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Author(s)	DI PROIETTO, Valentina
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Kernel of the monodromy operator for semistable curves

By

Valentina Di Proietto*

Abstract

This is an announcement of results, obtained in collaboration with B. Chiarellotto, R. Coleman and A. Iovita, whose proofs will be published elsewhere. We study the relation between the kernel of the monodromy operator acting on the first log crystalline cohomology group of the special fiber of a semistable curve and the first rigid cohomology group with non trivial coefficients.

§ 1. Notations and preliminaries

Let k be a perfect field of characteristic $p > 0$. We denote by W the ring of Witt vectors of k and by K its fraction field.

Let X be a projective curve over $\mathrm{Spec}(W)$ with semistable reduction. We denote by X_k the special fiber of X, which we suppose is a union of smooth irreducible components and by X_K its generic fiber. We suppose that X_k is connected and that it has at least two irreducible components. For simplicity, we suppose also that all the intersection points of irreducible components of X_k are k-rational.

In this situation, we consider the notion of convergent F-isocrystals on $X_k/\mathrm{Spf}(W)$, as defined by Berthelot in [1] and Ogus in [9]. Since X is projective over $\mathrm{Spec}(W)$, we can embed X_k into a p-adic formal scheme P which is formally smooth over $\mathrm{Spf}(W)$. Hence, as described in [1], we can construct the tube of X_k in P, denoted by $]X_k[_P$. Every convergent F-isocrystal E on $X_k/\mathrm{Spf(W)}$ naturally induces a module with integrable connection $(\mathcal{E}, \nabla)$ on $]X_k[_P$ and the first rigid cohomology group is defined as $H^1_{\mathrm{rig}}(X_k/K, E) := H^1_{\mathrm{dR}}(]X_k[_P, (\mathcal{E}, \nabla))$.

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Supported by JSPS Post-doctoral fellowship program
*Graduate School of Mathematical Sciences, University of Tokyo, Komaba 153-8914 Tokyo, Japan.
e-mail: `proietto@ms.u-tokyo.ac.jp`

One can also consider the rigid analytic space X_K^{rig} associated to the p-adic completion of X. Since we have an immersion $X_K^{\mathrm{rig}} \hookrightarrow]X_k[_P$, every convergent F-isocrystal induces a module with integrable connection on X_K^{rig} by pullback, which we also denote by $(\mathcal{E}, \nabla)$. Then we can consider its first de Rham cohomology $H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla))$.

Note that we can equip the curve X with a log structure associated to the special fiber X_k which is a divisor with simple normal crossing and we can equip $\mathrm{Spec}(W)$ with a log stucture associated to the closed point. By pulling them back, we can also equip X_k and $\mathrm{Spec}(k)$ with log structures, and we denote the resulting log schemes by $X_k^\times$ and $\mathrm{Spec}(k)^\times$, respectively. The log structure on $\mathrm{Spec}(W)$ induces a log structure on $\mathrm{Spf}(W)$; again, we denote the resulting formal log scheme by $\mathrm{Spf}(W)^\times$. We can consider the first log crystalline cohomology with coefficients in a convergent F-isocrystal E, which we denote by $H^1_{\text{log-crys}}(X_k^\times/\mathrm{Spf}(W)^\times, E)$; thanks to [7] Section 2 (see also [6] Section 2.1), we have an isomorphism $H^1_{\text{log-crys}}(X_k^\times/\mathrm{Spf}(W)^\times, E) \cong H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla))$ and we have the monodromy operator acting on this cohomology group, which we denote by $N_{\mathcal{E}}$ in this paper.

In [6] the authors describe the action of the monodromy operator $N_{\mathcal{E}}$ on $H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla))$ explicitly by using certain admissible covering of X_K^{rig}. In this paper, we recall their description and we announce some results on the relation between the kernel of the monodromy operator $N_{\mathcal{E}}$ and the rigid cohomology $H^1_{\mathrm{rig}}(X_k/K, E)$ which we prove by using their description: in the case of trivial coefficients, we give another proof of a theorem of B. Chiarellotto ([2]) saying that the kernel of the monodromy operator is isomorphic to the rigid cohomology $H^1_{\mathrm{rig}}(X_k/K)$. Moreover, we announce a theorem that, if E is a unipotent convergent F-isocrystal, the kernel of the monodromy operator $N_{\mathcal{E}}$ on $H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla))$ does not necessarily coincide with the rigid cohomology group $H^1_{\mathrm{rig}}(X_k/K, E)$.

§ 2. A Mayer-Vietoris exact sequence

First we recall the construction of the graph $Gr(X_k)$ attached to X_k (*e.g.* [5]): we associate a vertex v to every irreducible component C_v of X_k, and we associate an edge e connecting v and w to every intersection point C_e of the irreducible components C_v, C_w of X_k. We can regard $Gr(X_k)$ as an oriented graph: for any edge of the graph e we choose an orientation of it and given two vertices of the graph v and w, we denote the set of edges connecting v and w, with origin v and end w, by $[v, w]$. We denote by $\mathscr{V}$ the set of vertices and by $\mathscr{E}$ the set of oriented edges.

As described in [1] there is a specialization map $\mathrm{sp} : X_K^{\mathrm{rig}} \to X_k$. For every $v \in \mathscr{V}$ we define $X_v := \mathrm{sp}^{-1}(C_v)$ and for every $e \in \mathscr{E}$ we define $X_e := \mathrm{sp}^{-1}(C_e)$. Then the set X_e is an open annulus in X_K^{rig} and X_v is what Coleman calls a wide open subspace ([4] proposition 3.3), namely an open of X_K^{rig} isomorphic to X_K^{rig} minus a finite number of closed disks contained in residue classes.

The covering $\{X_v\}_{v\in\mathscr{V}}$ is an admissible covering of X_K^{rig} and wide opens are Stein spaces, since every wide open X_v can be written as a union of an affinoid (inverse image via the specialization map of an affine subscheme of the special fiber) and a finite number of open annuli. So we can use the covering $\{X_v\}_{v\in\mathscr{V}}$ to calculate the de Rham cohomology of X_K^{rig} via a Čech complex. Note also that every intersection of three distinct elements in $\{X_v\}_{v\in\mathscr{V}}$ is empty. Hence, for a convergent F-isocrystal E, we have the Mayer-Vietoris exact sequence:

(2.1)

$$\oplus_{v\in\mathscr{V}} H^0_{\mathrm{dR}}(X_v,(\mathcal{E},\nabla)) \xrightarrow{\alpha} \oplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)) \longrightarrow H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla)) \longrightarrow \oplus_{v\in\mathscr{V}} H^1_{\mathrm{dR}}(X_v,(\mathcal{E},\nabla)) \xrightarrow{\beta} \oplus_{e\in\mathscr{E}} H^1_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)).$$

We remark that we can calculate the cohomologies in the above exact sequence except $H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla))$ as the cohomology of the complex of global sections, since wide opens are Stein spaces. We remark also that, since there is a comparison theorem between the de Rham cohomology of wide open subspaces and the de Rham cohomology of certain algebraic curves ([4] theorem 4.2), the cohomologies in the above exact sequences are finite dimensional. From the exact sequence (2.1), we can deduce the short exact sequence

$$0 \longrightarrow \mathrm{Coker}(\alpha) \xrightarrow{\gamma} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla))) \longrightarrow \mathrm{Ker}(\beta) \longrightarrow 0. \tag{2.2}$$

When E is the trivial isocrystal, then Coker(α) coincides with the first Betti cohomology of the graph $Gr(X_k)$.

§ 3. Monodromy operator

Let the notations be as in the previous section. We recall here the explicit definition of the monodromy operator $N_{\mathcal{E}}$ on the first de Rham cohomology $H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla))$ given by Coleman and Iovita in [6] section 2.2.

Since wide opens are Stein spaces, every cohomology class $[\omega]$ in $H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla))$ is represented by a hypercocycle $((\omega_v)_{v\in\mathscr{V}},(f_e)_{e\in\mathscr{E}})$, where $\omega_v\in\Gamma(X_v,\Omega^1_{X_K^{\mathrm{rig}}}\otimes\mathcal{E})$ and $f_e\in\Gamma(X_e,\mathcal{E})$, which satisfies $\omega_{v|X_e}-\omega_{w|X_e}=\nabla(f_e)$ for any v,w with $e\in[v,w]$.

The orientation on e induces an orientation on X_e. Using this, we define a residue map

$$\mathrm{Res}: H^1_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)) \longrightarrow H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla))$$

as follows. The module with connection $(\mathcal{E}, \nabla)$ has a basis of horizontal sections $e_1, \ldots, e_n$ on X_e, because X_e is a residue class (lemma 2.2 of [6]). Hence if z is an uniformizer of the oriented annulus X_e every differential form $\mu_e \in H^1_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla))$ can be written as $\mu_e = \sum_{i=1}^{n}(e_i \otimes \sum_j a_{i,j} z^j dz)$ with $a_{i,j} \in K$. Then we define the morphism Res by $\mathrm{Res}(\mu_e) := \sum_{i=1}^{n} a_{i,-1} e_i$.

One can prove that the morphism

$$\tilde{N}_{\mathcal{E}} : H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)) \longrightarrow \oplus_{e \in \mathscr{E}} H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla))$$

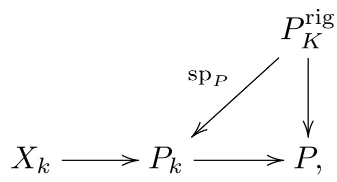
given by $[\omega] = ((\omega_v)_{v \in \mathscr{V}}, (f_e)_{e \in \mathscr{E}}) \mapsto (\mathrm{Res}(\omega_{v|X_e})_{e \in \mathscr{E}})$ is well-defined. Also, we have the morphism $i : \oplus_{e \in \mathscr{E}} H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)) \longrightarrow \mathrm{Coker}(\alpha) \longrightarrow H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla))$ induced by the Mayer-Vietoris sequence, which can be explicitly written as

$$i((f_e)_{e \in \mathscr{E}}) = \text{the class of } (0, (f_e)_{e \in \mathscr{E}}).$$

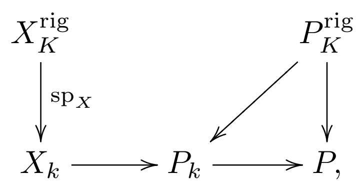
Then the monodromy operator $N_{\mathcal{E}}$ on $H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla))$ is defined as $N_{\mathcal{E}} = i \circ \tilde{N}_{\mathcal{E}}$.

§ 4. Monodromy and rigid cohomology

Keep the same hypothesis as before. Since X is projective, X_k can be embedded in a p-adic formal scheme P smooth over $\mathrm{Spf}(W)$. Let us denote by P_k its special fiber and by P_K^{rig} the rigid analytic space associated to P. We have the following diagram:



where the map between P_K^{rig} and P_k is the specialization. We put $]X_k[_P := \mathrm{sp}_P^{-1}(X_k)$ and define the first rigid cohomology group with coefficients in a convergent F-isocrystal E as $H^1_{\mathrm{rig}}(X_k/K, E) := H^1_{\mathrm{dR}}(]X_k[_P, (\mathcal{E}, \nabla))$. On the other hand, we can complete the above diagram as follows:



where the arrow $\mathrm{sp}_X : X_K^{\mathrm{rig}} \to X_k$ is again the specialization map, as in section 2. We have a morphism

$$\varphi : X_K^{\mathrm{rig}} \longrightarrow]X_k[_P$$

induced by the immersion of X into P. It induces a map

$$\varphi^* : H^1_{\mathrm{rig}}(X_k/K, E) = H^1_{\mathrm{dR}}(]X_k[_P, (\mathcal{E}, \nabla)) \longrightarrow H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)). \tag{4.1}$$

Using φ^* and $N_{\mathcal{E}}$ we can form the following sequence

$$H^1_{\mathrm{rig}}(X_k/K, E) \xrightarrow{\varphi^*} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)) \xrightarrow{N_{\mathcal{E}}} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)). \tag{4.2}$$

In [2] (proposition 4.10), Chiarellotto proved the following theorem, with the additional hypothesis that k is a finite field. (We remark also that higher dimensional versions of theorem 4.1 are proven by Nakkajima in [8] theorem 8.3 and theorem 9.1.)

Theorem 4.1. *If $(\mathcal{E}, \nabla)$ is the module with connection $(\mathcal{O}_{X_K^{\mathrm{rig}}}, d)$ induced by the trivial F-isocrystal $\mathcal{O}$, then, in the sequence (4.2), the map φ^* is injective and* $\mathrm{Im}(\varphi^*) = \mathrm{Ker}(N_{\mathcal{O}})$.

We study the sequence (4.2) when E is not necessarily trivial. We can prove the following proposition for any convergent F-isocrystal E.

Proposition 4.2. *If E is a convergent F-isocrystal and $(\mathcal{E}, \nabla)$ is the module with integrable connection induced by it, then the map φ^* in the sequence (4.2) is injective and*

$$N_{\mathcal{E}} \circ \varphi^* = 0.$$

To study the exactness of the sequence (4.2) we find a criterion to detect when an element in $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla))$ is in the image of φ^*.

Proposition 4.3. *Let us take $[\omega] \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla))$ and, as before, a representative $((\omega_v)_{v \in \mathscr{V}}, (f_e)_{e \in \mathscr{E}})$. Then $\mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$ for every $e \in \mathscr{E}$ if and only if $[\omega] \in \mathrm{Im}(\varphi^*)$.*

When $E = \mathcal{O}$ is the trivial F-isocrystal, we prove that if $[\omega] \in H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{O}_{X_K^{\mathrm{rig}}}, d))$ belongs to $\mathrm{Ker}(N_{\mathcal{O}})$, then $\mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$ for every $e \in \mathscr{E}$, *i.e.* the condition in proposition 4.3 is fulfilled. We prove this fact using graph theory. This implies that the sequence (4.2) is exact, giving a new proof of theorem 4.1.

§ 5. Main result: the case of unipotent coefficients

In this section, we announce our result that (4.2) fails to remain exact in general when E is a unipotent convergent F-isocrystal.

Consider the following non-split exact sequence of convergent F-isocrystals

$$0 \longrightarrow E \longrightarrow G \longrightarrow \mathcal{O} \longrightarrow 0, \tag{5.1}$$

and assume E is a unipotent convergent F-isocrystal for which the sequence (4.2) is exact. Let

$$0 \longrightarrow (\mathcal{E}, \nabla_{\mathcal{E}}) \xrightarrow{\iota} (\mathcal{G}, \nabla_{\mathcal{G}}) \longrightarrow (\mathcal{O}_{X_K^{\mathrm{rig}}}, d) \longrightarrow 0, \tag{5.2}$$

be the exact sequence of modules with integrable connections on X_K^{rig} induced by (5.1) and let $0 \neq x \in H^1_{\mathrm{rig}}(X_k/K, E)$ be the element corresponding to the exact sequence (5.1). The next result shows that, although the sequences (4.2) attached to E and $\mathcal{O}$ are exact, this may fail for that attached to G.

Theorem 5.1. *Let us suppose that there exists* $y \in H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla_{\mathcal{E}}))$ *such that* $\varphi^*(x) = N_{\mathcal{E}}(y)$. *Let*

$$\iota_* : H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla_{\mathcal{E}})) \longrightarrow H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{G}, \nabla_{\mathcal{G}}))$$

be the map in cohomology induced by the map ι *in the exact sequence (5.2). Then*

$$\mathrm{Ker}(N_{\mathcal{G}}) = \mathrm{Im}(\varphi^*) \oplus K \cdot \iota_*(y).$$

In fact, we have an explicit example of a semistable curve X and a unipotent convergent F-isocrystal E on $X_k/\mathrm{Spf}(W)$ for which the sequence (4.2) is not exact.

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