Crystals of Relative Displays and Grothendieck-Messing Deformation Theory

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I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

Signature: . . . .

Abstract

Displays can be thought of as relative versions of Fontaine's notion of *strongly divisible lattice* from integral p-adic Hodge theory. In favourable circumstances, the crystalline cohomology of a smooth projective R-scheme X is endowed with a display-structure coming from complexes associated with the relative de Rham-Witt complex $W\Omega^{\bullet}_{X/R}$ of [LZ04], and can be thought of as a kind of mixed characteristic Hodge structure.

In this article, we show that under certain geometric conditions, deforming X over PD-thickenings of R gives a crystal of relative displays. We then apply the crystal of relative displays to prove Grothendieck-Messing type results for the deformation theory of Calabi-Yau threefolds. We also show that primitive crystalline cohomology often carries a display-structure, and we prove a Grothendieck-Messing type result for the deformation theory of smooth cubic fourfolds in terms of the crystal of relative displays on primitive crystalline cohomology. Finally, we investigate the deformation theory of ordinary smooth cubic fourfolds in terms of the displays on the cohomology of their Fano schemes of lines.

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Contents

1 Introduction

Displays were introduced in [Zin02] in order to extend the Dieudonné module classification of p-divisible groups over perfect fields of characteristic $p > 0$ to more general base rings R in which p is nilpotent. Roughly, a display is a pair of finitely generated projective $W(R)$-modules (P_0, P_1) and a pair of Frobenius-linear maps $(F : P_0 \to P_0, F_1 : P_1 \to P_0)$ satisfying certain compatibility requirements. By work of Zink and of Lau, there is an equivalence between the category of displays over R and the category of p-divisible groups over R. As one might expect - in light of Serre-Tate theory - there are applications to the theory of PEL-type Shimura varieties and in particular to problems involving their various strata.

In [LZ04], the relative de Rham-Witt $W\Omega^\bullet_{X/R}$ of a scheme X over a $\mathbb{Z}_{(p)}$-algebra R was defined. In the case that X/R is smooth and proper, it was shown that the hypercohomology of $W\Omega^\bullet_{X/R}$ recovers the crystalline cohomology of the crystalline structure sheaf $\mathcal{O}_{X/W(R)}$

$$\mathbb{H}^n(X, W\Omega^\bullet_{X/R}) \cong H^n_{\text{cris}}(X/W(R))$$

This generalises the theory over perfect base fields due to Bloch and Illusie. Langer and Zink in loc. cit. were then able to give a de Rham-Witt construction of the (dual of the) display associated to the p-divisible group of an abelian scheme A/R. Indeed, (P_0, F_0) is the F-crystal given by $H^1_{\text{cris}}(A/R)$ together with the Frobenius endomorphism induced from the special fibre. The $W(R)$-module P_1 is the hypercohomology of the *"Nygaard (pro-)complex"*

$$\mathcal{N}^1 W\Omega^\bullet_{X/R} := W\mathcal{O}_X \xrightarrow{dV} W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R} \xrightarrow{d} W\Omega^3_{X/R} \xrightarrow{d} \cdots$$

where we view $W\mathcal{O}_X$ as a $W(R)$-module via restriction of scalars along $W(R) \xrightarrow{F} W(R)$. The *"divided Frobenius"* map $F_1 : P_1 \to P_0$ is induced on hypercohomology by the morphism of complexes

$$\begin{array}{ccccccccc}
W\mathcal{O}_X & \xrightarrow{dV} & W\Omega^1_{X/R} & \xrightarrow{d} & W\Omega^2_{X/R} & \xrightarrow{d} & W\Omega^3_{X/R} & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle pF} & & \downarrow{\scriptstyle p^2F} & & \\
W\mathcal{O}_X & \xrightarrow{d} & W\Omega^1_{X/R} & \xrightarrow{d} & W\Omega^2_{X/R} & \xrightarrow{d} & W\Omega^3_{X/R} & \xrightarrow{d} & \cdots
\end{array}$$

Using this new perspective, [LZ07] introduced the notion of n-displays; the previous theory of displays being the $n = 1$ incarnation. Very roughly, n-displays are tuples $(P_i, F_i)_{i=0,\dots,n}$ consisting of finitely generated projective $W(R)$-modules P_i and Frobenius-linear maps $F_i : P_i \to P_0$ satisfying compatibility relationships including a relative version of Fontaine's *"strong divisibility condition"* (see [Fon83], §1, for example). In good circumstances, the F-crystal $H^n_{\mathrm{cris}}(X/W(R))$ of a smooth projective scheme X/R will have the additional structure of an n-display, with the $W(R)$-modules P_i being the hypercohomology of the Nygaard complexes

$$\mathcal{N}^i W\Omega^\bullet_{X/R} := W\mathcal{O}_X \xrightarrow{d} W\Omega^1_{X/R} \xrightarrow{d} \cdots \xrightarrow{d} W\Omega^{i-1}_{X/R} \xrightarrow{dV} W\Omega^i_{X/R} \xrightarrow{d} \cdots$$

and the divided Frobenius maps $F_i : P_i \to P_0$ being induced by the morphisms

$$\begin{array}{ccccccccccccc}
W\mathcal{O}_X & \xrightarrow{d} & \cdots & \xrightarrow{d} & W\Omega^{i-1}_{X/R} & \xrightarrow{dV} & W\Omega^i_{X/R} & \xrightarrow{d} & W\Omega^{i+1}_{X/R} & \xrightarrow{d} & W\Omega^{i+2}_{X/R} & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle pF} & & \downarrow{\scriptstyle p^2F} & & \\
W\mathcal{O}_X & \xrightarrow{d} & \cdots & \xrightarrow{d} & W\Omega^{i-1}_{X/R} & \xrightarrow{d} & W\Omega^i_{X/R} & \xrightarrow{d} & W\Omega^{i+1}_{X/R} & \xrightarrow{d} & W\Omega^{i+2}_{X/R} & \xrightarrow{d} & \cdots
\end{array}$$

For example, it is known that abelian varieties, K3 surfaces and smooth relative complete intersections have display-structures on their crystalline cohomology.

In [LZ15] the deformation theory of 2-displays is developed, and then used to study the deformation theory of schemes of K3-type. More specifically, if k is a perfect field of characteristic $p > 0$ and R is an artinian local $W(k)$-algebra with residue field k, Langer and Zink prove that the deformations to Spec R of an F_0-étale K3-type scheme X_0/k which lifts to $X/W(k)$ correspond to the self-dual 2-displays over the small Witt vectors $\hat{W}(R)$ of the 2-display-structure on

$H^2_{\text{cris}}(X_0/W(k))$. Here the self-duality refers to the perfect pairing induced by the Beauville-Bogomolov form. In the case of an ordinary K3 surface X_0/k, the Beauville-Bogomolov form is just the cup-product, and the result says that there is an equivalence of categories

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{deformation classes of}\\ X_0/k \text{ over Spec } R\end{array}\right\} & \xrightarrow{\sim} & \left\{\begin{array}{c}\text{isomorphism classes of self-dual}\\ 2\text{-displays over } \hat{W}(R)\\ \text{deforming } H^2_{\text{cris}}(X_0/W(k))\end{array}\right\} \\ X/R & \longmapsto & (H^2_{\text{cris}}(X/\hat{W}(R)), \lambda_{cup}) \end{array}$$

One of the main tools in [LZ15] and in our work is the notion of relative display. Let $S \twoheadrightarrow R$ be a PD-thickening with kernel $\mathfrak{a}$, and let X'/S be a deformation of X/R. Then, following [GL16], these are displays $(P_i, F_i)_{i=0,\dots,n}$ with the P_i given by the hypercohomology of relative versions of the Nygaard complexes

$$\begin{aligned} W\mathcal{O}_{X'} \oplus \tilde{\mathfrak{a}}^i W\mathcal{O}_{X'} \xrightarrow{d\oplus d} W\Omega^1_{X'/S} \oplus \tilde{\mathfrak{a}}^{i-1} W\Omega^1_{X'/S} \xrightarrow{d\oplus d} \cdots \\ \cdots \xrightarrow{d\oplus d} W\Omega^{i-1}_{X'/S} \oplus \tilde{\mathfrak{a}} W\Omega^{i-1}_{X'/S} \xrightarrow{dV+d} W\Omega^i_{X'/S} \xrightarrow{d} \cdots \end{aligned}$$

where $\tilde{\mathfrak{a}} := \log^{-1}(\mathfrak{a})$ is the logarithmic Teichmüller ideal. The partial Frobenius maps are defined on the first summands as for the usual Nygaard complexes, and is the zero map on the other summands. Roughly, our main technical result is the following

Theorem. *Let k be a perfect field of characteristic $p > 0$, and $S \twoheadrightarrow R$ a PD-thickening of local artinian $W(k)$-algebras with residue field k. Let $f : X \to R$ be a smooth projective scheme satisfying "good" cohomological properties and suppose that its special fibre X_0/k admits a smooth versal deformation space. Let X' be a deformation of X/R to Spec S. Then the relative display on $H^n_{\text{cris}}(X'/W(S)) \cong H^n_{\text{cris}}(X/W(S))$ only depends upon X and S. In other words, the crystal $R^n f_{\text{cris}*}\mathcal{O}_{X/W(k)}$ on Cris$(R/W(k))$ can be refined to have the extra struture of a crystal of relative displays.*

The "good" cohomological properties alluded to in the theorem are made precise in §4.

In the usual way there is a connection associated to the crystal of relative displays, and we show that this connection satisfies a display-version of Griffiths transversality and is compatible with the Gauss-Manin connection.

The crystal of relative displays is a natural framing for crystalline deformation problems arising in mixed characteristic algebraic geometry. As a first application, we study Calabi-Yau threefolds X/R. We use an infinitesimal Torelli style argument to prove the following Grothendieck-Messing type result

Theorem. *Let $S \twoheadrightarrow R$ be a nilpotent PD-thickening of artinian local $W(k)$-algebras with residue field k, and let X/R be a Calabi-Yau threefold. Let X' be a lifting of X/R to Spec S. Then there is a bijection*

$$\left\{\begin{array}{c}\textit{deformation classes of}\\ X/R \textit{ over Spec } S\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\textit{lines } E \subseteq \mathrm{Fil}^2 H^3_{dR}(X'/S)\\ \textit{lifting } \mathrm{Fil}^3 H^3_{dR}(X/R)\end{array}\right\}$$

We then use the connection associated to the crystal of relative displays to reinterpret this theorem in terms of the display-structures on $H^3_{\mathrm{cris}}(X/W(S))$. Indeed, we show that there is a bijection between the deformations of X/R to Spec S and those display-structures on $H^3_{\mathrm{cris}}(X/W(S))$ which arise *"geometrically"*, i.e. are given by the Nygaard complex construction via some deformation Y/S. A priori this latter set still requires the input of the set of deformations of X/R over Spec S in order to understand it, but we show how to construct this set using only the knowledge of the relative display; namely its Hodge filtration and the connection associated to the crystal. More specifically, we use the connection to define *"CY-type"* liftings of the Hodge filtration and prove the following:

Theorem. *Let $S \twoheadrightarrow R$ be a nilpotent PD-thickening of artinian local $W(k)$-algebras with residue field k, and let X/R be a Calabi-Yau threefold which lifts to Spec S. Then there are bijections*

$$\left\{\begin{array}{c}\textit{deformation classes of}\\ X/R \textit{ over Spec } S\end{array}\right\} \overset{\text{1:1}}{\longleftrightarrow} \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{geometric displays on}\\ H^3_{\textit{cris}}(X/W(S))\end{array}\right\}$$

$$\overset{\text{1:1}}{\longleftrightarrow} \left\{\begin{array}{c}\textit{CY-type liftings of the}\\ \textit{Hodge filtration of the}\\ \textit{relative display on}\\ H^3_{\textit{cris}}(X/W(S))\end{array}\right\}$$

This can be viewed as a sort of infinitesimal Torelli theorem in terms of displays rather than Hodge structures.

One can re-write this as the following analogue of the Grothendieck-Messing theorem for deformations of p-divisible groups:

Theorem. *There is an equivalence of categories*

$$\underline{\mathcal{CY}3}_S \xrightarrow{\sim} \underline{\mathcal{DCY}3}_{S/R}$$

where $\underline{\mathcal{CY}3}_S$ is the category whose objects are Calabi-Yau threefolds $X' \to$ Spec S, and whose morphisms are isomorphisms, and $\underline{\mathcal{DCY}3}_{S/R}$ is the category whose objects are pairs $(X, E^\bullet)$ where $X \to$ Spec R is a Calabi-Yau threefold and $E^\bullet$ is a CY-type lifting of the Hodge filtration of the relative display on $H^3_{\textit{cris}}(X/W(S))$, and the morphisms are isomorphisms with the obvious compatibilities.

Next we turn our attention to smooth cubic fourfolds X/R. We first show that the deformations of X/R over a nilpotent PD-thickening $S \twoheadrightarrow R$ of artinian local $W(k)$-algebras with perfect residue field k of characteristic $p > 0$ correspond to the S-lines lifting $\mathrm{Fil}^3 P^4_{\mathrm{dR}}(X/R)$ which are isotropic with respect to cup-product. This was proved in [Lev01] for the special case $R = k$, $S = k[\epsilon]/(\epsilon^2)$ and used in [MP15] to establish that the Kuga-Satake period map is étale.

We then show that, in favourable circumstances, the primitive crystalline cohomology $P^n_{\mathrm{cris}}(X/W(R))$ of a smooth projective scheme X over a ring R in which p is nilpotent is also endowed with a display-structure. If R is an artinian local $W(k)$-algebra with perfect residue field k of characteristic $p > 0$, and if the special fibre

X_0/k of X/R has a smooth versal deformation, one also has a crystal of relative displays refining the primitive crystalline cohomology crystal on Cris$(X/W(k))$. Arguing in the same way as before, we prove

Theorem. *There is an equivalence of categories*

$$\underline{\mathcal{S}m\mathcal{C}ub4}_S \xrightarrow{\sim} \underline{\mathcal{D}\mathcal{S}m\mathcal{C}ub4}_{S/R}$$

where $\underline{\mathcal{S}m\mathcal{C}ub4}_S$ is the category whose objects are smooth cubic fourfolds $X' \to$ Spec S, and whose morphisms are isomorphisms, and $\underline{\mathcal{D}\mathcal{S}m\mathcal{C}ub4}_{S/R}$ is the category whose objects are pairs $(X, E^\bullet)$ where $X \to$ Spec R is a smooth cubic fourfold and $E^\bullet$ is a self-dual lifting of the Hodge filtration of the relative display on $P^4_{\textit{cris}}(X/W(S))$, and the morphisms are isomorphisms with the obvious compatibilities.

Now let k be an algebraically closed field of characteristic $p \geq 5$ and X_0/k a smooth cubic fourfold. Then the Fano scheme of lines $F(X_0)$ is a K3-type fourfold with a Beauville-Bogomolov form, and by [BD85] there is an isomorphism of self-dual F-isocrystals

$$H^4_{\text{cris}}(X_0/K) \xrightarrow{\sim} H^2_{\text{cris}}(F(X_0)/K)(1)$$

where $K :=$ Frac $W(k)$. When both X_0 and $F(X_0)$ are ordinary in the sense of Bloch-Illusie-Kato, we show that this extends to an isomorphism of self-dual F-crystals

$$H^4_{\text{cris}}(X_0/W(k)) \xrightarrow{\sim} H^2_{\text{cris}}(F(X_0)/W(k))(1)$$

We prove

Theorem. *Let X_0 be an ordinary smooth cubic fourfold over an algebraically closed field k of characteristic $p \geq 5$, and suppose that its Fano scheme of lines $F(X_0)$ is ordinary. Then for a local artinian $W(k)$-algebra R with residue field k, we have an injection*

$$\left\{\begin{array}{c}\textit{deformation classes of}\\ X_0 \textit{ over Spec } R\end{array}\right\} \hookrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of self-dual}\\ 2\textit{-displays over } \hat{W}(R)\\ \textit{deforming } H^2_{\textit{cris}}(F(X_0)/W(k))\end{array}\right\}$$

$$X/R \longmapsto (H^2_{\textit{cris}}(F(X)/\hat{W}(R)), \lambda_{cup})$$

The right-hand side is well-understood by [LZ15].

1.1 Outline of the paper

Section 2 contains the foundational theory of higher displays, as developed in [LZ07] and [LZ15]. This section is only for later reference and contains nothing original, although we have endeavoured to provide examples so as to demystify some of the constructions.

In Section 3 we first recall the display-structures of [LZ07] and [GL16] on crystalline cohomology arising from Nygaard complexes. In addition to the known cases of abelian varieties and K3 surfaces, we show that the crystalline cohomology of an Enriques surface is endowed with a display-structure (provided $p \geq 3$). We then construct displays for primitive crystalline cohomology.

Section 4 contains the proof that the functor taking a PD-thickening to the associated relative display is in fact a crystal on the crystalline site whenever X has a smooth versal deformation space. This is the main tool for the rest of the paper. We also recall the explicit description of [GL16] of a relative display in terms of a relative version of Nygaard complex. We prove that the connection associated to the crystal of relative displays is compatible with the Gauss-Manin connection on the de Rham cohomology of the lifting, and observe that the connection satisfies a form of Griffiths transversality on the underlying modules of the relative display.

We apply the crystal of relative displays in Section 5 to investigate the infinitesimal deformation theory of Calabi-Yau threefolds in mixed characteristic. We first discuss the works [ESB05] and [KvdG00] establishing a sufficient and "generic" criteria for the special fibre to have a smooth versal deformation space. We then use an infinitesimal Torelli argument to prove a Grothendieck-Messing-style result associating liftings over nilpotent PD-thickenings with certain admissible liftings of the Hodge filtration. This is then rephrased in terms of the deformation theory of the display-structures inside the crystal of relative displays. As an aside, we also prove that ordinary Calabi-Yau threefolds always have a (canonical) lift over the Witt vectors.

The first part of section 6 gives a similar description of the deformation theory of smooth cubic fourfolds, but this time we use the crystal of relative displays on primitive crystalline cohomology. The second part also discusses the deformation theory of smooth cubic fourfolds, albeit under a restrictive ordinarity hypothesis, but this time in terms of the display on the second crystalline cohomology of their Fano schemes of lines; these displays are well-understood by [LZ15].

2 Displays

Fix a prime number p. In this preliminary section we shall collect some aspects of the theory of displays, mainly drawing from [LZ07], [LZ15] and [Wid12].

2.1 Frames with Verjüngung

Definition. *A* frame $\mathscr{F}$ *is a quintuple* $(W, J, R, \sigma, \dot{\sigma})$*, where* R *and* W *are rings and* $J \subseteq W$ *is an ideal such that* $W/J = R$*. The map* $\sigma : W \to W$ *is an endomorphism and* $\dot{\sigma} : J \to W$ *is a* σ*-linear* W*-module homomorphism. We require the following conditions to be satisfied:-*

(i) $J + pW \subseteq \mathcal{R}ad(W)$

(ii) $\sigma(a) \equiv a^p \mod pW$ *for all* $a \in W$

(iii) $\dot{\sigma}(J)$ *generates* W *as a* W*-module.*

Remark. By definition, condition (iii) says that the linearisation $\dot{\sigma}^{\#} : J^{(\sigma)} \to W$ is surjective. Choose $b \in J^{(\sigma)}$ such that $\dot{\sigma}^{\#}(b) = 1$ and set $\theta = \sigma^{\#}(b)$. Then for any $a \in J$, we compute

$$
\begin{aligned}
\sigma(a) &= \dot{\sigma}^{\#}(b)\sigma(a) \\
&= \dot{\sigma}^{\#}(ab) \\
&= \sigma^{\#}(b)\dot{\sigma}(a) \\
&= \theta\dot{\sigma}(a)
\end{aligned}
$$

so we see that there is a unique $\theta \in W$ such that $\sigma(a) = \theta\dot{\sigma}(a)$ for every $a \in J$. We will always assume $\theta = p$.

Example. We will use two types of frame. The first is the relative Witt frame of a surjection $S \twoheadrightarrow R$ of p-adic rings, with kernel $\mathfrak{a}$ equipped with a PD-structure, and such that $\mathfrak{a}$ is nilpotent modulo pS. Indeed, let $W(S)$ denote the Witt vectors of S, and write $I_S = {}^{V}W(S)$ for the image of Verschiebung. The PD-structure on $\mathfrak{a}$

allows us to define the divided Witt polynomials $\mathbf{w_n}/\mathbf{p^n}$ on the ideal $W(\mathfrak{a}) \subset W(S)$. The divided Witt polynomials give an isomorphism of additive groups

$$\log : W(\mathfrak{a}) \xrightarrow{\sim} \prod_{\mathbb{N}} \mathfrak{a}$$

Then $\tilde{a} := \log^{-1}(\mathfrak{a}) = (\mathfrak{a}, 0, 0, \dots)$ is an ideal in $W(S)$. In this way, we can view $\mathfrak{a}$ as an ideal in $W(S)$.

Writing $\mathcal{J} = \ker(W(S) \to S \to R)$, we have a direct sum decomposition $\mathcal{J} = \tilde{a} \oplus I_S$, with $\sigma(\tilde{a}) = 0$ and $\tilde{a} \cdot I_S = 0$, where $\sigma : W(S) \to W(S)$ is the Witt vector Frobenius. We define $\dot{\sigma} : \mathcal{J} \to W(S)$ using this decomposition; $\dot{\sigma}(a + {}^V\xi) = \xi$. Since S is p-adically complete, $W(S)$ is both p-adically and I_S-adically complete *([Zin02], Prop. 3)*, and it follows that $\mathcal{J} + pW(S) \subseteq \mathcal{R}ad(W(S))$.

The frame $\mathscr{W}_{S/R} = (W(S), \mathcal{J}, R, \sigma, \dot{\sigma})$ is called the *relative Witt frame* of S/R. If $S = R$, we simply write $\mathscr{W}_R$.

Now suppose that S and R are artinian local rings with perfect residue fields, then we have the *small Witt rings* $\hat{W}(S) \subset W(S)$ and $\hat{W}(R) \subset W(R)$ introduced in [Zin01]. We define the *small relative Witt frame* $\hat{\mathscr{W}}_{S/R} = (\hat{W}(S), \hat{\mathcal{J}}, R, \sigma, \dot{\sigma})$ of S/R in the same way as before, but replacing the usual Witt vectors with the small Witt ring everywhere. As usual, if $S = R$ then we write $\hat{\mathscr{W}}_R$. One should note that if $R = k$ is a perfect field then $\hat{W}(k) = W(k)$.

The relative Witt frames $\mathscr{W}_{S/R}$ carry an extra structure which will be important later on; they are endowed with a *Verjüngung*. By definition, a Verjüngung for a frame $\mathscr{F} = (W, J, R, \sigma, \dot{\sigma})$ is the data of two W-module homomorphisms

$$\nu : J \otimes_W J \to J \quad \text{and} \quad \pi : J \to J$$

satisfying the following four conditions for all $\eta, \eta' \in J$:-

(i) $\pi(\nu(\eta, \eta')) = \eta\eta'$

(ii) $\dot{\sigma}(\nu(\eta, \eta')) = \dot{\sigma}(\eta)\dot{\sigma}(\eta')$

(iii) $\dot{\sigma}(\pi(\eta)) = \sigma(\eta)$

(iv) $\ker \dot{\sigma} \cap \ker \pi = 0$

In the case of a relative Witt frame $\mathscr{W}_{S/R}$, the Verjüngung (ν, π) is defined as

$$\nu(a_1 + {}^V\xi_1, a_2 + {}^V\xi_2) = a_1a_2 + {}^V(\xi_1\xi_2)$$
$$\pi(a + {}^V\xi) = a + p^V\xi$$

In general the iterations $\nu_k : J \otimes_W \cdots \otimes_W J \to J$ are well-defined, and we denote the image by J_k. By convention, we set $\nu_1 = \mathrm{id}_J$ and $\nu_2 = \nu$. For the relative Witt frames, we have a decomposition:-

$$J_k = \tilde{\mathfrak{a}}^k + I_S$$

This works identically for the small relative Witt frames as well.

2.2 $\mathscr{F}$-displays

Before defining what a display over a frame with Verjüngung $\mathscr{F}$ is, we shall first introduce the intermediate concept of a predisplay; the category of $\mathscr{F}$-displays will then be a certain well-behaved full subcategory of the category of $\mathscr{F}$-predisplays.

Notation. *In the above situation, given a σ-linear map of W-modules $f : M \to N$, we will denote by $\tilde{f} : J \otimes_W M \to N$ the σ-linear map given by $\tilde{f}(\eta \otimes m) = \dot{\sigma}(\eta) f(m)$.*

Definition. *Let $\mathscr{F} = (W, J, R, \sigma, \dot{\sigma})$ be a frame. An $\mathscr{F}$-*predisplay *$\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ is the following data: for each $i \geq 0$ we have* *(i) a W-module P_i*

(ii) two W-module homomorphisms

$$\iota_i : P_{i+1} \to P_i \quad \textit{and} \quad \alpha_i : J \otimes_W P_i \to P_{i+1}$$

(iii) a σ-linear map $F_i : P_i \to P_0$.

The data is required to satisfy the following properties:-

(P1) $\iota_i \circ \alpha_i = \alpha_{i-1} \circ (\mathrm{id}_J \otimes \iota_{i-1})$ for $i \geq 1$

(P2) $\iota_i \circ \alpha_i : J \otimes P_i \to P_i$ is the multiplication map for each $i \geq 0$

(P3) $F_{i+1} \circ \alpha_i = \tilde{F}_i$ for all $i \geq 0$.

An immediate consequence of the definition is that $F_i \circ \iota_i = pF_{i+1}$. Indeed, let $\eta \in J$ and $x \in P_{i+1}$. Then we compute

$$p\tilde{F}_i(\eta \otimes \iota_i(x)) = pF_{i+1} \circ \alpha_i(\eta \otimes \iota_i(x)) = F_{i+1}(\eta x) = p\sigma(\eta)F_{i+1}(x)$$

but

$$p\tilde{F}_i(\eta \otimes \iota_i(x)) = p\dot{\sigma}(\eta)F_i(\iota_i(x)) = \sigma(\eta)F_i(\iota_i(x))$$

The idea is that we should think of the F_i as divided Frobenius maps; indeed, we shall often refer to them as such.

Definition. *Let $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ and $\mathcal{P}' = (P_i', \iota_i', \alpha_i', F_i')$ be two $\mathscr{F}$-predisplays. A* morphism of predisplays *$\psi : \mathcal{P} \to \mathcal{P}'$ is a sequence of W-module homomorphisms $\psi_i : P_i \to P_i'$ such that the following squares commute for each $i \geq 0$:-*

$$\begin{tikzcd} P_{i+1} \arrow[r, "\iota_i"] \arrow[d, "\psi_{i+1}"'] & P_i \arrow[d, "\psi_i"] \\ P_{i+1}' \arrow[r, "\iota_i'"] & P_i' \end{tikzcd} \qquad \begin{tikzcd} J \otimes_W P_i \arrow[r, "\alpha_i"] \arrow[d, "\mathrm{id}_J \otimes \psi_i"'] & P_{i+1} \arrow[d, "\psi_{i+1}"] \\ J \otimes_W P_i' \arrow[r, "\alpha_i'"] & P_{i+1}' \end{tikzcd} \qquad \begin{tikzcd} P_i \arrow[r, "F_i"] \arrow[d, "\psi_i"'] & P_0 \arrow[d, "\psi_0"] \\ P_i' \arrow[r, "F_i'"] & P_0' \end{tikzcd}$$

In this way we have the category $\mathscr{F}$-$\underline{\mathcal{P}redisp}$ of $\mathscr{F}$-predisplays.

Let $\mathcal{P}$ and $\mathcal{P}'$ be $\mathscr{F}$-predisplays as in the definition above. Then we define the direct sum of $\mathcal{P}$ and $\mathcal{P}'$ as

$$\mathcal{P} \oplus \mathcal{P}' := (P_i \oplus P'_i, \iota_i \oplus \iota'_i, \alpha_i \oplus \alpha'_i, F_i \oplus F'_i)$$

Clearly this gives another $\mathscr{F}$-predisplay.

Given a morphism of predisplays $\psi : \mathcal{P} \to \mathcal{P}'$, we can define the *kernel* of ψ as

$$\mathcal{K}er\ \psi := (\ker \psi_i, \bar{\iota}_i, \bar{\alpha}_i, \bar{F}_i)$$

where $\bar{\iota}_i = \iota_i|_{\ker \psi_{i+1}}$, $\bar{\alpha}_i = \alpha_i|_{J \otimes_W \ker \psi_i}$ and $\bar{F}_i = F_i|_{\ker \psi_i}$. Similarly we can also define $\mathcal{C}oker\ \psi$. It is immediate that $\mathcal{K}er\ \psi$ and $\mathcal{C}oker\ \psi$ are both $\mathscr{F}$-predisplays.

***Corollary* 2.2.1.** *$\mathscr{F}$-$\underline{\mathcal{P}redisp}$ is an abelian category.*

When $\mathscr{F}$ is a frame with Verjüngung, we can build $\mathscr{F}$-predisplays using *standard data*:-

Definition. *Let $\mathscr{F} = (W, J, R, \sigma, \dot{\sigma}, \nu, \pi)$ be a frame with Verjüngung. A* standard datum *for $\mathscr{F}$ is a a set of finitely generated projective W-modules $L_0, \dots, L_d$, and σ-linear homomorphisms*

$$\Phi_i : L_i \to L_0 \oplus \cdots \oplus L_d$$

for each $i = 0, \dots, d$, such that the induced map $\Phi_0 \oplus \cdots \oplus \Phi_d$ is a σ-linear isomorphism.

Given a standard datum (L_i, Φ_i), we set

$$P_i := J_i L_0 \oplus \cdots \oplus J L_{i-1} \oplus \cdots \oplus L_d$$

Then the maps ι_i, α_i and F_i are defined by the following diagrams:-

$$\begin{array}{ccccccccccccccc}
P_{i+1} & = & J_{i+1}L_0 & \oplus & J_iL_1 & \oplus & \cdots & \oplus & JL_i & \oplus & L_{i+1} & \oplus & \cdots & \oplus & L_d \\
\downarrow{\scriptstyle \iota_i} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\
P_i & = & J_iL_0 & \oplus & J_{i-1}L_1 & \oplus & \cdots & \oplus & L_i & \oplus & L_{i+1} & \oplus & \cdots & \oplus & L_d
\end{array}$$

$$\begin{array}{ccccccccccccccc}
J \otimes P_i & = & J \otimes J_iL_0 & \oplus & J \otimes J_{i-1}L_1 & \oplus & \cdots & \oplus & J \otimes L_i & \oplus & J \otimes L_{i+1} & \oplus & \cdots & \oplus & J \otimes L_d \\
\downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} & & & & \downarrow{\scriptstyle \nu_1 = \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{mult}} & & & & \downarrow{\scriptstyle \mathrm{mult}} \\
P_{i+1} & = & J_{i+1}L_0 & \oplus & J_iL_1 & \oplus & \cdots & \oplus & JL_i & \oplus & L_{i+1} & \oplus & \cdots & \oplus & L_d
\end{array}$$

$$\begin{array}{ccccccccccccccccc}
P_i & = & J_iL_0 & \oplus & \cdots & \oplus & JL_{i-1} & \oplus & L_i & \oplus & L_{i+1} & \oplus & L_{i+2} & \oplus & \cdots & \oplus & L_d \\
\downarrow{\scriptstyle F_i} & & \downarrow{\scriptstyle \tilde{\Phi}_0} & & & & \downarrow{\scriptstyle \tilde{\Phi}_{i-1}} & & \downarrow{\scriptstyle \Phi_i} & & \downarrow{\scriptstyle p\Phi_{i+1}} & & \downarrow{\scriptstyle p^2\Phi_{i+2}} & & & & \downarrow{\scriptstyle p^{d-i}\Phi_d} \\
P_0 & = & L_0 & \oplus & \cdots & \oplus & L_{i-1} & \oplus & L_i & \oplus & L_{i+1} & \oplus & L_{i+2} & \oplus & \cdots & \oplus & L_d
\end{array}$$

***Lemma* 2.2.2.** *The tuple $(P_i, \iota_i, \alpha_i, F_i)$ constructed from the standard datum (L_i, Φ_i) in the above fashion is an $\mathscr{F}$-predisplay.*

Proof. We verify the properties (P1),(P2) and (P3) in turn. The strategy is to look at the individual direct summands:-

(P1) This is immediate from the definition of Verjüngung.

(P2) We are required to prove that $\iota_i \circ \alpha_i =$ mult. This is obvious for the j^{th}-summand with $j > i$, and also true for $j = i$ by the definition of Verjüngung. For $j < i$, we need to check that $\pi\nu =$ mult. So choose $x \in L_j$ and $\eta \in J$, $\eta' \in J_{i-j}$. Then the map $\iota_i \circ \alpha_i$ on the j^{th}-summand is

$$J \otimes_W J_{i-j}L_j \xrightarrow{\nu} J_{i-j+1}L_j \xrightarrow{\pi} J_{i-j}L_j$$

$$\eta \otimes \eta' \otimes x \;\mapsto\; \nu(\eta, \eta') \otimes x \;\mapsto\; \pi(\nu(\eta, \eta')) \otimes x$$

but $\pi(\nu(\eta, \eta')) = \eta\eta'$.

(P3) We are required to show $F_{i+1} \circ \alpha_i = \tilde{F}_i$ for $i \geq 0$. Consider the j^{th}-summand. Then we have three cases:-

The case $j = i$ is the trivial statement $\tilde{\Phi}_j = \tilde{\Phi}_j$. If $j > i$ then the map $\tilde{F}_i$ on the j^{th}-summand is $p^{j-i}\tilde{\Phi}_{j-i}$, F_{i+1} is $p^{j-i-1}\Phi_{j+1}$ and α_i is mult. Let $x \in L_j$ and $\eta \in J$. Then we see that

$$\begin{aligned} F_{i+1} \circ \alpha_i(\eta \otimes x) &= F_{i+1}(\eta x) \\ &= p^{j-i-1}\Phi_{j-i}(\eta x) \\ &= p^{j-i-1}\sigma(\eta)\Phi_{j-i}(x) \\ &= p^{j-i}\dot{\sigma}(\eta)\Phi_{j-i}(x) \\ &= p^{j-1}\tilde{\Phi}_{j-i}(\eta \otimes x) \\ &= \tilde{F}_i(\eta \otimes x) \end{aligned}$$

The final case is when $j < i$. In this case, the map $\tilde{F}_i$ on the j^{th}-summand is $\tilde{\tilde{\Phi}}_j$, F_{i+1} is $\tilde{\Phi}_j$ and α_i is ν. Let $x \in L_j$, $\eta \in J$ and $\eta' \in J_{i-j}$. Then we see that

$$\begin{aligned} F_{i+1} \circ \alpha_i(\eta \otimes \eta' \otimes x) &= \tilde{\Phi}_j(\nu(\eta, \eta') \otimes x) \\ &= \dot{\sigma}(\nu(\eta, \eta'))\Phi_j(x) \\ &= \dot{\sigma}(\eta)\dot{\sigma}(\eta')\Phi_j(x) \\ &= \dot{\sigma}(\eta)\tilde{\Phi}_j(\eta' \otimes x) \\ &= \tilde{\tilde{\Phi}}_j(\eta \otimes \eta' \otimes x) \\ &= \tilde{F}_i(\eta \otimes \eta' \otimes x) \end{aligned}$$

□

Definition. *The $\mathscr{F}$-predisplay constructed from the standard datum (L_i, Φ_i) $i = 0, \ldots, d$ as above is called the d-$\mathscr{F}$-*display *of the standard datum (L_i, Φ_i). A d-$\mathscr{F}$-*display *is an $\mathscr{F}$-predisplay which is isomorphic to the d-$\mathscr{F}$-display of a standard datum. Such an isomorphism is called a* normal decomposition*.*

A *morphism of displays* is simply a morphism of predisplays, and we denote the full subcategory of $\mathscr{F}$-displays by $\mathscr{F}\text{-}\underline{\mathcal{D}isp}$. One can define the usual linear algebra operations such as tensor product, duals, base change and so forth for $\mathscr{F}$-displays by appealing to the normal decomposition. These definitions turn out to be independent of the choice (up to canonical isomorphism). In this way, one may show that $\mathscr{F}\text{-}\underline{\mathcal{D}isp}$ is an exact tensor category. Note that it is not an abelian category, since "quotients" of displays will fail to be displays in general.

Example. The *unit $\mathscr{W}_R$-display $\mathscr{U}$* has underlying $W(R)$-modules $P_0 = W(R)$ and $P_i = I_R$ for $i \geq 1$. The maps are

$$\iota_0 : I_R \overset{\text{inc.}}{\hookrightarrow} W(R), \qquad \alpha_0 : I_R \otimes W(R) \xrightarrow{\text{mult.}} I_R, \qquad F_0 : W(R) \xrightarrow{F} W(R)$$

and

$$\iota_i : I_R \xrightarrow{\cdot p} I_R, \qquad \alpha_i : I_R \otimes I_R \xrightarrow{\nu} I_R, \qquad F_i : I_R \xrightarrow{V^{-1}} W(R)$$

for $i \geq 1$.

Example. Given any $\mathscr{F}$-display $\mathscr{D} = (P_i, \iota_i, \alpha_i, F_i)$, we define the *Tate twist* $\mathscr{D}(1) = (P_i', \iota_i', \alpha_i', F_i')$ as follows:

$$\begin{aligned}
&P_0' = P_0 \text{ and } P_i' = P_{i-1} \text{ for } i \geq 1 \text{ (so } P_1' = P_0' = P_0\text{)}\\
&\iota_0' = \mathrm{id}_{P_0} \text{ and } \iota_i' = \iota_{i-1} \text{ for } i \geq 1\\
&\alpha_0' = \text{mult. and } \alpha_i' = \alpha_{i-1} \text{ for } i \geq 1\\
&F_0' = pF_0 \text{ and } F_i' = F_{i-1} \text{ for } i \geq 1
\end{aligned}$$

Repeating this process n-times gives $\mathscr{D}(n)$.

Example. Let $\mathscr{F} = (W, J, R, \sigma, \dot{\sigma}, \nu, \pi)$ and $\mathscr{F}' = (W', J', R', \sigma', \dot{\sigma}', \nu', \pi')$ be two frames with Verjüngung, and suppose that $u : \mathscr{F}' \to \mathscr{F}$ is a frame homomorphism. Let $\mathscr{D} = (P_i, \iota_i, \alpha_i, F_i)$ be an $\mathscr{F}$-predisplay. Then the *pullback* $u^\bullet \mathscr{D} = (P_i', \iota_i', \alpha_i', F_i')$ *of $\mathscr{D}$ along u* is the $\mathscr{F}'$-predisplay where $P_i' = P_i$ but considered as

W'-modules, $\iota'_i = \iota_i$, $F'_i = F_i$ and α'_i is the composite

$$J' \otimes_{W'} P'_i \to J \otimes_W P'_i \xrightarrow{\alpha_i} P'_{i+1}$$

The functor $u^\bullet : \mathscr{F}\text{-}\underline{\mathcal{D}isp} \to \mathscr{F}'\text{-}\underline{\mathcal{D}isp}$ has a left adjoint

$$u_\bullet : \mathscr{F}'\text{-}\underline{\mathcal{D}isp} \to \mathscr{F}\text{-}\underline{\mathcal{D}isp}$$

(see [LZ15], Prop. 6). If $\mathscr{D}'$ is an $\mathscr{F}'$-display, then the $\mathscr{F}$-display $u_\bullet \mathscr{D}'$ is called the *base-change of $\mathscr{D}'$ along u*. If (L'_i, Φ'_i) is a standard datum for $\mathscr{D}'$, then $(W \otimes_{W'} L'_i, \sigma \otimes \Phi_i)$ is a standard datum for $u_\bullet \mathscr{D}'$.

Example. We give a less trivial example coming from geometry. This example is a special case of the display structure on the crystalline cohomology of a nice enough variety due to [LZ07] (see Section 3.2 for a discussion), but it seems instructive to provide this example here before introducing the de Rham-Witt machinery.

Let A be an abelian scheme over a reduced local ring R such that $pR = 0$, and let $A'/W(R)$ be a lifting of A. Then $H^1_{\text{cris}}(A/W(R))$ is a finitely generated projective $W(R)$-module of rank $2 \dim A$. Define

$$P_0 := H^1_{\text{cris}}(A/W(R)) = H^1_{\text{dR}}(A'/W(R))$$

and

$$P_1 := \ker \left[H^1_{\text{cris}}(A/W(R)) \to H^1_{\text{cris}}(A/R) = H^1_{\text{dR}}(A/R) \to H^1(A, \mathcal{O}_A) \right]$$

where the last arrow comes from the Hodge-de Rham spectral sequence for A.

We claim that (P_0, P_1) together with the semi-linear Frobenius $F : H^1_{\text{cris}}(A/W(R)) \to H^1_{\text{cris}}(A/W(R))$ induced by the absolute Frobenius on A forms a 1-$\mathscr{W}_R$-display. Indeed, the Hodge decomposition for A' gives $W(R)$-modules

$$P_0 \cong H^1(A', \mathcal{O}_{A'}) \oplus H^0(A', \Omega^1_{A'/W(R)}) =: L_0 \oplus L_1$$
$$P_1 \cong I_R H^1(A', \mathcal{O}_{A'}) \oplus H^0(A', \Omega^1_{A'/W(R)}) =: I_R L_0 \oplus L_1$$

We must set $\Phi_0 := F|_{L_0}$. We now consider Φ_1:-

Let $A^{(p)}$ denote the base-change of A along the absolute Frobenius. Then, by definition, $F : H^1_{\mathsf{dR}}(A/R) \to H^1_{\mathsf{dR}}(A/R)$ is the composition

$$H^1_{\mathsf{dR}}(A/R) \hookrightarrow H^1_{\mathsf{dR}}(A^{(p)}/R) \xrightarrow{\bar{F}} H^1_{\mathsf{dR}}(A/R)$$

where $\bar{F}$ is induced by the relative Frobenius $F_{A/R} : A \to A^{(p)}$. Consider the commutative square

$$\begin{array}{ccc} H^1_{dR}(A^{(p)}/R) & \xrightarrow{proj} & H^1(A^{(p)}, \mathcal{O}_{A^{(p)}}) \\ \downarrow{\scriptstyle \bar{F}} & & \downarrow{\scriptstyle \wr} \\ H^1_{dR}(A/R) & \longleftarrow & H^1(A, \mathcal{H}^0(\Omega^\bullet_{A/R})) \end{array}$$

where the right vertical arrow is the Cartier operator isomorphism. The bottom vertical arrow comes from the conjugate spectral sequence

$$E_2^{i,j} = H^i(A, \mathcal{H}^j(\Omega^\bullet_{A/R})) \Rightarrow H^{i+j}_{\mathsf{dR}}(A/R)$$

This degenerates at E_2 by using the degeneration of the Hodge-de Rham spectral sequence at E_1 and the Cartier operator isomorphism. In particular, the bottom vertical arrow of the square is an injection and we deduce that

$$\ker \bar{F} = \ker\left(H^1_{\mathsf{dR}}(A^{(p)}/R) \overset{proj}{\twoheadrightarrow} H^1(A^{(p)}, \mathcal{O}_{A^{(p)}})\right)$$
$$= H^0(A^{(p)}, \Omega^1_{A^{(p)}/R})$$

This gives that $F(H^0(A, \Omega^1_{A/R})) = pH^1_{dR}(A/R)$, and as a result

$$F(H^0(A', \Omega^1_{A'/W(R)})) = pH^1_{dR}(A'/W(R))$$

Let $x \in L_1 = H^0(A', \Omega^1_{A'/W(R)})$, and suppose that $F(x) = py \in H^1_{\mathrm{dR}}(A'/W(R))$. Then we define $\Phi_1(x) := y$.

We see that $(L_0, L_1, \Phi_0, \Phi_1)$ is a standard datum for $(H^1_{\mathrm{cris}}(A/W(R)), F)$. The display we obtain is what will later be called $\mathscr{D}_R(A)$.

We remark that, *a priori*, this display-structure depends upon the lifting of A that we chose. The de Rham-Witt complex construction outlined in Section 3.2 gives the independence immediately. We also remark that the display above is actually the dual of the display of the p-divisible group associated to A, as constructed in [Zin02]. This is the content of ([LZ04], Prop. 3.7).

2.3 The Hodge filtration of a display

Definition. *Let $\mathscr{D} = (P_i, \iota_i, \alpha_i, F_i)$ be a $\mathscr{F}$-display. The* Hodge filtration *of $\mathscr{D}$ is the decreasing filtration of P_0/JP_0 by the (direct summand) R-modules*

$$\mathit{Fil}^i\mathscr{D} := \mathit{im}\left(P_i \xrightarrow{\iota_i^{\mathit{iter}}} P_0 \to P_0/JP_0\right)$$

where $\iota_i^{\mathit{iter}} := \iota_{i-1} \circ \cdots \circ \iota_0$.

Example. If $\mathscr{D} = \mathscr{D}_R(A)$ is the 1-$\mathscr{W}_R$-display on $H^1_{\mathrm{cris}}(A/W(R))$ of an abelian scheme A/R as in the example above, we see that the Hodge filtration of $\mathscr{D}_R(A)$ is identified with the Hodge filtration of A:-

$$\mathrm{Fil}^\bullet\mathscr{D}_R(A) = \mathrm{Fil}^\bullet H^1_{\mathrm{dR}}(A/R).$$

In Section 3.2 we will see that this is true more generally.

Now restrict attention to the relative Witt frames $\mathscr{W}_{S/R}$, although we could also write down the same definitions for the small relative Witt frames $\hat{\mathscr{W}}_{S/R}$.

Definition. *Let $\mathscr{D} = (P_i, \iota_i, \alpha_i, F_i)$ be a $\mathscr{W}_{S/R}$-display. Then a filtration by direct summands $E^\bullet \subseteq P_0/I_S P_0$ lifting Fil$^\bullet \mathscr{D}$ is called* admissible *if*

$$E^i \subseteq \mathit{im}\left(P_i \xrightarrow{\iota_i^{\mathit{iter}}} P_0 \to P_0/I_S P_0\right)$$

for each $i \geq 0$.

***Proposition* 2.3.1.** *Let $u : \mathscr{W}_S \to \mathscr{W}_{S/R}$. Then there is an equivalence of categories*

$$\begin{array}{ccc} \left\{\mathcal{W}_S\text{-}\mathit{displays}\right\} & \xrightarrow{\sim} & \left\{\begin{array}{c}\mathcal{W}_{S/R}\text{-}\mathit{displays\ together\ with\ an} \\ \mathit{admissible\ lifting\ of\ the\ Hodge\ filtration}\end{array}\right\} \\ \mathscr{D} & \longmapsto & (u_\bullet \mathscr{D}, \mathit{Fil}^\bullet \mathscr{D}) \end{array}$$

Given a $\mathscr{W}_{S/R}$-display $\mathscr{D}$ and an admissible lifting $E^\bullet$ of Fil$^\bullet\mathscr{D}$, we shall write $\mathscr{D}_{E^\bullet}$ for the $\mathscr{W}_S$-display obtained from the above equivalence. By construction we have Fil$^\bullet \mathscr{D}_{E^\bullet} = E^\bullet$.

Proof. This is ([LZ15], Corollary 11). $\square$

3 Displays and cohomology

3.1 Crystalline cohomology over $W(R)$

For p a prime, let R be a noetherian $\mathbb{Z}_{(p)}$-algebra in which p is nilpotent. For each $n \in \mathbb{N}$ let $W_n(R)$ be the ring of truncated Witt vectors of length n, so in particular $R = W_1(R)$ and $W(R) \cong \varprojlim_n W_n(R)$.

For any $\mathbb{Z}_{(p)}$-algebra S, the ideal $pS \subset S$ has a PD-structure γ given on $x = py \in pS$ by

$$\gamma_m(x) := \frac{p^m}{m!} y^m \qquad \forall m \in \mathbb{N}$$

(this makes sense because $\frac{p^{m-1}}{m!} \in \mathbb{Z}_{(p)}$, and is independent of the choice of y). In particular, we have a PD-structure on $pW_n(R) \subset W_n(R)$ and this is compatible with the PD-structure γ on $I_R := {}^V W_{n-1}(R) \subset W_n(R)$ by

$$\gamma_m({}^V\xi) := \frac{p^{m-1}}{m!} {}^V(\xi^m) \qquad \forall m \in \mathbb{N},\ \xi \in W_{n-1}(R)$$

Let Spec $W_n(R)$ be the PD-scheme with respect to the PD-ideal I_R. For a quasi-compact R-scheme X we write Cris$(X/W_n(R))$ for the crystalline site of X relative to Spec $W_n(R)$. The objects are PD-thickenings $(U \hookrightarrow T, \delta)$ of Zariski open subsets $U \subset X$, that is $U \hookrightarrow T$ is a closed $W_n(R)$-immersion defined by an ideal sheaf $\mathcal{J}$ whose PD-structure δ is compatible with γ, and such that p is nilpotent on T. The morphisms $(U \hookrightarrow T, \delta) \to (U' \hookrightarrow T', \delta')$ are commutative diagrams

$$\begin{tikzcd} U' \arrow[r, hook] & T' \\ U \arrow[u] \arrow[r, hook] & T \arrow[u] \end{tikzcd}$$

where $U \to U'$ is a Zariski inclusion and $T \to T'$ is a PD-morphism. The coverings are the sets of morphisms $\{(U_i \hookrightarrow T_i, \delta_i) \to (U \hookrightarrow T, \delta)\}$ where the $T_i \to T$ are open immersions and $T = \cup T_i$.

Let $(X/W_n(R))_{\text{cris}}$ be the associated topos. Then for each $n \geq m$ we have a canonical morphism of topoi

$$\iota_{m,n} : (X/W_m(R))_{cris} \to (X/W_n(R))_{\text{cris}}$$

Given a sheaf $\mathcal{E}_n$ on Cris$(X/W_n(R))$, let $\mathbb{R}\Gamma_{\text{cris}}(X/W_n(R), \mathcal{E}_n)$ denote the object in the derived category of $W_n(R)$-modules computing the cohomology of $(X/W_n(R))_{\text{cris}}$ with coefficients in $\mathcal{E}_n$.

Now suppose that X/R is proper and smooth, and let $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ be a compatible system of locally free crystals on $\{\text{Cris}(X/W_n(R))\}_{n\in\mathbb{N}}$, in the sense that for all $n \geq m$ we have

$$\iota_{m,n}^* \mathcal{E}_n \simeq \mathcal{E}_m$$

***Proposition* 3.1.1.** *There is a perfect object $\mathbb{R}\Gamma_{\textit{cris}}(X/W(R), \mathcal{E})$ in the derived category of $W(R)$-modules such that*

$$\mathbb{R}\Gamma_{\textit{cris}}(X/W(R), \mathcal{E}) \otimes^{\mathbb{L}}_{W(R)} W_n(R) \simeq \mathbb{R}\Gamma_{\textit{cris}}(X/W_n(R), \mathcal{E}_n)$$

for each $n \in \mathbb{N}$.

Proof. By ([Stacks], §15.65), it suffices to prove that for all $n \in \mathbb{N}$ we have

$$\mathbb{R}\Gamma_{\text{cris}}(X/W_{n+1}(R), \mathcal{E}_{n+1}) \otimes^{\mathbb{L}}_{W_{n+1}(R)} W_n(R) \simeq \mathbb{R}\Gamma_{\text{cris}}(X/W_n(R), \mathcal{E}_n)$$

and that $\mathbb{R}\Gamma_{\text{cris}}(X/R, \mathcal{E}_1)$ is perfect. The first fact is the base-change theorem for crystalline cohomology ([BO78], 7.8), and the second fact follows from the comparison with de Rham cohomology:

$$\mathbb{R}\Gamma_{\text{cris}}(X/R, \mathcal{E}_1) \simeq \mathbb{R}\Gamma_{\text{Zar}}(X, \mathcal{E}_{1,X} \otimes_{\mathcal{O}_X} \Omega^{\bullet}_{X/R})$$

This is ([BO78],7.1) in the special case that X is smooth. The right hand side is perfect since we assumed R to be noetherian and the sheaves of differentials

$\Omega^n_{X/R}$ are locally free of finite type. Then

$$\mathbb{R}\Gamma_{\text{cris}}(X/W(R), \mathcal{E}) := \mathbb{R} \varprojlim_n \mathbb{R}\Gamma_{\text{cris}}(X/W_n(R), \mathcal{E}_n)$$

is the object we seek. $\square$

Definition. *We define the crystalline cohomology of X over $W(R)$ with coefficients in $\mathcal{E}$ to be*

$$H^m_{\textit{cris}}(X/W(R), \mathcal{E}) := \mathbb{R}^m\Gamma_{\textit{cris}}(X/W(R), \mathcal{E})$$

If $\mathcal{E} = \{\mathcal{O}_{X/W_n(R)}\}_{n\in\mathbb{N}} = \mathcal{O}_{X/W(R)}$, then the crystalline cohomology of X over $W(R)$ is

$$H^m_{\textit{cris}}(X/W(R)) := H^m_{\textit{cris}}(X/W(R), \mathcal{O}_{X/W(R)})$$

Let S be a $\mathbb{Z}_{(p)}$-scheme and X an S-scheme. In [LZ04], the relative de Rham-Witt complex $W\Omega^\bullet_{X/S} = \{W_n\Omega^\bullet_{X/S}\}_{n\in\mathbb{N}}$ is constructed as the initial object in the category of F-V-procomplexes. In the case that $S = \text{Spec } k$ is the spectrum of a perfect field of characteristic p the relative de Rham-Witt complex recovers the de Rham-Witt complex of Bloch-Illusie. In our situation, the hypercohomology of $W\Omega^\bullet_{X/R}$ computes the crystalline cohomology of proper smooth schemes X/R:-

***Theorem* 3.1.2.** *Let R be a noetherian $\mathbb{Z}_{(p)}$-algebra in which p is nilpotent, and let $X/$Spec R be a proper smooth scheme. Then there is a canonical isomorphism*

$$H^m_{\textit{cris}}(X/W(R)) \cong \mathbb{H}^m(X, W\Omega^\bullet_{X/R})$$

Proof. This is ([LZ04], Theorem 3.5). $\square$

3.2 Display-structure on crystalline cohomology

For smooth projective schemes whose cohomology behaves well with respect to base-change, [LZ07] showed that the Nygaard complexes endow crystalline cohomology, which *a priori* are F-crystals, with the richer structure of a display. We will briefly recall the construction.

Let $X/\mathrm{Spec}\, R$ be a smooth projective scheme. Let $\mathcal{N}^i W_n\Omega^\bullet_{X/R}$ denote the complex of $W_n(R)$-modules

$$W_{n-1}\mathcal{O}_X \xrightarrow{d} \cdots \xrightarrow{d} W_{n-1}\Omega^{i-1}_{X/R} \xrightarrow{dV} W_n\Omega^i_{X/R} \xrightarrow{d} W_n\Omega^i_{X/R} \xrightarrow{d} \cdots$$

where we consider $W_{n-1}\Omega^j_{X/R}$ as a $W_n(R)$-module via restriction of scalars along ${}^F : W_n(R) \to W_{n-1}(R)$. Then $\mathcal{N}^0 W_n\Omega^\bullet_{X/R} = W_n\Omega^\bullet_{X/R}$. Write $I_{R,n} := {}^V W_{n-1}(R)$. Then there are morphisms of complexes

$$\hat{\iota}_i : \mathcal{N}^{i+1} W_n\Omega^\bullet_{X/R} \to \mathcal{N}^i W_n\Omega^\bullet_{X/R}$$
$$\hat{F}_i : \mathcal{N}^i W_n\Omega^\bullet_{X/R} \to W_{n-1}\Omega^\bullet_{X/R}$$
$$\hat{\alpha}_i : I_{R,n} \otimes_{W_n(R)} \mathcal{N}^i W_n\Omega^\bullet_{X/R} \to \mathcal{N}^{i+1} W_n\Omega^\bullet_{X/R}$$

given by

$$\begin{tikzcd}
W_{n-1}\mathcal{O}_X \arrow[r, "d"] \arrow[d, "p"] & \cdots \arrow[r, "d"] & W_{n-1}\Omega^{i-1}_{X/R} \arrow[r, "d"] \arrow[d, "p"] & W_{n-1}\Omega^i_{X/R} \arrow[r, "dV"] \arrow[d, "V"] & W_n\Omega^{i+1}_{X/R} \arrow[r, "d"] \arrow[d, "\mathrm{id}"] & W_n\Omega^{i+2}_{X/R} \arrow[r, "d"] \arrow[d, "\mathrm{id}"] & \cdots \\
W_{n-1}\mathcal{O}_X \arrow[r, "d"] & \cdots \arrow[r, "d"] & W_{n-1}\Omega^{i-1}_{X/R} \arrow[r, "dV"] & W_n\Omega^i_{X/R} \arrow[r, "d"] & W_n\Omega^{i+1}_{X/R} \arrow[r, "d"] & W_n\Omega^{i+2}_{X/R} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
W_{n-1}\mathcal{O}_X \arrow[r, "d"] \arrow[d, "\mathrm{id}"] & \cdots \arrow[r, "d"] & W_{n-1}\Omega^{i-1}_{X/R} \arrow[r, "dV"] \arrow[d, "\mathrm{id}"] & W_n\Omega^i_{X/R} \arrow[r, "d"] \arrow[d, "F"] & W_n\Omega^{i+1}_{X/R} \arrow[r, "d"] \arrow[d, "pF"] & W_n\Omega^{i+2}_{X/R} \arrow[r, "d"] \arrow[d, "p^2F"] & \cdots \\
W_{n-1}\mathcal{O}_X \arrow[r, "d"] & \cdots \arrow[r, "d"] & W_{n-1}\Omega^{i-1}_{X/R} \arrow[r, "d"] & W_{n-1}\Omega^i_{X/R} \arrow[r, "d"] & W_{n-1}\Omega^{i+1}_{X/R} \arrow[r, "d"] & W_{n-1}\Omega^{i+2}_{X/R} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

$$\begin{array}{ccccccccccccc}
I_R\otimes W\mathcal{O}_X & \to & \cdots & \to & I_R\otimes W\Omega^{i-1}_{X/R} & \to & I_R\otimes W\Omega^{i}_{X/R} & \to & I_R\otimes W\Omega^{i+1}_{X/R} & \to & I_R\otimes W\Omega^{i+2}_{X/R} & \to & \cdots \\
\downarrow & & & & \downarrow & & \downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle \text{mult.}} & & \downarrow{\scriptstyle \text{mult.}} & & \\
W\mathcal{O}_X & \xrightarrow{d} & \cdots & \xrightarrow{d} & W\Omega^{i-1}_{X/R} & \xrightarrow{d} & W\Omega^{i}_{X/R} & \xrightarrow{dV} & W\Omega^{i+1}_{X/R} & \xrightarrow{d} & W\Omega^{i+2}_{X/R} & \xrightarrow{d} & \cdots
\end{array}$$

respectively. We omitted the subscript in the final diagram for space, and in that diagram the first unlabelled arrows are given by

$$I_{R,n}\otimes W_{n-1}\Omega^j_{X/R} \to W_{n-1}\Omega^j_{X/R}$$
$${}^V\xi\otimes\omega \mapsto \xi\omega$$

Definition. *The* i-th Nygaard complex *of* X/R *is*

$$\mathcal{N}^iW\Omega^\bullet_{X/R} := \varprojlim_n \mathcal{N}^iW_n\Omega^\bullet_{X/R}$$

Now suppose that R is a ring in which p is nilpotent. Then the hypercohomology of the relative de Rham-Witt complex computes crystalline cohomology:-

$$\mathbb{H}^n(X, W\Omega^\bullet_{X/R}) \cong H^n_{\text{cris}}(X/W(R))$$

Write ι_i, α_i, F_i for the maps on hypercohomology induced by the morphisms $\hat{\iota}_i, \hat{\alpha}_i, \hat{F}_i$. Then the data $(\mathbb{H}^n(X, \mathcal{N}^iW\Omega^\bullet_{X/R}), \iota_i, \alpha_i, F_i)$ is a predisplay structure on $P_0 = H^n_{cris}(X/W(R))$ and we seek a natural condition on X to ensure that this is a display structure.

Let A be a torsion-free p-adic PD-thickening of R and $\mathscr{A} = (A, J, R, \sigma, \dot{\sigma})$ a frame for R. Write $A_n := A/p^nA$. Suppose that X/R admits a compatible system of smooth and projective liftings $(X_n/A_n)_{n\in\mathbb{N}}$ and suppose moreover that for each $n \geq 0$, X_n satisfies the following condition:-

(LZ1) $H^q(X_n, \Omega^p_{X_n/A_n})$ is a locally free finitely generated A_n-module for all p, q.

(LZ2) The Hodge-de Rham spectral sequence

$$E_1^{p,q} = H^q(X_n, \Omega^p_{X_n/A_n}) \Rightarrow H^{p+q}_{\mathrm{dR}}(X_n/A_n)$$

degenerates at E_1.

***Theorem* 3.2.1.** *Let R be a noetherian ring and p be nilpotent in R. Then:-*

(i) Let X/R be a smooth projective scheme with a with a compatible system of liftings $(X_n/W_n(R))_{n\in\mathbb{N}}$ satisfying the analogue of (LZ1) and (LZ2). Then for $m < p$ the data $(P_i = \mathbb{H}^m(X, \mathcal{N}^i W\Omega^\bullet_{X/R}), \iota_i, \alpha_i, F_i)$ is a display-structure $\mathscr{D}_R(X)$ on $H^m_{\mathit{cris}}(X/W(R))$. If R is a reduced ring, then we have isomorphisms

$$P_i = \mathbb{H}^m(X, \mathcal{N}^i W\Omega^\bullet_{X/R}) \cong H^m_{\mathit{cris}}(X, \mathcal{J}^{[i]}_{X/W(R)})$$

where $\mathcal{J}^{[i]}_{X/W(R)}$ is the i-th divided power of the usual crystalline ideal sheaf $\mathcal{J}_{X/W(R)} := \ker(\mathcal{O}_{X/W(R)} \to \mathcal{O}_X)$.

(ii) Let X/R be a smooth projective scheme with a smooth formal lifting $\mathcal{X}/\mathit{Spf}\ A$. For each $n \in \mathbb{N}$ write $X_n := \mathcal{X} \times_{\mathit{Spf}\ A} \mathit{Spec}\ A_n$ and suppose that the system $(X_n/A_n)_{n\in\mathbb{N}}$ satisfies (LZ1) and (LZ2) as above. Then, for $m < p$, there is an $\mathscr{A}$-m-window structure (see [LZ07] §5 for the definition) on $H^m_{\mathit{cris}}(X/A)$, with underlying A-modules

$$P_i := H^m_{\mathit{cris}}(X, \mathcal{J}^{[i]}_{X/A})$$

and divided Frobenius maps $F_i := \frac{1}{p^i}F$, where F is the composite

$$H^m_{\mathit{cris}}(X/A, \mathcal{J}^{[i]}_{X/A}) \to H^m_{\mathit{cris}}(X/A) \xrightarrow{F} H^m_{\mathit{cris}}(X/A)$$

The base-change of the the $\mathscr{A}$-m-window on $H^m_{cris}(X/A)$ along the frame homo-

morphism $\mathscr{A} \to \mathscr{W}_R$ is the $\mathscr{W}_R$-m-display $\mathscr{D}_R(X)$ of part (i).

Proof. In this generality, this is ([GL16], 1.1), although it was shown first for reduced rings R in ([LZ07], §5). □

***Remark*.** Let X/R be as above and $n < p$, and consider the display $\mathscr{D}_R(X)$ on $H^n_{\text{cris}}(X/W(R))$ as in Theorem 3.2.1. Then the filtered comparison theorem ([LZ07], Theorem 4.6) shows that the Hodge filtration of $\mathscr{D}_R(X)$ is identified with the Hodge filtration of X:-

$$\text{Fil}^\bullet \mathscr{D}_R(X) = \text{Fil}^\bullet H^n_{\text{dR}}(X/R).$$

3.3 Example: Enriques surfaces

We give some examples of schemes which satisfy the conditions (LZ1) and (LZ2). Recall that abelian schemes, K3 surfaces and complete intersections satisfy (LZ1) and (LZ2), by ([BBM82], Prop. 2.5.2), [Del81] and ([Del73], Theorem 1.5) respectively. In [LZ15] it shown the schemes of K3-type are also endowed with display-structures.

Let $p \geq 3$ be a prime, and let R be a ring such that p is nilpotent in R. Recall that an Enriques surface $f : X \to \text{Spec } R$ is smooth and proper morphism of relative dimension 2 satisfying the following two conditions:-

(i) $K_X \neq \mathcal{O}_X$ but $K_X^2 = \mathcal{O}_X$

(ii) for each point $s \to \text{Spec } R$, the irregularity of the fibre X_s is zero.

Let

$$\pi : Y := \underline{\text{Spec}}_X(\mathcal{O}_X \oplus K_X) \to X$$

be the canonical étale double cover corresponding to the non-trivial 2-torsion line bundle K_X. Since $c_2(Y) = 2c_2(X) = 24$ and $K_Y = \pi^* K_X = \mathcal{O}_Y$, Y is a K3-surface; it is called the K3-cover of X. The presence of the K3-cover is a useful tool in the study of X. Indeed, we recall below how the presence of Y allows us to easily deduce the existence of a formal lifting of X over Spf $W(R)$:-

By deformation theory, the obstruction to lifting X formally over Spf $W(R)$ is an element of $H^2(X, \mathcal{T}_{X/R})$; we will show that this group vanishes. Since $H^3(X_s, \mathcal{T}_{X_s/k(s)}) = 0$ for all closed points $s \in$ Spec R, we have

$$R^2 f_* \mathcal{T}_{X/R} \otimes_R k(s) \xrightarrow{\sim} H^2(X_s, \mathcal{T}_{X_s/k(s)}) \quad \forall s \in \text{Spec } R$$

by ([Mum70], §5. Cor. 3), i.e. $H^2(X, \mathcal{T}_{X/R})$ commutes with base-change. We may therefore reduce to the case that $R = k$ is an algebraically closed field of characteristic p. But in this case $H^2(X, \mathcal{T}_{X/k}) = 0$ by ([Lan83], Theorem 1.1). Indeed, by Serre duality

$$\dim_k H^2(X, \mathcal{T}_{X/k}) = \dim_k H^0(X, \Omega^1_{X/k} \otimes K_X).$$

Suppose for a contradiction that this is non-zero. Then

$$\dim_k H^0(Y, \Omega^1_{Y/k} \otimes K_Y) \neq 0.$$

Now Y is a K3-surface, so $K_Y \cong \mathcal{O}_Y$, and therefore $h^{1,0}(Y) \neq 0$. But this is impossible since a K3-surface has no global 1-forms (the Rudakov-Shafarevich theorem, see [RS76]). We conclude that there exists a formal lifting $\mathcal{X}$ of X over Spf $W(R)$.

For each $n \in \mathbb{N}$ let $X_n := \mathcal{X} \times_{\text{Spf } W(R)} \text{Spec } W_n(R)$. Then we have a compatible system of liftings $(X_n/W_n(R))_{n \in \mathbb{N}}$ of $X = X_1/R$. From the Hodge diamond

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
0 & & 10 & & 0 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

of an Enriques surface over an algebraically closed field of characteristic $p \geq 3$, one sees that the Hodge cohomology groups commute with arbitrary base-change and are locally free ([Mum70], §5). Then the Hodge-de Rham spectral sequences

$$E_1^{i,j} = H^j(X_n, \Omega^j_{X_n/W_n(R)}) \Rightarrow H^{i+j}_{\mathsf{dR}}(X_n/W_n(R))$$

degenerate at E_1, since the all differentials $d_r^{i,j} : E_r^{i,j} \to E_r^{i+r,j-r+1}$ are forced to be zero for every $r \geq 1$. In particular, we conclude that X satisfies the conditons (LZ1) and (LZ2) and has m-$\mathscr{W}_R$-display structure $\mathscr{D}_R(X)$ on $H^m_{\mathsf{cris}}(X/W(R))$ as in Theorem 3.2.1, for all $m < p$.

Remarks.

- The obstruction to lifting line bundles from X to $\mathcal{X}$ is an element of $H^2(X, \mathcal{O}_X)$, and we saw that this group is zero. In particular, one may always lift an ample line bundle to $\mathcal{X}$, so the formal lifting $\mathcal{X} \to \mathsf{Spf}\, W(R)$ algebraises to a lifting $X' \to \mathsf{Spec}\, W(R)$ of X by Grothendieck's algebraisation theorem ([Gro61], Thm. 5.4.5.-).

- Note that we must assume $p \geq 3$ since in characteristic 2 the crystalline cohomology of an Enriques surface is not torsion-free in general.

3.4 Display-structure on primitive cohomology

Let p be a prime and consider a ring R in which p is nilpotent. We prove the analogue of Theorem 3.2.1 for primitive crystalline cohomology:-

***Theorem* 3.4.1.** *Let X be a smooth and projective R-scheme and let $\mathscr{L}$ be an ample line bundle on X. Suppose that $\mathscr{A} = (A \to R)$ is a frame for R and assume* *that X lifts to a smooth and projective p-adic formal scheme $\mathcal{X} \to$ Spf A satisfying the following two conditions:-*

(i) For each i, j, n, $H^j(X_n, \Omega^i_{X_n/A_n})$ is a locally free A_n-module of finite type.

(ii) For each n, the Hodge-de Rham spectral sequence

$$E_1^{i,j} = H^j(X_n, \Omega^i_{X_n/A_n}) \Rightarrow H^{i+j}_{\textit{dR}}(X_n/A_n)$$

degenerates at E_1.

(Here $X_n := \mathcal{X} \times_{\textit{Spf}\, A}$ Spec A_n). Let $F : H^m_{\textit{cris}}(X/A) \to H^m_{\textit{cris}}(X/A)$ denote the Frobenius endomorphism induced by the absolute Frobenius on $X \times_{\textit{Spec}\, R}$ Spec R/pR. Then for each $m < p$, the data

$$\begin{aligned} Q_i &:= H^m_{\textit{cris}}(X/A, \mathcal{J}^{[i]}_{X/A}) \cap P^m_{\textit{cris}}(X/A) \\ \bar{F}_i &:= \frac{1}{p^i} F|_{Q_i} \end{aligned}$$

form an $\mathscr{A}$-m-window structure on the primitive cohomology $P^m_{\textit{cris}}(X/A) := \langle c_1^{\textit{cris}}(\mathscr{L})\rangle^{\perp}$.

Proof. By Theorem 3.2.1(b), there is an $\mathscr{A}$-window structure given by

$$(P_i := H^m_{\text{cris}}(X/A, \mathcal{J}^{[i]}_{X/A}), F_i := \frac{1}{p^i} F|_{P_i})$$

Since $Q_i \subseteq P_i$, we simply have $\bar{F}_i = F_i|_{Q_i}$. We must first check that the σ-linear homomorphisms $\bar{F}_i$, which *a priori* have image in $P_0 = H^m_{\text{cris}}(X/A)$, actually have image in $Q_0 = P^m_{\text{cris}}(X/A)$. Since A is p-torsion free, it suffices to check that im$(\bar{F}_0 = F|_{Q_0}) \subseteq Q_0$, in other words that $\langle F(x), c_1^{\text{cris}}(\mathscr{L})\rangle = 0$ for $x \in Q_0$. But, for $x \in Q_0$, $\langle x, c_1^{\text{cris}}(\mathscr{L})\rangle = 0$, and we see that

$$\langle F(x), F(c_1^{\text{cris}}(\mathscr{L}))\rangle = p^m \sigma(\langle x, c_1^{\text{cris}}(\mathscr{L})\rangle) = 0$$

But the left-hand side is

$$
\begin{aligned}
0 = \langle F(x), F(c_1^{\text{cris}}(\mathscr{L}))\rangle &= \langle F(x), c_1^{\text{cris}}(\mathscr{L}^p)\rangle \\
&= \langle F(x), pc_1^{\text{cris}}(\mathscr{L})\rangle \\
&= p\langle F(x), c_1^{\text{cris}}(\mathscr{L})\rangle
\end{aligned}
$$

Therefore the homomorphisms $\bar{F}_i : Q_i \to Q_0$ are well-defined.

In light of the proof of Theorem 3.2.1(a) ([LZ07], 5.5), the only non-trivial criterion left to prove is that $\bigcup_i \bar{F}_i(Q_i)$ generates Q_0 as an A-module. Let

$$
L_i := P^{m-i}(X, \Omega^i_{\mathcal{X}/A}) := H^{m-i}(X, \Omega^i_{\mathcal{X}/A}) \cap P^m_{\text{dR}}(\mathcal{X}/A)
$$

and

$$
\Phi_i := \frac{1}{p^i} F|_{L_i}
$$

Then it is the same to show that $\bigoplus_{i=0}^m \Phi_i : L_0 \oplus \cdots \oplus L_n \to Q_0$ is a σ-linear isomorphism, or equivalently that $\det(\bigoplus_{i=0}^m \Phi_i) \in A^\times$ is a unit. But each Φ_i is the just restriction of $\Phi_i' := \frac{1}{p^i} F|_{L_i'}$ to L_i, where $L_i' = H^{m-i}(X, \Omega^i_{\mathcal{X}/A})$, and we have that $\det(\bigoplus_{i=0}^m \Phi_i') \in A^\times$ because (P_i, F_i) is an $\mathscr{A}$-window. □

***Corollary* 3.4.2.** *Suppose that X/R satisfies the conditions of Theorem 3.4.1. Then for $m < p$, base-change along the frame homomorphism $\mathscr{A} \to \mathscr{W}_R$ induced by the composition*

$$
A \to W(A) \to W(R)
$$

gives $P^m_{\text{cris}}(X/W(R))$ the structure of an m-$\mathcal{W}_R$-display. This is independent of the choice of frame $\mathscr{A}$ for R and the choice of formal lifting $\mathcal{X} \to$ Spf A if $pR = 0$.

Proof. This is the primitive analogue of ([LZ07], Corollary 5.6), and their proof carries over verbatim. I repeat it here for the convenience of the reader.

First, the non-dependence on the formal lifting $\mathcal{X}$ is clear since crystalline cohomology only depends on X and on A.

It is also clear that if $\mathscr{B} \to \mathscr{A}$ is a frame homomorphism, then the induced display-structure on $P^m_{\text{cris}}(X/W(R))$ does not depend on whether we base change from the $\mathscr{A}$-window structure on $P^m_{\text{cris}}(X/A)$ or the $\mathscr{B}$-window structure on $P^m_{\text{cris}}(X/B)$; indeed, if $\mathcal{X} \to$ Spf B is a formal lifting of X over B, then $\mathcal{X} \times_{\text{Spf } B} \text{Spf } A$ is a formal lifting of X over A, and the associated $\mathscr{A}$-window is obtained from the $\mathscr{B}$-window by base-change along $\mathscr{B} \to \mathscr{A}$.

In general, suppose that $pR = 0$ and let $\mathscr{A} = (A, R, \sigma)$, $\mathscr{B} = (B, R, \sigma')$ be two frames for R. Suppose also that $\mathcal{X} \to$ Spf A and $\mathcal{Z} \to$ Spf B are two formal liftings of X. Then $\mathscr{A} \times \mathscr{B} = (A \times_R B, R, \sigma \times \sigma')$ is a frame for R, because σ and σ' both agree when reduced to R, and $\mathcal{X} \times_R \mathcal{Z}$ is a formal lifting of X to $A \times_R B$. We can then conclude by the first two parts of the proof. $\square$

4 Crystals of relative displays

Let k be a perfect field of characteristic $p > 0$, and $R \in \underline{Art}_{W,k}$, where we write $\underline{Art}_{W,k}$ for the category of local artinian W-algebras with residue field k. Instructive examples of objects in $\underline{Art}_{W,k}$ are $W_n(k)$ and $k[t]/(t^n)$. Let X_0 be a smooth projective variety with smooth versal deformation space $\mathfrak{S}$ and versal family $\mathfrak{X}/\mathfrak{S}$. Then $\mathfrak{S} \cong$ Spf A, where $A = W(k)[\![t_1, \dots, t_h]\!]$ and $h = \dim_k H^1(X_0, \mathcal{T}_{X_0/k})$. Suppose that for $f : X \to$ Spec R a deformation of X_0/k, the following two conditions analogous to (LZ1) and (LZ2) of Section 3.2 hold:-

- $R^q f_* \Omega^p_{X/R}$ is a free R-module and commutes with base-change $R \to R'$ in $\underline{Art}_{W,k}$, for all p, q.
- The spectral sequence

$$E_1^{p,q} = R^q f_* \Omega^p_{X/R} \Rightarrow \mathbb{R}^{p+q} f_* \Omega^\bullet_{X/R}$$

degenerates at E_1.

Consider the Frobenius $\sigma : A \to A$ given by $\sigma = {}^F$ on $W(k)$ and by $T_i \mapsto T_i^p$. Set $C_m := A/(T_1^m, \cdots, T_h^m)$ and $R_m := C_m/p^m C_m$, and write $\sigma : C_m \to C_m$ for the endomorphism induced by σ. Then $(C_m, p^m C_m, R_m, \sigma, \sigma/p)$ is a frame for R_m. So, by Theorem 3.2.1 we have a n-$\mathscr{W}_{R_m}$-display $\mathscr{D}_{R_m}(\mathfrak{X}_{R_m})$ on $H^n_{\mathrm{cris}}(\mathfrak{X}_{R_m}/W(R_m))$, for $n < p$. For m large enough, there is a homomorphism $R_m \to R$, and hence a frame homomorphism $\mathscr{W}_{R_m} \to \mathscr{W}_R$. Base-changing along this frame homomorphism gives the n-$\mathscr{W}_R$-display structure $\mathscr{D}_R(X)$ on $H^n_{\mathrm{cris}}(X/W(R))$ on the deformation X/R of X_0/k corresponding to Spec $R \to \mathfrak{S}$.

***Proposition* 4.0.1.** *Suppose that X'/S is a deformation of X/R over a PD-thickening $S \twoheadrightarrow R$. Write*

$$u : \mathcal{W}_S \to \mathcal{W}_{S/R}$$

for the frame homomorphism. Then $u_\bullet \mathscr{D}_S(X')$ depends only on X and S.

Proof. Let X''/S be another deformation of X/R. For the purposes of the proof, write $\mathscr{D}_{S/R}(X;X')$ and $\mathscr{D}_{S/R}(X;X'')$ for the $\mathscr{W}_{S/R}$-displays on $H^n_{\text{cris}}(X/W(S))$ given by:-

$$\mathscr{D}_{S/R}(X;X') := u_\bullet \mathscr{D}_S(X')$$
$$\mathscr{D}_{S/R}(X;X'') := u_\bullet \mathscr{D}_S(X'')$$

Then we are required to show that $\mathscr{D}_{S/R}(X;X') = \mathscr{D}_{S/R}(X;X'')$.

By the versality of $\mathfrak{S}$, the deformations X' and X'' are induced by two $W(k)$-algebra homomorphisms

$$A \rightrightarrows S \tag{1}$$

Let $\mathcal{A}_{\text{triv}} = (A, 0, A, \sigma, \sigma/p)$ be the trivial frame for A. Then Theorem 3.2.1 gives an $\mathcal{A}_{\text{triv}}$-display structure $\mathscr{D}^{\text{triv}}_A(\mathfrak{X})$ on the versal family. Then $\mathscr{D}_{S/R}(X;X')$ and $\mathscr{D}_{S/R}(X;X'')$ are the base-change of $\mathscr{D}^{\text{triv}}_A(\mathfrak{X})$ along the two frame homomorphisms

$$\mathcal{A}_{\text{triv}} \overset{x}{\underset{y}{\rightrightarrows}} \mathcal{W}_{S/R}$$

induced by (1), that is

$$\mathscr{D}_{S/R}(X;X') = x_\bullet \mathscr{D}^{\text{triv}}_A(\mathfrak{X})$$
$$\mathscr{D}_{S/R}(X;X'') = y_\bullet \mathscr{D}^{\text{triv}}_A(\mathfrak{X})$$

Now consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] & B := A\hat{\otimes}_W A \arrow[r, "\text{mult.}"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & S \arrow[r, two heads] & R &
\end{tikzcd}$$

Write $D_B(J)$ for the PD-envelope of (B, J). Similarly, set $A_0 := W[T_1, \dots, T_h]$, $B_0 := A_0 \otimes_W A_0$, $J_0 := \ker(B_0 \xrightarrow{\text{mult.}} A_0)$ and write $D_{B_0}(J_0)$ for the PD-envelope of (B_0, J_0). We compute $D_B(J)$:-

Assume first that $h = 1$. Then

$$B_0 = A_0 \otimes_W A_0 \xrightarrow{\sim} W[s,t]$$
$$s \otimes 1 \leftarrow\!\shortmid s$$
$$1 \otimes t \leftarrow\!\shortmid t$$

so J_0 is the ideal generated by $\tau = t \otimes 1 - 1 \otimes t$. Therefore $D_{B_0}(J_0) \cong B_0\langle\tau\rangle$; the PD-polynomial algebra on B_0 in one generator. Since $A = W[\![t]\!]$ is flat over $A_0 = W[t]$, ([BO78], 3.21) gives us that

$$\begin{aligned} D_B(J) &\cong D_{B_0}(J_0) \otimes_{B_0} B \\ &\cong B_0\langle\tau\rangle \otimes_{B_0} B \\ &\cong B\langle\tau\rangle \end{aligned}$$

In general, we conclude that $D_B(J)$ is isomorphic to the PD-polynomial algebra on B in h variables (see, for example, ([Stacks], 47.2.5)). In particular, $D_B(J)$ is a flat $D_{B_0}(J_0)$-module, so is certainly p-torsion free. We get a diagram

$$\begin{tikzcd} D_B(J) \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r, two heads] & R \end{tikzcd}$$

Let $\widehat{D_B(J)} := \varprojlim D_B(J)/p^n$ denote the p-adic completion of $D_B(J)$. Then we have get a frame $\mathcal{A} = \left(\widehat{D_B(J)} \to A\right)$. By the universal property of PD-envelopes, (1) factors as

$$A \rightrightarrows \widehat{D_B(J)} \to S$$

and we get induced factorisations of the frame morphisms

$$\begin{tikzcd} \mathcal{A}_{\text{triv}} \arrow[r, "\bar{x}"] \arrow[rr, bend left, "x"] & \mathcal{A} \arrow[r, "\bar{z}"] & \mathcal{W}_{S/R} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_{\text{triv}} \arrow[r, "\bar{y}"] \arrow[rr, bend left, "y"] & \mathcal{A} \arrow[r, "\bar{z}"] & \mathcal{W}_{S/R} \end{tikzcd}$$

Now, Theorem 3.2.1 also associates an $\mathcal{A}$-display $\mathscr{D}_A(\mathfrak{X})$ to $\mathfrak{X}$. By compatibility with base-change, we have that $\mathscr{D}_A(\mathfrak{X})$ is the base-change of $\mathscr{D}_A^{\text{triv}}(\mathfrak{X})$ with respect to both $\bar{x}$ and $\bar{y}$, and hence $x_\bullet \mathscr{D}_A^{\text{triv}}(\mathfrak{X}) = y_\bullet \mathscr{D}_A^{\text{triv}}(\mathfrak{X})$. $\square$

Definition. *We define $\mathscr{D}_{S/R}(X) := u_\bullet \mathscr{D}_S(X')$, where X'/S is any deformation of X/R.*

In ([GL16], Thm. 1.2), it is shown that, much like $\mathscr{D}_R(X)$, the relative display $\mathscr{D}_{S/R}(X)$ also has an explicit description in terms of a relative version of the Nygaard complexes. Indeed, let $\mathcal{N}^i_{rel/R} W\Omega^\bullet_{X'/S}$ be defined as follows

$$W\mathcal{O}_{X'} \oplus \tilde{\mathfrak{a}}^i W\mathcal{O}_{X'} \xrightarrow{d\oplus d} W\Omega^1_{X'/S} \oplus \tilde{\mathfrak{a}}^{i-1} W\Omega^1_{X'/S} \xrightarrow{d\oplus d} \cdots$$
$$\cdots \xrightarrow{d\oplus d} W\Omega^{i-1}_{X'/S} \oplus \tilde{\mathfrak{a}} W\Omega^{i-1}_{X'/S} \xrightarrow{dV+d} W\Omega^i_{X'/S} \xrightarrow{d} \cdots$$

where $\tilde{\mathfrak{a}} := \log^{-1}(\mathfrak{a})$ is the logarithmic Teichmüller ideal of $\mathfrak{a} := \ker(S \twoheadrightarrow R)$. Then it is shown that, under the analogous conditions (LZ1) and (LZ2) for X', the underlying $W(S)$-modules of $\mathscr{D}_{S/R}(X)$ are

$$\tilde{P}_i = \mathbb{H}^n(X', \mathcal{N}^i_{rel/R} W\Omega^\bullet_{X'/S})$$

The maps F_i, ι_i, α_i are induced by the obvious morphisms of complexes

$$\hat{\alpha}_i : \mathcal{J} \otimes \mathcal{N}^i_{rel/R} W\Omega^\bullet_{X'/S} \to \mathcal{N}^{i+1}_{rel/R} W\Omega^\bullet_{X'/S}$$
$$\hat{\iota}_i : \mathcal{N}^{i+1}_{rel/R} W\Omega^\bullet_{X'/S} \to \mathcal{N}^i_{rel/R} W\Omega^\bullet_{X'/S}$$
$$\hat{F}_i : \mathcal{N}^i_{rel/R} W\Omega^\bullet_{X'/S} \to W\Omega^\bullet_{X'/S}$$

where multiplication by p on $W\Omega^j_{X'/S}$ is replaced by the map

$$\pi : W\Omega^j_{X'/S} \oplus \tilde{\mathfrak{a}}^{i-j} W\Omega^j_{X'/S} \to W\Omega^j_{X'/S} \oplus \tilde{\mathfrak{a}}^{i-j-1} W\Omega^j_{X'/S}$$

which is multiplication by p on $W\Omega^j_{X'/S}$ and the inclusion on the other summand. The divided Frobenius $\hat{F}_r$ is defined on the subcomplex $\mathcal{N}^i W\Omega^\bullet_{X'/S}$ as before and on $\tilde{\mathfrak{a}}^{i-j} W\Omega^j_{X'/S}$ it is defined to be the zero map.

Remark. It is easy to see from this explicit description in terms of hypercohomology of relative Nygaard complexes and the filtered comparison theorem ([LZ07], Theorem. 4.6), that the Hodge filtration of the relative display $\mathscr{D}_{S/R}(X)$ coincides with the Hodge filtration of X:-

$$\mathsf{Fil}^\bullet \mathscr{D}_{S/R}(X) = \mathsf{Fil}^\bullet H^n_{\mathsf{dR}}(X/R).$$

Remark. Let $\mathscr{D}^{\mathsf{prim}}_S(X')$ denote the $\mathscr{W}_S$-display on the primitive cohomology $P^n_{\mathsf{cris}}(X'/W(S))$ given by Theorem 3.4.2. Then the proof of Proposition 4.0.1 also gives a crystal of relative displays $S \mapsto \mathscr{D}^{\mathsf{prim}}_{S/R}(X)$, where $\mathscr{D}^{\mathsf{prim}}_{S/R}(X) := u_\bullet \mathscr{D}^{\mathsf{prim}}_S(X')$, for any deformation X'/S of X/R.

Remark. The proof of Proposition 4.0.1 remains unchanged if we instead work with displays over the small Witt frame $\hat{\mathscr{W}}_{S/R}$. In particular, one has crystals of relative displays $S \mapsto \hat{\mathscr{D}}_{S/R}(X)$. The crystal of $\hat{\mathscr{W}}_{S/R}$-displays on $H^2_{\mathsf{cris}}(X/\hat{W}(S))$ of a K3-type scheme X/R is the main tool used in [LZ15].

4.1 Griffiths transversality for $\mathscr{D}_{S/R}(X)$

Proposition 4.0.1 says that the functor $(S \twoheadrightarrow R) \mapsto \mathscr{D}_{S/R}(X)$ is a crystal of relative displays on Cris$(X/W(k))$, refining the crystal $R^n f_{\mathsf{cris}*}\mathcal{O}_{X/W(k)}$. In particular, one has an isomorphism

$$\psi : \mathsf{pr}_2^\bullet \mathscr{D}_{S/R}(X) \cong \mathscr{D}_{\Delta/R}(X) \cong \mathsf{pr}_1^\bullet \mathscr{D}_{S/R}(X)$$

where Δ denotes the p-adic completion of the PD-envelope of $S \otimes_{W(k)} S \xrightarrow{\mathsf{mult}} S \twoheadrightarrow R$, and $\mathsf{pr}_1, \mathsf{pr}_2 : \mathcal{W}_{\Delta/R} \rightrightarrows \mathcal{W}_{S/R}$ are the frame homomorphisms corresponding to the two projections of PD-thickenings:

$$\begin{array}{ccc} \Delta & \twoheadrightarrow & R \\ \downdownarrows & & \| \\ S & \twoheadrightarrow & R \end{array}$$

Write $\tilde{P}_i$ for the $W(S)$-modules underlying the n-$\mathcal{W}_{S/R}$-display $\mathscr{D}_{S/R}(X)$, i.e.

$$\tilde{P}_i = \mathbb{H}^n(X', \mathcal{N}^i_{rel/R} W\Omega^\bullet_{X'/S})$$

The isomorphism ψ induces a connection

$$\nabla : \tilde{P}_0 \to \tilde{P}_0 \otimes_{W(S)} \Omega^1_{W(S)/W(k)}$$

which is defined by the composition

$$\tilde{P}_0 \xrightarrow{\psi \mathrm{pr}_2^* - \mathrm{pr}_1^*} \tilde{P}_0 \otimes_{W(S), \mathrm{pr}_1} \ker(\mu) \to \tilde{P}_0 \otimes_{W(S)} \ker(\mu)/\ker(\mu)^2 \xrightarrow{\sim} \tilde{P}_0 \otimes_{W(S)} \Omega^1_{W(S)/W(k)}$$

where the maps $\mathrm{pr}_i : W(\Delta) \rightrightarrows W(S)$ and $\mu : W(\Delta) \to W(S)$ are those induced by the projections $\Delta \rightrightarrows S$ and $S \otimes_{W(k)} S \xrightarrow{\text{mult}} S \twoheadrightarrow R$, respectively.

***Proposition* 4.1.1.** *The connection ∇ on $\tilde{P}_0 \cong H^n_{cris}(X'/W(S))$ reduces to the Gauss-Manin connection on $\tilde{P}_0/I_S\tilde{P}_0 \cong H^n_{dR}(X'/S)$.*

Proof. It is a foundational theorem of Berthelot that the Gauss-Manin connection on $H^n_{\mathrm{dR}}(X'/S)$ is the connection corresponding to the crystal $S \mapsto H^n_{\mathrm{cris}}(X/S)$, (see [Ber74], IV.3.6). More precisely, it is given as above by the isomorphism

$$\mathrm{pr}_2^* H^n_{\mathrm{dR}}(X'/S) \cong \mathrm{pr}_1^* H^n_{\mathrm{dR}}(X'/S)$$

But this is the reduction of the isomorphism defining ∇ on $\tilde{P}_0$ via the reduction map

$$\tilde{P}_0 = H^n_{\mathrm{cris}}(X'/W(S)) \to H^n_{\mathrm{cris}}(X'/S) = H^n_{\mathrm{dR}}(X'/S)$$

$\square$

***Corollary* 4.1.2.** *The connection $\nabla : \tilde{P}_0 \to \tilde{P}_0 \otimes_{W(S)} \Omega^1_{W(S)/W(k)}$ on the relative $\mathcal{W}_{S/R}$-display $\mathscr{D}_{S/R}(X)$ satisfies a form of Griffiths transversality: for all i we have*

$$\nabla(\iota_{i-1}(\tilde{P}_i)) \subseteq \tilde{P}_{i-1} \otimes_{W(S)} \Omega^1_{W(S)/W(k)}$$

Proof. Recall that the underlying $W(S)$-modules $\tilde{P}_i$ of $\mathscr{D}_{S/R}(X)$ have a normal decomposition

$$\tilde{P}_i = J_i L_0 \oplus J_{i-1} L_1 \oplus \cdots \oplus J L_{i-1} \oplus L_i \oplus \cdots \oplus L_n$$

where the L_i are $W(S)$-modules lifting the Hodge decomposition

$$H^n_{\mathrm{dR}}(X'/S) = \bigoplus_{i=0}^{n} \bar{L}_i, \qquad \bar{L}_i = H^{n-i}(X', \Omega^i_{X'/S})$$

By Proposition 4.1.1, reducing modulo I_S and using Griffiths transversality of the Gauss-Manin connection on $H^n_{\mathrm{dR}}(X'/S)$, we see that

$$\begin{aligned}\nabla(\iota_{i-1}(\tilde{P}_i)) &\subseteq \left(\bigoplus_{j=0}^{i-2} I_S L_i \oplus \bigoplus_{j=i-1}^{n} L_i \right) \otimes_{W(S)} \Omega^1_{W(S)/W(k)} \\ &\subseteq \tilde{P}_{i-1} \otimes_{W(S)} \Omega^1_{W(S)/W(k)}\end{aligned}$$

□

5 Deformation theory of Calabi-Yau threefolds

In this chapter, k is a perfect field of characteristic $p > 0$ and $W = W(k)$. X_0/k will be a Calabi-Yau threefold, i.e. a smooth projective k-variety with trivial canonical bundle and $H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = 0$.

There are two immediate conseqeunces of the definition. First, triviality of the canonical bundle gives the following identification of the tangent sheaf

$$\Omega^2_{X_0/k} \cong \Omega^3_{X_0/k} \wedge \Omega^{-1}_{X_0/k} \cong \mathcal{O}_{X_0} \wedge \Omega^{-1}_{X_0/k} \cong \mathcal{T}_{X_0/k}$$

Second, Serre duality along with triviality of the canonical bundle shows that the Hodge diamond of X_0 is as follows:-

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & h^{1,0} & & 0 & & \\
 & h^{2,0} & & h^{1,1} & & 0 & \\
1 & & h^{1,2} & & h^{1,2} & & 1 \\
 & 0 & & h^{1,1} & & h^{2,0} & \\
 & & 0 & & h^{1,0} & & \\
 & & & 1 & & &
\end{array}$$

where $h^{i,j} := \dim_k H^j(X_0, \Omega^j_{X_0/k})$.

One should note that Hodge symmetry (i.e. $h^{i,j} = h^{j,i}$) fails in positive characteristic, in general. It is clear that the Calabi-Yau threefold X_0 satisfies Hodge symmetry if and only if $H^0(X_0, \Omega^1_{X_0/k}) = 0$ and $H^0(X_0, \mathcal{T}_{X_0/k}) = 0$.

In order to utilise Proposition 4.0.1 to talk about a crystal of relative displays, we require X_0 to have a smooth versal deformation space. In the first section will see that this is generically the case.

5.1 Smoothness of the deformation space $\mathfrak{S}$

We now consider the universal (formal) deformation $\mathfrak{X} \to \mathfrak{S}$ over W. In other words, any deformation X of X_0/k over $R \in \underline{Art}_{W,k}$ is given by pulling back $\mathfrak{X} \to \mathfrak{S}$ along a uniquely determined Spec $A \to \mathfrak{S}$. Our method will require that the deformation space be formally smooth, i.e. $\mathfrak{S} \simeq$ Spf A, where A is the adic ring

$$A \simeq W[\![t_1, \dots, t_h]\!], \quad h := \dim_k H^1(X_0, \mathcal{T}_{X_0/k}) = h^{2,1}$$

with ideal of definition $(p, t_1, \dots, t_h)$.

The Bogomolov-Tian-Todorov theorem guarantees that this is indeed the case in characteristic zero, but smoothness does not always hold in characteristic p. However, Ekedahl-Shepherd-Barron's *"PD tangent lifting"* technique gives a result in this direction; we write down a special case:-

***Theorem* 5.1.1.** *([ESB05], 4.2)*

Suppose that X_0/k satisfies the following condition:-

$$b_3(X_0) = \dim_k H^3_{dR}(X_0/k) = \sum_{i+j=3} h^{ij} \tag{†}$$

Then the universal deformation space is smooth.

Notice that condition (†) certainly holds if the Hodge-to-de Rham spectral sequence degenerates, and therefore will hold if X_0 lifts to $W_2(k)$ and $p \geq 5$, by Deligne-Illusie [DI87].

***Remark*.** There are now many examples of Calabi-Yau threefolds which do not lift to W, or indeed to any mixed characteristic DVR with residue field k. What is more, Ekedahl [Eke04] has shown that one such example, *"Hirokado's variety"* [Hir99], does not even lift to $W_2(k)$.

In light of the remark, it is natural to wonder whether condition (†) is particularly common. We will reproduce the argument of [Jos07] (attributed to unpublished work of V. B. Mehta) giving examples of Calabi-Yau threefolds satisfying condition (†), but first we will review the related notions of ordinariness and height of varieties:-

Suppose that k is now algebraically closed, and set

$$Z^n_{X_0/k} := \ker(\Omega^n_{X_0/k} \xrightarrow{d} \Omega^{n+1}_{X_0/k})$$
$$B^n_{X_0/k} := \mathrm{im}(\Omega^{n-1}_{X_0/k} \xrightarrow{d} \Omega^n_{X_0/k})$$

Definition. *A projective variety X_0/k is called ordinary if $H^i(X_0, B^j_{X_0/k}) = 0$ for all $i \geq 0$ and $j \geq 1$.*

We will show that ordinary Calabi-Yau threefolds satisfy condition (†) (one should note that ([Ill90], Proposition 1.2) says that ordinarity is an open condition). First, it will be useful to recall the height invariant of characteristic p projective varieties:-

Extending the notion of formal Picard group of a variety, Artin-Mazur [AM77] defined functors between the categories of artinian local k-algebras and abelian groups:-

$$\Phi^m_{X_0}(S) := \ker\left[H^m_{\text{ét}}(X_0 \otimes_k S, \mathbb{G}_m) \to H^m_{\text{ét}}(X_0, \mathbb{G}_m)\right]$$

In the special case that X_0/k is a Calabi-Yau threefold, their results show that $\Phi^3_{X_0}$ is pro-representable by a connected 1-dimensional formal group, also denoted by $\Phi^3_{X_0}$.

Definition. *The height $ht(X_0)$ of a Calabi-Yau threefold X_0/k is defined to be the height of its associated Artin-Mazur formal group $\Phi^3_{X_0}$.*

We now give an equivalent and less cumbersome definition of height for a Calabi-Yau threefold in terms of Serre's Witt vector cohomology $H^i(X_0, W\Omega^j_{X_0/k})$. The argument is only a superficial alteration of the K3-surface case found in [KvdG00]. First, a lemma:-

***Lemma* 5.1.2.** *Let X_0/k be a Calabi-Yau threefold. Then*

$$\frac{H^3(X_0, W\mathcal{O}_{X_0})}{VH^3(X_0, W\mathcal{O}_{X_0})} \cong k$$

Proof. We first use induction on n to show that $H^4(X_0, W_n\mathcal{O}_{X_0}) = 0$ for all $n \geq 1$. Indeed, $H^4(X_0, W_1\mathcal{O}_{X_0}) = H^4(X_0, \mathcal{O}_{X_0}) = 0$ for a Calabi-Yau threefold, and the short exact sequence

$$0 \to \mathcal{O}_{X_0} \to W_n\mathcal{O}_{X_0} \to W_{n-1}\mathcal{O}_{X_0} \to 0$$

yields

$$H^4(X_0, \mathcal{O}_{X_0}) \to H^4(X_0, W_n\mathcal{O}_{X_0}) \to H^4(X_0, W_{n-1}\mathcal{O}_{X_0})$$

and hence $H^4(X_0, W_n\mathcal{O}_{X_0}) = 0$ for all $n \geq 1$.

Using this and that $H^2(X_0, \mathcal{O}_{X_0}) = 0$, taking cohomology of the short exact sequence

$$0 \to W_{n-1}\mathcal{O}_{X_0} \xrightarrow{V} W_n\mathcal{O}_{X_0} \to \mathcal{O}_{X_0} \to 0$$

gives

$$0 \to H^3(X_0, W_{n-1}\mathcal{O}_{X_0}) \xrightarrow{V} H^3(X_0, W_n\mathcal{O}_{X_0}) \to H^3(X_0, \mathcal{O}_{X_0}) \to 0$$

Everything in sight satisfies the Mittag-Leffler condition, so taking limits gives

$$0 \to H^3(X_0, W\mathcal{O}_{X_0}) \xrightarrow{V} H^3(X_0, W\mathcal{O}_{X_0}) \to H^3(X_0, \mathcal{O}_{X_0}) \to 0$$

and we conclude the result, since $H^3(X_0, \mathcal{O}_{X_0}) \cong k$. $\square$

***Proposition* 5.1.3.** *A Calabi-Yau threefold X_0/k has $ht(X_0) \geq n+1$ if and only if $F : H^3(X_0, W_n\mathcal{O}_{X_0}) \to H^3(X_0, W_n\mathcal{O}_{X_0})$ is the zero map.*

Proof. First suppose that $F : H^3(X_0, W_n\mathcal{O}_{X_0}) \to H^3(X_0, W_n\mathcal{O}_{X_0})$ is the zero map. If $ht(X_0) = ht(\Phi^3_{X_0}) = \infty$ then we are done, so suppose $ht(X_0) = h < \infty$.

By ([AM77], Cor. 4.3), the Dieudonné module of $\Phi^3_{X_0}$ is

$$D(\Phi^3_{X_0}) \cong H^3(X_0, W\mathcal{O}_{X_0})$$

Lemma 5.1.2 and the height formula for p-divisible groups gives

$$\begin{aligned} h = ht(\Phi^3_{X_0}) &= \text{rank}_{W(k)} D(\Phi^3_{X_0}) \\ &= \dim_k \left(\frac{D(\Phi^3_{X_0})}{pD(\Phi^3_{X_0})} \right) \\ &= \dim_k \left(\frac{D(\Phi^3_{X_0})}{FD(\Phi^3_{X_0})} \right) + \dim_k \left(\frac{D(\Phi^3_{X_0})}{VD(\Phi^3_{X_0})} \right) \\ &= \dim_k \left(\frac{D(\Phi^3_{X_0})}{FD(\Phi^3_{X_0})} \right) + 1 \end{aligned}$$

Now, the surjection $H^3(X_0, W\mathcal{O}_{X_0}) \twoheadrightarrow H^3(X_0, W_n\mathcal{O}_{X_0})$ induces

$$\frac{H^3(X_0, W\mathcal{O}_{X_0})}{FH^3(X_0, W\mathcal{O}_{X_0})} \twoheadrightarrow \frac{H^3(X_0, W_n\mathcal{O}_{X_0})}{FH^3(X_0, W_n\mathcal{O}_{X_0})} \cong H^3(X_0, W_n\mathcal{O}_{X_0})$$

where the isomorphism is our assumption. Taking dimensions and using the short exact sequence at the end of the proof of Lemma 5.1.2 gives $h - 1 \geq n$.

Conversely, suppose that $ht(X_0) \geq n + 1$. If $ht(X_0) = ht(\Phi^3_{X_0}) = \infty$ then $\Phi^3_{X_0} = \hat{\mathbb{G}}_a$, so F is zero on $D(\Phi^3_{X_0}) \cong H^3(X_0, W\mathcal{O}_{X_0})$. So suppose that $ht(X_0) = h < \infty$, and consider the filtration

$$0 \subset V^{h-2}H^3(X_0, W_1\mathcal{O}_{X_0}) \subset \ldots \subset VH^3(X_0, W_{h-2}\mathcal{O}_{X_0}) \subset H^3(X_0, W_{h-1}\mathcal{O}_{X_0})$$

The short exact sequence

$$0 \to W_{n-1}\mathcal{O}_{X_0} \xrightarrow{V} W_n\mathcal{O}_{X_0} \to \mathcal{O}_{X_0} \to 0$$

shows that the all the inclusions are proper.

Using the classification of Dieudonné modules,

$$D(\Phi^3_{X_0}) \cong \frac{W(k)[F,V]}{W(k)[F,V](F - V^{h-1})}$$

for F written on the right. But in this non-commutative module we have the relation $FV = VF = p$, so we can write F on the left as:-

$$\begin{aligned} F\left(\sum a_{ij}F^iV^j\right) &= \sum \sigma(a_{ij})FF^iV^j \\ &= \sum (\sigma(a_{ij})F^iV^j)F \\ &= \sum (\sigma(a_{ij})F^iV^j)V^{h-1} \\ &= V^{h-1}\left(\sum \sigma^h(a_{ij})F^iV^j\right) \end{aligned}$$

Therefore $FH^3(X_0, W_{h-1}\mathcal{O}_{X_0}) = V^{h-1}H^3(X_0, W_{h-1}\mathcal{O}_{X_0}) = 0$. We conclude that F is zero on $H^3(X_0, W_n\mathcal{O}_{X_0})$ for all $n \leq h-1$ since F commutes with restriction.

□

***Remark* 5.1.4.** Notice that Proposition 5.1.3 gives a different characterisation of height for Calabi-Yau threefolds:-

$$\text{ht}(X_0) = \min\{i \geq 1 : [F : H^3(X_0, W_i\mathcal{O}_{X_0}) \to H^3(X_0, W_i\mathcal{O}_{X_0})] \neq 0\}$$

Recall that a projective variety X_0/k is called F*-split* if there is a splitting of the *Frobenius exact sequence* of $\mathcal{O}_{X_0}$-modules

$$0 \to \mathcal{O}_{X_0} \to F_*\mathcal{O}_{X_0} \to B^1_{X_0/k} \to 0$$

where $F : X_0 \to X_0$ denotes the absolute Frobenius.

***Proposition* 5.1.5.** *A Calabi-Yau threefold X_0/k has $ht(X_0) = 1$ if and only if it is F-split.*

Proof. This is well-known, but we recount the proof here for the reader's convenience.

Consider the Frobenius exact sequence

$$0 \to \mathcal{O}_{X_0} \to F_*\mathcal{O}_{X_0} \to B^1_{X_0/k} \to 0$$

Since, by definition, X_0 is smooth, these $\mathcal{O}_{X_0}$-modules are locally free, so the sequence splits if and only if the dual sequence splits:-

$$0 \to \left(B^1_{X_0/k}\right)^\vee \to \left(F_*\mathcal{O}_{X_0}\right)^\vee \to \mathcal{O}_{X_0} \to 0$$

and this is the case if and only if the identity id : $\mathcal{O}_{X_0} \to \mathcal{O}_{X_0}$ factors through $\left(F_*\mathcal{O}_{X_0}\right)^\vee$, i.e. if and only if $H^0(X_0, \left(F_*\mathcal{O}_{X_0}\right)^\vee) \to H^0(X_0, \mathcal{O}_{X_0})$ is not the zero map. But, by Serre duality (remember that the canonical bundle is trivial), this is not the zero map if and only if $H^3(X_0, \mathcal{O}_{X_0}) \to H^3(X_0, F_*\mathcal{O}_{X_0})$ is not the zero map.

Now, recall that the map on cohomology $F : H^3(X_0, \mathcal{O}_{X_0}) \to H^3(X_0, \mathcal{O}_{X_0})$ is the composite

$$F : H^3(X_0, \mathcal{O}_{X_0}) \to H^3(X_0, F_*\mathcal{O}_{X_0}) \xrightarrow{\sim} H^3(X_0, \mathcal{O}_{X_0})$$

where the isomorphism is because the absolute Frobenius is a finite morphism. So we have that X_0 is ordinary if and only if $F : H^3(X_0, \mathcal{O}_{X_0}) \to H^3(X_0, \mathcal{O}_{X_0})$ is not the zero map, and we conclude by the equivalent definition of height given above. □

We will now see that ordinary Calabi-Yau threefolds have height one.

***Proposition* 5.1.6.** *Let X_0/k be a Calabi-Yau threefold. Then $ht(X_0) = 1$ if and only if $H^2(B^1_{X_0/k}) = H^3(B^1_{X_0/k}) = 0$.*

Proof. Suppose first that $H^2(B^1_{X_0/k}) = H^3(B^1_{X_0/k}) = 0$. Consider the Frobenius exact sequence

$$0 \to \mathcal{O}_{X_0} \to F_*\mathcal{O}_{X_0} \to B^1_{X_0/k} \to 0$$

Taking cohomology gives an exact sequence

$$H^2(X_0, B^1_{X_0/k}) \to H^3(X_0, \mathcal{O}_X) \to H^3(X_0, F_*\mathcal{O}_X) \to H^3(X_0, B^1_{X_0/k})$$

The outer terms are zero by assumption, and the resulting isomorphism of the middle terms shows that $F : H^3(X_0, \mathcal{O}_X) \to H^3(X_0, \mathcal{O}_X)$ is not the zero map. We conclude that $ht(X_0) = 1$ by Remark 5.1.4.

Conversely, suppose that $ht(X_0) = 1$. Then Proposition 5.1.5 shows that the Frobenius exact sequence splits, so

$$\begin{aligned} & H^n(X_0, F_*\mathcal{O}_X) \cong H^n(X_0, \mathcal{O}_X) \oplus H^n(B^1_{X_0/k}) \\ \Rightarrow \quad & h^{0,n} = h^{0,n} + \dim_k H^n(B^1_{X_0/k}) \\ \Rightarrow \quad & H^n(B^1_{X_0/k}) = 0 \quad \text{for all } n \geq 0. \end{aligned}$$

□

Remark **5.1.7.** The concepts of ordinary and height one coincide for abelian varieties and K3-surfaces. For Calabi-Yau threefolds, we can easily show that having height one is “close” to being ordinary, however, it is not clear to the author that $H^1(X_0, B^2_{X_0/k}) \cong H^2(X_0, B^2_{X_0/k})$ necessarily vanishes.

Indeed, the proof of Lemma 5.1.6 shows that $H^n(B^1_{X_0/k}) = 0$ for all $n \geq 0$ if $ht(X_0) = 1$. Now, the Cartier operator pairing and Calabi-Yau condition induce a duality

$$(B^1_{X_0/k})^* \cong B^3_{X_0/k} \otimes \omega_{X_0} \cong B^3_{X_0/k}$$

so we see $H^n(X_0, B^3_{X_0/k}) = 0$ for all $n \geq 0$, by Serre duality.

Now consider the short exact sequence

$$0 \to Z^1_{X_0/k} \to \Omega^1_{X_0/k} \xrightarrow{d} B^2_{X_0/k} \to 0$$

Then taking cohomology and using the Calabi-Yau condition gives

$$H^3(X_0, B^2_{X_0/k}) \cong H^4(X_0, Z^1_{X_0/k})$$

Now consider the short exact sequence

$$0 \to B^1_{X_0/k} \to \Omega^1_{X_0/k} \to Z^1_{X_0/k} \to 0$$

and take cohomology to see that

$$H^4(X_0, \Omega^1_{X_0/k}) \to H^4(X_0, Z^1_{X_0/k}) \to H^5(X_0, B^1_{X_0/k})$$

The outer terms are zero, so $H^3(X_0, B^2_{X_0/k}) \cong H^4(X_0, Z^1_{X_0/k}) = 0$. Now notice that $H^0(X_0, B^2_{X_0/k}) \cong H^3(X_0, B^2_{X_0/k})$ by Serre duality and $(B^2_{X_0/k})^* \cong B^2_{X_0/k}$ under the Cartier operator pairing.

Notice that the second half of the argument holds for any Calabi-Yau threefold. In other words, the height only gives information about the cohomology of $B^1_{X_0/k}$ (and hence $B^3_{X_0/k}$), so it seems likely that being ordinary is a strictly stronger condition than having height one.

***Proposition* 5.1.8.** *Let X_0/k be a height one Calabi-Yau threefold, where k is an algebraically closed field of characteristic $p > 3$. Then X_0 satisfies condition* (†).

Proof. Consider the Frobenius exact sequence

$$0 \to \mathcal{O}_{X_0} \to F_*\mathcal{O}_{X_0} \to B^1_{X_0/k} \to 0$$

and write $\delta : \text{Ext}^1(\Omega^1_{X_0/k}, B^1_{X_0/k}) \to \text{Ext}^2(\Omega^1_{X_0/k}, \mathcal{O}_{X_0})$ for the induced connecting homomorphism on the Ext long exact sequence. Then, by [Sri90], the obstruction to lifting X to $W_2(k)$ is the class $\delta(\xi) \in \text{Ext}^2(\Omega^1_{X_0/k}, \mathcal{O}_{X_0})$, where $\xi \in \text{Ext}^1(\Omega^1_{X_0/k}, B^1_{X_0/k})$ is the obstruction to lifting the pair (X, F) to $(X^{(2)}, F^{(2)})$, where $X^{(2)}/W_2(k)$ is a lifting of X, and $F^{(2)} : X^{(2)} \to X^{(2)}$ is a lifting of the absolute

Frobenius $F : X \to X$ compatible with the Frobenius automorphism on $W_2(k)$.

Now, $ht(X_0) = 1$, so it is F-split by Proposition 5.1.5. Therefore

$$\text{Ext}^1(\Omega^1_{X_0/k}, F_*\Omega^1_{X_0/k}) \twoheadrightarrow \text{Ext}^1(\Omega^1_{X_0/k}, B^1_{X_0/k})$$

i.e. the connecting homomorphism δ is the zero map. In particular, the obstruction $\delta(\xi)$ to lifting X to $W_2(k)$ is zero. We assumed $p > 3 = \dim X_0$, so we may conclude by [DI87]. $\square$

5.2 Aside on liftability of ordinary Calabi-Yau threefolds

Although it has no real bearing on the rest of this thesis, it seems interesting to include the following stronger version of Proposition 5.1.8 in the case of ordinary Calabi-Yau threefolds. Presumably the result is well-known but the author has been unable to find a reference. In any case, the proof is just an adaptation of the argument for cubic fourfolds found in ([Lev01], Sections 3 and 4).

***Proposition* 5.2.1.** *Let X_0/k be an ordinary Calabi-Yau threefold. Then X_0 (canonically) lifts to $W = W(k)$.*

Proof. By Proposition 5.1.8, the universal deformation space of X_0 is smooth, i.e.

$$\mathfrak{S} = \text{Spf}\, W[\![t_1, \ldots, t_h]\!]$$

where $h = \dim_k H^1(X_0, \Omega^2_{X_0/k})$. We can describe the tangent space as

$$\begin{aligned} T_{\mathfrak{S}} &\cong H^1(X_0, \mathcal{T}_{X_0}) \\ &\cong \text{Hom}\left(H^0(X_0, \Omega^3_{X_0/k}), H^1(X_0, \Omega^2_{X_0/k})\right) \\ &\cong \text{Hom}\left(H^0(X_0, \mathcal{O}_{X_0}), H^1(X_0, \mathcal{T}_{X_0})\right) \end{aligned}$$

where the isomorphisms are given in order by the Kodaira-Spencer map, cup-product with a generator of $H^0(X_0, \Omega^3_{X_0/k})$ and the Calabi-Yau conditions (see

Proposition 5.3.1).

Since X_0 is ordinary, $H^3_{\text{cris}}(X_0/W)$ is an ordinary crystal (see [Del81], 1.3), so its Newton and Hodge polygons coincide and we get a *"Newton-Hodge decomposition"*, i.e. there is a subcrystal $P \subset H^3_{\text{cris}}(X_0/W)$ with Newton and Hodge slopes ≤ 1 and a quotient crystal Q of $H^3_{\text{cris}}(X_0/W)$ with Newton and Hodge slopes > 1 fitting into the following short exact sequence:-

$$0 \to P \to H^3_{\text{cris}}(X_0/W) \to Q \to 0 \tag{2}$$

See ([Kat79] Thm. 1.6.1) for the details. Since k is perfect, (2) splits canonically, so we may consider Q as a subcrystal of $H^3_{\text{cris}}(X_0/W)$. We read off from the Hodge polygon of X_0 that

$$\text{rank}_W P = \text{rank}_W Q = h + 1$$

and Q has slopes $2, 3$. Therefore $Q(2)$ has slopes $0, 1$, i.e. it is a Dieudonné module (see [Ogu77] 1.6.4) and hence there is a unique p-divisible group $\mathcal{P}_0$ over k with $\mathbb{D}(\mathcal{P}_0) = Q(2)$.

We consider the deformation functor

$$\text{Def}_{\mathcal{P}_0} : \underline{\mathcal{A}rt}_{W,k} \to \underline{\mathcal{S}ets}$$

sending $R \in \underline{\mathcal{A}rt}_{W,k}$ to isomorphism classes of p-divisible groups $\mathcal{P}$ over R with $\mathcal{P} \otimes_R k \cong \mathcal{P}_0$. As a result of Grothendieck-Messing theory, $\text{Def}_{\mathcal{P}_0}$ is pro-representable and formally smooth, i.e. has fine moduli space

$$\mathfrak{T} \cong \text{Spf}\, W[\![t_1, \dots, t_h]\!]$$

We will build an isomorphism $\mathfrak{S} \xrightarrow{\sim} \mathfrak{T}$ (a *"period map"*), in other words we make the assertion *"deforming X_0 over $\underline{Art}_{W,k}$ is the same as deforming $\mathcal{P}_0$ over $\underline{Art}_{W,k}$"*

First a simplifying remark:- by ([dJ95] 2.4.4), the categories of p-divisible groups on Spf $W[\![t_1, \dots, t_h]\!]$ and on Spec $W[\![t_1, \dots, t_h]\!]$ are equivalent. Further, since $H^2(X_0, \mathcal{O}_{X_0}) = 0$, there is no obstruction to lifting ample line bundles on X_0 to ample line bundles on $\mathfrak{X}$, so Grothendieck's algebraisation theorem tells us that $\mathfrak{X}$ algebraises, i.e. there is a smooth Calabi-Yau threefold

$$X \to S := \text{Spec } W[\![t_1, \dots, t_h]\!]$$

with $\mathfrak{X} \simeq X \times_S \mathfrak{S}$. As such, it suffices to construct a period map $S \to T$, i.e. construct a p-divisible group $\mathcal{P}$ on S which reduces to $\mathcal{P}_0$ on the special fibre, in order to build our period map $\mathfrak{S} \to \mathfrak{T}$.

Our construction of such a $\mathcal{P}$ will have two steps. Let $S_k := S \otimes_W k = k[\![t_1, \dots, t_h]\!]$. The first step is to construct a p-divisible group $\mathcal{P}_k$ on S_k which reduces to $\mathcal{P}_0$ on the special fibre. We will use the same argument as before to find $\mathcal{P}_0$. The second step will be showing that $\mathcal{P}_k$ lifts canonically to a p-divisible group on S; this is will be done via Grothendieck-Messing theory.

For the first step, write $X_k := X \times_S S_k$ for the base-change to S_k. Using base-change for crystalline cohomology, we see that $H^3_{\text{cris}}(X_k/S)$ specialises to $H^3_{\text{cris}}(X_0/W)$ at the closed point, so it is also an ordinary crystal (see [Del81], Prop. 1.3.2), i.e its Newton and Hodge polygons coincide. Therefore, just as before but now using ([Kat79], §2.4), there are P', Q' of Newton and Hodge slopes ≤ 1, > 1 respectively, lying in a short exact sequence of crystals

$$0 \to P' \to H^3_{\text{cris}}(X_k/S) \to Q' \to 0$$

By the uniqueness (see [Del81], Prop. 1.3.2 (iii)), these must pullback to P and Q via the specialisation morphism $S_k \to \text{Spec } k$. We see that $Q'(2)$ is a Dieudonné module over S (see [dJ95], 2.3.4 for the definition of Dieudonné modules over formal power series rings.). The Dieudonné functor is an equivalence of categories

$$\mathbb{D} : \{\text{Dieudonné modules over } S\} \xrightarrow{\sim} \{p\text{-divisible groups over } S_k\}$$

(this follows by composing the equivalences in [dJ95], 2.4.8, 4.1.1, 2.4.4.). In particular, there is a p-divisible group $\mathcal{P}_k$ over S_k with $\mathbb{D}(\mathcal{P}_k) = Q'(2)$. Since the Dieudonné functor $\mathbb{D}$ commutes with base-change, $\mathcal{P}_k$ specialises to $\mathcal{P}_0$ on the special fibre.

By Grothendieck-Messing theory, for the second step it is sufficient to put an admissible filtration on $Q'(2)$ in order to lift $\mathcal{P}_k$ to S. From now on we regard P', Q' as sub/quotient modules of $H^3_{\mathrm{dR}}(X/S)$ via the canonical isomorphism $H^3_{\mathrm{cris}}(X_k/S) \cong H^3_{\mathrm{dR}}(X/S)$. As we shall see, the admissible filtration on $Q'(2)$ comes from the Hodge filtration on $H^3_{\mathrm{dR}}(X/S)$.

First we note that $Q' \cong \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X/S)$. Indeed, all the cohomology modules are free and the Hodge-to-de Rham spectral sequence degenerates at E_1, and we know

$$H^3_{\mathrm{dR}}(X/S) \otimes_S S_k \cong H^3_{\mathrm{dR}}(X_k/S_k)$$

and

$$\mathrm{Fil}^2 H^3_{\mathrm{dR}}(X/S) \otimes_S S_k \cong \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X_k/S_k).$$

Since $H^3_{\mathrm{dR}}(X/S) \cong H^3_{\mathrm{cris}}(X_k/S)$ is an ordinary crystal, modulo p we have

$$H^3_{\mathrm{dR}}(X_k/S_k) \cong P'_k \oplus Q'_k \cong P'_k \oplus \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X_k/S_k).$$

Nayakama's lemma then gives that

$$H^3_{\mathrm{dR}}(X/S) \cong P' + \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X/S).$$

Now, both P' and $\mathrm{Fil}^2 H^3_{\mathrm{dR}}(X/S)$ are direct summands of $H^3_{\mathrm{dR}}(X/S)$ and the sum of their ranks equals the rank of $H^3_{\mathrm{dR}}(X/S)$ (that is, $(h+1)+(h+1) = 2h+2$).

But we also have $P' \cap \mathsf{Fil}^2 H^3_{\mathsf{dR}}(X/S) = \emptyset$ (tensor by the fraction field and look at the slopes), so the sum is direct, i.e. $Q' \cong \mathsf{Fil}^2 H^3_{\mathsf{dR}}(X/S)$.

Let H be the image of $\mathsf{Fil}^3 H^3_{\mathsf{dR}}(X/S)$ in Q'. This is a direct summand by the previous observation. Let H_k be the reduction of H modulo p. Since the Hodge filtration on $H^3_{\mathsf{dR}}(X_k/S_k)$ is a 4-step filtration, we have that the entire (2-step) Hodge filtration on Q' is

$$0 \subseteq H \subseteq Q'$$

where $H = \mathsf{Fil}^3_{\mathsf{Hodge}} Q'$. Twisting by 2 lowers the Hodge slopes by 2, i.e.

$$0 \subseteq H \subseteq Q'(2)$$

where $H = \mathsf{Fil}^1_{\mathsf{Hodge}} Q'(2)$. Hence $0 \subseteq H \subseteq Q'(2)$ is an admissible filtration of $Q'(2)$, i.e. there is a p-divisible group $\mathcal{P}$ on S with Dieudonné module $\mathbb{D}(\mathcal{P}) = Q'(2)$, and furthermore, $\mathcal{P}$ specialises to $\mathcal{P}_k$ on the special fibre of S. This completes the construction of the period map $\mathfrak{S} \to \mathfrak{T}$.

To show that $\mathfrak{S} \to \mathfrak{T}$ is an isomorphism, it suffices to show that it is smooth, i.e. that the induced map $T_{\mathfrak{S}} \to T_{\mathfrak{T}}$ on tangent spaces is a surjection. We know from the discussion at the beginning (or Proposition 5.3.1) that

$$T_{\mathfrak{S}} \cong \mathsf{Hom}\left(H^0(X_0, \mathcal{O}_{X_0}), H^1(X_0, \mathcal{T}_{X_0})\right).$$

But on the other hand

$$T_{\mathfrak{T}} \cong \mathsf{Hom}\left(\omega_{\mathcal{P}_0^*}, \mathsf{Lie}\ \mathcal{P}_0\right).$$

Now, by the construction of $\mathcal{P}_0$, we have $\mathbb{D}(\mathcal{P}_0) = Q(2)$ and

$$\omega_{\mathcal{P}_0^*} \cong \mathsf{Fil}^1_{\mathsf{Hodge}} Q(2) = \mathsf{Fil}^3 H^3_{\mathsf{dR}}(X_0/k) \cong H^0(X_0, \mathcal{O}_{X_0})$$

and

$$\mathsf{Lie}\ \mathcal{P}_0 \cong \mathsf{gr}^0_{\mathsf{Hodge}} Q(2) = \mathsf{gr}^2 H^3_{\mathsf{dR}}(X_0/k) \cong H^1(X_0, \mathcal{T}_{X_0})$$

and the induced map $T_{\mathfrak{S}} \to T_{\mathfrak{T}}$ is the obvious one under these identifications.

Consider the Dieudonné module $Q(2)$ of $\mathcal{P}_0$. It is ordinary and has Hodge slopes $0, 1$, so we can once again invoke [Kat79] to find subcrystals $M, N \subset Q(2)$ with Hodge slopes 0 and 1 respectively, such that $Q(2) = M \oplus N$. This corresponds to a direct sum decomposition on the p-divisible group side, i.e.

$$\mathcal{P}_0 = \mathcal{P}_0^{\text{conn}} \oplus \mathcal{P}_0^{\text{ét}}$$

with $\mathbb{D}(\mathcal{P}_0^{\text{conn}}) = M$ and $\mathbb{D}(\mathcal{P}_0^{\text{ét}}) = N$.

To lift $\mathcal{P}_0^{\text{conn}}$ and $\mathcal{P}_0^{\text{ét}}$, it suffices to put an admissible filtration on M, N. But

$$\text{Fil}^1_{\text{Hodge}} M_k = 0 \qquad \text{Fil}^1_{\text{Hodge}} N_k = N_k$$

so we must have

$$\text{Fil}^1_{\text{Hodge}} M = 0 \qquad \text{Fil}^1_{\text{Hodge}} N = N$$

Therefore $\mathcal{P}_0^{\text{conn}}$ and $\mathcal{P}_0^{\text{ét}}$ lift to W; call their liftings $\mathcal{P}^{\text{conn}}$ and $\mathcal{P}^{\text{ét}}$ respectively. Set $\mathcal{P} := \mathcal{P}^{\text{conn}} \oplus \mathcal{P}^{\text{ét}}$. This is clearly a lifting of $\mathcal{P}_0$ to W. But the period map $\mathfrak{S} \to \mathfrak{T}$ is an isomorphism, so $\mathcal{P}$ corresponds to a Calabi-Yau threefold X/W lifting X_0/k. $\square$

***Remark*.** In light of Proposition 5.1.8 and Proposition 5.2.1, it is natural to ask whether height one Calabi-Yau threefolds lift to characteristic zero. Furthermore, the examples of Calabi-Yau threefolds (known to the author) which do not lift all have infinite height. Perhaps having infinite height is the only obstruction to lifting?

5.3 A Grothendieck-Messing type result for Calabi-Yau threefolds

Let k be a perfect field of characteristic p, and consider a Calabi-Yau threefold X_0/k. We write $\mathfrak{X} \to \mathfrak{S}$ for the universal family, and always assume that $\mathfrak{S}$ is smooth; $\mathfrak{S} \simeq \text{Spf}\, A = \text{Spf}\, W[\![t_1, \ldots, t_h]\!]$, for $h = h^{21}$. See Section 5.1 for a discussion of this condition.

Let $X_n := \mathfrak{X} \times_A A/p^{n+1}A$ for $n \geq 0$. From the long exact sequence on cohomology we have

$$H^1(X_0, \mathcal{O}_{X_0}) \to H^1(X_{n+1}, \mathcal{O}^*_{X_{n+1}}) \to H^1(X_n, \mathcal{O}^*_{X_n}) \to H^2(X_0, \mathcal{O}_{X_0})$$

The outer terms are zero and hence the natural restriction $\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ is an isomorphism. For each n let ω_{X_n} be a 3-form. Since $\omega_{X_{n+1}}|_{X_n} \simeq \omega_{X_n}$, induction gives $\Omega^3_{\mathfrak{X}/S} \simeq \mathcal{O}_{\mathfrak{X}}$, and so we see that $\mathcal{T}_{\mathfrak{X}/S} \simeq \Omega^2_{\mathfrak{X}/S}$ on $\mathfrak{X}$.

We denote by $\mathrm{Fil}^i H^n_{\mathrm{dR}}(\mathfrak{X}/A)$ the i^{th}-step in the Hodge filtration of $H^n_{\mathrm{dR}}(\mathfrak{X}/A)$, and write $\mathrm{gr}^i H^n_{\mathrm{dR}}(\mathfrak{X}/A)$ for the subquotients of the filtration, i.e.

$$\mathrm{gr}^i H^n_{\mathrm{dR}}(\mathfrak{X}/A) := \frac{\mathrm{Fil}^i H^n_{\mathrm{dR}}(\mathfrak{X}/A)}{\mathrm{Fil}^{i+1} H^n_{\mathrm{dR}}(\mathfrak{X}/A)}$$

Consider the Gauss-Manin connection

$$\nabla : H^n_{\mathrm{dR}}(\mathfrak{X}/A) \to H^n_{\mathrm{dR}}(\mathfrak{X}/A) \otimes_A \Omega^1_{A/W}$$

Then Griffiths transversality states that

$$\nabla \left(\mathrm{Fil}^i H^n_{\mathrm{dR}}(\mathfrak{X}/A)\right) \subseteq \mathrm{Fil}^{i-1} H^n_{\mathrm{dR}}(\mathfrak{X}/A) \otimes_A \Omega^1_{A/W}$$

and hence ∇ induces an A-linear Higgs field:-

$$\mathrm{gr}^i \nabla : \mathrm{gr}^i H^n_{\mathrm{dR}}(\mathfrak{X}/A) \to \mathrm{gr}^{i-1} H^n_{\mathrm{dR}}(\mathfrak{X}/A) \otimes_A \Omega^1_{A/W}$$

Taking duals and using local freeness, we get an A-linear map

$$\mathcal{T}_{\mathfrak{S}/W} \to \mathrm{Hom}_A \left(\mathrm{gr}^i H^n_{\mathrm{dR}}(\mathfrak{X}/A), \mathrm{gr}^{i-1} H^n_{\mathrm{dR}}(\mathfrak{X}/A)\right)$$

In particular, putting $n = i = 3$ gives

$$\mathcal{T}_{\mathfrak{S}/W} \to \mathrm{Hom}_A\left(H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}), H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/\mathfrak{S}})\right) \tag{3}$$

***Proposition* 5.3.1.** *The map (3) is an isomorphism.*

Proof. This is essentially the derivative of the period map. It is well-known (e.g. [Voi02], 10.4) that the map (3) is the composition of the cup-product with the Kodaira-Spencer map:-

$$\mathrm{KS}(\mathfrak{X}/\mathfrak{S}) : \mathcal{T}_{\mathfrak{S}/W} \to H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/\mathfrak{S}})$$

Since the formal deformation is certainly versal, $\mathrm{Kod}(\mathfrak{X}/\mathfrak{S})$ is an isomorphism. Thus we are in the following situation

$$\begin{tikzcd}
\mathcal{T}_{\mathfrak{S}/W} \arrow[r, "\mathrm{KS}(\mathfrak{X}/\mathfrak{S})", "\sim"'] \arrow[dr] & H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/\mathfrak{S}}) \arrow[d] \\
& \mathrm{Hom}_A\left(H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}), H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/\mathfrak{S}})\right)
\end{tikzcd}$$

where the right-hand arrow is given by cup-product. But choosing a generator $\omega \in H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and using that cup-product is a perfect pairing shows that this is an isomorphism. □

Composing the map $\mathrm{gr}^3\nabla$ with the natural maps

$$\partial/\partial t_i : \Omega^1_{A/W} \to A \qquad (i = 1, \dots, r)$$

gives maps

$$\nabla_i : H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/\mathfrak{S}})$$

and Proposition 5.3.1 shows that $\nabla_1(\omega), \dots, \nabla_r(\omega)$ is an A-basis for $H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/\mathfrak{S}})$.

We will now describe the deformation theory of X_0 over nilpotent PD-thickenings. Let $\alpha : (S, \mathfrak{m}_S) \twoheadrightarrow (R, \mathfrak{m}_R)$ be a surjection in $\underline{\mathcal{A}rt}_{W,k}$ with $\mathfrak{a}\mathfrak{m}_S = 0$, where $\mathfrak{a} = \ker \alpha$ (a so-called *'small surjection'*). Let X be a lifting of X_0/k to R and write

$$\mathsf{Def}_X : \underline{\mathcal{A}rt}_{W,k} \to \underline{\mathcal{S}ets}$$

for the deformation functor sending $S \in \underline{\mathcal{A}rt}_{W,k}$ to isomorphism classes of deformations Y over S with $Y \times_S R \cong X$. We shall first describe $\mathsf{Def}_X(S)$ in terms of liftings of the Hodge filtration, and then in §5.4 we will reinterpret this in terms of displays.

For now, endow α with the trivial PD-structure. and let X', Y be deformations of X/R over Spec S. Then, since $\mathfrak{X} \to \mathfrak{S} =$ Spf A is the universal family, X' and Y are given by uniquely determined W-algebra homomorphisms $f, g : A \to S$ respectively, and we have a commutative diagram:-

$$A \underset{g}{\overset{f}{\rightrightarrows}} S \xrightarrow{\alpha} R$$

Given $u \in H^3_{\mathsf{dR}}(X'/S)$, let $\tilde{u} \in H^3_{\mathsf{dR}}(\mathfrak{X}/A)$ be such that $f_*(\tilde{u}) = u$, and set $v = g_*(\tilde{u})$. Then the Gauss-Manin connection induces an isomorphism

$$\begin{aligned} \Psi : H^3_{\mathsf{dR}}(X'/S) &\xrightarrow{\sim} H^3_{\mathsf{dR}}(Y/S) \\ u &\mapsto v + \sum_{i=1}^{h} (f(t_i) - g(t_i)) \breve{\nabla}_i(\tilde{u}) \end{aligned} \tag{4}$$

where $\breve{\nabla}_i(\tilde{u})$ denotes the image of $\nabla_i(\tilde{u})$ in $H^3_{\mathsf{dR}}(X_0/k)$ (see [Del81]).

Write $F' = \mathsf{Fil}^3 H^3_{\mathsf{dR}}(X'/S)$. Let $F \subset H^3_{\mathsf{dR}}(X'/S)$ be a *"lifting of the Hodge filtration"* on X, i.e. a direct summand lifting $\mathsf{Fil}^3 H^3_{\mathsf{dR}}(X/R)$. Since the canonical map

$$F \to H^3_{\mathsf{dR}}(X'/S)/F'$$

factors through $F \to \mathsf{Fil}^3 H^3_{\mathsf{dR}}(X_0/k) = H^0(X_0, \Omega^3_{X_0/k})$ and has image in

$$\mathfrak{a}\left(H^3_{\mathsf{dR}}(X'/S)/F'\right) \cong \mathfrak{a} \otimes_k \left(H^3_{\mathsf{dR}}(X_0/k)/\mathsf{Fil}^3 H^3_{\mathsf{dR}}(X_0/k)\right)$$

we see that liftings of the Hodge filtration are classified by k-linear maps

$$\varkappa(F) : H^0(X_0, \Omega^3_{X_0/k}) \to \mathfrak{a} \otimes_k \left(H^3_{\mathsf{dR}}(X_0/k)/\mathsf{Fil}^3 H^3_{\mathsf{dR}}(X_0/k)\right)$$

***Proposition* 5.3.2.** *We have that* $\Psi^{-1}(\mathsf{Fil}^3 H^3_{\mathsf{dR}}(Y/S)) \subseteq \mathsf{Fil}^2 H^3_{\mathsf{dR}}(X'/S)$.

Proof. Set $F_Y := \Psi^{-1}(\mathsf{Fil}^3 H^3_{\mathsf{dR}}(Y/S)$. By the above discussion, it is equivalent to show that $\varkappa(F_Y)$ has image contained in $\mathfrak{a} \otimes_k \mathsf{gr}^2 H^3_{\mathsf{dR}}(X_0/k)$. We will do this using the explicit expression given in the isomorphism (4):-

As above, let $u \in H^3_{\mathsf{dR}}(X'/S)$, choose a generator $\tilde{u} \in \mathsf{Fil}^3 H^3_{\mathsf{dR}}(\mathfrak{X}/A)$ such that $f_*(\tilde{u}) = u$, and write u_0 for the image of $\tilde{u}$ in $H^0(X_0, \Omega^3_{X_0/k})$. Then the isomorphism (4) shows that

$$u - \sum_{i=1}^{r} (f(t_i) - g(t_i)) \breve{\nabla}_i(\tilde{u})$$

is a generator of F_Y. Then ([Del81], 1.1.2) says that the map $\varkappa(F_Y)$ on u_0 is given by

$$\varkappa(F_Y)(u_0) = -\sum_{i=1}^{r} (f(t_i) - g(t_i)) \otimes \breve{\nabla}_i(\tilde{u}) \in \mathfrak{a} \otimes_k \mathsf{gr}^2 H^3_{\mathsf{dR}}(X_0/k)$$

since the $\breve{\nabla}_i(\tilde{u})$ are in $H^1(X_0, \mathcal{T}_{X_0/k}) = \mathsf{gr}^2 H^3_{\mathsf{dR}}(X_0/k)$. $\square$

***Proposition* 5.3.3.** *In the above situation, we have an identification*

$$\begin{array}{rcl} \mathsf{Def}_X(S) & = & \left\{ \textit{lines } E \subseteq \mathsf{Fil}^2 H^3_{\mathit{dR}}(X'/S) \textit{ lifting } \mathsf{Fil}^3 H^3_{\mathit{dR}}(X/R) \right\} \\ Y & \mapsto & F_Y := \Psi^{-1}(\mathsf{Fil}^3 H^3_{\mathit{dR}}(Y/S)) \end{array}$$

Proof. Proposition 5.3.2 exactly tells us that F_Y is in the right-hand side. We provide an inverse map:-

As was already noted, Proposition 5.3.1 shows that the $\breve{\nabla}_i(\tilde{u})$ form a basis of

$\mathrm{gr}^2 H^3_{\mathrm{dR}}(X_0/k)$. Therefore, F_Y determines $a_i \in \mathfrak{a}$ ($i = 1, \ldots, r$) such that

$$u - \sum_{i=1}^{r} a_i \breve{\nabla}_i(\tilde{u})$$

generates F_Y. Then the desired Calabi-Yau threefold Y is given by pulling back the universal family along the W-algebra homomorphism

$$g : A \to S$$
$$t_i \mapsto f(t_i) - a_i$$

$\square$

We now extend Proposition 5.3.3 to arbitrary nilpotent PD-thickenings:-

***Theorem* 5.3.4.** *Let X_0/k be a Calabi-Yau threefold and let $\alpha : S \twoheadrightarrow R$ be a surjection in $\underline{Art}_{W,k}$ such that $\mathfrak{a} = \ker \alpha$ is endowed with a nilpotent PD-structure which is compatible with the canonical PD-structure on pW.*

Let X be a lifting of X_0/k to R, and let X' be a deformation of X/R over Spec S. Then if Y is a deformation of X/R over S, the Gauss-Manin connection induces an isomorphism

$$\Psi : H^3_{dR}(X'/S) \xrightarrow{\sim} H^3_{dR}(Y/S)$$

We have an identification

$$\begin{aligned} \mathrm{Def}_X(S) &= \left\{ \textit{lines } E \subseteq \mathrm{Fil}^2 H^3_{dR}(X'/S) \textit{ lifting } \mathrm{Fil}^3 H^3_{dR}(X/R) \right\} \\ Y \quad &\mapsto \quad F_Y := \Psi^{-1}(\mathrm{Fil}^3 H^3_{dR}(Y/S)) \end{aligned}$$

Proof. In the usual way, we first reduce the statement by decomposing α into a series “smaller” PD-thickenings:-

Write $\mathfrak{a}^{[n]}$ for the n^{th}-divided power ideal of $\mathfrak{a}$ and let t be such that

$$0 = \mathfrak{a}^{[t]} \subset \mathfrak{a}^{[t-1]} \subset \cdots \subset \mathfrak{a}^{[2]} \subset \mathfrak{a}$$

Then we can write α as the composition of two nilpotent PD-thickenings:-

$$\alpha : S \twoheadrightarrow R_1 := S/\mathfrak{a}^{[t-1]} \twoheadrightarrow R$$

Notice that, by induction, we can assume the theorem holds for the second PD-thickening (the base case $t = 2$ is exactly Proposition 5.3.3).

Now, given a line $E \subseteq \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X'/S)$ lifting $\mathrm{Fil}^3 H^3_{\mathrm{dR}}(X/R)$ we get an induced $E_{R_1} \subseteq \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X' \times_S R_1/R_1)$. But since the theorem holds for $R_1 \twoheadrightarrow R$, there corresponds to E_{R_1} a deformation Z of X/R over R_1. Choose some deformation Z' of Z/R_1 over R'. Then both X' and Z' lift X to Spec S, so we have an isomorphism

$$H^3_{\mathrm{dR}}(X'/R') \xrightarrow{\sim} H^3_{\mathrm{dR}}(Z'/R')$$

The image of E under this isomorphism is a line in $\mathrm{Fil}^2 H^3_{\mathrm{dR}}(Z'/S)$ lifting $\mathrm{Fil}^3 H^3_{\mathrm{dR}}(Z/R_1)$. In this way we are reduced to proving the theorem for the nilpotent PD-thickening

$$S \twoheadrightarrow R_1 = S/\mathfrak{a}^{[t-1]}$$

But the divided powers on $\mathfrak{b} = \ker(S \twoheadrightarrow R_1) = \mathfrak{a}^{[t-1]}$ are trivial, and $\mathfrak{b}^2 = 0$. Then we can decompose again into $S \twoheadrightarrow R_m \twoheadrightarrow \cdots \twoheadrightarrow R_1$ and conclude surjectivity by Proposition 5.3.3. It should also be noted that Proposition 5.3.3 at each step also gives injectivity. $\square$

5.4 Deformation theory of Calabi-Yau threefolds and 3-displays

As above, $\alpha : S \twoheadrightarrow R$ denotes a nilpotent PD-thickening of artinian local $W(k)$-algebras with residue field k, and we write $\mathfrak{a} = \ker \alpha$. We choose a Calabi-Yau threefold X_0/k with smooth universal deformation space $\mathfrak{S} = \text{Spf } A$ and universal family $\mathfrak{X}/\mathfrak{S}$. As before, $X/\text{Spec } R$ and $X'/\text{Spec } S$ are deformations.

***Lemma* 5.4.1.** *Let $E^3 = \langle e \rangle \subseteq \text{Fil}^2 H^3_{\text{dR}}(X'/S)$ be a line lifting $\text{Fil}^3 H^3_{\text{dR}}(X/R)$, and set*

$$E^1 := \left(E^3\right)^{\perp}$$
$$E^2 := E^3 \oplus \text{Span}(\nabla_1(e), \ldots, \nabla_h(e))$$

where ∇ is the connection on $H^3_{\text{dR}}(X'/S)$ induced by the Gauss-Manin connection on $H^3_{\text{dR}}(\mathfrak{X}/A)$. Then we have $E^2 \subseteq E^1$ and $\left(E^2\right)^{\perp} = E^2$.

Proof. We first show the result in the special case where $\mathfrak{a}^2 = 0$. Let $f \in E^2$, to show the inclusion $E^2 \subseteq E^1 = \left(E^3\right)^{\perp}$ we must show that $\langle e, f \rangle = 0$. We write

$$f = ce + \sum_{j=1}^{h} s_j \nabla_j(e) \qquad c, s_j \in S$$

and see then that

$$\langle e, f \rangle = c\langle e, e \rangle + \sum_{j=1}^{h} s_j \langle e, \nabla_j(e) \rangle = \sum_{j=1}^{h} s_j \langle e, \nabla_j(e) \rangle$$

Since $e \in E^3 \subseteq \text{Fil}^2 H^3_{\text{dR}}(X'/S)$ lifts a generator $\tilde{e}$ of $\text{Fil}^3 H^3_{\text{dR}}(X/R)$, we can write

$$e = d\tilde{e} + \sum_{i=1}^{h} a_i \nabla_i(\tilde{e}) \qquad d \in S \text{ and } a_i \in \mathfrak{a}$$

and hence

$$\nabla_j(e) = d\nabla_j(\tilde{e}) + \sum_{i=1}^{h} a_i \nabla_{ji}(\tilde{e}) \qquad \text{where } \nabla_{ji} := \nabla_j \circ \nabla_i$$

A simple calculation shows that

$$\langle e, \nabla_j(e) \rangle = \sum_{i=1}^{h} a_i \left\langle \nabla_i(\tilde{e}), \sum_{k=1}^{h} a_k \nabla_{jk}(\tilde{e}) \right\rangle$$

which vanishes since $\mathfrak{a}^2 = 0$. We conclude that $E^2 \subseteq E^1$ in this case. The calculation showing $\left(E^2\right)^{\perp} = E^2$ is similar.

Now we treat the general case. Write $\mathfrak{a}^{[n]}$ for the n^{th}-divided power ideal of $\mathfrak{a}$ and let t be such that

$$0 = \mathfrak{a}^{[t]} \subseteq \mathfrak{a}^{[t-1]} \subseteq \cdots \subseteq \mathfrak{a}^{[2]} \subseteq \mathfrak{a}$$

Consider the following factorisation of α into a composition of PD-thickenings:-

$$\alpha : S \xrightarrow{\beta} R_1 := S/\mathfrak{a}^{[t-1]} \xrightarrow{\gamma} R$$

If $t = 2$ then we recover the special case of $\mathfrak{a}^2 = 0$ treated above, and by induction we can assume that the proposition holds for $\gamma : R_1 \twoheadrightarrow R$.

Now let $E^3 \subseteq \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X'/S)$ be a line lifting $\mathrm{Fil}^3 H^3_{\mathrm{dR}}(X/R)$. This induces a line $E^3_{R_1} \subseteq \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X' \times_S R_1/R_1)$, which corresponds to a deformation $Z \to \mathrm{Spec}\ R_1$ of $X \to \mathrm{Spec}\ R$. Let Z' be an arbitrary lifting of Z to $\mathrm{Spec}\ S$. Since X' and Z' are both liftings of X, the Gauss-Manin connection on $H^3_{\mathrm{dR}}(\mathfrak{X}/A)$ induces an isomorphism

$$H^3_{\mathrm{dR}}(X'/S) \xrightarrow{\sim} H^3_{\mathrm{dR}}(Z'/S).$$

Write G^3 for the image of E^3 under this isomorphism. Then $G^3 \subseteq \mathrm{Fil}^2 H^3_{\mathrm{dR}}(Z'/S)$ is a line lifting $E^3_{R_1} = \mathrm{Fil}^3 H^3_{\mathrm{dR}}(Z/R_1)$. So we are reduced to proving the lemma for the PD-thickening $\beta : S \twoheadrightarrow R_1$. But here we have $\mathfrak{b}^2 = 0$ where $\mathfrak{b} := \ker \beta$, so we are done by the special case. □

Since X/R is a deformation of the Calabi-Yau threefold X_0 with smooth universal deformation, it satisfies the conditions of the first section. In particular, there is a crystal of relative displays $S \mapsto \mathscr{D}_{S/R}(X)$.

Definition. *Consider a lifting $E^\bullet$ of Fil$^\bullet \mathscr{D}_{S/R}(X) =$ Fil$^\bullet H^3_{dR}(X/R)$*

$$E^3 \subseteq E^2 \subseteq E^1 \subseteq \tilde{P}_0/I_S\tilde{P}_0 = H^3_{cris}(X/S) = H^3_{dR}(X'/S)$$

and write ∇ for the Gauss-Manin connection on $\tilde{P}_0/I_S\tilde{P}_0$ (it is induced by the connection on $\tilde{P}_0$, see Proposition 4.1.1). We say that $E^\bullet$ is of CY-type if the following three conditions hold:-

(i) $E^3 = \langle e \rangle \subseteq$ Fil$^2 H^3_{dR}(X'/S)$ is a line lifting Fil$^3 H^3_{dR}(X/R)$

(ii) $E^1 = \left(E^3\right)^\perp$

(iii) $E^2 = E^3 \oplus$ Span$(\nabla_1(e), \ldots, \nabla_h(e))$

We may now rephrase Theorem 5.3.4 in terms of liftings of the Hodge filtration of the associated relative $\mathcal{W}_{S/R}$-display:-

***Theorem* 5.4.2.** *There is a bijection*

$$\mathrm{Def}_X(S) \xrightarrow{1:1} \left\{\textit{CY-type liftings of Fil}^\bullet \mathscr{D}_{S/R}(X)\right\}$$

Proof. There is a bijection

$$\left\{\begin{matrix}\text{direct summands } E \subseteq \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X'/S) \\ \text{lifting } \mathrm{Fil}^3 H^3_{\mathrm{dR}}(X/R)\end{matrix}\right\} \xrightarrow{\textit{1:1}} \left\{\text{CY-type liftings of } \mathrm{Fil}^\bullet \mathscr{D}_{S/R}(X)\right\}$$

given by the obvious map:-

$$E = \langle e \rangle \mapsto 0 \subseteq E \subseteq E \oplus \mathrm{Span}(\nabla_1(e), \ldots, \nabla_h(e)) \subseteq E^\perp \subseteq \tilde{P}_0/I_S\tilde{P}_0 = H^3_{\mathrm{dR}}(X'/S)$$

Notice that this is well-defined by Lemma 5.4.1. It is clearly a bijection since the line determines the filtration and vice versa. We conclude by Theorem 5.3.4. □

Definition. *We call an n-$\mathcal{W}_S$-display structure $\mathscr{D}$ on $H^n_{\text{cris}}(X/W(S))$* geometric *if $\mathscr{D} = \mathscr{D}_S(X')$ for some smooth deformation X'/S of X.*

***Theorem* 5.4.3.** *We have a bijection*

$$\{\textit{geometric } \mathcal{W}_S\textit{-displays on } H^3_{\text{cris}}(X/W(S))\} \xrightarrow{1:1} \{\textit{CY-type liftings of Fil}^\bullet \mathscr{D}_{S/R}(X)\}$$

Proof. Let $\mathscr{D}$ be a geometric $\mathcal{W}_S$-display, i.e. $\mathscr{D} = \mathscr{D}_S(Y)$ for some smooth deformation Y of X. Write

$$\tilde{u}_\bullet : \{\mathcal{W}_S\text{-displays}\} \to \{\text{extended } \mathcal{W}_{S/R}\text{-displays}\}$$

for the functor induced by $u : \mathcal{W}_S \to \mathcal{W}_{S/R}$, and denote by $\mathscr{D}^{\text{ext}}_{S/R}(X)$ the extended $\mathcal{W}_{S/R}$-display associated to $\mathscr{D}_{S/R}(X)$. Then:-

$$\begin{aligned}\text{Hom}_{\mathcal{W}_S}(\mathscr{D}, \tilde{u}^\bullet \mathscr{D}^{\text{ext}}_{S/R}(X)) &= \text{Hom}_{\mathcal{W}_S}(\mathscr{D}_S(Y), \tilde{u}^\bullet \mathscr{D}^{\text{ext}}_{S/R}(X)) \\ &\cong \text{Hom}_{\mathcal{W}_{S/R}}(\tilde{u}_\bullet \mathscr{D}_S(Y), \mathscr{D}^{\text{ext}}_{S/R}(X)) \\ &= \text{Hom}_{\mathcal{W}_{S/R}}(\mathscr{D}^{\text{ext}}_{S/R}(X), \mathscr{D}^{\text{ext}}_{S/R}(X))\end{aligned}$$

where $\tilde{u}^\bullet \mathscr{D}^{\text{ext}}_{S/R}(X)$ is just $\mathscr{D}^{\text{ext}}_{S/R}(X)$ viewed as a $\mathcal{W}_S$-predisplay.

The canonical map

$$\mathscr{D} \to \tilde{u}^\bullet \mathscr{D}^{\text{ext}}_{S/R}(X) = \tilde{u}^\bullet \tilde{u}_\bullet \mathscr{D}$$

coming from the bijection gives $P_i \hookrightarrow \tilde{P}_i$, and hence a lifting of $\text{Fil}^\bullet \mathscr{D}^{\text{ext}}_{S/R}(X)$, i.e. an admissible lifting of $\text{Fil}^\bullet \mathscr{D}_{S/R}(X)$. Moreover, this lifting came from the deformation Y, so one sees that it is of CY-type. This gives a map

$$\{\text{geometric } \mathcal{W}_S\text{-displays on } H^3_{\text{cris}}(X/W(S))\} \xrightarrow{\xi} \{\text{CY-type liftings of Fil}^\bullet \mathscr{D}_{S/R}(X)\}$$

In order to prove that ξ is a bijection, we shall construct its inverse:-

By ([LZ15], Corollary 11) there is an equivalence of categories

$$\begin{array}{ccc} \{\mathcal{W}_S\text{-displays}\} & \xrightarrow{\sim} & \left\{\begin{array}{c}\mathcal{W}_{S/R}\text{-displays together with an} \\ \text{admissible lifting of the Hodge filtration}\end{array}\right\} \\ \mathscr{D} & \longmapsto & (u_\bullet \mathscr{D}, \mathsf{Fil}^\bullet \mathscr{D}) \\ \tilde{\mathscr{D}}_{E^\bullet} & \longleftarrow\!\shortmid & (\tilde{\mathscr{D}}, E^\bullet) \end{array}$$

Since CY-type liftings of $\mathsf{Fil}^\bullet \mathscr{D}_{S/R}(X)$ are admissible, the above functor induces a map

$$\begin{array}{ccc} \{\text{CY-type liftings of } \mathsf{Fil}^\bullet \mathscr{D}_{S/R}(X)\} & \to & \{\mathcal{W}_S\text{-displays on } H^3_{\mathsf{cris}}(X/W(S))\} \\ E^\bullet & \mapsto & \mathscr{D}_{S/R}(X)_{E^\bullet} \end{array}$$

First we show that the image of this map is geometric. Recall from Theorem 5.3.4 that there is a bijection

$$\begin{array}{ccc} \beta : \left\{\begin{array}{c}\text{isomorphism classes of} \\ \text{deformations of } X/R \text{ over } S\end{array}\right\} & \xrightarrow{1:1} & \left\{\begin{array}{c}\text{direct summands } E \subset \mathsf{Fil}^2 H^3_{\mathsf{dR}}(X'/S) \\ \text{lifting } \mathsf{Fil}^3 H^3_{\mathsf{dR}}(X/R)\end{array}\right\} \\ [Y] & \longmapsto & \beta([Y]) := \Psi^{-1}(\mathsf{Fil}^3 H^3_{\mathsf{dR}}(Y/S)) \end{array}$$

where $\Psi : H^3_{\mathsf{dR}}(X'/S) \xrightarrow{\sim} H^3_{\mathsf{dR}}(Y/S)$ is the "parallel transport" isomorphism coming from the Gauss-Manin connection. The preceding lemma then yields a one-to-one correspondence between isomorphism classes of deformations of X/R to Spec S and CY-type liftings of $\mathsf{Fil}^\bullet \mathscr{D}_{S/R}(X)$.

Given a CY-type lifting $E^\bullet$ of $\mathsf{Fil}^\bullet \mathscr{D}_{S/R}(X)$, corresponding to the isomorphism class $[Y]$, we claim that $\mathscr{D}_{S/R}(X)_{E^\bullet} = \mathscr{D}_S(Y)$. Indeed, by the construction of the bijection β, we have that $E^\bullet = \mathsf{Fil}^\bullet H^3_{\mathsf{dR}}(Y/S) = \mathsf{Fil}^\bullet \mathscr{D}_S(Y)$. Then the equivalence of categories gives

$$\mathscr{D}_{S/R}(X)_{E^\bullet} = (u_\bullet \mathscr{D}_S(Y))_{\mathsf{Fil}^\bullet \mathscr{D}_S(Y)} = \mathscr{D}_S(Y)$$

So we have two maps:-

$$\left\{\begin{array}{c}\text{CY-type liftings}\\ \text{of } \mathsf{Fil}^\bullet\mathscr{D}_{S/R}(X)\end{array}\right\} \underset{\xi}{\overset{\eta}{\rightleftarrows}} \left\{\begin{array}{c}\text{geometric } \mathcal{W}_S\text{-displays}\\ \text{on } H^3_{\mathsf{cris}}(X/W(S))\end{array}\right\}$$

That these are inverses follows easily from the equivalence of categories:-

$$\begin{aligned}\xi \circ \eta(\mathscr{D}_S(Y)) &= \eta(\mathsf{Fil}^\bullet\mathscr{D}_S(Y))\\ &= \mathscr{D}_{S/R}(X)_{\mathsf{Fil}^\bullet\mathscr{D}_S(Y)}\\ &= (u_\bullet\mathscr{D}_S(Y))_{\mathsf{Fil}^\bullet\mathscr{D}_S(Y)}\\ &= \mathscr{D}_S(Y)\end{aligned}$$

$$\begin{aligned}\eta \circ \xi(E^\bullet) &= \xi(\mathscr{D}_{S/R}(X)_{E^\bullet})\\ &= \mathsf{Fil}^\bullet(\mathscr{D}_{S/R}(X)_{E^\bullet})\\ &= E^\bullet\end{aligned}$$

□

Write $\underline{\mathcal{CY}3}_S$ for the category whose objects are Calabi-Yau threefolds $X' \to \operatorname{Spec} S$, and whose morphisms are isomorphisms. Write $\underline{\mathcal{DCY}3}_{S/R}$ for the category whose objects are pairs $(X, E^\bullet)$ where $X \to \operatorname{Spec} R$ is a Calabi-Yau threefold and $E^\bullet$ is a CY-type lifting of $\mathsf{Fil}^\bullet\mathscr{D}_{S/R}(X)$. The morphisms are isomorphisms with the obvious compatibilities.

Let $X' \in \underline{\mathcal{CY}3}_S$ and write X'_R for its reduction over $\operatorname{Spec} R$. The proof of Theorem 5.4.3 showed that $\mathsf{Fil}^\bullet\mathscr{D}_S(X')$ is a CY-type lifting of $\mathsf{Fil}^\bullet\mathscr{D}_{S/R}(X'_R)$. By the compatibility of crystalline cohomology with base-change, we get a functor

$$\underline{\mathcal{CY}3}_S \to \underline{\mathcal{DCY}3}_{S/R} \ ; \ X' \mapsto (X'_R, \mathsf{Fil}^\bullet\mathscr{D}_S(X'))$$

We can then summarise the results of this section as the following:-

***Theorem* 5.4.4.** *The above functor is an equivalence of categories.*

6 Deformation theory of smooth cubic fourfolds

6.1 A Grothendieck-Messing type result for smooth cubic fourfolds

We consider a smooth cubic hypersurface $X_0 \subset \mathbb{P}^5_k$, where k is a perfect field of characteristic $p > 0$. Then

$$H^0(X_0, \mathcal{T}_{X_0/k}) = H^2(X_0, \mathcal{T}_{X_0/k}) = 0$$

and

$$\dim_k H^1(X_0, \mathcal{T}_{X_0/k}) = 20$$

(e.g. by [Lev01], §3), so there exists a formally smooth universal family

$$\mathfrak{X} \to \mathfrak{S} = \mathsf{Spf}\ A$$

with $A = W(k)[\![t_1, \ldots, t_{20}]\!]$. Just as in Proposition 5.3.1, the derivative of the period map gives an isomorphism

$$\mathcal{T}_{\mathfrak{S}/W} \xrightarrow{\sim} H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/\mathfrak{S}}) \xrightarrow{\sim} \mathsf{Hom}_A\left(H^1(\mathfrak{X}, \Omega^3_{\mathfrak{X}/\mathfrak{S}}), H^2(\mathfrak{X}, \Omega^2_{\mathfrak{X}/\mathfrak{S}})_{\mathsf{prim}}\right)$$

and we conclude as before that $\{\nabla_1(\omega), \ldots, \nabla_{20}(\omega)\}$ is an A-basis of $H^2(\mathfrak{X}, \Omega^2_{\mathfrak{X}/\mathfrak{S}})_{\mathsf{prim}}$, where ω is a generator of the rank one A-module $H^1(\mathfrak{X}, \Omega^3_{\mathfrak{X}/\mathfrak{S}})$.

Now let $X/\mathsf{Spec}\ R$ be a smooth cubic fourfold deforming X_0 over $R \in \underline{Art}_{W,k}$. Then the middle de Rham cohomology has Hodge numbers

$$h^{4,0} = h^{0,4} = 0, \quad h^{3,1} = h^{1,3} = 1, \quad h^{2,2} = 21 \tag{4}$$

Fix a lifting X' of X over a nilpotent PD-thickening Spec R $\hookrightarrow$ Spec S. Then one mimics the argument of §5.3 to prove the following Grothendieck-Messing type result:-

***Theorem* 6.1.1.** *There is an identification*

$$\begin{aligned}\mathsf{Def}_X(S) &= \left\{\textit{lines } E \subseteq \mathsf{Fil}^2 P^4_{\textit{dR}}(X'/S) \textit{ lifting } \mathsf{Fil}^3 P^4_{\textit{dR}}(X/R)\right\} \\ Y \quad \mapsto &\quad F_Y := \Psi^{-1}(\mathsf{Fil}^3 P^4_{\textit{dR}}(Y/S))\end{aligned}$$

Notice that the cup-product on $H^4_{\mathsf{dR}}(\mathfrak{X}/\mathfrak{S})$ induces a self-dual structure on the primitive cohomology $P^4_{\mathsf{dR}}(\mathfrak{X}/\mathfrak{S})$. Using this one may re-write Theorem 6.1.1:-

***Theorem* 6.1.2.** *One has*

$$\begin{aligned}\mathsf{Def}_X(S) &= \left\{\textit{isotropic lines } E \subset P^4_{\textit{dR}}(X'/S) \textit{ lifting Fil}^3 P^4_{\textit{dR}}(X/R)\right\} \\ &= \left\{\textit{self-dual filtrations of } P^4_{\textit{dR}}(X'/S) \textit{ lifting Fil}^\bullet P^4_{\textit{dR}}(X/R)\right\} \\ &= \left\{\textit{self-dual filtrations of } P^4_{\textit{cris}}(X/S) \textit{ lifting Fil}^\bullet P^4_{\textit{cris}}(X/R)\right\}\end{aligned}$$

Proof. The first equality is because a line E lifting $\mathsf{Fil}^3 P^4_{\mathsf{dR}}(X/R)$ is isotropic in $P^4_{dR}(X'/S)$ if and only if E is contained in $\mathsf{Fil}^2 P^4_{dR}(X'/S)$; this follows easily by writing a generator of E in terms of the basis given by the Gauss-Manin connection. The second equality is because a self-dual filtration with Hodge numbers as in (4) is completely determined by its Fil^3, via $\mathsf{Fil}^2 = (\mathsf{Fil}^3)^\perp$. The final equality is the comparison isomorphism between crystalline cohomology and the de Rham cohomology of a lifting:-

$$P^4_{\mathsf{cris}}(X/S) \cong P^4_{\mathsf{cris}}(X'/S) \cong P^4_{\mathsf{dR}}(X'/S)$$

□

***Remark*.** One should note that Theorem 6.1.2 is explicitly stated as part of ([MP15], Theorem 6.14) in the special case $R = k$, $S = k[\epsilon]/(\epsilon^2)$, which cites ([Lev01],§3) for the origin of the statement.

Write $\mathscr{D}^{\mathsf{prim}}_R(X)$ for the $\mathscr{W}_R$-display structure on the primitive cohomology $P^4_{\mathsf{cris}}(X/W(R))$, as constructed in Corollary 3.4.2. Since X is a smooth projective hypersurface with smooth universal deformation, it satisfies the conditions of Proposition 4.0.1 and has a crystal of relative displays $S \mapsto \mathscr{D}^{\mathrm{prim}}_{S/R}(X)$, where $\mathscr{D}^{\mathrm{prim}}_{S/R}(X) := u_\bullet \mathscr{D}^{\mathrm{prim}}_{S}(X')$ for a (any) smooth deformation X'/S of X.

Let $\underline{\mathcal{S}m\mathcal{C}ub4}_S$ denote the category whose objects are smooth cubic fourfolds over Spec S, and whose morphisms are isomorphisms. Write $\underline{\mathcal{DS}m\mathcal{C}ub4}_{S/R}$ for the category whose objects are pairs $(X, E^\bullet)$, where X/R is a smooth cubic fourfold and $E^\bullet$ is a lifting of $\mathrm{Fil}^\bullet \mathscr{D}^{\mathrm{prim}}_{S/R}(X)$ such that $(E^3)^\perp = E^2$. Then copying the proof of Theorem 5.4.3, this time invoking Theorem 6.1.2, gives the following analogue of Theorem 5.4.4:-

***Theorem* 6.1.3.** *The functor*

$$\underline{\mathcal{S}m\mathcal{C}ub4}_S \to \underline{\mathcal{DS}m\mathcal{C}ub4}_{S/R} \ \ ; \ \ X' \mapsto (X'_R, \mathit{Fil}^\bullet \mathscr{D}^{\mathit{prim}}_{S}(X'))$$

is an equivalence of categories.

6.2 Fano scheme of lines and 2-displays

Now consider a smooth cubic fourfold X_0 over an algebraically closed field k of characteristic $p \geq 5$. Following [AK77], we will recall below the definition of the Fano scheme of lines $F(X_0)$ of X_0.

Let V be a six-dimensional k-vector space, so $X_0 = Z(f) \subset \mathbb{P}(V) = \mathbb{P}^5_k$ for some global section $f \in H^0(\mathbb{P}^5_k, \mathcal{O}(3)) = \mathrm{Sym}^3 V^\vee$. Write $G := \mathrm{Grass}(2, V)$ for the Grassmannian of lines in $\mathbb{P}^5_k$, and write

$$0 \to \mathcal{S} \to \mathcal{O}_G \otimes V \to \mathcal{Q} \to 0$$

for the tautological sequence on G. One sees then that $\mathcal{O}_G \otimes \mathrm{Sym}^3 V^\vee \twoheadrightarrow \mathrm{Sym}^3 \mathcal{S}^\vee$. In particular, f induces a global section $s_f \in H^0(G, \mathrm{Sym}^3 \mathcal{S}^\vee)$.

Definition. *The* Fano scheme of lines *of X_0 is defined to be the zero locus in G*

of s_f:-

$$F(X_0) := Z(s_f)$$
$$= \{[L] \in G : L \subset X_0\}$$

Under the Plücker embedding $G \hookrightarrow \mathbb{P}^{14}_k$, ([AK77], 1.3, 1.12) shows that $F(X_0)$ is a smooth projective fourfold. In [BD85] it is shown that $F(X_0)$ is a scheme of K3-type, and in particular has a Beauville-Bogomolov pairing (see [LZ15],§3).

There is also the notion of *relative* Fano scheme of lines, which will be useful to us when considering deformation theory over bases which are not fields. The construction proceeds in an analogous fashion using relative Grassmannians.

Consider the incidence correspondence

$$\begin{tikzcd} & I(X_0) = \{(x,[L]) \in X_0 \times F(X_0) : x \in L\} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ F(X_0) & & X_0 \end{tikzcd}$$

***Proposition* 6.2.1.** *Let $K :=$ Frac $W(k)$. The incidence correspondence induces an isomorphism of self-dual F-isocrystals:-*

$$a_K := a \otimes_W K : H^4_{\mathit{cris}}(X_0/K) \xrightarrow{\sim} H^2_{\mathit{cris}}(F(X_0)/K)(1)$$

where $a := q_* p^*$.

Proof. We need to check that the morphism of F-isocrystals

$$a_K : H^4_{\text{cris}}(X_0/K) \to H^2_{\text{cris}}(F(X_0)/K)$$

is in fact an isomporphism and respects the pairings given by cup-product on the source and by the Beauville-Bogomolov form (see ([LZ15], §3)) on the target. This is the p-adic analogue of the Beauville-Donagi isomorphism (see [BD85]) on

ℓ-adic étale cohomology $a : H^4_{\text{ét}}(X_0, \mathbb{Q}_\ell)(1) \xrightarrow{\sim} H^2_{\text{ét}}(F(X_0), \mathbb{Q}_\ell)$. For the reader's convenience we will repeat their proof in this setting, and will largely recycle their notation, in particular we will refrain from writing the intersection pairing each time.

First lift X_0 to a smooth cubic fourfold $X/W(k)$. By the Berthelot-Ogus comparison isomorphism, we can, and do, replace crystalline cohomology with de Rham cohomology of the lifting. The map $q : I(X) \to F(X)$ is a $\mathbb{P}^1$-bundle, so the projective bundle formula for de Rham cohomology over K gives

$$H^*_{\text{dR}}(I(X)/K) \cong \frac{q^* H^*_{\text{dR}}(F(X)/K)[p^*H]}{(p^*H^2 = q^*\sigma_1 p^*H + p^*\sigma_2)}$$

where H is a hyperplane section of X. As such, for $\xi \in H^4_{\text{dR}}(X/K)$ we can write

$$p^*\xi = q^*v_1(\xi)p^*H + q^*v_2(\xi) \in H^4_{\text{dR}}(I(X)/K) \tag{5}$$

for some $v_1(\xi) \in H^2_{\text{dR}}(F(X)/K)$ and $v_2(\xi) \in H^4_{\text{dR}}(F(X)/K)$, which we will compute. Firstly, the above equation gives

$$p^*H^2 = q^*v_1(H^2)p^*H + q^*v_2(H^2) \tag{6}$$

Projecting (5) down to $F(X)$, we see that

$$a_K(\xi) = v_1(\xi)a_K(H) = v_1(\xi) \tag{7}$$

Now suppose that $\xi \in P^4_{\text{dR}}(X/K)$, i.e. $\xi H = 0$. Then on the incidence variety we have

$$
\begin{aligned}
0 &= p^*(\xi H) \\
&= p^*\xi p^* H \\
&\overset{(5)}{=} q^* v_1(\xi) p^* H^2 + q^* v_2(\xi) p^* H \\
&\overset{(6)}{=} q^* v_1(\xi)(q^* v_1(H^2) p^* H + q^* v_2(H^2)) + q^* v_2(\xi) p^* H \\
&\overset{(7)}{=} q^*(a_K(\xi) a_K(H^2) + v_2(\xi)) p^* H + q^*(a_K(\xi) v_2(H^2))
\end{aligned}
$$

We then conclude that $v_2(\xi) = -a_K(\xi)a_K(H^2)$, and $a_K(\xi)v_2(H^2) = 0$ in $H^4_{\mathsf{dR}}(F(X)/K)$. From the first equality we see in particular that $v_2(H^2) = -a_K(H^2)^2$, and therefore a_K sends $P^4_{\mathsf{dR}}(X/K)$ to $P^2_{\mathsf{dR}}(F(X)/K)$, where the latter is the orthogonal complement of $a_K(H^2)^2$.

Now that we know $v_2(\xi)$ for $\xi \in P^4_{\mathsf{dR}}(X/K)$, equation (5) can be rewritten as

$$
p^*\xi = q^* a_K(\xi) p^* H - q^*(a_K(\xi) a_K(H^2)) \tag{8}
$$

from which we see that

$$
\begin{aligned}
p^*\xi^2 &= q^* a_K(\xi)^2 p^* H^2 - 2q^*(a_K(\xi)^2 a_K(H^2)) p^* H + q^*(a_K(\xi)^2 a_K(H^2)^2) \\
&\overset{(8)}{=} q^* a_K(\xi)^2 (q^* a_K(H^2) p^* H - q^* a_K(H^2)^2) - 2q^*(a_K(\xi)^2 a_K(H^2)) p^* H \\
&\quad + q^*(a_K(\xi)^2 a_K(H^2)^2) \\
&= -q^*(a_K(\xi)^2 a_K(H^2)) p^* H
\end{aligned}
$$

in $H^8_{\mathsf{dR}}(I(X)/K)$, and therefore

$$
p^*\xi^2 q^* a_K(H^2) = -q^*(a_K(\xi)^2 a_K(H^2)^2) p^* H \in H^{10}_{\mathsf{dR}}(I(X)/K) \cong K
$$

(The isomorphism is because $I(X)$ has dimension five, since $F(X)$ is a fourfold and the fibres of q are $\mathbb{P}^1$'s).

A (general) line in X intersects the hyperplane H at a point, so p^*H is a section of q. We are then left to compute the number of lines through a general point $x \in X$ which pass through a codimension 2 linear space S, that is the degree of $q^*\Sigma_1 \xrightarrow{p} X$ where $\Sigma_1 \subset F(X)$ denotes the space of lines passing through x. In other words, the number of lines through a general point x in a general hyperplane section $X \cap \langle x, S\rangle$. But a general hyperplane section of a smooth cubic fourfold is a smooth cubic threefold, and it is classical that there are six line passing through a general point on a smooth cubic threefold. Therefore

$$\xi^2 = -6^{-1} a_K(\xi^2) a_K(H^2)^2$$

This pairing known to be a (a scalar multiple of) the Beauville-Bogomolov pairing ([BD85], Prop. 6). □

***Proposition* 6.2.2.** *Suppose that X_0/k is an ordinary smooth cubic fourfold and that its Fano scheme of lines $F(X_0)$ is also ordinary. Then the isomorphism a_K of Proposition 6.2.1 extends to an isomorphism*

$$a : H^4_{\mathit{cris}}(X_0/W) \xrightarrow{\sim} H^2_{\mathit{cris}}(F(X_0)/W)(1)$$

of self-dual F-crystals. Moreoever, a is an isomorphism of self-dual filtered F-crystals when we consider the source and target with their Hodge filtrations.

Proof. By the Dieudonné-Manin decomposition (see [Man62]), $H^4_{\text{cris}}(X_0/W)$ and $H^2_{\text{cris}}(F(X_0)/W)(1)$ both admit canonical largest direct summands on which F is an isomorphism (the slope zero parts). We denote these summands by $\text{Fil}_0^{\text{slope}} H^4_{\text{cris}}(X_0/W)$ and $\text{Fil}_0^{\text{slope}} H^2_{\text{cris}}(F(X_0)/W)(1)$ respectively. They are rank one W-modules. Define increasing 3-step filtration on $H^4_{\text{cris}}(X_0/W)$ and $H^2_{\text{cris}}(F(X_0)/W)(1)$ by setting

$$\begin{aligned}\text{Fil}_1^{\text{slope}} H^4_{\text{cris}}(X_0/W) &:= \text{Fil}_0^{\text{slope}} H^4_{\text{cris}}(X_0/W)^{\perp} \\ \text{Fil}_2^{\text{slope}} H^4_{\text{cris}}(X_0/W) &:= H^4_{\text{cris}}(X_0/W)\end{aligned}$$

and

$$\begin{aligned}
\mathsf{Fil}_1^{\mathsf{slope}} H^2_{\mathsf{cris}}(F(X_0)/W)(1) &:= \mathsf{Fil}_0^{\mathsf{slope}} H^2_{\mathsf{cris}}(F(X_0)/W)(1)^{\perp} \\
\mathsf{Fil}_2^{\mathsf{slope}} H^2_{\mathsf{cris}}(F(X_0)/W)(1) &:= H^2_{\mathsf{cris}}(F(X_0)/W)(1)
\end{aligned}$$

where in the first definition we take orthogonal complement with respect to cup-product, and in the second we use the Beauville-Bogomolov pairing. These filtrations are the slope filtrations. In particular, a_K respects the filtrations since it is F-equivariant.

By ordinarity (see [Del81],§1.3), we have splittings

$$\begin{aligned}
H^4_{\mathsf{cris}}(X_0/W) =& \mathsf{Fil}^2 H^4_{\mathsf{cris}}(X_0/W) \\
& \oplus (\mathsf{Fil}^2 H^4_{\mathsf{cris}}(X_0/W) \oplus \mathsf{Fil}_0^{\mathsf{slope}} H^4_{\mathsf{cris}}(X_0/W))^{\perp} \\
& \oplus \mathsf{Fil}_0^{\mathsf{slope}} H^4_{\mathsf{cris}}(X_0/W)
\end{aligned}$$

and

$$\begin{aligned}
H^2_{\mathsf{cris}}(F(X_0)/W)(1) =& \mathsf{Fil}^2 H^2_{\mathsf{cris}}(F(X_0)/W)(1) \\
& \oplus (\mathsf{Fil}^2 H^2_{\mathsf{cris}}(F(X_0)/W)(1) \oplus \mathsf{Fil}_0^{\mathsf{slope}} H^2_{\mathsf{cris}}(F(X_0)/W))(1)^{\perp} \\
& \oplus \mathsf{Fil}_0^{\mathsf{slope}} H^2_{\mathsf{cris}}(F(X_0)/W)(1)
\end{aligned}$$

Since a_K respects slope zero parts, it respects these splittings and therefore a sends $H^4_{\mathsf{cris}}(X_0/W)$ to $H^2_{\mathsf{cris}}(F(X_0)/W)(1)$. To see that

$$a : \mathsf{Fil}^i H^4_{\mathsf{cris}}(X_0/W) = \mathsf{Fil}^i H^4_{\mathsf{cris}}(X_0/W) \xrightarrow{\sim} \mathsf{Fil}^{i-1} H^2_{\mathsf{cris}}(F(X_0)/W)$$

note that for ordinary F-crystals H we have

$$H = \mathsf{Fil}_i^{\mathsf{slope}} \oplus \mathsf{Fil}^{i+1}$$

(see ([Del81], Prop. 1.3.6)). □

***Remark*.** *Proposition 6.2.2 only used that the F-crystals $H^4_{\textit{cris}}(X_0/W)$ and $H^2_{\textit{cris}}(F(X_0)/W)$ were ordinary; it suffices to work with this weaker assumption for the rest of the discussion. The author is wary to suggest that $F(X_0)$ is an ordinary variety whenever X_0 is, since one has no control over $F(X_0)$ in cohomological degree 4. On the other hand, it does seem plausible that $H^2_{\textit{cris}}(F(X_0)/W)$ is an ordinary F-crystal whenever $H^4_{\textit{cris}}(X_0/W)$ is.*

We now recall that for a perfect field k, a $\mathscr{W}_k$-display $\mathscr{D} = (P_i, F_i)$ is just the same as the data (P_0, F_0). Indeed, by definition P_0 is a projective $W(k)$-module, and $W(k)$ is a principal ideal domain, so P_0 is free. Then, by the elementary divisor theorem, one always has a decomposition into a direct sum of $W(k)$-modules

$$P_0 = \bigoplus_{i=0}^{d} L_i$$

such that

$$F_0^{-1} P_0 = \bigoplus_{i=0}^{d} p^{-i} L_i \subset P_0 \otimes \mathbb{Q}$$

and then the data $(L_i, p^{-i} F_0|_{L_i})$ is a standard datum for $\mathscr{D}$.

***Corollary* 6.2.3.** *Suppose that X_0/k is an ordinary smooth cubic fourfold and that its Fano scheme of lines $F(X_0)$ is also ordinary. Then there is an isomorphism of self-dual $\mathscr{W}_k$-displays*

$$(\mathscr{D}_k(X_0), \lambda_{cup}) \xrightarrow{\sim} (\mathscr{D}_k(F(X_0))(1), \lambda_{BB})$$

where $\mathscr{D}_k(X_0)$ is the 4-display on $H^4_{\textit{cris}}(X_0/W)$ with the pairing induced by cup-product, and $\mathscr{D}_k(F(X_0))$ is the 2-display on $H^2_{\textit{cris}}(F(X_0)/W)$ with the Beauville-Bogomolov pairing.

Proof. This is an immediate consequence of the preceding discussion and Proposition 6.2.2 □

***Theorem* 6.2.4.** *Let $X_0 \subset \mathbb{P}^5_k$ be an ordinary smooth cubic fourfold over an algebraically closed field k of characteristic ≥ 5, and suppose that its Fano scheme of lines $F(X_0)$ is also ordinary. Then for R a local artinian ring with residue field k, we have an injection*

$$\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{deformations of } X_0\\ \textit{over Spec } R\end{array}\right\} \hookrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{deformations of } (\mathscr{D}_k(F(X_0)), \lambda_{BB})\\ \textit{to self-dual } \hat{\mathscr{W}}_R\textit{-}2\textit{-displays}\end{array}\right\}$$

Proof. The map is given by sending a deformation X/R to the self-dual $\hat{W}_R$-2-display $(\hat{\mathscr{D}}_R(F(X), \lambda_{BB})$ of its Fano scheme of lines. To see that this is an injection, we first note that the set of classes of self-dual $\hat{\mathscr{W}}_R$-2-displays deforming the self-dual $\mathscr{W}_k$-2-display $(\mathscr{D}_k(F(X_0), \lambda_{BB})$ is in bijection with the deformation classes of the K3-type scheme $F(X_0)$; this is ([LZ15], Theorem 36). It then suffices to show that the map $X/R \mapsto F(X)/R$ is injective, in other words that one can recover a smooth cubic fourfold from its Fano scheme of lines, up to deformation equivalence. This is possible, see ([Cha12], Prop. 4). □

***Corollary* 6.2.5.** *In the setting of the theorem, there is an injection*

$$\begin{array}{ccc}\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{deformations of } X_0\\ \textit{over Spec } R\end{array}\right\} & \hookrightarrow & \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{deformations of } (\mathscr{D}_k(X_0), \lambda_{cup})\\ \textit{to self-dual } \hat{\mathscr{W}}_R\textit{-}4\textit{-displays}\end{array}\right\}\\ X/R & \longmapsto & (\hat{\mathscr{D}}_R(X_0), \lambda_{cup})\end{array}$$

References

[AK77] A. Altman, S. Kleiman, *Foundations of the theory of Fano schemes*, Compositio Mathematica **34**, 3-47 (1977)

[AM77] M. Artin, B. Mazur, *Formal groups arising from algebraic varieties*, Ann. sci. de E.N.S., Vol. 4 **10**, 87-131 (1977)

[Ber74] P. Berthelot, *Cohomologie Cristalline des Schémas de Caractéristique* $p > 0$, Lecture Notes in Math., **407**, Springer-Verlag, (1974)

[BBM82] P. Berthelot, L. Breen, W. Messing, *Théorie Dieudonné cristalline II*, Lecture Notes in Math., **930**, Springer-Verlag, (1982)

[BD85] A. Beauville, R. Donagi, *La variété des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris. Sér. I Math. Vol. 14 **301**, 703-706 (1985)

[BO78] P. Berthelot, A. Ogus, *Notes on crystalline cohomology*, Princeton University Press (1978)

[Bou83] N. Bourbaki, *Algèbre commutative*, Éléments de mathématique, Chap. 8 & 9. Masson, Paris (1983)

[Cha12] F. Charles, *A remark on the Torelli theorem for cubic fourfolds*, preprint, arXiv:1209.4509, (2012)

[Del73] P. Deligne, *Cohomologie des intersectiones completes*, in: SGA 7, Groupes de Monodromie en Géométrie Algébrique, Lecture Notes in Math., **340**, Springer-Verlag, (1973)

[Del81] P. Deligne, *Cristaux ordinaires et coordonnées canoniques*, Lecture Notes in Math., **868**, Algebraic Surfaces, 80-137, Springer (1981)

[DI87] P. Deligne, L. Illusie, *Relevements modulo* p^2 *et decomposition du complexe de de Rham*, Invent. Math., Vol. 89 **2**, 247-270 (1987)

[Eke04] T. Ekedahl, *On non-liftable Calabi-Yau threefolds*, preprint, math.AG/0306435, (2004)

[ESB05] T. Ekedahl, N.I. Shepherd-Barron, *Tangent lifting of deformations in mixed characteristic*, J. Algebra, Vol. 291 **1**, 108-128 (2005)

[Fal89] G. Faltings, *Crystalline cohomology and p-adic Galois representations*, Algebraic analysis, geometry, and number theory (Baltimore MD, 1988), 25-80, Johns Hopkins Univ. Press (1989)

[Fon83] J.-M. Fontaine, *Cohomologie de de Rham, cohomologie cristalline et représentations p-adiques*, Lecture Notes in Math. **1016**, Algebraic Geometry (Tokyo/Kyoto, 1982) 86-108, Springer (1983)

[GL16] O. Gregory, A. Langer, *Higher displays arising from filtered de Rham-Witt complexes*, preprint (2016)

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I.*, Inst. Hautes Études Sci. Publ. Math., (11): 167, 1961.

[Hir99] M, Hirokada, *A non-liftable Calabi-Yau threefold in characteristic 3*, Tohoku. Math. J., Vol. 51 **4**, 479-487 (1999)

[Ill90] L. Illusie, *Ordinarité des intersections complètes générales*, The Grothendieck Festschrift, Vol. II, Progr. Math. **87**, Birkhäuser, 376-405 (1990)

[dJ95] A. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Pub. Mathématiques de I.H.É.S **82**, 5-96 (1995)

[Jos07] K. Joshi, *Exotic torsion, Frobenius splitting and the slope spectral sequence*, Canad. Math. Bull., Vol. 50, 567-578 (2007)

[KvdG00] T. Katsura, G. van der Geer, *On a Stratification of the Moduli of K3 Surfaces*, J. Eur. Math. Soc, Vol. 3, 259-290 (2000)

[Kat79] N. Katz, *Slope filtration of F-crystals*, Asterisque **63**, 113-164 (1979)

[Lan83] W. Lang, *On Enriques surfaces in characteristic p. I*, Math. Ann., **265**, 45-65 (1983)

[Lan16] A. Langer, *p-adic Deformation of motivic Chow groups*, preprint (2016)

[Lev01] N. Levin, *The Tate Conjecture for Cubic Fourfolds over a Finite Field*, Compositio Mathematica **127**, 1-21 (2001)

[LZ04] A. Langer and T. Zink, *De Rham-Witt cohomology for a proper and smooth morphism*, J. Inst. Math. Jussieu 3, 231-314 (2004)

[LZ07] A. Langer and T. Zink, *De Rham-Witt cohomology and displays*, Documenta Mathematica **12**, 147-191 (2007)

[LZ15] A. Langer, T. Zink, *Grothendieck-Messing deformation theory for varieties of $K3$-type*, preprint (2015)

[Lau08] E. Lau, *Displays and formal p-divisible groups*, Invent. Math. **171**, 617-628 (2008)

[Lau14] E. Lau, *Relations between Dieudonné displays and crystalline Dieudonné theory*, Algebra Number Theory 8, 2201-2262 (2014)

[MP15] K. Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math., Vol. 201 **2**, 625-668 (2015)

[Man62] J. Manin, *On the classification of formal Abelian groups*, Dokl. Akad. Nauk SSSR, **144**, 490-492 (1962)

[Mum70] D. Mumford, *Abelian varieties*, Published for the Tate Institute of Fundamental Research, Oxford University Press (1970)

[Ogu77] A. Ogus, *F-crystals and Griffiths transversality*, International Symposium on Algebraic Geometry, Kyoto, 15-44 (1977)

[RS76] A. Rudakov, J. Shafarevich, *Inseparable morphisms of algebraic surfaces*, Akad. Sc. SSSR. **40**, 1264-1307 (1976)

[Sri90] V. Srinivas, *Decomposition of the de Rham complex*, Proc. Indian. Acad. Sci. (Math. Sci.), Vol. 100 **2**, 103-106 (1990)

[Stacks] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu` (2015)

[Voi02] C. Voisin, *Hodge Theory and Complex Algebraic Geometry, I*, Cambridge Studies in Advanced Mathematics 76, Cambridge University Press (2002)

[Wid12] M. Wid, *Displays and Gauges*, Regensburg Univ. Dissertation, (2012)

[Zin01] T. Zink, *A Dieudonne theory for p-divisible groups*, in: Class field theory - its centenary and prospect, Adv. Stud. Pure Math. **30**, Math. Soc. Japan, 139-160 (2001)

[Zin02] T. Zink, *The display of a formal p-divisible group*, Astérisque **278**, 127-248 (2002)