, logdelta, T

! Define the state variables
x = U(1)
y = U(2)
lambda = U(3)

! Define the system parameters
mu = PAR(1)
lambda_max = PAR(2)
logepsilon = PAR(3)
logdelta = PAR(4)

! Define the integration time as a parameter
T = PAR(11)

! Define the right-hand sides
F(1) = T * (y + lambda + x * (x - 1))
F(2) = T * (10**logdelta * -1 * x)
F(3) = T * (10**logdelta * (10**logepsilon + 0.2) / lambda_max * (lambda_max**2 - lambda **2))

END SUBROUTINE FUNC

SUBROUTINE STPNT(NDIM, U, PAR, T)

IMPLICIT NONE

INTEGER, INTENT(IN) :: NDIM
DOUBLE PRECISION, INTENT(INOUT) :: U(NDIM)
DOUBLE PRECISION, INTENT(IN) :: T

DOUBLE PRECISION x, lambda, lambda_max, logepsilon

PAR(1) = 1.0 !mu
PAR(2) = 2.5 !lambda_max
PAR(3) = -3 !-1.796 epsilon = 10^-1.796+0.2 = 0.216
PAR(4) = -3.6 !-2 !delta = 0.01
PAR(11) = 0

!position of FN for epsilon=0
lambda_max = PAR(2)
logepsilon = PAR(3)

U(1) = 0.5 !x
U(3) = -SQRT(lambda_max * (lambda_max - 1/(2*(10**logepsilon + 0.2))) ) !lambda
x = U(1)
lambda = U(3)

U(2) = -x*(x-1) - lambda !y

!start point at the same point
PAR(5) = U(1) !x(1)
PAR(6) = U(3) !lambda(1)
PAR(7) = U(3) !lambda(1)

END SUBROUTINE STPNT

SUBROUTINE BCND(NDIM, PAR, ICP, NBC, U0, U1, FB, IJAC, DBC)

IMPLICIT NONE

INTEGER, INTENT(IN) :: NDIM, ICP(*), NBC

!...
B. Computer Programs

```fortran
! Define boundary conditions */
! Define the critical manifold S as \( \{ (x, y, \lambda) \mid y + \lambda + x(x-1) = 0 \} \) ! end point stays on S
FB(1) = U1(2) + U1(3) + U1(1) \( \times \) (U1(1) - 1) ! end point is on L_out, the intersection between S and the plane \( \{ x=1.5 \} \)
FB(3) = U1(3) - PAR(7)
FB(4) = U0(3) - PAR(6) ! initial point is in a cross-section containing ! the folded node: Sigma_FN := (lambda=const)
IF (NBC==4) RETURN
FB(5) = 0.5 - U1(1) ! FB(5) gives that eqn of the fold F x=0.5
END SUBROUTINE BCND

! Define external parameter which monitors the x- and the y-coordinate
! of the end point in section Sigma_FN
PAR(9) = GETP("BV0", 1, U)
PAR(10) = GETP("BV0", 2, U)
END SUBROUTINE PVLS

SUBROUTINE ICND
END SUBROUTINE ICND

SUBROUTINE FOPT
END SUBROUTINE FOPT
```

fsn.f90

```fortran
! Slow manifolds computation in the fast-slow shift system
! Continuation of canard orbits in parameter space
! Based on AUTO demo func

SUBROUTINE FUNC(NDIM, U, ICP, PAR, IJAC, F, DFDU, DFDP)
IMPLICIT NONE
INTEGER, INTENT(IN) :: NDM, ICP(*)
DOUBLE PRECISION, INTENT(IN) :: U(NDM)
DOUBLE PRECISION, INTENT(INOUT) :: PAR(*)
DOUBLE PRECISION, EXTERNAL :: GETP

DOUBLE PRECISION x, y, lambda, mu, lambda_max, logepsilond, logdelta, T

! Define the state variables
x = U(1)
y = U(2)
lambda = U(3)

! Define the system parameters
mu = PAR(1)
lambda_max = PAR(2)
```
B. Computer Programs

```fortran
logepsilond = PAR(3)
logdelta  = PAR(4)

! Define the integration time as a parameter
T   = PAR(11)

! Define the right-hand sides
F(1) = T * ( y + lambda + x * ( x - 1 ) )
F(2) = T * ( y + logdelta - 1 + x )
F(3) = T * ( 10**logepsilond + 0.2 ) / lambda_max * ( lambda_max**2 - lambda **2 )

END SUBROUTINE FUNC

SUBROUTINE STPNT
END SUBROUTINE STPNT

SUBROUTINE BCND(NDIM,PAR, ICP, NBC, U0, U1, FB, IJAC, DBC)

IMPLICIT NONE
INTEGER, INTENT(IN) :: NDIM, ICP( ), NBC, IJAC
DOUBLE PRECISION, INTENT(IN) :: PAR( ), U0(NDIM), U1(NDIM)
DOUBLE PRECISION, INTENT(OUT) :: FB(NBC)
DOUBLE PRECISION, INTENT(INOUT) :: DBC(NBC, )

! Define boundary conditions
FB(1) = U0(1) - PAR(5) ! Initial point is on the intersection between S
FB(2) = U0(3) - PAR(7) ! and the plane \{ x = -0.5 \}, param by some point lambda
FB(3) = U0(2)+U0(3)+U0(1)*( U0(1)-1 )
FB(4) = U1(1) - 1.5
FB(5) = U1(2)+U1(3)+U1(1)*( U1(1)-1 ) ! equiv to constraining U1(2) on S
FB(6) = U1(3) - PAR(6)

END SUBROUTINE BCND
```

For small $\log_{10}(\delta)$, or small $\log_{10}(\epsilon = 0.2)$ and canard trajectories with many rotations may need to use less strict convergence criteria $\text{EPSL}$, $\text{EPSU}$, $\text{EPSS}$ and bigger step sizes $\text{DS}$, $\text{DSMIN}$, $\text{DSMAX}$. Note, the convergence criteria are stricter than is normal in AUTO (see [21]), because the problem is very stiff. These convergence criteria are similar to those used in AUTO demo fnc.
B. Computer Programs

c.rep

usnames = {1: 'x', 2: 'y', 3: 'lambda'}
parnames = {1: 'mu', 2: 'lambda', 3: 'xrl0', 4: 'yrl0', 5: 'x0', 6: 'y0', 7: 'Tr'}

NDIM = 3 , IPS = 4 , IRS = 0 , ILP = 0
ICP = ['Tr', 'xrl0', 'lambda']
NST= 100 , NCOL= 500 , MXIP= 0 , IAD= 4 , ISP= 4 , ISW= 4 , NBC= 4 , NINT= 4
NT= 100 , NPR= 1000 , MXBF= 0 , IID= 2 , ITMX= 8 , NINT= 3 , JAC= 0
EPSL= 1e-07 , EPSU = 1e-07 , EPSS = 1e-05

DS = 0.1 , DSMIN= 5e-05 , DSMAX = 1.0 , IADS= 1

NPAR = 11 , THL = {}, THU = {}

UZR = {'xrl0': 0.0}, STOP = ['UZ1']
	note for large epsilon, may need to change UZR end point

---

pgm = "attr"

print "Demo %s is started" %pgm
r1 = run(c=pgm, e=pgm)

pgm = "rep"

print "Demo %s is started" %pgm
r1 = run(c=pgm, e=pgm)
B. Computer Programs

# for plotting attracting and repelling manifolds
# save(attr+rep, 'attrrep_364_pts') # so has precise pts for continuation
plot(attr+rep, top_title=\'log(\epsilon−0.2), log(\delta)=\((-3, -3.6)\)\', bifurcation_x=\[\'xa1', \'xr0'\],
bifurcation_y=\[\'ya1', \'yr0'\])

fsn_vareps.auto

# AUTO Demo fsn
# based on Inc
# Used to continue results found for eps=0.216 d=0.01 in epsilon plane
# For smaller eps and delta, may need to experiment with:
# relax relative convergence criteria: EPSS,EPSL,EPSU
# step size (decrease near fold): DSMIN,DS,DSMAX

attr = loadbd("attr")
rep = loadbd("rep")
pgm = "fsn"
dl("exis") # delete existing file

# Loop through 5 canard trajectories xi1 to xi5 and continue in epsilon
for i in [1, 2, 3, 4, 5]:
    print "calculating orbit for xi"+str(i)
    uxlabel = "UZ"+str(i)
    sola = attr(uxlabel)
solr = rep(uxlabel)

    Ta = sola.PAR[\"Ta\"]
    Tr = solr.PAR[\"Tr\"]
    Tc = Ta + Tr

    # concatenate manifold coordinates and rescale t
    t = [[t * Ta] / Tc for t in sola[\"t\"][: -1]] +
        [[t * Tr + Ta] / Tc for t in solr[\"t\"]]
    x = list(sola[\"x\"][: -1]) + list(solr[\"x\"])
    y = list(sola[\"y\"][: -1]) + list(solr[\"y\"])
    l = list(sola[\"lambda\"][: -1]) + list(solr[\"lambda\"])
    u = \[x, y, l\]

    p = {} # cont
    for j in \[
        \'mu\', \'lambda_max\', \'logepsilond\', \'logdelta\', \'xa0\', \'lambdar0\', 
        \'Tc\'
    \]:
        p[j] = sola.PAR[j]
p[\"lambdar1\"] = solr.PAR[\"lambdar1\"]
p[\"Tc\"] = Tc

    # continue canard trajectory with varying epsilon
    rmin = run(u, epgm, cm="fsn_vareps", PAR=p)
rplus = run(u, epgm, cm="fsn_vareps_deltaplus",PAR=p)
r = merge(rmin+rplus)
save(r, \"exis+str(i)\")
append(r, \"exis\")

cl()
B. Computer Programs

EPSL = 1e−07, EPSU = 1e−07, EPSS = 1e−05
DS = −0.01, DSMIN = 1e−8, DSMAX = 0.01, IADS = 1
NPAR = 11, THL = \{’Tc’: 0.0\}, THU = \{
UZR = \{’logepsilond’: [−3.5, −3, −2.5, −2, −1.5]\}
UZSTOP = \{’logepsilond’: [−4, −1]\}, STOP = [’LP4’]

B.5. AUTO codes for Chapter 4

B.5.1. Computing heteroclinic connection from a point to periodic orbit for \(\omega \neq 0\)

This uses Lin’s method and the code is an adaptation of AUTO demo pcl [21, Sec. 15.13]. This code is used to produce the results for system (4.22)–(4.24) shown in Fig. 4.9, the method is described in brief in Section 4.3 and in detail in Appendix A.2.2.

System (4.22)–(4.24) can equivalently be expressed as

\[
\begin{align*}
\frac{dx}{dt} &= -\alpha(x - \lambda) - \omega y + ((x - \lambda)^2 + y^2)(x - \lambda), \\
\frac{dy}{dt} &= -\alpha y + \omega(x - \lambda) + ((x - \lambda)^2 + y^2)y, \\
\frac{d\lambda}{dt} &= \rho\lambda(\Delta - \lambda),
\end{align*}
\]

where \(z = x + iy, \alpha = a, \rho = \epsilon/\lambda_{\text{max}}, \Delta = 2\lambda_{\text{max}}\).

Initially the system is solved as problem type 1, IPS=1, to detect steady states in the system – i.e. values of \(u(t)\) where the function \(f(u, \nu) = 0\). After the Hopf bifurcation occurs, the system of equations is solved to find the periodic orbit. The periodic orbit is expressed as a boundary value problem so problem type 4, IPS=4, is used.

**NtoPhopf.f90**

1 * NtoPhopf: Finding a node-to-p.o. (periodic orbit) heteroclinic connection with Lins method
2 * near the hopf normal form with shift forcing – after style of pcl
3 * Variables:
4 * For p.o.
5 * U(1) : xgamma (or real part z)
6 * U(2) : ygamma (or imaginary part z)
7 * U(3) : lambdagamma
8 * For eigenfunc around p.o.
9 * U(4) : x
10 * U(5) : y (or imaginary part z)
11 * U(6) : lambda
12 * For connection from section to p.o.
13 * U(7) : x+ (or real part z)
14 * U(8) : y+ (or imaginary part z)
15 * U(9) : lambdad+
16 * Note, BVP U0 is end at section, U1 is end at p.o.
17 * For connection from equilb to section
18 * U(10) : x− (or real part z)
19 * U(11) : y− (or imaginary part z)
20 * U(12) : lamdad−
21 * Note, BVP U0 is end at equilb, U1 is end at section
B. Computer Programs

Parameters:

PAR(1) : alpha  
PAR(2) : omega
PAR(3) : rho
PAR(4) : delta

PAR(11) : T: period of the cycle
PAR(12) : mu: log of the Floquet multiplier
PAR(13) : h : norm of eigenfunction for cycle at 0
PAR(14) : T^+ : time for connection from section to cycle (U(7:9))
PAR(15) : eps_c : distance from end connection to cycle
PAR(16) : T^- : time for connection from point to section (U(10:12))
PAR(17) : eps_p : distance from point to start connection
PAR(21) : sigma+: U0(9) - delta /2 (lambda - distance W^s(P) from section lambda =0.5)
PAR(22) : sigma^- : U1(12) - delta /2 (lambda - distance W^u(E) from section lambda =0.5)
PAR(23) : eta : gap size for Lin vector
PAR(24) : Zx : Lin vector (x-coordinate)
PAR(25) : Zy : Lin vector (y-coordinate)
PAR(26) : Zz : Lin vector (z-coordinate)

This subroutine defines the system of ODEs (B.7)–(B.9). U are the variables, F are the functions of U. If IPS=1 then solves for F = 0, or if IPS=4 then solves for F being tangent to U using the Jacobian matrix A.

IMPLICIT NONE
DOUBLE PRECISION, INTENT(IN) :: U(3), PAR(*)
LOGICAL, INTENT(IN) :: JAC
DOUBLE PRECISION, INTENT(OUT) :: F(3), A(3,3)

alpha = PAR(1)
omega = PAR(2)
rho = PAR(3)
delta = PAR(4)
x = U(1)
y = U(2)
lambda = U(3)
v = x - lambda ! makes easier to express functions

F(1) = - alpha*v - omega*y + (v*v + y*y) + v
F(2) = omega*v - y*alpha + (v*v + y*y) + y
F(3) = rho + delta + lambda - rho*lambda + lambda

The Jacobian matrix for (B.7)–(B.9) is

A = \begin{bmatrix}
-a + 3v^2 + y^2 & -\omega + 2vy & \alpha - 3v^2 - y^2 \\
\omega + 2vy & -\alpha + v^2 + 3y^2 & -\omega - 2vy \\
0 & 0 & \rho(\Delta - 2\lambda)
\end{bmatrix}

IF (JAC) THEN
A(1,1) = -alpha + 3v*v + y*y
A(1,2) = -omega + 2*y*v
A(1,3) = alpha - 3*v*v - y*y
A(2,1) = omega + 2*y*v
A(2,2) = -alpha + v*v + 3*y*y
A(2,3) = -omega - 2*y*v
A(3,1) = 0
A(3,2) = 0
A(3,3) = rho + delta - 2*rho + lambda
ENDIF

END SUBROUTINE RHS

SUBROUTINE FUNC(NDIM, U, ICP, PAR, JAC, F, DFDU, DDFP)

IMPLICIT NONE
INTEGER, INTENT(IN) :: NDIM, ICP, PAR(*)
DOUBLE PRECISION, INTENT(IN) :: U(NDIM)
DOUBLE PRECISION, INTENT(OUT) :: F(NDIM), DFDU(NDIM,NDIM), DDFP(NDIM,*)

END SUBROUTINE FUNC

149
DO Subroutine Func

DOUBLE PRECISION T, mu
DOUBLE PRECISION A(3,3)
CALL RHS(U, PAR, F, NDIM>3, A)  \( \text{Equation (A.15)} \)
IF (NDIM==3) RETURN
F(4:6) = MATMUL(A, U(4:6))
T = PAR(11)
F(1:6) = F(1:6) * T
\( ! \log \text{ of Floquet multiplier in PAR(12)} \)
mu = PAR(12)
F(4:6) = F(4:6) - mu*U(4:6)  \( \text{Equation (A.18)} \)
IF (NDIM==6) RETURN
CALL RHS(U(7:9), PAR, F(7:9), .FALSE., A)  \( \text{Equation (A.24)} \)
T = PAR(14)
F(7:9) = F(7:9) * T
IF (NDIM==9) RETURN
CALL RHS(U(10:12), PAR, F(10:12), .FALSE., A)  \( \text{Equation (A.21)} \)
T = PAR(16)
F(10:12) = F(10:12) * T
END SUBROUTINE FUNC

DO Subroutine STPNT(NDIM, U, PAR, T)
IMPLICIT NONE
INTEGER, INTENT(IN) :: NDIM
DOUBLE PRECISION, INTENT(INOUT) :: U(NDIM), PAR(*)
DOUBLE PRECISION, INTENT(IN) :: T
DOUBLE PRECISION, PARAMETER :: eps_c = 1d-7, eps_p = 1d-7
DOUBLE PRECISION alpha, omega, rho, delta, ev(3), nev
DOUBLE PRECISION, SAVE :: s(6)

IF (NDIM==9) THEN
  IF (T==0) THEN
    s(1:6) = U(1:6)
  ENDIF
  U(7:9) = s(1:3) - eps_c*s(4:6)
  RETURN
ELSEIF (NDIM==12) THEN
  alpha = PAR(1)
  omega = PAR(2)
  rho = PAR(3)
  delta = PAR(4)
  \( ! \text{unstable eigenvector at the 0 equilibrium} \)
  ev(1) = alpha * (alpha + rho*delta) + omega*omega
  ev(2) = -rho*delta + omega
  ev(3) = omega*omega + (alpha + rho*delta)*(alpha + rho*delta)
  nev = sqrt(DOT_PRODUCT(ev, ev))
  ev(1:3) = ev(1:3) / nev
  U(10:12) = eps_p*ev(1:3)
  RETURN
ENDIF

alpha = -1.0
omega = 2.0
rho = 0.8
delta = 2.0
PAR(1:4) = (/alpha, omega, rho, delta/)
PAR(15) = eps_c
PAR(17) = eps_p
PAR(21:22) = 0
U(1) = delta
U(2) = 0.0
U(3) = delta
END SUBROUTINE STPNT

DO Subroutine PVLS(NDIM, U, PAR)
IMPLICIT NONE

103  DOUBLE PRECISION T, mu
104  DOUBLE PRECISION A(3,3)
105
106  CALL RHS(U, PAR, F, NDIM>3, A)  \( \text{Equation (A.15)} \)
107  IF (NDIM==3) RETURN
108  F(4:6) = MATMUL(A, U(4:6))
109  T = PAR(11)
110  F(1:6) = F(1:6) * T
111  \( ! \log \text{ of Floquet multiplier in PAR(12)} \)
112  mu = PAR(12)
113  F(4:6) = F(4:6) - mu*U(4:6)  \( \text{Equation (A.18)} \)
114  IF (NDIM==6) RETURN
115  CALL RHS(U(7:9), PAR, F(7:9), .FALSE., A)  \( \text{Equation (A.24)} \)
116  T = PAR(14)
117  F(7:9) = F(7:9) * T
118  IF (NDIM==9) RETURN
119  CALL RHS(U(10:12), PAR, F(10:12), .FALSE., A)  \( \text{Equation (A.21)} \)
120  T = PAR(16)
121  F(10:12) = F(10:12) * T
122
123  END SUBROUTINE FUNC

124  SUBROUTINE STPNT(NDIM, U, PAR, T)
125  IMPLICIT NONE
126  INTEGER, INTENT(IN) :: NDIM
127  DOUBLE PRECISION, INTENT(INOUT) :: U(NDIM), PAR(*)
128  DOUBLE PRECISION, INTENT(IN) :: T
129  DOUBLE PRECISION, PARAMETER :: eps_c = 1d-7, eps_p = 1d-7
130  DOUBLE PRECISION alpha, omega, rho, delta, ev(3), nev
131  DOUBLE PRECISION, SAVE :: s(6)
132
133  IF (NDIM==9) THEN
134    IF (T==0) THEN
135      s(1:6) = U(1:6)
136    ENDIF
137    U(7:9) = s(1:3) - eps_c*s(4:6)
138    RETURN
139  ELSEIF (NDIM==12) THEN
140    alpha = PAR(1)
141    omega = PAR(2)
142    rho = PAR(3)
143    delta = PAR(4)
144    \( ! \text{unstable eigenvector at the 0 equilibrium} \)
145    ev(1) = alpha * (alpha + rho*delta) + omega*omega
146    ev(2) = -rho*delta + omega
147    ev(3) = omega*omega + (alpha + rho*delta)*(alpha + rho*delta)
148    nev = sqrt(DOT_PRODUCT(ev, ev))
149    ev(1:3) = ev(1:3) / nev
150    U(10:12) = eps_p*ev(1:3)
151    RETURN
152  ENDIF
153
154  alpha = -1.0
155  omega = 2.0
156  rho = 0.8
157  delta = 2.0
158  PAR(1:4) = (/alpha, omega, rho, delta/)
159  PAR(15) = eps_c
160  PAR(17) = eps_p
161  PAR(21:22) = 0
162  U(1) = delta
163  U(2) = 0.0
164  U(3) = delta
165
166  END SUBROUTINE STPNT

167  SUBROUTINE PVLS(NDIM, U, PAR)
168  IMPLICIT NONE

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B. Computer Programs

INTEGER, INTENT(IN) :: NDIM
DOUBLE PRECISION, INTENT(IN) :: U(NDIM)
DOUBLE PRECISION, INTENT(INOUT) :: PAR(*)
DOUBLE PRECISION, EXTERNAL :: GETP

INTEGER i , NBC
LOGICAL, SAVE :: FIRST = .TRUE.

IF (FIRST) THEN ! initialization for BCND
   FIRST = .FALSE.
ELSEIF (NDIM==9) THEN
   PAR(21) = GETP("BV0",9,U) - 0.5*PAR(4)/PAR(3)
ELSEIF (NDIM == 12) THEN
   NBC = AINT(GETP("NBC",0,U))
   IF (NBC == 15) THEN
      PAR(22) = GETP("BV1",12,U) - 0.5*PAR(4)/PAR(3)
   ELSE
      ! check if Lin vector initialized:
      IF (DOT_PRODUCT(PAR(24:26),PAR(24:26)) > 0) RETURN
      DO i =1,3
         d(i) = GETP("BV0",6+i,U) - GETP("BV1",9+i,U)
      ENDDO
      normlv = sqrt(DOT_PRODUCT(d,d))
      ! gap size in PAR(23)
      PAR(23) = normlv
      ! Lin vector in PAR(24) - PAR(26)
      PAR(24:26) = d(1:3)/normlv
   ENDIF
ENDIF
RETURN
END IF

END SUBROUTINE PVLS

END SUBROUTINE BCND(NDIM,PAR,ICP,NBC,U0,U1,FB,IJAC,DBC)

IMPLICIT NONE
INTEGER, INTENT(IN) :: NDIM, ICP(*), NBC, IJAC
DOUBLE PRECISION, INTENT(IN) :: PAR(*), U0(NDIM), U1(NDIM)
DOUBLE PRECISION, INTENT(OUT) :: FB(NBC)
DOUBLE PRECISION, INTENT(INOUT) :: DBC(NBC,*)

DOUBLE PRECISION alpha, omega, rho, delta, eps_c, eps_p, ev(3), nev, eta

! Periodicity boundary conditions on state variables
FB(1:3) = U0(1:3) - U1(1:3) Equation (A.16)

! Floquet boundary condition
FB(4:6) = U1(4:6) - U0(4:6) Equation (A.19)

! normalization
FB(7) = PAR(13) - DOT_PRODUCT(U0(4:6),U0(4:6)) Equation (A.20)
IF (NBC==7) RETURN
eps_c = PAR(15)
FB(8:10) = U1(7:9) - (U0(1:3) - eps_c*U0(4:6)) Equation (A.25)
FB(11) = U0(9) - 0.5*PAR(4)/PAR(3) - PAR(21)
IF (NBC==11) RETURN
alpha = PAR(1)
omega = PAR(2)
rho = PAR(3)
delta = PAR(4)
eps_p = PAR(17)

! unstable eigenvector at the 0 equilibrium
ev(1) = alpha * (alpha + rho*delta) + omega*omega
ev(2) = - rho*delta * omega
ev(3) = omega*omega + (alpha + rho*delta)*omega
ev = sqrt(DOT_PRODUCT(ev,ev))
ev(1:3) = ev(1:3) / ev
FB(12:14) = U0(10:12) - eps_p*ev(1:3) Equation (A.25)
IF (NBC==15) THEN
   FB(15) = U1(12) - 0.5*PAR(4)/PAR(3) - PAR(22)
   RETURN
ENDIF
eta = PAR(23)
FB(15:17) = U0(7:9) - U1(10:12) - eta*PAR(24:26) Equation (A.26)
This is the AUTO file listing constants for each stage (see [21] for details).

# Hopf shift system
# compute the NtoP connection via Lin's method
4 print "\n1st run - continue in alpha past Hopf bifurcation at 0"
5 ri = run(ri='NtoPhopf',ICP=['alpha'],NDIM=3,IPS=1,IRS=0,ILP=0,
6 NTST=20,NCOL=4,IAID=3,ISP=2,ISW=1,PLT=0,
7 NPAR=26,NBC=0,NINT=0,NMX=99999,
8 MNR=10,JCP=2,ITM=9,NITN=3,ICA=0,JN=7,JADS=1,
9 EPSL=1e-07,EPSU = 1e-07, EPS = 1e-05,
10 DS=0.1,DMIN = 0.005,DSMAX = 2.0,
11 UZSTOP=[\"alpha\":5.0], THL=[\"x\":0.0], TH=[\{}
12 ualpha={\"x\":\"x\", \"y\":\"y\", \"lambda\":\"lambda\"},
13 \n14 \n15 \n16 \n17 print "\n2nd run - switch to the periodic orbit and continue in alpha 1.0"
18 ri = run(ri='HLI',ICP=['alpha','T'],IPS=2,NTST=50,DS = 0.001,DMIN=0.001,DSMAX=0.1,
19 UZK=[\"alpha\":[0.1,0.2]],UZSTOP=[\"alpha\":1.0])
20 print "\n3rd run - extend the system"
21 ri = run(ri='UZI'),ICP=['mu','h','T'],IPS=4,NDIM=6,NBC=7,NINT=1,
22 STOP=[\"BP1\",UZK=[]],DS=-0.1,DMIN=1e-7,DSMAX=1.0)
23 print "\n4th run - normalize the Floquet bundle"
24 ri = run(ri='BP1'),ISW=1,
25 STOP=[],UZSTOP=['h':1.0],DS=0.01,DMIN=1e-5,DSMAX=0.1)
26 print "\n5th run - integrate backwards from the periodic orbit"
27 ri = run(ri='BP1'),ISW=1,
28 STOP=[],UZSTOP=['h':1.0],DS=0.01,DMIN=1e-5,DSMAX=0.1)
29 print "\n6th run - integrate away from the equilibrium up to Sigma"
30 ri = run(ri='UZI'),ICP=['T','mu','T','sigma +'],ISW=1,NDIM=9,NBC=11,
31 STOP=[],UZSTOP=['sigma +'],NPR=50,DS = 0.01,DMIN=1e-05,DSMAX=1)
32 print "\n7th run - put starting data for Lin vector and Lin gap in"
33 ri = run(ri='UZI'),ICP=['T','sigma -','T+','mu','T'],NDIM=12,NBC=15,
34 STOP=[],UZSTOP=['sigma -'],NPR=50,DS = 0.01,DMIN=1e-05,DSMAX=1)
35 print "\n8th run - integrate away from the equilibrium up to Sigma"
36 ri = run(ri='UZI'),ICP=['T','sigma -','T+','mu','T'],NDIM=12,NBC=15,
37 STOP=[],UZSTOP=['sigma -'],NPR=50,DS = 0.01,DMIN=1e-05,DSMAX=1)
38 ri = run(ri='UZI'),ICP=['T','sigma -','T+','mu','T'],NDIM=12,NBC=15,
39 STOP=[],UZSTOP=['sigma -'],NPR=50,DS = 0.01,DMIN=1e-05,DSMAX=1)
40 ri = run(ri='UZI'),ICP=['T','sigma -','T+','mu','T'],NDIM=12,NBC=15,
41 STOP=[],UZSTOP=['sigma -'],NPR=50,DS = 0.01,DMIN=1e-05,DSMAX=1)
42 ri = run(ri='UZI'),ICP=['T','sigma -','T+','mu','T'],NDIM=12,NBC=15,
43 STOP=[],UZSTOP=['sigma -'],NPR=50,DS = 0.01,DMIN=1e-05,DSMAX=1)
44 ri = run(ri='UZI'),ICP=['T','sigma -','T+','mu','T'],NDIM=12,NBC=15,
45 STOP=[],UZSTOP=['sigma -'],NPR=50,DS = 0.01,DMIN=1e-05,DSMAX=1)
46 ri = run(ri='UZI'),ICP=['T','sigma -','T+','mu','T'],NDIM=12,NBC=15,
47 STOP=[],UZSTOP=['sigma -'],NPR=50,DS = 0.01,DMIN=1e-05,DSMAX=1)
48 ri = run(ri='UZI'),ICP=['T','sigma -','T+','mu','T'],NDIM=12,NBC=15,
49 STOP=[],UZSTOP=['sigma -'],NPR=50,DS = 0.01,DMIN=1e-05,DSMAX=1)
50 ri = run(ri='UZI'),ICP=['T','sigma -','T+','mu','T'],NDIM=12,NBC=15,
51 STOP=[],UZSTOP=['sigma -'],NPR=50,DS = 0.01,DMIN=1e-05,DSMAX=1)
B. Computer Programs

print "\n9th run — Continue with fixed omega=1.5 in rho and Delta"
start9 = load(r8('UZ1'),ICP=['rho','delta','eps_c','eps_p','T-','T+','mu','T'],
STOP=[],UZSTOP={'rho':4},ILP=1,NPR=100,DS=0.001,DSMAX=0.1)
r9 = merge(run(start9)+run(start9,DS='-',UZSTOP={'rho':0.5}))
save(r9,'omega15')
plot(r8)
clean()
print "\nDone."
Bibliography


