

Parameter shifts for nonautonomous systems in low dimension: Bifurcation- and Rate-induced tipping

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Abstract

We discuss the nonlinear phenomena of irreversible tipping for non-autonomous systems where time-varying inputs correspond to a smooth “parameter shift” from one asymptotic value to another. We express tipping in terms of properties of local pullback attractors and present some results on how nontrivial dynamics for non-autonomous systems can be deduced from analysis of the bifurcation diagram for an associated autonomous system where parameters are fixed. In particular, we show that there is a unique local pullback point attractor associated with each linearly stable equilibrium for the past limit. If there is a smooth stable branch of equilibria over the range of values of the parameter shift, the pullback attractor will remain close to (track) this branch for small enough rates, though larger rates may lead to rate-induced tipping. More generally, we show that one can track certain stable paths that go along several stable branches by pseudo-orbits of the system, for small enough rates. For these local pullback point attractors, we define notions of bifurcation-induced and irreversible rate-induced tipping of the non-autonomous system. In one-dimension, we introduce the notion of forward basin stability and use this to give a number of sufficient conditions for the presence or absence of rate-induced tipping. We apply our results to give criteria for irreversible rate-induced tipping in a conceptual climate model example.

1 Introduction

In common language, a *tipping point* or *critical transition* is a sudden, large and irreversible change in output of a complex system in response to a small change of input. There have been a range of papers in the applied sciences that use the notion of a tipping point in applications such as climate systems [9, 18, 34] or ecosystems [30, 11]. Further work has attempted to find predictors or *early warning signals* in terms of changes in noise properties

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near a tipping point [9, 10, 33, 31, 28], other recent work on tipping includes [22, 23]. Indeed there is a large literature on catastrophe theory (for example [2] and references therein) that addresses related questions in the setting of gradient systems.

There seems to be no universally agreed rigorous mathematical definition of what a tipping point is, other than it is a type of bifurcation point, though this excludes both “noise-induced” [32, 10] and “rate-induced” [30, 34, 25] transitions that have been implicated in this sort of phenomenon. Other work on critical transitions includes for example [17] that examines the interaction of noise and bifurcation-related tipping in the framework of stochastically perturbed fast-slow systems. In an attempt to understand some of these issues, a recent paper [3] classified tipping points into three distinct mechanisms: bifurcation-induced (B-tipping), noise-induced (N-tipping) and rate-induced (R-tipping). As noted in [3], a tipping point in a real non-autonomous system will typically be a mixture of these effects but even in idealized cases it is a challenge to come up with a mathematically rigorous and testable definition of “tipping point”.

The aims of this paper are:

- (a) to suggest some definitions for B-tipping and R-tipping appropriate for asymptotically constant “parameter shifts” in terms of pullback attractors for non-autonomous systems.
- (b) to show how these definitions allow us to exploit the bifurcation diagram of the associated autonomous (i.e. quasi-static) system to obtain a lot of information about B- and R-tipping of the non-autonomous system associated with a parameter shift.
- (c) to give some illustrative examples of properties that allow us to predict or rule out B- and R-tipping in various cases.

Restricting to simple attractors and asymptotically constant “parameter shifts” allows us to obtain a number of rigorous results for a natural class of non-stationary parameter changes that may be encountered in natural sciences and engineering.

The paper is organized as follows: In Section 2 we consider arbitrary dimensional non-autonomous systems and introduce a class of parameter shifts $\mathcal{P}(\lambda_-, \lambda_+)$ that represent smooth (but not necessarily monotonic) changes between asymptotic values λ_- and λ_+ . For such parameter shifts, we relate stability properties of branches of solutions in the autonomous system with fixed parameters to properties of local pullback point attractors of the non-autonomous system with time-varying parameters. This means that the systems we consider are asymptotically autonomous [20, 27] both in forward and backward time.

After some discussion of (local) pullback attractors, we show in Theorem 2.2 that one can associate a unique pullback point attractor with each linearly stable equilibrium X_- for the past limit system (i.e. the autonomous system with $\lambda = \lambda_-$). Then, we define two types of tracking: *ϵ -close tracking* and *end-point tracking*. We show in Lemma 2.3 that if there is a stable branch of equilibria with no bifurcation points from this X_- to some stable equilibrium X_+ for the future limit system (i.e. the autonomous system with $\lambda = \lambda_+$), then this unique pullback point attractor will ϵ -closely track a path of stable equilibria for sufficiently slow rates.

On the other hand, there may be a number of stable paths connecting X_- to X_+ - each of these with an image that is a union of stable branches, but possibly passing through bifurcation points. In this case, the pullback point attractor can clearly only track at most one of these paths. Allowing arbitrarily small perturbations along the path, there will be pseudo-orbits that can track any stable path. In Theorem 2.4 we give such result by stating necessary conditions for ϵ -close tracking by pseudo-orbits.

Section 3 gives some definitions for B- and R-tipping that correspond to lack of end-point tracking for parameter shifts. In that section we prove several results for one-dimensional systems, including the following:

- Certain bifurcation diagrams or parameter ranges will not permit irreversible R-tipping (Theorem 3.2, case 1.).
- Other bifurcation diagrams will give irreversible R-tipping for certain parameter shifts (Theorem 3.2, cases 2. and 3.).
- There are restrictions on the regions in parameter space where irreversible R-tipping can occur (Section 3.3).

These results use some properties of the paths of attractors. We introduce the notion of *forward basin stability* where at each point in time the basin of attraction of an attractor for the associated autonomous system contains all earlier attractors on the path. We prove that:

- Forward basin stability guarantees tracking (Theorem 3.2 case 1.).
- Lack of forward basin stability, plus some additional assumptions, gives testable criteria for irreversible R-tipping (Theorem 3.2 cases 2. and 3.).

We also give an example application to a class of global energy-balance climate models in Section 4.

In Section 5 we discuss some of the challenges to extend the rigorous results to higher-dimensional state or parameter spaces, to cases where noise is present and to parameter paths that are more general than parameter shifts.

1.1 The setting: non-autonomous nonlinear systems

Suppose we have a non-autonomous system of the form

$$\frac{dx}{dt} = f(x, \Lambda(rt)), \tag{1}$$

where $x \in \mathbb{R}^n$, $\Lambda(s) \in C^2(\mathbb{R}, \mathbb{R}^d)$, $f \in C^2(\mathbb{R}^{n+d}, \mathbb{R}^n)$, $t \in \mathbb{R}$ and the “rate” $r > 0$ is fixed. For $x_0 \in \mathbb{R}^n$, $t, s \in \mathbb{R}$ and starting at $x(s) = x_0$ we write

$$\Phi_{t,s}(x_0) := x(t)$$

to denote the *solution cocycle* [16], where $x(t)$ is the corresponding solution of (1). We also assume $\Phi_{t,s}(x_0)$ is defined for all $t > s$ and x_0 , and note that it satisfies a cocycle equation

$$\Phi_{t,s} \circ \Phi_{s,u}(x_0) = \Phi_{t,u}(x_0)$$

for any $t > s > u$ and $x_0 \in \mathbb{R}^n$. Note that, in addition to x_0 , t and s , $\Phi_{t,s}$ depends on the shape of the parameter shift Λ , its rate of change r , as well as on the function f that defines the system. The key difference from the evolution operator of an autonomous system is that a cocycle depends on both the time elapsed ($t - s$) and the initial time s .

The model (1) can be extended to include stochastic effects, e.g. by considering a stochastic differential equation of the form

$$dx = f(x, \Lambda(rt))dt + \eta dW_t \quad (2)$$

where W_t represents an n -dimensional Wiener process and η represents a noise amplitude. A random dynamical systems approach [1] gives a solution cocycle that depends also on the choice of noise path.

One can understand a lot about the asymptotic behaviour of (1) [or (2)] from the bifurcation/attractor diagram of the associated autonomous system

$$\frac{dx}{dt} = f(x, \lambda), \quad (3)$$

where λ is fixed rather than time-dependent. General properties of dynamical systems suggest that if r is “small enough” then we can expect, for an appropriate definition of attractor, that trajectories of (1) will closely track a branch of attractors of (3) if the branch continues to be linearly stable over the range of the varying λ . In this paper we consider only the case of linearly stable equilibrium attractors for (3) where such statements can be made rigorous through use of geometric singular perturbation theory [13] or uniform hyperbolicity [5]. The control theory literature discusses related questions; see for example [14] that allows Λ to depend on state as well. However, close tracking of a continuous path that traverses several stable branches of (1) cannot always be guaranteed, even if r is small enough.

If the variation of Λ along a branch of attractors brings the system to a bifurcation point where no stable branches are nearby, we have the ingredients of a bifurcation-induced or B-tipping point; no matter how slowly we change the parameter, there will be a sudden and irreversible change in the state of the system x on, or near where the parameter passed through a dynamic bifurcation [4, 8, 23].

However, B-tipping is not the only way to get a sudden irreversible change. Even if there is a branch of attractors available for the system to track, a rate-induced tipping or R-tipping [3, 34] can occur; if there is a critical value of the rate r beyond which the system cannot track the branch of attractors then it may suddenly move to a different state. This happens even though there is no loss of stability in the autonomous system. Nonetheless, the bifurcation diagram of the associated autonomous system may give us a lot of information and indeed constraints on when R-tipping may or may not happen. Nontrivial behaviour that can appear as a result has been recently studied in [34, 21, 25] in terms of canard trajectories.

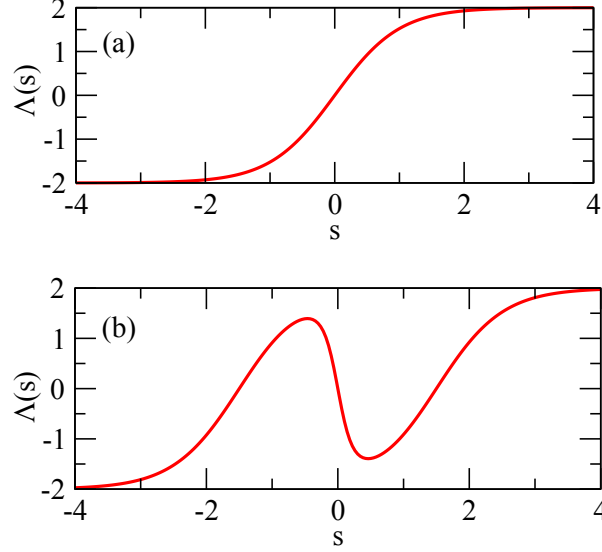


Figure 1: Examples of $\Lambda(s) \in \mathcal{P}(-2, 2)$. (a) shifts monotonically from -2 to 2 while (b) does this shift non-monotonically.

2 Tracking of attractors for non-autonomous systems

We consider the behaviour of (1) for one dimensional ($d = 1$) “parameter shifts”, i.e. choices of $\Lambda(s)$ that are C^2 -smooth (but not necessarily monotonic), bounded and satisfying

$$\lambda_- < \Lambda(s) < \lambda_+ \text{ for each (finite) value of } s,$$

that limit to the ends of the interval, $\Lambda(s) \rightarrow \lambda_{\pm}$ as $s \rightarrow \pm\infty$, where the limiting behaviour is asymptotically constant (i.e. $d\Lambda/ds \rightarrow 0$ as $s \rightarrow \pm\infty$). More precisely,

Assumption 2.1 *We assume*

$$\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$$

where we consider the subset of $C^2(\mathbb{R}, (\lambda_-, \lambda_+))$:

$$\mathcal{P}(\lambda_-, \lambda_+) = \left\{ \Lambda(s) : \lambda_- < \Lambda(s) < \lambda_+, \lim_{s \rightarrow \pm\infty} \Lambda(s) = \lambda_{\pm} \text{ and } \lim_{s \rightarrow \pm\infty} \frac{d\Lambda}{ds} = 0 \right\}.$$

Figure 1 gives two examples of parameter shifts that we consider, though clearly a parameter shift may have an arbitrary number of maxima and minima.

The *set of equilibria* for the bifurcation diagram of (3) is

$$\mathcal{X} = \{(x, \lambda) : f(x, \lambda) = 0 \text{ and } \lambda_- \leq \lambda \leq \lambda_+\} \subset \mathbb{R}^n \times \mathbb{R}.$$

Writing $df(x, \lambda)$ to denote the linearization of f with respect to x , we identify the subsets of \mathcal{X} that are linearly *stable*, *unstable* and *bifurcation equilibria*:

$$\begin{aligned} \mathcal{X}_{\text{stab}} &= \{(x, \lambda) \in \mathcal{X} : \max\{\text{Re}(\sigma(df(x, \lambda)))\} < 0\}, \\ \mathcal{X}_{\text{unst}} &= \{(x, \lambda) \in \mathcal{X} : \max\{\text{Re}(\sigma(df(x, \lambda)))\} > 0\}, \\ \mathcal{X}_{\text{bif}} &= \{(x, \lambda) \in \mathcal{X} : \max\{\text{Re}(\sigma(df(x, \lambda)))\} = 0\} \end{aligned}$$

where $\sigma(M)$ denotes the spectrum of the matrix M . Note that, in addition to generic bifurcations, \mathcal{X}_{bif} may include degenerate bifurcation points and even “bifurcations” that do not involve any topological change in the phase portrait, e.g. $\frac{d}{dt}x = -\lambda^2x - x^3$ at $\lambda = 0$.

We say \mathcal{X} is *regular* if (i) the set of bifurcation points are isolated (\mathcal{X}_{bif} has no accumulation point) and (ii) the only asymptotically stable recurrent sets (attractors) are equilibria. The first property of a regular \mathcal{X} is implied by the Kupka-Smale theorem for flows [15, Thm 7.2.13] under generic assumptions, though our assumption is less restrictive and allows the possibility of high codimension bifurcations in \mathcal{X}_{bif} . The second property of a regular \mathcal{X} holds for one-dimensional systems, but does not hold in general for higher dimensional systems.

Assumption 2.2 *We assume \mathcal{X} is regular. Furthermore, for ease of exposition, we assume w.l.o.g. that $\lambda_- < \lambda_+$ throughout and that there are no bifurcations at λ_{\pm} .*

Henceforth, we will denote the closure of U by \bar{U} and define:

Definition 2.1 *We say a C^2 -smooth curve $X(\lambda)$ in \mathcal{X} is a branch if it does not intersect with \mathcal{X}_{bif} except possibly at end points. We say it is a stable branch if it is contained in $\bar{\mathcal{X}}_{\text{stab}}$.*

Definition 2.2 *Given a parameter shift $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$, we say a continuous curve $(X(s), \Lambda(s))$ that limits to some (X_{\pm}, λ_{\pm}) as $s \rightarrow \pm\infty$ and whose image lies within \mathcal{X} is a path. We say it is a stable path if its image lies within $\bar{\mathcal{X}}_{\text{stab}}$.*

Note that branches and paths are defined in terms of solutions to the autonomous system (3) and, to some extent, the shape of the parameter shift $\Lambda(s)$, but are independent of the rate r . A path can traverse several branches that meet at isolated points in \mathcal{X}_{bif} , where the path may be nonsmooth. It may also traverse the same branch several times. As illustrated in Figure 1, some $\Lambda(s)$ can be monotonic functions of s and visit each point on a branch at just one value of s , others may have internal minima or maxima and visit the same point on a branch at more than one value of s . We mention a minor result that we use in a later theorem.

Lemma 2.1 *If \mathcal{X} is regular then a stable path traverses a finite number of smooth stable branches.*

Proof: Regularity of \mathcal{X} in particular means that there are no accumulations of bifurcation points \mathcal{X}_{bif} . Suppose that a stable path $(X(s), \Lambda(s))$ traverses an infinite number of smooth stable branches and hence passes through an infinite number of bifurcation points. As the path is continuous and bifurcation points are isolated on the path, there must be an accumulation as $s \rightarrow \infty$ or $-\infty$. However, as X has well-defined end-points in these limits $X_{\pm} = \lim_{s \rightarrow \pm\infty} X(s)$, one of X_{\pm} is an accumulation point, which gives a contradiction. **QED**

2.1 Local pullback attractors for the non-autonomous system

We are interested in the influence of parameter shifts $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$ and rates r on pullback attractors of the non-autonomous system (1). Note that the limit $r \rightarrow 0$ is singular in the sense that $r = 0$ does not exhibit any change in parameter values with time. In terms of the cocycle $\Phi_{s,t}$, the following definition is a local version of the pullback point attractor discussed in [16, Defn 3.48(ii)] that we have adapted to the context of parameter shifts:

Definition 2.3 *We say a solution $x(t)$ of (1) is a (local) pullback attractor if there is a bounded open set $U \subset \mathbb{R}^n$ and a time $T > 0$ such that (i) $x(t) \in U$ for all $t < -T$, and (ii) for any $y \in U$ and $t \in \mathbb{R}$ we have*

$$|\Phi_{t,t-s}(y) - x(t)| \rightarrow 0 \text{ as } s \rightarrow \infty. \quad (4)$$

In contrast to [16, Defn 3.48(ii)] we do not require that the pullback attractor attracts a non-autonomous open set $U(t)$. For the asymptotically autonomous case we consider here, we can relate each attractor for the past limit autonomous system to a pullback attractor for the non-autonomous system via the following theorem, whose proof is given in Appendix A.¹

Theorem 2.2 *For any X_- such that (X_-, λ_-) is in $\mathcal{X}_{\text{stab}}$, there is a unique trajectory $\tilde{x}_{pb}(t)$ of (1) with $\tilde{x}_{pb}(t) \rightarrow X_-$ as $t \rightarrow -\infty$, and this trajectory is a (local) pullback point attractor for (1). Moreover there is a neighbourhood U of X_- and $T > 0$ such that $\tilde{x}(t)$ is the only trajectory remaining in U for all $t < -T$.*

Theorem 2.2 shows that any trajectory that limits to a linearly stable equilibrium in the past is a pullback point attractor, and these pullback attractors are in one-to-one correspondence with choices of X_- such that $(X_-, \lambda_-) \in \mathcal{X}_{\text{stab}}$. Note that this result is similar to [16, Thm 3.54(i)], except that their assumption that $x = 0$ is invariant means that $\tilde{x}_{pb}(t) = 0$. Any small enough open neighbourhood U of X_- can be used to define the corresponding pullback attractor of (1), in particular by choosing small enough $\epsilon > 0$, the unique pullback attractor with past limit X_- can be expressed as

$$\tilde{x}_{pb}^{[\Lambda, r, X_-]}(t) := \bigcap_{s > 0} \bigcup_{u > s} \Phi_{t,t-u}(B_\epsilon(X_-)) \quad (5)$$

for a given parameter shift $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$, rate $r > 0$ and initial state X_- , where B_ϵ denotes a n -dimensional ball of radius ϵ . We do not use (5) in the following, but note that it may be useful as a basis for numerical approximation of the pullback attractor that limits to a given X_- for $t \rightarrow -\infty$.

¹We are indebted to an anonymous referee for suggesting this method of proof.

2.2 Tracking of stable branches by pullback attractors

We define two notions of tracking of branches of stable equilibria of (3) by attractors of (1). The strongest notion of tracking we consider is ϵ -close tracking:

Definition 2.4 For a given $\epsilon > 0$, we say a (piecewise) solution $x(t)$ of (1) for some $r > 0$ and $\Lambda(s) \in \mathcal{P}$ ϵ -close tracks the stable path $(X(s), \Lambda(s))$ if

$$|x(t) - X(rt)| < \epsilon \quad (6)$$

for all $t \in \mathbb{R}$.

The following is another, somewhat weaker notion of tracking, which we call *end-point tracking*:

Definition 2.5 Consider a stable path $(X(s), \Lambda(s))$ from (X_-, λ_-) to (X_+, λ_+) . We say a solution $x(t)$ of (1) end-point tracks this path if it satisfies

$$\lim_{t \rightarrow \pm\infty} x(t) = X_{\pm}.$$

Note that end-point tracking implies that for all $\epsilon > 0$ there is a $T > 0$ such that (6) holds for all $|t| > T$. The following result gives a sufficient condition for close- and end-point tracking of a stable branch.

Lemma 2.3 Suppose that $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$ and $(X(s), \Lambda(s))$ is a stable path that traverses a stable branch from (X_-, λ_-) to (X_+, λ_+) bounded away from \mathcal{X}_{bif} . Then for any $\epsilon > 0$ there is a $r_0 > 0$ such that for all $0 < r < r_0$ the pullback attractor $x_{\text{pb}}^{[\Lambda, r, X_-]}(t)$ satisfies ϵ -close tracking of $(X(s), \Lambda(s))$. The pullback attractor also end-point tracks the stable path $(X(s), \Lambda(s))$ for all sufficiently small $0 < r$.

Proof: Let $(X(s), \Lambda(s))$ be the stable path whose image is contained within the stable branch $X(\lambda)$ bounded away from \mathcal{X}_{bif} and we write $X(s) = X(\Lambda(s))$. Now consider the augmented system on \mathbb{R}^{n+1} given by

$$\begin{aligned} \frac{dx}{dt} &= f(x, \Lambda(s)) \\ \frac{ds}{dt} &= r. \end{aligned} \quad (7)$$

For $r = 0$ this has a compact invariant critical manifold

$$M_0 := \{(X(\Lambda(s)), s) : s \in \mathbb{R}\}$$

that is foliated with (neutrally stable) equilibria. Note that the section of M_0 with constant s limits to X_{\pm} for $s \rightarrow \pm\infty$. As the branch $X(\lambda)$ is bounded away from any bifurcation points, there is a $\rho > 0$ such that the normal Jacobian satisfies $\max\{\text{Re}(\sigma(df(X(s), \Lambda(s))))\} < -\rho <$

0 for all $s \in \mathbb{R}$. In consequence, the one-dimensional invariant manifold M_0 is uniformly normally hyperbolic. One can check that M_0 is a smooth Riemannian manifold of bounded geometry. Applying² [12, Theorem 3.1], there is an $r_0(\epsilon) > 0$ such that for any $r < r_0$ the system (7) has a unique normally hyperbolic invariant manifold M_r , where the C^1 distance from M_0 to M_r does not exceed ϵ .

Moreover, this manifold contains a single trajectory parametrized by $s \in \mathbb{R}$. In summary, for any $\epsilon > 0$ and for all $0 < r < r_0(\epsilon)$ there is a trajectory $(\tilde{x}(t), rt) \in M_r$ of (7) with $|\tilde{x}(t) - X(\Lambda(rt))| < \epsilon$ for all $rt \in \mathbb{R}$.

By Theorem 2.2 there is a neighbourhood U containing X_- such that only one trajectory $\tilde{x}(t)$ satisfies $\tilde{x}(t) \in U$ as $t \rightarrow -\infty$. Fixing $\epsilon > 0$ and $T > 0$ such that $\tilde{x}(t) \in B_\epsilon(X(t)) \subset U$ for all $t < -T$ we can see that the trajectory obtained is the pullback attractor

$$\tilde{x}(t) = x_{pb}^{[\Lambda, r, X_-]}(t).$$

In particular $\tilde{x}(t) \rightarrow X_-$ as $t \rightarrow -\infty$.

Now consider the behaviour of $\tilde{x}(t)$ in the limit $t \rightarrow \infty$. If we set $\tilde{x}(t) := X_+ + z(t)$ then

$$\frac{dz}{dt} = f(X_+ + z, \lambda_+) + G(z, t)$$

where $G(z, t) = f(X_+ + z, \Lambda(rt)) - f(X_+ + z, \lambda_+)$. Noting that

$$\frac{dz}{dt} = df(X_+, \lambda_+)z + G(z, t) + O(|z|^2) \tag{8}$$

where the error term is uniform in t . For any $\epsilon > 0$ define

$$g(t) := \sup\{|G(z, t')| : |z| < \epsilon \text{ and } t' > t\}$$

and continuity of f implies that $g(t) \rightarrow 0$ as $t \rightarrow \infty$. For any $\epsilon > 0$, $|\tilde{x}(t) - X(\Lambda(rt))| < \epsilon/2$ for all $r < r_0(\epsilon/2)$ and t . The fact that $\lim_{s \rightarrow \infty} X(\Lambda(s)) = X^+$ means there is a $T(\epsilon) > 0$ such that $|X_+ - X(\Lambda(rt))| < \epsilon/2$ for $t > T(\epsilon)$. Hence $|z(t)| < \epsilon$ for all $t > T(\epsilon)$. Finally, note that from the linear stability of $\dot{z} = df(X_+, \lambda_+)z$, there is an $\epsilon > 0$ such that if $|z(T)| < \epsilon$ then $z(t) \rightarrow 0$ as $t \rightarrow \infty$. It follows that $\tilde{x}(t) \rightarrow X_+$ as $t \rightarrow \infty$ and so the solution $\tilde{x}(t)$ end-point tracks the branch. **QED**

2.3 Tracking of stable paths by pseudo-orbits

One might wish to characterize a wider class of stable paths that can potentially be tracked by including stable paths that pass through bifurcation points where stable branches meet. To do this we have to widen our notion of tracking to consider tracking by pseudo-orbits.

Let us consider the possible stable paths that may be accessible for a given Λ .

²We use $k = 2$ and $\alpha = 0$ in the notation of [12] and note that because we restrict to $\mathcal{X}_{\text{stab}}$ there is an empty unstable bundle.

Definition 2.6 Given a parameter shift $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$, we say points (X_-, λ_-) and (X_+, λ_+) on $\mathcal{X}_{\text{stab}}$ are Λ -connected if they lie on the same stable path $(X(s), \Lambda(s))$.

Note that (X_-, λ_-) and (X_+, λ_+) are $\Lambda(t)$ -connected if and only if they are $\Lambda(rt)$ -connected for any $r > 0$, so the definition does not depend on r . For given (X_-, λ_-) and (X_+, λ_+) that are Λ -connected, it is not necessarily the case that there will be tracking by orbits of the non-autonomous system for some $r > 0$. In particular, there is no guarantee that the pullback attractor $x_{pb}^{[\Lambda, r, X_-]}(t)$ from Theorem 2.2 remains close to $X(rt)$, that

$$\lim_{t \rightarrow \infty} x_{pb}^{[\Lambda, r, X_-]}(t) = X_+$$

or even that $\lim_{t \rightarrow \infty} x_{pb}^{[\Lambda, r, X_-]}(t)$ is an attractor for the future limit system. Indeed the example in Section 2.4 shows that such a trajectory may not exist: achieving the appropriate branch switching may not be possible even on varying r and Λ .

As discussed later in Section 2.4 there may be several stable paths from X_- that end at different X_+ . We show that there will be tracking of any stable path by a *pseudo-orbit*, namely an orbit with occasional arbitrarily small adjustments. Recall the following definition of a pseudo-orbit (N.B. there are several possible definitions of pseudo-orbit for flows; e.g. see [24]). We say $x(t)$ is an (ϵ, T) -pseudo-orbit [29] of (1) on $[T_a, T_b]$ if there is a finite series $t_i \in [T_a, T_b]$ for $i = 1, \dots, M$, with $t_1 = T_a$ and $t_M = T_b$, such that for all $i = 1, \dots, M - 1$,

- $t_{i+1} > t_i + T$
- $x(t)$ is a trajectory of (1) on the time-interval $[t_i, t_{i+1})$
- $|x(t_{i+1}) - \lim_{t \rightarrow t_{i+1}^-} x(t)| < \epsilon$.

We say $x(t)$ defined for t in some (possibly infinite) interval is an (ϵ, T) -pseudo-orbit if it is an (ϵ, T) -pseudo-orbit on all finite sub-intervals. For convenience we refer to an $(\epsilon, 1)$ -pseudo-orbit as an ϵ -pseudo-orbit.

Theorem 2.4 Suppose that (X_-, λ_-) and (X_+, λ_+) are Λ -connected by a stable path $(X(s), \Lambda(s))$ for some $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$. Then for any $\epsilon > 0$ there is a $r_0 > 0$ such that for all $r < r_0$ there is an ϵ -pseudo-orbit $\tilde{x}(t)$ of (1) with

$$|\tilde{x}(t) - X(rt)| < \epsilon \tag{9}$$

for all $t \in \mathbb{R}$.

This Theorem (proved below) suggests that if we consider the system perturbed by noise of arbitrarily low amplitude, for any noise amplitude there will be an initial condition, a rate and a realization of additive noise such that the perturbed system has a trajectory that end-point tracks the given stable path. Under additional assumptions, we believe that stable paths will be ϵ -close tracked with positive probability. Before proving Theorem 2.4, we give a special case whose proof uses similar argument to that of Lemma 2.3.

Lemma 2.5 *Suppose that $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$ and $(X(\Lambda(s)), \Lambda(s))$ traverses a single stable branch that is bounded away from \mathcal{X}_{bif} in the interval $s \in [s_a, s_b]$ (we include the possibility $s_a = -\infty$ or $s_b = \infty$). Then for any $\epsilon > 0$ there is a $r_0 > 0$ such that for all $0 < r < r_0$ there is a trajectory $\tilde{x}(t)$ that satisfies ϵ -close tracking of this branch in the sense of (9) for $t \in [s_a/r, s_b/r]$.*

Proof: This is similar to that of Lemma 2.3 except that instead of $s \in \mathbb{R}$ we consider $s \in [s_a, s_b]$ and \tilde{x} is not necessarily a pullback attractor for the system, or even unique. Consider the case of s_a and s_b finite and note that if the closed set $\{(X(s), \Lambda(s)) : s \in [s_a, s_b]\}$ does not intersect the closed set \mathcal{X}_{bif} then they must be isolated by a neighbourhood. Hence there is a $\rho > 0$ such that $\max\{\text{Re}(\sigma(df(X(s), \Lambda(s))))\} < -\rho < 0$ for all $s \in S := [s_a, s_b]$. By considering an augmented system as in the proof of Lemma 2.3 we can apply Fenichel's theorem [13] to conclude that for small enough r there is a trajectory $\tilde{x}(t)$ that ϵ -tracks the branch for $rt = s \in S$. **QED**

Proof: (of Theorem 2.4) Let $(X(s), \Lambda(s))$ be the stable path that starts at (X_-, λ_-) and ends at (X_+, λ_+) . Because \mathcal{X} is regular, we can assume that there are at most finitely many bifurcation points on the path $(X(s), \Lambda(s))$. However, it is possible that $\Lambda(s)$ may remain at bifurcation points for a set of s that may include intervals: there is a finite set of points $\{s_1^\pm, \dots, s_k^\pm\}$ with

$$s_1^- \leq s_1^+ \cdots < s_k^- \leq s_k^+$$

such that $\Lambda(s)$ is constant and $(X(s), \Lambda(s)) \in \mathcal{X}_{\text{bif}}$ when $s \in [s_i^-, s_i^+]$ for some i . Note that the generic (transversal) case is $s_i^- = s_i^+$ but we allow a more general case where the parameter can “linger” at bifurcation points for a time.

We proceed to show that for any $\epsilon > 0$ there is an $r_0(\epsilon)$ such that it is possible to construct an ϵ -pseudo-orbit $\tilde{x}(t)$ that shadows the stable path $X(rt)$ for all $r < r_0$. All discontinuities of the constructed pseudo-orbit will be near $t \in [s_j^-/r, s_j^+/r]$. We show this in detail for the finite interval $s \in [s_{j-1}^+, s_j^-]$: similar proofs hold for the semi-infinite intervals $(-\infty, s_1^-]$ and $[s_k^+, \infty)$ by using uniformity of linear stability near the endpoints X_\pm . Let us define $t_j^\pm := s_j^\pm/r$ and $(X_j, \lambda_j) := (X(s_j^\pm), \Lambda(s_j^\pm))$.

Pick any $\epsilon > 0$. Continuity of $X(s)$ means we can find a δ such that

$$|X(s) - X_j| < \frac{\epsilon}{4}. \tag{10}$$

for all $s \in [s_j^- - \delta, s_j^-]$.

Pick any η with $0 < \eta < \delta$; Lemma 2.5 implies that there is a rate $r_0 > 0$ (depending on ϵ and η) and a trajectory $x(t)$ such that for all $r < r_0$ there is a trajectory $x_j(t)$ of (1) satisfying

$$|x(t) - X(rt)| < \frac{\epsilon}{4} \tag{11}$$

on the interval $rt \in [s_{j-1}^+ + \eta, s_j^- - \eta]$.

The fact that (X_j, λ_j) is a bifurcation point ($f(X_j, \lambda_j) = df(X_j, \lambda_j) = 0$) and $f \in C^2$ means that, by Taylor expansion, for any $\xi, \zeta > 0$ there are $M, N > 0$ such that

$$|f(x, \lambda)| < M|x - X_j|^2 + N|\lambda - \lambda_j|$$

whenever $|x - X_j| < \xi$ and $|\lambda - \lambda_j| < \zeta$. Similarly, $\Lambda(s) \in C^1$ means there is a $P > 0$ such that

$$|\Lambda(s) - \lambda_j| < Pd(s, [s_j^-, s_j^+])$$

for all $s \in [s_j^- - \eta, s_j^+ + \eta]$, where $d(s, A) = \inf\{|s - t| : t \in A\}$. Choosing $\zeta = P\eta$ we have

$$|f(x, \Lambda(rt))| < M\xi^2 + NP\eta \quad (12)$$

for all points in

$$I_j(\xi, \eta) := \{(x, s) : |x - X_j| < \xi \text{ and } s \in [s_j^- - \eta, s_j^+ + \eta]\}. \quad (13)$$

Considering a forward trajectory $x(t)$ of (1) we have

$$|x(t) - x(t')| < (M\xi^2 + NP\eta)|t - t'| \quad (14)$$

for $t > t'$, if $(x(t), rt) \in I_j(\xi, \eta)$ between t and t' . Hence by choosing ξ such that

$$\epsilon < \xi < \sqrt{\frac{\epsilon}{16M}} \text{ and } \eta < \frac{\epsilon}{16NP}$$

(which is possible as long as $\epsilon < \frac{1}{16M}$) we can ensure that:

- There is containment $I_j(\xi, \epsilon) \subset I_j(\xi, \eta)$.
- From (14), the inequality

$$|x(t) - x(u)| < \frac{\epsilon}{4} \text{ for all } t \in [u, u + 2] \quad (15)$$

holds for any trajectory $x(t)$ such that $|x(t) - X_j| < \epsilon$ for all $t \in [u, u + 2]$ where $(x(t), rt) \in I_j(\xi, \eta)$.

Because of (15), when $rt \in [s_j^- - \eta, s_j^+ + \eta]$ we can find a finite number of trajectory segments, each of time length $T \geq 1$, starting at $x(u) = X(ru)$, such that $|x(t) - x(u)| < \epsilon/4$ for all $t \in [u, u + 2]$.

Hence there is a choice of $\eta > 0$ and $r_j > 0$ such that, if $r < r_j$ then there is a pseudo-orbit $\tilde{x}(t)$ with the desired property on the interval $rt \in [s_j^- - \eta, s_{j+1}^- - \eta]$. More precisely, for any $rt \in S := [s_j^- - \eta, s_j^+ + \eta]$ we can find an increasing sequence of $s_{j,k}$ and a piecewise solution $\tilde{x}(t)$ with $s_{j,1} = s_j^- - \eta$, $s_{j,M} = s_j^+ + \eta$ such that $s_{j,k+1} > s_{j,k} + 1$. Combining the estimates (10,11) and (15) we have for this solution

$$\begin{aligned} |\tilde{x}(t) - X(rt)| &< |\tilde{x}(t) - \tilde{x}(u)| + |\tilde{x}(u) - X(ru)| + |X(ru) - X_j| + |X_j - X(rt)| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \epsilon. \end{aligned} \quad (16)$$

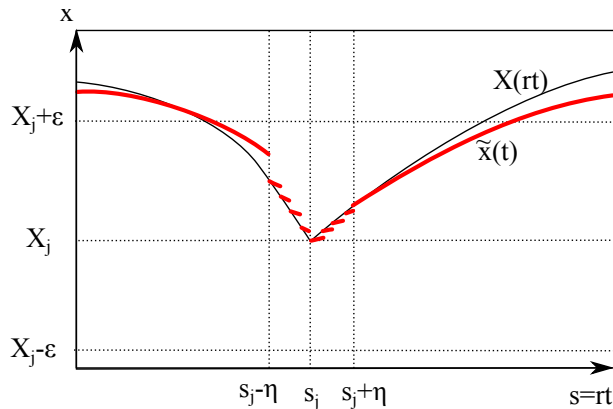


Figure 2: Construction of a pseudo-orbit $\tilde{x}(t)$ (red) that ϵ -close tracks a stable path $X(rt)$ (black) through a bifurcation point. We sketch the case where $s_j = s_j^- = s_j^+$ (i.e. the path does not linger at the bifurcation point). The short sections near the bifurcation point have duration between 1 and 2 time-units so that the width of the smaller intervals of continuity in s lie between $1/r$ and $2/r$. If $s_j^- < s_j^+$ then we choose a pseudo-orbit that remains at X_j over this interval.

for all $rt \in [s_j^- - \eta, s_{j+1}^- - \eta]$ except where $rt = s_{j,k}$ for some k .

On choosing $r < \min\{r_j : j = 1, \dots, k\}$ we have a piecewise defined path $\tilde{x}(t)$ which either satisfies (1) at t and (16) holds, or

$$\left| \lim_{u \rightarrow t^-} \tilde{x}(u) - \lim_{u \rightarrow t^+} \tilde{x}(u) \right| < \left| \lim_{u \rightarrow t^-} \tilde{x}(u) - X(rt) \right| < \frac{\epsilon}{4} < \epsilon.$$

By taking the upper limit at any discontinuity, we note that $\tilde{x}(t)$ is an ϵ -pseudo-orbit as required. **QED**

2.4 An example: dependence on Λ -connectedness

Let us consider the system

$$\frac{dx}{dt} = \sin(x\pi)(\Lambda(rt) + \cos(x\pi)) \quad (17)$$

for $x, r > 0$ and $\Lambda(s) \in \mathcal{P}(-2, 2)$ this is a simple example where we can explore which stable paths can be adiabatically tracked for a given $\Lambda(s)$. If we note that $\lambda_- = -2$ and $\lambda_+ = +2$ then the bifurcation diagram is as in Figure 3. Although there are branches that continue through for $x = k \in \mathbb{Z}$, there is an exchange of stability by a combination of pitchfork bifurcations. For this system we see that the set of stable branches is connected into one component and so for any $k, l \in \mathbb{Z}$ there is a stable path from $(2k, -2)$ to $(2l+1, 2)$. However, if $|k - l|$ is large then the path will require $\Lambda(s)$ with a large number of internal maxima and minima.

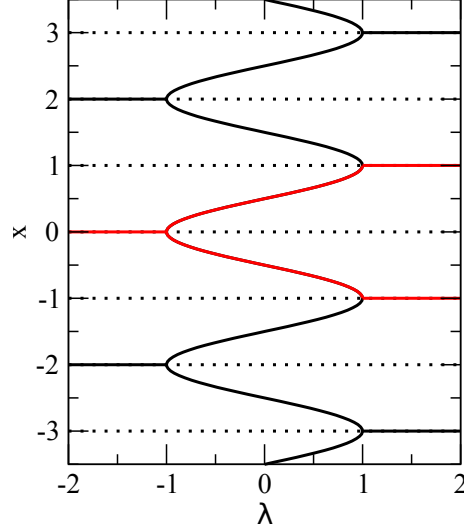


Figure 3: Bifurcation diagram for (17); the solid lines show the stable branches while the dashed lines are unstable. Observe that the only possible tracking by a pseudo-orbit that starts at $(X_-, \lambda_-) = (0, -2)$ for a monotonic $\Lambda(s)$ as in Figure 1(a) will be the red lines shown that finish at $(X_+, \lambda_+) = (\pm 1, 2)$. However, for $\Lambda(s)$ as in Figure 1(b) one can also track paths by pseudo-orbits that finish at $(X_+, \lambda_+) = (\pm 3, 2)$.

Now consider any trajectory $x(t)$ of (17). As the lines of equilibria at integer values of x remain flow-invariant subspaces for any choice of Λ and r , it follows that

$$\lfloor x(0) \rfloor \leq x(t) < \lfloor x(0) \rfloor + 1 \quad \text{for all } t,$$

where $\lfloor x \rfloor$ is the smallest integer $k \leq x$. This means that the pullback attractor from $X_- = 0$ cannot track a stable path from $(X_-, \lambda_-) = (0, -2)$ to (X_+, λ_+) where $X_+ = \pm 1, \pm 3, \dots$, and $\lambda_+ = 2$ in Figure 3. Indeed, it is easy to see that the pullback attractor in this case is simply the constant $\tilde{x}_{pb}^{[\Lambda, r, X_-]} = X_-$ and in all cases this is in fact a repeller for the future limit system.

However, Theorem 2.4 implies that a *pseudo-orbit* can track such a path - it allows the possibility of arbitrarily small adjustments along the way. Starting at $(X_-, \lambda_-) = (0, -2)$ and allowing arbitrarily small adjustments, there are two possible finishing points for a $\Lambda(s)$ such as Figure 1(a) and four possible finishing points for a $\Lambda(s)$ such as Figure 1(b).

2.5 Bifurcation-equivalent systems

Since the discussion so far only depends on branches of equilibria, their stabilities and bifurcations, many of the properties carry across to systems that have the same bifurcation diagram. Suppose that

$$\frac{dy}{d\tau} = \rho(y)f(y, \Lambda(r\tau)) \quad (18)$$

where $y \in \mathbb{R}^n$ and $\rho \in C^2(\mathbb{R}^n, \mathbb{R})$ is a strictly positive scalar function that is bounded below. The associated autonomous system

$$\frac{dy}{d\tau} = \rho(y)f(y, \lambda) \quad (19)$$

will have the same trajectories in the phase portrait, different magnitude but the same sign eigenvalues, and hence the same bifurcation diagram as (3). For the same reason systems (1) and (18) will have the same Λ -connected points X_- and X_+ . They also have orbits that are equivalent in the following (rather weak) sense:

Theorem 2.6 *Consider non-autonomous systems (1) and (18).*

- (a) *Suppose that for some $\Lambda(rt) \in \mathcal{P}(\lambda_-, \lambda_+)$ and $r > 0$ system (1) has a solution $x(t)$. Then there exists a (state-dependent) rescaling of time $T(\tau)$ with $T(0) = 0$, and a $\tilde{y}(\tau) = x(T(\tau))$ that solves the equivalent system (18) with $\tilde{\Lambda}(r\tau) = \Lambda(rT(\tau))$ replacing $\Lambda(r\tau)$.*
- (b) *Conversely, suppose that for some $\Lambda(r\tau) \in \mathcal{P}(\lambda_-, \lambda_+)$ and $r > 0$ system (18) has a solution $y(\tau)$. Then, this $y(\tau)$ defines a (state-dependent) time rescaling $T(\tau)$ with $T(0) = 0$ such that the equivalent system (1) has a solution $\tilde{x}(t) = y(T^{-1}(t))$ with $\tilde{\Lambda}(rt) = \Lambda(rT^{-1}(t))$ replacing $\Lambda(rt)$.*

Proof: To show (a), define a (state-dependent) time rescaling $T(\tau)$ through a coupled system

$$\begin{aligned} \frac{dT}{d\tau} &= \rho(y), \\ \frac{dy}{d\tau} &= \rho(y) f(y, \Lambda(rT)), \end{aligned}$$

with $T(0) = 0$ and $\Lambda(rT(\tau)) = \tilde{\Lambda}(r\tau)$. Using the assumption on $x(t)$ one can verify that $\tilde{y}(\tau) = x(T(\tau))$ together with a suitable monotonically increasing function $T(\tau)$ solve the coupled system above:

$$\begin{aligned} \frac{d\tilde{y}}{dt} &= \frac{dx}{dT} \frac{dT}{d\tau} = \rho(\tilde{y}(\tau)) f(x(T(\tau)), \Lambda(rT(\tau))) \\ &= \rho(\tilde{y}) f(\tilde{y}, \Lambda(rT)) = \rho(\tilde{y}) f(\tilde{y}, \tilde{\Lambda}(r\tau)). \end{aligned}$$

Hence, $\tilde{y}(\tau)$ also solves (18) with $\tilde{\Lambda}(r\tau) = \Lambda(rT(\tau))$ replacing $\Lambda(r\tau)$.

To show (b), for a given $y(\tau)$ define a state-dependent time rescaling through the integral

$$T(\tau) = \int_{u=0}^{\tau} g(y(u)) du,$$

that satisfies $T(0) = 0$ and is a monotonically increasing (hence invertible) function of τ whose first derivative $dT/d\tau = \rho(y(\tau))$ is bounded from below. We can now verify that $\tilde{x}(t = T(\tau)) = y(\tau)$ solves (1) with $\tilde{\Lambda}(rt) = \Lambda(rT^{-1}(t))$ replacing $\Lambda(rt)$. On the one hand

$$\frac{dy}{d\tau} = \frac{d\tilde{x}}{d\tau} = \frac{d\tilde{x}}{dT} \frac{dT}{d\tau} = \rho(y(\tau)) \frac{d\tilde{x}}{dT} = \rho(y(\tau)) \frac{d\tilde{x}}{dt}.$$

On the other hand, the assumption on $y(\tau)$ gives

$$\begin{aligned} \frac{dy}{d\tau} &= \rho(y(\tau)) f(y(\tau), \Lambda(r\tau)) = \rho(y(\tau)) f(\tilde{x}(t = T(\tau)), \Lambda(r\tau)) \\ &= \rho(y(\tau)) f(\tilde{x}(t), \Lambda(rT^{-1}(t))) \\ &= \rho(y(\tau)) f(\tilde{x}(t), \tilde{\Lambda}(rt)). \end{aligned}$$

Since $\rho(y) > 0$, we have that

$$\frac{d\tilde{x}}{dt} = f(\tilde{x}, \tilde{\Lambda}(rt)).$$

QED

In particular, this means that a tracking trajectory in an equivalent system implies that there is a tracking trajectory in the first system for a different Λ . Note that as the transformation from Λ to $\tilde{\Lambda}$ does depend on the trajectory chosen, one needs to be cautious about applying Theorem 2.6. We also note that varying r can lead to different parameter shifts $\tilde{\Lambda}$. Nonetheless, it does allow extension of some results to more general bifurcation equivalent systems.

3 Bifurcation- and rate-induced tipping

We now propose some definitions and results on bifurcation (B-) and rate (R-) induced tipping in the sense of [3]. Recall from [3] that B-tipping is associated with a bifurcation point of the corresponding autonomous system, while R-tipping appears only for fast enough variation of parameters and is associated with loss of tracking of a stable branch.

Definition 3.1 *Suppose that $(X_{\pm}, \lambda_{\pm}) \in \mathcal{X}_{\text{stab}}$ and fix $\Lambda(s) \in \mathcal{P}(\lambda_{-}, \lambda_{+})$. Suppose for all small enough $r > 0$ we have*

$$\lim_{t \rightarrow \infty} \tilde{x}_{pb}^{[\Lambda, r, X_{-}]}(t) = X_{+}.$$

If X_{+} and X_{-} are not Λ -connected then we say there is B-tipping from X_{-} for this Λ .

Definition 3.1 requires that the system reaches a point on a stable branch that is not accessible by following a stable path permitted by the chosen Λ . The following result shows that a bifurcation must indeed be responsible for B-tipping.

Corollary 3.1 *Suppose there is B-tipping for some $\Lambda(s)$ starting at X_- . Then there is at least one bifurcation point λ_b with $\lambda_- < \lambda_b < \lambda_+$ on the stable branch starting at X_- .*

Proof: If there is no such bifurcation point, then there must be a stable branch spanning from (X_-, λ_-) to (X_+, λ_+) . For any parameter shift $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$, Lemma 2.3 implies that we track this branch for all small enough $r > 0$ - a contradiction to the assumption that there is B-tipping. **QED**

We define irreversible R-tipping as follows:

Definition 3.2 *Suppose that $(X_-, \lambda_-) \in \mathcal{X}_{\text{stab}}$ and fix $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$. We say there is irreversible R-tipping from X_- on Λ if there is a $r_0 > 0$ and an X_+ that is Λ -connected to X_- such that the system end-point tracks a stable path from X_- to X_+ for $0 < r < r_0$ but not for $r = r_0$, i.e.*

$$\lim_{t \rightarrow \infty} \tilde{x}_{pb}^{[\Lambda, r, X_-]}(t) = \begin{cases} X_+ & \text{for } 0 < r < r_0 \\ Y_+ \neq X_+ & \text{for } r = r_0 \end{cases}$$

Although Theorem 2.4 guarantees ϵ -close tracking by pseudo-orbits for small enough r , an increase in r may lead to irreversible R-tipping as above, but then further increase of r may lead back to tracking again! Indeed the values of r that give tracking may be interspersed by several windows of rates that give tipping; see the example in Section 3.1 and Figure 4.

In addition to *irreversible R-tipping* defined above, there can be *transient R-tipping*. Recall that our definition of end-point tracking makes no assumption that $|\tilde{x}(t) - X(t)|$ is small, except in the limits $t \rightarrow \pm\infty$. This means that the system may end-point track the stable path for all $r > 0$, but depart from the path temporarily when $r > r_0$ to visit a different state/path at intermediate times [34]. Such transient R-tipping is not discussed here.

3.1 An example with rate-induced tipping for intermediate rates

Consider $x \in \mathbb{R}$ governed by

$$\frac{dx}{dt} = g(x, \Lambda(rt)) \tag{20}$$

where $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$ and

$$g(x, \lambda) = -((x + a + b\lambda)^2 + c \tanh(\lambda - d))(x - K / \cosh(e\lambda)) \tag{21}$$

for a, b, c, d, e and K constants. This example has a particular bifurcation structure as illustrated in Figure 4 if we choose

$$a = -0.25, \quad b = 1.2, \quad c = -0.4, \quad d = -0.3, \quad e = 3, \quad K = 2.$$

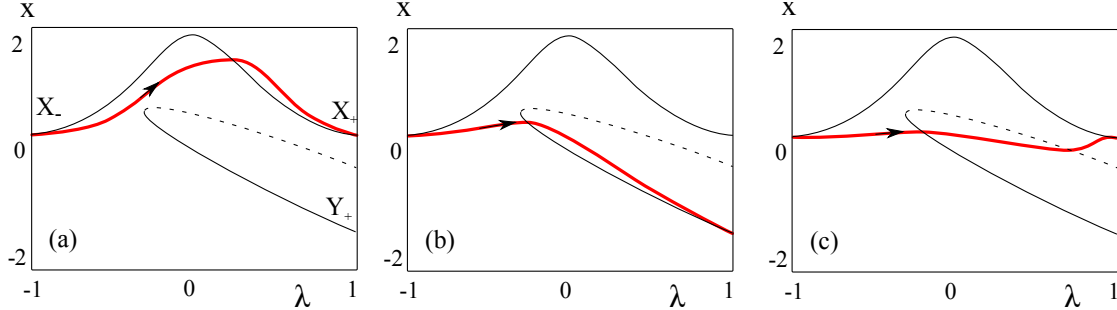


Figure 4: Bifurcation diagrams showing stable (black solid lines) and unstable (black dashed lines) equilibria for the system (20,21) with λ between $\lambda_- = -1$ and $\lambda_+ = +1$. The red lines show a numerical approximation of the pullback attractor starting at X_- for parameter shifts (22) on varying the rate r . (a) For $r < r_1$ the pullback attractor end-point tracks the branch through to X_+ . (b) There is a range of rates $r_1 < r < r_2$ where the pullback attractor R-tips to Y_+ . (c) For $r > r_2$ the pullback attractor again end-point tracks the branch through to X_+ .

Observe that this is a structurally stable bifurcation diagram and so qualitatively robust to small changes in any parameter, and indeed to any small changes in $g(x, \lambda)$. For $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$ and $r > 0$ we take solutions of

$$\frac{d\Lambda}{ds} = -(\Lambda - \lambda_-)(\Lambda - \lambda_+). \quad (22)$$

Figure 4 illustrates the appearance of critical rates $0 < r_1 < r_2$ such that R-tipping only occurs for a finite range $r \in (r_1, r_2)$.

3.2 Identifying regions of R-tipping in one-dimensional systems

In this section we assume one state space dimension ($n = 1$) throughout and show that geometric properties of the bifurcation diagram can be used to infer where R-tipping is or is not possible.

Consider a stable path $(X(s), \Lambda(s))$ for some $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$. For a fixed $v \in \mathbb{R}$, let $\mathbb{B}(X(v), \Lambda(v))$ be the basin of attraction of the stable equilibrium $X = X(v)$ for the autonomous system (3) with parameter $\lambda = \Lambda(v)$, i.e.

$$\mathbb{B}(X(v), \Lambda(v)) := \{x_0 : d(\phi_t(x_0, \lambda), X) \rightarrow 0 \text{ as } t \rightarrow \infty\},$$

where $\phi_t(x_0, \lambda)$ is the trajectory of the autonomous system (3) with initial condition x_0 . We define a notion of *forward basin stability* in terms of λ -dependent basin of attraction for the attractors of the associated autonomous system on a given stable path.

Definition 3.3 *We say a stable path $(X(s), \Lambda(s))$ for some $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$ is forward basin-stable if for every $v \in \mathbb{R}$ the basin of attraction of the stable equilibrium $X(v)$ on the*

path contains the closure of the set of all earlier positions of this equilibrium along the path:

$$\overline{\{X(u) : u < v\}} \subset \mathbb{B}(X(v), \Lambda(v)) \quad \text{for all } v \in \mathbb{R}.$$

Note that forward basin stability: (i) is defined in terms of solutions to the autonomous system (3) and the shape of the parameter shift $\Lambda(s)$, but is independent of the rate r , and (ii) is a property of a stable path rather than a stable branch traversed by the path. We remark that this is a somewhat different notion to basin stability as defined in [19]: this is a quantity that characterises the relative volume of a basin of attraction and is not a property of a branch of attractors.

In the following, we use the notion of forward basin stability to give sufficient conditions for irreversible R-tipping in one dimension to be absent (case 1) or present (cases 2, 3).

Theorem 3.2 *Suppose that (X_-, λ_-) and (X_+, λ_+) in $\mathcal{X}_{\text{stab}}$ are Λ -connected by the stable path $(X(s), \Lambda(s))$ for some $\Lambda \in \mathcal{P}(\lambda_-, \lambda_+)$:*

1. *If $(X(s), \Lambda(s))$ is forward basin-stable then $\tilde{x}_{pb}^{[\Lambda, r, X_-]}(t)$ end-point tracks $(X(s), \Lambda(s))$ for all $r > 0$. Hence in such a case there can be no irreversible R-tipping from X_- on Λ .*
2. *If the stable path $(X(s), \Lambda(s))$ traverses a single stable branch $X(\lambda)$, there is another stable branch³ $Y(\lambda)$ with $Y_+ := Y(\lambda_+) \neq X_+$ and there are $u < v$ such that*

$$X(u) \in \mathbb{B}(Y(\Lambda(v)), \Lambda(v)),$$

then $(X(s), \Lambda(s))$ is not forward basin-stable and there is a parameter shift $\tilde{\Lambda}(s) \in \mathcal{P}(\lambda_-, \lambda_+)$ such that there is irreversible R-tipping from X_- on this $\tilde{\Lambda}$.

3. *If the stable path $(X(s), \Lambda(s))$ traverses a single stable branch, and there is a $(Y_+, \lambda_+) \in \mathcal{X}_{\text{stab}}$ with X_- and Y_+ not Λ -connected such that*

$$X_- \in \mathbb{B}(Y_+, \lambda_+),$$

then $(X(s), \Lambda(s))$ is not forward basin-stable and there is irreversible R-tipping from X_- for this Λ .

Proof: We define

$$X_{\text{fut}}(u) := \overline{\bigcup_{v>u} X(v)} = \left[\inf_{v>u} X(v), \sup_{v>u} X(v) \right],$$

$$\mathbb{B}_{\text{fut}}(u) := \bigcap_{v>u} \mathbb{B}(X(v), \Lambda(v)).$$

and note that from these definitions, $X_{\text{fut}}(u) \supset X_{\text{fut}}(v)$ and $\mathbb{B}_{\text{fut}}(u) \subset \mathbb{B}_{\text{fut}}(v)$ for all $u < v$.

³NB this may be defined only on $[\lambda_0, \lambda_+]$ where $\lambda_- \leq \lambda_0 < \lambda_+$.

For case 1, let us fix any $r > 0$ and suppose that $(X(s), \Lambda(s))$ is forward basin-stable. This implies that $X(u) \in X_{\text{fut}}(u) \cap \mathbb{B}_{\text{fut}}(u)$ and also that $X_- \in \mathbb{B}_{\text{fut}}(u)$ for all $u \in \mathbb{R}$. For any t_0 and $x_0 \in \mathbb{B}_{\text{fut}}(rt_0)$ we define $x(t) := \Phi_{t,t_0}x_0$ and claim that

$$\Delta(t) := d(x(t), X_{\text{fut}}(rt_0))$$

satisfies $\Delta(t) \rightarrow 0$ as $t \rightarrow \infty$. To see this, note that if $\Delta(t) > 0$ for any $t > t_0$ then either

$$x(t) < \inf_{v > rt_0} X(v) \text{ or } x(t) > \sup_{v > rt_0} X(v).$$

In either case $\frac{d}{dt}\Delta(t) < 0$ and so $\Delta(t)$ is monotonic decreasing with t . If it converges to some $\Delta_0 > 0$ then we can infer there is another equilibrium (not equal to X_+) within $\mathbb{B}(X_+, \lambda_+)$ for the future limit system: this is a contradiction. Hence for any trajectory $x(t)$ such that $x(t) \rightarrow X_-$ as $t \rightarrow \infty$ we have

$$d(\Phi_{t,t_0}x, X_{\text{fut}}(rt_0)) \rightarrow 0.$$

Since this holds for all t_0 and $\bigcap_{u > 0} X_{\text{fut}}(u) = X_+$ we have $x(t) \rightarrow X_+$ as $t \rightarrow \infty$. By Theorem 2.2 this trajectory is the pullback attractor and hence the result.

For case 2 the existence of such u, v implies that $X(u) \notin \mathbb{B}(X(v), \Lambda(v))$ and hence $(X(s), \Lambda(s))$ is not forward basin stable. We construct a reparametrization

$$\tilde{\Lambda}(s) := \Lambda(\sigma(s))$$

using a smooth monotonic increasing $\sigma \in C^2(\mathbb{R}, \mathbb{R})$ that increases rapidly from $\sigma(s) = u$ to $\sigma(s) = v$ but increases slowly otherwise. More precisely, for any $M > 0$ and $\eta > 0$ we choose a smooth monotonic function $\sigma(s)$ such that

$$\begin{aligned} \sigma(s) &= s & \text{for } s < u, \\ 1 \leq \frac{d}{ds}\sigma(s) \leq M & \text{for } u \leq \sigma(s) \leq v \\ \frac{d}{ds}\sigma(s) &= M & \text{for } u + \eta < \sigma(s) < v - \eta, \quad \text{and} \\ \frac{d}{ds}\sigma(s) &= 1 & \text{for } \sigma(s) > v \end{aligned}$$

Using the assumption that the stable path traverses a single stable branch, we apply Lemma 2.3 to show that for all small enough rates $r > 0$ we have

$$\tilde{x}_{pb}^{[\Lambda, r, X_-]}(u/r) \in \mathbb{B}(Y(v), \Lambda(v)).$$

Note that by construction,

$$\tilde{x}_{pb}^{[\tilde{\Lambda}, r, X_-]}(t) = \tilde{x}_{pb}^{[\Lambda, r, X_-]}(t)$$

for any $t \leq u/r$. Indeed, by picking M large and η small, we can ensure that

$$|\tilde{x}_{pb}^{[\tilde{\Lambda}, r, X_-]}(v/r) - Y(\tilde{\Lambda}(v))|$$

is as small as desired. In particular this means that $\tilde{x}_{pb}^{[\tilde{\Lambda}, r, X_-]}(v/r)$ is in the basin of $Y(\tilde{\Lambda}(v))$. Hence $\tilde{x}_{pb}^{[\tilde{\Lambda}, r, X_-]}(t)$ will track $Y(\tilde{\Lambda}(t))$ for $t > v/r$ if r is small enough (recall that $s = rt$). Applying Lemma 2.5 again for the interval $s \in [v, \infty)$ means that reducing r , if necessary, will give a pullback attractor with $\tilde{x}_{pb}^{[\tilde{\Lambda}, r, X_-]}(t) \rightarrow Y_+$ as $t \rightarrow \infty$.

For case 3 a similar argument to case 2 implies that the path is not forward basin stable. Given any $\epsilon > 0$ there will be a $K > 0$ such that if

$$I := \overline{\{X(u') : u' < -K\}}$$

then

$$I \subset [X_- - \epsilon, X_- + \epsilon] \subset [X_- - 2\epsilon, X_- + 2\epsilon] \subset \bigcap_{v' > K} \mathbb{B}(Y(v'), \Lambda(v')).$$

By continuity, there is an M such that $|f(x, \lambda)| < M$ for all $x \in I$ and $\lambda \in [\lambda_-, \lambda_+]$. Now pick any

$$r > 2 \frac{MK}{\epsilon}$$

so that for the pullback attractor $x(t) := \tilde{x}_{pb}^{[\Lambda, r, X_-]}(t)$ we have $x(-K/r) \in I$ and

$$|x(K/r) - x(-K/r)| < 2 \frac{MK}{r} < \epsilon.$$

Therefore

$$x(K/r) \in \bigcap_{v' > K} \mathbb{B}(Y(v'), \Lambda(v'))$$

and so $x(t) \rightarrow Y_+$ as $t \rightarrow \infty$: there is R-tipping on this path for some $0 < r_0 < r$. **QED**

3.3 Isolation of B-tipping and R-tipping in one-dimensional systems

An interesting consequence of Theorem 3.2 is that rate-induced tipping will appear neither for too small a range of parameters, nor on a segment of a branch that lies too close to a saddle-node bifurcation.

Lemma 3.3 *If $(X_-, \lambda_-) \in \mathcal{X}_{\text{stab}}$ then there is a $\nu > 0$ such that for $\lambda_+ = \lambda_- + \nu$ there is a stable branch $X(\lambda)$ on (λ_-, λ_+) , and no irreversible R-tipping is possible from X_- for any $\Lambda \in \mathcal{P}(\lambda_-, \lambda_+)$.*

Proof: As X_- is a linearly stable equilibrium, for all small enough $\delta > 0$ there will be a $\nu > 0$ such that if $\lambda_+ = \lambda_- + \nu$ then

$$B_\delta(X_-) \subset \bigcup_{\lambda_- < \lambda < \lambda_+} \mathbb{B}(X(\lambda), \lambda).$$

Fixing such a δ , continuity of $X(\lambda)$ means that choosing a (possibly smaller) $\nu > 0$ we can ensure

$$\overline{\{X(\lambda) : \lambda_- < \lambda < \lambda_+\}} \subset B_\delta(X_-) \subset \bigcup_{\lambda_- < \lambda < \lambda_+} \mathbb{B}(X(\lambda), \lambda).$$

Hence any path on this branch from λ_- to λ_+ will be forward basin stable: by Theorem 3.2 case 1, R-tipping is not possible on such a path. **QED**

In fact, a stronger statement may be made near a saddle-node bifurcation. Suppose there is a saddle-node bifurcation at (λ_0, x_0) where the branches exist for $\lambda > \lambda_0$. The unstable and stable branches from the saddle-node bifurcation will be quadratically tangent to the line $\lambda = \lambda_0$ in the bifurcation diagram, meaning they will move monotonically in opposite senses near to the bifurcation. More precisely, suppose that there is a branch of stable equilibria $X^s(\lambda)$ and a branch of unstable equilibria $X^u(\lambda) < X^s(\lambda)$. Suppose that a $Y^u(\lambda)$ is the nearest branch of unstable equilibria above X^s . Then there is a $\lambda_1 > \lambda_0$ such that

$$\max\{X^u(\mu) : \lambda_0 < \mu < \lambda_1\} < X^s(\lambda) < \min\{Y^u(\mu) : \lambda_0 < \mu < \lambda_1\}$$

for all $\lambda_0 < \lambda < \lambda_1$. The branch $X^s(\lambda)$ is therefore forward basin-stable and by Theorem 3.2 case 1 there will be no R-tipping for any parameter shifts within this range of parameters: this is a stronger conclusion than Lemma 3.3.

As an example see Figure 5; this shows a bifurcation diagram along with three examples of choices of (λ_-, λ_+) . Two of these ranges (R1,R3) will not show R-tipping, while R2 can show R-tipping by Theorem 3.2 case 3.

4 Example: R-tipping in an energy balance climate model

We briefly indicate how we can characterise paths where there will be R-tipping for the Fraedrich energy-balance model of global mean temperature: see [3] for more details and an example of a path that gives R-tipping. We show how this can be deduced from application of Theorem 3.2. This model for the evolution of global mean surface temperature T is

$$\frac{dT}{dt} = f(T) = \tau^{-1}a(-T^4 + bT^2 - c), \quad (23)$$

where the parameters a, b, c, τ can be expressed in terms of physical quantities such as the solar output, the planetary albedo etc; note that $a > 0$ and $\tau > 0$ from physical arguments. We set $X = T^2$ and note that (23) transforms to

$$\frac{dX}{dt} = 2T \frac{dT}{dt} = \frac{2a}{\tau}(-X^2 + bX - c)\sqrt{X} \quad (24)$$

which (if we are bounded away from $X = T = 0$) is bifurcation-equivalent to

$$\frac{dX}{dt} = -X^2 + bX - c \quad (25)$$

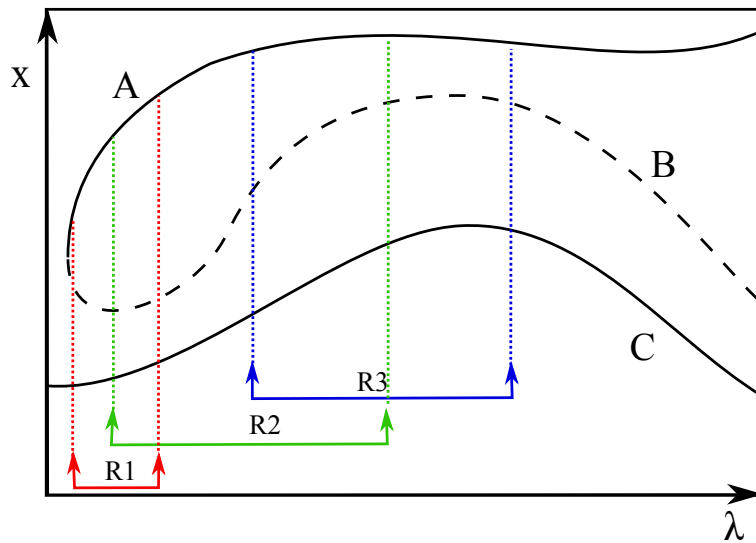


Figure 5: Bifurcation diagram for a one-dimensional system with stable branches A and C and unstable branch B. For ranges (λ_-, λ_+) with $\lambda_- < \lambda_+$ that are illustrated by R1 and R3, Theorem 3.2 case 1 applies: the paths are forward basin stable meaning there cannot be an R-tipping from branch A. For range R2, Theorem 3.2 case 3 implies there can be R-tipping from pullback attractors starting on the stable branch A to the stable branch C.

in the sense of Theorem 2.6: though we do not need to apply this theorem for the results below. Suppose now that $b(\lambda)$ and $c(\lambda)$ vary smoothly according to some parameter λ that undergoes a shift (without loss of generality) from 0 to 1. If $b(\lambda)^2 > 4c(\lambda)$ then there are two branches of equilibria for (25) – and hence also for (23); a stable branch

$$X^s(\lambda) = \frac{b(\lambda)}{2} + \frac{\sqrt{b(\lambda)^2 - 4c(\lambda)}}{2}$$

and an unstable branch

$$X^u(\lambda) = \frac{b(\lambda)}{2} - \frac{\sqrt{b(\lambda)^2 - 4c(\lambda)}}{2}.$$

These meet at saddle nodes whenever $b(\lambda)^2 = 4c(\lambda)$ and there are no equilibria for $b(\lambda)^2 < 4c(\lambda)$. The next statement summarizes a number of results that can be proved about this system for parameter shifts of λ from 0 to 1.

Theorem 4.1 *Consider the system (23) with $b(\lambda)$ and $d(\lambda)$ depending smoothly on a single real parameter $\lambda \in [0, 1]$. Suppose that λ varies with time by a parameter shift $\Lambda \in \mathcal{P}(0, 1)$.*

- (a) *If $b(\lambda)^2 > 4c(\lambda)$ for all $\lambda \in [0, 1]$ then there is a stable branch from $X_-^s = X^s(0)$ to $X_+^s = X^s(1)$ and there is no B-tipping from X_-^s for any $\Lambda \in \mathcal{P}(0, 1)$.*
- (b) *Suppose that (a) is satisfied and $X^s(\mu) > X^u(\nu)$ for all $\mu, \nu \in [0, 1]$. Then there is no irreversible R-tipping from X_-^s on any $\Lambda \in \mathcal{P}(0, 1)$.*
- (c) *Suppose that (a) is satisfied and there are $0 < \mu < \nu < 1$ such that $X^s(\mu) < X^u(\nu)$. Then there is a $\Lambda \in \mathcal{P}(0, 1)$ for which there is irreversible R-tipping from X_-^s .*
- (d) *Suppose that (a) is satisfied and let $X_-^u = X^u(0)$, $X_+^u = X^u(1)$ be the end-points of the unstable branch. If $X_-^s < X_+^u$ then there is a $\Lambda \in \mathcal{P}(0, 1)$ for which we have irreversible R-tipping from X_-^s .*
- (e) *If $b(0)^2 > 4c(0)$ and there is $0 < \mu < 1$ such that $b(\mu)^2 < 4c(\mu)$ then there is B-tipping for any $\Lambda \in \mathcal{P}(0, 1)$ starting at X_-^s .*

Proof: Conclusion (a) follows using Lemma 2.5. Conclusion (b,c,d) follow as applications of Theorem 3.2 cases 1, 2 and 3 respectively. Conclusion (e) follows on verifying Definition 3.1 of B-tipping. **QED**

Some consequences of this are that (i) if we fix b and vary just c , we cannot find an R-tipping in this system: this is because in that case

$$X^u(\lambda) < \frac{b}{2} < X^s(\lambda)$$

for all $\lambda \in [0, 1]$ so that we can apply Theorem 4.1(b), (ii) similarly we cannot get R-tipping if we fix $c > 0$ and vary just b . For (i) and (ii), note that it is still possible to get B-tipping in either case. (iii) Theorem 4.1(d) can be used to show that the parameter shift considered in [3] leads to R-tipping.

5 Discussion

In this paper we introduce a formalism that uses pullback attractors to describe the phenomena of B-tipping and irreversible R-tipping [3] in a class of non-autonomous systems. We restrict to a specific type of system, namely those undergoing what we call “parameter shifts” between asymptotically autonomous systems and focus on equilibrium attractors. In this setting we show in Theorem 2.2 that there is a unique pullback point attractor associated with each stable equilibrium for the past time asymptotic system, and we can give definitions of what it means to “ ϵ -close track”, to “end-point track” and to undergo some type of “tipping” depending on the details of the system. For one-dimensional systems, we obtain a number of results (in particular Theorem 3.2) that give necessary conditions for irreversible R-tipping to be present or absent in a system.

We consider smooth parameter shifts that remain within the end-points of their range and that go (possibly non-monotonically) from λ_- to λ_+ for some $\lambda_- < \lambda_+$. It should be possible to relax several of these assumptions and give similar results, as long as the parameters have well-defined limits for $t \rightarrow \pm\infty$, though this may lead to less intuitive statements.

We do not consider noise-induced tipping: for this, one needs to go beyond the purely topological/geometric setup here, and to discuss probabilities of tipping. We note that Theorem 2.4 gives criteria for tracking of stable branches by pseudo-orbits as well as useful conditions for the tracking of branches in the presence of low amplitude noise. Indeed, similar results can be made about the tracking of unstable branches by pseudo-orbits though presumably if these pseudo-orbits are the result of an occasional random process, there will not be tracking of unstable branches for long periods of time with any high probability. We also do not consider interaction of noise and rate-induced tipping - but see for example [28].

Our results rely on an assumption of simple dynamics in state space - in particular that the limiting behaviour of the asymptotically autonomous systems are just equilibria, and (for the later results) that the state space is one-dimensional and so well-ordered. Although many of the phenomena discussed will be present in higher dimensional systems (for example [30]), such systems will clearly display additional types of nonlinear behaviour. This includes more complex branches of attractors, periodic orbits, homoclinic and heteroclinic connections, chaotic dynamics and complex topological structures of attractor branches. Nonetheless, notions such as forward basin-stability should generalize to give similar results for tipping from equilibria in higher dimensions using parameter shifts of the type considered here.

As considered in [3], one particular type of tipping not present in one-dimensional systems may already appear in two dimensional state space where the system has a single globally attracting invariant set for all values of the parameters. Such a case of transient R-tipping, associated with a novel type of excitability threshold in [34, 21, 25], cannot be explained simply in terms of the branches of the bifurcation diagram even if we include all recurrent invariant sets. Instead it is due to the richer geometry of attraction that can appear in two or more dimensional systems.

We end with a brief discussion of an alternative setting that can be used in cases where the parameter shift behaviour is determined by an autonomous ODE, for example $d\lambda/dt = g(\lambda, r)$ for $\lambda \in \mathbb{R}$ where $g(\lambda_-, r) = g(\lambda_+, r) = 0$ and $g(\lambda, r) > 0$ for all $\lambda \in (\lambda_-, \lambda_+)$. In this

case we can consider an extended system of the form

$$\frac{d}{dt} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} f(x, \lambda) \\ g(\lambda, r) \end{pmatrix}.$$

As discussed in [26], the presence of irreversible R-tipping can be understood in terms of (heteroclinic) connections between saddle equilibria in this extended system and indeed this can be used as a tool to numerically locate rates corresponding to R-tipping. By contrast, in this paper we make no such assumption and consider the behaviour of the non-autonomous system (1).

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References

- [1] L. Arnold. Random Dynamical Systems. Springer Monographs in Mathematics, Springer Verlag, Berlin, 1998.
- [2] V. I. Arnold, V. S. Afrajmovich, Yu. S. Ilyashenko, and L. P. Shilnikov. *Bifurcation theory and catastrophe theory*. Springer-Verlag, Berlin, 1999.
- [3] P. Ashwin, S. Wieczorek, R. Vitolo, and P. Cox, Tipping points in open systems: bifurcation, noise-induced and rate-dependent examples in the climate system, *Phil Trans Roy Soc A* **370**:1166–1184 (2012). Correction co-authored with C. Perryman (Née Hobbs) **371**:20130098 (2013).
- [4] S. M. Baer, T. Erneux, and J. Rinzel. The slow passage through a Hopf bifurcation: Delay, memory effects and resonance. *SIAM J. Appl. Math.*, **49**:55–71, 1989.
- [5] Z. Bishnani and R.M. MacKay. Safety criteria for aperiodically forced systems. *Dynamical Systems* **18**:107–129, 2003.
- [6] W.A. Coppel. *Stability and asymptotic behaviour of differential equations*, Boston Heath. 1965.

- [7] W.A. Coppel. *Dichotomies in stability theory*, volume 629 of *Lecture Notes in Mathematics*. Springer, 1978.
- [8] E. Benoit. *Dynamic bifurcations*, volume 1498 of *Lecture Notes in Mathematics*. Springer, 1991.
- [9] V. Dakos, M. Scheffer, E. H. van Nes, V. Brovkin, V. Petoukhov, and H. Held. Slowing down as an early warning signal for abrupt climate change. *Proc. Natl Acad. Sci.*, **105**:14308-14312, 2008.
- [10] P. D. Ditlevsen and S. Johnsen. Tipping points: early warning and wishful thinking. *Geophys. Res. Letters*, **37**:L19703, 2010.
- [11] J. M. Drake and B. D. Griffen. Early warning signals of extinction in deteriorating environments. *Nature*, **467**:456–459, 2010.
- [12] J. Eldering. *Normally hyperbolic invariant manifolds - the noncompact case* Springer Atlantis Series in Dynamical Systems vol 2, 2013.
- [13] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J. Differential Equations*, **31**:53–98, 1979.
- [14] S. Hinrichsen and A.J. Pritchard. *Mathematical systems theory I: modelling, state space analysis, stability and robustness*, vol. 48 (corrected 3rd edn.). Berlin, Germany: Springer (2011).
- [15] A. Katok, and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems* Cambridge University Press, Cambridge (1995)
- [16] P. Kloeden and M. Rasmussen. *Nonautonomous Dynamical Systems*. AMS Mathematical Surveys and Monographs, **176**, 2011.
- [17] C. Kuehn. A mathematical framework for critical transitions: bifurcations, fast-slow systems and stochastic dynamics. *Physica D*, **240**:1020–1035, 2011.
- [18] T.M. Lenton, H. Held, E. Kriegler, J.W. Hall, W. Lucht, S. Rahmstorf, and H.J. Schellnhuber. Tipping elements in the earth’s climate system. *Proc. Natl Acad. Sci.*, **105**:1786-1793, 2008.
- [19] P.J. Menck, J. Heitzig, N. Marwan and J. Kurths. How basin stability complements the linear-stability paradigm. *Nature Physics*, **9**:8992, 2013.
- [20] K. Mischaikow, H. Smith and H.R. Thieme. Asymptotically autonomous semiflows: chain recurrence and lyapunov functions. *Trans. AMS*, **347**:1669-1685, 1995.
- [21] J. Mitry, M. McCarthy, N. Kopell, and M. Wechselberger. Excitable neurons, firing threshold manifolds and canards. *J. Math. Neurosci.* **3**:12 (doi:10.1186/2190-8567-3-12), 2013.

- [22] N.R. Nene and A. Zaikin. Interplay between Path and Speed in Decision Making by High-Dimensional Stochastic Gene Regulatory Networks. *PLoS one* **7**:e40085, 2012.
- [23] T. Nishikawa and E. Ott. Controlling systems that drift through a tipping point. *Chaos* **24**:033107, 2014.
- [24] K.J. Palmer. *Shadowing in dynamical systems. Theory and applications*. Kluwer, Dordrecht (2000)
- [25] C. Perryman and S. Wieczorek. Adapting to a changing environment: non-obvious thresholds in multi-scale systems. *Proceedings of the Royal Society A* **470**:20140226, 2014.
- [26] C. Perryman. *How Fast is Too Fast? Rate-induced Bifurcations in Multiple Time-scale Systems*. PhD thesis, University of Exeter (2015).
- [27] M. Rasmussen. Bifurcations of asymptotically autonomous differential equations. *Set-Valued Anal.* **16**:821849, 2008.
- [28] P. Ritchie and J. Sieber. Early-warning indicators for rate-induced tipping. Preprint *arXiv:1509.01696v3* (2015).
- [29] M. Shub. *Global stability of dynamical systems*. Springer-Verlag, New York, Berlin, Heidelberg (1987)
- [30] M. Scheffer, E.H. van Nes, M. Holmgren and T. Hughes. Pulse-driven loss of top-down control: the critical-rate hypothesis. *Ecosystems*, **11**: 226-237, 2008.
- [31] J. Shi and T. Li. Towards a critical transition theory under different temporal scales and noise strengths. *Physical Review E*, **93**: 032137, 2016.
- [32] A. Sutera. On stochastic perturbation and long-term climate behaviour. *Quarterly Journal of the Royal Meteorological Society*, **107**:137–151, 1981.
- [33] J.M.T. Thompson and J. Sieber. Predicting climate tipping as a noisy bifurcation: a review. *International Journal of Bifurcation and Chaos*, **76**:27–46, 2011.
- [34] S. Wieczorek, P. Ashwin, C. M. Luke, and P. M. Cox. Excitability in ramped systems: the compost-bomb instability. *Proceedings of the Royal Society A*, **467**:1243–1269, 2010.

Appendix A: Proof of Theorem 2.2

We thank an anonymous referee for suggesting the detailed proof that we reproduce below.

Proof: (of Theorem 2.2) Firstly, we fix $r > 0$ and claim there is a unique solution $x(t)$ of (1) with $x(t) \rightarrow X_-$ as $t \rightarrow -\infty$. To see this, let us define

$$\begin{aligned}\omega(\epsilon) &:= \sup \{ |df(x, \Lambda(rt)) - df(X_-, \Lambda(rt))| : t \in \mathbb{R}, |x - X_-| < \epsilon \} \\ \delta(T) &:= \max \left\{ \sup_{t < -T} |f(X_-, \Lambda(rt))|, \sup_{t < -T} |df(X_-, \Lambda(rt)) - df(X_-, \lambda_-)| \right\}\end{aligned}$$

and note, because $f \in C^2$ and $f(X_-, \lambda_-) = 0$, that $\omega(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ and $\delta(T) \rightarrow 0$ as $T \rightarrow \infty$. Linear stability means that the eigenvalues of

$$A := df(X_-, \lambda_-)$$

have negative real parts and so there are $K > 0$ and $\alpha > 0$ such that $|e^{tA}| \leq Ke^{-\alpha t}$ for $t \geq 0$ (in fact, one can choose any α such that $\max\{\operatorname{Re}(\sigma(A))\} < -\alpha < 0$): see for example [14, Lemma 3.3.19]).

Now set $x = X_- + y$ and note that

$$\dot{y} = Ay + h(y, t) \tag{26}$$

where

$$h(y, t) := f(X_- + y, \Lambda(rt)) - Ay$$

so that

$$\begin{aligned}dh(y, t) &= df(X_- + y, \Lambda(rt)) - A \\ &= [df(X_- + y, \Lambda(rt)) - df(X_-, \Lambda(rt))] + [df(X_-, \Lambda(rt)) - df(X_-, \lambda_-)] \\ &\quad + df(X_-, \lambda_-) - A \\ &= [df(X_- + y, \Lambda(rt)) - df(X_-, \Lambda(rt))] + [df(X_-, \Lambda(rt)) - df(X_-, \lambda_-)].\end{aligned}$$

Hence, for all $t < -T$ we have

$$|h(0, t)| \leq \delta(T), \quad |dh(y, t)| \leq \omega(|y|) + \delta(T)$$

and we seek a solution of (26) such that $|y(t)| \leq \Delta$ for $t \leq -T$ for any $T > T_0$. We choose $\Delta > 0$ and $T_0 > 0$ such that

$$4K\alpha^{-1}\omega(\Delta) \leq 1, \quad 2K\alpha^{-1}\delta(T_0) \leq \Delta, \quad \text{and} \quad 4K\alpha^{-1}\delta(T_0) \leq 1.$$

For any continuous $y(t)$ defined for $t < -T_0$ and $|y(t)| \leq \Delta$ we define \hat{y} by

$$\hat{y}(t) = \int_{u=-\infty}^t e^{(t-u)A} h(y(u), u) du$$

and note that the required solution of (26) satisfies $\hat{y} = y$. Note that

$$\begin{aligned} |\hat{y}(t)| &\leq \int_{u=-\infty}^t K e^{-\alpha(t-u)} [\delta(T) + (\omega(\Delta) + \delta(T)) |y(u)|] du \\ &\leq K \alpha^{-1} [\delta(T) + (\omega(\Delta) + \delta(T)) \Delta]. \end{aligned} \quad (27)$$

and hence $\|\hat{y}\| := \sup_{t \leq -T} |y(t)| \leq \Delta$.

If there are two such functions $y_{1,2}(t)$ one can verify that

$$\|\hat{y}_1 - \hat{y}_2\| \leq \frac{1}{2} \|y_1 - y_2\|$$

and so the integral operator is a contraction on the set of continuous $y(t)$ with $|y(t)| \leq \Delta$ for $t < -T$ for the norm $\|\cdot\|$. Hence there is a unique fixed point such that $|y(t)| \leq \Delta$ for $t \leq -T$ and this is a solution of (26). Note from (27) with $\hat{y} = y$ that for $t \leq -T$

$$|y(t)| \leq K \alpha^{-1} (\omega(\Delta) + \delta(T)) \|y\| + K \alpha^{-1} \delta(T) \leq \frac{1}{2} \|y\| + K \alpha^{-1} \delta(T).$$

Hence when $T \leq T_0$ we have $|y(t)| \leq 2K \alpha^{-1} \delta(T)$ for $t \leq -T$. Hence $y(t) \rightarrow 0$ as $t \rightarrow -\infty$. If we consider $U = B_\Delta(X_-)$, the Δ -ball around X_- , and for any $t < -T$ define

$$x_{pb}(t) := \bigcap_{s>0} \bigcup_{u>s} \Phi_{t,t-u}(U). \quad (28)$$

Then $x_{pb}(t)$ is the unique trajectory $X_- + y(t)$ for all $t < -T$ and we have $|x_{pb}(t) - X_-| < \Delta$ on $t < -T$ and $x_{pb} \rightarrow X_-$ as $t \rightarrow -\infty$. Note that (28) is well defined for all t and is invariant under (1) - hence it defines a unique trajectory for all t .

Moreover, there is a $\gamma > 0$ such that for any trajectory $z(t)$ with $z(t) \in U$ for $s < t < -T$ we have

$$|z(t) - x_{pb}(t)| \leq K e^{-\gamma(t-s)} |z(s) - x_{pb}(s)|$$

for $s \leq t \leq -T$.

To see this, note that $|x_{pb}(t) - X_-| \leq \Delta = 2K \alpha^{-1} \delta(T)$ for $t \leq -T$, and hence

$$|df(x_{pb}(t), \Lambda(rt)) - A| \leq \omega(\Delta) + \delta(T).$$

It follows from [7, Prop 1] that if $Y(t)$ is the fundamental matrix solution of

$$\dot{y} = A(t)y = df(x_{pb}(t), \Lambda(rt))y$$

then

$$|Y(t)Y^{-1}(s)| \leq K e^{-(\alpha - K(\omega(\Delta) + \delta(T)))(t-s)}$$

for $s \leq t \leq -T$. If we choose T large enough that

$$\alpha - K(\omega(\Delta) + \delta(T)) > 0$$

and put $x = x_{pb}(t) + y$ then (1) can be written

$$\dot{y} = A(t)y + h_1(y, t) \tag{29}$$

where

$$h_1(y, t) := f(x_{pb}(t) + y, \Lambda(rt)) - f(x_{pb}(t), \Lambda(rt)) - df(x_{pb}(t), \Lambda(rt))y.$$

and so if $|y| < \epsilon$ then

$$|h_1(y, t)| \leq \omega_1(\epsilon)|y|$$

where

$$\omega_1(\epsilon) := \sup\{|df(x + y, \Lambda(s)) - df(x, \Lambda(s))| : s \in \mathbb{R}, |x - X_-| \leq \Delta, |y| \leq \epsilon\}.$$

Uniform continuity of df in this region implies that $\omega_1 \rightarrow 0$ as $\epsilon \rightarrow 0$. If we fix ϵ small enough that

$$K\omega_1(\epsilon) < \alpha - K(\omega(\Delta) + \delta(T))$$

then it follows from the argument in [6, Thm 9] that if $y(t)$ is a solution of (29) with $|y(s)| < \epsilon/K$ for some $s < -T$ then

$$|y(t)| \leq K|y(s)|e^{-\gamma(t-s)}$$

for all t with $s \leq t \leq -T$ where $\gamma = \alpha - K(\omega(\Delta) + \delta(T)) - K\omega_1(\epsilon)$. Hence $|z(t) - x_{pb}(t)| \rightarrow 0$ as $t - s \rightarrow \infty$.

Moreover, note that if $y(s)$ is a solution of (29) where $|y(s)| < \epsilon/K$ for all $s < -T$ we have

$$|y(t)| \leq K|y(s)|e^{-\gamma(t-s)}$$

for all s, t with $s \leq t \leq -T$. Fixing any t and taking the limit $s \rightarrow -\infty$ implies that $y(t) = 0$. Hence the only solution that satisfies $|x(t) - x_{pb}(t)| < \epsilon/K$ for all $t < -T$ is the pullback attractor x_{pb} . **QED**