The Astrophysical Journal, 764:67 (4pp), 2013 February 10 doi:10.1088/0004-637X/764/1/67

ON THE GRAVITATIONAL FIELDS OF MACLAURIN SPHEROID MODELS OF ROTATING FLUID PLANETS

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Received 2012 October 15; accepted 2012 December 14; published 2013 January 28

ABSTRACT

Hubbard recently derived an important iterative equation for calculating the gravitational coefficients of a Maclaurin spheroid that does not require an expansion in a small distortion parameter. We show that this iterative equation, which is based on an incomplete solution of the Poisson equation, diverges when the distortion parameter is not sufficiently small. We derive a new iterative equation that is based on a complete solution of the Poisson equation and, hence, always converges when calculating the gravitational coefficients of a Maclaurin spheroid.

Key words: gravitation – hydrodynamics – planets and satellites: general – planets and satellites: interiors

Online-only material: color figure

1. INTRODUCTION

An oblate Maclaurin spheroid model of a rapidly rotating fluid planet or star is described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \quad \text{or} \quad r = \tilde{r}(\mu) = \frac{a}{\sqrt{1 + l^2\mu^2}}, \qquad (1)$$

where (x, y, z) are Cartesian coordinates with z along the axis of rotation, (r, θ, ϕ) are spherical polar coordinates with $\mu = \cos\theta$, and a and b $(a > b)$ are the equatorial and polar radii of the Maclaurin spheroid, respectively. The distortion parameter due to the effect of rotation is given by either α or l defined as $\alpha = (a - b)/a$ with $0 < \alpha < 1$ or $l = \sqrt{(a^2 - b^2)}/b$ with $0 < l < \infty$. Hubbard (2012) derived an important iterative equation for calculating the gravitational coefficients of oblate Maclaurin spheroids with constant density ρ that does not require an expansion in a small distortion parameter. It is suggested that the iterative method for constant density can be readily generalized to models with multiple interfaces of an "onion-skin" structure of constant-density layers.

In this paper, we show that the small distortion parameter is still implicitly required in the iterative equation derived by Hubbard (2012) because his gravitational potential expansion is based on an incomplete solution of the Poisson equation and, hence, diverges when the distortion parameter is not sufficiently small. We derive a new iterative equation that is based on a complete solution of the Poisson equation and, hence, always converges when calculating the gravitational coefficients of Maclaurin spheroids.

2. HUBBARD'S ITERATIVE EQUATION

It is well known (Lamb 1932) that the rotation rate ω of an oblate Maclaurin spheroid is related to l by the equation

$$\frac{3\omega^2}{4\pi G\rho} = \frac{3}{2l^3}[(3 + l^2)\ \arctan\ l - 3l], \qquad (2)$$

where G is the gravitational constant. The total potential U, the sum of the centrifugal potential V_c, and the gravitational potential V_g which are a solution of the Poisson equation,

$$\nabla^2 V_g = -4\pi G\rho, \qquad (3)$$

must remain constant on the bounding surface of the Maclaurin spheroid. In order to determine the shape (i.e., the size of α or l) and the external gravity field of a Maclaurin spheroid, an expression for the gravitational potential V_g is needed in the external domain $b \leqslant r \leqslant a$. Hubbard's (2012) iterative method is based on the expression for the total potential U in the form

$$\begin{aligned} U(r, \mu) &= V_g(r, \mu) + V_c(r, \mu) \\ &= \frac{GM}{r}\left[1 - \sum_{k=1}^{\infty}\left(\frac{a}{r}\right)^{2k} J_{2k} P_{2k}(\mu)\right] \\ &\quad + \frac{1}{3}r^2\omega^2\left[1 - P_2(\mu)\right], \end{aligned} \qquad (4)$$

where $P_{2k}(\mu)$ are Legendre polynomials and $J_2, J_4, J_6, \ldots$ are zonal gravitational coefficients given by

$$J_{2k} = -\left(\frac{3}{2k+3}\right)\frac{\int_0^1 P_{2k}(\mu)[\xi(\mu)]^{2k+3}\, d\mu}{\int_0^1 [\xi(\mu)]^3\, d\mu}, \qquad (5)$$

with

$$\xi(\mu) = \frac{\tilde{r}(\mu)}{a} = \frac{1}{\sqrt{1 + l^2\mu^2}}. \qquad (6)$$

By equating $U(r, \mu)$ to its equatorial value $U(a, 0)$ on the bounding surface $\tilde{r}(\mu)$ of the Maclaurin spheroid, Hubbard (2012) derived the iterative equation:

$$\begin{aligned} &\frac{GM}{\tilde{r}}\left[1 - \sum_{k=1}^{\infty}\left(\frac{a}{\tilde{r}}\right)^{2k} J_{2k} P_{2k}(\mu)\right] + \frac{1}{3}\tilde{r}(\mu)^2\omega^2\left[1 - P_2(\mu)\right] \\ &\quad = \frac{GM}{a}\left[1 - \sum_{k=1}^{\infty} J_{2k} P_{2k}(0)\right] + \frac{1}{2}a^2\omega^2. \end{aligned} \qquad (7)$$

For a given rotation rate ω, a total mass M, and an equatorial radius a, Equation (7) can be solved via an iterative process to determine both the shape of the Maclaurin spheroid $\tilde{r}(\mu)$ and the multipole moments J_{2k} for $k = 1, 2, 3, \ldots, k_{\max}$, where $k_{\max}$ is the truncation parameter in expansions (4) and (7). This is because the summation $\sum_{k=1}^{\infty}$ in Equation (4) or (7) must be replaced with $\sum_{k=1}^{k_{\max}}$ in any practical computation. For a convergent expansion, however, the solution of the problem, e.g., $U(r, \mu)$, would become independent of $k_{\max}$ for sufficiently large $k_{\max}$.

3. A NEW ITERATIVE EQUATION

We show that, although Equations (4) and (7) do not explicitly require an expansion in a small distortion parameter, the small-distortion condition is still implicitly required in the iterative equation (7) derived by Hubbard (2012). This is because the expansion of the gravitational potential V_g used in Equation (4) or (7) is based on an incomplete solution of the Poisson equation (3) in the external domain $b \leqslant r \leqslant a$ and, hence, diverges when the distortion parameter is not sufficiently small. We derive a new iterative equation that is based on a complete solution of the Poisson equation (3) and, thus, always converges when calculating the gravitational coefficients of Maclaurin spheroids.

We begin by expressing the gravitational potential $V_g(r, \mu, \phi)$, a solution of the Poisson equation (3), in the form

$$V_g(r, \mu, \phi) = G \int_V \frac{\rho(r', \mu') dV'}{|\mathbf{r} - \mathbf{r}'|}, \qquad (8)$$

where $\int_V$ denotes the volume integration over the Maclaurin spheroid, $\mathbf{r} = (r, \mu, \phi)$ is the position vector located in the exterior of the Maclaurin spheroid, $\mathbf{r}' = (r', \mu', \phi')$ denotes the position vector of the density $\rho(r', \mu', \phi')$ within the interior of the Maclaurin spheroid, and

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{4\pi}{2l+1}\right) \frac{(r')^l}{r^{l+1}} \times Y_l^m(\mu, \phi) Y_l^m(\mu', \phi') \quad \text{when} \quad r > r', \qquad (9)$$

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{4\pi}{2l+1}\right) \frac{r^l}{(r')^{l+1}} \times Y_l^m(\mu, \phi) Y_l^m(\mu', \phi') \quad \text{when} \quad r < r', \qquad (10)$$

with $Y_l^m(\mu, \phi)$ being the spherical harmonics defined as

$$Y_l^m(\mu, \phi) = \sqrt{\left(\frac{2l+1}{4\pi}\right)}\, P_l^m(\mu)\, e^{im\phi},$$

where $i = \sqrt{-1}$ and $P_l^m(\mu')$ are the associated Legendre polynomials. By assuming that rapidly rotating bodies are axisymmetric $(\partial/\partial\phi = 0)$, Equations (9) and (10) become

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{(r')^l}{r^{l+1}} P_l(\mu) P_l(\mu') \quad \text{when} \quad r > r', \qquad (11)$$

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r^l}{(r')^{l+1}} P_l(\mu) P_l(\mu') \quad \text{when} \quad r < r'. \qquad (12)$$

We have to consider the gravitational potential $V_g(r, \mu)$ in the two different external domains, marked by $r > a$ and $b < r < a$, separately because they have different forms of solution for the Poisson equation (3).

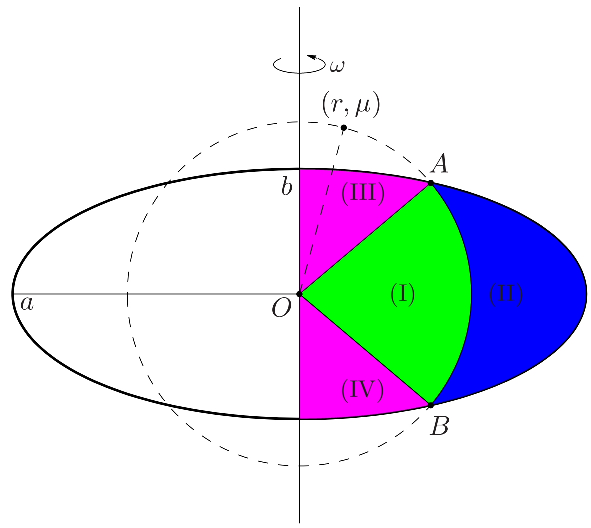

Figure 1. Sketch of different integration domains in a Maclaurin spheroid in which (r, μ) represents an external point in the domain $b \leqslant r \leqslant a$, the point A has coordinates $(r, \mu = \mu_r = \sqrt{a^2 - r^2}/(rl))$, and the point B has coordinates $(r, \mu = -\mu_r)$.

(A color version of this figure is available in the online journal.)

Consider first the gravitational potential V_g in the region $r > a$ for which only the expansion (11) is needed and, hence, the analysis is simple. It is straightforward, after making use of Equation (11), to show that

$$V_g(r, \mu) = \rho G \int_{-1}^{+1} \int_0^{\tilde{r}} \left[\sum_{l=0}^{\infty} \frac{(r')^l}{r^{l+1}} P_l(\mu) P_l(\mu') \right] \times (r')^2\, dr'\, d\mu' \quad \text{when} \quad r \geqslant a. \qquad (13)$$

Carrying out the integration in the radial direction and noting that

$$M = (4\pi\rho/3) \int_0^1 [\tilde{r}(\mu')]^3\, d\mu',$$

we obtain

$$V_g(r, \mu) = \frac{GM}{r} \left\{ 1 + \sum_{k=1}^{\infty} \left(\frac{a}{r}\right)^{2k} \left[\left(\frac{3}{2k+3}\right) \times \frac{\int_0^1 P_{2k}(\mu')[\xi(\mu')]^{2k+3}\, d\mu'}{\int_0^1 [\xi(\mu')]^3\, d\mu'} \right] P_{2k}(\mu) \right\}, \qquad (14)$$

which represents a complete solution of the Poisson equation (3) for the domain $r \geqslant a$.

Consider the gravitational potential $V_g(r, \mu)$ for the external domain in $b \leqslant r \leqslant a$ that is needed in deriving an iterative equation for determining both the shape and the gravity field of a Maclaurin spheroid. As a result of non-spherical geometry, the gravitational potential $V_g(r, \mu)$ in this domain is much more complicated than Equation (14). Let $\mathbf{r} = (r, \mu)$ be an external point in the domain $b \leqslant r \leqslant a$ which is displayed in Figure 1. There always exist two circles representing the intercept between the sphere of the radius r and the bounding surface $\tilde{r}$ of the Maclaurin spheroid. It is sufficient, because of the axisymmetry property, to illustrate various regions of the Maclaurin spheroid in a meridional plane. In Figure 1, the two circles in a meridional plane are marked by the point A whose coordinates are $(r, \mu_r = \sqrt{a^2 - r^2}/(rl))$ and by the point B whose coordinates are $(r, \mu_r = -\sqrt{a^2 - r^2}/(rl))$. It follows that, in contrast to the domain $r > a$, the gravitational potential $V_g(r, \mu)$ in $b < r < a$ requires a summation of the four different expansions

$$\begin{aligned} V_g(r,\mu) &= 2\pi\rho G \int_{-\mu_r}^{+\mu_r} \int_0^{r} \left[\sum_{l=0}^{\infty} \frac{(r')^l}{r^{l+1}} P_l(\mu)P_l(\mu')(r')^2\,dr'\right] d\mu' \\ &+ 2\pi\rho G \int_{-\mu_r}^{+\mu_r} \int_r^{\tilde{r}(\mu')} \left[\sum_{l=0}^{\infty} \frac{r^l}{(r')^{l+1}} P_l(\mu)P_l(\mu')(r')^2\,dr'\right] d\mu' \\ &+ 2\pi\rho G \int_{\mu_r}^{1} \int_0^{\tilde{r}(\mu')} \left[\sum_{l=0}^{\infty} \frac{(r')^l}{r^{l+1}} P_l(\mu)P_l(\mu')(r')^2\,dr'\right] d\mu' \\ &+ 2\pi\rho G \int_{-1}^{-\mu_r} \int_0^{\tilde{r}(\mu')} \left[\sum_{l=0}^{\infty} \frac{(r')^l}{r^{l+1}} P_l(\mu)P_l(\mu')(r')^2\,dr'\right] d\mu', \end{aligned} \quad (15)$$

where $\tilde{r}(\mu')$ is given by Equation (1), the integral in the first term of the right side of Equation (15) is over region (I) in Figure 1, the integral in the second term is over region (II), the integral in the third term is over region (III), and the integral in the fourth term is over region (IV). On making a proper rearrangement as well as recognizing the symmetry property, Equation (15) can be reduced to the three expansions

$$\begin{aligned} V_g(r,\mu) &= 4\pi\rho G \int_0^{1} \int_0^{\tilde{r}(\mu')} \left[\sum_{k=0}^{\infty} \frac{(r')^{2k}}{r^{2k+1}} P_{2k}(\mu)P_{2k}(\mu')(r')^2\,dr'\right] d\mu' \\ &- 4\pi\rho G \int_0^{\mu_r} \int_r^{\tilde{r}(\mu')} \left[\sum_{k=0}^{\infty} \frac{(r')^{2k}}{r^{2k+1}} P_{2k}(\mu)P_{2k}(\mu')(r')^2\,dr'\right] d\mu' \\ &+ 4\pi\rho G \int_0^{\mu_r} \int_r^{\tilde{r}(\mu')} \left[\sum_{k=0}^{\infty} \frac{r^{2k}}{(r')^{2k+1}} P_{2k}(\mu)P_{2k}(\mu')(r')^2\,dr'\right] d\mu', \end{aligned} \quad (16)$$

for which the first term is the same as Equation (4) for V_g in the external domain $r \geqslant a$. Upon performing integration in the radial direction, the second term on the right side of Equation (16) can be expressed as

$$\begin{aligned} 4\pi\rho G \int_0^{\mu_r} \int_r^{\tilde{r}(\mu')} &\left[\sum_{k=0}^{\infty} \frac{(r')^{2k}}{r^{2k+1}} P_{2k}(\mu)P_{2k}(\mu')(r')^2\,dr'\right] d\mu' \\ &= \frac{GM}{r}\left[K_0(\mu_r) - \sum_{k=1}^{\infty}\left(\frac{a}{r}\right)^{2k} K_{2k}(\mu_r)\; P_{2k}(\mu)\right], \end{aligned}$$

where K_0 and K_{2k} are functions of μ_r defined as

$$\begin{aligned} K_0(\mu_r) &= \frac{\int_0^{\mu_r}\left(\xi^3 - \xi_0^3\right) d\mu'}{\int_0^1 \xi^3\, d\mu'}, \\ K_{2k}(\mu_r) &= -\left(\frac{3}{2k+3}\right) \\ &\quad\times \frac{\int_0^{\mu_r}\left(\xi^{2k+3} - \xi_0^{2k+3}\right) P_{2k}(\mu')\, d\mu'}{\int_0^1 \xi^3\, d\mu'}, \quad k \geqslant 1. \end{aligned}$$

with $\xi_0 = r/a$ while $\xi = \xi(\mu')$ given by Equation (6). Similarly, the third term on the right side of Equation (16) can be written as

$$\begin{aligned} 4\pi\rho G \int_0^{\mu_r} \int_r^{\tilde{r}(\mu')} &\left[\sum_{k=0}^{\infty} \frac{r^{2k}}{(r')^{2k+1}} P_{2k}(\mu)P_{2k}(\mu')(r')^2\,dr'\right] d\mu' \\ &= \frac{GM}{r}\left[\left(\frac{r}{a}\right) N_0(\mu_r) + \left(\frac{r}{a}\right)^3 N_2(\mu_r)P_2(\mu) \right. \\ &\quad \left. - \sum_{k=2}^{\infty}\left(\frac{r}{a}\right)^{2k+1} N_{2k}(\mu_r)\; P_{2k}(\mu)\right], \end{aligned}$$

where N_0, N_2, and N_{2k} are functions of μ_r defined as

$$\begin{aligned} N_0(\mu_r) &= \left(\frac{3}{2}\right) \frac{\int_0^{\mu_r}\left(\xi^2 - \xi_0^2\right) d\mu'}{\int_0^1 \xi^3\, d\mu'}, \\ N_2(\mu_r) &= \frac{3\int_0^{\mu_r} \ln(\xi/\xi_0) P_2(\mu')\, d\mu'}{\int_0^1 \xi^3\, d\mu'}, \\ N_{2k}(\mu_r) &= -\left(\frac{3}{2k-2}\right) \\ &\quad\times \frac{\int_0^{\mu_r}\left(\xi_0^{-2k+2} - \xi^{-2k+2}\right) P_{2k}(\mu')\, d\mu'}{\int_0^1 \xi^3\, d\mu'}, \quad k \geqslant 2. \end{aligned}$$

It follows that the total potential in the external domain of a Maclaurin spheroid—which represents a complete solution of the Poisson equation—is given by

$$\begin{aligned} U(r,\mu) &= \frac{GM}{r}\left\{\left[1 - K_0(\mu_r) + \left(\frac{r}{a}\right) N_0(\mu_r)\right] \right. \\ &\quad - \sum_{k=1}^{\infty}\left[\left(\frac{a}{r}\right)^{2k}\left(J_{2k} - K_{2k}(\mu_r)\right) \right. \\ &\quad \left.\left. + \left(\frac{r}{a}\right)^{2k+1} N_{2k}(\mu_r)\right] P_{2k}(\mu)\right\} \\ &\quad + \frac{1}{3} r^2\omega^2\left[1 - P_2(\mu)\right] \quad \text{when } b \leqslant r \leqslant a. \end{aligned} \quad (17)$$

By equating $U(r, \mu)$ to its equatorial value $U(a, 0)$ on the bounding surface $\tilde{r}(\mu)$ of the Maclaurin spheroid we derive a new iterative equation from Equation (17) in the form

$$\begin{aligned} &\frac{GM}{\tilde{r}}\left\{\left[1 - K_0(\mu_r) + \left(\frac{\tilde{r}}{a}\right) N_0(\mu_r)\right] \right. \\ &\quad - \sum_{k=1}^{\infty}\left[\left(\frac{a}{\tilde{r}}\right)^{2k}\left(J_{2k} - K_{2k}(\mu_r)\right) \right. \\ &\quad \left.\left. + \left(\frac{\tilde{r}}{a}\right)^{2k+1} N_{2k}(\mu_r)\right] P_{2k}(\mu)\right\} \\ &\quad + \frac{1}{3}\tilde{r}^2\omega^2\left[1 - P_2(\mu)\right] \\ &\quad = \frac{GM}{a}\left[1 - \sum_{k=1}^{\infty} J_{2k} P_{2k}(0)\right] + \frac{1}{2}a^2\omega^2, \end{aligned} \quad (18)$$

which is valid for Maclaurin spheroids without any restriction. In comparison to the iterative equation (7) derived by Hubbard (2012), there are four extra terms in our new iterative equation (18).

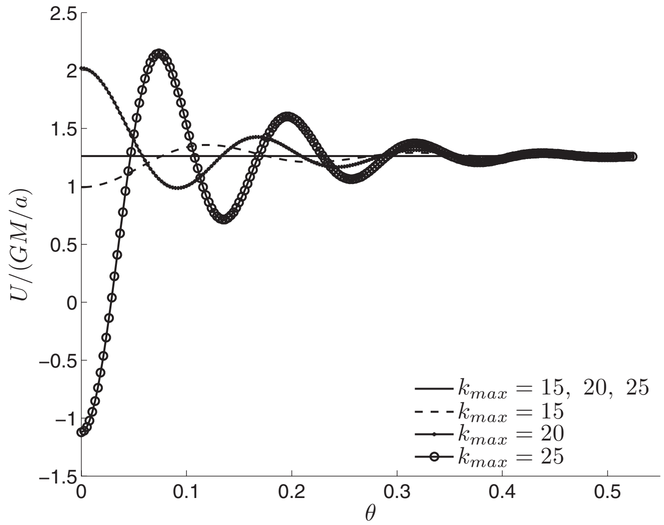

Figure 2. Scaled total potential U for a Maclaurin spheroid with $\alpha = 0.35$ is plotted on its bounding surface as a function of θ. Three solutions using three different truncation parameters $k_{\max} = 15, 20, 25$ are calculated. The horizontal solid line represents the solutions obtained from Equations (17) and (18) for $k_{\max} = 15$, 20, 25 while other lines (the dashed line for $k_{\max} = 15$, the dotted line $k_{\max} = 20$, and the circled line for $k_{\max} = 25$) are calculated from Equations (4) and (7).

4. DISCUSSION

It is important to point out that when the distortion parameter α or l is sufficiently small, our Equations (17) and (18) approach Equations (4) and (7) given by Hubbard (2012). This is because

$$K_0 \to 0,\ K_{2k} \to 0,\ N_0 \to 0,\ N_2 \to 0,\ N_{2k} \to 0 \text{ as } \alpha \to 0.$$

In other words, the iterative equation (7) derived by Hubbard (2012) mathematically represents the small flattening limit of our iterative equation (18). Our calculation shows that when the distortion parameter α is not sufficiently small ($\alpha \gtrsim 0.29$), Hubbard's (2012) expansions (4) and (7) diverge and, hence, cannot produce physically meaningful solutions.

Comparison of the convergent behavior of Equations (17) and (18) with the divergent behavior of Equations (4) and (7) is illustrated in Figure 2 for a Maclaurin spheroid with $\alpha = 0.35$, showing the total potential $U(\tilde{r}, \cos\theta)$ on the bounding surface of the Maclaurin spheroid as a function of θ. In the case of a convergent expansion, we anticipate that the scaled total potential $U(\tilde{r}, \cos\theta)/(GM/a)$ on the bounding surface of the Maclaurin spheroid not only remains constant (which is $U(a, 0)/(GM/a) = 1.2622$) but also is independent of the truncation parameter $k_{\max}$. Using three different truncations $k_{\max} = 15, 20, 25$ in our Equations (17) and (18), our calculation shows that the total scaled potential $U(\tilde{r}, \cos\theta)/(GM/a)$, which is depicted in Figure 2 on the solid line, remains constant and independent of the truncation parameter $k_{\max}$. The scaled total potential $U(\tilde{r}, \cos\theta)/(GM/a)$ calculated using Equations (4) and (7) given by Hubbard (2012) becomes strongly divergent in the polar regions where the departure from spherical geometry is the largest. At $k_{\max} = 15$, the value of $U(\tilde{r}, \cos\theta)/(GM/a)$ oscillates slightly in the polar regions with a small amplitude $|U(b, 1)/(GM/a) - 1.26| \approx 0.26$ which is shown by the dashed line in Figure 2. The amplitude of oscillation increases to $|U(b, 1)/(GM/a) - 1.25| \approx 2.3$ at $k_{\max} = 25$ which is shown by the circled line in Figure 2. When $k_{\max}$ increases further, the amplitude of divergent oscillation becomes too large to be shown in the figure. In fact, the divergent behavior of Equations (4) and (7) should be expected because they are based on an incomplete solution of the Poisson equation (see a relevant discussion in Section 38 of Zharkov & Trubitsyn 1978). However, our new iterative equation (18)—that is based on a complete solution of the Poisson equation and contains the four extra terms—always converges for calculating the gravitational fields of any Maclaurin spheroid at any practically large $k_{\max}$.

K.Z. is supported by UK NERC, STFC, and Leverhulme grants and G.S. is supported by the National Science Foundation under grant NSF AST-0909206.

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