A refined version of Grothendieck's birational anabelian conjecture for curves over finite fields

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Abstract. In this paper we prove a refined version of Uchida's theorem on isomorphisms between absolute Galois groups of global fields in positive characteristics, where one "ignores" the information provided by a "small" set of primes.

§**0. Introduction.** Let k be a finite field of characteristic $p > 0$ and X a proper, smooth, and geometrically connected algebraic curve over k. Let K be the function field of X, with separable closure K^{sep}, and let $\bar{k}$ be the algebraic closure of k in K^{sep}. We have the following exact sequence of profinite groups:

$$1 \to \overline{G}_K \to G_K \xrightarrow{\mathrm{pr}} G_k \to 1.$$

Here, G_k is the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ of k, G_K is the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K, and $\overline{G}_K$ is the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K\bar{k})$ of the function field $K\bar{k}$ of $\overline{X} \overset{\mathrm{def}}{=} X \times_k \bar{k}$. The following result is fundamental in the birational anabelian geometry of curves over finite fields.

Theorem A (Uchida). *Let X, Y be proper, smooth, and geometrically connected curves over finite fields k, l, respectively. Let K, L be the function fields of X, Y, respectively. Let $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$, $G_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ be the absolute Galois groups of K, L, respectively. Let*

$$\sigma : G_K \xrightarrow{\sim} G_L$$

be an isomorphism of profinite groups. Then σ arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} L^{\mathrm{sep}} & \xrightarrow{\sim} & K^{\mathrm{sep}} \\ \uparrow & & \uparrow \\ L & \xrightarrow{\sim} & K \end{array}$$

in which the horizontal arrows are isomorphisms, and the vertical arrows are the natural field extensions.

This theorem was proved by Uchida [Uchida]. A stronger result involving fundamental groups of hyperbolic curves over finite fields was proved by Tamagawa [Tamagawa] and Mochizuki [Mochizuki] (see also [Saïdi-Tamagawa2] for a survey of recent results in the anabelian geometry of hyperbolic curves over finite fields). Uchida's theorem implies in particular that one can embed a suitable category of curves over finite fields into the category of profinite groups via the absolute Galois group functor. It is essential in the anabelian philosophy of Grothendieck, as was formulated in [Grothendieck], to be able to determine the image of this functor. Recall that the full structure of the absolute Galois group G_K is unknown, though one knows the structure of the closed subgroup $\overline{G}_K$ of G_K by a result of Pop and Harbater. Namely $\overline{G}_K$ is a free profinite group on countably infinitely many generators (cf. [Pop1], [Harbater]), though one has no precise description of a free set of generators of $\overline{G}_K$. Thus, the problem of determining the image of the above functor seems to be quite difficult, at least for the moment. It is quite natural to address the following question:

Question 1. Is it possible to prove any result analogous to Theorem A where G_K is replaced by some (continuous) quotient of G_K whose structure is better understood?

The first quotients that come into mind are the following. Let $\mathfrak{Primes}$ denote the set of all prime numbers. Let $\Sigma \subset \mathfrak{Primes}$ be a set of prime numbers not containing the characteristic p. Let $\mathcal{C}$ be the full class of finite groups whose cardinality is divisible only by primes in Σ. Let $\overline{G}_K^{\Sigma}$ be the maximal pro-$\mathcal{C}$ quotient of $\overline{G}_K$. Here, the structure of $\overline{G}_K^{\Sigma}$ is well understood: $\overline{G}_K^{\Sigma}$ is isomorphic to the projective limit of the maximal pro-Σ quotients $\pi_1(\overline{U})^{\Sigma}$ of the fundamental groups $\pi_1(\overline{U})$, where $\overline{U}$ runs over all non-empty open subschemes of $\overline{X}$, and $\pi_1(\overline{U})^{\Sigma}$ is isomorphic to the pro-Σ completion of a certain well-known finitely generated discrete group (i.e., either a free group or a surface group).

Let $G_K^{(\Sigma)} \stackrel{\text{def}}{=} G_K/\operatorname{Ker}(\overline{G}_K \twoheadrightarrow \overline{G}_K^{\Sigma})$. (Note that $\operatorname{Ker}(\overline{G}_K \twoheadrightarrow \overline{G}_K^{\Sigma})$ is a normal subgroup of G_K since it is a characteristic subgroup of $\overline{G}_K$.) We shall refer to $G_K^{(\Sigma)}$ as the maximal geometrically pro-Σ quotient of the absolute Galois group G_K (or, in short, the geometrically pro-Σ Galois group of K).

Question 2. Is it possible to prove any result analogous to Theorem A where G_K is replaced by $G_K^{(\Sigma)}$, for a given set of prime numbers $\Sigma \subset \mathfrak{Primes}$ (not containing the characteristic p)?

The first set Σ to consider is the set $\Sigma \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \{\operatorname{char}(k)\}$. In this case we shall refer to $G_K^{(\prime)} \stackrel{\text{def}}{=} G_K^{(\Sigma)}$ as the maximal geometrically prime-to-characteristic quotient of the absolute Galois group G_K. We have the following result which was proved by Saïdi and Tamagawa (cf. [Saïdi-Tamagawa1], Corollary 3.10).

Theorem B (Prime-to-p Version of Uchida's Theorem). *Notations as in Theorem A, let $G_K^{(\prime)}$, $G_L^{(\prime)}$ be the maximal geometrically prime-to-characteristic quotients of G_K, G_L, respectively. Let*

$$\sigma : G_K^{(\prime)} \xrightarrow{\sim} G_L^{(\prime)}$$

be an isomorphism of profinite groups. Then σ arises from a uniquely determined *commutative diagram of field extensions:*

$$\begin{array}{ccc} L^{(\prime)} & \xrightarrow{\sim} & K^{(\prime)} \\ \uparrow & & \uparrow \\ L & \xrightarrow{\sim} & K \end{array}$$

in which the horizontal arrows are isomorphisms, and the vertical arrows are the extensions corresponding to the Galois groups $G_L^{(\prime)}$, $G_K^{(\prime)}$, *respectively. Thus,* $L^{(\prime)}/L$ *(resp.* $K^{(\prime)}/K$*) is the subextension of* L^{sep}/L *(resp.* K^{sep}/K*) with Galois group* $G_L^{(\prime)}$ *(resp.* $G_K^{(\prime)}$*).*

Let $\Sigma \subset \mathfrak{Primes}$ be a set of primes, and set $\Sigma' \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \Sigma$. We say that Σ is *k-large* if the following condition is satisfied: the Σ'-cyclotomic character $\chi_{\Sigma'} : G_k \to \prod_{\ell \in \Sigma' \setminus \{p\}} \mathbb{Z}_\ell^\times$ is not injective ($p = \mathrm{char}(k)$). We say that Σ satisfies the condition (ϵ_X) if the following holds:

(ϵ_X): For each finite extension k' of k in $\bar{k}$, there exists an (infinite or finite) extension k'' of k' in $\bar{k}$, such that $2\sharp(J_X(k'')\{\Sigma'\}) < \sharp(k'') \leq \infty$ (hence, in particular, $\sharp(J_X(k'')\{\Sigma'\}) < \infty$),

where J_X denotes the jacobian variety of X over k and $J_X(k'')\{\Sigma'\}$ denotes the Σ'-primary part of the torsion group $J_X(k'')$.

Sets of prime numbers $\Sigma \subset \mathfrak{Primes}$ which are k-large and satisfy the condition (ϵ_X) (in the above sense) include those such that $\mathfrak{Primes} \setminus \Sigma$ is a finite set. There exist sets of primes Σ which are k-large and satisfy (ϵ_X) such that $\mathfrak{Primes} \setminus \Sigma$ is an infinite set. However, a finite set of prime numbers is never k-large.

Our main result in this paper is the following refined version of the above Theorems A and B.

Theorem C (A Refined Version of Uchida's Theorem). *Notations as in Theorem A, let* $\Sigma_X, \Sigma_Y \subset \mathfrak{Primes}$ *be sets of primes. Assume that* Σ_X *is* k*-large and satisfies the condition* (ϵ_X)*. Let* $G_K^{(\Sigma_X)}$ *(respectively,* $G_L^{(\Sigma_Y)}$*) be the maximal geometrically pro-*Σ_X *quotient of* G_K *(respectively, the maximal geometrically pro-*Σ_Y *quotient of* G_L*). Let*

$$\sigma : G_K^{(\Sigma_X)} \xrightarrow{\sim} G_L^{(\Sigma_Y)}$$

be an isomorphism of profinite groups. Then σ *arises from a uniquely determined commutative diagram of field extensions:*

$$\begin{array}{ccc} L^{\sim} & \xrightarrow{\sim} & K^{\sim} \\ \uparrow & & \uparrow \\ L & \xrightarrow{\sim} & K \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the field extensions corresponding to the Galois groups $G_L^{(\Sigma_Y)}$, $G_K^{(\Sigma_X)}$, *respectively. Thus,* $L^{\sim}/L$ *(resp.* $K^{\sim}/K$*) is the subextension of* L^{sep}/L *(resp.* K^{sep}/K*) with Galois group* $G_L^{(\Sigma_Y)}$ *(resp.* $G_K^{(\Sigma_X)}$*).*

Note. When the authors announced the result of the present paper in [Saïdi-Tamagawa2] (cf. loc. cit. Theorem 1.5), they overlooked the necessity to assume condition (ϵ_X). For the time being, they do not know if one could remove this extra assumption in general. (It is not difficult to see that we can remove it at least when the genus of X is ≤ 1.)

Strategy of Proof. In what follows we explain the steps/ideas of the proof.

Step 1. Starting from an isomorphism

$$\sigma : G_K^{(\Sigma_X)} \xrightarrow{\sim} G_L^{(\Sigma_Y)}$$

between profinite groups, one can first, using well-known results on the group-theoretic characterization of decomposition groups in Galois groups (the so-called local theory), establish a set-theoretic bijection

$$\phi : X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$$

between the sets of closed points of X, Y, respectively, such that $\sigma(D_x) = D_{\phi(x)}$ where D_x, $D_{\phi(x)}$ denote the decomposition groups of x, $\phi(x)$ in $G_K^{(\Sigma_X)}$, $G_L^{(\Sigma_Y)}$, respectively (which are only defined up to conjugation).

Step 2. It is not difficult to prove that $p \overset{\mathrm{def}}{=} \mathrm{char}(k) = \mathrm{char}(l)$, that $\Sigma \overset{\mathrm{def}}{=} \Sigma_X = \Sigma_Y$, and that Σ is both k-large and l-large and satisfies both (ϵ_X) and (ϵ_Y).

Step 3. Using global class field theory (one could also use Kummer theory in this step) one can reconstruct, naturally from σ, finite index subgroups $\overline{H}_K$, $\overline{H}_L$ of the groups of principal divisors $K^\times/k^\times$, $L^\times/l^\times$, respectively, finite index subgroups H'_K, H'_L of the multiplicative groups $(K^\times)^{(\Sigma)} \overset{\mathrm{def}}{=} K^\times/(k^\times\{\Sigma'\})$, $(L^\times)^{(\Sigma)} \overset{\mathrm{def}}{=} L^\times/(l^\times\{\Sigma'\})$, respectively, and a commutative diagram:

$$\begin{array}{ccc} H'_K & \xrightarrow{\rho} & H'_L \\ \downarrow & & \downarrow \\ \overline{H}_K & \xrightarrow{\bar{\rho}} & \overline{H}_L \end{array}$$

where the vertical arrows are the natural surjective homomorphisms and the horizontal arrows are natural isomorphisms induced by σ. Here $\Sigma' \overset{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma$, and $k^\times\{\Sigma'\}$ (resp. $l^\times\{\Sigma'\}$) is the Σ'-primary part of the multiplicative group $k^\times$ (resp. $l^\times$). Using, among other facts, that the set Σ is k-large and satisfies (ϵ_X), we show that the equalities $\overline{H}_K = K^\times/k^\times$, $\overline{H}_L = L^\times/l^\times$, $H'_K = (K^\times)^{(\Sigma)}$, and $H'_L = (L^\times)^{(\Sigma)}$ hold. Thus, one deduces naturally from the isomorphism $\sigma : G_K^{(\Sigma)} \xrightarrow{\sim} G_L^{(\Sigma)}$ a commutative diagram:

$$\begin{array}{ccc} (K^\times)^{(\Sigma)} & \xrightarrow{\rho} & (L^\times)^{(\Sigma)} \\ \downarrow & & \downarrow \\ K^\times/k^\times & \xrightarrow{\bar{\rho}} & L^\times/l^\times \end{array}$$

where the vertical arrows are the natural surjective homomorphisms and the horizontal arrows are the isomorphisms induced by σ.

Step 4. One shows that the isomorphism $\bar{\rho} : K^\times/k^\times \xrightarrow{\sim} L^\times/l^\times$ between principal divisors has the property that it preserves the valuation of the functions, or equivalently divisors, with respect to the set-theoretic bijection $\phi : X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ established (in Step 1) between the sets of closed points of X, Y, respectively.

Step 5. We think of the elements of $(K^\times)^{(\Sigma)} = K^\times/(k^\times\{\Sigma'\})$ and $(L^\times)^{(\Sigma)} = L^\times/(l^\times\{\Sigma'\})$ as "*pseudo-functions*", i.e., classes of rational functions modulo Σ'-primary constants. In particular, given a pseudo-function $f' \in (K^\times)^{(\Sigma)}$ (resp. $g' \in (L^\times)^{(\Sigma)}$), and a closed point $x \in X^{\mathrm{cl}}$ (resp. $y \in Y^{\mathrm{cl}}$) it makes sense to consider the Σ-value $f'(x)$ (resp. $g'(y)$) of f' (resp. g') (cf. discussion before Lemma 4.6). Then the isomorphism $\rho : (K^\times)^{(\Sigma)} \xrightarrow{\sim} (L^\times)^{(\Sigma)}$ has the property that it preserves the Σ-values of the pseudo-functions with respect to the set-theoretic bijection $\phi : X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ established between the sets of closed points of X, Y, respectively.

Step 6. We think of the elements of $K^\times/k^\times$ (respectively, $L^\times/l^\times$) as points of the infinite-dimensional projective spaces associated to the k (resp. l)-vector spaces K (resp. L). Using again, in an essential way, the fact that the set Σ is k-large, and satisfies (ϵ_X), as well as the above property of the isomorphism $\rho : (K^\times)^{(\Sigma)} \xrightarrow{\sim} (L^\times)^{(\Sigma)}$, we show that the isomorphism $\bar{\rho} : K^\times/k^\times \xrightarrow{\sim} L^\times/l^\times$ in the above diagram, viewed as one between points of projective spaces, preserves colineations. Thus, by the fundamental theorem of projective geometry (cf. [Artin]), it arises from a uniquely determined ψ_0-isomorphism

$$\psi : (K, +) \xrightarrow{\sim} (L, +), \ \psi(1) = 1,$$

where $\psi_0 : k \xrightarrow{\sim} l$ is a uniquely determined field isomorphism and ψ is an isomorphism of abelian groups which is compatible with ψ_0. Finally, we show that the isomorphism $\psi : (K, +) \xrightarrow{\sim} (L, +)$ preserves multiplication so that it is a field isomorphism. By passing to open subgroups of $G_K^{(\Sigma)}$ and $G_L^{(\Sigma)}$ which correspond to each other via σ, one constructs a field isomorphism $K^\sim \xrightarrow{\sim} L^\sim$ which is compatible with ψ, and the inverse $L^\sim \xrightarrow{\sim} K^\sim$ of this isomorphism is the desired isomorphism.

Note that the above idea to resort to the fundamental theorem of projective geometry is not new in anabelian geometry (see, e.g., [Bogomolov], [Pop2]), while the above idea to consider "*pseudo-functions*" and "*k-largeness*" is (to the best of our knowledge) new in anabelian geometry.

This paper is divided in two main parts. Part I is mostly of local nature. In Part I, §1, we review some basic facts on the Galois theory of function fields of algebraic curves and the main (well-known) results of the so-called local theory on the characterization of the decomposition subgroups in Galois groups. In Part I, §2, we reconstruct, using the local theory in §1, various information encoded in the geometrically pro-Σ absolute Galois group of a function field of a curve over a finite field. Part II is of global nature. In Part II, §3, we define and give various characterizations of the notions of small and large sets of primes, and we also prove the main Proposition 3.11 which plays a crucial role in the proof of our main result. Finally, in Part II, §4, we state and prove our main result Theorem 4.1.

Part I

In this first part we describe the local information encoded in the geometrically pro-Σ absolute Galois group of the function field of a curve over a finite field, and how much of this information is preserved under isomorphisms between geometrically pro-Σ absolute Galois groups.

§1. Generalities on Galois Groups of Function Fields of Curves. In this section we fix some notations that we will use in this paper, and review some basic facts on Galois groups of function fields of algebraic curves. Let k be a finite field of characteristic $p > 0$. Let X be a proper, smooth, and geometrically connected curve over k. Let K be the function field of X. Let $\eta = \operatorname{Spec} K$ be the generic point of X and $\bar{\eta} = \operatorname{Spec} \Omega$ a geometric point of X above η. Write K^{sep} (resp. $\bar{k} = k^{\mathrm{sep}}$) for the separable closure of K (resp. k) in Ω. Write $G = G_K \overset{\mathrm{def}}{=} \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}/k)$ for the absolute Galois groups of K and k, respectively. We have the following exact sequence of profinite groups:

$$1 \to \overline{G} \to G \xrightarrow{\mathrm{pr}} G_k \to 1, \tag{1.1}$$

where $\overline{G}$ is the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K\bar{k})$ of $K\bar{k}$, and $\mathrm{pr} : G \twoheadrightarrow G_k$ is the canonical projection. It is well-known that the kernel $\overline{G}$ of the projection $\mathrm{pr} : G \twoheadrightarrow G_k$ is a free profinite group of countably infinite rank (cf. [Pop1] and [Harbater]). However, the structure of the extension (1.1) is not known.

We shall consider a variant of (1.1) above. Let $\mathcal{C}$ be a full class of finite groups, i.e., $\mathcal{C}$ is closed under taking subgroups, quotients, finite products, and extensions. For a profinite group H, denote by $H^{\mathcal{C}}$ the maximal pro-$\mathcal{C}$ quotient of H. Given a profinite group H and a closed normal subgroup $\overline{H}$ of H, we set $H^{(\mathcal{C})} \overset{\mathrm{def}}{=} H/\operatorname{Ker}(\overline{H} \twoheadrightarrow \overline{H}^{\mathcal{C}})$. (Observe that $\operatorname{Ker}(\overline{H} \twoheadrightarrow \overline{H}^{\mathcal{C}})$ is a normal subgroup of H since it is a characteristic subgroup of $\overline{H}$.) Note that $H^{(\mathcal{C})}$ coincides with $H^{\mathcal{C}}$ if and only if the quotient $A \overset{\mathrm{def}}{=} H/\overline{H}$ is a pro-$\mathcal{C}$ group. Let $\mathfrak{Primes}$ denote the set of all prime numbers. When $\mathcal{C}$ is the class of finite Σ-groups, where $\Sigma \subset \mathfrak{Primes}$ is a set of prime numbers, write H^{Σ} and $H^{(\Sigma)}$, instead of $H^{\mathcal{C}}$ and $H^{(\mathcal{C})}$, respectively. (In later sections, the notation $H^{(\Sigma)}$ is used for a slightly more general setting where H is a (not necessarily profinite) topological group and $\overline{H}$ is a closed normal subgroup of H which is profinite.) By definition, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{H} & \longrightarrow & H & \longrightarrow & A & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & \overline{H}^{\mathcal{C}} & \longrightarrow & H^{(\mathcal{C})} & \longrightarrow & A & \longrightarrow & 1
\end{array}$$

where the rows are exact and the columns are surjective.

Lemma 1.1. *Let ℓ be a prime number and $i \geq 0$. Assume either $\mathrm{cd}_\ell(N) \leq 1$ or $\mathbb{F}_\ell \notin \mathcal{C}$, where $N \overset{\mathrm{def}}{=} \operatorname{Ker}(\overline{H} \twoheadrightarrow \overline{H}^{\mathcal{C}})$.*

(i) *Let M be a finite discrete ℓ-primary $\overline{H}^{\mathcal{C}}$-module. Then*

$$H^i(\overline{H}^{\mathcal{C}}, M) = \begin{cases} H^i(\overline{H}, M), & \text{if } \mathbb{F}_\ell \in \mathcal{C} \text{ or } i = 0, \\ 0, & \text{if } \mathbb{F}_\ell \notin \mathcal{C} \text{ and } i > 0. \end{cases}$$

In particular,

$$\mathrm{cd}_\ell(\overline{H}^{\mathcal{C}}) \begin{cases} \leq \mathrm{cd}_\ell(\overline{H}), & \mathbb{F}_\ell \in \mathcal{C}, \\ = 0, & \mathbb{F}_l \notin \mathcal{C}. \end{cases}$$

(ii) *Let M be a finite discrete ℓ-primary $H^{(\mathcal{C})}$-module. Then*

$$H^i(H^{(\mathcal{C})}, M) = \begin{cases} H^i(H, M), & \text{if } \mathbb{F}_\ell \in \mathcal{C}, \\ H^i(A, M^{\overline{H}}), & \text{if } \mathbb{F}_\ell \notin \mathcal{C}. \end{cases}$$

In particular,

$$\mathrm{cd}_\ell(H^{(\mathcal{C})})\begin{cases} \leq \mathrm{cd}_\ell(H), & \mathbb{F}_\ell \in \mathcal{C}, \\ = \mathrm{cd}_\ell(A), & \mathbb{F}_\ell \notin \mathcal{C}. \end{cases}$$

Proof. (i) If $\mathbb{F}_\ell \notin \mathcal{C}$, then $\overline{H}^{\mathcal{C}}$ is of order prime to ℓ, hence

$$H^i(\overline{H}^{\mathcal{C}}, M) = \begin{cases} 0, & i > 0, \\ M^{\overline{H}^{\mathcal{C}}} = M^{\overline{H}} = H^0(\overline{H}, M), & i = 0, \end{cases}$$

as desired. If $\mathbb{F}_\ell \in \mathcal{C}$, then $\mathrm{cd}_\ell(N) \leq 1$ by assumption, hence $H^j(N, M) = 0$ for $j > 1$. Further, $H^1(N, M) = \mathrm{Hom}(N, M) = 0$, as $\overline{H}^{\mathcal{C}}$ is the maximal pro-$\mathcal{C}$ quotient of $\overline{H}$. From these, we have

$$H^i(\overline{H}, M) = H^i(\overline{H}^{\mathcal{C}}, H^0(N, M)) = H^i(\overline{H}^{\mathcal{C}}, M),$$

as desired (cf. [Neukirch-Schmidt-Winberg], (1.6.6)Proposition). The second assertion follows from the first.
(ii) If $\mathbb{F}_\ell \notin \mathcal{C}$, then, by (i),

$$H^i(H^{(\mathcal{C})}, M) = H^i(A, H^0(\overline{H}^{\mathcal{C}}, M)) = H^i(A, M^{\overline{H}}),$$

as desired. If $\mathbb{F}_\ell \in \mathcal{C}$, then, similarly to the proof of (i),

$$H^i(H, M) = H^i(H^{(\mathcal{C})}, H^0(N, M)) = H^i(H^{(\mathcal{C})}, M),$$

as desired. The second assertion follows from the first. (For the inequality $\mathrm{cd}_\ell(H^{(\mathcal{C})}) \geq \mathrm{cd}_\ell(A)$, note that any finite discrete ℓ-primary A-module M can be regarded naturally as a finite discrete ℓ-primary $H^{(\mathcal{C})}$-module with $M^{\overline{H}} = M$.) $\square$

Applying the above construction to (1.1) we obtain the exact sequence:

$$1 \to \overline{G}^{\mathcal{C}} \to G^{(\mathcal{C})} \xrightarrow{\mathrm{pr}} G_k \to 1.$$

We shall refer to the quotient $G^{(\mathcal{C})}$ of G as the maximal geometrically pro-$\mathcal{C}$ quotient of G. Let U be an open subgroup of $G^{(\mathcal{C})}$ and H the inverse image of U in G via the canonical map $G \twoheadrightarrow G^{(\mathcal{C})}$. Then H is an open subgroup of G corresponding to a finite extension K'/K of K. Further, H (resp. U) is naturally identified with the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ of K' (resp. with $H^{(\mathcal{C})} = \mathrm{Gal}(K^{\mathrm{sep}}/K')^{(\mathcal{C})}$).

Let

$$\Sigma \subset \mathfrak{Primes}$$

be a set of prime numbers. Set

$$\Sigma^\dagger \overset{\mathrm{def}}{=} \Sigma \setminus \{p\}$$

and

$$\Sigma' \overset{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma.$$

Write

$$\hat{\mathbb{Z}}^{\Sigma} \stackrel{\text{def}}{=} \prod_{\ell \in \Sigma} \mathbb{Z}_{\ell}.$$

For a field κ of characteristic $p > 0$, with a separable closure κ^{sep}, we shall write

$$M^{\Sigma}_{\kappa^{\text{sep}}} \stackrel{\text{def}}{=} \text{Hom}(\mathbb{Q}/\mathbb{Z}, (\kappa^{\text{sep}})^{\times}) \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}^{\Sigma^{\dagger}}.$$

Thus, $M^{\Sigma}_{\kappa^{\text{sep}}}$ is a free $\hat{\mathbb{Z}}^{\Sigma^{\dagger}}$-module of rank one. Further, $M^{\Sigma}_{\kappa^{\text{sep}}}$ has a natural structure of $G_{\kappa} \stackrel{\text{def}}{=} \text{Gal}(\kappa^{\text{sep}}/\kappa)$-module, which is isomorphic to the Tate twist $\hat{\mathbb{Z}}^{\Sigma^{\dagger}}(1)$, i.e., G_{κ} acts on $M^{\Sigma}_{\kappa^{\text{sep}}}$ via the Σ-part of the cyclotomic character $\chi_{\Sigma} : G_{\kappa} \to (\hat{\mathbb{Z}}^{\Sigma^{\dagger}})^{\times}$. In particular, we write

$$M^{\Sigma}_{\bar{k}} = \text{Hom}(\mathbb{Q}/\mathbb{Z}, \bar{k}^{\times}) \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}^{\Sigma^{\dagger}}.$$

Similarly, we shall write

$$M^{\Sigma}_{X} \stackrel{\text{def}}{=} M^{\Sigma}_{K^{\text{sep}}} = \text{Hom}(\mathbb{Q}/\mathbb{Z}, (K^{\text{sep}})^{\times}) \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}^{\Sigma^{\dagger}}.$$

Note that M^{Σ}_{X} has a natural structure of G-module, which is naturally identified with the G-module $M^{\Sigma}_{\bar{k}}$ (the natural G- and G_k-module structures of $M^{\Sigma}_{\bar{k}}$ are compatible with respect to the natural projection $\text{pr} : G \twoheadrightarrow G_k$).

For a scheme T denote by T^{cl} the set of closed points of T. Let $K^{\sim}/K$ be the subextension of K^{sep}/K corresponding to the subgroup $\text{Ker}(G \twoheadrightarrow G^{(\Sigma)})$ of G, and let $\tilde{X}$ be the normalization of X in $K^{\sim}$. The Galois group $G^{(\Sigma)}$ acts naturally on the set $\tilde{X}^{\text{cl}}$ and the quotient of $\tilde{X}^{\text{cl}}$ by this action is naturally identified with X^{cl}. For a point $\tilde{x} \in \tilde{X}^{\text{cl}}$, with residue field $k(\tilde{x})$ (which is an algebraic closure of the residue field $k(x)$ of x), we define its decomposition group $D_{\tilde{x}}$ and inertia group $I_{\tilde{x}}$ by

$$D_{\tilde{x}} \stackrel{\text{def}}{=} \{\gamma \in G^{(\Sigma)} \mid \gamma(\tilde{x}) = \tilde{x}\}$$

and

$$I_{\tilde{x}} \stackrel{\text{def}}{=} \{\gamma \in D_{\tilde{x}} \mid \gamma \text{ acts trivially on } k(\tilde{x})\},$$

respectively. We have a canonical exact sequence:

$$1 \to I_{\tilde{x}} \to D_{\tilde{x}} \to G_{k(x)} \stackrel{\text{def}}{=} \text{Gal}(k(\tilde{x})/k(x)) \to 1.$$

For a profinite group H we write $\text{Sub}(H)$ for the set of closed subgroups of H. Let $\Sigma \subset \mathfrak{Primes}$ be a set of prime numbers. The following are well-known facts concerning the decomposition and inertia subgroups of the geometrically pro-Σ Galois group $G^{(\Sigma)}$.

Proposition 1.2. *(Properties of Decomposition and Inertia Subgroups) Let $\tilde{x} \in \tilde{X}^{\text{cl}}$, and x the image in X^{cl} of $\tilde{x} \in \tilde{X}^{\text{cl}}$.*
(i) *Let X' be the normalization of X in K^{sep} and x' a point of $(X')^{\text{cl}}$ above $\tilde{x}$. Let $I_{x'} \subset D_{x'} \subset G$ be the inertia and the decomposition subgroups of G at x'. (Thus, $D_{x'} \stackrel{\text{def}}{=} \{\gamma \in G \mid \gamma(x') = x'\}$, and $I_{x'} \stackrel{\text{def}}{=} \{\gamma \in D_{x'} \mid \gamma \text{ acts trivially on } k(x')\}$.)*

Then we have $D_{\tilde{x}} = D_{x'}^{(\Sigma)}$. More precisely, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \to & I_{x'} & \to & D_{x'} & \to & G_{k(x)} & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & I_{x'}^{\Sigma} & \to & D_{x'}^{(\Sigma)} & \to & G_{k(x)} & \to & 1 \\
 & & \| & & \| & & \| & & \\
1 & \to & I_{\tilde{x}} & \to & D_{\tilde{x}} & \to & G_{k(x)} & \to & 1
\end{array}$$

where the horizontal rows are exact and the vertical arrows are surjective.
(ii) *The inertia subgroup $I_{\tilde{x}}$ possesses a unique p-Sylow subgroup $I_{\tilde{x}}^{\mathrm{w}}$. The quotient $I_{\tilde{x}}^{\mathrm{t}} \overset{\mathrm{def}}{=} I_{\tilde{x}}/I_{\tilde{x}}^{\mathrm{w}}$ is isomorphic to $\hat{\mathbb{Z}}^{\Sigma^{\dagger}}$, and is naturally identified with the Galois group $\mathrm{Gal}(K_x^{\mathrm{t}}/K_x^{\mathrm{ur}})$, where K_x^{ur} (resp. K_x^{t}) is the maximal unramified (resp. tamely ramified) extension of the x-adic completion K_x of K. We have a natural exact sequence:*

$$1 \to I_{\tilde{x}}^{\mathrm{t}} \to D_{\tilde{x}}^{\mathrm{t}} \to G_{k(x)} \to 1,$$

where $D_{\tilde{x}}^{\mathrm{t}} \overset{\mathrm{def}}{=} \mathrm{Gal}(K_x^{\mathrm{t}}/K_x)$.

In particular, $I_{\tilde{x}}^{\mathrm{t}}$ has a natural structure of $G_{k(x)}$-module. Further, there exists a natural identification $I_{\tilde{x}}^{\mathrm{t}} \xrightarrow{\sim} M_{k(\tilde{x})}^{\Sigma}$ of $G_{k(x)}$-modules.

Proof. (i) The only nontrivial point in the assertion is that the natural homomorphism $I_{x'}^{\Sigma} \to \overline{G}^{\Sigma}$ (whose image coincides with $I_{\tilde{x}}$) is injective. A proof of this fact is as follows.

Step 1. First, if $p \notin \Sigma$, this follows from [Saïdi-Tamagawa3], Lemma 1.3.

Step 2. Next, consider the special case $\Sigma = \{p\}$. Then we have

$$\mathrm{cd}_p(I_{\tilde{x}}) \leq \mathrm{cd}_p(\overline{G}^{\{p\}}) \leq \mathrm{cd}_p(\overline{G}) \leq 1.$$

Indeed, the first inequality follows from the fact that $I_{\tilde{x}} \subset \overline{G}^{\{p\}}$ ([Serre1]), the second inequality follows from Lemma 1.1 (i) (and the fact that $\mathrm{Ker}(\overline{G} \twoheadrightarrow \overline{G}^{\{p\}})$ is the absolute Galois group of a field of characteristic p ([Serre1])), and the third inequality follows from the fact that $\overline{G}$ is the absolute Galois group of a field of characteristic p ([Serre1]). In particular, the surjective homomorphism $I_{x'}^{\{p\}} \twoheadrightarrow I_{\tilde{x}}$ of pro-p groups admits a section $s : I_{\tilde{x}} \to I_{x'}^{\{p\}}$. Now, the homomorphism $I_{x'}^{\{p\}} \to \overline{G}^{\{p\}}$ induces a homomorphism $(I_{x'}^{\{p\}})^{\mathrm{ab}}/p \to (\overline{G}^{\{p\}})^{\mathrm{ab}}/p$ of pro-p abelian groups killed by p. By Artin-Schreier theory, the (Pontryagin) dual of this last homomorphism is identified with $K\bar{k}/\wp(K\bar{k}) \to (K\bar{k})_{\bar{x}}/\wp((K\bar{k})_{\bar{x}})$. Here, $\bar{x}$ denotes the image in $(X \times_k \bar{k})^{\mathrm{cl}}$ of $\tilde{x} \in \tilde{X}^{\mathrm{cl}}$, $(K\bar{k})_{\bar{x}}$ denotes the $\bar{x}$-adic completion of $K\bar{k}$, and $\wp : \alpha \mapsto \alpha^p - \alpha$. Observe that $\wp((K\bar{k})_{\bar{x}}) \supset \wp(\mathcal{O}_{\bar{x}}) = \mathcal{O}_{\bar{x}}$ by Hensel's lemma, where $\mathcal{O}_{\bar{x}}$ denotes the ring of integers of $(K\bar{k})_{\bar{x}}$. Now, since the natural homomorphism $K\bar{k} \to (K\bar{k})_{\bar{x}}/\mathcal{O}_{\bar{x}}$ is surjective (as follows from the Riemann-Roch theorem), the homomorphism $K\bar{k}/\wp(K\bar{k}) \to (K\bar{k})_{\bar{x}}/\wp((K\bar{k})_{\bar{x}})$ is also surjective. Thus, we have $(I_{x'}^{\{p\}})^{\mathrm{ab}}/p \hookrightarrow (\overline{G}^{\{p\}})^{\mathrm{ab}}/p$, hence, a fortiori, $(I_{x'}^{\{p\}})^{\mathrm{ab}}/p \hookrightarrow (I_{\tilde{x}})^{\mathrm{ab}}/p$. Thus,

$(I_{x'}^{\{p\}})^{\mathrm{ab}}/p \xrightarrow{\sim} (I_{\tilde{x}})^{\mathrm{ab}}/p$. In particular, $s(I_{\tilde{x}})$ must surject onto $(I_{x'}^{\{p\}})^{\mathrm{ab}}/p$. By the Frattini property, this implies that $s(I_{\tilde{x}}) = I_{x'}^{\{p\}}$, hence $I_{x'}^{\{p\}} \xrightarrow{\sim} I_{\tilde{x}}$, as desired.

Step 3. Finally, consider a general Σ. By Step 1, we may assume that $p \in \Sigma$. For a profinite group H, denote by $H^{\Sigma^\dagger\text{-by-}\{p\}}$ the (unique) maximal quotient of H which is an extension of a pro-$\Sigma^\dagger$ group by a (normal) pro-p group. Let $H(p)$ denote the kernel of $H^{\Sigma^\dagger\text{-by-}\{p\}} \twoheadrightarrow H^{\Sigma^\dagger}$. In general, the natural homomorphism $H^{\Sigma} \twoheadrightarrow H^{\Sigma^\dagger\text{-by-}\{p\}}$ is not an isomorphism. But we have $I_{x'}^{\Sigma} \xrightarrow{\sim} I_{x'}^{\Sigma^\dagger\text{-by-}\{p\}}$, since the wild inertia subgroup (i.e. the p-Sylow subgroup of the inertia group) is normal. Thus, we have the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & I_{x'}(p) & \to & I_{x'}^{\Sigma} & \to & I_{x'}^{\Sigma^\dagger} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \overline{G}(p) & \to & \overline{G}^{\Sigma^\dagger\text{-by-}\{p\}} & \to & \overline{G}^{\Sigma^\dagger} & \to & 1
\end{array}$$

where the horizontal rows are exact. Here, the right vertical arrow is injective by Step 1, as $p \notin \Sigma^\dagger$. On the other hand, the left vertical arrow can be obtained as the projective limit of homomorphisms $(I_{x'} \cap \overline{H})^{\{p\}} \to \overline{H}^{\{p\}}$, where $\overline{H}$ runs over all open subgroups of $\overline{G}$ that contain the kernel of $\overline{G} \twoheadrightarrow \overline{G}^{\Sigma^\dagger}$. Thus, the left vertical arrow is injective by Step 2, hence the middle vertical arrow $I_{x'}^{\Sigma} \to \overline{G}^{\Sigma^\dagger\text{-by-}\{p\}}$ is also injective. Now, the homomorphism $I_{x'}^{\Sigma} \to \overline{G}^{\Sigma}$ is, a fortiori, injective, as desired.
(ii) This follows from (i), together with well-known facts on ramification theory (cf. [Serre2], Chapitre IV). $\square$

In fact, the decomposition subgroups of $G^{(\Sigma)}$ are completely determined by their group-theoretic structure. More precisely, we have the following fundamental result.

Proposition 1.3. *(Galois Characterization of Decomposition Subgroups) Consider the natural map* $D = D_{\tilde{X}}^{(\Sigma)} : \tilde{X}^{\mathrm{cl}} \to \mathrm{Sub}(G^{(\Sigma)}),\ \tilde{x} \mapsto D_{\tilde{x}}$.
(i) *The map D is Galois-equivariant. More precisely, for $g \in G^{(\Sigma)}$ and $\tilde{x} \in \tilde{X}^{\mathrm{cl}}$, we have $D_{g\tilde{x}} = gD_{\tilde{x}}g^{-1}$.*
(ii) *Assume $\Sigma \neq \emptyset$. Then D is injective. More precisely, let $\tilde{x} \neq \tilde{x}_1$ be two elements of $\tilde{X}^{\mathrm{cl}}$, then $D_{\tilde{x}} \cap D_{\tilde{x}_1}$ is pro-Σ' and is of infinite index both in $D_{\tilde{x}}$ and in $D_{\tilde{x}_1}$.*
(iii) *Assume $\Sigma^\dagger \neq \emptyset$ and $\ell \in \Sigma^\dagger$. Let $\mathrm{Dec}_\ell(G^{(\Sigma)}) \subset \mathrm{Sub}(G^{(\Sigma)})$ be the set of closed subgroups $\mathfrak{D}$ of $G^{(\Sigma)}$ satisfying the following property: There exists an open subgroup $\mathfrak{D}_0$ of $\mathfrak{D}$ such that for any open subgroup $\mathfrak{D}' \subset \mathfrak{D}_0$, $\dim_{\mathbb{F}_\ell} H^2(\mathfrak{D}', \mathbb{F}_\ell) = 1$. Define $\mathrm{Dec}_\ell^{\max}(G^{(\Sigma)}) \subset \mathrm{Dec}_\ell(G^{(\Sigma)})$ to be the set of maximal elements of $\mathrm{Dec}_\ell(G^{(\Sigma)})$ with respect to the inclusion relation. Then the image of D coincides with $\mathrm{Dec}_\ell^{\max}(G^{(\Sigma)})$. (In particular, $\mathrm{Dec}_\ell^{\max}(G^{(\Sigma)})$ does not depend on the choice of $\ell \in \Sigma^\dagger$.)*

Thus, $D : \tilde{X}^{\mathrm{cl}} \to \mathrm{Sub}(G^{(\Sigma)})$ induces a natural, Galois-equivariant bijection $\tilde{X}^{\mathrm{cl}} \xrightarrow{\sim} \mathrm{Dec}_\ell^{\max}(G^{(\Sigma)})$.

Proof. (i) This follows from the definition of decomposition group.
(ii) Let $\ell \in \Sigma$. If $\ell \neq p$, then $D_{\tilde{x}} \cap D_{\tilde{x}_1}$ is of order prime to ℓ by [Saïdi-Tamagawa3], Proposition 1.5 (i). The case $\ell = p$ can be treated along the same lines, by resorting to Artin-Schreier theory instead of Kummer theory. More precisely, let D_p be a p-Sylow subgroup of $D_{\tilde{x}} \cap D_{\tilde{x}_1}$, and suppose that $D_p \neq 1$. We have

$$\mathrm{cd}_p(D_p) \leq \mathrm{cd}_p(G^{(\Sigma)}) \leq \mathrm{cd}_p(G) \leq 1 < \infty.$$

Indeed, the first inequality follows from the fact that $D_p \subset G^{(\Sigma)}$ ([Serre1]), the second inequality follows from Lemma 1.1 (ii) (and the fact that $\mathrm{Ker}(\overline{G} \twoheadrightarrow \overline{G}^{\Sigma}) = \mathrm{Ker}(G \twoheadrightarrow G^{(\Sigma)})$ is the absolute Galois group of a field of characteristic p ([Serre1])), and the third inequality follows from the fact that G is the absolute Galois group of a field of characteristic p ([Serre1]). In particular, D_p is torsion-free, hence is infinite. Thus, one may replace $G^{(\Sigma)}$ by any open subgroup, and assume that the images x, x_1 in X^{cl} of $\tilde{x}, \tilde{x}_1 \in \tilde{X}^{\mathrm{cl}}$ are distinct, and that the image of D_p in $(G^{(\Sigma)})^{\mathrm{ab}}/p$ is nontrivial. In particular, this implies that the natural map

$$D_{\tilde{x}}^{\mathrm{ab}}/p \times D_{\tilde{x}_1}^{\mathrm{ab}}/p \to (G^{(\Sigma)})^{\mathrm{ab}}/p$$

induced by the group operation of $(G^{(\Sigma)})^{\mathrm{ab}}/p$ is not injective. By Artin-Schreier theory and Proposition 1.2 (i), this last condition is equivalent to saying that the natural map

$$K/\wp(K) \to K_x/\wp(K_x) \times K_{x_1}/\wp(K_{x_1})$$

is not surjective, where $\wp : \alpha \mapsto \alpha^p - \alpha$. (Observe that by Hensel's lemma $\wp(K_x)$ (resp. $\wp(K_{x_1})$) contains the maximal ideal of the ring of integers of K_x (resp. K_{x_1}), hence is open in K_x (resp. K_{x_1}).) This contradicts the approximation theorem (cf. [Neukirch], Lemma 8).

Thus, $D_{\tilde{x}} \cap D_{\tilde{x}_1}$ is pro-Σ', which also implies that it is of infinite index both in $D_{\tilde{x}}$ and in $D_{\tilde{x}_1}$. (Note that for $\ell \in \Sigma$ the pro-ℓ-Sylow subgroups of $D_{\tilde{x}}$ and $D_{\tilde{x}_1}$ are infinite.) In particular, D must be injective.

(iii) This is a special case of [Saïdi-Tamagawa3], Proposition 1.5 (ii). (This result goes back to [Uchida], where the case $\Sigma = \mathfrak{Primes}$ is treated.) $\square$

Remark 1.4. For other characterizations of decomposition groups, see [Saïdi-Tamagawa3], Remark 1.6.

§**2. Isomorphisms between Geometrically Pro-Σ Galois Groups.** In this section we follow the notations in §1. Let k, l be finite fields of characteristic p_k, p_l, respectively, and of cardinality q_k, q_l, respectively.

Let X, Y be smooth, proper, and geometrically connected curves over k, l, respectively. Let K, L be the function fields of X, Y, respectively. We will write $G_K \overset{\mathrm{def}}{=} \mathrm{Gal}(K^{\mathrm{sep}}/K)$, $G_L \overset{\mathrm{def}}{=} \mathrm{Gal}(L^{\mathrm{sep}}/L)$ for the absolute Galois groups of K, L, respectively.

Let $\Sigma_X, \Sigma_Y \subset \mathfrak{Primes}$ be sets of prime numbers. We assume that the Σ_X-cyclotomic character $\chi_{\Sigma_X} : G_k \to \prod_{\ell \in \Sigma_X \setminus \{p_k\}} \mathbb{Z}_\ell^\times$ is injective. (In the terminology of §3 (cf. Definition/Proposition 3.1), this is equivalent to saying that Σ_X is not k-small.) Write $G_K^{(\Sigma_X)}$ (resp. $G_L^{(\Sigma_Y)}$) for the maximal geometrically pro-Σ_X (resp. Σ_Y) quotient of G_K (resp. G_L). Thus, we have exact sequences:

$$1 \to \overline{G}_K^{\Sigma_X} \to G_K^{(\Sigma_X)} \xrightarrow{\mathrm{pr}} G_k \to 1,$$

and

$$1 \to \overline{G}_L^{\Sigma_Y} \to G_L^{(\Sigma_Y)} \xrightarrow{\mathrm{pr}} G_l \to 1,$$

where $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}/k)$ (resp. $G_l \overset{\mathrm{def}}{=} \mathrm{Gal}(\bar{l}/l)$) is the absolute Galois group of k (resp. l), and $\overline{G}_K^{\Sigma_X}$ (resp. $\overline{G}_L^{\Sigma_Y}$) is the maximal pro-$\Sigma_X$ (resp. pro-Σ_Y) quotient of the absolute Galois group $\overline{G}_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(K^{\mathrm{sep}}/K\bar{k})$ (resp. $\overline{G}_L \stackrel{\mathrm{def}}{=} \mathrm{Gal}(L^{\mathrm{sep}}/L\bar{l})$) of $K\bar{k}$ (resp. $L\bar{l}$).

For the rest of this section we will consider an isomorphism of profinite groups

$$\sigma : G_K^{(\Sigma_X)} \xrightarrow{\sim} G_L^{(\Sigma_Y)}$$

between the maximal geometrically pro-Σ_X (resp. pro-Σ_Y) quotient of the absolute Galois group G_K (resp. G_L). We write $\tilde{X}$ (resp. $\tilde{Y}$) for the normalization of X (resp. Y) in $K^{\sim}$ (resp. $L^{\sim}$). Here, $K^{\sim}/K$ (resp. $L^{\sim}/L$) is the subextension of K^{sep}/K (resp. L^{sep}/L) with Galois group $G_K^{(\Sigma_X)}$ (resp. $G_L^{(\Sigma_Y)}$).

Recall $\Sigma_X^{\dagger} = \Sigma_X \setminus \{p_k\}$, and $\Sigma_Y^{\dagger} = \Sigma_Y \setminus \{p_l\}$.

Lemma 2.1. *(Invariance of Sets of Primes)*
(i) *We have* $\Sigma_X^{\dagger} = \Sigma_Y^{\dagger}$, $\Sigma_X \cap \{p_k\} = \Sigma_Y \cap \{p_l\}$, *and* $\Sigma_X = \Sigma_Y$. *Set* $\Sigma^{\dagger} \stackrel{\mathrm{def}}{=} \Sigma_X^{\dagger} = \Sigma_Y^{\dagger}$, $\Sigma \stackrel{\mathrm{def}}{=} \Sigma_X = \Sigma_Y$, *and* $\Sigma' \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma$.
(ii) Σ *is infinite.*

Proof. (i) It follows from global class field theory for K that (for a prime number ℓ) the maximal pro-ℓ quotient $(G_K^{(\Sigma_X)})^{\mathrm{ab},\ell}$ of the maximal abelian quotient $(G_K^{(\Sigma_X)})^{\mathrm{ab}}$ of $G_K^{(\Sigma_X)}$ is described as follows:

$$(G_K^{(\Sigma_X)})^{\mathrm{ab},\ell} \simeq \begin{cases} \mathbb{Z}_\ell, & \ell \notin \Sigma_X, \\ \mathbb{Z}_\ell \times \overline{(\text{infinite torsion group})}, & \ell \in \Sigma_X^{\dagger}, \\ \mathbb{Z}_\ell^{\aleph_0}, & \ell \in \Sigma_X \cap \{p_k\}. \end{cases}$$

Here $\overline{(\text{infinite torsion group})}$ denotes the closure of the torsion subgroup (which is infinite) of $(G_K^{(\Sigma_X)})^{\mathrm{ab},\ell}$. A similar description holds for $(G_L^{(\Sigma_Y)})^{\mathrm{ab},\ell}$. This implies $\Sigma_X^{\dagger} = \Sigma_Y^{\dagger}$, $\Sigma_X \cap \{p_k\} = \Sigma_Y \cap \{p_l\}$, and $\Sigma_X = \Sigma_Y$.
(ii) This follows immediately from the assumption that the Σ_X-cyclotomic character $\hat{\mathbb{Z}} \simeq G_k \to \prod_{\ell \in \Sigma_X \setminus \{p_k\}} \mathbb{Z}_\ell^{\times}$ is injective. $\square$

Lemma 2.2. *(Set-Theoretic Correspondence between Points) The isomorphism* σ *induces naturally a bijection:*

$$\tilde{\phi} : \tilde{X}^{\mathrm{cl}} \xrightarrow{\sim} \tilde{Y}^{\mathrm{cl}}, \ \tilde{x} \mapsto \tilde{y},$$

such that

$$\sigma(D_{\tilde{x}}) = D_{\tilde{y}}, \ \forall \tilde{x} \in \tilde{X}^{\mathrm{cl}},$$

where $D_{\tilde{x}}$ *(resp.* $D_{\tilde{y}}$*) is the decomposition subgroup of* $G_K^{(\Sigma)}$ *(resp.* $G_L^{(\Sigma)}$*) at* $\tilde{x}$ *(resp.* $\tilde{y}$*), and* $\tilde{\phi}$ *induces a bijection*

$$\phi : X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}, \ x \mapsto y,$$

where x *(resp.* y*) is the image of* $\tilde{x}$ *(resp.* $\tilde{y}$*) in* X^{cl} *(resp.* Y^{cl}*). Thus, in particular,* σ *induces naturally a bijection:*

$$\phi : \mathrm{Div}_X \xrightarrow{\sim} \mathrm{Div}_Y$$

between the groups of divisors of X and Y, respectively.

Proof. This follows from Proposition 1.3 and Lemma 2.1 (to ensure $\Sigma^\dagger = \Sigma^\dagger_X = \Sigma^\dagger_Y \neq \emptyset$). $\square$

Let $x \in X^{\mathrm{cl}}$, and $y \stackrel{\mathrm{def}}{=} \phi(x) \in Y^{\mathrm{cl}}$. Write K_x (resp. L_y) for the completion of K (resp. L) at x (resp. y). Denote the ring of integers of K_x (resp. L_y) by $\mathcal{O}_x$ (resp. $\mathcal{O}_y$). Write $D_x \stackrel{\mathrm{def}}{=} D_{\tilde{x}}$ (resp. $D_y \stackrel{\mathrm{def}}{=} D_{\tilde{y}}$) and $I_x \stackrel{\mathrm{def}}{=} I_{\tilde{x}}$ (resp. $I_y \stackrel{\mathrm{def}}{=} I_{\tilde{y}}$) for the decomposition and the inertia subgroups of $G_K^{(\Sigma)}$ (resp. $G_L^{(\Sigma)}$) at $\tilde{x}$ (resp. $\tilde{y}$), where $\tilde{x} \in \tilde{X}^{\mathrm{cl}}$ (resp. $\tilde{y} \in \tilde{Y}^{\mathrm{cl}}$) is a point above x (resp. y). Thus, D_x (resp. D_y) is defined only up to conjugation. By Proposition 1.2 (i) and local class field theory (cf., e.g., [Serre3]), we have natural isomorphisms

$$(K_x^\times)^{\wedge,(\Sigma)} \xrightarrow{\sim} D_x^{\mathrm{ab}},$$

and

$$(L_y^\times)^{\wedge,(\Sigma)} \xrightarrow{\sim} D_y^{\mathrm{ab}},$$

where $(K_x^\times)^\wedge$ (resp. $(L_y^\times)^\wedge$) is the profinite completion of the topological group $K_x^\times$ (resp. $L_y^\times$), and we set

$$(K_x^\times)^{\wedge,(\Sigma)} \stackrel{\mathrm{def}}{=} (K_x^\times)^\wedge / \operatorname{Ker}(\mathcal{O}_x^\times \twoheadrightarrow (\mathcal{O}_x^\times)^\Sigma)$$

and

$$(L_y^\times)^{\wedge,(\Sigma)} \stackrel{\mathrm{def}}{=} (L_y^\times)^\wedge / \operatorname{Ker}(\mathcal{O}_y^\times \twoheadrightarrow (\mathcal{O}_y^\times)^\Sigma),$$

where $(\mathcal{O}_x^\times)^\Sigma$, $(\mathcal{O}_y^\times)^\Sigma$ stand for the maximal pro-Σ quotients of the profinite groups $\mathcal{O}_x^\times$, $\mathcal{O}_y^\times$, respectively.

More concretely, we have

$$(K_x^\times)^{\wedge,(\Sigma)} = (K_x^\times)^\wedge / N_x,\ (\mathcal{O}_x^\times)^\Sigma = \mathcal{O}_x^\times / N_x$$

with

$$N_x \stackrel{\mathrm{def}}{=} \operatorname{Ker}(\mathcal{O}_x^\times \twoheadrightarrow (\mathcal{O}_x^\times)^\Sigma) = \begin{cases} U_x^1(\mathcal{O}_x^{\times,\mathrm{tor}}\{\Sigma'\}), & \Sigma = \Sigma^\dagger, \\ \mathcal{O}_x^{\times,\mathrm{tor}}\{\Sigma'\}, & \Sigma \neq \Sigma^\dagger, \end{cases}$$

and we have a similar description for $(L_y^\times)^{\wedge,(\Sigma)}$. Here, U_x^1 is the group of principal units in $\mathcal{O}_x^\times$, and $\mathcal{O}_x^{\times,\mathrm{tor}}\{\Sigma'\}$ is the group of Σ'-primary torsion of $\mathcal{O}_x^\times$. (Observe that $\mathcal{O}_x^{\times,\mathrm{tor}}\{\Sigma'\} \xrightarrow{\sim} k(x)^\times\{\Sigma'\}$.)

We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_x^\times & \to & K_x^\times & \to & \mathbb{Z} & \to & 0 \\ & & \downarrow & & \downarrow & & \cap & & \\ 0 & \to & (\mathcal{O}_x^\times)^\Sigma & \to & (K_x^\times)^{\wedge,(\Sigma)} & \to & \hat{\mathbb{Z}} & \to & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \to & \operatorname{Im}(I_x) & \to & D_x^{\mathrm{ab}} & \to & G_k & \to & 0, \end{array}$$

where the horizontal rows are exact. Here, the map $K_x^\times \to \mathbb{Z}$ is the x-adic valuation, $\mathrm{Im}(I_x)$ is the image of I_x in D_x^{ab}, and the map $\hat{\mathbb{Z}} \xrightarrow{\sim} G_k$ sends $1 \in \hat{\mathbb{Z}}$ to the q_k-th power Frobenius element in G_k.

Further, the natural filtration

$$(k(x)^\times)^\Sigma, (U_x^1)^\Sigma \subset (\mathcal{O}_x^\times)^\Sigma \subset (K_x^\times)^{(\Sigma)} \subset (K_x^\times)^{\wedge,(\Sigma)},$$

where $(U_x^1)^\Sigma$ denotes the maximal pro-Σ quotient of U_x^1 and $(K_x^\times)^{(\Sigma)}$ is the image of $K_x^\times$ in $(K_x^\times)^{\wedge,(\Sigma)}$, induces, via the above isomorphism $(K_x^\times)^{\wedge,(\Sigma)} \xrightarrow{\sim} D_x^{\mathrm{ab}}$, a filtration

$$\mathrm{Im}((k(x)^\times)^\Sigma), \mathrm{Im}((U_x^1)^\Sigma) \subset \mathrm{Im}((\mathcal{O}_x^\times)^\Sigma) \subset \mathrm{Im}((K_x^\times)^{(\Sigma)}) \subset \mathrm{Im}((K_x^\times)^{\wedge,(\Sigma)}) = D_x^{\mathrm{ab}}.$$

Here, $\mathrm{Im}((\mathcal{O}_x^\times)^\Sigma)$ coincides with the image $\mathrm{Im}(I_x)$ in D_x^{ab} of I_x. Similar statements and filtrations hold for $(L_y^\times)^{\wedge,(\Sigma)}$ and D_y^{ab}.

Let

$$\sigma_{x,y} : D_x \xrightarrow{\sim} D_y$$

be the isomorphism of profinite groups induced by σ (which is only defined up to conjugation) (cf. Lemma 2.2). Write

$$\sigma_{x,y}^{\mathrm{ab}} : D_x^{\mathrm{ab}} \xrightarrow{\sim} D_y^{\mathrm{ab}}$$

for the induced isomorphism between the maximal abelian quotients of D_x and D_y, respectively.

Lemma 2.3. *(Invariants of Isomorphisms between Geometrically Pro-Σ Decomposition Groups)*
(i) *The isomorphism $\sigma_{x,y}^{\mathrm{ab}} : D_x^{\mathrm{ab}} \to D_y^{\mathrm{ab}}$ preserves the images* $\mathrm{Im}((k(x)^\times)^\Sigma)$ *and* $\mathrm{Im}((k(y)^\times)^\Sigma)$ *, hence it induces naturally an isomorphism*

$$\tau_{x,y} : (k(x)^\times)^\Sigma \xrightarrow{\sim} (k(y)^\times)^\Sigma$$

between the maximal pro-Σ quotients of the multiplicative groups of the residue fields at x and y, respectively.
(ii) *The isomorphism $\sigma_{x,y}$ induces naturally an isomorphism $M^\Sigma_{\overline{k(x)}} \xrightarrow{\sim} M^\Sigma_{\overline{k(y)}}$, which is Galois-equivariant with respect to $\sigma_{x,y}$. In particular, $\sigma_{x,y}$ commutes with the Σ-parts of the cyclotomic characters $\chi_x : D_x \to (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$ (resp. $\chi_y : D_y \to (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$) of D_x (resp. D_y), i.e., we have a commutative diagram:*

$$\begin{tikzcd}
(\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times \arrow[r, equal] & (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times \\
D_x \arrow[u, "\chi_x"] \arrow[r, "\sigma_{x,y}"] & D_y \arrow[u, "\chi_y"']
\end{tikzcd}$$

(iii) *The isomorphism $\sigma_{x,y}$ preserves I_x and I_y.*

Proof. The proofs of (i)(ii)(iii) are similar to those of [Saïdi-Tamagawa3], Proposition 2.1 (iii)(iv)(v), respectively. More precisely:

(i) By Proposition 1.2 (i) and local class field theory, $\mathrm{Im}((k(x)^\times)^\Sigma) \subset D_x^{\mathrm{ab}}$ coincides with the torsion subgroup $D_x^{\mathrm{ab,tor}}$ of D_x^{ab}, and a similar statement holds for $\mathrm{Im}((k(y)^\times)^\Sigma) \subset D_y^{\mathrm{ab}}$. From this, the assertion follows.
(ii) By applying (i) to open subgroups of D_x, D_y, (which correspond to each other via $\sigma_{x,y}$), and passing to the projective limit, we obtain a natural isomorphism $M^\Sigma_{\overline{k(x)}} \overset{\sim}{\to} M^\Sigma_{\overline{k(y)}}$ between the modules of roots of unity. More precisely, let E be a finite extension of K_x corresponding to an open subgroup H of D_x and h the residue field of E. Then the following diagram commutes:

$$\begin{array}{ccccc} (h^\times)^\Sigma & \subset & (E^\times)^{\wedge,(\Sigma)} & \overset{\sim}{\to} & H^{\mathrm{ab}} \\ \downarrow & & \downarrow & & \downarrow \\ (k(x)^\times)^\Sigma & \subset & (K_x^\times)^{\wedge,(\Sigma)} & \overset{\sim}{\to} & D_x^{\mathrm{ab}}, \end{array}$$

where the map $H^{\mathrm{ab}} \to D_x^{\mathrm{ab}}$ is induced by the natural inclusion $H \subset D_x$ and the map $(E^\times)^{\wedge,(\Sigma)} \to (K_x^\times)^{\wedge,(\Sigma)}$ is induced by the norm map $E^\times \to K_x^\times$. The map $(h^\times)^\Sigma \to (k(x)^\times)^\Sigma$ is induced by the (norm) map $(E^\times)^{\wedge,(\Sigma)} \to (K_x^\times)^{\wedge,(\Sigma)}$, hence coincides with the e-th power of the map $(h^\times)^\Sigma \to (k(x)^\times)^\Sigma$ induced by the norm map $h^\times \to k(x)^\times$, where e denotes the ramification index of E/K_x. Thus, if we consider the projective subsystem formed by the open subgroups of D_x that are obtained as the inverse image of an open subgroup of $D_x^{\mathrm{ab}}/(\mathrm{torsion})$, we get a projective system $((h^\times)^\Sigma)$ with surjective transition homomorphisms (as all the ramification indices are powers of p) whose limit is identified with $M^\Sigma_{\overline{k(x)}}$. Indeed, (again as all the ramification indices are powers of p) the limit is unchanged if the projective system is replaced with the subsystem indexed by the open subgroups $H \subseteq D_x$ obtained as inverse image of open subgroups of $D_x/I_x = G_{k(x)}$. Then the above norm map $h^\times \to k(x)^\times$ is just the a-th power map, where $a = \sum_{i=0}^{[h:k(x)]-1} |k(x)|^i = |h^\times|/|k(x)^\times|$. [This sort of precise argument involving suitable projective subsystems should have been inserted also in the proof of [Saïdi-Tamagawa3], Proposition 2.1 (iv).] Further, this identification is (by construction) Galois-compatible with respect to the isomorphism $\sigma_{x,y}$, as desired. The second assertion follows from this Galois-compatibility.
(iii) The character $\chi_x : D_x \to (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$ (resp. $\chi_y : D_y \to (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$) factors as $D_x \twoheadrightarrow D_x/I_x = G_{k(x)} \overset{\chi_{k(x)}}{\to} (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$ (resp. $D_y \twoheadrightarrow D_y/I_y = G_{k(y)} \overset{\chi_{k(y)}}{\to} (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$), where $\chi_{k(x)}$ (resp. $\chi_{k(y)}$) is the Σ-cyclotomic character of $G_{k(x)}$ (resp. $G_{k(y)}$). Further, since the Σ-cyclotomic character of G_k is assumed to be injective, $\chi_{k(x)}$ is also injective. Thus, I_x coincides with the kernel of χ_x and I_y is included in the kernel of χ_y.

Now, it follows from (ii) that $\sigma_{x,y}(I_x) \supset I_y$, hence

$$\hat{\mathbb{Z}} \simeq G_{k(x)} = D_x/I_x \overset{\sigma_{x,y}}{\overset{\sim}{\to}} D_y/\sigma_{x,y}(I_x) \twoheadleftarrow D_y/I_y = G_{k(y)} \simeq \hat{\mathbb{Z}}.$$

As any surjective homomorphism $\hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$ is an isomorphism, this shows $\sigma_{x,y}(I_x) = I_y$, as desired. $\square$

2.4. *Invariants of Isomorphisms between Geometrically Pro-Σ Galois Groups.*

Lemma 2.4.1. *The following diagram is commutative:*

$$
\begin{array}{ccc}
(\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times & = & (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times \\
\chi_k \uparrow & & \chi_l \uparrow \\
G_k & & G_l \\
\mathrm{pr}_K \uparrow & & \mathrm{pr}_L \uparrow \\
G_K^{(\Sigma)} & \xrightarrow{\sigma} & G_L^{(\Sigma)}
\end{array}
\tag{2.1}
$$

where χ_k (resp. χ_l) is the Σ-part of the cyclotomic character of G_k (resp. G_l).

Proof. For each $\tilde{x} \in \tilde{X}^{\mathrm{cl}}$, with $\tilde{y} \stackrel{\mathrm{def}}{=} \tilde{\phi}(\tilde{x}) \in \tilde{Y}^{\mathrm{cl}}$, we have the following diagram:

$$
\begin{array}{ccc}
(\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times & = & (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times \\
\chi_K \uparrow & & \chi_L \uparrow \\
G_K^{(\Sigma)} & \xrightarrow{\sigma} & G_L^{(\Sigma)} \\
\uparrow & & \uparrow \\
D_{\tilde{x}} & \xrightarrow{\sigma_{\tilde{x},\tilde{y}}} & D_{\tilde{y}}
\end{array}
\tag{2.2}
$$

where the maps $D_{\tilde{x}} \to G_K^{(\Sigma)}$ and $D_{\tilde{y}} \to G_L^{(\Sigma)}$ are the natural inclusions, the lower square is commutative, and χ_K (resp. χ_L) is the Σ-part of the cyclotomic character of $G_K^{(\Sigma)}$ (resp. $G_L^{(\Sigma)}$). Since the restriction of χ_K (resp. χ_L) to $D_{\tilde{x}}$ (resp. $D_{\tilde{y}}$) coincides with the Σ-part of the cyclotomic character of $D_{\tilde{x}}$ (resp. $D_{\tilde{y}}$), the exterior square of (2.2) is commutative by Lemma 2.3 (ii). Hence the upper square of (2.2) is also commutative, since $G_K^{(\Sigma)}$ is (topologically) generated by the decomposition subgroups $D_{\tilde{x}}$, $\forall \tilde{x} \in \tilde{X}^{\mathrm{cl}}$, as follows from Chebotarev's density theorem. The commutativity of the diagram (2.1) follows from this, since $\chi_k \circ \mathrm{pr}_K = \chi_K$ and $\chi_l \circ \mathrm{pr}_L = \chi_L$. $\square$

Lemma 2.4.2. *The isomorphism σ commutes with the canonical surjections $G_K^{(\Sigma)} \twoheadrightarrow \pi_1(X)^{(\Sigma)}$ (resp. $G_L^{(\Sigma)} \twoheadrightarrow \pi_1(Y)^{(\Sigma)}$), where $\pi_1(X)^{(\Sigma)} \stackrel{\mathrm{def}}{=} \pi_1(X)/\operatorname{Ker}(\pi_1(\overline{X}) \twoheadrightarrow \pi_1(\overline{X})^{\Sigma})$ (resp. $\pi_1(Y)^{(\Sigma)} \stackrel{\mathrm{def}}{=} \pi_1(Y)/\operatorname{Ker}(\pi_1(\overline{Y}) \twoheadrightarrow \pi_1(\overline{Y})^{\Sigma})$) is the maximal geometrically pro-Σ quotient of the fundamental group $\pi_1(X)$ (resp. $\pi_1(Y)$) of X (resp. Y) (with respect to the geometric point* $\mathrm{Spec}(K^{\mathrm{sep}}) \to X$ *(resp.* $\mathrm{Spec}(L^{\mathrm{sep}}) \to Y$*)). More precisely, we have a commutative diagram:*

$$
\begin{array}{ccc}
G_K^{(\Sigma)} & \longrightarrow & \pi_1(X)^{(\Sigma)} \\
\sigma \downarrow & & \downarrow \\
G_L^{(\Sigma)} & \longrightarrow & \pi_1(Y)^{(\Sigma)}
\end{array}
$$

where the vertical arrows are isomorphisms.

Proof. Let $\mathcal{I}_X$ (resp. $\mathcal{I}_Y$) denote the closed normal subgroup of $G_K^{(\Sigma)}$ (resp. $G_L^{(\Sigma)}$) (topologically) generated by the inertia subgroups. Then the isomorphism σ maps $\mathcal{I}_X$ onto $\mathcal{I}_Y$ by Lemma 2.2 and Lemma 2.3 (iii). Since $\pi_1(X)^{(\Sigma)} = G_K^{(\Sigma)}/\mathcal{I}_X$ and $\pi_1(Y)^{(\Sigma)} = G_L^{(\Sigma)}/\mathcal{I}_Y$, the assertion follows. $\square$

Lemma 2.4.3. *The isomorphism σ commutes with the canonical projections $\mathrm{pr}_K : G_K^{(\Sigma)} \twoheadrightarrow G_k$ and $\mathrm{pr}_L : G_L^{(\Sigma)} \twoheadrightarrow G_l$, i.e., we have a commutative diagram:*

$$\begin{array}{ccc} G_K^{(\Sigma)} & \xrightarrow{\mathrm{pr}_K} & G_k \\ \sigma\downarrow & & \downarrow \\ G_L^{(\Sigma)} & \xrightarrow{\mathrm{pr}_L} & G_l \end{array} \tag{2.3}$$

where the vertical arrows are isomorphisms.

Proof. This follows from Lemma 2.4.2, since we have $G_k = \pi_1(X)^{(\Sigma),\mathrm{ab}}/(\text{torsion})$ as a quotient of $G_K^{(\Sigma)}$ (cf. [Tamagawa], Proposition 3.3 (ii)). Similarly, $G_l = \pi_1(Y)^{(\Sigma),\mathrm{ab}}/(\text{torsion})$ as a quotient of $G_L^{(\Sigma)}$. Here, $\pi_1(X)^{(\Sigma),\mathrm{ab}}$ (resp. $\pi_1(Y)^{(\Sigma),\mathrm{ab}}$) is the maximal abelian quotient of $\pi_1(X)^{(\Sigma)}$ (resp. $\pi_1(Y)^{(\Sigma)}$). (Alternatively, this follows from Lemma 2.4.1. Indeed, since $\chi_k \circ \mathrm{pr}_K = \chi_K$, $\chi_l \circ \mathrm{pr}_L = \chi_L$, and pr_K, pr_L are surjective, $\mathrm{Im}(\chi_K) \subset (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$ coincides with $\mathrm{Im}(\chi_k)$ (similarly, $\mathrm{Im}(\chi_L) \subset (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$ coincides with $\mathrm{Im}(\chi_l)$). By assumption χ_k is injective. Since this injectivity condition is equivalent to requiring $\mathrm{Im}(\chi_k) \simeq \hat{\mathbb{Z}}$ (as abstract profinite groups), we see that both χ_k and χ_l are injective. In summary, we have

$$G_k \xrightarrow{\sim} \mathrm{Im}(\chi_k) = \mathrm{Im}(\chi_K) \subset (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$$

and

$$G_l \xrightarrow{\sim} \mathrm{Im}(\chi_l) = \mathrm{Im}(\chi_L) \subset (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times.$$

Now, the assertion follows from the commutativity of the diagram (2.1).) $\square$

Lemma 2.4.4. *For each subset $T \subset \Sigma$, the isomorphism σ commutes with the canonical surjections $G_K^{(\Sigma)} \twoheadrightarrow G_K^{(T)}$, and $G_L^{(\Sigma)} \twoheadrightarrow G_L^{(T)}$, i.e., we have a commutative diagram:*

$$\begin{array}{ccc} G_K^{(\Sigma)} & \longrightarrow & G_K^{(T)} \\ \sigma\downarrow & & \downarrow \\ G_L^{(\Sigma)} & \longrightarrow & G_L^{(T)} \end{array}$$

where the vertical arrows are isomorphisms.

Proof. This follows from Lemma 2.4.3, since the quotient $G_K^{(\Sigma)} \twoheadrightarrow G_K^{(T)}$ can be characterized as

$$G_K^{(T)} = G_K^{(\Sigma)} / \mathrm{Ker}(\mathrm{Ker}(\mathrm{pr}_K) \twoheadrightarrow (\mathrm{Ker}(\mathrm{pr}_K))^T),$$

and a similar statement holds for $G_L^{(\Sigma)} \twoheadrightarrow G_L^{(T)}$. $\square$

Lemma 2.4.5. *The bijection $\phi : X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ commutes with the degree functions $\deg_X : X^{\mathrm{cl}} \to \mathbb{Z}_{>0}$, $x \mapsto [k(x) : k]$, and $\deg_Y : Y^{\mathrm{cl}} \to \mathbb{Z}_{>0}$, $y \mapsto [k(y) : l]$.*

Proof. This follows from Lemmas 2.2 and 2.4.3. Indeed, for each $x \in X^{\mathrm{cl}}$, take $\tilde{x} \in \tilde{X}^{\mathrm{cl}}$ above x and set $y = \phi(x)$ and $\tilde{y} = \tilde{\phi}(\tilde{x})$ (which is above y). Then we have

$$\deg_X(x) = (G_k : \mathrm{pr}_K(D_{\tilde{x}})) = (G_l : \mathrm{pr}_L(D_{\tilde{y}})) = \deg_Y(y). \quad \square$$

Lemma 2.4.6. *For each integer $n > 0$, let $k \subset k_n \subset \bar{k}$ (resp. $l \subset l_n \subset \bar{l}$) denote the unique extension with $[k_n : k] = [l_n : l] = n$. Then we have $\sharp(X(k_n)) = \sharp(Y(l_n))$ for all $n > 0$.*

Proof. This follows from Lemma 2.4.5, since

$$\sharp(X(k_n)) = \sum_{0<d|n} d \cdot \sharp(\deg_X^{-1}(d)),$$

and

$$\sharp(Y(l_n)) = \sum_{0<d|n} d \cdot \sharp(\deg_Y^{-1}(d)). \quad \square$$

Lemma 2.4.7. (i) *We have $q_k = q_l$. In particular, $p_k = p_l$.*
(ii) *Notations as in Lemma 2.4.6, we have $\sharp(k_n) = \sharp(l_n)$ for all $n > 0$.*

Proof. (i) This follows from Lemma 2.4.6 (cf. [Pop2], Lemma 2.3). More precisely, by the Weil estimate, we have

$$1 + q_k^n - 2g_X q_k^{\frac{n}{2}} \leq \sharp(X(k_n)) \leq 1 + q_k^n + 2g_X q_k^{\frac{n}{2}},$$

where g_X denotes the genus of X, hence

$$\frac{\sharp(X(k_n))}{q_k^n} \to 1 \ (n \to \infty),$$

and similarly $\frac{\sharp(Y(l_n))}{q_l^n} \to 1$ $(n \to \infty)$. Now, by Lemma 2.4.6, we obtain

$$\left(\frac{q_k}{q_l}\right)^n = \frac{q_k^n}{q_l^n} \to 1 \ (n \to \infty),$$

which implies $q_k = q_l$, as desired.
(ii) This follows immediately from (i), as $\sharp(k_n) = q_k^n$ and $\sharp(l_n) = q_l^n$. $\square$

Set $p \stackrel{\text{def}}{=} p_k = p_l$ and $q \stackrel{\text{def}}{=} q_k = q_l$.

Lemma 2.4.8. *The bijection $\phi : X^{\text{cl}} \xrightarrow{\sim} Y^{\text{cl}}$ commutes with the norm functions $N_X : X^{\text{cl}} \to \mathbb{Z}_{>0}$, $x \mapsto \sharp(k(x))$, and $N_Y : Y^{\text{cl}} \to \mathbb{Z}_{>0}$, $y \mapsto \sharp(k(y))$.*

Proof. This follows from Lemmas 2.4.5 and 2.4.7 (i). $\square$

Lemma 2.4.9. *The isomorphism $G_k \to G_l$ induced naturally by σ (cf. Lemma 2.4.3) maps the q-th power Frobenius element φ_k of G_k to the q-th power Frobenius element φ_l of G_l.*

Proof. As shown in the (alternative) proof of Lemma 2.4.3, $\chi_k : G_k \to (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$ and $\chi_l : G_l \to (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$ are injective. Thus, $\varphi_k \in G_k$ can be characterized by the property $\chi_k(\varphi_k) = q \in \hat{\mathbb{Z}}^{\Sigma^\dagger}$, and similarly $\chi_l(\varphi_l) = q \in \hat{\mathbb{Z}}^{\Sigma^\dagger}$. Now, the assertion follows from the commutativity of the diagram (2.1) in Lemma 2.4.1, and the diagram (2.3) in Lemma 2.4.3. $\square$

Lemma 2.4.10. *In the notation of Lemma 2.4.6, we have $\sharp(J_X(k_n)) = \sharp(J_Y(l_n))$ for all $n > 0$, where J_X (resp. J_Y) denotes the jacobian variety of X (resp. Y).*

Proof. By Lemmas 2.4.2 and 2.4.3, the isomorphism $\pi_1(X)^{(\Sigma)} \xrightarrow{\sim} \pi_1(Y)^{(\Sigma)}$ induced by σ preserves $\pi_1(\overline{X})^{\Sigma} = \mathrm{Ker}(\pi_1(X)^{(\Sigma)} \twoheadrightarrow G_k)$ and $\pi_1(\overline{Y})^{\Sigma} = \mathrm{Ker}(\pi_1(Y)^{(\Sigma)} \twoheadrightarrow G_l)$. It follows from this that σ induces an isomorphism

$$T(J_X)^{\Sigma} = (\pi_1(\overline{X})^{\Sigma})^{\mathrm{ab}} \xrightarrow{\sim} (\pi_1(\overline{Y})^{\Sigma})^{\mathrm{ab}} = T(J_Y)^{\Sigma}$$

(where $T(J_X) \stackrel{\mathrm{def}}{=} \prod_{\ell\in\mathfrak{Primes}} T_\ell(J_X)$ (resp. $T(J_Y) \stackrel{\mathrm{def}}{=} \prod_{\ell\in\mathfrak{Primes}} T_\ell(J_Y)$) is the full Tate module of J_X (resp. J_Y) and $T(J_X)^{\Sigma}$ (resp. $T(J_Y)^{\Sigma}$) is its maximal pro-Σ quotient), which is Galois-equivariant with respect to the isomorphism $G_k \xrightarrow{\sim} G_l$ in Lemma 2.4.3. Thus, it follows from Lemma 2.4.9 that $P_{X,n} = P_{Y,n}$ for all $n > 0$, where $P_{X,n}$ (resp. $P_{Y,n}$) denotes the characteristic polynomial for the action of φ_k^n (resp. φ_l^n) on the free $\hat{\mathbb{Z}}^{\Sigma^\dagger}$-module $T(J_X)^{\Sigma^\dagger}$ (resp. $T(J_Y)^{\Sigma^\dagger}$). Now, we have

$$\sharp(J_X(k_n)) = P_{X,n}(1) = P_{Y,n}(1) = \sharp(J_Y(l_n)),$$

as desired. □

Remark 2.4.11. Let ℓ be a prime $\neq p = p_k$. When $\Sigma = \{\ell\}$, most of the results presented in Lemmas 2.4.1-2.4.9 are proved in [Pop2], Part I, 2, without resorting to Lemma 2.3 which relies heavily on local class field theory. (In fact, in [Pop2], function fields with arbitrary transcendence degree are also treated.)

Further, when $\Sigma = \Sigma^\dagger$ (i.e., $\Sigma \not\ni p$), the quotient $G_K^{(\Sigma)} \twoheadrightarrow G_k$ can be identified with $G_K^{(\Sigma)} \twoheadrightarrow (G_K^{(\Sigma)})^{\mathrm{ab}}/\overline{(G_K^{(\Sigma)})^{\mathrm{ab,tor}}}$, where $(G_K^{(\Sigma)})^{\mathrm{ab,tor}}$ is the torsion subgroup of $(G_K^{(\Sigma)})^{\mathrm{ab}}$ and $\overline{(G_K^{(\Sigma)})^{\mathrm{ab,tor}}}$ is its closure in $(G_K^{(\Sigma)})^{\mathrm{ab}}$ (cf. the proof of Lemma 2.1 (i)). It follows from this that for each $\ell \in \Sigma$, the quotient $G_K^{(\Sigma)} \twoheadrightarrow G_K^{(\{\ell\})}$ can be recovered group-theoretically from $G_K^{(\Sigma)}$. Thus, most of the results presented in Lemmas 2.4.1-2.4.9 for this case could be reduced to the case $\Sigma = \{\ell\}$ basically.

However, in the general case where Σ may contain p, the authors do not know any quick way (without establishing Lemma 2.3 first) of reconstructing the quotient $G_K^{(\Sigma)} \twoheadrightarrow G_k$ and reducing to the case $\Sigma = \{\ell\}$.

Lemma 2.5. *(Invariance of Filtrations of Geometrically Pro-Σ Decomposition Groups) Let the notations be as in Lemma 2.3 and the discussion before Lemma 2.3. Then the isomorphism $\sigma_{x,y}^{\mathrm{ab}} : D_x^{\mathrm{ab}} \to D_y^{\mathrm{ab}}$ preserves the filtrations*

$$\mathrm{Im}((k(x)^\times)^\Sigma), \mathrm{Im}((U_x^1)^\Sigma) \subset \mathrm{Im}((\mathcal{O}_x^\times)^\Sigma) \subset \mathrm{Im}((K_x^\times)^{(\Sigma)}) \subset \mathrm{Im}((K_x^\times)^{\wedge,(\Sigma)}) = D_x^{\mathrm{ab}},$$

and

$$\mathrm{Im}((k(y)^\times)^\Sigma), \mathrm{Im}((U_y^1)^\Sigma) \subset \mathrm{Im}((\mathcal{O}_y^\times)^\Sigma) \subset \mathrm{Im}((L_y^\times)^{(\Sigma)}) \subset \mathrm{Im}((L_y^\times)^{\wedge,(\Sigma)}) = D_y^{\mathrm{ab}}.$$

Proof. First, $\sigma_{x,y}^{\mathrm{ab}}$ preserves $\mathrm{Im}((k(x)^\times)^\Sigma)$ and $\mathrm{Im}((k(y)^\times)^\Sigma)$ by Lemma 2.3 (i). Next, $\sigma_{x,y}^{\mathrm{ab}}$ preserves $\mathrm{Im}((\mathcal{O}_x^\times)^\Sigma) = \mathrm{Im}(I_x)$ and $\mathrm{Im}((\mathcal{O}_y^\times)^\Sigma) = \mathrm{Im}(I_y)$ by Lemma 2.3 (iii), and preserves $\mathrm{Im}((U_x^1)^\Sigma)$ and $\mathrm{Im}((U_y^1)^\Sigma)$ by Lemma 2.4.7 (i), since $\mathrm{Im}((U_x^1)^\Sigma)$ (resp. $\mathrm{Im}((U_y^1)^\Sigma)$) is the pro-$p$ Sylow group of $\mathrm{Im}((\mathcal{O}_x^\times)^\Sigma)$ (resp. $\mathrm{Im}((\mathcal{O}_y^\times)^\Sigma)$). Finally, $\sigma_{x,y}^{\mathrm{ab}}$ preserves $\mathrm{Im}((K_x^\times)^{(\Sigma)}) = \mathrm{pr}_X^{-1}(\varphi_k^{\mathbb{Z}})$ and $\mathrm{Im}((L_y^\times)^{(\Sigma)}) = \mathrm{pr}_Y^{-1}(\varphi_l^{\mathbb{Z}})$ by Lemma 2.4.3 and lemma 2.4.9. □

Part II

In this part we introduce the notion of "small" and "large" sets of primes, and we state and prove our main results.

§3. Small and Large Sets of Primes. Let $\mathfrak{Primes}$ be the set of all prime numbers and $\Sigma \subset \mathfrak{Primes}$ a subset. Set $\Sigma' \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \Sigma$. Let k be a finite field of characteristic $p > 0$ and set $\Sigma^\dagger = \Sigma \setminus \{p\}$. Write

$$\hat{\mathbb{Z}}^\Sigma \stackrel{\text{def}}{=} \prod_{\ell \in \Sigma} \mathbb{Z}_\ell.$$

For a prime number $\ell \in \mathfrak{Primes} \setminus \{p\}$ let

$$\chi_\ell : G_k \to \mathbb{Z}_\ell^\times$$

be the ℓ-adic cyclotomic character of k, and define the Σ-part of the cyclotomic character of k by:

$$\chi_\Sigma \stackrel{\text{def}}{=} (\chi_\ell)_{\ell \in \Sigma^\dagger} : G_k \to (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times = \prod_{\ell \in \Sigma^\dagger} \mathbb{Z}_\ell^\times.$$

Thus, we have

$$\bar{k}^{\mathrm{Ker}(\chi_\Sigma)} = k_\Sigma \stackrel{\text{def}}{=} k(\zeta_{\ell^j} \mid \ell \in \Sigma^\dagger, j \in \mathbb{Z}_{\geq 0}).$$

For a prime number $\ell \in \mathfrak{Primes}$, let $G_{k,\ell} \subset G_k$ be the pro-ℓ-Sylow subgroup of G_k. (Recall that $G_k \simeq \hat{\mathbb{Z}}$ and $G_{k,\ell} \simeq \mathbb{Z}_\ell$.)

Definition/Proposition 3.1. (Small Set of Primes) *Let $\Sigma \subset \mathfrak{Primes}$ be a set of prime numbers. We say that the set Σ is k-small if the following equivalent conditions are satisfied:*
(i) $k_\Sigma \neq \bar{k}$.
(ii) *The Σ-part χ_Σ of the cyclotomic character is not injective.*
(iii) *There exists a prime number $\ell_0 \in \mathfrak{Primes}$, such that $\sharp(\chi_\Sigma(G_{k,\ell_0})) < \infty$.*
(iii′) *There exists a prime number $\ell_0 \in \mathfrak{Primes}$, such that $\ell_0 \notin \Sigma^\dagger$ and that there exists $N_0 \in \mathbb{Z}_{\geq 0}$ satisfying that for any $\ell \in \Sigma^\dagger$, the order of $p \bmod \ell \in \mathbb{F}_\ell^\times$ is not divisible by $\ell_0^{N_0}$.*
(iv) *There exists a subfield $k \subset k' \subset \bar{k}$ such that $(G_k : G_{k'}) = \infty$ and that $(\chi_\Sigma(G_k) : \chi_\Sigma(G_{k'})) < \infty$.*

Proof. Easy. $\square$

Definition 3.2. (Large Set of Primes) Let $\Sigma \subset \mathfrak{Primes}$ be a set of prime numbers. We say that the set Σ is *k-large* if the set $\Sigma' = \mathfrak{Primes} \setminus \Sigma$ is k-small.

Proposition 3.3. *Let $\Sigma \subset \mathfrak{Primes}$ be a set of prime numbers. Consider the following conditions:*
(i) *Σ is cofinite, i.e., Σ' is finite.*
(ii) *Σ is k-large.*
(ii′) *Σ is not k-small.*
(i′) *Σ is infinite.*
Then we have the following implications: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (ii′) $\Longrightarrow$ (i′).

Proof. To prove the implication (ii) $\implies$ (ii$'$), suppose that Σ is k-large and k-small at a time, or, equivalently, that both Σ and Σ' are k-small. This contradicts [Grunewald-Segal], Theorem A, as $\mathfrak{Primes} = \Sigma \cup \Sigma'$. To prove the implication (ii$'$) $\implies$ (i$'$), suppose that Σ is finite, then there is no injective homomorphism $\hat{\mathbb{Z}} \to (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$. In particular, $\chi_\Sigma : G_k \to (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times$ is not injective, i.e., Σ is k-small, which is a contradiction. The implication (i) $\implies$ (ii) is obtained by applying the implication (ii$'$) $\implies$ (i$'$) to Σ'. $\square$

Remarks 3.4.

3.4.1. Consider the following conditions:
(i) Σ is cofinite.
(ii) Σ is k-large.
(iii) Σ is $\mathbb{F}_p$-large
(iv) Σ is of (natural) density 1.
(iv$'$) Σ is of (natural) density $\neq 0$.
(iii$'$) Σ is not $\mathbb{F}_p$-small.
(ii$'$) Σ is not k-small.
(i$'$) Σ is infinite.
Then we have the following implications:

$$\begin{array}{ccccccc}
\text{(ii)} & \iff & \text{(iii)} & \implies & \text{(iii}'\text{)} & \iff & \text{(ii}'\text{)} \\
\Uparrow & & & & & & \Downarrow \\
\text{(i)} & \implies & \text{(iv)} & \implies & \text{(iv}'\text{)} & \implies & \text{(i}'\text{)},
\end{array}$$

(iii) $\implies$ (iv$'$), and (iv) $\implies$ (iii$'$).

Indeed, the implications (i) $\implies$ (ii), (ii$'$) $\implies$ (i$'$), and (iii) $\implies$ (iii$'$) are proved in Proposition 3.3. The implications (ii) $\iff$ (iii) and (i) $\implies$ (vi) $\implies$ (vi$'$) $\implies$ (i$'$) are immediate. To prove the implication (iii) $\implies$ (iv$'$), suppose that Σ is of density 0. Then by [Grunewald-Segal], Theorem A, $\chi_{\Sigma'} : G_{\mathbb{F}_p} \to (\hat{\mathbb{Z}}^{(\Sigma')^\dagger})^\times$ is injective. This is equivalent to saying that Σ' is not $\mathbb{F}_p$-small, or that Σ is not $\mathbb{F}_p$-large. The implications (iv) $\implies$ (iii$'$) $\iff$ (ii$'$) are obtained by applying the implications (ii) $\iff$ (iii) $\implies$ (iv$'$) to Σ'.

3.4.2. The implication (ii) $\implies$ (i) in 3.4.1 does not always hold. To construct such an example, set $k = \mathbb{F}_p$ (for simplicity), consider a prime number $r \neq p$, $r \nmid p-1$, and define Σ to be the set of prime numbers which do not divide $p^{r^m} - 1$ for any $m \geq 0$. Then Σ' is infinite. Indeed, we have

$$(p-1) \mid (p^r - 1) \mid \cdots \mid (p^{r^m} - 1) \mid (p^{r^{m+1}} - 1) \mid \cdots,$$

$$\frac{p^{r^{m+1}} - 1}{p^{r^m} - 1} > 1,$$

and

$$\left(p^{r^m} - 1, \frac{p^{r^{m+1}} - 1}{p^{r^m} - 1} \right) = (p^{r^m} - 1, r) = (p-1, r) = 1$$

by the Euclidean algorithm. (Here, to prove the second equality, use the fact that $p^{r^m} \equiv p \pmod{r}$.) Thus, for each $m \geq 0$, there exists a prime number

ℓ_{m+1} such that $\ell_{m+1} \mid (p^{r^{m+1}} - 1)$ and that $\ell_{m+1} \nmid (p^{r^m} - 1)$. This implies that $\Sigma' \ni \ell_1, \ell_2, \dots, \ell_m, \dots$ is infinite. On the other hand, Σ' is k-small. Indeed, take any $\ell_0 \in \Sigma \setminus \{r\}$ (e.g., $\ell_0 = p$). Then we claim that $\chi_{\Sigma'}(G_{k,\ell_0}) = \{1\}$. To prove this, it suffices to show that $\chi_s(G_{k,\ell_0}) = \{1\}$ for all prime number $s \in (\Sigma')^\dagger$. Further, since $\mathrm{Ker}((\mathbb{Z}_s)^\times \to (\mathbb{F}_s)^\times)$ is pro-s and $\ell_0 \neq s$, it suffices to show that $\chi_s(G_{k,\ell_0}) \bmod s = \{1\}$. But $\chi_s(G_{k,\ell_0}) \bmod s$ is the ℓ_0-Sylow subgroup of $\chi_s(G_k) \bmod s = \langle p \bmod s\rangle \subset (\mathbb{F}_s)^\times$. Since $s \mid (p^{r^m} - 1)$ for some $m \geq 0$, the order of $\langle p \bmod s\rangle \subset (\mathbb{F}_s)^\times$ is a power of r. Now, as $\ell_0 \neq r$, $\chi_s(G_{k,\ell_0}) \bmod s = \{1\}$, as desired.

Taking Σ' in this example as Σ, we also see that the implication (i$'$) $\Longrightarrow$ (ii$'$) in 3.4.1 does not always hold.

3.4.3. The implication (iii) $\Longrightarrow$ (iv) in 3.4.1 does not always hold. In fact, for any $\varepsilon > 0$, there exists a set of prime numbers Σ of density $< \varepsilon$, such that Σ is $\mathbb{F}_p$-large. Indeed, for each $N \in \mathbb{Z}_{>0}$, set $\Sigma_i(N) \stackrel{\text{def}}{=} \{\ell \in \mathfrak{Primes} \mid \ell \equiv i \pmod N\}$, $i = 1, \dots, N$. Then the density of $\Sigma_i(N)$ is $1/\varphi(N)$ (resp. 0) if $(i, N) = 1$ (resp. $(i, N) \neq 1$). Now, choose a prime number N such that $\varphi(N) = N - 1 > 1/\varepsilon$, and set $\Sigma \stackrel{\text{def}}{=} \Sigma_1(N) \cup \Sigma_0(N) = \Sigma_1(N) \cup \{N\}$, whose density is $1/\varphi(N) < \varepsilon$. We claim that Σ is $\mathbb{F}_p$-large. To see this, we have to prove that $\chi_{\Sigma'} : G_{\mathbb{F}_p} \to (\hat{\mathbb{Z}}^{(\Sigma')^\dagger})^\times$ is not injective. But this follows from the fact that $G_{\mathbb{F}_p}$ $(\simeq \hat{\mathbb{Z}})$ has a nontrivial pro-N-Sylow group $(\simeq \mathbb{Z}_N)$, while $(\hat{\mathbb{Z}}^{(\Sigma')^\dagger})^\times$ $(= \prod_{\ell \in (\Sigma')^\dagger} \mathbb{Z}_\ell^\times)$ has trivial pro-N-Sylow group. (For the latter, observe that, for each $\ell \in \Sigma'$, $\mathbb{Z}_\ell^\times$ has trivial pro-N-Sylow group, since $\ell \not\equiv 1 \pmod N$, and $\ell \neq N$.)

Taking Σ' in this example as Σ, we also see that the implication (iv$'$) $\Longrightarrow$ (iii$'$) in 3.4.1 does not always hold.

3.4.4. The implication (iii$'$) $\Longrightarrow$ (iv$'$) in 3.4.1 does not always hold. In fact, there exists a set of prime numbers Σ of density 0, such that Σ is not $\mathbb{F}_p$-small. To see this, take a sequence of positive integers $N_1 \mid N_2 \mid \cdots \mid N_k \mid \cdots$ such that $N_k \to \infty$ $(k \to \infty)$, hence $\varphi(N_k) \to \infty$ $(k \to \infty)$. Identify $\{1, \dots, N\} \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$ by $i \mapsto i \bmod N$ and set

$$I(N) \stackrel{\text{def}}{=} \{i \in \mathbb{Z}/N\mathbb{Z} \mid \Sigma_i(N) \text{ is not } \mathbb{F}_p\text{-small}\}.$$

(See 3.4.3 for the definition of $\Sigma_i(N)$.) Then, by [Grunewald-Segal], Theorem A, $I(N) \neq \emptyset$. It is easy to see that $(I(N_k))_{k=1,2,\dots}$ is a projective subsystem of $(\mathbb{Z}/N_k\mathbb{Z})_{k=1,2,\dots}$. As $I(N_k) \neq \emptyset$ for all $k \geq 1$, we see that $\varprojlim I(N_k) \neq \emptyset$. Fix any element $(i_k)_{k=1,2,\dots}$ of this projective limit. Then, for each $k \geq 1$, $\Sigma_{i_k}(N_k)$ is not $\mathbb{F}_p$-small, hence there exist $r_k \geq 1$, $\ell_{k,1}, \dots, \ell_{k,r_k} \in \Sigma_{i_k}(N_k)^\dagger$, and $e_{k,1}, \dots, e_{k,r_k} \in \mathbb{Z}_{>0}$, such that the order of $p \bmod \ell_{k,1}^{e_{k,1}} \cdots \ell_{k,r_k}^{e_{k,r_k}}$ in $(\mathbb{Z}/\ell_{k,1}^{e_{k,1}} \cdots \ell_{k,r_k}^{e_{k,r_k}}\mathbb{Z})^\times$ is divisible by $k!$. Now, set

$$\Sigma \stackrel{\text{def}}{=} \{\ell_{k,j} \mid k \geq 1, 1 \leq j \leq r_k\}.$$

Then it follows easily from the construction that Σ is not $\mathbb{F}_p$-small. On the other hand, for each $k \geq 1$, we have

$$\Sigma \subset \Sigma_{i_k}(N_k) \cup \{\ell_{k',j} \mid 1 \leq k' < k, 1 \leq j \leq r_k\}.$$

Since the density of $\Sigma_{i_k}(N_k)$ is (at most) $1/\varphi(N_k)$, we see that Σ must be of density 0, as desired.

Taking Σ' in this example as Σ, we also see that the implication (iv) $\Longrightarrow$ (iii) in 3.4.1 does not always hold.

In particular, the implication (iii$'$) $\Longrightarrow$ (iii) in 3.4.1 does not always hold. (Namely, there exists a set of prime numbers which is neither k-large nor k-small.) Indeed, if it held, then, combining it with the implication (iii) $\Longrightarrow$ (iv$'$), we would have the implication (iii$'$) $\Longrightarrow$ (iv$'$), which is absurd.

Next, and throughout the paper, for a subfield $\kappa \subset \bar{k}$, we write $\kappa^\times\{\Sigma'\} \subset \kappa^\times$ for the Σ'-primary part of the (torsion) multiplicative group $\kappa^\times$ and $(\kappa^\times)^\Sigma \stackrel{\text{def}}{=} \kappa^\times/\kappa^\times\{\Sigma'\}$ for the maximal Σ-primary quotient of $\kappa^\times$.

Let X be a proper, smooth, and geometrically connected curve over k. In the following discussion f, g: $X \to \mathbb{P}^1$ will be non-constant k-morphisms. Define the open subschemes $U \stackrel{\text{def}}{=} X \setminus (f^{-1}(\infty) \cup g^{-1}(\infty))$ and $U' \stackrel{\text{def}}{=} U \setminus (f^{-1}(0) \cup g^{-1}(0))$ of X. We have the following commutative diagram:

$$\begin{array}{cccc}
(f,g): & X & \to & \mathbb{P}^1_k \times \mathbb{P}^1_k \\
 & \cup & & \cup \\
 & U & \to & \mathbb{A}^1_k \times \mathbb{A}^1_k \\
 & \cup & & \cup \\
 & U' & \to & \mathbb{G}_{m,k} \times \mathbb{G}_{m,k}
\end{array}$$

where $(f,g) : X \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ is the natural morphism determined by f and g, the vertical inclusions are the natural open immersions, and the squares are fiber products.

Definition/Proposition 3.5. *We say that the pair (f,g) has the property P_Σ (respectively, Q_Σ, $Q_{0,\Sigma}$, $Q_{1,\Sigma}$ and $Q_{\infty,\Sigma}$) if the following holds:*

$P_\Sigma(f,g)$: *$\exists a, b \in k^\times\{\Sigma'\}$, such that $f = a + bg$.*

$Q_\Sigma(f,g)$: *$\forall x \in U^{\mathrm{cl}}$, $\exists a_x, b_x \in k(x)^\times\{\Sigma'\}$, such that $f(x) = a_x + b_x g(x)$.*

$Q_{0,\Sigma}(f,g)$: *$\forall x \in (U')^{\mathrm{cl}}$, $\exists a_x, b_x \in k(x)^\times\{\Sigma'\}$, such that $f(x) = a_x + b_x g(x)$.*

$Q_{1,\Sigma}(f,g)$: *$\forall' x \in U^{\mathrm{cl}}$, $\exists a_x, b_x \in k(x)^\times\{\Sigma'\}$, such that $f(x) = a_x + b_x g(x)$.*

$Q_{\infty,\Sigma}(f,g)$: *$\exists\infty\ x \in U^{\mathrm{cl}}$, for which $\exists a_x, b_x \in k(x)^\times\{\Sigma'\}$ such that $f(x) = a_x + b_x g(x)$.*

Here the sign $\forall'$ means "for all but finitely many" and the sign $\exists\infty$ means "there exist infinitely many".

Further, We say that the pair (f,g) has the property $\overline{P}_\Sigma$ (respectively, $\overline{Q}_\Sigma$, $\overline{Q}_{0,\Sigma}$, $\overline{Q}_{1,\Sigma}$ and $\overline{Q}_{\infty,\Sigma}$) if the following holds:

$\overline{P}_\Sigma(f,g)$: *$\exists a, b \in \bar{k}^\times\{\Sigma'\}$, such that $f = a + bg$.*

$\overline{Q}_\Sigma(f,g)$: *$\forall x \in U^{\mathrm{cl}}$, $\exists a_x, b_x \in \bar{k}^\times\{\Sigma'\}$, such that $f(x) = a_x + b_x g(x)$.*

$\overline{Q}_{0,\Sigma}(f,g)$: *$\forall x \in (U')^{\mathrm{cl}}$, $\exists a_x, b_x \in \bar{k}^\times\{\Sigma'\}$, such that $f(x) = a_x + b_x g(x)$.*

$\overline{Q}_{1,\Sigma}(f,g)$: *$\forall' x \in U^{\mathrm{cl}}$, $\exists a_x, b_x \in \bar{k}^\times\{\Sigma'\}$, such that $f(x) = a_x + b_x g(x)$.*

$\overline{Q}_{\infty,\Sigma}(f,g)$: *$\exists\infty\ x \in U^{\mathrm{cl}}$, for which $\exists a_x, b_x \in \bar{k}^\times\{\Sigma'\}$, such that $f(x) = a_x + b_x g(x)$.*

Then we have the following implications:

$$\begin{array}{ccc}
P_\Sigma(f,g) & \Longleftrightarrow & \overline{P}_\Sigma(f,g) \\
\Downarrow & & \Downarrow \\
Q_\Sigma(f,g) & \Longrightarrow & \overline{Q}_\Sigma(f,g) \\
\Downarrow & & \Downarrow \\
Q_{0,\Sigma}(f,g) & \Longrightarrow & \overline{Q}_{0,\Sigma}(f,g) \\
\Downarrow & & \Downarrow \\
Q_{1,\Sigma}(f,g) & \Longrightarrow & \overline{Q}_{1,\Sigma}(f,g) \\
\Downarrow & & \Downarrow \\
Q_{\infty,\Sigma}(f,g) & \Longrightarrow & \overline{Q}_{\infty,\Sigma}(f,g)
\end{array}$$

Proof. For the proof of " $\Longleftarrow$ " in the first row, consider the action of $\mathrm{Gal}(K\bar{k}/K) \simeq G_k$, where K denotes the function field of X, and resort to the fact that g is non-constant. The remaining implications are immediate. $\square$

Proposition 3.6. *Assume that* $\Sigma \cup \{p\} \subsetneq \mathfrak{Primes}$. *Then the property* $Q_{\infty,\Sigma}(f,g)$ *(hence also the property* $\overline{Q}_{\infty,\Sigma}(f,g)$*) always holds.*

Proof. First note that the condition $\Sigma \cup \{p\} \subsetneq \mathfrak{Primes}$ is equivalent to saying that $\bar{k}^\times\{\Sigma'\}$ is an infinite set. For the proof it suffices to consider the following three cases: 1) $f-g$ is a non-constant function. 2) $\frac{f-1}{g}$ is a non-constant function, and finally 3) both $f-g$ and $\frac{f-1}{g}$ are constant functions. In case 1, consider the non-constant (hence dominant) k-morphism $f-g: U \to \mathbb{A}^1_k$. For all but finitely many $a \in \bar{k}^\times\{\Sigma'\} \subset \bar{k} = \mathbb{A}^1(\bar{k})$, there exists $\bar{x} \in U(\bar{k})$ that maps to a. Then, for the image x of $\bar{x}$ in U, we have $f(x)-g(x) = a$, or $f(x) = a + 1 \cdot g(x)$. This completes the proof of case 1, as the equality $f(x) - g(x) = a$ also shows $a \in k(x)$. The proof of case 2 is similar to that of case 1: consider the non-constant k-morphism $\frac{f-1}{g}: U' \to \mathbb{A}^1_k$ and take a point in the fiber at $b \in \bar{k}^\times\{\Sigma'\} \subset \bar{k} = \mathbb{A}^1(\bar{k})$. In case 3, we have $f = a_0 + g = 1 + b_0 g$ for some $a_0, b_0 \in k$. As g is non-constant, the second equality forces $a_0 = b_0 = 1$, or, equivalently, $f = 1+g$. Thus, $P_\Sigma(f,g)$ holds, hence, a fortiori, $Q_{\infty,\Sigma}(f,g)$ holds $\square$

Proposition 3.7. (i) *Assume that* Σ *is* k*-small. Then the property* $\overline{Q}_{0,\Sigma}(f,g)$ *holds.*
(ii) *Assume that* Σ *is finite. Then the property* $Q_{1,\Sigma}(f,g)$ *holds.*

Proof. Fix $x \in U'$ and write $c = f(x), d = g(x) \in k(x)^\times \subset \bar{k}^\times$.
(i) Since Σ is k-small, we have $k \subset \exists k' \subset \bar{k}$, such that $(G_k : G_{k'}) = \infty$ and that $(\chi_\Sigma(G_k) : \chi_\Sigma(G_{k'})) < \infty$. Here, the first property says that $[k' : k] = \infty$, while the second implies that $N' \stackrel{\text{def}}{=} \sharp((k')^\times\{\Sigma\}) < \infty$. Replacing k' by the finite extension $k'(c,d)$, we may assume that $k(c,d) \subset k'$. Consider the $k(c,d)$-curve

$$Z_{N'} \stackrel{\text{def}}{=} \{(u,v) \mid c = u^{N'} + dv^{N'}\} \subset \mathbb{G}_m \times \mathbb{G}_m.$$

This is a twist of the N'-th Fermat curve (minus cusps), hence, in particular, it is smooth and geometrically connected. Thus, (by means of the Weil bound) we have $\sharp(Z_{N'}(k')) = \infty$. Take $(u_0, v_0) \in Z_{N'}(k')$ and set $a \stackrel{\text{def}}{=} u_0^{N'}, b \stackrel{\text{def}}{=} v_0^{N'}$. Then we have $c = a + bd$. This completes the proof, since we have $a, b \in (k')^\times\{\Sigma'\} (\subset \bar{k}^\times\{\Sigma'\})$ by the definition of N'.

(ii) The proof of (ii) is similar to (i) but a little bit more subtle. First, for each $n \in \mathbb{Z}_{>0}$, we define n_Σ to be the greatest divisor of n all of whose prime divisors belong to Σ. Next, set $q \stackrel{\text{def}}{=} q_{c,d} \stackrel{\text{def}}{=} \sharp(k(c,d))$ and $N \stackrel{\text{def}}{=} N_{c,d} \stackrel{\text{def}}{=} \sharp(k(c,d)^\times\{\Sigma\})$. Thus, we have $N = (q-1)_\Sigma$.

As in (i), consider the $k(c,d)$-curve

$$Z_N \stackrel{\text{def}}{=} \{(u,v) \mid c = u^N + dv^N\} \subset \mathbb{G}_m \times \mathbb{G}_m.$$

This is a twist of the N-th Fermat curve (minus cusps), hence, in particular, it is smooth and geometrically connected. The genus g of Z_N equals $(N-1)(N-2)/2$, and the cardinality r of the set of geometric points which are cusps is $3N$. Thus, by means of the Weil bound, we have $\sharp(Z_N(k(c,d))) > 0$, if $1 + q - 2g\sqrt{q} - r > 0$. This last inequality can be rewritten as:

$$q > N^2\sqrt{q} - \{(3N-2)(\sqrt{q}-1) - 1\}.$$

Thus, it holds if $q \geq 4$ and $q > N^2\sqrt{q}$ hold, or, equivalently, if $q \geq 4$ and $q > N^4$ hold. By Lemma 3.8 below, these inequalities are satisfied (hence $\sharp(Z_N(k(c,d)) > 0)$ for all but finitely many $q = p^m$. (Here, we resort to the fact that Σ is finite.)

Thus, for all but finitely many pairs (c,d), $\sharp(Z_N(k(c,d))) > 0$ holds. For such (c,d), take $(u_0, v_0) \in Z_N(k(c,d))$ and set $a \stackrel{\text{def}}{=} u_0^N, b \stackrel{\text{def}}{=} v_0^N$. Then we have $c = a + bd$. This completes the proof, since we have $a, b \in k(c,d)^\times\{\Sigma'\}(\subset k(x)^\times\{\Sigma'\})$ by the definition of $N = N_{c,d}$. $\square$

Lemma 3.8. *Let p be a prime number, and Σ a finite subset of $\mathfrak{Primes}$. Then there exists a constant $C > 0$ depending on p and Σ, such that, for all $m \in \mathbb{Z}_{>0}$, $(p^m - 1)_\Sigma \leq Cm$ holds. (For the notation n_Σ, see the proof of Proposition 3.7 (ii).)*

Proof. Let f be the order of the image of p in the multiplicative group $\prod_{\ell \in \Sigma^\dagger}(\mathbb{Z}/\ell^{\epsilon_\ell}\mathbb{Z})^\times$, where $\Sigma^\dagger \stackrel{\text{def}}{=} \Sigma \setminus \{p\}$ and $\epsilon_\ell \stackrel{\text{def}}{=} 1$ (resp. 2) for $\ell \neq 2$ (resp. $\ell = 2$). Then

$$(p^m - 1)_\Sigma \leq (p^{fm} - 1)_\Sigma = (p^f - 1)_\Sigma \cdot m_{\Sigma^\dagger} \leq (p^f - 1)_\Sigma \cdot m.$$

Here, the first inequality follows from the fact that $p^m - 1$ divides $p^{fm} - 1$ and the equality is obtained by considering the structure of the multiplicative group $\mathbb{Z}_\ell^\times$ for $\ell \in \Sigma^\dagger$. (More precisely, we have an isomorphism $1 + \ell^{\epsilon_\ell}\mathbb{Z}_\ell \xrightarrow{\sim} \ell^{\epsilon_\ell}\mathbb{Z}_\ell$ (say, the ℓ-adic logarithm), which maps $1 + \ell^e\mathbb{Z}_\ell$ onto $\ell^e\mathbb{Z}_\ell$ for each $e \geq \epsilon_\ell$. It follows from this that $a \in (1 + \ell^e\mathbb{Z}_\ell) \setminus (1 + \ell^{e+1}\mathbb{Z}_\ell)$ implies $a^m \in (1 + m\ell^e\mathbb{Z}_\ell) \setminus (1 + m\ell^{e+1}\mathbb{Z}_\ell)$, as desired.) Thus, $C \stackrel{\text{def}}{=} (p^f - 1)_\Sigma$ satisfies the desired property. $\square$

Remark 3.9. The proof of Proposition 3.7 (i) can be viewed as a down-to-earth, (2-dimensional) torus version of the proof of [Raynaud], Proposition 2.2.1.

Remark 3.10. (i) The proof of Proposition 3.7 (ii) shows that we may replace the assumption that Σ is finite by the following: For all $m \gg 0$, $(p^m - 1)_\Sigma < p^{m/4}$ holds.
(ii) Under the weaker assumption that Σ is k-small, $Q_{1,\Sigma}(f,g)$ does not always hold. To construct a counterexample, set $k = \mathbb{F}_p$ and consider a prime number $r \neq p$, $r \nmid p - 1$ and define Σ to be the set of prime numbers dividing $p^{r^m} - 1$ for some $m \geq 0$. Then, as in 3.4.2, Σ is (infinite and) k-small. We define k' to be the

union of the finite fields $\mathbb{F}_{p^{r^m}}$ ($m \in \mathbb{Z}_{\geq 0}$). (Namely, k' is the unique $\mathbb{Z}_r$-extension of the finite field k.) By definition, we have $(k')^\times\{\Sigma'\} = \{1\}$. Now, take any X, f, g as above such that $f \neq 1+g$. Then $U_1 \stackrel{\text{def}}{=} \{x \in U \mid f(x) \neq 1+g(x)\}$ is a non-empty open subset of X. Thus, (by the Weil bound) we have $\sharp(U_1(k')) = \infty$. Moreover, for any x in the image of $U_1(k')$ in U, there does not exist $a_x, b_x \in k(x)^\times\{\Sigma'\}$ such that $f(x) = a_x + b_x g(x)$. (Observe $k(x)^\times\{\Sigma'\} \subset (k')^\times\{\Sigma'\} = \{1\}$.) Thus, $Q_{1,\Sigma}(f,g)$ does not hold.

The following is the main result in this section, which plays a crucial role in the proof of the main Theorem 4.1 of this paper.

Proposition 3.11. *Assume that Σ is k-large. Then the implication*

$$\overline{Q}_{1,\Sigma}(f,g) \implies \overline{P}_{\Sigma}(f,g)$$

holds.

Proof. For each non-constant rational function h on $\overline{X} \stackrel{\text{def}}{=} X \times_k \bar{k}$, we define $\deg(h)$ to be the degree of the non-constant $\bar{k}$-morphism $h : \overline{X} \to \mathbb{P}^1$ associated with h (or, equivalently, the degree of the pole divisor $(h)_\infty$). Set $d = \deg(f) + \deg(g)$. Then, for any $a, b \in \bar{k}$, either $f - (a + bg)$ is a constant function or $\deg(f - (a + bg)) \leq d$. Assume that the property $\overline{Q}_{1,\Sigma}(f,g)$ holds. Then there exists a non-empty open subscheme $U_2 \subset U$, such that $\forall x \in (U_2)^{\mathrm{cl}}$, $\exists a_x, b_x \in \bar{k}^\times\{\Sigma'\}$ such that the equality $f(x) = a_x + b_x g(x)$ holds. First, consider the case where $f - (a_x + b_x g)$ is constant for some x. Then, by evaluating at x, we see that this constant must be 0. Namely, $f = a_x + b_x g$ holds, which implies that the property $\overline{P}_\Sigma(f,g)$ holds, as desired. So, suppose that $f - (a_x + b_x g)$ is non-constant for any x. Then the non-constant morphism $f - (a_x + b_x g) : \overline{X} \to \mathbb{P}^1$ is defined over $k(a_x, b_x) \subset k_{\Sigma'}$. (For the last inclusion, note that $\overline{k}^\times\{\Sigma'\} \subset k_{\Sigma'}$ by definition.) Considering the fiber at 0 of this non-constant morphism over $k(a_x, b_x) \subset k_{\Sigma'}$, we deduce:

$$[k_{\Sigma'}(x) : k_{\Sigma'}] \leq [k(a_x, b_x)(x) : k(a_x, b_x)] \leq \deg(f - (a_x + b_x g)) \leq d.$$

Now, since $G_{k_{\Sigma'}}$ is (pro)cyclic as a closed subgroup of $G_k \simeq \hat{\mathbb{Z}}$, we conclude that there exists a finite extension k' of $k_{\Sigma'}$, such that $k(x) \subset k_{\Sigma'}(x) \subset k'$ holds for any $x \in U_2^{\mathrm{cl}}$. By the assumption that Σ is k-large, we have $k_{\Sigma'} \subsetneq \bar{k}$, hence $[\bar{k} : k_{\Sigma'}] = \infty$. (Observe that G_k does not admit a nontrivial finite subgroup.) So, we also have $k' \subsetneq \bar{k}$. This contradicts the previous conclusion. Indeed, since U_2 is an affine curve over k, it admits a finite k-morphism $\phi : U_2 \to \mathbb{A}^1$. Take $a \in \bar{k} \setminus k' \subset \bar{k} = \mathbb{A}^1(\bar{k})$ and $x \in \phi^{-1}(a)$. Then we have $a \in k(a) \subset k(x) \subset k'$, which is absurd. $\square$

We will also use the following slight generalization of Proposition 3.11 later.

Definition/Proposition 3.12. *For a pair (f,g) as in the discussion before Definition/Proposition 3.5, a positive integer m, and a set of prime numbers $\Sigma \subset \mathfrak{Primes}$, we define the following properties:*

$P_\Sigma^{(m)}(f,g)$: *$\exists a, c \in k^\times\{\Sigma'\}$, such that $f = a(1+cg)^m$.*

$\overline{P}_\Sigma^{(m)}(f,g)$: *$\exists a, c \in \bar{k}^\times\{\Sigma'\}$, such that $f = a(1+cg)^m$.*

$\overline{Q}_{1,\Sigma}^{(m)}(f,g)$: *$\forall' x \in U$, $\exists a_x, c_x \in \bar{k}^\times\{\Sigma'\}$, such that $f(x) = a_x(1 + c_x g(x))^m$.*

Then:
(i) The implications

$$P_{\Sigma}^{(m)}(f,g) \iff \overline{P}_{\Sigma}^{(m)}(f,g) \implies \overline{Q}_{1,\Sigma}^{(m)}(f,g)$$

hold.
(ii) If Σ is k-large, then the implication

$$\overline{Q}_{1,\Sigma}^{(m)}(f,g) \implies \overline{P}_{\Sigma}^{(m)}(f,g)$$

holds.

Proof. (i) Similar to the proof of Definition/Proposition 3.5.
(ii) Similar to the proof of Proposition 3.11. □

§**4. The Main Theorem.** In this section we state and prove our main result. We follow the notations in §1, §2, and §3.

Let k, l be finite fields of characteristic p_k, p_l, respectively, and of cardinality q_k, q_l, respectively. Let X, Y be smooth, proper, and geometrically connected curves over k, l, respectively. Let K, L be the function fields of X, Y, respectively. We will write $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(K^{\mathrm{sep}}/K)$ for the absolute Galois group of K, and similarly $G_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ for the absolute Galois group of L.

Let $\Sigma_X, \Sigma_Y \subset \mathfrak{Primes}$ be sets of prime numbers. Write $G_K^{(\Sigma_X)}$ (resp. $G_L^{(\Sigma_Y)}$) for the maximal geometrically pro-Σ_X (resp. pro-Σ_Y) quotient of G_K (resp. G_L). Thus, we have exact sequences:

$$1 \to \overline{G}_K^{\Sigma_X} \to G_K^{(\Sigma_X)} \xrightarrow{\mathrm{pr}} G_k \to 1,$$

resp.

$$1 \to \overline{G}_L^{\Sigma_Y} \to G_L^{(\Sigma_Y)} \xrightarrow{\mathrm{pr}} G_l \to 1,$$

where $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}/k)$ (resp. $G_l \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\bar{l}/l)$) is the absolute Galois group of k (resp. l), and $\overline{G}_K^{\Sigma_X}$ (resp. $\overline{G}_L^{\Sigma_Y}$) is the maximal pro-$\Sigma_X$ (resp. pro-Σ_Y) quotient of the absolute Galois group $\overline{G}_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(K^{\mathrm{sep}}/K\bar{k})$ (resp. $\overline{G}_L \stackrel{\mathrm{def}}{=} \mathrm{Gal}(L^{\mathrm{sep}}/L\bar{l})$ of $K\bar{k}$ (resp. $L\bar{l}$). Our aim in this section is to prove the following Theorem:

Theorem 4.1. *Assume that Σ_X is k-large (cf. Definition 3.2). Assume also that Σ_X satisfies condition (ϵ_X) (cf. the discussion before Theorem C in §0). Let*

$$\sigma : G_K^{(\Sigma_X)} \xrightarrow{\sim} G_L^{(\Sigma_Y)}$$

be an isomorphism between profinite groups. Then σ arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} L^{\sim} & \xrightarrow{\sim} & K^{\sim} \\ \uparrow & & \uparrow \\ L & \xrightarrow{\sim} & K \end{array}$$

in which the horizontal arrows are isomorphisms, and the vertical arrows are the field extensions corresponding to the groups $G_L^{(\Sigma_Y)}$ and $G_K^{(\Sigma_X)}$, respectively. Thus, $L^{\sim}/L$ (resp. $K^{\sim}/K$) is the subextension of L^{sep}/L (resp. K^{sep}/K) with Galois group $G_L^{(\Sigma_Y)}$ (resp. $G_K^{(\Sigma_X)}$).

For the rest of this section we will consider an isomorphism of profinite groups

$$\sigma : G_K^{(\Sigma_X)} \xrightarrow{\sim} G_L^{(\Sigma_Y)}.$$

We write $\tilde{X}$ (resp. $\tilde{Y}$) for the normalization of X (resp. Y) in $K^{\sim}$ (resp. $L^{\sim}$).

We already know the following: $\Sigma \stackrel{\mathrm{def}}{=} \Sigma_X = \Sigma_Y$ (cf. Lemma 2.1), $p \stackrel{\mathrm{def}}{=} p_k = p_l$, and $q \stackrel{\mathrm{def}}{=} q_k = q_l$ (cf. Lemma 2.4.7 (i)). Moreover, there exists a bijection $\tilde{\phi} : \tilde{X}^{\mathrm{cl}} \xrightarrow{\sim} \tilde{Y}^{\mathrm{cl}}$, $\tilde{x} \mapsto \tilde{y}$, such that $\sigma(D_{\tilde{x}}) = D_{\tilde{y}}$, which naturally induces a bijection $\phi : X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ and an isomorphism $\phi : \mathrm{Div}_X \xrightarrow{\sim} \mathrm{Div}_Y$ (cf. Lemma 2.2).

Lemma 4.2. *(Invariance of Global Modules of Roots of Unity) The isomorphism σ induces naturally an isomorphism:*

$$M_X^{\Sigma} \xrightarrow{\sim} M_Y^{\Sigma}$$

between the (global) modules of roots of unity which is Galois-equivariant with respect to σ.

Proof. Let $J_K \stackrel{\mathrm{def}}{=} \prod'_{x \in X^{\mathrm{cl}}} K_x^{\times}$ be the idèle group of K and $J_K^{(\Sigma)} \stackrel{\mathrm{def}}{=} \prod'_{x \in X^{\mathrm{cl}}} (K_x^{\times})^{(\Sigma)}$ (cf. discussion before Lemma 2.3 for the definition of $(K_x^{\times})^{(\Sigma)}$ and the various notations below) which is a quotient of J_K. The Artin reciprocity map $\psi_K : J_K \to G_K^{\mathrm{ab}}$ of global class field theory induces naturally a map $\psi_K^{(\Sigma)} : J_K^{(\Sigma)} \to G_K^{(\Sigma),\mathrm{ab}}$. (When H is a profinite group, H^{ab} denotes the maximal abelian quotient of H.) The exact sequence

$$1 \to K^{\times} \to J_K \xrightarrow{\psi_K} G_K^{\mathrm{ab}}$$

from global class field theory induces naturally an exact sequence

$$1 \to k^{\times} \to \prod_{x \in X^{\mathrm{cl}}} \mathcal{O}_x^{\times} \to G_K^{\mathrm{ab}} \to \pi_1(X)^{\mathrm{ab}} \to 0,$$

where the map $G_K^{\mathrm{ab}} \to \pi_1(X)^{\mathrm{ab}}$ is the natural one, the map $\prod_{x \in X^{\mathrm{cl}}} \mathcal{O}_x^{\times} \to G_K^{\mathrm{ab}}$ is the restriction of the reciprocity map ψ_K, and the map $k^{\times} \to \prod_{x \in X^{\mathrm{cl}}} \mathcal{O}_x^{\times}$ is the natural diagonal embedding. (Here we recall that for each $x \in X^{\mathrm{cl}}$, the map $\psi_X : J_K \to G_K^{\mathrm{ab}}$ maps the component $K_x^{\times}$ of J_K into the decomposition group $D_x^{\mathrm{ab}} \subset G_K^{\mathrm{ab}}$ associated to x via the local reciprocity map $K_x^{\times} \to D_x^{\mathrm{ab}}$.) This latter sequence induces naturally an exact sequence

$$1 \to (k^{\times})^{\Sigma} \to \prod_{x \in X^{\mathrm{cl}}} (\mathcal{O}_x^{\times})^{\Sigma} \to G_K^{(\Sigma),\mathrm{ab}} \to \pi_1(X)^{(\Sigma),\mathrm{ab}} \to 0,$$

where the map $\prod_{x \in X^{\mathrm{cl}}} (\mathcal{O}_x^{\times})^{\Sigma} \to G_K^{(\Sigma),\mathrm{ab}}$ is naturally induced by the above map $\psi_K^{(\Sigma)} : J_K^{(\Sigma)} \to G_K^{(\Sigma),\mathrm{ab}}$, and the map $(k^{\times})^{\Sigma} \to \prod_{x \in X^{\mathrm{cl}}} (\mathcal{O}_x^{\times})^{\Sigma}$ is the natural diagonal embedding. Further, we have the following commutative diagram:

(4.1)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (k^\times)^\Sigma & \longrightarrow & \prod_{x\in X^{\rm cl}} (\mathcal{O}_x^\times)^\Sigma & \longrightarrow & G_K^{(\Sigma),\rm ab} & \longrightarrow & \pi_1(X)^{(\Sigma),\rm ab} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & (l^\times)^\Sigma & \longrightarrow & \prod_{y\in Y^{\rm cl}} (\mathcal{O}_y^\times)^\Sigma & \longrightarrow & G_L^{(\Sigma),\rm ab} & \longrightarrow & \pi_1(Y)^{(\Sigma),\rm ab} & \longrightarrow & 0 \end{array}$$

where the map $G_K^{(\Sigma),\rm ab} \to G_L^{(\Sigma),\rm ab}$ is naturally induced by $\sigma : G_K^{(\Sigma)} \xrightarrow{\sim} G_L^{(\Sigma)}$ (hence is an isomorphism), and the map $\prod_{x\in X^{\rm cl}} (\mathcal{O}_x^\times)^\Sigma \to \prod_{y\in Y^{\rm cl}} (\mathcal{O}_y^\times)^\Sigma$ maps each component $(\mathcal{O}_x^\times)^\Sigma$ isomorphically onto $(\mathcal{O}_y^\times)^\Sigma$, where $y \overset{\rm def}{=} \phi(x)$ (cf. Lemma 2.2 and Lemma 2.5). In particular, this map is an isomorphism since $\phi : X^{\rm cl} \xrightarrow{\sim} Y^{\rm cl}$ is a set-theoretic bijection. Thus, the far left vertical map in the diagram (4.1) gives an isomorphism $(k^\times)^\Sigma \xrightarrow{\sim} (l^\times)^\Sigma$. Passing to the open subgroups of $G_K^{(\Sigma)}$ and $G_L^{(\Sigma)}$, corresponding to extensions of the constant fields, to corresponding diagrams (4.1), and to the projective limits via the natural maps, we obtain the desired isomorphism $M_X^\Sigma \xrightarrow{\sim} M_Y^\Sigma$, which is Galois-equivariant with respect to σ as is easily verified by construction. $\square$

Lemma 4.3. *(Rigidity of Inertia) Let $x \in X^{\rm cl}$ and $y \overset{\rm def}{=} \phi(x)$. The following diagram is commutative:*

$$\begin{array}{ccc} M_X^\Sigma & \longrightarrow & M_{\overline{k(x)}}^\Sigma \\ \downarrow & & \downarrow \\ M_Y^\Sigma & \longrightarrow & M_{\overline{k(y)}}^\Sigma \end{array}$$

where the left vertical arrow is the isomorphism in Lemma 4.2, the right vertical arrow is the isomorphism in Lemma 2.3 (ii), and the horizontal maps are the natural identifications. Further, this diagram is Galois-equivariant with respect to the commutative diagram:

$$\begin{array}{ccc} D_{\tilde{x}} & \longrightarrow & G_K^{(\Sigma)} \\ \downarrow & & \sigma\downarrow \\ D_{\tilde{y}} & \longrightarrow & G_L^{(\Sigma)} \end{array}$$

where $\tilde{x} \in \tilde{X}^{\rm cl}$ is a point above $x \in X$, $\tilde{y} \overset{\rm def}{=} \tilde{\phi}(\tilde{x})$, and the horizontal maps are the natural inclusions.

Proof. Indeed, the far left square in the diagram (4.1) induces a diagram:

$$\begin{array}{ccc} (k^\times)^\Sigma & \longrightarrow & (k(x)^\times)^\Sigma \\ \downarrow & & \downarrow \\ (l^\times)^\Sigma & \longrightarrow & (k(y)^\times)^\Sigma \end{array}$$

where the vertical arrows are the isomorphisms induced by σ and the horizontal arrows are the natural inclusions (cf. Lemma 2.3 (i) and Lemma 4.2). Passing to the open subgroups of $G_K^{(\Sigma),\mathrm{ab}}$ and $G_L^{(\Sigma),\mathrm{ab}}$, corresponding to extensions of the constant fields, and corresponding diagrams as above, and to the projective limits, we obtain the desired Galois-equivariant diagram. (Observe that the above diagram is commutative cofinally.) $\square$

For an abelian group A, write $A\{\Sigma'\}$ for the Σ'-primary part of the torsion subgroup of A, and set $A^n \stackrel{\mathrm{def}}{=} \{a^n \mid a \in A\}$ for each $n \in \mathbb{Z}_{>0}$. Applying the notation $H^{(\Sigma)}$ in the beginning of §1 to the (discrete) group $H = K^\times$ (resp. $H = L^\times$) and a (pro)finite subgroup $\overline{H} = k^\times$ (resp. $\overline{H} = l^\times$), we have $(K^\times)^{(\Sigma)} = K^\times/(k^\times\{\Sigma'\})$ (resp. $(L^\times)^{(\Sigma)} = L^\times/(l^\times\{\Sigma'\})$).

Lemma 4.4. *(A Power of the Multiplicative Group modulo Σ'-primary Torsion).*
(i) *We have* $m \stackrel{\mathrm{def}}{=} \sharp(\pi_1(X)^{\mathrm{ab,tor}}\{\Sigma'\}) = \sharp(\pi_1(Y)^{\mathrm{ab,tor}}\{\Sigma'\})$.
(ii) *The isomorphism σ induces naturally an injective homomorphism*

$$\gamma' : ((K^\times)^{(\Sigma)})^m \hookrightarrow (L^\times)^{(\Sigma)}$$

between multiplicative groups.
(iii) *The homomorphism γ' fits into the following natural commutative diagram:*

$$\begin{array}{ccc} ((K^\times)^{(\Sigma)})^m & \xrightarrow{\gamma'} & (L^\times)^{(\Sigma)} \\ \downarrow & & \downarrow \\ (K^\times/k^\times)^m & \xrightarrow{\bar{\gamma}'} & L^\times/l^\times \end{array}$$

where the vertical maps are the natural surjective homomorphisms and $\bar{\gamma}' : (K^\times/k^\times)^m \hookrightarrow L^\times/l^\times$ is an injective homomorphism naturally induced by γ'.

Proof. (i) As $\pi_1(X)^{\mathrm{ab,tor}} \xrightarrow{\sim} J_X(k)$, we have

$$\sharp(\pi_1(X)^{\mathrm{ab,tor}}\{\Sigma'\}) = \sharp(J_X(k)\{\Sigma'\}) = \sharp(J_X(k))_{\Sigma'},$$

where, for each $n \in \mathbb{Z}_{>0}$, we define $n_{\Sigma'}$ to be the greatest divisor of n all of whose prime divisors belong to Σ'. Similarly

$$\sharp(\pi_1(Y)^{\mathrm{ab,tor}}\{\Sigma'\}) = \sharp(J_Y(l)\{\Sigma'\}) = \sharp(J_Y(l))_{\Sigma'}.$$

Thus, the assertion follows from Lemma 2.4.10.
(ii) We have the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Ker}(\psi_K^{(\Sigma)}) & \longrightarrow & J_K^{(\Sigma)} & \xrightarrow{\psi_K^{(\Sigma)}} & G_K^{(\Sigma),\mathrm{ab}} \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{Ker}(\psi_L^{(\Sigma)}) & \longrightarrow & J_L^{(\Sigma)} & \xrightarrow{\psi_L^{(\Sigma)}} & G_L^{(\Sigma),\mathrm{ab}} \end{array} \tag{4.2}$$

where the horizontal rows are exact. Here, $J_K^{(\Sigma)} \stackrel{\mathrm{def}}{=} \prod'_{x \in X^{\mathrm{cl}}} (K_x^\times)^{(\Sigma)}$ (resp. $J_L^{(\Sigma)} \stackrel{\mathrm{def}}{=} \prod'_{y \in Y^{\mathrm{cl}}} (L_y^\times)^{(\Sigma)}$) is a quotient of the idèle group J_K (resp. J_L) of K (resp. L), and the map $\psi_K^{(\Sigma)} : J_K^{(\Sigma)} \to G_K^{(\Sigma),\mathrm{ab}}$ (resp. the map $\psi_L^{(\Sigma)} : J_L^{(\Sigma)} \to G_L^{(\Sigma),\mathrm{ab}}$) is naturally induced by Artin's reciprocity map in global class field theory (cf. proof of Lemma 4.2). The far right vertical map is naturally induced by σ, and the middle vertical map $J_K^{(\Sigma)} \to J_L^{(\Sigma)}$ maps each component $(K_x^\times)^{(\Sigma)}$ isomorphically onto $(L_y^\times)^{(\Sigma)}$; where $y \stackrel{\text{def}}{=} \phi(x)$, via the natural identification in Lemma 2.5, which is induced by σ. In particular, the map $J_K^{(\Sigma)} \to J_L^{(\Sigma)}$ is an isomorphism. Thus, the far left vertical map in the diagram (4.2) is a natural isomorphism $\mathrm{Ker}(\psi_K^{(\Sigma)}) \to \mathrm{Ker}(\psi_L^{(\Sigma)})$ between kernels. We claim:

Claim 1. There exists a canonical exact sequence:

$$1 \to (K^\times)^{(\Sigma)} \to \mathrm{Ker}(\psi_K^{(\Sigma)}) \to \pi_1(X)^{\mathrm{ab,tor}}\{\Sigma'\} \to 0$$

(resp.

$$1 \to (L^\times)^{(\Sigma)} \to \mathrm{Ker}(\psi_L^{(\Sigma)}) \to \pi_1(Y)^{\mathrm{ab,tor}}\{\Sigma'\} \to 0).$$

Assuming this claim, we then have a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & (K^\times)^{(\Sigma)} \arrow[r] & \mathrm{Ker}(\psi_K^{(\Sigma)}) \arrow[r] \arrow[d] & \pi_1(X)^{\mathrm{ab,tor}}\{\Sigma'\} \arrow[r] & 0 \\
1 \arrow[r] & (L^\times)^{(\Sigma)} \arrow[r] & \mathrm{Ker}(\psi_L^{(\Sigma)}) \arrow[r] & \pi_1(Y)^{\mathrm{ab,tor}}\{\Sigma'\} \arrow[r] & 0
\end{tikzcd}$$

where the horizontal rows are exact, and the vertical arrow is the above isomorphism. This isomorphism has, a priori, no reason to map $(K^\times)^{(\Sigma)}$ into $(L^\times)^{(\Sigma)}$. However, since $\pi_1(Y)^{\mathrm{ab,tor}}\{\Sigma'\}$ is a finite abelian group of exponent dividing m, we can conclude that the above isomorphism $\mathrm{Ker}(\psi_K^{(\Sigma)}) \to \mathrm{Ker}(\psi_L^{(\Sigma)})$ maps $((K^\times)^{(\Sigma)})^m$ injectively into $(L^\times)^{(\Sigma)}$. Thus, we obtain a natural injective map $\gamma' : ((K^\times)^{(\Sigma)})^m \to (L^\times)^{(\Sigma)}$. It remains to prove Claim 1. We will only prove the assertion concerning $\mathrm{Ker}(\psi_K^{(\Sigma)})$ (the assertion concerning $\mathrm{Ker}(\psi_L^{(\Sigma)})$ is proved in a similar way).

We have the following commutative diagram:

$$\begin{tikzcd}
 & & 1 & 1 & \\
1 \arrow[r] & \mathrm{Ker}(\rho_K) \arrow[r] & \mathrm{Im}(\psi_K)(\subset G_K^{\mathrm{ab}}) \arrow[r, "\rho_K"] \arrow[u] & \mathrm{Im}(\psi_K^{(\Sigma)})(\subset G_K^{(\Sigma),\mathrm{ab}}) \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & \prod_{x \in X^{\mathrm{cl}}} N_x \arrow[r] \arrow[u] & J_K \arrow[r] \arrow[u, "\psi_K"] & J_K^{(\Sigma)} \arrow[r] \arrow[u, "\psi_K^{(\Sigma)}"] & 1 \\
1 \arrow[r] & k^\times\{\Sigma'\} \arrow[r] \arrow[u] & K^\times \arrow[r] \arrow[u] & \mathrm{Ker}(\psi_K^{(\Sigma)}) \arrow[u] & \\
 & 1 \arrow[u] & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}$$

where the vertical and horizontal rows are exact. Here, the map $\psi_K : J_K \to G_K^{\mathrm{ab}}$ is Artin's reciprocity map in global class field theory, and the map ρ_K :

$\mathrm{Im}(\psi_K) \to \mathrm{Im}(\psi_K^{(\Sigma)})$ is the restriction of the natural map $G_K^{\mathrm{ab}} \twoheadrightarrow G_K^{(\Sigma),\mathrm{ab}}$. Further, $J_K \to J_K^{(\Sigma)}$ is the natural map which maps each component $K_x^\times$ canonically onto $(K_x^\times)^{(\Sigma)} = K_x^\times/N_x$ (cf. the discussion before Lemma 2.3 for the definition of N_x). In particular, we deduce that the cokernel of the injective map $(\prod_{x\in X^{\mathrm{cl}}} N_x)/(k^\times\{\Sigma'\}) \hookrightarrow \mathrm{Ker}(\rho_K)$ is naturally isomorphic to the cokernel of the injective map $(K^\times)^{(\Sigma)} \hookrightarrow \mathrm{Ker}(\psi_K^{(\Sigma)})$. Observe that $\mathrm{Ker}(\rho_K)$ is naturally identified with the kernel $\mathrm{Ker}(G_K^{\mathrm{ab}} \twoheadrightarrow G_K^{(\Sigma),\mathrm{ab}})$. Further, we claim:

Claim 2. The cokernel of the above injective homomorphism $(\prod_{x\in X^{\mathrm{cl}}} N_x)/(k^\times\{\Sigma'\}) \hookrightarrow \mathrm{Ker}(\rho_K)$ is naturally isomorphic to $\pi_1(X)^{\mathrm{ab},\mathrm{tor}}\{\Sigma'\}$.

Indeed, we have the following commutative diagram:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & k^\times\{\Sigma'\} \arrow[r] \arrow[d] & k^\times \arrow[r] \arrow[d] & (k^\times)^{\Sigma} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \prod_{x\in X^{\mathrm{cl}}} N_x \arrow[r] \arrow[d] & \prod_{x\in X^{\mathrm{cl}}} \mathcal{O}_x^\times \arrow[r] \arrow[d] & \prod_{x\in X^{\mathrm{cl}}} (\mathcal{O}_x^\times)^{\Sigma} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Ker}(\rho_K) \arrow[r] \arrow[d] & G_K^{\mathrm{ab}} \arrow[r, "\rho_K"] \arrow[d] & G_K^{(\Sigma),\mathrm{ab}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Ker}(\nu_X) \arrow[r] \arrow[d] & \pi_1(X)^{\mathrm{ab}} \arrow[r, "\nu_X"] \arrow[d] & \pi_1(X)^{(\Sigma),\mathrm{ab}} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

where the vertical and horizontal rows are exact. Here, the maps $G_K^{\mathrm{ab}} \to \pi_1(X)^{\mathrm{ab}}$ and $G_K^{(\Sigma),\mathrm{ab}} \to \pi_1(X)^{(\Sigma),\mathrm{ab}}$ are the natural maps, the map $\prod_{x\in X^{\mathrm{cl}}} \mathcal{O}_x^\times \to G_K^{\mathrm{ab}}$ is the restriction of Artin's reciprocity map, and the map $k^\times \to \prod_{x\in X^{\mathrm{cl}}} \mathcal{O}_x^\times$ is the natural diagonal embedding. Further, the kernel $\mathrm{Ker}(\nu_X)$ of ν_X is canonically isomorphic to $\pi_1(X)^{\mathrm{ab},\mathrm{tor}}\{\Sigma'\}$, as follows from the structure of $\pi_1(X)^{\mathrm{ab}}$. Note that the maximal pro-Σ quotient $\pi_1(X)^{\mathrm{ab},\mathrm{tor},\Sigma}$ of $\pi_1(X)^{\mathrm{ab},\mathrm{tor}}$ is naturally isomorphic to the torsion subgroup $\pi_1(X)^{(\Sigma),\mathrm{ab},\mathrm{tor}}$ of $\pi_1(X)^{(\Sigma),\mathrm{ab}}$. Thus, Claim 2, hence Claim 1, are proved. This completes the proof of (ii).

(iii) This follows from the fact that $(K^\times)^{(\Sigma)}$ (resp. $(L^\times)^{(\Sigma)}$) modulo its torsion subgroup (which is naturally identified with $(k^\times)^{\Sigma}$ (resp. $(l^\times)^{\Sigma}$)) is naturally identified with $K^\times/k^\times$ (resp. $L^\times/l^\times$). $\square$

We have a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & (K^\times)^{(\Sigma)} \arrow[r] & \mathrm{Ker}(\psi_K^{(\Sigma)}) \arrow[r] \arrow[d, "\rho"'] & \pi_1(X)^{\mathrm{ab},\mathrm{tor}}\{\Sigma'\} \arrow[r] & 0 \\
1 \arrow[r] & (L^\times)^{(\Sigma)} \arrow[r] & \mathrm{Ker}(\psi_L^{(\Sigma)}) \arrow[r] & \pi_1(Y)^{\mathrm{ab},\mathrm{tor}}\{\Sigma'\} \arrow[r] & 0
\end{tikzcd}$$

where the horizontal rows are exact, and the vertical arrow is an isomorphism naturally induced by σ (cf. proof of Lemma 4.4). Let

$$R_K \overset{\text{def}}{=} \mathrm{Ker}(\psi_K^{(\Sigma)})/(\mathrm{Ker}(\psi_K^{(\Sigma)})^{\mathrm{tor}}\{\Sigma\})$$

where $\mathrm{Ker}(\psi_K^{(\Sigma)})^{\mathrm{tor}}\{\Sigma\}$ is the group of Σ-primary torsion of $\mathrm{Ker}(\psi_K^{(\Sigma)})$, which is contained in $(K^\times)^{(\Sigma)}$ (since $\pi_1(X)^{\mathrm{ab,tor}}\{\Sigma'\}$ is Σ'-primary), and is naturally identified with $(k^\times)^{\Sigma}$. Thus, R_K naturally inserts in the following exact sequence:

$$1 \to K^\times/k^\times \to R_K \to \pi_1(X)^{\mathrm{ab,tor}}\{\Sigma'\} \to 0.$$

We define R_L in a similar way which sits in the following exact sequence:

$$1 \to L^\times/l^\times \to R_L \to \pi_1(Y)^{\mathrm{ab,tor}}\{\Sigma'\} \to 0.$$

The above isomorphism

$$\rho : \mathrm{Ker}(\psi_K^{(\Sigma)}) \xrightarrow{\sim} \mathrm{Ker}(\psi_L^{(\Sigma)})$$

induces naturally a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^\times/k^\times & \longrightarrow & R_K & \longrightarrow & \pi_1(X)^{\mathrm{ab,tor}}\{\Sigma'\} & \longrightarrow & 0 \\ & & & & \bar{\rho}\big\downarrow & & & & \\ 1 & \longrightarrow & L^\times/l^\times & \longrightarrow & R_L & \longrightarrow & \pi_1(Y)^{\mathrm{ab,tor}}\{\Sigma'\} & \longrightarrow & 0 \end{array}$$

where the horizontal rows are exact, and the vertical arrow is an isomorphism. Further, define $\overline{H}_K \subset K^\times/k^\times$ to be the kernel of the composite homomorphism $K^\times/k^\times \hookrightarrow R_K \xrightarrow{\sim} R_L \twoheadrightarrow \pi_1(Y)^{\mathrm{ab,tor}}\{\Sigma'\}$, and set $\overline{H}_L \overset{\text{def}}{=} \bar{\rho}(\overline{H}_K)$. Then it is easy to see that $\overline{H}_L \subset L^\times/l^\times$ and that $(K^\times/k^\times : \overline{H}_K) = (L^\times/l^\times : \overline{H}_L)$ divides $m \overset{\text{def}}{=} \sharp(\pi_1(X)^{\mathrm{ab,tor}}\{\Sigma'\}) = \sharp(\pi_1(Y)^{\mathrm{ab,tor}}\{\Sigma'\})$. In particular, we have $(K^\times/k^\times)^m \subset \overline{H}_K$.

In the following, we will think of the elements of

$$\mathrm{PDiv}_X \overset{\text{def}}{=} K^\times/k^\times$$

(resp. $\mathrm{PDiv}_Y \overset{\text{def}}{=} L^\times/l^\times$) as principal divisors of rational functions on X (resp. Y), and denote them by $\bar{f}, \bar{g}, \dots$, where $f, g, \dots$ are rational functions on X (resp. Y). We will also denote the elements of $(K^\times)^{(\Sigma)}$ (resp. $(L^\times)^{(\Sigma)}$) by $f', g', \dots$, and refer to them as "*pseudo-functions*" $\overset{\text{def}}{=}$ classes of rational functions on X (resp. Y) modulo constants in $k^\times\{\Sigma'\}$ (resp. $l^\times\{\Sigma'\}$). We define

$$H'_K \overset{\text{def}}{=} \{f' \in (K^\times)^{(\Sigma)} \mid \bar{f} \in \overline{H}_K\},$$

and

$$H_K^\times \overset{\text{def}}{=} \{f \in K^\times \mid \bar{f} \in \overline{H}_K\}.$$

We define $H'_L \subset (L^\times)^{(\Sigma)}$, and $H_L^\times \subset L^\times$, in a similar way. Since $\overline{H}_K$ is a finite index subgroup of $K^\times/k^\times$, H'_K (resp. $H_K^\times$) is a finite index subgroup of $(K^\times)^{(\Sigma)}$ (resp. $K^\times$). Note that $(k^\times)^{\Sigma} \subset H'_K$ and $k^\times \subset H_K^\times$ by definition. Similar statements also hold for L. Moreover, the isomorphism

$$\rho : \mathrm{Ker}(\psi_K^{(\Sigma)}) \xrightarrow{\sim} \mathrm{Ker}(\psi_L^{(\Sigma)})$$

restricts to an isomorphism

$$\rho : H'_K \xrightarrow{\sim} H'_L.$$

In summary, we have the following:

Lemma 4.5. *(Almost-Recovering the Group of Principal Divisors). The isomorphism σ naturally induces isomorphisms:*

$$\rho : H'_K \xrightarrow{\sim} H'_L$$

and

$$\bar{\rho} : \overline{H}_K \xrightarrow{\sim} \overline{H}_L$$

where $\overline{H}_K$ (resp. $\overline{H}_L$) and H'_K (resp. H'_L) are defined as above, which fit into the following commutative diagram:

$$\begin{array}{ccc} H'_K & \xrightarrow{\rho} & H'_L \\ \downarrow & & \downarrow \\ \overline{H}_K & \xrightarrow{\bar{\rho}} & \overline{H}_L \end{array} \tag{4.3}$$

where the vertical maps are the natural surjective homomorphisms. Further, ρ induces naturally an isomorphism:

$$\tau : (k^\times)^\Sigma \xrightarrow{\sim} (l^\times)^\Sigma,$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} (k(x)^\times)^\Sigma & \xrightarrow{\tau_{x,y}} & (k(y)^\times)^\Sigma \\ \uparrow & & \uparrow \\ (k^\times)^\Sigma & \xrightarrow{\tau} & (l^\times)^\Sigma \end{array} \tag{4.4}$$

where $x \in X^{\mathrm{cl}}$, $y \stackrel{\mathrm{def}}{=} \phi(x) \in Y^{\mathrm{cl}}$, $\tau_{x,y}$ is the isomorphism in Lemma 2.3 (i), and the vertical maps are the natural ones.

Proof. For the last assertion, observe that $(k^\times)^\Sigma$ (resp. $(l^\times)^\Sigma$) is naturally identified with the torsion subgroup of H'_K (resp. H'_L). $\square$

Given a principal divisor $\bar{f} \in K^\times/k^\times$ and $x \in X^{\mathrm{cl}}$, we define $v_x(\bar{f}) \in \mathbb{Z}$ to be the order $v_x(f)$ at x of a representative $f \in K^\times$ of the class $\bar{f} \in K^\times/k^\times$. Thus, $v_x(\bar{f})$ is well-defined and does not depend on the choice of the representative f of the class $\bar{f}$. We shall refer to $v_x(\bar{f})$ as the valuation at x of the principal divisor $\bar{f}$. Similarly, we define the valuation $v_y(\bar{g})$ of a principal divisor $\bar{g}$ on Y at a point $y \in Y^{\mathrm{cl}}$.

Given a pseudo-function $f' \in (K^\times)^{(\Sigma)}$ and $x \in X^{\mathrm{cl}}$ with $v_x(\bar{f}) = 0$ where $\bar{f}$ is the image of f' in $K^\times/k^\times$, we will denote by $f'(x)$ the image of $f(x)$ in $(k(x)^\times)^\Sigma$, where $f \in K^\times$ is a representative of the class of $f' \in (K^\times)^{(\Sigma)} = K^\times/(k^\times\{\Sigma'\})$, via the natural surjective map $k(x)^\times \twoheadrightarrow (k(x)^\times)^\Sigma$. Thus, $f'(x)$ is well-defined and does not depend on the choice of the representative f of the class f'. We shall refer to $f'(x)$ as the Σ-value at x of the pseudo-function f'. We define the Σ-value $g'(y) \in (k(y)^\times)^\Sigma$ of a pseudo-function $g' \in (L^\times)^{(\Sigma)}$ at a point $y \in Y^{\mathrm{cl}}$ with $v_y(\bar{g}) = 0$ in a similar way.

Further, for $x \in X^{\mathrm{cl}}$ (resp. $y \in Y^{\mathrm{cl}}$) we will think of elements of $(k(x)^\times)^\Sigma$ (resp $(k(y)^\times)^\Sigma$) as classes of elements of $k(x)^\times$ (resp. $k(y)^\times$) modulo elements of $k(x)^\times\{\Sigma'\}$ (resp. $k(y)^\times\{\Sigma'\}$) and denote them by $\eta', \zeta', \dots$, where $\eta, \zeta, \dots \in k(x)^\times$ (resp. $\in k(y)^\times$) are elements of multiplicative groups of residue fields.

Lemma 4.6. *(Recovering the Valuations and the Σ-Values of Pseudo-Functions) Consider the commutative diagram (4.3) in Lemma 4.5. Let $x \in X^{\mathrm{cl}}$, and $y \stackrel{\mathrm{def}}{=} \phi(x)$. Then the following implications hold:*
(i) *For $\bar{f} \in \overline{H}_K$ and $\bar{g} \in \overline{H}_L$:*

$$\bar{\rho}(\bar{f}) = \bar{g} \Longrightarrow v_x(\bar{f}) = v_y(\bar{g}).$$

In particular, in terms of divisors, if:

$$\bar{f} = x_1 + x_2 + \cdots + x_n - x'_1 - \cdots - x'_{n'},$$

then:

$$\bar{g} = y_1 + y_2 + \cdots + y_n - y'_1 - \cdots - y'_{n'},$$

where $y_i \stackrel{\mathrm{def}}{=} \phi(x_i)$ (resp. $y'_{i'} \stackrel{\mathrm{def}}{=} \phi(x'_{i'})$) for $i \in \{1, \ldots, n\}$ (resp. $i' \in \{1, \ldots, n'\}$). In other words the map $\bar{\rho}$ preserves the valuations of the classes of functions in $\overline{H}_K$ with respect to the bijection $\phi : X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ between points.
(ii) *For $f' \in H'_K$ and $g' \in H'_L$:*

$$v_x(\bar{f}) = 0 \text{ and } \rho(f') = g' \Longrightarrow v_y(\bar{g}) = 0 \text{ and } \tau_{x,y}(f'(x)) = g'(y),$$

where

$$\tau_{x,y} : (k(x)^\times)^\Sigma \xrightarrow{\sim} (k(y)^\times)^\Sigma$$

is the isomorphism in Lemma 2.3 (i) and $\bar{f}$ (resp. $\bar{g}$) is the image of f' (resp. g') in $K^\times/k^\times$ (resp. $L^\times/l^\times$). In other words the map ρ preserves the Σ-values of the pseudo-functions in H'_K with respect to the bijection $\phi : X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ between points.

Proof. As shown in (the proofs of) Lemmas 4.4 and 4.5, we have the commutative diagram

$$\begin{array}{ccccccc}
H'_K & \subset & (K^\times)^{(\Sigma)} & \to & J_K^{(\Sigma)} & \stackrel{\mathrm{def}}{=} & \prod'_{x \in X^{\mathrm{cl}}} (K_x^\times)^{(\Sigma)} \\
\downarrow & & & & \downarrow & & \\
H'_L & \subset & (L^\times)^{(\Sigma)} & \to & J_L^{(\Sigma)} & \stackrel{\mathrm{def}}{=} & \prod'_{y \in Y^{\mathrm{cl}}} (L_y^\times)^{(\Sigma)}
\end{array}$$

where the vertical arrows are the isomorphisms induced by σ. More precisely, $H'_K \xrightarrow{\sim} H'_L$ is ρ', and $J_K^{(\Sigma)} \xrightarrow{\sim} J_L^{(\Sigma)}$ maps each component $(K_x^\times)^{(\Sigma)}$ isomorphically onto $(L_y^\times)^{(\Sigma)}$, where $y \stackrel{\mathrm{def}}{=} \phi(x)$. Further, the isomorphism $(K_x^\times)^{(\Sigma)} \xrightarrow{\sim} (L_y^\times)^{(\Sigma)}$ arises from Lemma 2.5. It follows from this that $\bar{\rho}$ preserves the valuations by Lemmas 2.4.3 and 2.4.9 and that ρ preserves the Σ-values by Lemma 2.5. $\square$

Let U be an open subgroup of $G_K^{(\Sigma)}$, and let $V \stackrel{\mathrm{def}}{=} \sigma(U)$. Let K'/K (resp. L'/L) be the finite subextension of $K^\sim/K$ (resp. $L^\sim/L$) corresponding to U (resp. V), k' (resp. l') the constant field of K' (resp. L'), and X' (resp. Y') the normalization of X (resp. Y) in K' (resp. L'). Then σ induces, by restriction to U, an isomorphism

$$\sigma : U (= G_{K'}^{(\Sigma)}) \xrightarrow{\sim} V (= G_{L'}^{(\Sigma)}),$$

which naturally induces by Lemma 4.5 the following commutative diagram:

$$\begin{array}{ccc}
H'_{K'} & \longrightarrow & H'_{L'} \\
\Big\downarrow & & \Big\downarrow \\
\overline{H}_{K'} & \longrightarrow & \overline{H}_{L'}
\end{array} \tag{4.5}$$

where the horizontal arrows are the isomorphisms induced by σ, and the vertical arrows are the natural surjective homomorphisms.

Lemma 4.7. *The above diagram (4.5) is compatible with the diagram (4.3) in Lemma 4.5. More precisely, the natural injective homomorphisms* $(K^\times)^{(\Sigma)} \to ((K')^\times)^{(\Sigma)}$, $(L^\times)^{(\Sigma)} \to ((L')^\times)^{(\Sigma)}$ *(resp.* $K^\times/k^\times \to (K')^\times/(k')^\times$, $L^\times/l^\times \to (L')^\times/(l')^\times$*) map* H'_K *into* $H'_{K'}$, H'_L *into* $H'_{L'}$ *(resp.* $\overline{H}_K$ *into* $\overline{H}_{K'}$, *and* $\overline{H}_L$ *into* $\overline{H}_{L'}$*) and the resulting diagrams*

$$\begin{array}{ccc} H'_{K'} & \longrightarrow & H'_{L'} \\ \uparrow & & \uparrow \\ H'_K & \longrightarrow & H'_L \end{array} \tag{4.6}$$

and

$$\begin{array}{ccc} \overline{H}_{K'} & \longrightarrow & \overline{H}_{L'} \\ \uparrow & & \uparrow \\ \overline{H}_K & \longrightarrow & \overline{H}_L \end{array} \tag{4.7}$$

are commutative.

Proof. First, consider the diagram

$$\begin{array}{ccc} J_{K'}^{(\Sigma)} & \longrightarrow & J_{L'}^{(\Sigma)} \\ \uparrow & & \uparrow \\ J_K^{(\Sigma)} & \longrightarrow & J_L^{(\Sigma)}, \end{array} \tag{4.8}$$

where the horizontal arrows are the natural isomorphisms induced by σ and the vertical arrows are induced by the natural inclusions $J_K \hookrightarrow J_{K'}$, $J_L \hookrightarrow J_{L'}$ of idèle groups. This diagram is commutative, since the vertical arrows arise from the (local) transfer maps. Now, the diagram (4.6) commutes as a subdiagram of (4.8), and the diagram (4.7) commutes as a quotient diagram of (4.6). $\square$

From now on, we shall assume that Σ *satisfies condition* (ϵ_X) (cf. discussion before Theorem C in §0). Then, by Lemmas 2.4.7 (ii) and 2.4.10, Σ also satisfies condition (ϵ_Y). We shall use the following lemma.

Lemma 4.8. *Let* $k \subset \kappa \subset \bar{k}$ *be an (infinite or finite) extension of* k, *and* $\mathcal{K} \stackrel{\text{def}}{=} K\kappa$. *Let* $\overline{U} \subset \mathcal{K}^\times/\kappa^\times$ *be a finite index subgroup and assume* $\sharp(\kappa) > 2(\mathcal{K}^\times/\kappa^\times : \overline{U})$. *Then there exists* $f \in \mathcal{K}^\times \setminus \kappa^\times$, *such that* $\bar{f}, \overline{1+f} \in \overline{U}$, *and that* $\deg(f) = \mathrm{gon}(X \times_k \kappa)$, *where* $\mathrm{gon}(X \times_k \kappa)$ *denotes the gonality of* $X \times_k \kappa$ *over* κ *and* $\deg(f)$ *is the degree of the finite map* $f : X \times_k \kappa \to \mathbb{P}^1_\kappa$ *(equivalently,* $\deg(f)$ *is the degree of the pole divisor of* f*).*

Proof. Take any $g \in \mathcal{K}$ that attains the gonality: $\deg(g) = \mathrm{gon}(X \times_k \kappa)$, and consider the set $\{g - a \mid a \in \kappa\} \subset \mathcal{K}^\times$. Since $\sharp(\kappa) > 2(\mathcal{K}^\times/\kappa^\times : \overline{U})$ by assumption, there exist three distinct values $a, b, c \in \kappa$ such that the images of $g-a, g-b, g-c$ in the quotient group $(\mathcal{K}^\times/\kappa^\times)/\overline{U}$ are the same. Now, define U to be the inverse image of $\overline{U}$ in $\mathcal{K}^\times$ and set

$$f \stackrel{\text{def}}{=} \frac{a-b}{b-c} \cdot \frac{g-c}{g-a} \in U.$$

Then we have

$$1+f=\frac{a-c}{b-c}\cdot\frac{g-b}{g-a}\in U.$$

Finally, as f is a linear fractional transformation of g, we have $\deg(f)=\deg(g)=\mathrm{gon}(X\times_k\kappa)$, as desired. $\square$

Lemma 4.9. *Let*

$$\tau:(k^\times)^\Sigma\xrightarrow{\sim}(l^\times)^\Sigma$$

be the isomorphism in Lemma 4.5 between the maximal Σ-primary quotients of the multiplicative groups of the constant fields, which, by Lemma 4.7, extends to

$$\tau:(\bar{k}^\times)^\Sigma\xrightarrow{\sim}(\bar{l}^\times)^\Sigma$$

naturally (by passing to the open subgroups of $G_K^{(\Sigma)}$ and $G_L^{(\Sigma)}$, corresponding to each other via σ). For $\eta\in\bar{k}^\times$ and $\zeta\in\bar{l}^\times$, if

$$1+\eta\neq 0 \text{ and } \tau(\eta')=\zeta',$$

where η' (resp. ζ') is the image of η (resp. ζ) in $(\bar{k}^\times)^\Sigma$ (resp. $(\bar{l}^\times)^\Sigma$), then there exist $\alpha,\beta\in\bar{l}^\times\{\Sigma'\}$, such that

$$\alpha+\beta\zeta\neq 0 \text{ and } \tau((1+\eta)')=(\alpha+\beta\zeta)'.$$

Proof. Take a finite extension k' of k (resp. l' of l) such that $\mathrm{gon}(X\times_k\bar{k})=\mathrm{gon}(X\times_k k')$ (resp. $\mathrm{gon}(Y\times_l\bar{l})=\mathrm{gon}(Y\times_l l')$). By replacing k' and l' with suitable finite extensions, we may and shall assume that $\sigma:G_K^{(\Sigma)}\xrightarrow{\sim}G_L^{(\Sigma)}$ induces an isomorphism $G_{Kk'}^{(\Sigma)}\xrightarrow{\sim}G_{Ll'}^{(\Sigma)}$ (cf. Lemma 2.4.3). Since Σ satisfies condition (ϵ_X), there exists an extension k'' of k' in $\bar{k}$, such that $\sharp(k'')>2\sharp(J_X(k'')\{\Sigma'\})$. In particular, $\sharp(J_X(k'')\{\Sigma'\})<\infty$, hence, by replacing k'' by a suitable subfield containing k' if necessary, we may and shall assume that k'' is a finite extension of k'. Let l'' be the finite extension of l' corresponding to k'' via σ: $\sigma(G_{Kk''}^{(\Sigma)})=G_{Ll''}^{(\Sigma)}$. Set $\mathcal{K}\stackrel{\mathrm{def}}{=}Kk''$ and $\mathcal{L}\stackrel{\mathrm{def}}{=}Ll''$. Now, by Lemma 4.8, there exists $f\in\mathcal{K}^\times\setminus(k'')^\times$, such that $\bar{f}$, $\overline{1+f}\in\mathrm{Ker}(\mathcal{K}^\times/(k'')^\times\to\pi_1(Y\times_l l'')^{\mathrm{ab,tor}}\{\Sigma'\})$ and that $\deg(f)=\mathrm{gon}(X\times_k k'')=\mathrm{gon}(X\times_k\bar{k})$. Similarly, there exists $g_1\in\mathcal{L}^\times\setminus(l'')^\times$, such that $\bar{g}_1$, $\overline{1+g_1}\in\mathrm{Ker}(\mathcal{L}^\times/(l'')^\times\to\pi_1(X\times_k k'')^{\mathrm{ab,tor}}\{\Sigma'\})$ and that $\deg(g_1)=\mathrm{gon}(Y\times_l l'')=\mathrm{gon}(Y\times_l\bar{l})$ (use again Lemma 4.8). Here we used the fact that $\pi_1(Y\times_l l'')^{\mathrm{ab,tor}}\{\Sigma'\}\xrightarrow{\sim}J_Y(l'')\{\Sigma'\}$ and $\pi_1(X\times_k k'')^{\mathrm{ab,tor}}\{\Sigma'\}\xrightarrow{\sim}J_X(k'')\{\Sigma'\}$.

We may write $\rho(f')=g'$ for some $g\in\mathcal{L}^\times/(l'')^\times$ and $\rho^{-1}(g_1')=f_1'$ for some $f_1\in\mathcal{K}^\times/(k'')^\times$. Thus, we have

$$\mathrm{gon}(X\times_k\bar{k})=\deg(f)=\deg(g)\geq\mathrm{gon}(Y\times_l\bar{l})$$

and

$$\mathrm{gon}(Y\times_l\bar{l})=\deg(g_1)=\deg(f_1)\geq\mathrm{gon}(X\times_k\bar{k}),$$

where the second equalities follow from Lemmas 2.4.5 and 4.6 (i), hence

$$n\stackrel{\mathrm{def}}{=}\mathrm{gon}(X\times_k\bar{k})=\mathrm{gon}(Y\times_l\bar{l}).$$

Further, again by replacing k'' and l'' by suitable finite extensions corresponding to each other, we may assume that the zeros ($\subset X^{\mathrm{cl}}$) of $1+f$ are k''-rational and that $\eta \in (k'')^\times$ and $\zeta \in (l'')^\times$.

From now on, we may and shall assume that $k'' = k$, and $l'' = l$, by replacing K and L by $\mathcal{K}$ and $\mathcal{L}$, respectively, and $\sigma : G_K^{(\Sigma)} \xrightarrow{\sim} G_L^{(\Sigma)}$ by the isomorphism $G_{\mathcal{K}}^{(\Sigma)} \xrightarrow{\sim} G_{\mathcal{L}}^{(\Sigma)}$ induced by σ. Thus, $f \in H_K^\times \subset K^\times$, $g \in H_L^\times \subset L^\times$, $\rho(f') = g'$ and $\bar{\rho}(\bar{f}) = \bar{g}$.

Claim. We have

$$\rho((1+f)') = (\alpha + \beta g)'$$

for some $\alpha, \beta \in \bar{l}^\times\{\Sigma'\}$.

Indeed, as $\overline{1+f} \in \overline{H}_K$, we have $(1+f)' \in H_K'$, hence we may write $\rho((1+f)') = h'$ for some $h \in H_L^\times \subset L^\times$. Write $\overline{1+f} = x_1 + \cdots + x_n - (f)_\infty$ as a divisor, where $(f)_\infty$ denotes the pole divisor of f which is equal to the pole divisor of $1+f$. Note that $x_1, \ldots, x_n$ are k-rational by our choice. Then $\bar{h} = y_1 + \cdots + y_n - (g)_\infty$, where $y_i \stackrel{\text{def}}{=} \phi(x_i)$ (observe that g and h have the same pole divisor by Lemma 4.6 (i), since f and $1+f$ do). Note that $y_1, \ldots, y_n$ are l-rational by Lemma 2.4.5. Let $c \stackrel{\text{def}}{=} g(y_1) \in l^\times$. Note that $-c \in l^\times\{\Sigma'\}$ by the preservation of the Σ-value of pseudo-functions (cf. Lemma 4.6 (ii)) and the fact that $f(x_1) = -1$. (Observe that $\tau_{x,y}(-1) = -1$.) Thus, $g - c$ has a zero at y_1 and $\overline{g-c} = y_1 + E - (g)_\infty$ as a divisor, where E is an effective divisor of degree $n-1$. Consider the function $h_1 \stackrel{\text{def}}{=} h/(g-c)$. Then $\overline{h_1} = y_2 + \cdots + y_n - E$ as a divisor. Thus, $\deg(h_1) < n$, which implies that $h_1 = \beta \in l$ is a constant (by the minimality of n as the degree of a non-constant function), and $h = \beta g + \beta(-c)$. Further, let $w \in X^{\mathrm{cl}}$ be a zero of f and set $z \stackrel{\text{def}}{=} \phi(w) \in Y^{\mathrm{cl}}$. Then z is a zero of g by Lemma 4.6 (i), and

$$\beta = h_1(z) = h(z)/(-c) \in \bar{l}^\times\{\Sigma'\}$$

by Lemma 4.6 (ii) and the fact that $(1+f)(w) = 1$. Thus, $\alpha \stackrel{\text{def}}{=} \beta(-c) \in \bar{l}^\times\{\Sigma'\}$, and the above claim is proved.

Let $f \in H_K^\times$ and $g \in H_L^\times$ be as above. In particular, $\rho(f') = g'$ and $\rho((1+f)') = (\alpha + \beta g)'$ for some $\alpha, \beta \in \bar{l}^\times\{\Sigma'\}$. Now, let $\eta \in k^\times$, $\zeta \in l^\times$ such that $1 + \eta \neq 0$ and $\tau(\eta') = \zeta'$. Let $x \in X^{\mathrm{cl}}$ be a zero of $f - \eta$ and set $y \stackrel{\text{def}}{=} \phi(x)$. We have $f \equiv \eta \pmod{x}$ which implies that $1 + f \equiv 1 + \eta \pmod{x}$, and $g \equiv \xi \pmod{y}$ where $\zeta' = \xi' \in (\bar{l}^\times)^\Sigma$ by the preservation of the Σ-values of pseudo-functions (cf. Lemma 4.6 (ii)), i.e. there exists $\epsilon \in \bar{l}^\times\{\Sigma'\}$ such that $\xi = \epsilon\zeta$. Then

$$\tau((1+\eta)') = \tau((1+f)'(x)) = \rho((1+f)')(y) = (\alpha+\beta g)'(y) = (\alpha+\beta\xi)' = (\alpha+\beta\epsilon\zeta)',$$

where the second equality results from the preservation of the Σ-values of pseudo-functions (cf. Lemma 4.6 (ii)). As $\alpha, \beta\epsilon \in \bar{l}^\times\{\Sigma'\}$, the assertion follows. $\square$

Lemma 4.10. *The isomorphisms*

$$\tau_{x,y} : (k(x)^\times)^\Sigma \xrightarrow{\sim} (k(y)^\times)^\Sigma,$$

in Lemma 2.3 (i) satisfy the following property: For $\eta \in k(x)^\times$ and $\zeta \in k(y)^\times$, if

$$1 + \eta \neq 0 \text{ and } \tau_{x,y}(\eta') = \zeta',$$

then there exist $\alpha, \beta \in \bar{l}^{\times}\{\Sigma'\}$, *such that*

$$\alpha + \beta\zeta \neq 0 \text{ and } \tau_{x,y}((1+\eta)') = (\alpha + \beta\zeta)'.$$

Proof. After passing to finite extensions of scalars, this follows directly from Lemma 4.9 and the commutativity of the diagram (4.4) in Lemma 4.5. $\square$

Next, recall the definition of $H_K^{\times}$:

$$H_K^{\times} \stackrel{\text{def}}{=} \{f \in K^{\times} \mid \bar{f} \in \overline{H}_K\}.$$

Then $H_K^{\times}$ is a finite index subgroup of $K^{\times}$, and the (finite) quotient $K^{\times}/H_K^{\times}$ is killed by $m \stackrel{\text{def}}{=} \sharp(\pi_1(X)^{\text{ab,tor}}\{\Sigma'\}) = \sharp(\pi_1(Y)^{\text{ab,tor}}\{\Sigma'\})$, hence is Σ'-primary.

Lemma 4.11. *Let* $f \in H_K^{\times}$, *and assume that* $1 + f \neq 0$. *Then* $1 + f \in H_K^{\times}$.

Proof. Write $\rho(f') = g'$ with $g \in H_L^{\times}$. First, we have $((1+f)')^m \in H'_K$. Thus, we may write $\rho(((1+f)')^m) = h'$ with $h \in H_L^{\times}$. Next, let $x \in X^{\text{cl}}$ such that x is neither a pole of f nor a zero of $1+f$ and set $y \stackrel{\text{def}}{=} \phi(x)$. Then we have

$$\begin{aligned} &h'(y) = \rho(((1+f)')^m)(y) \\ =&\tau_{x,y}(((1+f)')^m(x)) = \tau_{x,y}((1+f(x))')^m \\ =&((\alpha_y + \beta_y g(y))')^m \end{aligned}$$

for some $\alpha_y, \beta_y \in \bar{l}^{\times}\{\Sigma'\}$, by Lemma 4.6 (ii) and Lemma 4.10. Equivalently, for some $\alpha_y, \beta_y, \gamma_y \in \bar{l}^{\times}\{\Sigma'\}$, we have

$$h(y) = \gamma_y(\alpha_y + \beta_y g(y))^m.$$

Thus, we have

$$h(y) = a_y(1 + c_y g(y))^m,$$

where $a_y \stackrel{\text{def}}{=} \gamma_y \alpha_y^m \in \bar{l}^{\times}\{\Sigma'\}$ and $c_y \stackrel{\text{def}}{=} \beta_y/\alpha_y \in \bar{l}^{\times}\{\Sigma'\}$. By Definition/Proposition 3.12 (i)(ii), this implies that $h = a(1+cg)^m$ for some $a, c \in l^{\times}\{\Sigma'\}$. Accordingly, $\rho(((1+f)')^m) = h' = ((1+cg)')^m$ in $H'_Y \subset (L^{\times})^{(\Sigma)}$, hence $\rho((1+f)')^m = ((1+cg)')^m$ in $\text{Ker}(\psi_L^{(\Sigma)})$, where $\psi_L^{(\Sigma)} : J_L^{(\Sigma)} \to G_Y^{(\Sigma),\text{ab}}$ is naturally induced by Artin's reciprocity map in global class field theory.

Now, since $\text{Ker}(\psi_L^{(\Sigma)}) \subset J_L^{(\Sigma)} = \prod'_{y \in Y^{\text{cl}}} (L_y^{\times})^{(\Sigma)}$ does not admits a nontrivial Σ'-primary torsion, we conclude $\rho((1+f)') = (1+cg)'$ in $\text{Ker}(\psi_L^{(\Sigma)})$. As $(1+cg)' \in (L^{\times})^{(\Sigma)}$, we have $(1+f)' \in H'_K$ by definition, as desired. $\square$

We set

$$H_K \stackrel{\text{def}}{=} H_K^{\times} \cup \{0\} \subset K,$$

and

$$H_L \stackrel{\text{def}}{=} H_L^{\times} \cup \{0\} \subset L.$$

Lemma 4.12. (i) *The subset H_K of K is a subfield.*
(ii) *We have $H_K = K$, $H_L = L$, $H'_K = (K^\times)^{(\Sigma)}$, $H'_L = (L^\times)^{(\Sigma)}$, $\overline{H}_K = K^\times/k^\times$, and $\overline{H}_L = L^\times/l^\times$.*

Proof. (i) First note that $k \subset H_K$, and H_K is closed under multiplication by its definition. Also, H_K is closed under addition. Indeed, let us show $f + g \in H_K$ for any $f, g \in H_K$. This is clear if either one of $f, g, f+g$ is zero. So, assume that none of $f, g, f+g$ is zero. Then, as $\frac{g}{f} \in H_K^\times$ and $1 + \frac{g}{f} = \frac{f+g}{f} \neq 0$, Lemma 4.11 implies that $1 + \frac{g}{f} \in H_K^\times$, hence $f + g = f(1 + \frac{g}{f}) \in H_K^\times \subset H_K$. Thus, H_K is a k-subfield of K.
(ii) Since $H_K^\times$ is a finite index subgroup of $K^\times$, there exist finitely many $f_1, \ldots, f_r \in K^\times$ such that $K = H_K f_1 \cup \cdots \cup H_K f_r$ holds. In particular, $K = H_K f_1 + \cdots + H_K f_r$, hence $H_K \subset K$ is a finite field extension. This implies that H_K is an infinite field. Then K cannot be covered by finitely many proper H_K-vector subspaces. Thus, the equality $K = H_K f_1 \cup \cdots \cup H_K f_r$ implies that K is 1-dimensional over H_K, i.e., $K = H_K$, as desired. In particular, $H'_K = (K^\times)^{(\Sigma)}$ and $\overline{H}_K = K^\times/k^\times$.

As $((K^\times)^{(\Sigma)} : H'_K) = (\mathrm{Ker}(\psi_K^{(\Sigma)}) : H'_K)/\sharp(\pi_1(X)^{\mathrm{ab,tor}}\{\Sigma'\})$ and $((L^\times)^{(\Sigma)} : H'_L) = (\mathrm{Ker}(\psi_L^{(\Sigma)}) : H'_L)/\sharp(\pi_1(Y)^{\mathrm{ab,tor}}\{\Sigma'\})$, we have $((K^\times)^{(\Sigma)} : H'_K) = ((L^\times)^{(\Sigma)} : H'_L)$. Thus, $H'_L = L^\times/\Sigma$ also holds, from which $H_L = L$ and $\overline{H}_L = L^\times/l^\times$ follow. $\square$

It follows from Lemma 4.12 above that $\bar\rho$ is an isomorphism:

$$\bar\rho : K^\times/k^\times \xrightarrow{\sim} L^\times/l^\times$$

which is naturally induced by σ.

Next, we will think of elements of $K^\times/k^\times$ (resp. $L^\times/l^\times$) as points of the infinite-dimensional projective space over k (resp. l) associated to the vector space K (resp. L) over k (resp. l). In particular, points of this projective space correspond to one-dimensional k-linear (resp. l-linear) subspaces in K (resp. L), and lines correspond to two-dimensional k-linear (resp. l-linear) subspaces of K (resp. L).

Lemma 4.13. *(Recovering the Additive Structure of Function Fields) The natural isomorphism $\bar\rho : K^\times/k^\times \xrightarrow{\sim} L^\times/l^\times$ which follows from Lemmas 4.5 and 4.12, viewed as a set-theoretic bijection between points of projective spaces, preserves colineations. Accordingly, $\bar\rho$ arises from a ψ_0-isomorphism*

$$\psi : (K, +) \xrightarrow{\sim} (L, +),$$

where $\psi_0 : k \xrightarrow{\sim} l$ is a field isomorphism. Namely, ψ is an isomorphism of abelian groups which is compatible with ψ_0 in the sense that $\psi(ax) = \psi_0(a)\psi(x)$ for $a \in k$ and $x \in K$. Further, ψ_0 is uniquely determined and ψ is uniquely determined up to scalar multiplication.

Proof. In order to show the first assertion that the map $\bar\rho$ preserves colineations, it suffices to show that for a non-constant function $f \in K^\times \setminus k^\times$, if $\bar\rho(\bar f) = \bar g$, then $\bar\rho(\overline{1+f}) = \overline{\alpha + \beta g}$, where $\alpha, \beta \in l$. By replacing $g \in L^\times$ if necessary, we may and shall assume that $\rho(f') = g'$ holds. Write $\rho((1+f)') = h'$ with $h \in L^\times$. Let $x \in X^{\mathrm{cl}}$ with $f(x) \notin \{\infty, 0, -1\}$, and set $y \stackrel{\mathrm{def}}{=} \phi(x)$. Then $\tau_{x,y}(f'(x)) = g'(y)$ and $\tau_{x,y}((1+f)'(x)) = h'(y)$ by Lemma 4.6 (ii). Let $\eta \stackrel{\mathrm{def}}{=} f(x)$ and $\zeta \stackrel{\mathrm{def}}{=} g(y)$. Then $\tau_{x,y}(\eta') = \zeta'$.

But $\tau_{x,y}((1+\eta)') = (\alpha_y + \beta_y\zeta)'$ by Lemma 4.10, where $\alpha_y, \beta_y \in \bar{l}^\times\{\Sigma'\}$. Thus, $h(y) = a_y + b_y g(y)$, where $a_y, b_y \in \bar{l}^\times\{\Sigma'\}$, $\forall' x \in X^{\mathrm{cl}}$. But this implies that $h = a + bg$ for some $a, b \in l^\times\{\Sigma'\}$ by Proposition 3.11 and Definition/Proposition 3.5, as required.

The second and the third assertions follow from the first assertion and the fundamental theorem of projective geometry (cf. [Artin]). $\square$

Lemma 4.14. *(Recovering the Field Structure of Function Fields) If we normalize the isomorphism*

$$\psi : (K, +) \stackrel{\sim}{\to} (L, +)$$

in Lemma 4.13 by the condition $\psi(1) = 1$, it becomes a field isomorphism such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{\psi} & L \\ \uparrow & & \uparrow \\ k & \xrightarrow{\psi_0} & l \end{array}$$

commutes.

Proof. (See also the end of the proof of Theorem 5.11 in [Pop2].) Take any $f \in K^\times$, then $\psi \circ \mu_f$ and $\mu_{\psi(f)} \circ \psi$ are ψ_0-isomorphisms $(K, +) \stackrel{\sim}{\to} (L, +)$, where μ_g denotes the g-multiplication map. The isomorphisms $K^\times/k^\times \stackrel{\sim}{\to} L^\times/l^\times$ they induce coincide with each other:

$$\overline{\psi \circ \mu_f} = \bar{\rho} \circ \mu_{\bar{f}} = \mu_{\bar{\rho}(\bar{f})} \circ \bar{\rho} = \overline{\mu_{\psi(f)} \circ \psi},$$

where the second equality follows from the multiplicativity of $\bar{\rho}$. Further, we have

$$\psi \circ \mu_f(1) = \psi(f) = \mu_{\psi(f)}(1) = \mu_{\psi(f)} \circ \psi(1).$$

Thus, the equality $\psi \circ \mu_f = \mu_{\psi(f)} \circ \psi$ follows from the uniqueness in the fundamental theorem of projective geometry, which shows the multiplicativity of ψ.

For any $a \in k$, we have

$$\psi(a) = \psi(a \cdot 1) = \psi_0(a)\psi(1) = \psi_0(a) \cdot 1 = \psi_0(a),$$

which shows the commutativity of the diagram. $\square$

Let U be an open subgroup of $G_K^{(\Sigma)}$, and let $V \stackrel{\mathrm{def}}{=} \sigma(U)$. Let K'/K (resp. L'/L) be the subextension of $K^\sim/K$ (resp. $L^\sim/L$) corresponding to U (resp. V). Then σ induces, by restriction to U, an isomorphism

$$\sigma : U \stackrel{\sim}{\to} V.$$

Note that it is unclear in general if Σ satisfies condition $(\epsilon_{X'})$, where X' denotes the normalization of X in K'. However, this condition is only used to establish Lemma 4.9 by resorting to Lemma 4.8. Since the assertion of Lemma 4.9 for $\sigma : U(= G_{K'}^{(\Sigma)}) \stackrel{\sim}{\to} V(= G_{L'}^{(\Sigma)})$ is just the same as that for $\sigma : G_K^{(\Sigma)} \stackrel{\sim}{\to} G_L^{(\Sigma)}$, we can deduce from $\sigma : U \stackrel{\sim}{\to} V$, by Lemma 4.14 (without the need to assume that Σ satisfies condition $(\epsilon_{X'})$), a natural field isomorphism:

$$\psi' : K' \stackrel{\sim}{\to} L'.$$

Lemma 4.15. (i) *The following diagram is commutative:*

$$\begin{array}{ccc} K' & \xrightarrow{\psi'} & L' \\ \uparrow & & \uparrow \\ K & \xrightarrow{\psi} & L \end{array}$$

where the vertical arrows are the natural inclusions and ψ, ψ' are the field isomorphisms induced by σ.
(ii) *If, moreover, U is normal in $G_K^{(\Sigma)}$, then V is normal in $G_L^{(\Sigma)}$ and the above diagram is Galois-equivariant with respect to the isomorphism $G_K^{(\Sigma)}/U \xrightarrow{\sim} G_L^{(\Sigma)}/V$ induced by σ.*

Proof. (i) Let k' (resp. l') denote the constant field of K' (resp. L'). Then, the commutativity of the diagram (4.7) in Lemma 4.7 implies that the diagram

$$\begin{array}{ccc} (K')^\times/(k')^\times & \xrightarrow{\bar{\rho}'} & (L')^\times/(l')^\times \\ \uparrow & & \uparrow \\ K^\times/k^\times & \xrightarrow{\bar{\rho}} & L^\times/l^\times \end{array} \tag{4.9}$$

commutes.

Now, write $i : K \to K'$ and $j : L \to L'$ for the natural inclusions. To prove $\psi' \circ i = j \circ \psi$, we shall first show that the image $\psi'(K)$ of the left-hand side map $\psi' \circ i$ and the image $j(\psi(K)) = L$ of the right-hand side map $j \circ \psi$ coincide with each other. But by the commutativity of (4.9), we have at least: $\psi'(K^\times) \cdot (l')^\times = L^\times \cdot (l')^\times$. Set $H \stackrel{\text{def}}{=} \psi'(K) \cap L$, which is a subfield of $\psi'(K)$ and a subfield of L at a time. Further, $H^\times = \psi'(K^\times) \cap L^\times$ is of finite index (dividing $\sharp((l')^\times)$) both in $\psi'(K^\times)$ and in $L^\times$. Thus, as in the proof of Lemma 4.12, we deduce $\psi'(K) = H = L$.

Finally, the desired equality $\psi' \circ i = j \circ \psi$ follows from the uniqueness in the fundamental theorem of projective geometry, since the diagram (4.9) commutes and $\psi' \circ i(1) = 1 = j \circ \psi(1)$.
(ii) Assume that U is normal in $G_K^{(\Sigma)}$, then $V = \sigma(U)$ is normal in $G_L^{(\Sigma)}$ and σ induces an isomorphism $G_K^{(\Sigma)}/U \xrightarrow{\sim} G_L^{(\Sigma)}/V$.

Since the action of $G_K^{(\Sigma)}/U$ (resp. $G_L^{(\Sigma)}/V$) on $J_{K'}^{(\Sigma)}$ (resp. $J_{L'}^{(\Sigma)}$) arises from the conjugation on the decomposition groups, the isomorphism $J_{K'}^{(\Sigma)} \xrightarrow{\sim} J_{L'}^{(\Sigma)}$ is Galois-equivariant.

Further, since the diagram

$$\begin{array}{ccccc} (K')^\times/(k')^\times & \twoheadleftarrow & ((K')^\times)^{(\Sigma)} & \hookrightarrow & J_{K'}^{(\Sigma)} \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ (L')^\times/(l')^\times & \twoheadleftarrow & ((L')^\times)^{(\Sigma)} & \hookrightarrow & J_{L'}^{(\Sigma)} \end{array}$$

is commutative, the isomorphisms $\rho' : ((K')^\times)^{(\Sigma)} \xrightarrow{\sim} ((L')^\times)^{(\Sigma)}$ and $\bar{\rho}' : (K')^\times/(k')^\times \xrightarrow{\sim} (L')^\times/(l')^\times$ are Galois-equivariant.

Finally, it follows from the uniqueness in the fundamental theorem of projective geometry that the isomorphism $\psi' : K' \xrightarrow{\sim} L'$ is Galois-equivariant, as desired. $\square$

By considering various open subgroups of $G_K^{(\Sigma)}$ and $G_L^{(\Sigma)}$ as above, corresponding to each other via σ, and using Lemmas 4.14 and 4.15 (i), we obtain naturally a field isomorphism

$$\tilde{\psi} : K^{\sim} \xrightarrow{\sim} L^{\sim}.$$

Lemma 4.16. *The following diagram is commutative:*

$$\begin{array}{ccc} K^{\sim} & \xrightarrow{\tilde{\psi}} & L^{\sim} \\ \uparrow & & \uparrow \\ K & \xrightarrow{\psi} & L \end{array}$$

where the vertical arrows are the natural inclusions and ψ, $\tilde{\psi}$ are field isomorphisms induced by σ. Further, $\tilde{\psi}$ is Galois-equivariant with respect to $\sigma : G_K^{(\Sigma)} \xrightarrow{\sim} G_L^{(\Sigma)}$.

Proof. This follows directly from Lemma 4.15. $\square$

Thus, from Lemma 4.16 we deduce a commutative diagram

$$\begin{array}{ccc} L^{\sim} & \xrightarrow{\tilde{\psi}^{-1}} & K^{\sim} \\ \uparrow & & \uparrow \\ L & \xrightarrow{\psi^{-1}} & K \end{array}$$

which is Galois-equivariant with respect to $\sigma : G_K^{(\Sigma)} \xrightarrow{\sim} G_L^{(\Sigma)}$. This completes the proof of the existence part of Theorem 4.1. For the uniqueness part of Theorem 4.1, suppose that for $j = 1, 2$ there is a commutative diagram

$$\begin{array}{ccc} L^{\sim} & \xrightarrow{\tilde{\psi}_j^{-1}} & K^{\sim} \\ \uparrow & & \uparrow \\ L & \xrightarrow{\psi_j{}^{-1}} & K \end{array}$$

as above, where the horizontal maps are isomorphisms, and which is Galois-equivariant with respect to $\sigma : G_K^{(\Sigma)} \xrightarrow{\sim} G_L^{(\Sigma)}$. Set $\tilde{\alpha} \stackrel{\text{def}}{=} \tilde{\psi}_2^{-1} \circ \tilde{\psi}_1 \in \mathrm{Aut}(K^{\sim})$ and $\alpha \stackrel{\text{def}}{=} \psi_2^{-1} \circ \psi_1 \in \mathrm{Aut}(K)$. Then they fit into the following commutative diagram

$$\begin{array}{ccc} K^{\sim} & \xrightarrow{\tilde{\alpha}} & K^{\sim} \\ \uparrow & & \uparrow \\ K & \xrightarrow{\alpha} & K \end{array}$$

which is Galois-equivariant with respect to $\mathrm{id} : G_K^{(\Sigma)} \xrightarrow{\sim} G_K^{(\Sigma)}$. Namely, $\tilde{\alpha}$ commutes with $G_K^{(\Sigma)} (= \mathrm{Gal}(K^{\sim}/K))$ in $\mathrm{Aut}(K^{\sim})$, or, equivalently, the conjugation action of $\tilde{\alpha}$ on $G_K^{(\Sigma)}$ is trivial. Then, in particular, every finite Galois extension $K \subset K' \subset K^{\sim}$

is preserved by $\tilde{\alpha}$. Further, considering the action of $\tilde{\alpha}$ on $J_{K'}^{(\Sigma)} \hookleftarrow ((K')^\times)^{(\Sigma)} \twoheadrightarrow (K')^\times/(k')^\times$, we conclude that the action of $\tilde{\alpha}$ on $(K')^\times/(k')^\times$ is trivial. Now, it follows from the uniqueness in the fundamental theorem of projective geometry that the action of $\tilde{\alpha}$ on $(K')^\times$ is trivial. Since $K \subset K' \subset K^\sim$ is an arbitrary finite Galois extension, we conclude that the action of $\tilde{\alpha}$ on $K^\sim$ is trivial, i.e., $\tilde{\alpha} = 1$, as desired. This completes the proof of Theorem 4.1. $\square$

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