Electromagnetic interactions in a pair of coupled split-ring resonators: Supplementary material

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The Green's functions for studying electromagnetic interactions in free space have been discussed in detail in many past works. In this supplementary section, we derive the Green's functions of a rectangular waveguide. In the Lorentz gauge, the scalar and the vector potentials of the electromagnetic field satisfy the Helmholtz equation

$$\begin{aligned} \left[\nabla^2 + k_0^2\right]\phi &= -\frac{\rho(\mathbf{r})}{\epsilon_0} \\ \left[\nabla^2 + k_0^2\right]\mathbf{A} &= -\mu_0 \mathbf{J}(\mathbf{r}) \end{aligned} \tag{1}$$

The corresponding free-space scalar and tensor Green's functions are

$$\begin{aligned} g(\mathbf{r},\mathbf{r}') &= \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \\ \underline{\underline{G}}(\mathbf{r},\mathbf{r}') &= \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\mathbf{I} \end{aligned} \tag{2}$$

where the boundary is set at ∞, $\mathbf{I}$ is the unit tensor and $k_0 = \frac{\omega}{c}$. Since the wave is confined inside a rectangular waveguide in our experiment, we want to deform the boundary from infinity to the walls of the rectangular waveguide. Let us set the origin at one of the corners of the waveguide's cross-section. The radiation is confined to $0 < x < a$, $0 < y < b$ and is infinitely extended in the z direction. The walls of the waveguide are modelled as Perfect Electric Conductors (PEC), so the continuity of the tangential component of the electric field across the boundary may be written as

$$\mathbf{n} \times \mathbf{E} = 0 \qquad \text{on the boundaries} \tag{3}$$

In terms of the vector potential in the Lorentz gauge

$$\mathbf{n} \times (k_0^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A})) = 0 \qquad \text{on the boundaries} \tag{4}$$

As a result of the isotropy of the dielectric and the rectangular geometry of the waveguide, the Green's tensor will be diagonal in Cartesian coordinates. Hence (4) leads to the following boundary conditions at the waveguide walls

$$\begin{aligned} G_{xx}\ |_{y=0,b} &= 0 \\ \frac{\partial}{\partial x}G_{xx}\ |_{x=0,a} &= 0 \\ G_{yy}\ |_{x=0,a} &= 0 \\ \frac{\partial}{\partial y}G_{yy}\ |_{y=0,b} &= 0 \\ G_{zz}\ |_{x=0,a} &= 0 \\ G_{zz}\ |_{y=0,b} &= 0 \\ g\ |_{x=0,a} &= 0 \\ g\ |_{y=0,b} &= 0 \end{aligned} \tag{5}$$

Such boundary value problems admit unique solutions. Therefore, since g satisfies the same Helmholtz equation and the same boundary conditions as G_{zz}, they will be the same, $g = G_{zz}$.

Applying these boundary conditions to the Helmholtz equation of the Green's tensor

$$\left[\nabla^2 + k_0^2\right] G_{r_i r_i}(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{6}$$

we can readily obtain the final Green's tensor components as

$$\begin{aligned} G_{xx} &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} i\frac{\epsilon_m\epsilon_n}{2abk_I}\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\cdot \\ &\quad \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{ik_I|z-z'|} \\ G_{yy} &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} i\frac{\epsilon_m\epsilon_n}{2abk_I}\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)\cdot \\ &\quad \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{ik_I|z-z'|} \\ G_{zz} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} i\frac{\epsilon_m\epsilon_n}{2abk_I}\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\cdot \\ &\quad \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{ik_I|z-z'|} \end{aligned} \tag{7}$$

where

$$\epsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m > 0 \end{cases} \tag{8}$$

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and k_I is the propagating component of the wave-vector given by

$$k_I^2 = k_0^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \tag{9}$$

and m and n are non-negative integers that describe the different modes that could be accommodated inside the waveguide.