ÉTALE FUNDAMENTAL GROUPS OF AFFINOID p-ADIC CURVES

Mohamed Saïdi

In memory of Si M'hamed, my father.

Abstract. We prove that the *geometric* étale fundamental group of a (geometrically connected) *rigid smooth p-adic affinoid curve* is a *semi-direct factor* of a certain profinite *free* group. We prove that the maximal pro-p (resp. maximal prime-to-p) quotient of this geometric étale fundamental group is pro-p *free of infinite rank* (resp. (pro-)prime-to-p *free of finite computable rank*).

Contents

§0. Introduction/Main Results. A classical result in the theory of étale fundamental groups is the description of the structure of the geometric étale fundamental group of an affine, smooth, and geometrically connected curve over a field of characteristic 0 (cf. [Grothendieck], Exposé XIII, Corollaire 2.12). In this paper we investigate the structure of the geometric étale fundamental group of a smooth affinoid p-adic curve.

Let R be a complete discrete valuation ring, $K \overset{\text{def}}{=} \mathrm{Fr}(R)$ the quotient field of R, and k its residue field which is algebraically closed of characteristic $p \geq 0$. Let X_K be a smooth, proper, and geometrically connected rigid K-curve, $\mathcal{U} \hookrightarrow X_K$ a K-*affinoid* rigid subspace with $\mathcal{U}$ geometrically connected and $X_K \setminus \mathcal{U}$ is the disjoint union of K-rigid open unit discs $\{\mathcal{D}_i^o\}_{i=1}^m$ with centres $\{x_i\}_{i=1}^m$, $x_i \in X_K(K)$ (cf. §3 for more details, as well as Theorem 3.1 which asserts that any K-affinoid smooth curve can be embedded, after possibly a finite extension of K, into a proper and smooth rigid K-curve whose complement is as above).

Let $S \subset \mathcal{U}$ be a (possibly empty) finite set of points and $T \subset \bigcup_{i=1}^m \mathcal{D}_i^o$ a finite set of points of X_K. (We also denote, when there is no risk of confusion, by X_K the projective, smooth, and geometrically connected algebraic K-curve associated to the rigid curve X_K via the rigid GAGA functor.) We have an exact sequence of étale fundamental groups

$$1 \to \pi_1(X_K \setminus (T \cup S))^{\mathrm{geo}} \to \pi_1(X_K \setminus (T \cup S)) \to \mathrm{Gal}(\overline{K}/K) \to 1,$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where $\pi_1(X_K \setminus (T \cup S))$ is the arithmetic fundamental group of the (affine) curve $X_K \setminus (T \cup S)$, and by passing to the projective limit over all finite sets of points $T \subset \bigcup_{i=1}^m \mathcal{D}_i^o$ we obtain an exact sequence

$$1 \to \varprojlim_T \pi_1(X_K \setminus (T \cup S))^{\mathrm{geo}} \to \varprojlim_T \pi_1(X_K \setminus (T \cup S)) \to \mathrm{Gal}(\overline{K}/K) \to 1.$$

The profinite group $\varprojlim_T \pi_1(X_K \setminus (T \cup S))^{\mathrm{geo}}$ is *free* if $\mathrm{char}(K) = 0$ as follows from the well-known structure of the geometric étale fundamental groups of (affine) curves in characteristic zero (cf. loc. cit.). Write $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo}}$ for the geometric étale fundamental group (in the sense of Grothendieck) of $\mathcal{U} \setminus S$ (cf. 2.1 for a precise Definition). One of our main results is the following (cf. Theorem 3.4, Proposition 3.5, and Theorem 3.7).

Theorem A. *Assume* $\mathrm{char}(K) = 0$ *and* $\mathrm{char}(k) = p > 0$. *Let ℓ be a prime integer (possibly equal to p). Then the morphism $\mathcal{U} \to X_K$ induces (via the rigid GAGA functor) a continuous homomorphism $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo}} \to \varprojlim_T \pi_1(X_K \setminus (T \cup S))^{\mathrm{geo}}$ (resp. $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo},\ell} \to \varprojlim_T \pi_1(X_K \setminus (T \cup S))^{\mathrm{geo},\ell}$ between the maximal pro-ℓ quotients) which makes $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo}}$ (resp. $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo},\ell}$) into a semi-direct factor of $\varprojlim_T \pi_1(X_K \setminus (T \cup S))^{\mathrm{geo}}$ (resp. a direct factor of $\varprojlim_T \pi_1(X_K \setminus (T \cup S))^{\mathrm{geo},\ell}$) (cf. Definitions 1.2 and 1.4 for the meaning of the terms direct factor and semi-direct factor). Moreover, the pro-ℓ group $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo},\ell}$ is free of infinite rank if $\ell = p$, and of finite computable rank if $\ell \neq p$.*

In Theorem 3.3 we prove an analog of Theorem A in equal characteristic $p > 0$ in which the infinite set of points consisting of all T as above is replaced by the finite set $\{x_i\}_{i=1}^m$. Further, we prove the following (cf. Theorem 3.7) which, in case $\mathrm{char}(k) = 0$, gives a description of the structure of (the full) $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo}}$ in the equal characteristic 0 case.

Theorem B. *Assume* $\mathrm{char}(k) = p \geq 0$ *with no restriction on* $\mathrm{char}(K)$. *Then the morphism $\mathcal{U} \to X_K$ induces (via the rigid GAGA functor) a continuous homomorphism $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo},p'} \to \pi_1(X_K \setminus (\{x_i\}_{i=1}^m \cup S))^{\mathrm{geo},p'}$ between the maximal prime-to-p quotients of $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo}}$ and $\pi_1(X_K \setminus (\{x_i\}_{i=1}^m \cup S))^{\mathrm{geo}}$; respectively, which is an isomorphism. In particular, if $S(\overline{K}) = \{y_1, \cdots, y_r\}$ has cardinality $r \geq 0$ then $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo},p'}$ is (pro-)prime-to-p free on $2g + m + r - 1$ generators and can be generated by $2g + m + r$ generators $\{a_1, \cdots, a_g, b_1, \cdots, b_g, \sigma_1, \cdots, \sigma_m, \tau_1, \cdots, \tau_r\}$ subject to the unique relation $\prod_{j=1}^g [a_j, b_j] \prod_{i=1}^m \sigma_i \prod_{t=1}^r \tau_t = 1$, where σ_i (resp τ_t) is a generator of inertia at x_i (resp. y_t) and $g \overset{\mathrm{def}}{=} g_{X_K}$ is the arithmetic genus of X_K (also called the genus of the affinoid $\mathcal{U}$).*

Note that unless $\mathrm{char}(k) = 0$ the profinite group $\pi_1(\mathcal{U} \setminus S)^{\mathrm{geo}}$ is *not free* (neither is it finitely generated) as the ranks of its maximal pro-ℓ quotients can be different for different primes ℓ (cf. Theorem A, and its analog Theorem 3.3 in equal characteristic $p > 0$). In this sense Theorem A (and Theorem 3.3) is an optimal result one can prove regarding the structure of the *full* geometric fundamental group of a p-adic smooth affinoid curve. Also there is no analog to Theorem A if $\mathrm{char}(k) = p > 0$, for the full π_1^{geo}, where one replaces the infinite union of the finite sets of points T (as in the statement of Theorem A) by a single fixed finite set of points $\widetilde{T} \subset \bigcup_{i=1}^m \mathcal{D}_i^o$ (cf. Remark 3.8(ii)).

Next, we outline the content of the paper. In §1 we collect some well-known background material. In §2 we explain how one defines the étale fundamental group of a rigid analytic K-affinoid space (cf. 2.1), and recall the rigid analog of Runge's theorem proven by Raynaud (cf. 2.2). We then investigate in 2.3 the structure of a certain quotient of the geometric étale fundamental group of an annulus of thickness 0. In §3 we investigate the structure of the geometric étale fundamental group of a smooth affinoid p-adic curve and prove Theorem A (as well as its analog Theorem 3.3 if $\mathrm{char}(K) = p > 0$), and Theorem B.

In [Garuti] Garuti investigated, among others, the structure of the pro-p geometric fundamental group of a rigid closed p-adic annulus of thickness 0 and proved an analogue of Theorem A in this special case.

Acknowledgment. I would like to thank the referee for his/her careful reading of the paper and the many suggestions which helped improve the presentation of the paper.

Notations. In this paper K is a complete discrete valuation ring, R its valuation ring, π a uniformising parameter, and $k \overset{\text{def}}{=} R/\pi R$ the residue field of characteristic $p \geq 0$ which we assume to be algebraically closed.

We refer to [Raynaud], 3, for the terminology we will use concerning K-rigid analytic spaces, R-formal schemes, as well as the link between formal and rigid geometry. For an R-(formal) scheme X we will denote by $X_K \overset{\text{def}}{=} X \times_R K$ (resp. $X_k \overset{\text{def}}{=} X \times_R k$) the generic (resp. special) fibre of X (the generic fibre is understood in the rigid analytic sense in the case where X is a formal scheme). Moreover, if $X = \mathrm{Spf}\, A$ is an affine formal R-scheme of finite type we denote by $X_K \overset{\text{def}}{=} \mathrm{Sp}(A \otimes_R K)$ the associated K-rigid affinoid space and will also denote, when there is no risk of confusion, by X_K the affine scheme $X_K \overset{\text{def}}{=} \mathrm{Spec}(A \otimes_R K)$.

A formal (resp. algebraic) R-curve is an R-formal scheme of finite type (resp. scheme of finite type) flat and separated whose special fibre is equidimensional of dimension 1. For a K-scheme (resp. K-rigid analytic space) X and L/K a field extension (resp. a finite extension) we write $X_L \overset{\text{def}}{=} X \times_K L$ which is an L-scheme (resp. an L-rigid analytic space). If X is a proper and normal formal R-curve we also denote, when there is no risk of confusion, by X the algebraisation of X which is an algebraic normal and proper R-curve and by X_K the proper and normal algebraic K-curve associated to the rigid K-curve X_K via the rigid GAGA functor.

For a profinite group H and a prime integer ℓ we denote by H^ℓ the maximal pro-ℓ quotient of H, and $H^{\ell'}$ the maximal prime-to-ℓ quotient of H.

All scheme cohomology groups $H^1_{\text{et}}(\ ,\mathbb{Z}/\ell\mathbb{Z})$ in this paper are étale cohomology groups.

§1 Background.

1.1. Let $p > 1$ be a prime integer. We recall some well-known facts on profinite pro-p groups. First, we recall the following characterisations of free pro-p groups.

Proposition 1.1. *Let G be a profinite pro-p group. Then the following properties are equivalent.*
(i) G is a free pro-p group.

(ii) The p-cohomological dimension of G satisfies $\mathrm{cd}_p(G) \leq 1$.
In particular, a closed subgroup of a free pro-p group is free.

Proof. Well-known (cf. [Serre], and [Ribes-Zalesskii], Theorem 7.7.4). $\square$

Next, we recall the notion of a *direct factor* of a free pro-p group (cf. [Garuti], 1, the discussion preceding Proposition 1.8).

Definition/Lemma 1.2 (Direct factors of free pro-p groups). *Let F be a free pro-p group, $H \subseteq F$ a closed subgroup, and $\iota : H \to F$ the natural homomorphism. We say that H is a direct factor of F if there exists a continuous homomorphism $s : F \to H$ such that $s \circ \iota = \mathrm{id}_H$ (s is necessarily surjective). There exists then a (non unique) closed subgroup N of F such that F is isomorphic to the free direct product $H \star N$. We will refer to such a subgroup N as a supplement of H.*

Proof. In what follows we consider $\mathbb{Z}/p\mathbb{Z}$ as the trivial discrete module. Let $s : F \to H$ be a left inverse to ι as above which induces a retraction $h^1(s) : H^1(H, \mathbb{Z}/p\mathbb{Z}) \to H^1(F, \mathbb{Z}/p\mathbb{Z})$ of the map $h^1(\iota) : H^1(F, \mathbb{Z}/p\mathbb{Z}) \to H^1(H, \mathbb{Z}/p\mathbb{Z})$ induced by ι; in particular $h^1(s)$ is injective. Let M be a supplement of (the image via $h^1(s)$ of) $H^1(H, \mathbb{Z}/p\mathbb{Z})$ in $H^1(F, \mathbb{Z}/p\mathbb{Z})$ and $M^\wedge$ the corresponding subgroup of $F/F^\star$ where $F^\star$ is the Frattini subgroup of F (recall that $F/F^\star$ is the Pontrjagin dual of $H^1(F, \mathbb{Z}/p\mathbb{Z})$). Let $\{\tilde{g}_i\}_i$ be a minimal set of generators of $M^\wedge$ and $\{g_i\}_i \subset F$ a lift of the $\{\tilde{g}_i\}_i$. Write N for the closed subgroup of F generated by the $\{g_i\}_i$ (which is free pro-p) and $H \star N$ the free product of H and N. Then the natural morphism $H \star N \to F$ is an isomorphism as it is an isomorphism on the cohomology with coefficients in $\mathbb{Z}/p\mathbb{Z}$ (cf. [Ribes-Zalesskii], Proposition 7.7.2). $\square$

Note that in the discussion before Proposition 1.8 in [Garuti], and with the notations in Definition/Lemma 1.2, it is stated that F is isomorphic to the free product $H \star \mathrm{Ker}(s)$. This is not necessarily the case as the induced map on cohomology is not necessarily an isomorphism as claimed in loc. cit. A similar inaccurate statement occurs in the proof of Proposition 1.8 of loc. cit., but this doesn't affect the validity of this Proposition.

One has the following cohomological characterisation of direct factors of free pro-p groups.

Proposition 1.3. *Let H be a pro-p group, F a free pro-p group, and $\sigma : H \to F$ a continuous homomorphism. Assume that the map induced by σ on cohomology*

$$h^1(\sigma) : H^1(F, \mathbb{Z}/p\mathbb{Z}) \to H^1(H, \mathbb{Z}/p\mathbb{Z})$$

is surjective, where $\mathbb{Z}/p\mathbb{Z}$ is considered as a trivial discrete module. Then σ induces an isomorphism $H \xrightarrow{\sim} \sigma(H)$ and $\sigma(H)$ is a direct factor of F. We say that σ makes H into a direct factor of F.

Proof. cf. [Garuti], Proposition 1.8. $\square$

Next, we consider the notion of a *semi-direct factor* of a profinite group.

Definition 1.4 (Semi-direct factors of profinite groups). Let G be a profinite group, $H \subseteq G$ a closed subgroup, and $\iota : H \to G$ the natural homomorphism. We say that H is a *semi-direct factor* of G if there exists a continuous homomorphism $s : G \to H$ such that $s \circ \iota = \mathrm{id}_H$ (s is necessarily surjective).

Note that G is the semi-direct product of $\mathrm{Ker}\, s$ and H. Unlike the pro-p case, a semi-direct factor of a free profinite group is not necessarily free.

Lemma 1.5. *Let $\tau : H \to G$ be a continuous homomorphism between profinite groups. Write $H = \varprojlim_{j \in J} H_j$ as the projective limit of the inverse system $\{H_j, \phi_{j'j}, J\}$ of finite quotients H_j of H with index set J. Suppose there exists, $\forall j \in J$, a surjective homomorphism $\psi_j : G \twoheadrightarrow H_j$ such that $\psi_j \circ \tau : H \twoheadrightarrow H_j$ is the natural map and $\psi_j = \phi_{j'j} \circ \psi_{j'}$ whenever this makes sense. Then τ induces an isomorphism $H \overset{\sim}{\to} \tau(H)$ and $\tau(H)$ is a semi-direct factor of G. We say that τ makes H into a semi-direct factor of G.*

Proof. Indeed, the $\{\psi_j\}_{j \in J}$ give rise to a continuous (necessarily surjective) homomorphism $\psi : G \to H$ which is a left inverse of τ. $\square$

§**2. Geometric fundamental groups of annuli of thickness zero.** In this section we explain how one defines the étale fundamental group of a rigid analytic K-affinoid space (cf. 2.1), and recall the rigid analog of Runge's Theorem proven by Raynaud (cf. 2.2). We then investigate, in 2.3, the structure of a certain quotient of the geometric étale fundmantal group of an annulus of thickeness 0. The main results in this section are inspired from [Garuti], §2.

2.1. First, we explain how one defines the étale fundamental group of a rigid analytic K-affinoid space. Let $U = \operatorname{Spf} A$ be an affine R-formal scheme which is topologically of finite type. Thus, A is a π-adically complete noetherian R-algebra. Let $\mathcal{A} \overset{\text{def}}{=} A \otimes_R K$ be the corresponding Tate algebra and $\mathcal{U} \overset{\text{def}}{=} \operatorname{Sp} \mathcal{A}$ the associated K-rigid analytic affinoid space which is the generic fibre of U in the sense of Raynaud (cf. [Raynaud], 3). Assume that the affine scheme $\operatorname{Spec} \mathcal{A}$ is (geometrically) normal and geometrically connected. Let η be a geometric point of $\operatorname{Spec} \mathcal{A}$ above its generic point. Then η determines an algebraic closure $\overline{K}$ of K and a geometric point of $\operatorname{Spec}(\mathcal{A} \times_K \overline{K})$ which we will also denote η.

Definition 2.1.1 (Étale Fundamental Groups of Affinoid Spaces). (See also [Garuti], Définition 2.2 and Définition 2.3). We define the *étale fundamental group* of $\mathcal{U}$ with base point η by

$$\pi_1(\mathcal{U}, \eta) \overset{\text{def}}{=} \pi_1(\operatorname{Spec} \mathcal{A}, \eta),$$

where $\pi_1(\operatorname{Spec} \mathcal{A}, \eta)$ is the étale fundamental group of the connected scheme $\operatorname{Spec} \mathcal{A}$ with base point η in the sense of Grothendieck (cf. [Grothendieck], V). Thus, $\pi_1(\mathcal{U}, \eta)$ classifies finite coverings $\operatorname{Spec} \mathcal{B} \to \operatorname{Spec} \mathcal{A}$ where $\mathcal{B}$ is a finite étale $\mathcal{A}$-algebra. There exists a continuous surjective homomorphism $\pi_1(\mathcal{U}, \eta) \twoheadrightarrow \operatorname{Gal}(\overline{K}/K)$. We define the *geometric étale fundamental group* $\pi_1(\mathcal{U}, \eta)^{\text{geo}}$ of $\mathcal{U}$ so that the following sequence is exact

$$1 \to \pi_1(\mathcal{U}, \eta)^{\text{geo}} \to \pi_1(\mathcal{U}, \eta) \to \operatorname{Gal}(\overline{K}/K) \to 1.$$

Remark 2.1.2. If L/K is a finite field extension contained in $\overline{K}/K$, and $\mathcal{U}_L \overset{\text{def}}{=} \mathcal{U} \times_K L$ is the affinoid L-rigid analytic space obtained from $\mathcal{U}$ by extending scalars, then we have a commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\mathcal{U}_L, \eta)^{\text{geo}} & \longrightarrow & \pi_1(\mathcal{U}_L, \eta) & \longrightarrow & \operatorname{Gal}(\overline{K}/L) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{U}, \eta)^{\text{geo}} & \longrightarrow & \pi_1(\mathcal{U}, \eta) & \longrightarrow & \operatorname{Gal}(\overline{K}/K) & \longrightarrow & 1
\end{array}$$

where the two right vertical maps are injective homomorphisms and the left vertical map is an isomorphism. The geometric fundamental group $\pi_1(\mathcal{U},\eta)^{\rm geo}$ is strictly speaking *not* the fundamental group of a rigid analytic space (since $\overline{K}$ is not complete). It is, however, the projective limit of fundamental groups of rigid affinoid spaces. More precisely, there exists an isomorphism

$$\pi_1(\mathcal{U},\eta)^{\rm geo} \xrightarrow{\sim} \varprojlim_{L/K} \pi_1(\mathcal{U}\times_K L,\eta),$$

where the limit is taken over all finite extensions L/K contained in $\overline{K}$.

Similarly, if $\mathcal{U}$ above is a geometrically connected and (geometrically) normal *affinoid* K*-curve*, and S is a finite set of points of $\mathcal{U}$ (cf. [Raynaud], 3.1, for the definition of points of a rigid analytic space), we define the étale fundamental group $\pi_1(\mathcal{U}\setminus S,\eta)$ of $\mathcal{U}\setminus S$ with base point η which is a profinite group and classifies finite coverings $\operatorname{Spec}\mathcal{B}\to\operatorname{Spec}\mathcal{A}$, where $\mathcal{B}$ is a finite $\mathcal{A}$-algebra which is étale above $\mathcal{U}\setminus S$. In this case we have an exact sequence

$$1\to\pi_1(\mathcal{U}\setminus S,\eta)^{\rm geo}\to\pi_1(\mathcal{U}\setminus S,\eta)\to\operatorname{Gal}(\overline{K}/K)\to 1,$$

where $\pi_1(\mathcal{U}\setminus S,\eta)^{\rm geo}\overset{\rm def}{=}\operatorname{Ker}\big(\pi_1(\mathcal{U}\setminus S,\eta)\twoheadrightarrow\operatorname{Gal}(\overline{K}/K)\big)$, and a similar description of $\pi_1(\mathcal{U}\setminus S,\eta)^{\rm geo}$ to that of $\pi_1(\mathcal{U},\eta)^{\rm geo}$ given in Remark 2.1.2.

2.2. Next, we recall the rigid analog of Runge's Theorem proven by Raynaud. Let X_K be a proper, smooth, and geometrically connected algebraic K-curve. We denote by $X_K^{\rm rig}$ the associated K-rigid analytic proper and smooth curve. Let $\mathcal{U}\hookrightarrow X_K^{\rm rig}$ be an open affinoid subspace of $X_K^{\rm rig}$ (cf. [Raynaud], 3.1). The following is well-known (cf. [Raynaud], Proposition 3.5.1).

Proposition 2.2.1. *The complement $\mathcal{W}\overset{\rm def}{=}X_K^{\rm rig}\setminus\mathcal{U}$ has a natural structure of an (non quasi-compact) open rigid subspace of $X_K^{\rm rig}$ which is an increasing union of open quasi-compact rigid subspaces of $X_K^{\rm rig}$. The rigid space $\mathcal{W}$ has a finite number of connected components $\{\mathcal{W}_i\}_{i\in I}$. For each $i\in I$, let $x_i\in\mathcal{W}_i$ be a point (in the sense of [Raynaud], 3.1) and write $U_K\overset{\rm def}{=}X_K\setminus\{x_i\}_{i\in I}$ which is an affine K-curve. Then there exists a canonical affine and normal R-scheme $U^{\rm alg}$ of finite type such that $(U^{\rm alg})_K=U_K$, and if $\widetilde{U}=\operatorname{Spf}A$ denotes the formal completion of $U^{\rm alg}$ for the π-adic topology then the generic fibre $\widetilde{U}_K=\operatorname{Sp}\mathcal{A}$ of $\widetilde{U}$ (in the sense of [Raynaud], 3.1), where $\mathcal{A}\overset{\rm def}{=}A\otimes_R K$, is the rigid affinoid K-curve $\mathcal{U}$.*

As a consequence one obtains the following version of Runge's Theorem for rigid K-curves (cf. [Raynaud], Corollaire 3.5.2).

Proposition 2.2.2 (Runge's Theorem). *We use the same notations as in Proposition 2.2.1. Then the ring of regular functions on the affine curve U_K has a dense image in the ring of holomorphic functions on $\mathcal{U}$. More generally, a coherent sheaf M_K on U_K induces a coherent sheaf $\mathcal{M}$ on $\mathcal{U}$ and the image of the sections of M_K on U_K is dense in the space of sections of $\mathcal{M}$ on $\mathcal{U}$.*

We will refer to a pair $(\mathcal{U},U_K)$ as in Proposition 2.2.1 as a *Runge pair.*

2.3. In this section we investigate the structure of a certain quotient of the geometric étale fundamental group of an annulus of thickness 0. Let $D = \operatorname{Spf} R < Z >$ be the formal standard closed disc and $\mathcal{D} \overset{\text{def}}{=} D_K = \operatorname{Sp} K < Z >$ its generic fibre which is the standard closed rigid analytic disc centred at the point "$Z = 0$". Given an integer $n \geq 0$ consider the formal closed disc $D_n \overset{\text{def}}{=} \operatorname{Spf} \frac{R<Z,Y>}{(Z-\pi^n Y)}$ and its generic fibre $\mathcal{D}_n \overset{\text{def}}{=} D_{n,K} = \operatorname{Sp} \frac{K<Z,Y>}{(Z-\pi^n Y)}$ (recall π is a uniformiser of R). The natural embedding $\mathcal{D}_n \subset \mathcal{D}_0 = \mathcal{D}$ induces an identification between the points of $\mathcal{D}_n$ and the closed disc $\{x \in \mathcal{D},\ |Z(x)| \leq |\pi|^n\}$. We also consider the formal annulus $C_n \overset{\text{def}}{=} \operatorname{Spf} \frac{R<Z,Y,W>}{(Z-\pi^n Y, YW-1)}$ and its generic fibre $\mathcal{C}_n \overset{\text{def}}{=} C_{n,K} = \operatorname{Sp} \frac{K<Z,Y,W>}{(Z-\pi^n Y, YW-1)}$. The natural embedding $\mathcal{C}_n \subset \mathcal{D}_n$ induces an identification between the points of $\mathcal{C}_n$ and the closed annulus of thickeness zero $\{x \in \mathcal{D},\ |Z(x)| = |\pi|^n\}$.

Let $U_K \overset{\text{def}}{=} \mathbb{G}_{m,K} = \operatorname{Spec} \frac{K[Z,V]}{(ZV-1)}$ and $X_K = \mathbb{P}^1_K$ its smooth compactification, with function field $K(Z)$. We have natural embeddings $\mathcal{C}_n \subset \mathcal{D}_n \subset (X_K)^{\text{rig}}$. (Here we consider the rigid analytic structure on X_K arising from the admissible covering $\{x \in \mathbb{P}^1_K,\ |Z(x)| \leq |\pi|^n\} \bigcup \{x \in \mathbb{P}^1_K,\ |Z(x)| \geq |\pi|^n\}$.) Let η be a geometric point of $\mathcal{C}_n$ as in 2.1 which induces a geometric point of $\mathcal{D}_n$, U_K, X_K and $X_{\overline{K}} \overset{\text{def}}{=} X_K \times_K \overline{K}$ (which we also denote η). There exist continuous homomorphisms $\phi_n : \pi_1(\mathcal{C}_n, \eta) \to \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)$ and $\psi_n : \pi_1(\mathcal{D}_n \setminus \{0\}, \eta) \to \pi_1(U_K, \eta)$ (via the rigid GAGA functor) which induce continuous homomorphisms $\phi_n^{\text{geo}} : \pi_1(\mathcal{C}_n, \eta)^{\text{geo}} \to \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo}}$ and $\psi_n^{\text{geo}} : \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo}} \to \pi_1(U_K, \eta)^{\text{geo}}$.

Proposition 2.3.1. *Let $p = \operatorname{char}(k) \geq 0$ with no restriction on $\operatorname{char}(K)$. Then the homomorphisms $\phi_n^{\text{geo},p'} : \pi_1(\mathcal{C}_n, \eta)^{\text{geo},p'} \to \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p'}$ and $\psi_n^{\text{geo},p'} : \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p'} \to \pi_1(U_K, \eta)^{\text{geo},p'}$ (induced by ϕ_n^{geo} and ψ_n^{geo}; respectively) are isomorphisms. In particular, both $\Gamma \overset{\text{def}}{=} \pi_1(\mathcal{C}_n, \eta)^{\text{geo},p'}$ and $\widetilde{\Gamma} \overset{\text{def}}{=} \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p'}$ are isomorphic to the maximal prime-to-p quotient $\hat{\mathbb{Z}}^{p'}$ of $\hat{\mathbb{Z}}$.*

Proof. The last assertion follows from the first, and the well-known fact (since $U_K = \mathbb{G}_{m,K}$) that $\pi_1(U_K, \eta)^{\text{geo},p'}$ is isomorphic to $\hat{\mathbb{Z}}^{p'}$. Let $C_{n,k} \overset{\text{def}}{=} \operatorname{Spec} \frac{k[y,w]}{(yw-1)} = \mathbb{G}_{m,k}$ be the special fibre of C_n and β a geometric point of $C_{n,k}$ which induces a geometric point of C_n noted also β. There exist continuous homomorphisms $\pi_1(\mathcal{C}_n, \eta)^{\text{geo}} \to \pi_1(C_n, \beta)^{\text{geo}} \to \pi_1(C_{n,k}, \beta)^{\text{geo}}$, where the first map is surjective (a geometrically connected étale cover of C_n induces by passing to the generic fibres a geometrically connected étale cover of $\mathcal{C}_n$), and the second map is the inverse of the natural map $\pi_1(C_{n,k}, \beta)^{\text{geo}} \to \pi_1(C_n, \beta)^{\text{geo}}$ which is an isomorphism (cf. [SGA1], Exposé I, Corollaire 8.4). The composite map $\pi_1(\mathcal{C}_n, \eta)^{\text{geo}} \twoheadrightarrow \pi_1(C_{n,k}, \beta)^{\text{geo}}$ is a surjective specialisation homomorphism, which induces a surjective specialisation homomorphism $\pi_1(\mathcal{C}_n, \eta)^{\text{geo},p'} \twoheadrightarrow \pi_1(C_{n,k}, \beta)^{\text{geo},p'}$ between the respective prime-to-p parts. One can show, using Abhyankar's lemma (cf. loc. cit. Exposé X, Lemma 3.6) and the theorem of purity of Zariski (cf. loc. cit. Exposé X, Théorème 3.1), that this latter map is an isomorphism (similar arguments used in loc. cit., Exposé X, in order to prove Théorème 3.8 and Corollaire 3.9). On the other hand $\pi_1(C_{n,k}, \beta)^{\text{geo},p'} \xrightarrow{\sim} \pi_1(\mathbb{G}_{m,k})^{\text{geo},p'} \xrightarrow{\sim} \hat{\mathbb{Z}}^{p'}$. Thus, $\pi_1(\mathcal{C}_n, \eta)^{\text{geo},p'} \xrightarrow{\sim} \hat{\mathbb{Z}}^{p'}$.

Further, the composite homomorphism $\pi_1(\mathcal{C}_n, \eta)^{\text{geo},p'} \to \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p'} \to \pi_1(U_K, \eta)^{\text{geo},p'}$ is an isomorphism. Indeed, this composite map is surjective since a finite Galois étale cover $V_K \to U_K$ of order prime-to-p, with V_K geometrically connected, extends to a finite (cyclic) Galois cover $Z \to X_K$ (totally) ramified above 0 and ∞ and the corresponding Galois cover $Z^{\text{rig}} \to (X_K)^{\text{rig}}$ of rigid curves restricts (after possibly a finite extension of K) to an étale cover $\mathcal{V}_n \to \mathcal{C}_n$ with $\mathcal{V}_n$ geometrically connected. More precisely, the rigid cover $Z^{\text{rig}} \to (X_K)^{\text{rig}}$ induces in reduction, via suitable formal models, a finite étale cover $V_k \to C_{n,k}$ with V_k geometrically connected, hence $\mathcal{V}_n$ is geometrically connected. As both $\pi_1(\mathcal{C}_n, \eta)^{\text{geo},p'}$ and $\pi_1(U_K, \eta)^{\text{geo},p'}$ are isomorphic to $\hat{\mathbb{Z}}^{p'}$, the composite surjective map $\pi_1(\mathcal{C}_n, \eta)^{\text{geo},p'} \to \pi_1(U_K, \eta)^{\text{geo},p'}$ is an isomorphism. Further, the map $\pi_1(\mathcal{C}_n, \eta)^{\text{geo},p'} \to \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p'}$ is surjective (same argument as the one used above for the surjectivity of the map $\pi_1(\mathcal{C}_n, \eta)^{\text{geo},p'} \to \pi_1(U_K, \eta)^{\text{geo},p'}$). We then deduce that the maps $\pi_1(\mathcal{C}_n, \eta)^{\text{geo},p'} \to \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p'}$ and $\pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p'} \to \pi_1(U_K, \eta)^{\text{geo},p'}$ are isomorphisms as claimed. $\square$

Proposition 2.3.2. *Assume* $\text{char}(K) = p > 0$. *Then the homomorphism* $\psi_n^{\text{geo},p} : \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p} \to \pi_1(U_K, \eta)^{\text{geo},p}$ *(induced by* ψ_n^{geo}*) makes* $\pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p}$ *into a direct factor of* $\pi_1(U_K, \eta)^{\text{geo},p}$ *and* $\pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p}$ *is a free pro-p group of infinite rank. Furthermore, the homomorphism* $\phi_n^{\text{geo},p} : \pi_1(\mathcal{C}_n, \eta)^{\text{geo},p} \to \pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p}$ *(induced by* ϕ_n^{geo}*) makes* $\pi_1(\mathcal{C}_n, \eta)^{\text{geo},p}$ *into a direct factor of* $\pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p}$ *and* $\pi_1(\mathcal{C}_n, \eta)^{\text{geo},p}$ *is a free pro-p group of infinite rank.*

Proof. First, note that $\pi_1(U_K, \eta)^{\text{geo},p}$ is free since U_K is an affine scheme of characteristic $p > 0$ (cf. [Serre1], Proposition 1). Next, we prove the first assertion. Using Proposition 1.3, we need to show that the map $H^1(\pi_1(U_K, \eta)^{\text{geo}}, \mathbb{Z}/p\mathbb{Z}) \to H^1(\pi_1(\mathcal{D}_n \setminus 0, \eta)^{\text{geo}}, \mathbb{Z}/p\mathbb{Z})$ induced by ψ_n^{geo} on cohomology is surjective. Let $f : \mathcal{Z} \to \mathcal{D}_n$ be a generically $\mathbb{Z}/p\mathbb{Z}$-torsor which is étale outside 0 with $\mathcal{Z}$ geometrically connected. Let $n' > n$ an integer and $\mathcal{X} \overset{\text{def}}{=} \mathcal{D}_n \setminus \mathcal{D}_{n'}^o$; where $\mathcal{D}_{n'}^o \overset{\text{def}}{=} \mathcal{D}_{n'} \setminus \mathcal{C}_{n'}$ is an open disc, which is an affinoid subdomain of $\mathcal{D}_n$. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be the restriction of f which is an étale $\mathbb{Z}/p\mathbb{Z}$-torsor. For $n' >> n$, $f^{-1}(\mathcal{D}_{n'}^o)$ is geometrically connected since f is totally ramified above 0, and $\mathcal{Y}$ is then geometrically connected, which we will assume from now on. (More precisely, the pre-image of $0 \in \mathcal{D}_n(K)$ in $\mathcal{Z}$ consists of a single point $z \in \mathcal{Z}(K)$ as f is totally ramified above 0. By passing to a formal model of $\mathcal{Z}$, its minimal desingularisation, and the quotient of the latter by the action of the Galois group $\mathbb{Z}/p\mathbb{Z}$ of the covering f, one sees that $f^{-1}(\mathcal{D}_{n'}^o)$ is an open disc for $n' >> n$.) By Artin-Schreier theory the torsor $\tilde{f}$ is given by an Artin-Schreier equation $\alpha^p - \alpha = g$ where g is a holomorphic function on $\mathcal{X}$. The pair $(\mathcal{X}, U_K)$ is a Runge pair. The function g can be approximated by a regular function $\tilde{g}$ on U_K (cf. Proposition 2.2.2). For $\tilde{g}$ close to g the equation $\alpha^p - \alpha = \tilde{g}$ defines a $\mathbb{Z}/p\mathbb{Z}$-étale torsor $f' : Z_K \to U_K$ whose pull-back to $\mathcal{X}$ is isomorphic to $\tilde{f}$. In particular, Z_K is geometrically connected. More precisely, for $\tilde{g}$ close to g (for example if $||\tilde{g} - g|| < 1$, where $||\ \ ||$ is the supremum norm on $\mathcal{X}$) then $\sum_{t \geq 0} (\tilde{g} - g)^{p^t}$ converges to a holomorphic function h on $\mathcal{X}$ and $g = h^p - h + \tilde{g}$ in $\mathcal{X}$. Hence the class of f' in $H^1(\pi_1(U_K, \eta)^{\text{geo}}, \mathbb{Z}/p\mathbb{Z})$ maps to the class of f in $H^1(\pi_1(\mathcal{D}_n \setminus 0, \eta)^{\text{geo}}, \mathbb{Z}/p\mathbb{Z})$. The assertion that $\pi_1(\mathcal{D}_n \setminus \{0\}, \eta)^{\text{geo},p}$ is free follows from loc. cit. (cf. Proposition 1.1). Similarly, using the fact that $(\mathcal{C}_n, U_K)$ is a Runge pair, one proves that the natural map $H^1(\pi_1(U_K, \eta)^{\text{geo}}, \mathbb{Z}/p\mathbb{Z}) \to H^1(\pi_1(\mathcal{C}_n, \eta)^{\text{geo}}, \mathbb{Z}/p\mathbb{Z})$ is surjective, hence the second assertion follows from Proposition 1.3. Finally, the assertions on infinite rank follow from the facts that $H^1(\pi_1(\mathcal{D}_n \setminus 0, \eta)^{\text{geo}}, \mathbb{Z}/p\mathbb{Z})$ and $H^1(\pi_1(\mathcal{C}_n, \eta)^{\text{geo}}, \mathbb{Z}/p\mathbb{Z})$ are infinite dimensional $\mathbb{F}_p$-vector spaces. $\square$

Next, we investigate the structure of the maximal pro-p quotients of $\pi_1(\mathcal{C}_n, \eta)^{\text{geo}}$

and $\pi_1(\mathcal{D}_n\backslash T,\eta)^{\text{geo}}$ ($T\subset\mathcal{D}_n\backslash\mathcal{C}_n$ is a finite set of points) in the mixed characteristic case. Let T be a finite set of points of $\mathcal{D}_n\backslash\mathcal{C}_n$ and S a finite set of points of $X_K^{\text{rig}}\backslash\mathcal{D}_n$. We view $T\cup S\subset X_K$ as a closed subscheme of X_K and write $(T\cup S)_L\overset{\text{def}}{=}(T\cup S)\times_K L$ if L/K is a sub-extension of $\overline{K}/K$. We also denote by $\pi_1(X_L\backslash(T\cup S)_L,\eta)$ the étale fundamental group of $X_L\setminus(T\cup S)_L$ with base point η. The natural embedding $\mathcal{D}_{n,L}\overset{\text{def}}{=}\mathcal{D}_n\times_K L\to X_L^{\text{rig}}$ induces (via the rigid GAGA functor) a continuous homomorphism $\pi_1(\mathcal{D}_{n,L}\setminus T_L,\eta)\to\pi_1(X_L\setminus(T\cup S)_L,\eta)$, and by passing to the projective limit a homomorphism (cf. Remark 2.1.2)

$$\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo}}\to\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)\overset{\text{def}}{=}\varprojlim_{L/K}\pi_1(X_L\setminus(T\cup S)_L,\eta),$$

where L/K runs over all finite extensions contained in $\overline{K}$.

Let ℓ be a prime integer. The above homomorphism $\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo}}\to\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)$ induces homomorphisms $\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},\ell}\to\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{\ell}$ and (by passing to the projective limit over all S as above)

$$\phi_{n,T}:\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},\ell}\to\varprojlim_{S}\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{\ell},$$

which induces a homomorphism

$$\phi_n\overset{\text{def}}{=}\varprojlim_{T}\phi_{n,T}:\varprojlim_{T}\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},\ell}\to\varprojlim_{(T,S)}\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{\ell},$$

where the limit is taken over all finite sets of points T and S as above and $\varprojlim_{(T,S)}\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{\ell}\overset{\text{def}}{=}\varprojlim_{T}\left(\varprojlim_{S}\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{\ell}\right)$. The profinite groups $\varprojlim_{S}\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{\ell}$ and $\varprojlim_{(T,S)}\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{\ell}$ are *free* pro-ℓ groups if $\ell\neq\text{char}(K)$ (as follows from [Grothendieck], Exposé XIII, Corollaire 2.12). Note that for each finite set $T\subset\mathcal{D}_n\setminus\mathcal{C}_n$ as above we have a continuous homomorphism $\pi_1(\mathcal{C}_n,\eta)^{\text{geo}}\to\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo}}$ which induces homomorphisms $\psi_{n,T}:\pi_1(\mathcal{C}_n,\eta)^{\text{geo},\ell}\to\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},\ell}$ and (by passing to the projective limit over all T as above)

$$\psi_n\overset{\text{def}}{=}\varprojlim_{T}\psi_{n,T}:\pi_1(\mathcal{C}_n,\eta)^{\text{geo},\ell}\to\varprojlim_{T}\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},\ell}.$$

Proposition 2.3.3. *Assume* $\text{char}(K)=0$ *and* $\text{char}(k)=p>0$. *Then the continuous homomorphism* $\phi_{n,T}:\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},p}\to\varprojlim_{S}\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{p}$, *(resp.* $\phi_n\overset{\text{def}}{=}\varprojlim_{T}\phi_{n,T}:\varprojlim_{T}\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},p}\to\varprojlim_{(T,S)}\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{p}$*) makes* $\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},p}$ *(resp.* $\varprojlim_{T}\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},p}$*) into a direct factor of* $\varprojlim_{S}\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{p}$ *(resp.* $\varprojlim_{(T,S)}\pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta)^{p}$*). In particular, both* $\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},p}$ *and* $\varprojlim_{T}\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},p}$ *are free pro-p groups of infinite ranks. Furthermore, the homomorphism* $\psi_n:\pi_1(\mathcal{C}_n,\eta)^{\text{geo},p}\to\varprojlim_{T}\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},p}$

makes $\pi_1(\mathcal{C}_n,\eta)^{\text{geo},p}$ into a direct factor of $\varprojlim_T \pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo},p}$, and $\pi_1(\mathcal{C}_n,\eta)^{\text{geo},p}$ is a free pro-p group of infinite rank.

Proof. First, we prove the assertion regarding the homomorphism $\phi_{n,T}$ by proving that the map $H^1(\varprojlim_S \pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta),\mathbb{Z}/p\mathbb{Z})\to H^1(\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo}},\mathbb{Z}/p\mathbb{Z})$ induced by $\phi_{n,T}$ on cohomology is surjective, the result will then follow from Proposition 1.3. We can (after possibly passing to a finite extension of K) assume that K contains the p-th roots of unity and all points in T are K-rational. (In this, and other, proofs we will often use this argument. This is permissible because the various (pro-p, pro-ℓ, and full) fundamental groups under consideration are geometric fundamental groups.) Let $f:\mathcal{U}\to\mathcal{D}_n$ be a generically μ_p-torsor with $\mathcal{U}$ geometrically connected and which is ramified above T. Let $\{\mathcal{D}_s^o\}_{s\in T}$ be pairwise disjoint open discs centred at the points $s\in T$, $\mathcal{X}\overset{\text{def}}{=}\mathcal{D}_n\setminus(\cup_s\mathcal{D}_s^o)$ an affinoid subdomain, and $\tilde{f}:\mathcal{Y}\to\mathcal{X}$ the restriction of f which is a μ_p-torsor. When $\mathcal{D}_s^o$ is small enough $f^{-1}(\mathcal{D}_s^o)$ is geometrically connected as f is totally ramified above s, $\forall s\in T$, and $\mathcal{Y}$ is then geometrically connected, which we will assume form now on (cf. the argument in the proof of Proposition 2.3.2). We can assume after possibly a finite extension of R that $\mathcal{X}$ has a (canonical) R-formal model $Z=\operatorname{Spf}A$ with Z_k reduced (cf. [Bosch-Lütkebohmert-Raynaud], Theorem 1.3).

Let $h:Y\to Z$ be the finite morphism where Y is the normalisation of Z in $\mathcal{Y}$. After possibly passing to a finite extension of K we can assume that Y_k is reduced (cf. [Epp]). The μ_p-torsor $\tilde{f}$ is given by a Kummer equation $\beta^p=g$ where g is a unit on $\mathcal{X}$. Let $V_K\overset{\text{def}}{=}X_K\setminus(T\cup\{\infty\})$, then $(\mathcal{X},V_K)$ is a Runge pair. The function g can be approximated by a regular function $\tilde{g}$ on V_K (cf. Proposition 2.2.2). For $\tilde{g}$ close to g the equation $\beta^p=\tilde{g}$ defines a (possibly ramified) Galois covering $f_1:W_K\to X_K$ of degree p, with W_K geometrically connected, whose pull-back to $\mathcal{X}$ (via the rigid GAGA functor) is isomorphic to $\tilde{f}$. More precisely, one can write $g=\pi^t g_0$ where $g_0\in A$ is a unit and $0\le t<p$ an integer. One verifies easily that $t=0$ since Y_k and Z_k are reduced. Let $\tilde{g}\in A^{\text{alg}}$ such that $\tilde{g}-g\in\pi^r A$ where $U^{\text{alg}}=\operatorname{Spec}A^{\text{alg}}$ (cf. Propositions 2.2.1, Proposition 2.2.2, and the notations therein). Then for r large enough $\tilde{g}g^{-1}\in 1+\pi^r A$ is a p-th power in A and the Galois covering $f_1:Z_K\to X_K$ generically defined by the equation $\beta^\ell=\tilde{g}$ satisfies the above property. (More precisely, let $f\in A$, $r\ge 1$ large enough, then we can find $g\in A$ and $t\ge 1$ such that $1+\pi^r f=(1+\pi^t g)^p$. If $r>\frac{pv_K(p)}{p-1}$, set $t=r-v_K(p)\ge 1$. Then $1+\pi^r f=1+p\pi^t(g+u_2\pi^t g^2+\ldots+u_p\pi^{t(p-1)-v_K(p)}g^p)$, where $u_i\in R$ are units, and by Hensel's lemma we can find $g\in A$ such that $vf=g+u_2\pi^t g^2+\ldots+\pi^{t(p-1)}g^p$ where $v=\pi^{r-t}p^{-1}\in R$ is a unit (cf. [Bourbaki], Chapter III, §4.3, Theorem 1).) Further, f_1 induces (via the rigid GAGA functor) a μ_p-torsor $f_2:\mathcal{V}\to\mathcal{X}$ which is isomorphic to $\tilde{f}$, and a generically Galois cover $f_3:\mathcal{U}\to X_K^{\text{rig}}\setminus(\cup_s\mathcal{D}_s^o)$. One can then glue the generically μ_p-torsors f and f_3 along f_2 and construct (via the rigid GAGA functor) a Galois covering $Y_K\to X_K$ which is ramified above T and possibly a finite set $\tilde{S}\subseteq X_K\setminus\mathcal{D}_n$, and whose class in $H^1(\varprojlim_S \pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta),\mathbb{Z}/\ell\mathbb{Z})$ maps to the class of f in $H^1(\pi_1(\mathcal{D}_n\setminus T,\eta)^{\text{geo}},\mathbb{Z}/\ell\mathbb{Z})$. The assertion regarding the homomorphism ϕ_n is proven in a similar way. Similarly, using the fact that $(\mathcal{C}_n,U_K)$ is a Runge pair, one proves that the natural map $H^1(\varprojlim_{(T,S)} \pi_1(X_{\overline{K}}\setminus(T\cup S)_{\overline{K}},\eta),\mathbb{Z}/p\mathbb{Z})\to H^1(\pi_1(\mathcal{C}_n,\eta)^{\text{geo}},\mathbb{Z}/p\mathbb{Z})$ is surjective, hence the second assertion follows from Proposition 1.3. Finally, the assertions on infinite rank follow from the fact that $H^1(\pi_1(\mathcal{C}_n,\eta)^{\mathrm{geo}},\mathbb{Z}/p\mathbb{Z})$ is an infinite dimensional $\mathbb{F}_p$-vector space. $\square$

Recall the notations in Proposition 2.3.1, and consider the following exact sequence

$$1\to\mathcal{H}_n\to\pi_1(\mathcal{C}_n,\eta)^{\mathrm{geo}}\to\Gamma\to 1,$$

where $\mathcal{H}_n\overset{\mathrm{def}}{=}\mathrm{Ker}(\pi_1(\mathcal{C}_n,\eta)^{\mathrm{geo}}\twoheadrightarrow\Gamma)$. Further, let $P_n\overset{\mathrm{def}}{=}\mathcal{H}_n^p$ be the maximal pro-p quotient of $\mathcal{H}_n$. By pushing out the above sequence by the characteristic quotient $\mathcal{H}_n\twoheadrightarrow P_n$ we obtain an exact sequence

$$1\to P_n\to\Delta_n\to\Gamma\to 1.$$

Similarly, consider the following exact sequence

$$1\to\mathcal{H}'_n\to\pi_1(\mathcal{D}_n\setminus\{0\},\eta)^{\mathrm{geo}}\to\widetilde{\Gamma}\to 1,$$

where $\mathcal{H}'_n\overset{\mathrm{def}}{=}\mathrm{Ker}(\pi_1(\mathcal{D}_n\backslash\{0\},\eta)^{\mathrm{geo}}\twoheadrightarrow\widetilde{\Gamma})$. Further, let $\widetilde{P}'_n\overset{\mathrm{def}}{=}(\mathcal{H}'_n)^p$ be the maximal pro-p quotient of $\mathcal{H}'_n$. By pushing out the above sequence by the characteristic quotient $\mathcal{H}'_n\twoheadrightarrow\widetilde{P}'_n$ we obtain an exact sequence

$$1\to\widetilde{P}'_n\to\widetilde{\Delta}'_n\to\widetilde{\Gamma}\to 1.$$

Proposition 2.3.4. *Assume K of equal characteristic $p\geq 0$. Then the natural morphism $\mathcal{C}_n\to\mathcal{D}_n$ induces a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P_n & \longrightarrow & \Delta_n & \longrightarrow & \Gamma & \longrightarrow & 1\\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \widetilde{P}'_n & \longrightarrow & \widetilde{\Delta}'_n & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & 1
\end{array}$$

where the right vertical homomorphism $\Gamma\to\widetilde{\Gamma}$ is an isomorphism (cf. Lemma 2.3.1) and the middle vertical homomorphism $\Delta_n\to\widetilde{\Delta}'_n$ makes Δ_n into a semi-direct factor of $\widetilde{\Delta}'_n$ (cf. Lemma 1.5).

Proof. We only present the proof in the case $p>0$, the proof in the case $p=0$ is the same except for obvious simplifications. Let $\Delta_n\twoheadrightarrow G$ be a finite quotient which sits in an exact sequence $1\to Q\to G\to\Gamma_e\to 1$ where Γ_e is the unique quotient of Γ of cardinality e; for some integer e prime-to-p (cf. Proposition 2.3.1), with Q a p-group. We will show there exists a surjective homomorphism $\widetilde{\Delta}'_n\twoheadrightarrow G$ whose composition with $\Delta_n\to\widetilde{\Delta}'_n$ is the above homomorphism. We can assume (without loss of generality) that the corresponding Galois covering $\mathcal{Y}\to\mathcal{C}_n$ with group G; $\mathcal{Y}$ is normal and geometrically connected, is defined over K. This covering factorizes as $\mathcal{Y}\to\mathcal{Y}'\to\mathcal{C}_n$ where $\mathcal{Y}'\to\mathcal{C}_n$ is Galois with group $\Gamma_e\overset{\sim}{\to}\mu_e$ and $\mathcal{Y}\to\mathcal{Y}'$ is Galois with group Q. After possibly a finite extension of K we can assume that n is divisible by e, the μ_e-torsor $\mathcal{Y}'\to\mathcal{C}_n$ is generically defined by an equation $\widetilde{Z}^e=Z$ for a suitable choice of the parameter Z, and $\mathcal{Y}'=\mathcal{C}_{\frac{n}{e}}=\mathrm{Sp}\,\frac{K<\widetilde{Z},\widetilde{Y},\widetilde{W}>}{(\widetilde{Z}-\pi^{\frac{n}{e}}\widetilde{Y},\widetilde{Y}\widetilde{W}-1)}$ ($\widetilde{Y}^e=Y$) is an annulus of thickeness 0. The μ_e-torsor $\mathcal{Y}'\to\mathcal{C}_n$ extends then to a generically μ_e-torsor $\mathcal{X}'\to\mathcal{D}_n$ generically defined by an equation $\widetilde{Z}^e=Z$, which is (totally) ramified only above 0, and $\mathcal{X}' = \mathcal{D}_{\frac{n}{e}} = \mathrm{Sp}\, \frac{K<\widetilde{Z},\widetilde{Y}>}{(\widetilde{Z}-\pi^{\frac{n}{e}}\widetilde{Y})}$ is a closed disc centred at the unique point above $0 \in \mathcal{D}_n$; which we denote also 0.

By Proposition 2.3.2 applied to $\mathcal{Y}' \to \mathcal{X}'$ there exists (after possibly a finite extension of K) a Galois covering $\mathcal{X} \to \mathcal{X}'$ with group Q, ramified only above 0, with $\mathcal{X}$ geometrically connected and such that we have a commutative diagram of cartesian squares.

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{C}_n \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{D}_n \end{array}$$

Next, we borrow some ideas from [Garuti] (preuve du Théorème 2.13). We claim one can choose the above (geometric) covering $\mathcal{X} \to \mathcal{X}'$ such that the composite covering $\mathcal{X} \to \mathcal{D}_n$ is Galois with group G. Indeed, consider the quotient $\Delta_n \twoheadrightarrow \Delta_{\mathcal{Y}'}$ (resp. $\widetilde{\Delta}'_n \twoheadrightarrow \Delta_{\mathcal{X}'}$) of Δ_n (resp. $\widetilde{\Delta}'_n$) which sits in the following exact sequence $1 \to P_{\mathcal{Y}'} \to \Delta_{\mathcal{Y}'} \to \Gamma_e \to 1$ (resp. $1 \to P_{\mathcal{X}'} \to \Delta_{\mathcal{X}'} \to \widetilde{\Gamma}_e \to 1$) where $P_{\mathcal{Y}'} \stackrel{\mathrm{def}}{=} \pi_1(\mathcal{Y}', \eta)^{\mathrm{geo},p}$ (resp. $P_{\mathcal{X}'} \stackrel{\mathrm{def}}{=} \pi_1(\mathcal{X}' \setminus \{0\}, \eta)^{\mathrm{geo},p}$) and $\widetilde{\Gamma}_e$ is the unique quotient of $\widetilde{\Gamma}$ of cardinality e. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_{\mathcal{Y}'} & \longrightarrow & \Delta_{\mathcal{Y}'} & \longrightarrow & \Gamma_e & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & P_{\mathcal{X}'} & \longrightarrow & \Delta_{\mathcal{X}'} & \longrightarrow & \widetilde{\Gamma}_e & \longrightarrow & 1 \end{array}$$

where the right vertical map is an isomorphism (cf. Proposition 2.3.1). The choice of a splitting of the upper sequence in the above diagram (which splits since $P_{\mathcal{Y}'}$ is pro-p and Γ_e is (pro-)prime-to-p) induces an action of Γ_e on $P_{\mathcal{X}'}$, and $P_{\mathcal{Y}'}$ is a direct factor of $P_{\mathcal{X}'}$ (cf. Proposition 2.3.2) which is stable by this action of Γ_e. Further, $P_{\mathcal{Y}'}$ has a supplement E in $P_{\mathcal{X}'}$ which is invariant under the action of Γ_e by [Garuti], Corollaire 1.11. The existence of this supplement E implies that one can choose $\mathcal{X} \to \mathcal{X}'$ as above such that the finite composite covering $\mathcal{X} \to \mathcal{D}_n$ is Galois with group G. More precisely, if the Galois covering $\mathcal{Y} \to \mathcal{Y}'$ corresponds to the surjective homomorphism $\rho : P_{\mathcal{Y}'} \twoheadrightarrow Q$ (which is stable by Γ_e since $\mathcal{Y} \to \mathcal{C}_n$ is Galois) then we consider the Galois covering $\mathcal{X} \to \mathcal{X}'$ corresponding to the surjective homomorphism $\widetilde{P}_{\mathcal{X}'} = P_{\mathcal{Y}'} \star E \twoheadrightarrow Q$ which is induced by ρ and the trivial homomorphism $E \to Q$, which is stable by Γ_e.

The above construction can be performed in a functorial way with respect to the various finite quotients of Δ_n. More precisely, let $\{\phi_j : \Delta_n \twoheadrightarrow G_j\}_{j \in J}$ be a cofinal system of finite quotients of Δ_n where G_j sits in an exact sequence $1 \to Q_j \to G_j \to \Gamma_{e_j} \to 1$, for some integer e_j prime-to-p, and Q_j a p-group. Assume we have a factorisation $\Delta_n \twoheadrightarrow G_{j'} \twoheadrightarrow G_j$ for $j', j \in J$. Thus, e_j divides $e_{j'}$, and we can assume without loss of generality (after replacing the group extension G_j by its pull-back via $\Gamma_{e'_j} \twoheadrightarrow \Gamma_{e_j}$) that $e \stackrel{\mathrm{def}}{=} e_j = e_{j'}$. With the above notations we then have surjective homomorphisms $\rho_{j'} : P_{\mathcal{Y}'} \twoheadrightarrow Q_{j'}$, $\rho_j : P_{\mathcal{Y}'} \twoheadrightarrow Q_j$ (which are stable by Γ_e), and ρ_j factorises through $\rho_{j'}$. Then we consider the Galois covering(s) $\mathcal{X}_{j'} \to \mathcal{X}'$ (resp. $\mathcal{X}_j \to \mathcal{X}'$) corresponding to the surjective homomorphism(s) $\psi_{j'} : P_{\mathcal{X}'} = P_{\mathcal{Y}'} \star E \twoheadrightarrow Q_{j'}$ (resp. $\psi_j : P_{\mathcal{X}'} = P_{\mathcal{Y}'} \star E \twoheadrightarrow Q_j$) which are induced by $\rho_{j'}$ (resp. ρ_j) and the trivial homomorphism $E \to Q$, which are stable by Γ_e and

ψ_j factorises through $\psi_{j'}$. Finally, we deduce from this construction the existence of a surjective continuous homomorphism $\widetilde{\Delta}'_n \twoheadrightarrow \Delta_n$ which is a left inverse to the natural homomorphism $\Delta_n \to \widetilde{\Delta}'_n$ (cf. Lemma 1.5). $\square$

Next, recall the discussion and notations before Proposition 2.3.3, and consider the following exact sequence

$$1 \to \widetilde{\mathcal{H}}_n \to \varprojlim_T \pi_1(D_n \setminus T, \eta)^{\text{geo}} \to \widetilde{\Gamma} \to 1,$$

where $\widetilde{\mathcal{H}}_n \overset{\text{def}}{=} \operatorname{Ker}(\varprojlim_T \pi_1(D_n \setminus T, \eta)^{\text{geo}} \twoheadrightarrow \widetilde{\Gamma})$. Further, let $\widetilde{P}_n \overset{\text{def}}{=} \widetilde{\mathcal{H}}_n^p$ be the maximal pro-p quotient of $\widetilde{\mathcal{H}}_n$. By pushing out the above sequence by the characteristic quotient $\widetilde{\mathcal{H}}_n \twoheadrightarrow \widetilde{P}_n$ we obtain an exact sequence

$$1 \to \widetilde{P}_n \to \widetilde{\Delta}_n \to \widetilde{\Gamma} \to 1.$$

Proposition 2.3.5. *Assume* $\operatorname{char}(K) = 0$ *with no restriction on* $\operatorname{char}(k) = p \geq 0$. *Then the morphism $\mathcal{C}_n \to \mathcal{D}_n$ induces a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P_n & \longrightarrow & \Delta_n & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \widetilde{P}_n & \longrightarrow & \widetilde{\Delta}_n & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & 1
\end{array}$$

where the right vertical homomorphism $\Gamma \to \widetilde{\Gamma}$ is an isomorphism (cf. Lemma 2.3.1) and the middle vertical homomorphism $\Delta_n \to \widetilde{\Delta}_n$ makes Δ_n into a semi-direct factor of $\widetilde{\Delta}_n$ (cf. Lemma 1.5).

Proof. We only explain the proof in the case $p > 0$, the proof in the case $p = 0$ is the same except for obvious simplifications. The proof is entirely similar to the proof of Proposition 2.3.4 using Proposition 2.3.3 instead of Proposition 2.3.2. With the notations in the proof of loc. cit. one applies Proposition 2.3.3 to $\mathcal{Y}' \to \mathcal{X}'$ to ensure the existence (after possibly a finite extension of K) of a Galois covering $\mathcal{X} \to \mathcal{X}'$ with group Q, ramified above a finite set $T' \subset \mathcal{X}'$, with $\mathcal{X}$ geometrically connected and such that we have a commutative diagram as in loc. cit. One then considers the quotients $\Delta_n \twoheadrightarrow \Delta_{\mathcal{Y}'}$ as in loc. cit., and $\widetilde{\Delta}_n \twoheadrightarrow \Delta_{\mathcal{X}'}$ which sits in the following exact sequence $1 \to P_{\mathcal{X}'} \to \Delta_{\mathcal{X}'} \to \widetilde{\Gamma}_e \to 1$ where $P_{\mathcal{X}'} \overset{\text{def}}{=} \pi_1(\mathcal{X}' \setminus T', \eta)^{\text{geo},p}$ and follow the same arguments as in loc. cit. $\square$

Remark 2.3.6. With the same notations as in Propositions 2.3.4 and 2.3.5 the pro-p group $\widetilde{P}'_n$ (resp. $\widetilde{P}_n$) is free and the homomorphism $P_n \to \widetilde{P}'_n$ (resp. $P_n \to \widetilde{P}_n$) makes P_n into a direct factor of $\widetilde{P}'_n$ (resp. $\widetilde{P}_n$).

Remark 2.3.7. The proofs of Propositions 2.3.4 and 2.3.5 rely on Corollaire 1.11 in [Garuti] which relies on Lemme 1.9 and Lemme 1.10 in loc. cit. We take this opportunity to make precise some steps in the proof of these Lemmas. Using the notations in loc. cit., in the proof of Lemme 1.9; the definition of the set of pairs (S, φ_S), one should require $S \subseteq L^\star$ and the given action φ_S of Γ on L/S to lift the given action of Γ on $L/L^\star$. Also the group Γ_1 should be defined as the subgroup of $\operatorname{Aut}(L)$ generated by lifts of the elements of Γ viewed as acting on L/N

(automorphisms of L/N which lift automorphisms of $L/L^{\star}$ lift to automorphisms of L cf. Corollaire 1.7 in loc. cit.) so that Γ_1 acts on $L/(N\cap U)$ stabilising $N/(N\cap U)$, Γ_2 should then be defined by first taking the image of Γ_1 in $\mathrm{Aut}(L/(N\cap U))$ (this image maps to $\mathrm{Aut}(L/N)$) and then take the pull back of this image via the map $\Gamma \twoheadrightarrow \mathrm{Im}(\Gamma_1 \to \mathrm{Aut}(L/N))$. Also in the proof of Lemme 1.10, the definition of the pair (S,α_S), with the notations in loc. cit. one should require $S\subseteq L^{\star}$, and the automorphism $\alpha_S\in \mathrm{Aut}(L/S)$ should lift the identity of $\mathrm{Aut}(L/L^{\star})$.

§3 Geometric fundamental groups of affinoid p-adic curves. In this section we investigate the structure of the geometric fundamental group of rigid affinoid K-curves which are embedded in a proper K-curve. Let X be a proper and normal formal R-curve with X_K smooth, $U\hookrightarrow X$ an R-formal affine sub-scheme, and $\mathcal{U}\overset{\text{def}}{=}U_K\hookrightarrow X_K$ the associated K-rigid analytic affinoid space (which is an affinoid rigid subspace of X_K). We assume that the special fibre U_k of U is connected, reduced, and $X_k\setminus U_k=\{\bar{x}_i\}_{i=1}^m$ consists of a finite set of closed points where $\bar{x}_i\in X_k(k)$ is a smooth point of X_k, $1\le i\le m$. Let $\mathcal{F}_i\overset{\text{def}}{=}\mathrm{Spf}\,\hat{\mathcal{O}}_{X,\bar{x}_i}$ be the formal germ of X at the point $\bar{x}_i$, $1\le i\le m$. Thus, $\hat{\mathcal{O}}_{X,\bar{x}_i}\overset{\sim}{\to}R[[T_i]]$ is the completion of the local ring $\mathcal{O}_{X,\bar{x}_i}$. Let $\mathcal{D}_i\overset{\text{def}}{=}\mathrm{Sp}\,K<T_i>$ be the rigid closed unit disc and $\mathcal{C}_i\overset{\text{def}}{=}\mathrm{Sp}\,K<T_i,\frac{1}{T_i}>$ the rigid standard annulus of thickness 0. The formal fibre of $\bar{x}_i$ in X_K is isomorphic to the open disc $\mathcal{D}_i^o\overset{\text{def}}{=}\mathcal{D}_i\setminus\mathcal{C}_i$ and $X_K\setminus U_K$ is isomorphic to the disjoint union $\bigcup_{i=1}^m\mathcal{D}_i^o$. In what follows we identify the formal fibre of $\bar{x}_i$ in X_K with the open disc $\mathcal{D}_i^o$, $1\le i\le m$. Write $x_i\in\mathcal{D}_{i,K}(K)$ for the zero point $T_i=0$ of $\mathcal{D}_{i,K}$ which we view, via the above identification, as a point in $X_K(K)$. The above conditions on the affinoid curve $\mathcal{U}$ are not too restrictive. More precisely, we have the following.

Theorem 3.1. *Let Y_K be a smooth and geometrically connected affinoid K-curve. Then, after possibly a finite extension of K, one can embed Y_K into a proper, geometrically connected, and smooth K-curve X_K such that the complement $X_K\setminus Y_K$ consists of a disjoint union of finitely many open unit K-discs as in the above discussion where $Y_K=\mathcal{U}$.*

Proof. See [Van Der Put], Theorem 1.1. □

We use the notations in 2.3. For an integer $n\ge 0$ we write $\mathcal{V}_n\overset{\text{def}}{=}X_K\setminus(\bigcup_{i=1}^m\{\mathcal{D}_{i,n}^o\})$ where $\mathcal{D}_{i,n}\overset{\text{def}}{=}\mathcal{D}_n\subseteq\mathcal{D}_i$ and $\mathcal{D}_{i,n}^o\overset{\text{def}}{=}\mathcal{D}_n^o\overset{\text{def}}{=}\mathcal{D}_n\setminus\mathcal{C}_n$ are as in loc. cit. Thus, $\mathcal{U}\subseteq\mathcal{V}_n$ is a quasi-compact rigid analytic subspace of X_K. We also write $\mathcal{C}_{i,n}\overset{\text{def}}{=}\mathcal{C}_n\subset\mathcal{D}_{i,n}$ which is a closed annulus of thickness 0.

Proposition 3.2. *Let $S\subset\mathcal{U}$ be a finite set of points and $f:\mathcal{Z}\to\mathcal{U}$ a finite Galois covering with Galois group G which is étale above $\mathcal{U}\setminus S$. Then, after possibly a finite extension of K, there exists $n>0$ such that f extends to a finite Galois covering $f_n:\mathcal{Z}_n\to\mathcal{V}_n$ which is Galois with Galois group G and étale above $\mathcal{U}\setminus S$. Moreover, let $f_i\overset{\text{def}}{=}f_{i,n}:\mathcal{W}_i=\bigcup_j\mathcal{W}_{i,j}\to\mathcal{C}_{i,n}$ be the restriction of f_n to the annulus $\mathcal{C}_{i,n}$, where $\{\mathcal{W}_{i,j}\}_j$ are the connected components of $\mathcal{W}_i$, $1\le i\le m$. Then the decomposition group $G_{i,j}\subseteq G$ of each connected component $\mathcal{W}_{i,j}$ is a solvable group which is an extension of a cyclic group of order prime-to-p by a p-group.*

Proof. This can be proven using similar arguments used by Raynaud in [Raynaud] to prove a similar result in the case where $\mathcal{U}$ is the closed unit disc centred at

0 which is embedded in $(\mathbb{P}^1_K)^{\mathrm{rig}}$ as the complement of the open disc centred at ∞ (see Remarques 3.4.12(i) in loc. cit.). We briefly explain the outline of proof. First, one can assume (without loss of generality) that $S=\emptyset$, $m=1$, $x\overset{\mathrm{def}}{=}x_1$, and $\mathcal{D}\overset{\mathrm{def}}{=}\mathcal{D}_1$. Using similar arguments as in [Raynaud] Proposition 3.4.1 one can extend the étale covering f to an étale covering $f_{n'}:\mathcal{Z}_{n'}\to\mathcal{U}_{n'}\overset{\mathrm{def}}{=}X_K\setminus\mathcal{D}^o_{\frac{1}{n'}}$ where $\mathcal{D}^o_{\frac{1}{n'}}\overset{\mathrm{def}}{=}\{x\in\mathcal{D}\ :\ |Z(x)|<|\pi|^{\frac{1}{n'}}\}$ for some integer $n'\geq 1$ and the germ of such an extension is unique. (More precisely, one establishes the analogs of Lemme 3.4.2 and Lemme 3.4.3 in loc. cit. using similar arguments.) In particular, there exists n' as above such that $f_{n'}$ is Galois with Galois group G (cf. loc. cit. Corollaire 3.4.8). There exists a formal model $U_{n'}$ of $\mathcal{U}_{n'}$ whose (reduced) special fibre consists of the special fibre U_k of $\mathcal{U}$ which is linked to an affine line at a double point $\bar{x}$ in which specialise the points of the open annulus $\mathcal{A}_{\frac{1}{n'}}\overset{\mathrm{def}}{=}\{x\in\mathcal{D}\ :\ 1>|Z(x)|>|\pi|^{\frac{1}{n'}}\}$ (cf. [Raynaud] 3.3.3 in the special case where $\mathcal{U}$ is the unit closed disc). Let $g_{n'}:Z_{n'}\to U_{n'}$ be the morphism of normalisation of $U_{n'}$ in $\mathcal{Z}_{n'}$ which is Galois with group G, $g_{n'}^{-1}(U_k)$ the reduced inverse image of U_k in $Z_{n'}$, and $\{y_i\}_i$ the points of $g_{n'}^{-1}(U_k)$ above $\bar{x}$. Then one proves that for n' large enough $g_{n'}^{-1}(U_k)$ is normal at the points $\{y_i\}_i$ (cf. loc. cit. Lemme 3.4.2), there is a one-to-one correspondence between the $\{y_i\}_i$ and the connected components $\{\mathcal{Y}_i\}_i$ of the inverse image $f_{n'}^{-1}(\mathcal{A}_{\frac{1}{n'}})$ in $\mathcal{Z}_{n'}$ of the open annulus $\mathcal{A}_{\frac{1}{n'}}$, as well as a one-to-one correspondence between the corresponding decomposition groups (cf. loc. cit. Proposition 3.4.6(ii)). Moreover, the decomposition group of such a component $\mathcal{Y}_i$ is solvable (cf. loc. cit. Corollaire 3.4.8). Let Z be the normalisation of U in $\mathcal{Z}$. After possibly a finite extension of K we can assume that the special fibre Z_k of $\mathcal{Z}$ is reduced (cf. [Epp]). In this case the decomposition group of a connected component $\mathcal{Y}_i$ as above is an extension of a cyclic group of order prime-to-p by a p-group. Indeed, in this case with the notations of loc. cit., the proof of Corollaire 3.4.8, the group I_i is a p-group as we assumed Z_k is reduced. Finally, after a finite extension of K we can assume the above open annulus $\mathcal{A}_{\frac{1}{n'}}$ of thickness $\frac{1}{n'}$ (a rational) is an annulus $\{x\in\mathcal{D}\ :\ 1>|Z(x)|>|\pi|^n\}$ of thickness n for some integer $n>0$. $\square$

For the remaining of this section, let $S\subset\mathcal{U}$ be a (possibly empty) *finite* set of points.

Theorem 3.3. *Assume K of equal characteristic $p>0$, and let ℓ be a prime integer (possibly equal to p). Then the morphism $\mathcal{U}\to X_K$ induces (via the rigid GAGA functor) a continuous homomorphism $\pi_1(\mathcal{U}\setminus S,\eta)^{\mathrm{geo}}\to\pi_1(X_K\setminus(\{x_i\}_{i=1}^m\cup S),\eta)^{\mathrm{geo}}$ (resp. $\pi_1(\mathcal{U}\setminus S,\eta)^{\mathrm{geo},\ell}\to\pi_1(X_K\setminus(\{x_i\}_{i=1}^m\cup S),\eta)^{\mathrm{geo},\ell}$) which makes $\pi_1(\mathcal{U}\setminus S,\eta)^{\mathrm{geo}}$ (resp. $\pi_1(\mathcal{U}\setminus S,\eta)^{\mathrm{geo},\ell}$) into a semi-direct factor of $\pi_1(X_K\setminus(\{x_i\}_{i=1}^m\cup S),\eta)^{\mathrm{geo}}$ (resp. direct factor of $\pi_1(X_K\setminus(\{x_i\}_{i=1}^m\cup S),\eta)^{\mathrm{geo},\ell}$). Moreover, $\pi_1(\mathcal{U}\setminus S,\eta)^{\mathrm{geo},\ell}$ is a free pro-ℓ group of infinite (resp. finite) rank if $\ell=p$ (resp. if $\ell\neq p$).*

Proof. We show the criterion in Lemma 1.5 is satisfied. Let $\pi_1(\mathcal{U}\setminus S,\eta)^{\mathrm{geo}}\twoheadrightarrow G$ be a finite quotient which we assume (without loss of generality) corresponding to a finite Galois covering $f:\mathcal{V}\to\mathcal{U}$ with group G, étale above $\mathcal{U}\setminus S$, with $\mathcal{V}$ normal and geometrically connected. We will show the existence of a surjective homomorphism $\pi_1(X_K\setminus(\{x_i\}_{i=1}^m\cup S),\eta)^{\mathrm{geo}}\twoheadrightarrow G$ whose composite with $\pi_1(\mathcal{U}\setminus S,\eta)^{\mathrm{geo}}\to\pi_1(X_K\setminus(\{x_i\}_{i=1}^m\cup S),\eta)^{\mathrm{geo}}$ is the above homomorphism. We assume the existence of an extension $f_n:\mathcal{Z}_n\to\mathcal{V}_n$ of f as in Proposition 3.2. For $1\leq i\leq m$, let $f_i \overset{\text{def}}{=} f_{i,n} : \mathcal{W}_i = \bigcup_{j=1}^{t_i} \mathcal{W}_{i,j} \to \mathcal{C}_{i,n}$ be the restriction of f_n to the annulus $\mathcal{C}_{i,n}$ with $\{\mathcal{W}_{i,j}\}_{j=1}^{t_i}$ the connected components of $\mathcal{W}_i$, and $G_{i,j} \subseteq G$ the decomposition group of $\mathcal{W}_{i,j}$ which is an extension of a cyclic group of order prime-to-p by a p group. Fix $1 \le j_0 \le t_i$, then $f_i \overset{\sim}{\to} \operatorname{Ind}_{G_{i,j_0}}^{G} f_{i,j_0}$ is an induced covering (cf. [Raynaud 4.1) where $f_{i,j_0} : \mathcal{W}_{i,j_0} \to \mathcal{C}_{i,n}$ is the restriction of f_i to $\mathcal{W}_{i,j_0}$.

By Proposition 2.3.4 (the equal characteristic $p > 0$ case) there exists (after possibly a finite extension of K) a finite Galois covering $\tilde{f}_{i,j_0} : \mathcal{Y}_{i,j_0} \to \mathcal{D}_{i,n}$ with group G_{i,j_0}, $\mathcal{Y}_{i,j_0}$ is normal and geometrically connected, whose pull-back to $\mathcal{C}_{i,n}$ via the embedding $\mathcal{C}_{i,n} \hookrightarrow \mathcal{D}_{i,n}$ is isomorphic to f_{i,j_0}, and $\tilde{f}_{i,j_0}$ is ramified only above x_i. Let $\tilde{f}_i : \mathcal{Y}_i \overset{\text{def}}{=} \operatorname{Ind}_{G_{i,j_0}}^{G} \mathcal{Y}_{i,j_0} \to \mathcal{D}_{i,n}$ be the induced coverings (cf. loc. cit.), $1 \le i \le m$. One can then patch the covering f_n with the coverings $\{\tilde{f}_i\}_{i=1}^{m}$ along the restrictions of these coverings above the annuli $\mathcal{C}_{i,n}$ (the restriction of f_n and $\tilde{f}_{i,n}$ to $\mathcal{C}_{i,n}$ are isomorphic by construction) to construct a finite Galois covering $\tilde{f} : Y_K \to X_K$ between smooth and proper rigid K-curves with group G, Y_K is geometrically connected, which gives rise (via the rigid GAGA functor) to a homomorphism $\pi_1(X_K \setminus (\{x_i\}_{i=1}^{n} \cup S), \eta)^{\text{geo}} \twoheadrightarrow G$ as required.

Moreover, one verifies easily that the above construction can be performed in a functorial way with respect to the various quotients of $\pi_1(\mathcal{U} \setminus S, \eta)^{\text{geo}}$ (in the sense of Lemma 1.5) using Proposition 2.3.4, so that it induces a continuous homomorphism $\pi_1(X_K \setminus (\{x_i\}_{i=1}^{m} \cup S), \eta)^{\text{geo}} \to \pi_1(\mathcal{U} \setminus S, \eta)^{\text{geo}}$ which is left inverse to $\pi_1(\mathcal{U} \setminus S, \eta)^{\text{geo}} \to \pi_1(X_K \setminus (\{x_i\}_{i=1}^{m} \cup S), \eta)^{\text{geo}}$. The second assertion is proven in a similar way. Note that $\pi_1(X_K \setminus (\{x_i\}_{i=1}^{m} \cup S), \eta)^{\text{geo},\ell}$ is pro-ℓ free (cf. [Serre1], Proposition 1, and Proposition 1.1.1, in the case $\ell = p$, and [Grothendieck], Exposé XIII, Corollaire 2.12, otherwise), the assertion that $\pi_1(\mathcal{U} \setminus S, \eta)^{\text{geo},\ell}$ is free follows then from the above discussion. Finally, the assertion on the rank follows from Proposition 3.5 below if $\ell = p$, and from the fact that $\pi_1(X_K \setminus (\{x_i\}_{i=1}^{m} \cup S), \eta)^{\text{geo},\ell}$ is finitely generated if $\ell \neq p$ (cf. loc. cit.) $\square$

Let $T \subset \bigcup_{i=1}^{m} \mathcal{D}_i^o$ be a *finite* set of closed points of X_K. We view $T \subset X_K$ as a closed subscheme of X_K. We have an exact sequence of profinite groups

$$1 \to \pi_1(X_K \setminus (T \cup S), \eta)^{\text{geo}} \to \pi_1(X_K \setminus (T \cup S), \eta) \to \operatorname{Gal}(\overline{K}/K) \to 1.$$

By passing to the projective limit over all finite sets of closed points $T \subset \bigcup_{i=1}^{m} \mathcal{D}_i^o$ we obtain an exact sequence

$$1 \to \varprojlim_{T} \pi_1(X_K \setminus (T \cup S), \eta)^{\text{geo}} \to \varprojlim_{T} \pi_1(X_K \setminus (T \cup S), \eta) \to \operatorname{Gal}(\overline{K}/K) \to 1.$$

The profinite group $\varprojlim_{T} \pi_1(X_K \setminus (T \cup S), \eta)^{\text{geo}}$ is *free* if $\operatorname{char}(K) = 0$ as follows from the well-known structure of the geometric étale fundamental groups of (affine) curves in characteristic zero (cf. [Grothendieck], Exposé XIII, Corollaire 2.12).

Theorem 3.4. *Assume* $\operatorname{char}(K) = 0$ *with no restriction on* $\operatorname{char}(k) = p \ge 0$*. Let* ℓ *be a prime integer (possibly equal to* $\operatorname{char}(k)$ *if* $\operatorname{char}(k) > 0$*). Then the morphism* $\mathcal{U} \to X_K$ *induces (via the rigid GAGA functor) a continuous homomorphism* $\pi_1(\mathcal{U} \setminus S, \eta)^{\text{geo}} \to \varprojlim_{T} \pi_1(X_K \setminus (T \cup S), \eta)^{\text{geo}}$ *(resp.* $\pi_1(\mathcal{U} \setminus S, \eta)^{\text{geo},\ell} \to \varprojlim_{T} \pi_1(X_K \setminus (T \cup S), \eta)^{\text{geo},\ell}$*) which makes* $\pi_1(\mathcal{U} \setminus S, \eta)^{\text{geo}}$ *(resp.* $\pi_1(\mathcal{U} \setminus S, \eta)^{\text{geo},\ell}$*) into a semi-direct* *factor of* $\varprojlim_T \pi_1(X_K\backslash(T\cup S),\eta)^{\rm geo}$ *(resp. direct factor of* $\varprojlim_T \pi_1(X_K\backslash(T\cup S),\eta)^{{\rm geo},\ell}$*). In particular, the pro-ℓ group $\pi_1(\mathcal{U}\setminus S,\eta)^{{\rm geo},\ell}$ is free.*

Proof. The proof is similar, almost word by word, to the proof of Theorem 3.3. One has to use Proposition 2.3.5 instead of the use of Proposition 2.3.4 made in the proof of Theorem 3.3. $\square$

Proposition 3.5. *Assume* $\mathrm{char}\, k = p > 0$ *with no restriction on* $\mathrm{char}(K)$. *Then the pro-p group $\pi_1(\mathcal{U}\setminus S,\eta)^{{\rm geo},p}$ is free of infinite rank.*

Proof. The first assertion follows from Theorem 3.3 (resp. Theorem 3.4) if $\mathrm{char}(K) = p$ (resp. $\mathrm{char}(K) = 0$). For the second assertion it suffices to show that the $\mathbb{F}_p$-vector space $H^1_{\rm et}(\mathrm{Spec}\,\mathcal{A}_{\overline{K}},\mathbb{Z}/p\mathbb{Z})$, where $\mathcal{U}=\mathrm{Sp}\,\mathcal{A}$ and $\mathcal{A}_{\overline{K}}\overset{\rm def}{=}\mathcal{A}\otimes_K\overline{K}$, is infinite, which follows easily from the structure of $\mathcal{A}$ as an affinoid algebra. $\square$

Proposition 3.6. *Let* $\mathrm{char}(k) = p \geq 0$ *with no restrictions on* $\mathrm{char}(K)$. *Then the morphism $\mathcal{U}\to X_K$ induces a continuous homomorphism $\pi_1(\mathcal{U}\setminus S,\eta)^{{\rm geo},p'}\to \pi_1(X_K\backslash(\{x_i\}_{i=1}^m\cup S),\eta)^{{\rm geo},p'}$ which makes $\pi_1(\mathcal{U}\backslash S,\eta)^{{\rm geo},p'}$ into a semi-direct factor of $\pi_1(X_K\setminus(\{x_i\}_{i=1}^m\cup S),\eta)^{{\rm geo},p'}$.*

Proof. The proof follows by using similar arguments to the ones used in the proofs of Theorems 3.3 and 3.4. More precisely, with the notations in the proofs of loc. cit. the morphism $\mathcal{W}_{i,j}\to\mathcal{C}_{i,n}$ in this case is a μ_e-torsor, where e is an integer prime-to-p, and extends (uniquely, after possibly a finite extension of K) to a cyclic Galois cover $\mathcal{Y}_{i,j}\to\mathcal{D}_{i,n}$ of degree e ramified only above x_i (cf. Lemma 2.3.1 and the isomorphism $\Gamma\tilde{\to}\widetilde{\Gamma}$ therein).

In what follows let $g\overset{\rm def}{=} g_{X_K}$ be the arithmetic genus of X_K ($g_{\mathcal{U}}\overset{\rm def}{=} g$ is also called the genus of the affinoid $\mathcal{U}$).

Theorem 3.7. *Let* $\mathrm{char}(k) = p \geq 0$ *with no restriction on* $\mathrm{char}(K)$. *Let $S(\overline{K}) = \{y_1,\cdots,y_r\}$ of cardinality $r\geq 0$. Then the homomorphism $\pi_1(\mathcal{U}\setminus S,\eta)^{{\rm geo},p'}\to \pi_1(X_K\backslash(\{x_i\}_{i=1}^m\cup S),\eta)^{{\rm geo},p'}$ (as in Proposition 3.6) is an isomorphism. In particular, $\pi_1(\mathcal{U}\backslash S,\eta)^{{\rm geo},p'}$ is (pro-)prime-to-p free on $2g+m+r-1$ generators and further can be generated by $2g+m+r$ generators $\{a_1,\cdots,a_g,b_1,\cdots,b_g,\sigma_1,\cdots,\sigma_m,\tau_1,\cdots,\tau_r\}$ subject to the unique relation $\prod_{j=1}^g[a_j,b_j]\prod_{i=1}^m\sigma_i\prod_{t=1}^r\tau_t=1$, where σ_i (resp τ_t) is a generator of inertia at x_i (resp. y_t).*

Proof. The second assertion follows from [Grothendieck], Exposé XIII, Corollaire 2.12. The homomorphism $\pi_1(\mathcal{U}\setminus S,\eta)^{{\rm geo},p'}\to\pi_1(X_K\setminus(\{x_i\}_{i=1}^n\cup S),\eta)^{{\rm geo},p'}$ is injective by Proposition 3.6. We show it is surjective. To this end it suffices to show that given a finite Galois covering $f: Y\to X$ with group G of cardinality prime-to-p, with Y normal and geometrically connected, which is étale above $X_K\setminus(\{x_i\}_{i=1}^m\cup S)$, and $\tilde{f}:\mathcal{V}\to\mathcal{U}$ its restriction to $\mathcal{U}$, then $\mathcal{V}$ is geometrically connected. We can assume, without loss of generality, that Y_k is reduced (cf. Abhyankar's Lemma, [Grothendieck], Exposé X, Lemme 3.6). First, note that $f^{-1}(\mathcal{D}_i^o)$ is a disjoint union of finitely many formal open unit discs (cf. [Raynaud, Lemma 6.3.2), $1\leq i\leq m$. Let V be the normalisation of U in $\mathcal{V}$. Suppose that $\mathcal{V}$ is disconnected, then V_k is disconnected, and a fortiori Y_k is also disconnected as $Y_k\setminus V_k$ is regular (cf. loc. cit.), but this contradicts the fact that Y_K is connected. $\square$

Remark 3.8. **(i)** If $\mathrm{char}(k) = 0$ the profinite group $\pi_1(\mathcal{U} \setminus S, \eta)^{\mathrm{geo}}$ is *free* and finitely generated as follows from Theorem 3.7. Apart from this case the profinite group $\pi_1(\mathcal{U} \setminus S, \eta)^{\mathrm{geo}}$ is *not free* (neither is it finitely generated) as the ranks of its maximal pro-ℓ quotients can be different for different primes ℓ (cf. Theorems 3.3, 3.4, and 3.7). In this sense Theorem 3.3 and Theorem 3.4 are optimal results one can prove regarding the structure of the *full* geometric fundamental group of a p-adic affinoid curve.

(ii) There is no analog in mixed characteristics to Theorem 3.4, for the full π_1^{geo}, where one replaces the infinite union of the finite sets of points T (as in loc. cit.) by a single fixed finite set of points $\widetilde{T} \subset \bigcup_{i=1}^m \mathcal{D}_i^o$. More precisely, in this case one can not control the ramification arising from an extension of an étale covering $\mathcal{V} \to \mathcal{U}$ of the affinoid $\mathcal{U}$ to a ramified covering $Y_K \to X_K$.

For example, suppose $\mathrm{char}(K) = 0$ and $\mathrm{char}(k) = p > 0$. let $\mathcal{U} = \mathrm{Sp}\, K < T, \frac{1}{T} >$ be the closed annulus of thickness 0, assume K contains a primitive p-th root of unity ζ and set $\lambda = \zeta - 1$. Consider the étale μ_p-torsor $f : \mathcal{V} \to \mathcal{U}$ given by the equation $Z^p = 1 + \lambda^p T^{-m}$ where $m \geq 1$ is an integer prime-to-p. Consider $X_K = \mathbb{P}^1_K$ as the generic fibre of the formal R-projective line X obtained by glueing the formal closed discs $\mathrm{Sp}\, R < T >$ and $\mathrm{Sp}\, R < \frac{1}{T} >$ along the formal annulus $U = \mathrm{Spf}\, R < T, \frac{1}{T} >$. Then any Galois extension $\tilde{f} : Y_K \to X_K$ of f is ramified inside the closed disc $\mathrm{Sp}\, K < T >$ (which is embedded in X_K) above at least m points. Indeed, let $g : Y \to X$ be the finite generically μ_p-torsor where Y is the normalisation of X in Y_K. Then the finite morphism $Y_k \to X_k$ is generically étale and (generically) defined by an equation $h^p - h = t^{-m}$, where h (resp. t) is the reduction of H defined by $Z = 1 + \lambda H$ (resp. of T) modulo π, which is a generically étale Artin-Schreier cover with conductor m at 0. An easy verification using the Riemman-Hurwitz formula (and an argument reducing to the case where the extension $\tilde{f}$ is étale above $\mathrm{Sp}\, K < \frac{1}{T} >$) shows that any Galois extension $\tilde{f} : Y_K \to X_K$ of f as above is ramified inside the closed disc $\mathrm{Sp}\, K < T >$ above at least m points. As m increases one sees that it is not possible to bound the number of additional branched points in general.

Examples 3.9. Suppose $\mathrm{char}(K) = 0$ and $\mathrm{char}(k) = p > 0$. Let $U = \mathrm{Spf}\, R < T >$ (resp. $U = \mathrm{Spf}\, R < T_1, T_2 > /(T_1T_2 - 1)$) be the standard formal closed unit disc (resp. formal closed annulus of thickness 0) embedded in the R-projective line $X = \mathbb{P}^1_R$ and $X_K \setminus U_K$ is an open unit disc centred at ∞ (resp. embedded in the R-formal model of the projective line $\mathbb{P}^1_K$ consisting of two standard formal closed unit discs D_1 and D_2 centred at 0 and ∞; respectively, which are patched with U along their boundaries ($|T_1| = |T_2| = 1$) and $X_K \setminus U_K$ is the disjoint union of two open unit discs). Let $\mathcal{U} \overset{\mathrm{def}}{=} U_K$ and $S = \{y_1, \cdots, y_r\} \subset \mathcal{U}(K)$ a set of $r \geq 0$ distinct K-rational points. The results of §3 in this case read as follows. First, the homomorphism $\pi_1(\mathcal{U} \setminus S, \eta)^{\mathrm{geo}} \to \varprojlim_T \pi_1(\mathbb{P}^1_K \setminus (T \cup S), \eta)^{\mathrm{geo}}$, where the projective limit is over all finite sets of points $T \subset X_K \setminus \mathcal{U}$, makes $\pi_1(\mathcal{U} \setminus S, \eta)^{\mathrm{geo}}$ into a semi-direct factor of $\varprojlim_T \pi_1(\mathbb{P}^1_K \setminus (T \cup S), \eta)^{\mathrm{geo}}$, the maximal pro-$p$ quotient $\pi_1(\mathcal{U} \setminus S, \eta)^{\mathrm{geo},p}$ is pro-p free of infinite rank, and the maximal prime-to-p quotient $\pi_1(\mathcal{U} \setminus S, \eta)^{\mathrm{geo},p'}$ is (pro-)-prime-to-p free of rank r (resp. $r + 1$).

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Mohamed Saïdi
College of Engineering, Mathematics and Physical Sciences
University of Exeter
Harrison Building
North Park Road
EXETER EX4 4QF
United Kingdom
M.Saidi@exeter.ac.uk