RESEARCH ARTICLE

Adaptive Dual Layer Super-Twisting Control and Observation

Christopher Edwards[1] and Yuri Shtessel[2]

[1] *College of Engineering, Mathematics and Physical Sciences, University of Exeter, Exeter, EX4 4QF, U.K.*

[2] *Department of Electrical Engineering University of Alabama in Huntsville Huntsville, AL 35899, USA*

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In this paper a super-twisting-like structure with adaptive gains is proposed. The structure is parameterized by two scalar gains, both of which adapt, and by an additional time-varying term. The magnitudes of the adaptive terms are allowed to both increase and decrease as appropriate so that they are as small as possible, in the sense that they do not unnecessarily over-bound the uncertainty, and yet are large enough to sustain a sliding motion. In the paper, a new time varying gain is incorporated into the traditional super-twisting architecture. The proposed adaption law has a dual-layer structure which is formally analyzed using Lyapunov techniques. The additional term has the effect of simplifying the stability analysis whilst guaranteeing the second order sliding mode properties of the traditional super-twisting scheme.

Keywords: sliding modes

1 Introduction

Of all the second order sliding mode (2-SM) approaches in the literature, the super-twisting algorithm (Levant 1998) has probably proved to be the most popular. It is usually employed in two quite distinct situations: firstly as a way of enforcing a 2-SM in relative degree one systems using a smooth control signal; and secondly as a dynamical system for robust exact differentiation of bounded (smooth) signals. The super-twisting structure as originally presented in Levant (1998) has two tunable scalar gains, which govern its transient performance. Traditionally these have been fixed quantities and the stability proofs have relied on geometric bounding arguments. However recently (Moreno and Osorio 2008) classes of Lyapunov functions have been discovered which has led to renewed interest in this area – particularly in terms of developing adaptive versions of these schemes. Early work in the area of adaptive super twisting control (prior to the discovery of Lyapunov functions) appears in Kobayashi and Furuta (2007). The recent work has been predominantly based around the Lyapunov functions proposed by Moreno and Osorio (2008). The existing work on adaptive 2-SM schemes can be broadly split into two broad categories. The first of these categories, and by far the most extensively researched, involves developing adaptive laws to *increase* the gains of the super-twisting algorithm so that a 2-SM is obtained. Once sliding is obtained, adaptation stops and the gains remain fixed. If sliding is then subsequently broken – caused for example by the sudden occurrence of more aggressive disturbances – adaptation resumes and the gains once more increase. The gains are therefore subject

Corresponding author. Email: c.edwards@exeter.ac.uk

to a 'ratcheting' process and *are non-decreasing*. For a wide range of examples of such structures see Plestan et al. (2010), Shtessel et al. (2012), Alwi and Edwards (2013), Bartolini et al. (2013), Taleb et al. (2013). The second category of literature attempts to find the smallest gain values which will maintain sliding: specifically ensuring there is no unnecessarily conservative over-bounding of the uncertainty whilst still guaranteeing sliding will take place. This is conceptually different and has some advantages over the 'ratcheting' methods – especially from a chattering mitigation viewpoint. (Although the super-twisting algorithm yields a smooth control signal, chattering is still possible– for example in the presence of unmodelled dynamics in the feedback loop (Boiko and Fridman 2005).) In a differentiator problem setting, unnecessarily large gains amplify the transmission of noise on the measurement signal and so are again detrimental to performance. The volume of literature associated with the 'gain minimization' viewpoint is considerably smaller – perhaps because researchers have found it a more elusive problem. Recent references adopting this standpoint are Lee and Utkin (2007), Utkin and Poznyak (2013a,b). Of specific relevance to this paper, the work described in Utkin and Poznyak (2013b) uses information about the disturbance/uncertainty extracted from the equivalent control to adapt *one* of the gains of the super-twisting structure. This paper further research along these lines, but its main contribution is a small modification to the structure of the super-twisting algorithm itself. This change is very beneficial and yields a significant simplification in terms the Lyapunov analysis employed to guarantee a 2-SM. Compared to the work in Utkin and Poznyak (2013b), Edwards and Shtessel (2016), the scheme proposed in this paper provides a mechanism for adapting *both* gains of the super-twisting structure and provides global convergence guarantees (which Utkin and Poznyak (2013b), Edwards and Shtessel (2016) do not).

The structure of the paper is as follows: Section 2 describes the main results and in particular describes the newly proposed modified super-twisting scheme. A result is proved which indicates that the proposed scheme induces a 2-SM providing one of the gains in the structure is always greater in magnitude than the uncertainty. The dual-layer adaptive concept proposed in Edwards and Shtessel (2016) is then employed to create an adaptive law in conjunction with the novel modified super-twisting scheme. The notation used in the paper is quite standard. In particular $\|\cdot\|$ is used to represent the Euclidean norm for vectors and the induced spectral norm for matrices. The symbol $\mathbb{R}$ is used to represent the set of real numbers and $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ will be used to denote the maximum and minimum eigenvalues of a (symmetric) matrix.

2 Problem Formulation

A well known 2-SM scheme, known as the 'super-twisting' structure (Levant 1998), can be written in the form

$$\dot{e}_1(t) = -\bar{\alpha}\,\mathrm{sign}(e_1(t))|e_1(t)|^{1/2} + e_2(t) \tag{1}$$

$$\dot{e}_2(t) = -\bar{\beta}\,\mathrm{sign}(e_1(t)) + f(t) \tag{2}$$

where the states $e_1, e_2 \in \mathbb{R}$. The gains $\bar{\alpha}$ and $\bar{\beta}$ represent (positive) design scalars, whilst $f(t)$ corresponds to unknown bounded uncertainty. Assume $|f(t)| \leq \bar{L}$ where $\bar{L}$ is *fixed and known*. If the two gains are selected as $\bar{\alpha} = 1.5\sqrt{\bar{L}}$ and $\bar{\beta} = 1.1\bar{L}$, then it can be shown that $e_1 = \dot{e}_1 = 0$ in finite time (Levant 1998). The choice of the gains $\bar{\alpha}$ and $\bar{\beta}$ is not unique, but their selection affects the transient performance of the system (Levant 1998, Shtessel et al. 2013). The system structure in (1)-(2) can be arrived at from considering a nonlinear feedback controller applied to a first order system; or as a mechanism for exact differentiation of a measured signal. Other variations exist in which the uncertainty appears in (1) and its derivative is assumed to be bounded. Through a simple change of variable associated with $e_2(t)$ it is easy to see that both formulations are equivalent (Shtessel et al. 2013). Crucially a necessary condition is that $\bar{\beta} > |f(t)|$ which can be intuitively interpreted as the requirement that the switching

term $\bar{\beta}\,\text{sign}(e_1(t))$ in (2) dominates $f(t)$. If a tight upper-bound on $|f(t)|$ is known, then the choice of $\bar{\alpha}$ and $\bar{\beta}$ above can be applied to achieve sliding. Although in a feedback control framework the super-twisting structure in (1)-(2) results in a smooth control law, chattering can still occur in the presence of unmodelled dynamics for example, and so conservative (large) choices for the gains $\bar{\alpha}$ and $\bar{\beta}$ can be detrimental to closed-loop performance (Boiko and Fridman 2005). Furthermore in certain engineering circumstances, see for example Alwi and Edwards (2013), the disturbance $f(t)$ can have quite different characteristics at different periods of time, requiring very different gain levels. This has motivated the development of adaptive (second-order) sliding mode strategies (Plestan et al. 2010, Alwi and Edwards 2013, Shtessel et al. 2012, Bartolini et al. 2013, Taleb et al. 2013). In certain situations it is very desirable to allow the gains $\bar{\alpha}$ and $\bar{\beta}$ to be functions of time and to increase or decrease as appropriate. If tight time varying bounds on the uncertainty are *available and known*, the approach in Gonzalez et al. (2012) can be adopted. However if such bounds are not available or are conservative, adaptive approaches such as those proposed in (Plestan et al. 2010, Alwi and Edwards 2013, Shtessel et al. 2012, Bartolini et al. 2013, Taleb et al. 2013) must be employed. This problem is considered in this paper and in the following sections.

3 Main Results

The main result proposed in this paper involves a new variation on the super-twisting structure given by

$$\dot{e}_1(t) = -\alpha(t)\text{sign}(e_1(t))|e_1(t)|^{1/2} + e_2(t) + \phi(e_1, L) \tag{3}$$

$$\dot{e}_2(t) = -\beta(t)\text{sign}(e_1(t)) + f(t) \tag{4}$$

where $e_1, e_2 \in \mathbb{R}$, and $f(t)$ is unknown but bounded so that $|f(t)| < a_0$ *where* a_0 *is unknown*. The gains $\alpha(t)$ and $\beta(t)$ are positive scalar functions of a time-varying scalar $L(t) > l_0 > 0$ where l_0 is a positive fixed design scalar. *Compared to the traditional super-twisting structure in (1)-(2), an additional term* $\phi(\cdot)$ *is proposed in (3)-(4).* In this paper it is suggested that the gains

$$\alpha(t) = \sqrt{L(t)}\alpha_0 \tag{5}$$

$$\beta(t) = L(t)\beta_0 \tag{6}$$

where α_0 and β_0 are fixed positive scalars and the new term

$$\phi(e_1, L) = -\frac{\dot{L}(t)}{L(t)}e_1(t) \tag{7}$$

Remark 1: Clearly in the special case when $L(t)$ is fixed, then $\dot{L} = 0$ and $\phi(e_1, L) = 0$. Consequently the newly proposed scheme in (3)-(4) reverts to the traditional (fixed) super-twisting structure discussed in (1)-(2). Also note, during sliding, $\phi(e_1, L) = 0$ since $e_1 = 0$.

In this paper the objective is to derive an adaptive rule for creating a non-overestimated value of $L(t)$ without knowledge of the bound a_0, whilst at the same time ensuring a 2-SM takes place forcing $e_1 = \dot{e}_1 = 0$ to zero in finite time, despite the presence of the uncertainty in (3)-(4). For the newly proposed structure in (3)-(4), first a stability analysis is undertaken, assuming a scheme has already been devised for updating $L(t)$ so that $L(t)$ is differentiable, bounded and satisfies $L(t) > \max\{|f(t)|, l_0\}$ for all time.

Define three matrices as

$$A_0 = \begin{bmatrix} -\frac{1}{2}\alpha_0 & \frac{1}{2} \\ -\beta_0 & 0 \end{bmatrix} \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{8}$$

and also a (symmetric positive definite) matrix

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \tag{9}$$

where the scalars $p_1, p_3 > 0$ and $p_2^2 < p_1 p_3$.

Proposition 1: Suppose $L(t)$ is bounded and chosen so as to enforce $L(t) \geq \max\{|f(t)|, l_0\}$, then a 2-SM occurs making $\dot{e}_1 = e_1 = 0$ in finite time if the gains α_0 and β_0 are chosen so that there exists a P of the form in (9) such that

$$PA_0 + A_0^T P + \epsilon_0 P + PB_0 B_0^T P + C_0^T C_0 < 0 \tag{10}$$

where the design scalar $\epsilon_0 > 0$.

Proof: Consider a candidate Lyapunov function for the system in (3)-(4) of the form

$$V(t, e_1, e_2) = p_1 L(t)|e_1| + 2p_2 L(t)^{1/2} e_2 \text{sign}(e_1)|e_1|^{1/2} + p_3 e_2^2 \tag{11}$$

where p_1, p_2, p_3 relate to (9). Employing the notation

$$x = \text{col}(x_1, x_2) := \text{col}(L^{1/2}\text{sign}(e_1)|e_1|^{1/2}, e_2) \tag{12}$$

similar to that used in Gonzalez et al. (2012), the Lyapunov function $V(t, e_1, e_2)$ in (11) can be written as $V = x^T P x$ where the s.p.d matrix P is defined in (9). Since by assumption $L(t)$ is bounded and $L(t) > l_0 > 0$, then exploiting the positivity of P implies $V(t, e_1, e_2)$ is positive definite with respect to (e_1, e_2), continuous and radially unbounded. The expression for the time derivative $\dot{V}$ will now be obtained in terms of (x_1, x_2), which therefore requires expressions for $\dot{x}_1$ and $\dot{x}_2$. From (3), for $e_1 \neq 0$, x_1 from (12) satisfies

$$\begin{aligned} \dot{x}_1 &= \frac{\sqrt{L}(t)}{2|e_1|^{1/2}}\Big(-\alpha(t)\text{sign}(e_1)|e_1|^{1/2} + e_2 + \phi \Big) + \frac{\dot{L}(t)}{2\sqrt{L}(t)}\text{sign}(e_1)|e_1|^{1/2} \\ &= -\frac{\alpha(t)}{2|e_1|^{1/2}} x_1 + \frac{\sqrt{L}(t)}{2|e_1|^{1/2}} e_2 + \frac{\sqrt{L}(t)}{2|e_1|^{1/2}}\phi + \frac{\dot{L}(t)}{2\sqrt{L}(t)}\text{sign}(e_1)|e_1|^{1/2} \\ &= -\frac{\alpha(t)}{2|e_1|^{1/2}} x_1 + \frac{\sqrt{L(t)}}{2|e_1|^{1/2}} x_2 \end{aligned} \tag{13}$$

since

$$\frac{\sqrt{L}(t)}{2|e_1|^{1/2}}\phi(e_1, L) + \frac{\dot{L}(t)}{2\sqrt{L}(t)}\text{sign}(e_1)|e_1|^{1/2} = 0 \tag{14}$$

from the definition $\phi(e_1, L)$ in (7). Furthermore when $e_1 \neq 0$, x_2 from (12) satisfies

$$\dot{x}_2 = \frac{1}{|e_1|^{1/2}}\Big(-\beta(t)\text{sign}(e_1)|e_1|^{1/2} + |e_1|^{1/2} f \Big) = \frac{\sqrt{L}}{|e_1|^{1/2}}\Big(-\frac{\beta(t)}{L} x_1 + \tilde{f} \Big) \tag{15}$$

where the (re-defined) uncertainty in (15) is given by

$$\tilde{f}(t) = \frac{|e_1(t)|^{1/2}}{\sqrt{L}(t)} f(t) \tag{16}$$

Since $|x_1| = \sqrt{L}|e_1|^{1/2}$ it follows from (16) that the re-defined uncertainty satisfies

$$|\tilde{f}(t)| \leq \frac{|f(t)|}{L(t)}|x_1| \tag{17}$$

Since by assumption $|f(t)| < L(t)$, it follows that $|\tilde{f}(t)| \leq |x_1|$ i.e the uncertainty $\tilde{f}(t)$ lies in the sector $[-1 \;\; 1]$ (see Khalil (1992)). Using the definitions of $\alpha(t)$ and $\beta(t)$ in (5)-(6) it follows that (13) and (15) can be written in concise form as

$$\dot{x}(t) = \frac{\sqrt{L(t)}}{|e_1(t)|}\Big(A_0 x(t) + B_0 \tilde{f}(t)\Big), \quad e_1 \neq 0 \tag{18}$$

where A_0 and B_0 are defined in (8). Therefore along any solution of (3)-(4), when $e_1 \neq 0$

$$\dot{V} = \frac{\sqrt{L}}{|e_1|}\Big(x^T(PA_0 + A_0^T P_0)x + 2x^T P_0 B_0 \tilde{f}\Big) \tag{19}$$

where x is defined in (12). Applying Young's inequality to (19) yields

$$\dot{V} \leq \frac{\sqrt{L}}{|e_1|}\Big(x^T(PA_0 + A_0^T P_0 + P_0 B_0 B_0^T P)x + |\tilde{f}|^2\Big) \tag{20}$$

Using the definition of C_0 from (8) it follows the state component $x_1 = C_0 x$. Since $|\tilde{f}| \leq |x_1|$, it follows from (20) that

$$\begin{aligned}
\dot{V} &\leq \frac{\sqrt{L}}{|e_1|}\Big(x^T(PA_0 + A_0^T P_0 + P_0 B_0 B_0^T P)x + |x_1|^2\Big) \\
&= \frac{\sqrt{L}}{|e_1|} x^T\Big(PA_0 + A_0^T P_0 + P_0 B_0 B_0^T P + C_0^T C_0\Big)x \\
&\leq -\epsilon_0 \frac{\sqrt{L}}{|e_1|} V \qquad \text{(from (10))}
\end{aligned} \tag{21}$$

It follows from Rayleigh's inequality that $V > \lambda_{min}(P)\|x\|^2 > \lambda_{min}(P)|x_1|^2$ and therefore

$$\sqrt{V} > \sqrt{\lambda_{min}(P)}|x_1|$$

Since by definition $L(t) > l_0$ where l_0 is a positive scalar, from (21)

$$\dot{V} \leq -\epsilon_0 \frac{L(t)}{|x_1|} V \leq -\frac{\epsilon_0 l_0 V}{|x_1|} \leq -\eta\sqrt{V} \tag{22}$$

where the positive scalar $\eta = \epsilon_0 l_0 \sqrt{\lambda_{min}(P)}$. It follows from the arguments above that along the solution of (3)-(4), whenever $e_1 \neq 0$, $\dot{V} \leq -\eta\sqrt{V}$. The function $V(t)$ defined in (11) is continuous and differentiable except on the set $\mathcal{S} = \{(e_1, e_2) : e_1 = 0\}$. The trajectories of (3)-(4) cannot stay on $\mathcal{S}/\{\mathbf{0}\}$ since any point in this set takes the form $(0, e_2)$ where $e_2 \neq 0$, and

from (3), $\dot{e}_1 = e_2 \neq 0$. Consequently $V(t)$ is a continuously decreasing function and therefore from the 'Lyapunov' result for differential inclusions (Proposition 14.1 in Deimling (1992)), the equilibrium point $(e_1, e_2) = (0, 0)$ is reached in finite time. Substituting for $e_1 = e_2 = 0$ in (3) it follows that $\dot{e}_1 = 0$ in finite time. QED

Remark 2: Using the Bounded Real Lemma (Boyd et al. 1994) inequality (10) is equivalent to the frequency domain constraint $\|G_0(s)\|_\infty < 1$ where the transfer function

$$G_0(s) := C_0(sI - A_0)^{-1}B_0 = \frac{1}{(2s^2 + \alpha_0 s + \beta_0)}$$

The constant positive scalars α_0 and β_0 can always be chosen to ensure $\|G_0(s)\|_\infty < 1$. A necessary condition is that $\beta_0 > 1$ (because the dc gain of $G_0(s) = 1/\beta_0$). Furthermore it is easy to verify that choosing the gain $\alpha_0 = 2\sqrt{2\beta_0}$ guarantees $\|G_0(s)\|_\infty < 1$ and results in $G_0(s)$ having repeated real poles at $-\sqrt{\beta/2}$.

Note: the fact that $\beta_0 > 1$ will be exploited in the subsequent analysis.

Remark 3: The expression in (19) for the derivative of $V(t, e_1, e_2)$ from (11) is only valid when $e_1 \neq 0$. Consequently "standard" Lyapunov arguments are not valid here because $\dot{V}$ does not exist everywhere, and the approach adopted in the proof of Proposition 1 follows the one employed in Gonzalez et al. (2012), based on the differential inclusion results in Deimling (1992), which only require continuity of $V(t, e_1, e_2)$.

From the arguments above the problem thus becomes one of selecting $L(t)$ so that $L(t) > |f(t)|$. In this paper this constraint will be ensured by using a variation of the dual-layer adaptive structure proposed in Edwards and Shtessel (2016).

The dual-layer approach in Edwards and Shtessel (2016) relies on the concept of equivalent control (Utkin 1992) and permits the gain $L(t)$ to be increased and decreased. To analyze the role it plays, it is convenient to consider the system representation in (3)-(4). In system (3)-(4), during sliding, the discontinuous term $\beta(t)\text{sign}(e_1(t))$ in (4) must compensate for the uncertainty $f(t)$: formally

$$\beta(t)\text{sign}(e_1(t))|_{eq} = f(t) \tag{23}$$

where $\beta(t)\text{sign}(e_1(t))|_{eq}$ is the equivalent control which represents the average value the signal $\beta(t)\text{sign}(e_1(t))$ must take to maintain sliding (Utkin 1992). The concept of equivalent control as proposed by Utkin (1992) is a theoretical abstraction and usually used to analyze the reduced order dynamics associated with the sliding motion (Utkin 1992). However this quantity can be approximated in real-time by low pass filtering of the switched signal i.e.

$$\dot{\bar{u}}_{eq}(t) = \frac{1}{\tau}(\beta(t)\,\text{sign}(e_1(t)) - \bar{u}_{eq}(t)) \tag{24}$$

where τ is a (small) positive constant. Furthermore, at least in principle, during sliding the difference between $\bar{u}_{eq}(t)$ and the true value of the equivalent control $u_{eq}(t)$ can be made arbitrary small by making τ small (Shtessel et al. 2013). Consequently, during the sliding motion, by filtering the discontinuous injection signal $\beta(t)\,\text{sign}(\sigma(t))$, a good estimate of $f(t)$ can be obtained in real-time. This information will be subsequently exploited to adapt $L(t)$ to make it as small as possible, whilst still guaranteeing that $L(t) > |f(t)|$. Prior to sliding taking place, $\bar{u}_{eq}(t)$ does not represent the equivalent control, but nonetheless is an available signal which will be employed in the adaptive scheme. The notation $\bar{u}_{eq}(t)$ will be used throughout to represent the output of the filter in (24), but from the context, it will be clear under which circumstances it represents (an approximation) of the true equivalent control $u_{eq}(t)$.

Define a new scalar variable δ as

$$\delta(t) = L(t) - \frac{1}{a\beta_0}|\bar{u}_{eq}(t)| - \epsilon \tag{25}$$

In (25), as in Edwards and Shtessel (2016), the scalar a is chosen to satisfy $0 < a < 1/\beta_0 < 1$ and ϵ is a small positive scalar chosen to ensure

$$\frac{1}{a\beta_0}|\bar{u}_{eq}(t)| + \epsilon/2 > |u_{eq}(t)| \tag{26}$$

This definition of δ is a variation on the corresponding one in Edwards and Shtessel (2016) and is necessary to accommodate the different formulation in this paper. The design scalars a and ϵ represent 'safety margins' (at the expense of introducing conservatism). The proposed adaptive control element $L(t)$ on which the gains $\alpha(t)$ and $\beta(t)$ depend is given by

$$L(t) = l_0 + l(t) \tag{27}$$

where l_0 is a (small) positive design constant and the time varying term $l(t)$ satisfies

$$\dot{l}(t) = -\rho(t)\text{sign}(\delta(t)) \tag{28}$$

The time varying scalar in (28) is by definition

$$\rho(t) = r_0 + r(t) \tag{29}$$

where r_0 is a fixed positive design scalar and the time varying component $r(t)$ satisfies

$$\dot{r}(t) = \gamma|\delta(t)| \tag{30}$$

where γ is a fixed positive scalar design constant.

Proposition 2. Consider the system in (3)-(4) subject to uncertainty $f(t)$ which satisfies the two constraints $|f(t)| < a_0$, $|\dot{f}(t)| < a_1$ where the positive scalars a_0 and a_1 are finite <u>but unknown</u>. Then the dual-layer adaption scheme in (28) – (30) ensures $L(t) > |f(t)|$ in finite time.

Proof: Define a new variable

$$e(t) = qa_1/(a\beta_0) - r(t) \tag{31}$$

where $q > 1$ and represents a safety margin to guarantee $|\frac{d}{dt}|(\bar{u}_{eq})|| < qa_1$. From the definition of $e(t)$ and the expression for $\dot{r}(t)$ from (30), it follows that

$$\dot{e}(t) = -\dot{r}(t) = -\gamma|\delta(t)| \tag{32}$$

Arguing as in Utkin and Poznyak (2013b) the variable $\delta(t)$ evolves according to

$$\dot{\delta}(t) = \dot{l}(t) - \frac{1}{a\beta_0}\frac{d}{dt}|\bar{u}_{eq}(t)| \tag{33}$$

Consequently

$$\dot{\delta}(t) = -\underbrace{(r_0 + qa_1/(a\beta_0) - e(t))}_{\rho(t)}\text{sign}(\delta(t)) - 1/(a\beta_0)\phi(t) \tag{34}$$

where $\phi(t) = \frac{d}{dt}|\bar{u}_{eq}(t)|$ and $|\phi(t)| < qa_1$ where $q > 1$. The dynamical system formed from the variables $\delta(t)$ and $e(t)$, evolving according to (32) and (34), will now be analyzed using the Lyapunov function candidate

$$V(\delta, e) = \frac{1}{2}\delta^2 + \frac{1}{2\gamma}e^2 \tag{35}$$

From (34) it follows that

$$\delta\dot{\delta} \leq \delta(t)\dot{l}(t) + |\delta(t)|\frac{qa_1}{a\beta_0} = -r_0|\delta(t)| - r(t)|\delta(t)| + |\delta(t)|\frac{qa_1}{a\beta_0} = -r_0|\delta(t)| + e(t)|\delta(t)| \tag{36}$$

from the definition of $e(t)$ in (31). Therefore the derivative of (35) along the trajectories of $\delta(t)$ and $e(t)$ satisfies

$$\begin{aligned}\dot{V} &\leq -r_0|\delta(t)| + |\delta(t)|e(t) - \tfrac{1}{\gamma}e\dot{r}(t)\\ &= -r_0|\delta(t)| + |\delta(t)|e(t) - |\delta(t)|e(t)\\ &= -r_0|\delta(t)|\end{aligned} \tag{37}$$

Since $\dot{V} \leq 0$, and $V(\delta, e)$ is radially unbounded, it follows that both $e(t)$ and $\delta(t)$ remain bounded. Consequently since $r(t) = qa_1/(a\beta_0) - e(t)$ it follows that $r(t)$ remains bounded since $e(t)$ is bounded. Likewise since $|L(t)| \leq |\delta(t)| + qa_1/(a\beta_0) + \epsilon$ and $\delta(t)$ is bounded, the gain $L(t)$ remains bounded. Since $e(t)$ and $\delta(t)$ remain bounded, from (34), the derivative $\dot{\delta}(t)$ remains bounded and therefore $\delta(t)$ is absolutely continuous. It follows from (37) that

$$r_0\int_0^t |\delta(t)|dt \leq V(0)$$

where $|\delta(t)|$ is absolutely continuous. Therefore from Barbalat's Lemma (Khalil (1992)), $\delta(t) \to 0$ as $t \to \infty$. Consequently there exists a finite time t_0 such that $|\delta(t)| \leq \epsilon/2$ for all time $t > t_0$ (where ϵ is from the definition of δ in (25)). From the definition of $\delta(t)$ in (25) it follows

$$|\delta(t)| = |L(t) - \frac{1}{a\beta_0}|\bar{u}_{eq}(t)| - \epsilon| < \epsilon/2$$

and thus

$$L(t) - \frac{1}{a\beta_0}|\bar{u}_{eq}(t)| - \epsilon > -\epsilon/2$$

Since by definition $a\beta_0 < 1$, it follows using (26) the gain $L(t)$ satisfies

$$L(t) > \frac{1}{a\beta_0}|\bar{u}_{eq}(t)| + \frac{\epsilon}{2} > |u_{eq}(t)| = |f(t)| \tag{38}$$

From (38) it follows that the claim in the proposition statement is proved.

QED

Remark 4: Propositions 1 and 2 together provide the main results of this paper and demonstrate that using the dual-layer adaptive scheme from (25)-(30) which means choosing $L(t) = l(t) + l_0$

where

$$\dot{l}(t) = -\rho(t)\text{sign}(\delta(t)) \tag{39}$$

$$\dot{\rho}(t) = \gamma|\delta(t)| \tag{40}$$

where $\delta(t)$ is defined in (25), guarantees a 2-SM will take place in the system (3)-(4) and will force $e_1 = \dot{e}_1 = 0$ in finite time despite the uncertainty $f(t)$. Furthermore since the adaptive scheme forces $\delta(t)$ to become 'small', $L(t)$ 'tracks' the magnitude of the uncertainty $|f(t)|$ so that in situations when $f(t)$ is small, $L(t)$ will be small.

Remark 5: From equality (38)

$$L(t) > |\bar{u}_{eq}(t)| + \underbrace{\tfrac{1-a\beta_0}{a\beta_0}|\bar{u}_{eq}(t)| + \frac{\epsilon}{2}}_{\text{safety margin}} \tag{41}$$

This establishes a cone around the equivalent control $|u_{eq}(t)|$ involving both a multiplicative gain $\frac{(1-a\beta_0)}{a\beta_0}$ and a fixed gain $\frac{\epsilon}{2}$. These introduce a safety margin and robustness into the adaptive scheme at the cost of introducing conservatism. This is crucial since the value of $|u_{eq}(t)|$ can only be estimated by $\bar{u}_{eq}(t)$ through the low-pass filtering process in (24). The level of safety (and hence conservatism) is a function of the parameters a and ϵ which are to be selected by the designer (subject to $a\beta_0 < 1$ and $\epsilon > 0$). In this way a higher level of "safety" can be introduced by making a smaller and ϵ bigger – although this must be traded-off against how much larger $L(t)$ becomes with respect to $|f(t)|$.

Remark 6: At the start of the simulation/implementation, during the time interval $[0,\ t_2]$, or indeed, later during a more general time interval $[t_1,\ t_2]$ during which time sliding is not taking place, then $|\text{sign}(e_1)(t)| = 1$ for almost all $t \in [t_1,\ t_2]$. It follows from (24) that $\bar{u}_{eq}(t)$ is a high bandwidth low-pass filtered version of $\beta(t)$ and it follows there exists a time interval $[\bar{t}_1,\ t_2] \subset [t_1,\ t_2]$ such that

$$|\bar{u}_{eq}(t)| \geq a\beta(t) \tag{42}$$

for almost all $t \in [\bar{t}_1,\ t_2]$ since $0 < a < 1$. Consequently, from (42) and the definition of $\beta(t)$ in (6), it follows that

$$|\bar{u}_{eq}(t)| \geq a\beta(t) = a\beta_0 L(t) \tag{43}$$

and then utilizing (43) in (25) yields

$$\delta(t) = L(t) - \frac{1}{a\beta_0}|\bar{u}_{eq}(t)| - \epsilon \leq -\epsilon < 0 \tag{44}$$

Therefore from (44), for almost all $t \in [\bar{t}_1,\ t_2]$, $\text{sign}(\delta(t)) = -1$, and from (39) it follows

$$\dot{L}(t) = \dot{l}(t) = \rho(t) = r(t) + r_0 > 0$$

and so the gain $L(t)$ monotonically increases at a rate greater than a positive lower bound r_0. Over any interval of time of length a_0/r_0 the gain $L(t)$ will increase to a magnitude above a_0, after which a 2-SM will take place in finite time since by assumption $a_0 > |f(t)|$. The scheme in (39)-(40) is therefore still a valid adaptive mechanism prior to the establishment of a 2-SM in finite time (Utkin and Poznyak (2013b)). During the interval $[\bar{t}_1,\ t_2]$ the output of the low-pass filter, $\bar{u}_{eq}(t)$, no longer has the formal interpretation of 'equivalent control', and in fact in this

situation, as argued above, turns the adaptive scheme into one which produces monotonically increasing gains. This is precisely what is required: namely (monotonically) increasing the gains $\alpha(t)$ and $\beta(t)$ to induce sliding. Such behaviour is clearly seen in the example which follows in the next section.

Remark 7: For practical implementation the adaptive scheme in (39)-(40) can be replaced by

$$\dot{l}(t) = -\rho(t)\text{sign}(\delta(t)) \tag{45}$$

$$\dot{\rho}(t) = \begin{cases} \gamma|\delta(t)| & \text{if } |\delta(t)| > \delta_0 \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

where δ_0 is a (small) positive design scalar. The structure in (46) introduces a dead-zone in which no integration takes place, and counteracts the effects of practical implementation limitations involving noise and imperfections in the numerical integration schemes. Whilst the introduction of the deadzone creates limits on the choice of δ_0, it does not impact on the performance of the adaptive scheme proposed in this paper since in the proof of Proposition 2, $\delta(t)$ is only required to be 'practically stable' i.e. guaranteed to satisfy $|\delta(t)| < \epsilon/2$ in finite time, and is not required to converge to zero.

4 Simulation study

A simple example will now be used to demonstrate the theory. The system in (3)-(4) has been considered with the disturbance term $f(t) = \sin(t+0.57)+0.5\sin 3t$. This choice is for simulation purposes only, and is unknown to the controller. In all the simulations which follow the coefficients α_0 and β_0 in equations (5) and (6) have been selected to satisfy the conditions $\beta_0 > 1$ and $\alpha_0 = 2\sqrt{2\beta_0}$. Specifically here, $\beta_0 = 1.1$ and $\alpha_0 = 2.97$. The small time constant τ in equation (24) is taken as τ =0.001. The coefficients γ in equation (40), $a\beta_0$, l_0, r_0, and ϵ have been taken as: $\gamma = 8$, $a\beta_0 = 0.95$, $l_0 = 0.1$, $r_0 = 0.1$, $\epsilon = 0.01$. The system was simulated using the Euler integration algorithm with a fixed step size equal to 10^{-3}sec. The results are presented in Figures 1-4 shown below.

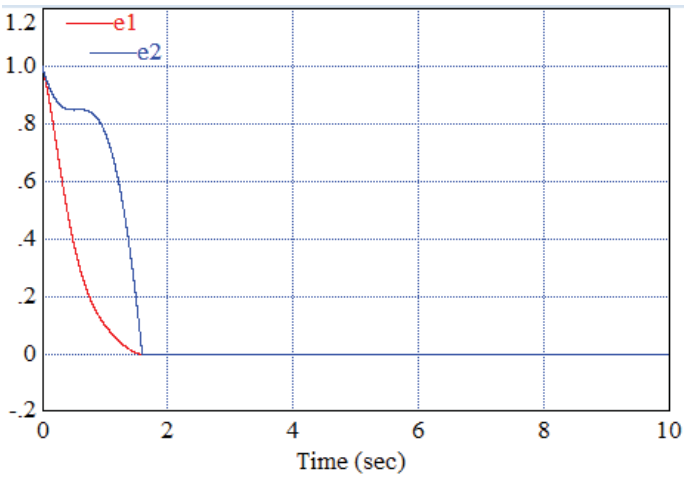

Figure 1.: Time history of the variables e_1 and e_2

It is clear from Figure 1 that e_1 and e_2 are driven to zero in finite time. The time histories of $\alpha(t)$ and $\beta(t)$ shown in Figure 2 demonstrate that the adaptive gain $\beta(t)$ closely follows the profile of $|f(t)|$, and the gain $\alpha(t)$ is adapted accordingly. The evolution of the new term $\phi(e_1, L)$ introduced

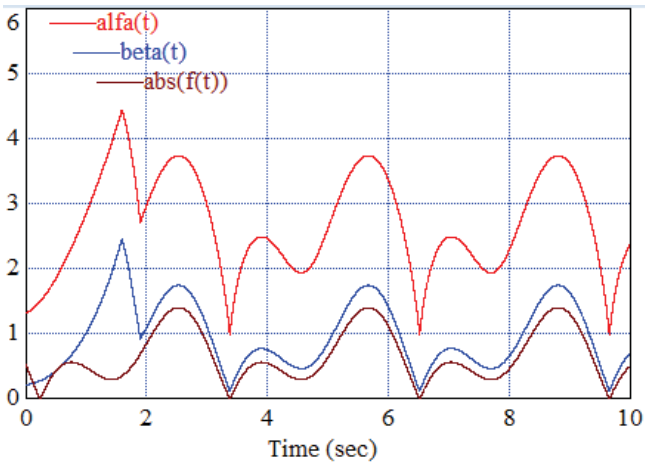

Figure 2.: Time histories of the gains $\alpha(t)$, $\beta(t)$ and the absolute value of the disturbance $f(t)$

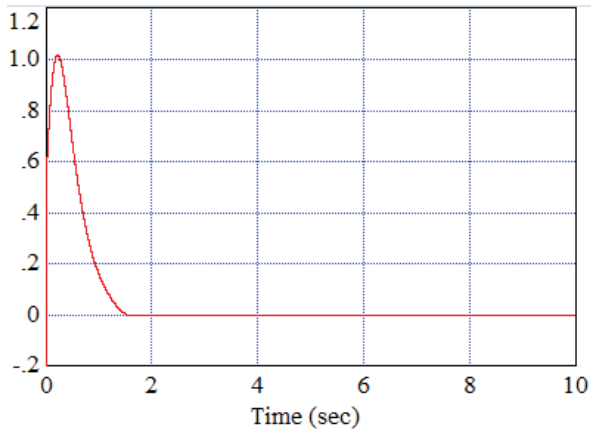

Figure 3.: Time history of the term $\phi(e_1, L) = -\frac{\dot{L}(t)}{L(t)} e_1(t)$

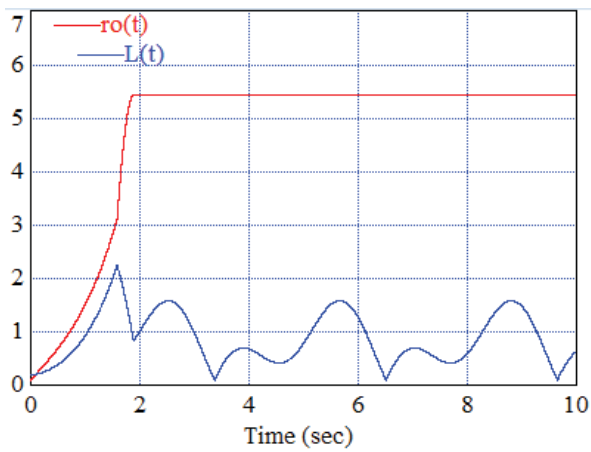

Figure 4.: Time histories of the adaptive parameters $\rho(t)$ and $L(t)$

in this paper is presented in Figure 3. It is shown to be continuous and clearly once sliding occurs $\phi = 0$ and the traditional super-twisting structure, albeit with adaptive gains, is recovered. The time history of the other adaptive parameters $\rho(t)$ and $L(t)$ can be observed in Figure 4. It is clear during the time interval $[0, 1.75]$, prior to the establishment of a sliding motion, that $L(t)$ is strictly increasing. During this period $\bar{u}_{eq}(t)$ does not represent the equivalent control and is simple the output signal of a filter. After this point in time, once sliding has been established, $\bar{u}_{eq}(t)$ does take on the connotation of 'equivalent control', and the dual layer scheme forces $L(t)$ to 'track' an upper bound on the magnitude of the disturbance. An additional benefit of the proposed adaptive dual-layer second order sliding mode control algorithm is that it can be used as an adaptive disturbance observer. From equation (24) it is clear $\bar{u}_{eq}$ accurately estimates $f(t)$ as soon as e_1 and e_2 converge to zero. This property is illustrated in Figure 5.

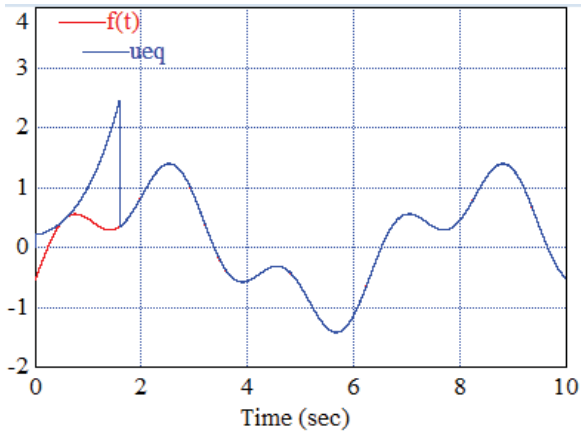

Figure 5.: The estimation of the disturbance $f(t)$

5 Conclusions

This paper has proposed a modification to the usual fixed gain super-twisting control structure to include a new time-varying term. This structural modification together with the use of a dual-layer adaptive scheme to update the two gains usually present in the super twisting structure has created a scheme which tries to minimize the degree of over-estimation of the bounds on the uncertainty, and yet still ensures a 2-SM will take place. The dual-layer adaptive scheme exploits knowledge of the equivalent control which contains information about the uncertainty. When there is no adaption and the gains become constant, the traditional super-twisting scheme is 'recovered'. Compared to earlier work the revised formulation allows both gains to adapt. Furthermore the inclusion of the new time-varying term simplifies the formal Lyapunov based proofs.

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