A theory of passive linear systems with no assumptions

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Abstract

We present two linked theorems on passivity: the passive behavior theorem, parts 1 and 2. Part 1 provides necessary and sufficient conditions for a general linear system, described by a set of high order differential equations, to be passive. Part 2 extends the positive-real lemma to include uncontrollable and unobservable state-space systems.

Key words: Passive system; Positive-real lemma; Linear system; Controllability; Observability; Behavior.

1 Introduction

A system is called passive if there is an upper bound on the net energy that can be extracted from the system from the present time onwards. This is a fundamental property of many physical systems. In systems and control theory, the concept of passivity has its origins in the study of electric networks comprising resistors, inductors, capacitors, transformers, and gyrators (RLCTG networks). In contemporary systems theory, passive systems are more familiar through their role in the positive-real lemma. This lemma proves the equivalence of: (i) an integral condition related to the energy exchanged with the system; (ii) a condition on the transfer function for the system (the positive-real condition); and (iii) a linear matrix inequality involving the matrices in a state-space realization for the system. As well as being relevant to passive systems, the lemma also gives necessary and sufficient conditions for the existence of a non-negative definite solution to an important linear matrix inequality (and algebraic Riccati equation) considered in the positive-real lemma are unknown when the state-space realization under consideration is uncontrollable. There have been many papers in the literature that have aimed to relax the assumption of controllability in the positive-real lemma, e.g., Pandolfi (2001); Collado et al. (2001); Kunimatsu et al. (2008) (and many papers have studied uncontrollable cyclo-dissipative systems, e.g., Ferrante and Pandolfi (2002); Camlibel et al. (2003); Ferrante (2005); Pal and Belur (2008)), but all of these papers contain other a-priori assumptions. The objective of this paper is to provide a complete theory of passive linear systems with no superfluous assumptions. Our main contributions are: 1. a new trajectory-based definition of passivity (Definition 5); and 2. two linked theorems that we call the passive behavior theorem, parts 1 and 2. Part 1 (Theorem 9) provides necessary and sufficient conditions for the passivity of a general linear system (described by a differential equation of the form \[ P \frac{d}{dt} \mathbf{y} = Q \frac{d}{dt} \mathbf{v} \] for some square polynomial matrices \( P \) and \( Q \)). This generalizes classical results that are restricted to controllable behaviors (where \( P \) and \( Q \) are left coprime). Part 2 (Theorem 13) extends the positive-real lemma by removing the a-priori controllability and observability assumptions. As a corollary of these results, we find that any passive (not necessarily controllable) behavior can be realized as the...
driving-point behavior of an electric (RLCTG) network.

The notation is as follows. \( \mathbb{R} (\mathbb{C}) \) denotes the real (complex) numbers; \( \mathbb{C}_+ \) (\( \mathbb{C}_- \)) denotes the open (closed) right-half plane; \( \mathbb{C}_- \) (\( \mathbb{C}_+ \)) denotes the open (closed) left-half plane. \( \mathbb{R} [\xi] (\mathbb{R} (\xi)) \) denotes the polynomials (rational functions) in the indeterminate \( \xi \) with real coefficients. \( \mathbb{R}^{m \times n} \) (resp., \( \mathbb{C}^{m \times n} \), \( \mathbb{R}^{m \times n} [\xi] \), \( \mathbb{R}^{m \times n} (\xi) \)) denotes the matrices with \( m \) rows and \( n \) columns with entries from \( \mathbb{R} \) (resp., \( \mathbb{C}, \mathbb{R}[\xi], \mathbb{R}(\xi) \)), and the number \( n \) is omitted whenever \( n = 1 \). If \( H \in \mathbb{R}^{m \times n} \), then \( H(\xi) \) denotes its real (imaginary) part, and \( H^T \) its complex conjugate. If \( H \in \mathbb{R}^{m \times n} \), \( \mathbb{R}^{m \times n} [\xi] \) or \( \mathbb{R}^{m \times n} (\xi) \), then \( H^T \) denotes its transpose; and if \( H \) is nonsingular (i.e., det \( H \) \( \neq 0 \)), then \( H^{-1} \) denotes its inverse. We let \( \text{col}(H_1 \ldots H_n) \) (\( \text{diag}(H_1 \ldots H_n) \)) denote the block column (block diagonal) matrix with entries \( H_1, \ldots, H_n \). If \( M \in \mathbb{R}^{m \times n} \), then \( M > 0 \) (\( M \geq 0 \)) indicates that \( M \) is Hermitian positive (non-negative) definite, and spec \( (M) \doteq \{ \lambda \in \mathbb{C} \mid \det (\lambda I - M) = 0 \} \). If \( G \in \mathbb{R}^{m \times n} (\xi) \), then normalrank \( (G) \doteq \max_{\lambda \in \mathbb{C}} \text{rank}(G(\lambda)) \). \( G^*(\xi) \doteq G(-\xi)^T \), \( G \) is called para-Hermitian if \( G = G^* \), and proper if \( \lim_{\xi \to \infty} G(\xi) \) exists. \( \mathcal{L}^{\text{loc}} (\mathbb{R}, \mathbb{R}^k) \) and \( \mathcal{C}_\infty (\mathbb{R}, \mathbb{R}^k) \) denote the (k-vector-valued) locally integrable and infinitely-often differentiable functions (Pol- derman and Willems, 1998, Definitions 2.3.3, 2.3.4). We equate any two locally integrable functions that differ only on a set of measure zero. If \( w \in \mathcal{L}^{\text{loc}} (\mathbb{R}, \mathbb{R}^k) \), then \( w^T \) denotes the function satisfying \( w^T(t) = w(t)^T \) for all \( t \in \mathbb{R} \). We also consider the function space

\[
\mathcal{E}_{C_-}(\mathbb{R}, \mathbb{R}^k) \doteq \{ w \mid w(t) = \mathbb{R}^{N \cdot n_i - 1} \sum_{i=1}^{N} \sum_{j=0}^{n_i} \begin{bmatrix} \vec{w}_{ij} \end{bmatrix} e^{\lambda_it} \} \quad \text{for all } t \in \mathbb{R}
\]

with \( \vec{w}_{ij} \in \mathbb{C}^k, \lambda_i \in \mathbb{C}_-, \) and \( N, n_i \) integers}, and note that \( \mathcal{E}_{C_-}(\mathbb{R}, \mathbb{R}^k) \subset \mathcal{C}_\infty (\mathbb{R}, \mathbb{R}^k) \subset \mathcal{L}^{\text{loc}}_1 (\mathbb{R}, \mathbb{R}^k) \).

We consider behaviors (systems) defined as the set of weak solutions to a linear differential equation:

\[
\mathbf{B} = \{ w \in \mathcal{L}^{\text{loc}}_1 (\mathbb{R}, \mathbb{R}^k) \mid R(\frac{dx}{dt})w = 0 \}, \quad R \in \mathbb{R}^{d \times k}[\xi]. \quad (1.1)
\]

Here, if \( R(\xi) = R_0 + R_1 \xi + \ldots + R_L \xi^L \) and \( w \in \mathcal{C}_\infty (\mathbb{R}, \mathbb{R}^k) \), then \( R(\frac{dx}{dt})w = R_0 w + R_1 \frac{dw}{dt} + \ldots + R_L \frac{d^Lw}{dt^L} \) (see Polderman and Willems, 1998, Definition 3.2.7 for the meaning of a weak solution to \( R(\frac{dx}{dt})w = 0 \) when \( w \) is not necessarily differentiable). Particular attention is paid to the special class of state-space systems:

\[
\mathbf{B}_s = \{ (u, y, x) \in \mathcal{L}^{\text{loc}}_1 (\mathbb{R}, \mathbb{R}^n) \times \mathcal{L}^{\text{loc}}_1 (\mathbb{R}, \mathbb{R}^n) \times \mathcal{L}^{\text{loc}}_1 (\mathbb{R}, \mathbb{R}^d) \mid \begin{array}{l}
\forall \xi \in \mathbb{R} (\xi), R(\frac{dx}{dt})w = 0 \Rightarrow x(t_1) = x(t_2) \\
A \in \mathbb{R}^{d \times d}, B \in \mathbb{R}^{d \times n}, C \in \mathbb{R}^{n \times d}, D \in \mathbb{R}^{n \times n}
\end{array} \} \quad (1.2)
\]

Several properties of state-space systems are listed in Appendix D. In particular, from note D1, if \( (u, y, x) \in \mathbf{B}_s \), then \( x \) satisfies the variation of the constants formula almost everywhere, which determines the value \( x(t_1) \) of \( x \) at an instant \( t_1 \in \mathbb{R} \). Finally, we also consider behaviors obtained by permuting and/or eliminating variables in a behavior \( \mathbf{B} \) as in (1.1). For example, associated with the state-space system \( \mathbf{B}_s \) in (1.2) is the corresponding external behavior \( \mathbf{B}_s^{(u,y)} = \{ (u, y) \mid \exists x \text{ with } (u, y, x) \in \mathbf{B}_s \} \).

More generally, for any given \( T_1 \in \mathbb{R}_{1 \times k}, \ldots, T_n \in \mathbb{R}^{l \times k} \) such that \( \text{col}(T_1 \ldots T_n) \in \mathbb{R}^{l \times k} \) is a permutation matrix, and integer \( 1 \leq n \leq \xi \), we denote the projection of \( \mathbf{B} \) onto \( T_1 w, \ldots, T_n w \) by

\[
\mathbf{B}(T_1 w, \ldots, T_n w) = \{ (T_1 w, \ldots, T_n w) \mid \exists (T_{m+1} w, \ldots, T_n w) \text{ such that } w \in \mathbf{B} \}.
\]

2 The positive-real lemma

The central role of passivity in systems and control is exemplified by the positive-real lemma (see Lemma 1). The name positive-real (PR) describes a function \( G(\xi) \in \mathbb{R}^{n \times n}(\xi) \) with the properties: (i) \( G \) is analytic in \( \mathbb{C}_+ \); and (ii) \( G(\lambda)^T + G(\lambda) > 0 \) for all \( \lambda \in \mathbb{C}_+ \) (see Anderson and Vongpanitlerd, 1973, Theorem 2.7.2 for a well-known equivalent condition). The positive-real lemma makes a state-space system as in (1.2) and provides necessary and sufficient conditions for the transfer function \( G(\xi) = D + C(\xi I - A)^{-1} B \) to be PR. Notably,
it is assumed that \((A, B)\) is controllable and \((C, A)\) is observable (see notes D2 and D4).

**Lemma 1 (Positive-real lemma)** Let \(B_s\) be as in (1.2) and let \((A, B)\) be controllable and \((C, A)\) observable. Then the following are equivalent:

1. Given any \(x_0 \in \mathbb{R}^d\), there exists \(S_a(x_0) \in \mathbb{R}\) with
   \[
   S_a(x_0) := \sup_{t_1 \geq t_0 \in \mathbb{R}} \sup_{(u, y, x) \in \mathbb{E}} \left(-\int_{t_0}^{t_1} u^T(t)y(dt)\right).
   \]
2. \(G(X) := 0 + \text{spec}(A+B(D+D^T)^{-1}(B^T X + C)) \in \mathbb{C}_+\).
3. There exist real matrices \(X, L_X, \hat{W}_X\) such that \(X > 0, -A^TX-XA=(C^T-XB)(D+D^T)^{-1}(C-B^TX) = 0\) and \(\text{spec}(A+(B+D^T)(D+D^T)^{-1}(B^TX-C)) \in \mathbb{C}_+\).
4. \(G(X) := D + C[(I-A)^{-1}]B \in \mathbb{R}\).

If, in addition, \(D + D^T > 0\), then the above conditions are equivalent to:

5. There exists a real \(X > 0\) such that \(\Pi(X) := -A^TX-XA-(C^T-XB)(D+D^T)^{-1}(C-B^TX) = 0\) and \(\text{spec}(A+B(D+D^T)^{-1}(B^TX-C)) \in \mathbb{C}_+\).

For a proof of the positive-real lemma, we refer to Willems (1972b); Anderson and Vongpanitlerd (1973). These references also describe links with spectral factorization, which is the concern of the following well-known result (Youla, 1961; Theorem 2):

**Lemma 2 (Youla’s spectral factorisation result)** Let \(H \in \mathbb{R}^{n \times n}(\xi)\) be para-Hermitian; let \(H(\omega) > 0\) for all \(\omega \in \mathbb{R}\), \(\omega\) not a pole of \(H\); and let normalrank\((H) = r\). There exists a \(Z \in \mathbb{R}^{n \times n}(\xi)\) such that (i) \(H = Z^* Z\); (ii) \(Z\) is analytic in \(C_+\); and (iii) \(Z(\xi)\) has full row rank for all \(\lambda \in C_+\). Moreover, if \(H \in \mathbb{R}^{n \times n}(\xi)\), then \(Z \in \mathbb{R}^{n \times n}(\xi)\); if \(H(\omega)\) is analytic for all \(\omega \in \mathbb{R}\), then \(Z\) is analytic in \(\mathbb{C}_+\); and if \(Z_1 \in \mathbb{R}^{n \times n}(\xi)\) also satisfies (i)–(iii), then there exists a \(T \in \mathbb{R}^{n \times r}\) such that \(Z_1 = TZ\) and \(T^* T = I\). We call any \(Z \in \mathbb{R}^{n \times n}(\xi)\) that satisfies (i)–(iii) a spectral factor of \(H\).

**Remark 3** When \(G\) is as in Lemma 1 with \(D + D^T > 0\), there exists \(W_X \in \mathbb{R}^{n \times n}\) with \(D + D^T = W_X^T W_X\). Then, with \(X\) as in condition 5 of Lemma 1, it can be shown that \(Z_X(\xi) := W_X + W_X^{-1}(C - B^TX)(C - B^TX)^{-1}B\) is a spectral factor of \(G + G^*\) (see Willems 1972b).

The assumptions in Lemma 1 can be relaxed in three particularly notable ways. First, from Willems (1971, Theorems 1, 3, 8), conditions 1–4 of Lemma 1 are equivalent even if \((C, A)\) is not observable, but \(X\) may then be singular in condition 3. Second, the following are equivalent irrespective of whether \((A, B)\) is controllable or \((C, A)\) is observable: (i) \(\text{spec}(A) \in \mathbb{C}_+\) and \(G(-j\omega)^T + G(j\omega) > 0\) for all \(\omega \in \mathbb{R} \cup \infty\); and (ii) the existence of a real symmetric \(X > 0\) such that \(\Pi(X) = 0\) and \(\text{spec}(A + B(D+D^T)^{-1}(B^TX - C)) \in \mathbb{C}_+\) (Zhou et al. 1996; Corollary 13.27). Third, if \(\text{spec}(A) \in \mathbb{C}_+\), then condition 3 in Lemma 1 is equivalent to condition 4 together with the additional condition (Pandolfi 2001, equation (4)) (this condition will be discussed in Remark 22).

Nevertheless, the results in these references, and other similar results in the literature (e.g., Collado et al. 2001; Kunimatsu et al. 2008), do not cover several important systems. In particular, they do not consider systems whose transfer functions possess imaginary axis poles. We consider one such system in Example 4. Other important examples include conservative systems, whose transfer functions are lossless PR (see Anderson and Vongpanitlerd 1973, Chapter 2).

**Example 4** Let \(B_s\) be as in (1.2) with

\[
A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \ 0], \quad \text{and } D = 1.
\]

Here, \((A, B)\) is not controllable. We now show that conditions 2 and 4 of Lemma 1 hold for this example, yet condition 1 does not. First, direct calculation verifies that \(G(\xi) = 1 + 1/\xi\), and so condition 4 is satisfied. Second, from the variation of the constants formula (see note D1),

\[
y(t) = x_1(t_0) + (2\cos(\xi-t_0) - 1)x_2(t_0) + 2\sin(\xi-t_0)x_3(t_0) + u(t) + \int_0^t u(\tau)d\tau \\
\]

for all \(t \geq t_0\). Hence, if \(x(t_0) = 0\) and \(t_1 \geq t_0\), then \(f_{t_1}^t u(t)y(dt) = f_{t_1}^t u^2(dt) + \frac{1}{2}(f_{t_1}^t u(\tau)d\tau)^2 \geq 0\), and so condition 2 is satisfied. Third, with \(x_1(0) = x_2(0) = 0, x_3(0) = -1, \) and \(u(t) = \sin(t)\) for all \(t \geq 0\), then \(y(t) = -\sin(t) - \cos(t)\) for all \(t \geq 0\). Thus, for any given positive integer \(n\),

\[
\int_0^{\xi n\pi} y(t)u(dt)dt = \int_0^{\xi n\pi} \sin^2(t)dt + \int_0^{\xi n\pi} \sin(t)\cos(t)dt = \frac{\xi n\pi}{4}.
\]

It follows that condition 1 does not hold. Furthermore, it will follow from Theorem 13 of this paper that condition 3 of Lemma 1 does not hold for this system.

One of the main contributions of this paper is a generalization of the positive-real lemma to include state-space systems that are not necessarily controllable or observable (Theorem 13). In contrast to other papers on this subject, we do not introduce any superfluous assumptions. However, as we will argue in the next section, a state-space system is not a natural starting point for the study of passive systems. Thus, a second major contribution of this paper is a necessary and sufficient condition for the passivity of a general linear system, described by a set of high order differential equations (Theorem 9).
that $-\int_{t_0}^{t_1} \dot{V}(t) v(t) dt$ is the net energy extracted from the system in the interval from $t_0$ to $t_1$. Passivity has its origins in the study of electric RLCTG networks, for which $i$ represents the driving-point currents and $v$ the corresponding driving-point voltages. As shown in [Hughes 2017a], for any given RLCTG network, the driving-point currents and voltages are related by a linear differential equation of the form:

$$B = \{ (i, v) \in L^1_{\text{loc}}(R, R^n) \times L^1_{\text{loc}}(R, R^n) \mid P(\frac{d}{dt}) v = Q(\frac{d}{dt}) i, \text{for some } P, Q \in \mathbb{R}^{n \times n}[\xi] \}. \quad (3.1)$$

Note that $(i, v)$ need not be an input-output partition in the sense of [Polderman and Willems 1998]. For example: (i) $Q$ is singular for a transformed $1$ and (ii) $Q^{-1}P$ is not proper for an inductor. Yet it is common for passivity to be defined for systems described using a state-space or input-output representation. This implies assumptions that (i) $Q$ is nonsingular; and (ii) $Q^{-1}P$ is proper. Accordingly, we provide a new definition of passivity for the general system in (3.1) that does not depend on such assumptions. Note that this definition extends naturally to non-linear and time-varying systems.

**Definition 5 (Passive system)** The system $B$ in (3.1) is called passive if, for any given $(i, v) \in B$ and $t_0 \in \mathbb{R}$, there exists a $K \in \mathbb{R}$ (dependent on $(i, v)$ and $t_0$) such that if $(i, \dot{v}) \in B$ satisfies $(i(t), \dot{v}(t)) = (i(t), v(t))$ for all $t < t_0$, then $-\int_{t_0}^{t_1} \hat{V}(t) \dot{v}(t) dt < K$ for all $t_1 \geq t_0$.

In words, a system is passive if there is an upper bound to the net energy that can be extracted from the system from $t_0$ onwards. The upper bound depends on the past of the trajectory, but, given this past, the same upper bound applies to all possible future trajectories.

A detailed discussion of the issues with existing definitions of passivity (and dissipativity) was provided in [Willems 2007, Section 8]. However, for reasons detailed at the end of this section, our definition differs from a similar definition proposed by [Willems 2007]. First, we compare Definition 5 to the conditions of the positive-real lemma. Note that it is not necessary to follow the discussion in the remainder of this section to understand the main results in the paper.

Condition 2 of Lemma 1 is sometimes stated as the definition of passivity for the system in (1.2) (e.g., [Anderson and Vongpanitlerd 1973, Section 2.3]). However, the system in Example 4 satisfies this condition but is not passive in the sense of Definition 5. In other words, condition 1 of Lemma 1 is stated as the definition for passivity (e.g., [Willems 1972b]). It is shown in [Hughes 2017b] that this is consistent with Definition 5 when considering systems with a state-space realization as in (1.2), where $i = u$ and $v = y$. However, as mentioned earlier, there are systems that are passive in the sense of Definition 5 that cannot be represented in this form. Specifically, as will be shown in Lemma 12, condition 1 of Lemma 1 only applies to systems of the form:

$$\tilde{B} = \{ (u, y) \in L^1_{\text{loc}}(R, R^n) \times L^1_{\text{loc}}(R, R^n) \mid P(\frac{d}{dt}) u = Q(\frac{d}{dt}) y, \text{where } \tilde{P}, \tilde{Q} \in \mathbb{R}^{n \times n}[\xi], \tilde{Q} \text{ is nonsingular, and } \tilde{Q}^{-1}\tilde{P} \text{ is proper.} \} \quad (3.2)$$

Thus, this condition does not cover systems of the form (5) for which either $Q$ is singular or $Q^{-1}P$ is not proper.

Definition 5 is similar to a definition for dissipativity proposed in [Willems 2007 (Section 8)] and used by [Hughes and Smith 2017] (note that it is straightforward to generalize Definition 5 to the framework of dissipative systems). In [Hughes and Smith 2017], the system $B$ in (3.1) was called passive if, given any $(i, v) \in B$ and any $t_0 \in \mathbb{R}$, there exists a $K \in \mathbb{R}$ (dependent on $(i, v)$ and $t_0$) such that $\int_{t_0}^{t_1} \hat{V}(t) v(t) dt < K$ for all $t_1 \geq t_0$. Evidently, if $B$ in (3.1) is passive in the sense of Definition 5, then $B$ is also passive in the sense of [Willems 2007, Hughes and Smith 2017]. It can also be shown that the converse is true. However, Definition 5 is a more accurate statement of the physical property of passivity (when extended to time-varying and non-linear systems), as the following example demonstrates.

**Example 6** Consider the behavior $B = \{ (u, y) \in L^1_{\text{loc}}(R, R) \times L^1_{\text{loc}}(R, R) \mid \exists x \in L^1_{\text{loc}}(R, R) \text{ with } (i) \ x(t) = 0 \text{ and } y(t) = 0 \text{ for all } t < 0; \ (ii) \ \frac{dx}{dt}(t) = u(t) \text{ and } y(t) = 0 \text{ for all } 0 \leq t < 1; \ (iii) \ \frac{dx}{dt}(t) = u(t) \text{ and } y(t) = 2x(t) \text{ for all } 1 \leq t < 2; \text{ and } (iv) \ \frac{dx}{dt}(t) = 0 \text{ and } y(t) = 0 \text{ for all } t \geq 2 \}$. Thus, if either $t_0 \geq 2$ or $t_1 \leq 1$, then $\int_{t_0}^{t_1} u(t) y(t) dt = 0$; if instead $t_1 > t_0$, $t_0 < 2$, and $t_1 > 1$, then $\int_{t_0}^{t_1} u(t) y(t) dt = -[x^2(t) - x^2(1)]_{max(1,t_0)}^{min(2,t_1)} = -[(0)_0 u(\tau) \text{ d}t]_{max(1,t_0)}^{min(2,t_1)} \leq (j_0)_0 u(\tau) \text{ d}t \text{ d}x^2$. It follows that, given any $x(t) \in \mathbb{R}$, there exists a $K \in \mathbb{R}$ depending on $t_0$ and $(u, y)$ such that $\int_{t_0}^{t_1} u(t) y(t) dt < K$ for all $t_1 > t_0$, and so $B$ is passive in the sense of [Hughes and Smith 2017]. On the other hand, for any given $(u, y) \in B$, $t_0 < 1$, and $K > 0$, there exists a $x(t) \in \mathbb{R}$ with $(u(t), y(t)) = (u(t), y(t))$ for all $t < t_0$ such that $\int_{t_0}^{t_1} u(t) y(t) dt > K$. (e.g., let $u(t) = (\sqrt{K} - j_0)_0 u(\tau) \text{ d}t/(1 - t_0)$ for all $0 \leq t < 1$, and $u(t) = -\sqrt{K}$ for all $t \geq 1$). Thus, if $t_0 < 1$, then an arbitrarily large amount of energy can be extracted.

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1. The behavior of a transformer with turns-ratio matrix $T \in \mathbb{R}^{n_1 \times n_2}$ is determined by the equations $V_1 = T^T V_2$, and $i_2 = -T \lambda_1$, with $v = \text{col}(v_1, v_2)$ and $i = \text{col}(i_1, i_2)$.

2. For an inductor with inductance $L$, then $Q^{-1}P(\xi) = L \xi$. Minor adjustments can be made to the proof given in this paper to show that if $B$ is passive in the sense of [Hughes and Smith 2017], then condition 2 of Theorem 13 holds.
from this system from $t_0$ onwards, and this system is not passive in the sense of Definition 5.

Motivated by electric (RLCTG) networks, we have introduced a definition for passivity for the system in (3.1). The classical theory of electric networks provides necessary and sufficient conditions on $P$ and $Q$ for the system in (3.1) to be realized by an RLCTG network providing $P$ and $Q$ are left coprime. Yet, as emphasised in [Camlibel et al., 2003], such conditions are unknown in cases when $P$ and $Q$ are not left coprime. More fundamentally, in these cases, necessary and sufficient conditions on $P$ and $Q$ for the system in (3.1) to be passive are also unknown. Such conditions are provided in Theorem 9 of this paper.

4 The passive behavior theorem

In this section, we present our new passive behavior theorem in two parts. The theorems use our new concept of a positive-real pair, which we define as follows:

**Definition 7** Let $P, Q \in \mathbb{R}^{n \times n}[\lambda]$. We call $(P, Q)$ a positive-real pair if the following conditions hold:

1. $P(\lambda)Q(\lambda) + Q(\lambda)P(\lambda) = 0$ for all $\lambda \in \mathbb{C}_+$.
2. $\text{rank}([P - Q(\lambda)]) = n$ for all $\lambda \in \mathbb{C}_+$.
3. If $p \in \mathbb{R}^n[\lambda]$ and $\lambda \in \mathbb{C}$ satisfy $p^T (PQ^* + QP^*) = 0$ and $p(\lambda)^T [P - Q(\lambda)] = 0$, then $p(\lambda) = 0$.

**Remark 8** A key result in behavioral theory is that any behavior $\mathcal{B}$ as in (1.1) has a controllable part ($B_c$ in Lemma 17) and an autonomous part ($B_a$ in Lemma 17). As will be shown in Section 5, the conditions in Definition 7 can be understood in terms of $B_c$ and $B_a$. Roughly speaking, the passivity of $B_a$ implies condition 1; the stability of $B_a$ implies condition 2, as does the stabilizability of $\mathcal{B}$ (see note B3); and condition 3 is a coupling condition between the trajectories in $B_a$ and the so-called lossless trajectories in $B_c$. In particular, if the transfer function from $u$ to $v$ is lossless PR (see [Anderson and Vongpanitlerd, 1973] Chapter 2), then $PQ^* + QP^* = 0$, and condition 3 implies that $P$ and $Q$ are left coprime, so $\mathcal{B}$ is controllable (see note B3).

We note that condition 1 of Definition 7 is a natural generalization of a positive-real transfer function $Q^{-1}P$ to the case with $Q$ singular. Yet, as discussed in Section 3, this condition is not sufficient for the behavior $\mathcal{B}$ in (3.1) to be passive. As the following theorem demonstrates, conditions 2 and 3 are also required to obtain a necessary and sufficient condition for passivity.

**Theorem 9** (Passive behavior theorem, Part 1)

Let $\mathcal{B}$ be as in (3.1). Then the following are equivalent:

1. $\mathcal{B}$ is passive.
2. $(P, Q)$ is a positive-real pair.
3. There exist compatible partitions $i = (i_1, i_2)$ and $v = (v_1, v_2)$ such that $\tilde{B} : = B_{(\text{col}(i_1, v_2), \text{col}(v_1, i_2))}$ takes the form of (3.2), and $\tilde{B}$ is passive.

**Remark 10** It is also the case that the conditions in Theorem 9 hold if and only if $\mathcal{B}$ is the driving-point behavior of an electric RLCTG network ([Hughes, 2017a]).

**Remark 11** In the terminology of behavioral theory, condition 3 of Theorem 9 implies that if $\mathcal{B}$ is as in (3.1) and normalrank($[P - Q]$) = $n$, then there exists an input-output partitioning of col($i$) into $u \in L_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$ and $v \in L_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$, for which $\tilde{B} := B_{(u, v)}$ takes the form of (3.2) ([Polderman and Willems, 1998], Section 3.3). However, this does not suffice to show condition 3 in Theorem 9. For example, for the system

$$
\begin{bmatrix}
0 & \frac{d}{dt} + 1 & i_1 \\
0 & 0 & i_2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
0 & \frac{d}{dt} + 2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix},
$$

it can be shown that there is no input-output partition with the property that $i_1v_1 + i_2v_2 = u^Tv$.

Theorem 9 allows us to apply the following results from [Willems, 1986; Rapisarda and Willems, 1997; Hughes, 2016a] on state-space realizations of behaviors.

**Lemma 12** Let $B_s$ be as in (1.2). Then there exist polynomial matrices $M, N, \tilde{P}$ and $\tilde{Q}$ such that

1. $\tilde{M} \in \mathbb{R}^{n \times n}[\lambda]$ and $\tilde{N} \in \mathbb{R}^{n \times d}[\lambda]$ are left coprime;
2. $M(\lambda)C = N(\lambda)(\xi I - A)$;
3. $\tilde{P} := NB + MD$ and $\tilde{Q} := \tilde{M}$.

Furthermore, if $\tilde{M}, \tilde{N}, \tilde{P}$ and $\tilde{Q}$ satisfy conditions 1–3, then $\tilde{B} := B_{(u, v)}^s$ takes the form of (3.2).

Now, let $\tilde{B}$ take the form of (3.2). Then there exists $B_s$ as in (1.2) such that $\tilde{B} = B_{(u, v)}^s$. Also, for any such $B_s$, there exist $\tilde{M}$ and $\tilde{N}$ such that conditions 1–3 hold.

In the next theorem, we consider the state-space system $B_s$ in (1.2), and we provide necessary and sufficient conditions for $\tilde{B}_{(u, v)}^s$ to be passive. This generalizes the positive-real lemma (Lemma 1) to state-space systems that need not be controllable or observable.

**Theorem 13** (Passive behavior theorem, Part 2)

Let $B_s$ be as in (1.2); let $\tilde{P}, \tilde{Q}$ be as in Lemma 12; and let $G(\xi) := D + C(\xi I - A)^{-1}B$. Then the following are equivalent:
1. $\mathcal{B} := \mathcal{B}_s^{(u,y)}$ is passive.
2. $(\mathcal{P}, \mathcal{Q})$ is a positive-real pair.
4. There exist real matrices $X, L_X, W_X$ as in condition 3 that have the additional property that $W_X + L_X(\xi I - A)^{-1} B$ is a spectral factor of $G + G^*$.

If, in addition, $D + D^T > 0$, then the above conditions are equivalent to:

5. There exists a real $X \geq 0$ such that $\Pi(X) := -A^T X - XA - (C^T - X B)(D + D^T)^{-1}(B^T - C) = 0$.

Now, suppose conditions 1–4 hold. Then:

(i) If $(C, A)$ is observable and $X$ is as in condition 3, then $(a) X > 0$; and $(b) \text{spec}(A) \in \mathbb{C}$.

(ii) If $D + D^T > 0$ and $X$ is as in condition 4, then $(a) \Pi(X) = 0$; and $(b) \text{spec}(A + B(D + D^T)^{-1}(B^T - C)) \in \mathbb{C}$ if and only if $\text{spec}(A) \in \mathbb{C}$.

**Remark 14** Note that, if the conditions in Theorem 13 hold for one state-space realization $B_s$ of $\mathcal{B} := \mathcal{B}_s^{(u,y)}$, then they hold for all state-space realizations of $\mathcal{B}$. Note also that $\mathcal{P}$ and $\mathcal{Q}$ are not uniquely defined in that theorem, but it is straightforward to show that condition 2 is invariant of the specific choice of matrices.

**Remark 15** Let $X, L_X, W_X$ be as in condition 3 of Theorem 13, let $(u, y, x) \in \mathcal{B}_s$, and let $t_0 \leq t_1 \in \mathbb{R}$. Since $x$ is absolutely continuous, then integration by parts gives

$$
\int_{t_0}^{t_1} u^T(t) y(t) + y^T(t) u(t) dt - [x^T(t) x(t)]_{t_0}^{t_1} = \int_{t_0}^{t_1} (L_X x + W_X u) x(t) + W_X u(t) dt \geq 0.
$$

With the notation $S(x) := \frac{1}{2} x^T X x$ for all $x \in \mathbb{R}^d$, it is straightforward to verify that $S$ is a storage function with respect to the supply rate $u^T y$ in the sense of (Willems [1972a, Definition 2]). It follows from Theorem 13 that, if $\mathcal{B} := \mathcal{B}_s^{(u,y)}$ is passive (in accordance with the trajectory-based Definition 5), then $B_s$ has a (non-negative) quadratic state storage function.

**Remark 16** It is instructive to compare Theorems 9 and 13 with papers by Camlibel et al. [2003]; Pal and Belur [2008], which consider cyclo-dissipativity in the behavioral framework. The reader who is unfamiliar with these papers may prefer to skip straight to Section 5.

In Camlibel et al. [2003]; Pal and Belur [2008], cyclo-dissipativity is defined using the formalism of quadratic differential forms (see Appendix C). With $B$ as in (3.1), then $\mathcal{B}_{cyc} := \mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^n) \times C^\infty(\mathbb{R}, \mathbb{R}^n)$ is called cyclo-dissipative with respect to the supply rate $i^T y$ (or cyclo-passive) if there exists a quadratic differential form $Q_e$ such that $i^T y \geq \frac{\gamma}{2} Q_e(c(i, v))$ for all $(i, v) \in \mathcal{B}_{cyc}$ (Pal and Belur [2008] Definition 3.1). Also, $\mathcal{B}_{cyc}$ is called strictly cyclo-dissipative with respect to the supply rate $i^T y$ (or strictly cyclo-passive) if there exists a quadratic differential form $Q_e$ and an $\epsilon > 0$ such that $i^T y \geq \frac{\gamma}{2} Q_e(c(i, v)) + \epsilon(i^T i + v^Tv)$ for all $(i, v) \in \mathcal{B}_{cyc}$ (Pal and Belur [2008] Definition 3.2). In these definitions, $Q_e$ is called a storage function (Trentelman and Willems [1997], Definition 4.2), which is called non-negative if $Q_e(c(i, v))(t) \geq 0$ for all $(i, v) \in \mathcal{B}_{cyc}$ and all $t \in \mathbb{R}$. Camlibel et al. [2003] considered cyclo-passive single-input single-output systems, while Pal and Belur [2008] considered a class of strictly cyclo-dissipative systems that includes the strictly cyclo-passive systems.

It can be shown that there are cyclo-passive systems that are not passive, and there are passive systems that are not strictly cyclo-passive. Thus the problems considered in Camlibel et al. [2003]; Pal and Belur [2008] are not equivalent to the problem considered in this paper. It can also be shown from Theorems 9 and 13 and Remark 15 that $\mathcal{B}_{cyc}$ is passive in accordance with Definition 5 if and only if $\mathcal{B}_{cyc}$ is cyclo-passive with a non-negative storage function. However, there are two notable reasons why we have not defined a passive system as a cyclo-passive system with a non-negative storage function. First, as discussed in Willems [2004], it is preferable to define passivity without invoking an a-priori assumption of the existence of a quadratic storage function. This is one of the main benefits of Definition 5. Second, we note that there is no consensus on the appropriate definition of a cyclo-dissipative system. This concerns the issue of whether to allow for unobservable storage functions, as arise in electric networks (see Willems [2004]). As shown in that paper, there are systems that are not cyclo-dissipative (with respect to a given supply rate), but do possess an unobservable storage function with respect to that supply rate (Willems [2004], Section VI). This issue does not arise with the definition of passivity given in this paper.

We also note that Camlibel et al. [2003]; Pal and Belur [2008] invoke assumptions that are not present in this paper. In Camlibel et al. [2003], only single-input single-output systems are considered (i.e., $n = 1$ for $B$ in (3.1)), for which condition 3 in Definition 7 takes the much simpler form: if $PQ + QP = 0$, then $|P - Q(\lambda)|$ has full row rank for all $\lambda \in \mathbb{C}$. Also, Camlibel et al. [2003] assume that there are no uncontrollable imaginary axis modes (i.e., rank$(P - Q(\omega i))$ is constant for all $\omega \in \mathbb{R}$). In contrast, we prove that this condition must hold if

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4 Note that these papers use the word dissipative for what we call cyclo-dissipative systems. We reserve the word dissipative for systems that have a non-negative storage function, as in Willems [1972a].
Theorem 13.}

5 Passive behaviors and positive-real pairs

In Section 2, we showed that the system in Example 4 has a positive-real transfer function, yet is not passive. For that system, it can be shown that $E_0^{(0,0)} := \tilde{B} = \{(u,y) \in L_1^{loc}(\mathbb{R},\mathbb{R}) \times L_1^{loc}(\mathbb{R},\mathbb{R}) | \left(\frac{d}{dt} + 1\right)u = \left(\frac{d}{dt} + 1\right)y\}$. In particular, if $\tilde{B}$ is passive, then $E_0^{(0,0)} = \{(u,y) \in L_1^{loc}(\mathbb{R},\mathbb{R}) \times L_1^{loc}(\mathbb{R},\mathbb{R}) | \left(\frac{d}{dt} + 1\right)u = \frac{d}{dt}y\}$ must be passive, and it follows that the transfer function $G(\xi) = 1 + \frac{1}{\xi}$ must be PR. But this condition is not sufficient for $\tilde{B}$ to be passive since there are trajectories in $\tilde{B}$ with $\left(\frac{d}{dt} + 1\right)\left(\frac{d}{dt} + 1\right)y = 0$ but $\left(\frac{d}{dt} + 1\right)y \neq \frac{d}{dt}y$.

As the preceding example indicates, the transfer function does not always determine the behavior of the system. In contrast, the behavior is always determined by the polynomial matrices corresponding to the differential equations governing the system (i.e., by $P$ and $Q$ in (3.1)). Thus, passivity will impose requirements on these polynomial matrices. The purpose of this section is to determine these requirements, resulting in Lemma 21. We will first prove some alternative requirements in Lemma 18, which we then show to be equivalent to the conditions in Lemma 21. These alternative requirements relate to the following decomposition of the behavior $\tilde{B}$ in (3.1) into controllable and autonomous parts:

**Lemma 17** Let $\tilde{B}$ in (3.1) satisfy normalrank($[P - Q]$) = $n$. Then there exist $F, \tilde{P}, \tilde{Q}, M, N, U, V, X, Y \in \mathbb{R}^{n \times n}$ such that

\[
P = F \tilde{P}, \quad Q = F \tilde{Q}, \quad \text{and} \quad \begin{bmatrix} \tilde{P} - \tilde{Q} \\ X M \\ U V \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} X M \\ Y N \\ U V \end{bmatrix}.
\]

(5.1)

Now, let $F, \tilde{P}, \tilde{Q}, M, N, U, V, X, Y \in \mathbb{R}^{n \times n}$ satisfy (5.1)–(5.2), and let $B := \{(i, v) \in L^{loc}(\mathbb{R}, \mathbb{R}) | \left(\frac{d}{dt} + 1\right)u = \eta \}

\begin{align*}
P^{loc}(\mathbb{R}, \mathbb{R}) \times P(\frac{d}{dt} + 1)u = \left(\frac{d}{dt} + 1\right)v; \\
\text{(ii) } B := \{(i, v) \in L^{loc}(\mathbb{R}, \mathbb{R}) \times L^{loc}(\mathbb{R}, \mathbb{R}) | \left(\frac{d}{dt} + 1\right)u = \left(\frac{d}{dt} + 1\right)v; \quad \text{and} \quad \{(i, v, i_1, v_1, i_2, v_2) | (i_1, v_1) \in B_1, (i_2, v_2) \in B_2, i = i_1 + i_2 \text{ and } v = v_1 + v_2\}.
\end{align*}

(5.3)

**PROOF.** The decomposition in the first part of the lemma statement is not unique, but one such decomposition is obtained by computing a lower echelon form for $[P - Q]$ (see note A4). This gives a unimodular $W \in \mathbb{R}^{2n \times 2n}$ such that $[F(\frac{d}{dt})] = [P - Q]W$. Then $W^{-1} := W \in \mathbb{R}^{2n \times 2n}$, and by suitably partitioning $W$ (resp., $W$) we obtain the polynomial matrices in the first (resp., second) block matrix in (5.2).

To show the second part of the lemma, we note initially that (5.3)–(5.4) are easily shown from (5.1)–(5.2) and (Polderman and Willems 1998 Theorem 3.2.15). Now, consider the compatible partitioned matrices

\[
Z := \begin{bmatrix} F P - F \tilde{P} \tilde{Q} & 0 & 0 & 0 \\ U V & 0 & 0 & 0 \\ 0 & 0 & P - \tilde{Q} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad R := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \end{bmatrix}, \quad W := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \end{bmatrix}.
\]

(5.5)

and note that $B$ is the set of locally integrable solutions to $R(\frac{d}{dt}) \text{col}(i, v) = Z(\frac{d}{dt}) \text{col}(i, v_1, v_2)$. Next, let $Z_2 \in \mathbb{R}^{2n \times 2n}$ be formed from the last four block rows of $Z$. It is straightforward to verify from (5.2) that $Z_2$ is unimodular. As $W$ is unimodular, then by pre-multiplying $R$ and $Z$ by $W$ we conclude that $B$ is the set of locally integrable solutions to $P(\frac{d}{dt}) i = Q(\frac{d}{dt}) v$ and $\text{col}(0 0 0 0) = Z_2(\frac{d}{dt}) \text{col}(i, v_1, v_2)$ (see note B1). In particular, $(i, v) \in B$, and it remains to show that, for any given $(i, v) \in B$, there exist locally integrable $(i_1, v_1, i_2, v_2)$ such that $\text{col}(0 0 0 0) = Z_2(\frac{d}{dt}) \text{col}(i_1, i_2, v_1, v_2)$. Accordingly, for any given $H \in \mathbb{R}^{n \times n}$ with normalrank($H$) = $m$, we let $H$ denote the maximum degree of all determinants composed of $m$ columns of $H$. Then, from (Polderman 1997 Theorem 2.8), it suffices to show that there exists a determinant of degree $H(\tilde{R} \tilde{Z})^2$ formed from the columns in $Z$ together with some of the columns in $R$.

It can be shown that $\Delta(\tilde{R} \tilde{Z}) = \Delta(\tilde{P} - \tilde{Q}) + \deg(\det(F))$ (this follows since any non-zero determinant formed from the first two block rows of $[R Z]$ which form a nonsingular matrix whose determinant is $\det(F)$; and (ii) at least $n$ non-zero columns from
the third block row). Next, let $\Delta((\tilde{P} - \tilde{Q}))$ be the degree of the determinant formed from columns $i_1, \ldots, i_n$ of $[\tilde{P} - \tilde{Q}]$. It can be shown that the degree of the determinant formed from columns $i_1, \ldots, i_n$ and $2n + 1, \ldots, 6n$ of $[R| Z]$ equals that of the determinant formed from columns $i_1, \ldots, i_n$ and $2n + 1, \ldots, 6n$ of $W[R| Z]$, which equals $\Delta((\tilde{P} - \tilde{Q})) + \deg(\det(F)) = \Delta([R| Z])$. \hfill \Box

Equations (5.3)-(5.4) represent the infinitely-often differentiable part of the behavior $B$ in terms of the five matrices $M, N, X, Y$ and $F \in \mathbb{R}^{n \times n}[\tau]$. In the next lemma, we provide three conditions on these matrices for $B$ to be passive. These correspond to the conditions:

1. $B_0$ is passive.
2. $B_1$ is stable, i.e., $(i_1, v_a) \in B_0 \Rightarrow i_a(t) \to 0$ and $v_a(t) \to 0$ as $t \to \infty$.
3. If $t_0 \leq t_1 \in \mathbb{R}$, $(i_1, v_a) \in B_0$, and $(i_2, v_b) \in B_0 \cap C^\infty([0, t_1])$ with $i_1(t) = v_a(t) = 0$ for all $t < t_0$ and $f_{t_0}^{t_1} \mathbf{I}^T(t) v_b(t) dt = 0$, then $f_{t_0}^{t_1} \mathbf{I}^T(t) v_a(t) + \nabla_t^+ h_1(t) dt = 0$.

Condition 1 is to be expected since $B_0 \subseteq B$. Condition 2 is equivalent to $B$ being stabilizable. Condition 3 is a coupling condition between the lossless trajectory $(i_1, v_a)$ and the autonomous trajectory $(i_2, v_b)$. In fact, this condition also holds if $B_0$ is replaced by $B \cap \mathbb{E}_{\mathbb{R}^n \times \mathbb{E}_{\mathbb{R}^n}}$ (an observation which is used in the proof of Theorem 13), and provides the intuition behind the third condition of the following lemma:

**Lemma 18** Let $B$ be as in (3.1) and let $B$ be passive. Then $\text{normrank}(P - Q) = n$. Furthermore, with $M, N$ and $F$ as in Lemma 17, then

1. $M(i)^T N + N(i)^T M(i) \geq 0$ for all $i \in \mathbb{C}_+$.
2. $F(i)$ is nonnegative for all $i \in \mathbb{C}_+$.
3. If $(i_1, v_a) \in B \cap \mathbb{E}_{\mathbb{R}^n \times \mathbb{E}_{\mathbb{R}^n}}$ and $b \in \mathbb{R}^{n}[\tau]$ satisfies $b^* (M^* N + N^* M) = 0$, then $b^* (\frac{d}{dt} M^* N + N^* M) = 0$.

**PROOF.** We first show that $n \equiv \text{rank}((P - Q)(i)) = \text{rank}(F(i)(P - Q))$ for all $i \in \mathbb{C}_+$. This implies that $\text{normrank}(P - Q) = n$ and condition 2 holds. We then show condition 1, and finally condition 3.

**Proof that rank**$(P - Q)(i)) = n$ **for all** $i \in \mathbb{C}_+$. Suppose instead that there exists $i \in \mathbb{C}_+$ such that $\text{rank}((P - Q)(i)) < n$. Then rank$(P(i) + Q(i)) < n$, and so there exists $0 \neq z \in \mathbb{C}^n$ such that $(P(i) + Q(i)) z = 0$. Then, with the notation $v(t) = ze^\lambda t + z e^{-\lambda t}$ and $i(t) = -v(t)$ for all $t \in \mathbb{R}$, we find that $(i, v) \in B$. Also, for any given $t_1 \geq t_0 \in \mathbb{R}$, then $-f_{t_0}^{t_1} \mathbf{I}^T(t) v(t) dt = 2R(\lambda z)^2 f_{t_0}^{t_1} e^{2\lambda t} dt + 2 z^2 f_{t_0}^{t_1} e^{2\lambda t} dt$. By considering separately the cases $\lambda = 0$ and $\lambda \neq 0$, it can be shown that for any given $K \in \mathbb{R}$ there exists $t_1 \geq t_0 \in \mathbb{R}$ such that $-f_{t_0}^{t_1} \mathbf{I}^T(t) v(t) dt \geq K$, whence $B$ is not passive. Thus, if $B$ is passive, then rank$(P - Q)(i)) = n$ for all $i \in \mathbb{C}_+$.

**Proof of condition 1.** Consider a fixed but arbitrary $\lambda \in \mathbb{C}_+$ and $c \in \mathbb{C}^n$; let $z(t) = ce^{\lambda t} + ce^{-\lambda t}$ for all $t \in \mathbb{R}$; let $i := M(\frac{d}{dt}) z$ and $v := N(\frac{d}{dt}) z$; let $\Psi(\eta, \xi) = M(\eta)^T N(\xi) + N(\eta)^T M(\xi)$; and let $\alpha = e^{\lambda T}(\lambda, \lambda) c$. Then $(i, v) \in B$ by Lemma 17, and

$$\int_{t_0}^{t_1} \mathbf{I}^T(t) v(t) dt = \Re \left( \alpha \int_{t_0}^{t_1} e^{2\lambda t} dt \right) + \beta \int_{t_0}^{t_1} e^{2\lambda t} dt.$$  \hspace{1cm} (5.6)

We will show that if there exists a $\lambda \in \mathbb{C}_+$ and $c \in \mathbb{C}^n$ such that $\beta = e^{\lambda T}(\lambda, \lambda) c < 0$, then for any given $K \in \mathbb{R}$ there exists a $t_1 \geq t_0$ with $-f_{t_0}^{t_1} \mathbf{I}^T(t) v(t) dt \geq K$. This will prove condition 1.

Let $\lambda = \sigma + j \omega$ for some $\sigma, \omega \in \mathbb{R}$ with $\sigma \geq 0$, and consider a fixed but arbitrary $K \in \mathbb{R}$. We consider the cases (i) $\omega = 0$; and (ii) $\omega \neq 0$. In case (i), let $\lambda = \lambda$ and $c = c$, so $\alpha = \beta$. Then, from (5.6), $f_{t_0}^{t_1} \mathbf{I}^T(t) v(t) dt = 2\beta f_{t_0}^{t_1} e^{2\lambda t} dt$, and $f_{t_0}^{t_1} e^{2\lambda t} dt = (1/2)(e^{2\lambda t_1} - e^{2\lambda t_0})$ if $\Re(\lambda) \neq 0$, and $t_1 - t_0$ otherwise. In case (ii), for any given integer $n$, we let $T(n) \in \mathbb{R}$ satisfy $2\omega T(n) = 2\pi(n + 1/4) - \arg(\alpha/\sigma + j\omega))$ (note, if $T(n) \geq t_0$, then $n \geq \omega t_0 / \pi + 1/4$ when $\omega > 0$, and $n \leq \omega t_0 / \pi - 3/4$ when $\omega < 0$). Then $\arg(\Re(e^{2\lambda T(n)})) = \pi/2$, so from (5.6) we find that $f_{t_0}^{t_1} \mathbf{I}^T(t) v(t) dt = (\beta e^{2\lambda T(n)} - e^{-2\lambda T(n)})/2(\Re(\lambda)) - \Re(\alpha e^{2\lambda T(n)})/2\lambda$ if $\Re(\lambda) \neq 0$, and $\beta(T(n) - t_0) - \Re(\alpha e^{2\lambda T(n)})/2\lambda$ otherwise. In both cases (i) and (ii), if $\beta < 0$, then by taking $t_1$ sufficiently large (and letting $t_1 = T(n)$ in case (ii)) we obtain $-f_{t_0}^{t_1} \mathbf{I}^T(t) v(t) dt \geq K$.

**Proof of condition 3.** Let $b \in \mathbb{R}^n[\tau]$ satisfy $b^* (M^* N + N^* M) = 0$, let $t_0 \leq t_1 \in \mathbb{R}$, and consider a fixed but arbitrary $(i_1, v_a) \in B \cap \mathbb{E}_{\mathbb{R}^n \times \mathbb{E}_{\mathbb{R}^n}} \times \mathbb{E}_{\mathbb{R}^n}$ and $z \in \mathbb{C}^n(\mathbb{R}, \mathbb{R})$. Then, with the notation $i := M(\frac{d}{dt}) b(\frac{d}{dt}) z + i_1$ and $v := N(\frac{d}{dt}) b(\frac{d}{dt}) z + v_8$, it follows that $(i, v) \in B$ by Lemma 17. Also, with

$$J_1 := \int_{t_0}^{t_1} ((M(\frac{d}{dt}) b(\frac{d}{dt}) z)^T(N(\frac{d}{dt}) b(\frac{d}{dt}) z) v_a) dt dt,$$

$$J_2 := \int_{t_0}^{t_1} ((M(\frac{d}{dt}) b(\frac{d}{dt}) z)^T v_a N(\frac{d}{dt}) b(\frac{d}{dt}) z) i_1 dt dt,$$

then $\int_{t_0}^{t_1} (i^T v(t) dt dt = J_1 + J_2 + \int_{t_0}^{t_1} (i_1^T v_a) dt dt.$  \hspace{1cm} (5.7)
Since $b^*(M^*N + N^*M) = 0$ then, from note C3, 
\[ J_1 = \int_{t_0}^{t_1} \left[ L_{\Phi_{\alpha}(z, (\frac{dd}{dt})z)}(t) + L_{\Phi_{\beta}(z, (\frac{dd}{dt})z)}(t) \right] dt, \]
and \[ J_2 = \int_{t_0}^{t_1} \left( z(b^*(\frac{dd}{dt})(M^*(\frac{dd}{dt})v_s + N^*(\frac{dd}{dt})i_s))(t) dt \right. \]
\[ + \left. [L_{\Phi_{\alpha}(z, v_s)}(t) + L_{\Phi_{\beta}(z, i_s)}(t)] \right] dt. \]

Now, let $g := b^*(\frac{dd}{dt})M^*(\frac{dd}{dt})v_s + N^*(\frac{dd}{dt})i_s$; let $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ and $t_0 \leq t_1 \in \mathbb{R}$ satisfy $\psi(t) = 0$ for all $t \leq t_0$, and $\frac{d^2\psi}{dt^2}(t_1) = 0$ for $t \geq t_0$. \[ J_1 \times \psi(t)dt \leq \psi(t)dt \leq K \times \psi(t)dt \]
for all $t \leq t_0$. Thus, from (5.7), there exists $M \in \mathbb{R}$ such that $J_1(f_0(\psi(t))dt < M$ for all $t \geq t_0$. Moreover, since $(i_s, v_s) \in B \cap E \times E \cap \mathbb{R}^n$, then it is straightforward to show that $I_s^2 v_s \in E \cap \mathbb{R}$, and that there exists an $M \in \mathbb{R}$ such that $f_0(\psi(t))dt < M$ for all $t \geq t_0$. Finally, we will show that, for any given $K \in \mathbb{R}$, there exist $\psi$ and $t_1$ with the properties outlined above that satisfy $-f_0(\psi(t))dt > K + M$. \[ \Box \]

In the next lemma, we present several equivalent conditions to the third condition in Lemma 18. This leads to two algebraic tests for this condition (see Remark 20), and the main result in this section (see Lemma 21).

**Lemma 19** Let $B$ be as in (3.1); let $\text{rank}(P - Q) = n$ for all $\lambda \in \mathbb{C}_+$; and let $F, \tilde{P}, \tilde{Q}, M, N, U, V, X, Y$, and $S_n$ be as in Lemma 17. Then the following are equivalent:

1. Condition 3 of Lemma 18 holds.
2. If $(i_s, v_s) \in S_n$ and $b \in \mathbb{R}^n[\lambda]$ satisfies $b^*(M^*N + N^*M) = 0$, then $b^*(\frac{dd}{dt})(M^*(\frac{dd}{dt})v_s + N^*(\frac{dd}{dt})i_s) = 0$.
3. If $b \in \mathbb{R}^n[\lambda]$ satisfies $b^*(M^*N + N^*M) = 0$, then there exists $p \in \mathbb{R}^n[\lambda]$ such that $b^* = p^T F$.
4. If $e \in \mathbb{R}^n[\lambda]$ satisfies $e^T(\tilde{P}Q^* + Q^*\tilde{P}) = 0$, then there exists $p \in \mathbb{R}^n[\lambda]$ such that $e^T = p^T F$.
5. If $p \in \mathbb{R}^n[\lambda]$ and $\lambda \in \mathbb{C}$ satisfy $p^T(\tilde{P}Q^* + Q^*\tilde{P}) = 0$ and $p(\lambda)^T[P - Q](\lambda) = 0$ then $p(\lambda) = 0$.

**Proof.** 1 $\iff$ 2. That 1 $\implies$ 2 follows since \[ \text{rank}(P - Q)(\lambda) = n \] for all $\lambda \in \mathbb{C}_+$ implies that $F(\lambda)$ is non-singular for all $\lambda \in \mathbb{C}_+$, and so $S_n \subseteq B \cap E \times E \cap \mathbb{R}^n)$ by Lemma 17 and [Polderman and Willems 1998, Section 3.2.2]. Then 2 $\implies$ 1 since, by Lemma 17, if $(i_s, v_s) \in B \cap E \times E \cap \mathbb{R}^n)$, then there exists $(i_s, v_s) \in B \cap E \times E \cap \mathbb{R}^n)$ such that $i_s = M(\frac{dd}{dt})z + i_s$ and $v_s = N(\frac{dd}{dt})z + v_s$.

2 $\implies$ 3. To see that 2 $\implies$ 3, note initially from Lemma 17 that condition 2 implies that if $b \in \mathbb{R}^n[\lambda]$ satisfies $b^*(M^*N + N^*M)$, and $z \in \mathbb{R}^n[\lambda]$ satisfies $F(\frac{dd}{dt})z = 0$, then $b^*(\frac{dd}{dt})(M^*Y + N^*X)(\frac{dd}{dt})z = 0$. From note B2, this implies that there exists $p \in \mathbb{R}^n[\lambda]$ such that $b^*(M^*Y + N^*X) = p^T F$. Similarly, from Lemma 17, it is straightforward to show that 3 $\implies$ 2.

3 $\implies$ 4. Note initially from (5.2) that \[ \begin{bmatrix} \tilde{P} + \tilde{Q}^* & V^* \\ \tilde{P}^* + U^* & M^* \end{bmatrix} \begin{bmatrix} Y^* & X^* \\ X M \\ Y N \end{bmatrix} = \begin{bmatrix} I_0 \\ 0 \end{bmatrix}. \]

To prove that 3 $\implies$ 4, note that if $c \in \mathbb{R}^n[\lambda]$ satisfies $c^T(\tilde{P}Q^* + \tilde{Q}^*\tilde{P}) = 0$, then $b^* := c^T(\tilde{P}V^* - \tilde{Q}^*U^*)$ satisfies $b^*(M^*N + N^*M) = 0$. Thus, from condition 3, there exists $p \in \mathbb{R}^n[\lambda]$ such that $b^*(M^*Y + N^*X) = p^T F$. But $b^*(M^*Y + N^*X) = c^T(\tilde{P}V^* - \tilde{Q}^*U^*)(M^*Y + N^*X)$, and $c^T = c^T(\tilde{P}V^* - \tilde{Q}^*U^*)(M^*Y + N^*X) = p^T F$. By (5.8). The proof of 4 $\implies$ 3 is similar.

4 $\iff$ 5. To see that 4 $\iff$ 5, we let $r := \text{normalrank}(\tilde{P}Q^* + \tilde{Q}^*\tilde{P})$, and we let the rows of $V_1 \in \mathbb{R}^{(n-r) \times n[\lambda]}$ be a basis for the left syzygy of $\tilde{P}Q^* + \tilde{Q}^*\tilde{P}$ (see note A3). Then condition 4 implies that there exists $V_1 \in \mathbb{R}^{(n-r) \times n[\lambda]}$ such that $V_1 = V_1 F$. Since $V_1(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, then so too must $V_1(\lambda)$.

Since, in addition, $PQ^* + Q^*P = F(\tilde{P}Q^* + \tilde{Q}^*\tilde{P})F^*$, then $V_1(PQ^* + Q^*P) = V_1(\tilde{P}Q^* + \tilde{Q}^*\tilde{P})F^* = 0$ and normalrank($PQ^* + Q^*P$) = normalrank($\tilde{P}Q^* + \tilde{Q}^*\tilde{P}$), and we conclude that the rows of $V_1$ are a basis for the left syzygy of $PQ^* + Q^*P$. It follows that if $p^T(PQ^* + Q^*P) = 0$, then there exists $g \in \mathbb{R}^{(n-r) \times n[\lambda]}$ such that $g^T = g^T V_1$. If, in addition, $p(\lambda)^T[P - Q](\lambda) = 0$, then $p(\lambda)^T F(\lambda) = 0$ since $\tilde{P}$ and $\tilde{Q}$ are left coprime, whence $g(\lambda)^T V_1(\lambda) F(\lambda) = g(\lambda)^T V_1(\lambda) = 0$. But $V_1(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, and we conclude that $g(\lambda) = 0$ and so $p(\lambda) = 0$. Finally, to show that 5 $\implies$ 4, we let the rows of $V_1 \in \mathbb{R}^{(n-r) \times n[\lambda]}$ be a basis for the left syzygy of $PQ^* + Q^*P$. Then, from condition 5, we conclude that $c(\lambda)^T V_1(\lambda) F(\lambda) = 0 \Rightarrow c(\lambda)^T = 0$, and it follows that $V_1(\lambda) F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. In a similar manner to before, it can then be shown that the rows of $V_1 F$ are a basis for the left syzygy of $PQ^* + Q^*P$. Hence, if $c \in \mathbb{R}^n[\lambda]$ satisfies $c^T(\tilde{P}Q^* + \tilde{Q}^*\tilde{P}) = 0$, then there exists $g \in \mathbb{R}^{(n-r) \times n[\lambda]}$ such that $c^T = g^T V_1$, and by letting $p^T := g^T V_1$ we obtain condition 4. $\Box$
Remark 20 Let $\mathcal{B}, F, \hat{P}, \hat{Q}, M, N, U, V, X, Y$, and $B_0$ be as in Lemma 19 (with $\text{rank}([P - Q](\lambda)) = n$ for all $\lambda \in \mathbb{C}_+$. Lemma 19 leads to two tests that can be implemented by a standard symbolic algebra program (using exact arithmetic if the polynomial matrix coefficients are rational). As in the proof of Lemma 19, let $r := \text{normalrank}(PQ^* + QP^*)$, and note that it is easily shown from the proof that $r = \text{normalrank}(M^*N + N^*M)$. The two tests are as follows.

1. Using the matrices $M, N, X, Y$ and $F$:
   (a) Compute a $V \in \mathbb{R}^{(n-r) \times n}$ whose rows are a basis for the left syzygy of $M^*N + N^*M$.
   (b) Condition 3 of Lemma 19 holds if and only if $V(M^*N + N^*M)$ is divisible on the right by $F$.
2. Using the matrices $P$ and $Q$:
   (a) Compute a $V \in \mathbb{R}^{(n-r) \times n}$ whose rows are a basis for the left syzygy of $PQ^* + QP^*$.
   (b) Condition 5 of Lemma 19 holds if and only if $V(\lambda)[P - Q](\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$.

Lemma 21 Let $\mathcal{B}$ be as in (3.1). If $\mathcal{B}$ is passive, then $(P, Q)$ is a positive-real pair.

PROOF. That $(P, Q)$ satisfy condition 2 in Definition 7 follows from condition 2 of Lemma 18, by noting from (5.1)–(5.2) that $[P - Q]\text{col}(X, Y) = F$. Then condition 3 in Definition 7 follows from Lemmas 18–19. It remains to show that condition 1 of Definition 7 holds. To see this, first note from condition 1 of Lemma 18 that $(M + N)(\lambda)T(M + N)(\lambda) - (N - M)(\lambda)T(N - M)(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$. Next, let $\lambda \in \mathbb{C}_+$, and note that $(M + N)(\lambda)z = 0$. Then $M\lambda \leq (N - M)(\lambda)z \leq 0$, which implies that $(N - M)(\lambda)z = 0$, and so $M(\lambda)z = 0$ and $N(\lambda)z = 0$. Then from (5.2) we obtain $z = U(\lambda)M(\lambda)z + V(\lambda)N(\lambda)z = 0$. The result then follows.

We note that $\hat{T} := \text{col}(\hat{T}_1, \hat{T}_2) = \text{diag}(\hat{T}, \hat{T})$ is nonsingular. Then, with $A := \hat{T}\hat{T}^{-1}$, $B := TB$, $C := \hat{T}\hat{T}^{-1}$, $X := \text{diag}(X, X, 0)$, $L_X := [L, 0]$, and $W_X := W$, it can be verified that $X \leq 0$, $-A^T\hat{T} - \hat{T}A = \text{diag}(-A^T\hat{T} - X, \lambda)I$, and $\hat{T}X - X\hat{T} = L_X$, where $\hat{T} := \text{diag}(\hat{T}, \hat{T})$ and $W_X := W$. Finally, note that $(\hat{T}X - X\hat{T})^{-1}$ is a spectral factor of $G + G^*$. We can then verify that $X, L_X, W_X$ satisfy condition 4.

We first prove (i). Direct calculation verifies that $G = Q^{-1}P$. Since $(\hat{P}, \hat{Q})$ is a positive-real pair and $Q$ is non-singular, then $G$ is PR. To see this, note that if $G$ is analytic in $\mathbb{C}_+$ then $\text{det}(G + \lambda I) = Q^{-1}(\lambda)\text{det}(\hat{P}(\lambda)Q(\lambda)^T + Q(\lambda)P(\lambda)^T) \geq 0$ for all $\lambda \in \mathbb{C}_+$, so $G$ is PR. But suppose instead that $G$ has a pole at some $\lambda \in \mathbb{C}_+$. By considering the Laurent series for $G$ about $\lambda$, it can be shown that, for any $\epsilon > 0$, there exists $\lambda \in \mathbb{C}_+$ and an $\eta \in \mathbb{C}$ with $|\eta| \leq \epsilon$ such that $\text{det}(G(\lambda + \eta) + G(\lambda + \eta)^T) \geq 0$: a contradiction.

Since $G$ is analytic in $\mathbb{C}_+$ and $G = G_u + G_s$ with $G_s(\lambda) = D + C_s(\lambda - A_s)^{-1}B_s$ (whose poles are all in $\mathbb{C}_+$) and $G_u(\lambda) = C_u(\lambda - A_u)^{-1}B_u$ (whose poles are all in $\mathbb{C}_+$), then the poles of $G_u$ must all be on the imaginary axis. Since, in addition, $G$ is PR, then $G_u$ and $G_s$ are both PR and $G_u + G_s = 0$. This can alternatively be shown using the real Jordan form. Here, letting $d_r$ denote the number of columns (and rows) of $A_r$, then the first $d_r$ rows (resp., last $d_r$ rows) of $\tilde{T}$ span the stable (resp., unstable) left eigenspace of $\tilde{A}_1$. 

6 This can alternatively be shown using the real Jordan form. Here, letting $d_r$ denote the number of columns (and rows) of $A_r$, then the first $d_r$ rows (resp., last $d_r$ rows) of $\tilde{T}$ span the stable (resp., unstable) left eigenspace of $\tilde{A}_1$. 

2 ⇒ 4. To prove this implication, we will show conditions (i) and (ii) below. The notation in those conditions is as follows. We let $T := \text{col}(T_1, T_2)$ be such that $C = [C_1 | 0] = CT^{-1}$ and $\hat{T} = TAT^{-1}$ have the observer staircase form indicated in note 22, and we let $TB := \hat{B}$ and $\hat{T}B := \hat{B}_1$. Then, with $A_1 \in \mathbb{R}^{d_1 \times d_1}$, as in note 22, we let $\hat{T} := (\hat{T}_1, \hat{T}_2)$ be such that $\hat{T}\hat{A}_1\hat{T}^{-1} = \text{diag}(A_1, A_2)$.

8 Chapter VII [9]. We then let $G_s(\lambda) = D + C_s(\lambda - A_s)^{-1}B_s$ and $G_u(\lambda) = C_u(\lambda - A_u)^{-1}B_u$, and direct calculation shows that $\gamma(\hat{T}) = \hat{T}_1\hat{A}_1\hat{T}^{-1} = \gamma(T_1)$. We will show the following. 

(i) There exists a real $X_u > 0$ such that $-A^T\hat{T} - \hat{T}A = \text{diag}(-A^T\hat{T} - X, \lambda)I$, and $\hat{T}X - X\hat{T} = L_X$, where $\hat{T} := \text{diag}(\hat{T}, \hat{T})$ is a spectral factor of $G + G^*$. 

We set $\hat{T} := \text{col}(\hat{T}_1, \hat{T}_2) = \text{diag}(\hat{T}, \hat{T})$ is nonsingular. Then, with $A := \hat{T}\hat{T}^{-1}$, $B := TB$, $C := \hat{T}\hat{T}^{-1}$, $X := \text{diag}(X, X, 0)$, $L_X := [L, 0]$, and $W_X := W$, it can be verified that $X \leq 0$, $-A^T\hat{T} - \hat{T}A = \text{diag}(-A^T\hat{T} - X, \lambda)I$, and $\hat{T}X - X\hat{T} = L_X$, where $\hat{T} := \text{diag}(\hat{T}, \hat{T})$ and $W_X := W$. Finally, note that $(\hat{T}X - X\hat{T})^{-1}$ is a spectral factor of $G + G^*$. We can then verify that $X, L_X, W_X$ satisfy condition 4.

We first prove (i). Direct calculation verifies that $G = Q^{-1}P$. Since $(\hat{P}, \hat{Q})$ is a positive-real pair and $Q$ is non-singular, then $G$ is PR. To see this, note that if $G$ is analytic in $\mathbb{C}_+$ then $\text{det}(G + \lambda I) = Q^{-1}(\lambda)\text{det}(\hat{P}(\lambda)Q(\lambda)^T + Q(\lambda)P(\lambda)^T) \geq 0$ for all $\lambda \in \mathbb{C}_+$, so $G$ is PR. But suppose instead that $G$ has a pole at some $\lambda \in \mathbb{C}_+$. By considering the Laurent series for $G$ about $\lambda$, it can be shown that, for any $\epsilon > 0$, there exists $\lambda \in \mathbb{C}_+$ and an $\eta \in \mathbb{C}$ with $|\eta| \leq \epsilon$ such that $\text{det}(G(\lambda + \eta) + G(\lambda + \eta)^T) \geq 0$: a contradiction.

Since $G$ is analytic in $\mathbb{C}_+$ and $G = G_u + G_s$ with $G_s(\lambda) = D + C_s(\lambda - A_s)^{-1}B_s$ (whose poles are all in $\mathbb{C}_+$) and $G_u(\lambda) = C_u(\lambda - A_u)^{-1}B_u$ (whose poles are all in $\mathbb{C}_+$), then the poles of $G_u$ must all be on the imaginary axis. Since, in addition, $G$ is PR, then $G_u$ and $G_s$ are both PR and $G_u + G_s = 0$.
Section 5.1). Next, note that $\tilde{B} := B_3^{(u,v)}$ is stabilizable (by condition 2 of Definition 7), and has the observable realization in note D3, whence $[I - A_{11} \; \tilde{B}_1]$ has full row rank for all $\lambda \in \mathbb{C}_+$ (this follows from note D4). It is then easily shown that $[I - A_{n} \; \tilde{B}_n]$ has full row rank for all $\lambda \in \mathbb{C}_+$, so that $(A_n, \tilde{B}_n)$ is controllable. It can then be shown that $(C_n, A_n)$ is observable since $(C_1, A_1)$ is. Thus, $G_n(\xi) = C_n(\xi - I)^{-1}B_n$ is PR with $G_n + \tilde{G}_n = 0$ and with $(A_n, \tilde{B}_n)$ controllable and $(C_n, A_n)$ observable, and so (i) holds by [Willems 1972b; Theorem 5].

Next, let $A_1(\xi) := I - A_1$; let $M$ and $N$ be as in Lemma 17 (so, in particular, $M$ is invertible, and $M^{-1} = Q^{-1}P = G$, which is PR); let $r$ = normalrank$(M_N + N^*)$; and let $K \in \mathbb{R}^{r \times n}[\xi]$ be a spectral factor for $M^*N + N^*M$ (i.e., $K(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}_+$, and $K^*K = M^*N + N^*M$). To prove condition (ii), we will show the following four conditions.

(a) There exist $J \in \mathbb{R}^{n \times d}[\xi]$ and $L \in \mathbb{R}^{r \times d}_\xi$ such that $K^*L + JA_1 = M_C^*$.

(b) With $L$ as in (a), there exists $X_1 \in \mathbb{R}^{d \times d}_\xi$ such that $-A_1^TJX_1 - X_1A_1 = L^TL$.

(c) Define $Z := KM^{-1}$ to be a spectral factor of $G + G^*$, where

\[ W := \lim_{\xi \to -\infty} Z(\xi), \quad Z = W + L\tilde{A}_1B_1. \]

In particular, $D + Dt = W^TW$.

(d) With $Z, W, L$ as in (a)–(c), then $G^*_1 = \lim_{\xi \to -\infty} Z(\xi)$, and we will show that:

(i) $W + L\tilde{A}_1B_1 - Z$ has no poles in $\mathbb{C}_+$, and

(ii) $W + L\tilde{A}_1B_1 - Z$ has no poles in $\mathbb{C}_+$. Since, in addition, $K$ is a spectral factor of $M^*N + N^*M$, then it is straightforward to show that $Z$ is a spectral factor of $G + G^* = M^{-1}N + N^*(M^{-1})$.

Next let $W := \lim_{\xi \to -\infty} Z(\xi)$, and we will show that:

(i) $W + L\tilde{A}_1B_1 - Z$ has no poles in $\mathbb{C}_+$, and

(ii) $W + L\tilde{A}_1B_1 - Z$ has no poles in $\mathbb{C}_+$. Since $K^* = J$ has no poles in $\mathbb{C}_+$, then we conclude that $W + L\tilde{A}_1B_1 - Z$ has no poles in $\mathbb{C}_+$.

Finally, to show condition (d), note initially from (b) that $X_1A_1^{-1} + (A_1^*)^{-1} X_1 = (A_1^*)^{-1}L^TLA_1^{-1}$. Next, note that $M^*(W^TL + B_1^T X_1 = C_1^*A_1^{-1} = M^*(W^TL + B_1^T (A_1^*)^{-1}L)A_1^{-1} = -M^*B_1^T (A_1^*)^{-1}X_1$, and $\tilde{A}_1 = (K^*L + M_C^*A_1^{-1} - M^*B_1^T (A_1^*)^{-1}X_1 = -\tilde{J} - M^*B_1^T (A_1^*)^{-1}X_1$. It follows that $M^*(W^TL +
$B^T X_n - C_n A_n^{-1}$ has no poles in $\mathbb{C}$. But $M^*(\lambda)$ is nonsingular for all $\lambda \in \mathbb{C}$, and we conclude that $(W^T L + B^T X_n - C_n) A_n^{-1}$ has no poles in $\mathbb{C}$. It is then straightforward to show that $W^T L + B^T X_n - C_n = 0$.

4) $\Rightarrow$ Immediate.

3) $\Rightarrow$ Consider a fixed but arbitrary $(u, y, x) \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, and let $(\tilde{u}, \tilde{y}) \in \mathbb{B} = \mathbb{B}(\lambda, y)$ satisfy $\tilde{u}(t) = u(t)$ and $\tilde{y}(t) = y(t)$ for all $t < t_0$. Then, from note D3, there exists $(\tilde{u}, \tilde{y}, \tilde{x}) \in \mathbb{B}$, with $\tilde{x}(t_0) = x(t_0)$. From Remark 15, since $X^T(t_1) X \tilde{x}(t_1) \geq 0$ and $\tilde{x}(t_0) = x(t_0)$, then $-f(t_1) \tilde{u}^T(t) \tilde{y}(t) dt \leq \frac{1}{T^2} X^T(t_0) X \tilde{x}(t_0)$. This inequality holds for all $(\tilde{u}, \tilde{y}) \in \mathbb{B}$ that satisfy $(\tilde{u}(t), \tilde{y}(t)) = (u(t), y(t))$ for all $t < t_0$, so $\mathbb{B}$ is passive.

We next assume that $D + D^T > 0$, and we prove that 4) $\Rightarrow$ 3) holds. Let $X, L, X_W$ and $Z_X$ be as in condition 3. Since $n \geq \text{normalrank}(G + G^T) \geq \text{rank}(D + D^T) = n$, then $\text{normalrank}(G + G^T) = n$, so $Z_Y \in \mathbb{R}^{n \times n}(\xi)$, and $W^T X = D + D^T$, which is nonsingular, then $W^T X$ is nonsingular. Then we find that $-A^T X - X A - (C^T - X B)(D + D^T)^{-1}(C - B^T X) = L^T X - L^T X_W X_W^{-1}(L^T - L^T X_W X_W^{-1})^{-1} W^T X = 0$. Also, suppose $X \geq 0$ is real and satisfies $\Pi(X) = 0$; let $W_X$ be a real nonsingular matrix with $D + D^T = W^T X W$; and let $X : L, X_W$ and $W_X$ satisfy condition 3.

We now prove condition (i). Accordingly, suppose condition 3 holds, and let $X, L, X_W$ and $W_X$ be as in that condition. To show condition (i)(a), suppose that $(C, A)$ is observable and there exists $z \in \mathbb{R}^d$ with $X z = 0$. Since $X$ is symmetric, then $X^T z = 0$. Thus, $X^T (X - X A) z = (X^T - X A) z = 0$. Also, $X^T (X - X A) z = 0$. By replacing $z$ with $A z$ in the preceding argument, we find that $L_X z = 0$, $C z = 0$, and $X A z = 0$. Proceeding inductively gives $C A^k z = 0$ for $k = 0, 1, 2, \ldots$. Since $(C, A)$ is observable, then $z = 0$, and we conclude that $X > 0$.

To show condition (i)(b), let $\lambda \in \mathbb{C}_+$ and $z \in \mathbb{C}^d$ satisfy $(\lambda - A) z = 0$. Then $z^T L^T X_W X_W^{-1} L^T X_W X_W^{-1} (X - X A) z = - (\lambda + z^T X A z) \leq 0$, whence $L_X z = 0$ and $z^T X z = 0$. Since $X > 0$ by condition (i)(a), then $z = 0$, and we conclude that $\text{spec}(A) \subset \mathbb{C}_-$.

It remains to prove condition (ii). Condition (ii)(a) was shown in the proof of 4) $\Rightarrow$ 5). To see condition (ii)(b), note that $A + B(D + D^T)^{-1}(B^T X - C) = A - B W^{-1} X_L$ and consider a fixed but arbitrary $\lambda \in \mathbb{C}_+$. From the proof of condition (i)(b), if $z \in \mathbb{C}^d$ satisfies $(\lambda - A) z = 0$, then $L_X z = 0$, whence $(\lambda - A - B W^{-1} X_L) z = (\lambda - A) z = 0$. It remains to show that if $(\lambda - A)$ is nonsingular, then $(\lambda - A - B W^{-1} X_L)$ is nonsingular accordingly. Suppose that $(\lambda - A)$ is nonsingular and $y \in \mathbb{C}^d$ satisfies $y^T (\lambda - A - B W^{-1} X_L) = 0$.

Remark 22 We note that the matrix $L$ in the above theorem can be obtained by considering the Jordan chains of $A_n$. Specifically, let $A_n$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ with Jordan chains $(v_1, 1, \ldots, v_1, N(\lambda_1), \ldots, v_n, 1, \ldots, v_n, N(\lambda_n))$. Also, for any given $H \in \mathbb{R}^{m \times n}$ and $\lambda \in \mathbb{C}$ such that $\lambda$ is not a pole of $H$, let $H_{\lambda,j}$ denote the $j$-th term in the Taylor expansion for $H$ about $\lambda$, i.e., $H_{\lambda,j} = (\lambda^j/ j!)(\lambda_{\gamma})$ for $j = 0, 1, \ldots$. Then $L$ can be obtained by solving the equations

$$\sum_{j=0}^{k-1} (K_{\lambda,j} L v_{k-j} - M_{\lambda,j} C v_{k-j}) = 0,$$

for $i = 1, \ldots, n$ and $k = 1, \ldots, N(\lambda_i)$. It can then be shown that, if spec$(A) \subset \mathbb{C}_-$ and condition 3 of Theorem 13 holds, then [Pandolfo 2001 equation (4)] must hold.

To show (6.1), we let $A_{\lambda}(\xi) := \xi I - A_n$, and we consider the Jordan chain for $A_n$ corresponding to an eigenvalue $\lambda; A_n(\lambda) v_1 = 0, \lambda A_n(\lambda) v_{j-1} + v_j = 0 (j = 2, \ldots, N(\lambda))$. If $J \in \mathbb{C}^{n \times d}$ and $L \in \mathbb{R}^{n \times d}$ satisfy $K^T L + J A_n = M^* C_n$, then $K_{\lambda,j} L - M_{\lambda,j} C_n = - \frac{1}{\lambda} (J_{\lambda,j})(A_n(\lambda)) = - J_{\lambda,j} A_n(\lambda) - J_{\lambda,j-1}$ (where $J_{\lambda,j} = 0$). Thus, for $k = 1, \ldots, N(\lambda_i)$, we have

$$\sum_{j=0}^{k-1} (K_{\lambda,j} L v_{k-j} - M_{\lambda,j} C v_{k-j}) = - \sum_{j=0}^{k-1} (J_{\lambda,j} A_n(\lambda) v_{k-j}) - \sum_{j=0}^{k-1} (J_{\lambda,j-1} v_{k-j}) = - \sum_{j=0}^{k-1} (J_{\lambda,j} A_n(\lambda) v_{k-j} + v_{k-j-1}) = - J_{\lambda,k-1} A_n(\lambda) v_{k-1} = 0.$$

PROOF OF THEOREM 9 (see p. 5). That 1) $\Rightarrow$ 2 was shown in Lemma 21. Here, prove that 2) $\Rightarrow$ 3) $\Rightarrow$.

Let $\hat{P} := P - Q$ and $\hat{Q} := P + Q$. Since $(P, Q)$ is a positive-real pair, then $\tilde{Q}(\lambda) \tilde{Q}^T(\lambda) - \tilde{T}(\lambda) \tilde{T}^T(\lambda) \geq 0$ and rank$[(\tilde{P} - \tilde{Q}) (\lambda)] = n$ for all $\lambda \in \mathbb{C}_+$. We will show that: (i) $\tilde{Q}(\lambda)$ is nonsingular for all $\lambda \in \mathbb{C}_+$; and (ii) $\tilde{Q}^T \tilde{P}$ is proper. To see (i), suppose $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}_+$ satisfy $z^T \tilde{Q}(\lambda) = 0$. Then $z^T \tilde{P}(\lambda) \tilde{T}(\lambda) \geq 0$, which implies that $z^T \tilde{P} \tilde{T}(\lambda) = 0$. Since rank$[(\tilde{P} - \tilde{Q}) (\lambda)] = n$, then this implies that $z = 0$. To see (ii), note that, since $\tilde{Q}(\lambda)$ is nonsingular for all $\lambda \in \mathbb{C}_+$, then $I - \tilde{Q}^T \tilde{P}(\lambda) (\lambda) \tilde{Q}^T(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$, and it is then easily shown that $\tilde{Q}^T \tilde{P}$ is proper.

Let $R \in \mathbb{R}^{m \times n}[\xi]$ with normalrank$(R) = m$, and recall the notation $\Delta(\xi)$ from the proof of Lemma 17. If $R$ is partitioned as $R = [R_1 R_2]$ where $R_2 \in \mathbb{R}^{m \times m}[\xi]$ is nonsingular, then $R_2^{-1} R_1$ is proper.
if and only if \( \text{deg}(\det(R_2)) = \Delta(R) \) \cite{Polderman_1998}. Thus, \( \text{deg}(\det(Q)) = \Delta(\lfloor \hat{P} - \hat{Q} \rfloor) \). But \( \det(\hat{Q}) = \det(P + Q) \), which is the sum of all the determinants composed of columns of \( P \) together with the complementary columns of \( Q \) (i.e., \( \det([p_1, p_2 \ldots]) + \det([q_1, p_2 \ldots]) + \det([p_1, q_2 \ldots]) + \ldots \), where \( p_k \) (resp., \( q_k \)) denotes the \( k \)th column of \( P \) (resp., \( Q \)). From among the determinants in this sum, we pick one of greatest degree, we let \( T = \text{col}(T_1, T_2) \) be a permutation matrix such that \( T_1^T \) (resp., \( T_2^T \)) selects the columns from \( Q \) (resp., \( P \)) appearing in this determinant, and we define \( \hat{Q} = \lfloor QT_1^T - PT_2^T \rfloor \) and \( \hat{P} = \lfloor PT_1^T - QT_2^T \rfloor \). Then \( \text{deg}(\det(\hat{Q})) \geq \text{deg}(\det(P + Q)) = \Delta(\lfloor \hat{P} - \hat{Q} \rfloor) \). Furthermore, \( T_1^T T_1 + T_2^T T_2 = I \) as \( T \) is a permutation matrix, and with the notation

\[
S_1 := \begin{bmatrix} T_1^T & 0 & 0 \\ 0 & T_2^T & T_1^T \\ T_2^T & 0 & 0 \end{bmatrix}, \quad \text{and} \quad S_2 := \frac{1}{2} \begin{bmatrix} I & I \\ -I & I \end{bmatrix},
\]

we find that \( S_1 S_1^T = 2 S_2 S_2^T = I \), and \( \lfloor \hat{P} - \hat{Q} \rfloor = \lfloor P - Q \rfloor S_1 = \lfloor P - Q \rfloor S_2 S_1 \). Then, from the Binet-Cauchy formula, we obtain \( \Delta(\lfloor \hat{P} - \hat{Q} \rfloor) = \Delta(\lfloor P - Q \rfloor) \) \cite{Hughes_2016a}. Since, in addition, \( \Delta(\lfloor P - Q \rfloor) \geq \text{deg}(\det(\hat{Q})) \geq \Delta(\lfloor \hat{P} - \hat{Q} \rfloor) \), then \( \text{deg}(\det(\hat{Q})) = \Delta(\lfloor \hat{P} - \hat{Q} \rfloor) \), so \( \hat{Q}^{-1} \hat{P} \) is proper.

Since \( S_1 S_1^T = I \), then \( \lfloor \hat{P} - \hat{Q} \rfloor \begin{bmatrix} \frac{1}{m} \end{bmatrix} S_1^T \text{col}(i, v) = \lfloor P - Q \rfloor \begin{bmatrix} \frac{1}{m} \end{bmatrix} \text{col}(i, v) \). Thus, with \( i_1 = T_1 i_1, i_2 = T_2 i_1 \), \( v_1 = T_1 v, \) and \( v_2 = T_2 v \), it follows that \( i \) and \( v \) have the compatible partitions \( i := (i_1, i_2) \) and \( v := (v_1, v_1) \). In this case, we define \( \hat{B} := \hat{B}(\text{col}(i_1, v_2), \text{col}(v_1, i_2)) \). Then, from Lemma 12, there exists a state-space system \( \mathcal{B}_s \) as in (1.2) such that \( \hat{B} = \hat{B}(u, y) \). Moreover, it is easily verified that \( \lfloor \hat{P}, \hat{Q} \rfloor \) is a positive-real pair since \( (P, Q) \) is, so \( \hat{B} \) is passive by Theorem 13.

3. \( \Rightarrow \) 1. Note from the preceding discussion that \( \hat{B} \) takes the form of (3.1) where \( [P - Q] = [\hat{P} - \hat{Q}] S_1^T \). Now, let \( i, v \in \mathcal{B} \) and \( t_0 \in \mathbb{R} \); let \( i, \nu \in \mathcal{B} \) be a fixed but arbitrary trajectory satisfying \( (i(t), \nu(t)) = (i(t), v(t)) \) for all \( t < t_0 \); and let \( u := \text{col}(T_1 T_1^{\nu} v), y := \text{col}(T_1 T_2^{\nu} v), \) and \( y := \text{col}(T_1 T_2^{\nu} \nu) \). Then \( (u, y) \in \mathcal{B}, (u, y) \in \mathcal{B}, \) and \( (u(t), y(t)) \) is \( \text{col}(u(t), v(t)) \) for all \( t < t_0 \). Since \( \mathcal{B} \) is passive, there exists a \( K \in \mathbb{R} \) such that \( -f_{t_0}^{t} u(t) \nu(t) dt < K \) for all \( t_0 \geq t_0 \). Since, in addition \( T_1^T T_1 + T_2^T T_2 = I \), then \( -f_{t_0}^{t} T_1^T T_1 \nu(t) dt = -f_{t_0}^{t} T_2^T T_2 \nu(t) dt < K \), and we conclude that \( \hat{B} \) is passive. □

7. Conclusions

The positive-real lemma links the concepts of passivity, positive-real transfer functions, spectral factorisation, linear matrix inequalities, and algebraic Riccati equations. However, the lemma only considers systems described by a controllable state-space realization, which leaves important questions unanswered. For example, it does not specify which uncontrollable systems are passive. In this paper, we sought to answer this question and others by proving two new theorems: the passive behavior theorem, parts 1 and 2.

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A Polynomial and rational matrices

Several of the results in this paper depend on the properties of polynomial matrices that we describe here.

A1 \( U \in \mathbb{R}^{l \times l}[\xi] \) is called unimodular if there exists \( V \in \mathbb{R}^{l \times l}[\xi] \) such that \( UV = I \) (whence \( VU = I \)). \( U \) is unimodular if and only if \( \det(U) \) is a non-zero constant.

A2 Let \( R_1 \in \mathbb{R}^{l \times n_1}[\xi] \) and \( R_2 \in \mathbb{R}^{l \times n_2}[\xi] \). We say that \( R_1 \) and \( R_2 \) are left coprime if \( [R_1, R_2] [\lambda] \) has full row rank for all \( \lambda \in \mathbb{C} \).

A3 Let \( R \in \mathbb{R}^{l \times n}[\xi] \). The left syzygy of \( R \) is the set of \( c \in \mathbb{R}^{l \times l}[\xi] \) that satisfy \( c^T R = 0 \). If normalrank(\( R \)) = \( m \), then there exists \( V \in \mathbb{R}^{l \times m}[\xi] \) such that \( V(\lambda) \) has full row rank for all \( \lambda \in \mathbb{C} \); and \( (ii) \) \( VR = 0 \). If \( V \in \mathbb{R}^{l \times m}[\xi] \) satisfies (i) and (ii), then \( c \in \mathbb{R}^{l \times l}[\xi] \) is in the left syzygy of \( R \) if and only if there exists a \( p \in \mathbb{R}^{l \times m}[\xi] \) such that \( p^T V = c^T \); and we say that the rows of \( V \) are a basis for the left syzygy of \( R \).

A4 Given any \( R \in \mathbb{R}^{l \times l}[\xi] \) with normalrank(\( R \)) = \( m \), there exists a unimodular \( U \in \mathbb{R}^{l \times l}[\xi] \) (resp., \( V \in \mathbb{R}^{n \times n}[\xi] \) such that \( UR = \text{col}(\hat{R}, 0_{l \times (m - l)}) \) (resp., \( RV = \text{col}(\hat{R}, 0_{n \times (n - m)}) \)), where \( \hat{R} \in \mathbb{R}^{m \times n}[\xi] \) is in either (i) upper echelon form, or (ii) row reduced form (resp., \( \hat{R} \in \mathbb{R}^{l \times m}[\xi] \) is in either (ib) lower echelon form, or (iih) column reduced form) (see, e.g., Gantmacher \( 1959 \)) Chapter VI and Wolovich \( 1974 \). The last \( l \times m \) rows of \( U \) are a basis for the left syzygy of \( R \). Evidently, if \( R \) is para-Hermitian, then \( URU^* = \text{diag}(\Phi[0]) \) where \( \Phi \in \mathbb{R}^{m \times n}[\xi] \) is para-Hermitian and nonsingular.

B Linear systems and behaviors

Here, we provide relevant results from behavioral theory (see Polderman and Willems \( 1998 \)).
B1 Let $B_1 = \{w \in L^1_{loc}(\mathbb{R}, \mathbb{R}^k) \mid R_1 (\frac{d}{dt}w) = 0\}$ and $B_2 = \{w \in L^1_{loc}(\mathbb{R}, \mathbb{R}^k) \mid R_2 (\frac{d}{dt}w) = 0\}$ for some $R_1, R_2 \in \mathbb{R}^{k \times k}$. Then $B_1 = B_2$ if and only if there exists a unimodular $U \in C_{\mathbb{R}^{k \times k}}$ such that $R_1 = UR_2$ (Polderman and Willems 1998 Theorem 3.6.2). The requirement that $R_1$ and $R_2$ have the same number of rows is of little consequence since the addition or deletion of rows of zeros doesn’t alter the behavior.

B2 Let $F \in \mathbb{R}^{m \times n}[\xi]$, $G \in \mathbb{R}^{m \times n}[\xi]$, and $B := \{z \in L^1_{loc}(\mathbb{R}, \mathbb{R}^n) \mid F(\frac{d}{dt})z = 0\}$. If $z \in B$ implies $G(\frac{d}{dt})z = 0$, then there exists $H \in \mathbb{R}^{m_2 \times m_1}[\xi]$ such that $G = HF$. To see this, note that $B = \{z \in L^1_{loc}(\mathbb{R}, \mathbb{R}^n) \mid \text{col}(F \ G)(\frac{d}{dt})z = 0\}$. It follows from note B1 that there exists a unimodular matrix $U$ with $U\text{col}(F \ 0_{m_2 \times n}) = \text{col}(F \ G)$. We form $H$ from the last $m_2$ rows and first $m_1$ columns of $U$ to obtain $G = HF$.

B3 Consider a system $B$ as in (1.1). $B$ is called controllable if, for any two trajectories $w_1(t), w_2(t) \in B$ and $t_0 \in \mathbb{R}$, there exists $w_3(t) \in B$ such that $w(t) = w_3(t)$ for all $t \leq t_0$ and $w(t) = w_2(t)$ for all $t \geq t_0$ [Polderman and Willems 1998 Definition 5.2.2]; and stabilizable if for any $w_1(t) \in B$ there exists $w(t) \in B$ such that $w(t) = w_1(t)$ for all $t \leq t_0$ and $\lim_{\infty} w(t) = 0$ [Polderman and Willems 1998 Definition 5.2.29]. $B$ is controllable (resp., stabilizable) if and only if the rank of $R(\Lambda)$ is the same for all $\Lambda \in C$ (resp., $\Lambda \in \mathbb{C}_+$) [Polderman and Willems 1998 Theorems 5.2.10, 5.2.30].

C Bilinear and quadratic differential forms

Bilinear and quadratic differential forms were introduced in Willems and Trentelman [1998], and are useful for studying dissipation. Some relevant definitions and results are presented here.

C1 A bilinear differential form is a mapping from $C_{\mathbb{R}}(\mathbb{R}^{m \times n}) \times C_{\mathbb{R}}(\mathbb{R}^{m \times n})$ to $C_{\mathbb{R}}(\mathbb{R}, \mathbb{R})$ of the form $L_{\Phi}(w, x) := \sum_{i=1}^{M} \sum_{j=1}^{N} (\frac{d}{dt})^{i-j} \Phi_{ij}(\frac{d}{dt})^{j} x$. It is naturally associated with the two variable polynomial matrix $\Phi \in \mathbb{R}^{m \times n}[\xi, \eta]$ defined as $\Phi(\xi, \eta) := \sum_{i=1}^{M} \sum_{j=1}^{N} \Phi_{ij}(\xi-i-j^{j})$. If $\Phi \in \mathbb{R}^{m \times n}[\xi, \eta]$, then $Q_{\Phi}(w) := L_{\Phi}(w, w)$ is called a quadratic differential form.

C2 Let $\Phi \in \mathbb{R}^{m \times n}[\xi, \eta]$ and let $\Psi(\xi, \eta) := (\xi + \eta)Q_{\Phi}(\xi, \eta)$. Then the product rule of differentiation gives $\frac{d}{dt}L_{\Phi}(w, x) = L_{\Phi}(w, x) + \frac{df}{dt}$. 

C3 Associated with a given $R \in \mathbb{R}^{m \times n}[\xi]$ is the bilinear differential form $L_{\Phi_R}$, where $\Phi_R(\xi, \eta) := (R(\xi) - R(-\eta))/(\xi + \eta)$. Since $\xi + \eta$ implies $R(\xi) = R(-\eta) = 0$, then $\Phi_R \in \mathbb{R}^{m \times n}[\xi, \eta]$ from the factor theorem. Furthermore, from note C2, $(R(\frac{d}{dt})w)^{T}x = -w^{T}(R^{T} - \frac{df}{dt})x = \int_{t_0}^{t_1} (R(\frac{d}{dt})w)^{T}x(t)dt = \int_{t_0}^{t_1} w^{T}(R^{T} - \frac{df}{dt})x(t)dt + [L_{\Phi_R}(w, x)]_{t_0}^{t_1}$. Note that if $R(\frac{df}{dt}) = \frac{df}{dt}$, then $\Phi_R = 1$, and this becomes the formula for integration by parts.

D States and state-space systems

In this final appendix, we provide several useful definitions and results concerning state-space systems.

D1 Let $B_s$ be as in (1.2). Then, for any given $u \in L^1_{loc}(\mathbb{R}, \mathbb{R}^n)$, $w_0 \in \mathbb{R}^n$, and $t_0 \in \mathbb{R}$, there exists a unique $(u, y, x) \in B_s$ with $x(t_0) = x_0$, which is given by the variation of the constants formula: $x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-t)}B \sigma(u)dt$ for all $t \geq t_0$; $x(t) = e^{A(t-t_0)}x_0 - \int_{t_0}^{t} e^{A(t-t)}B \sigma(u)dt$ for all $t < t_0$, and $y = Cx + Du$. 

D2 Let $B_s$ be as in (1.2). We call the pair $(C, A)$ observable if $(u, y, x) \in B_s$ imply $x = \tilde{x}$ [Polderman and Willems 1998 Definition 5.3.2]. With the notation $V_o := \text{col}(C \ A \ C A^{d-1})$, then $(C, A)$ is observable if and only if $\text{rank}(V_o) = d$ [Polderman and Willems 1998 Theorem 5.3.9]. Now, let $\text{rank}(V_o) = d_1 < d$, let the columns of $S_2$ be a basis for the set $\{z \in \mathbb{R}^d \mid V_o z = 0\}$; let $S = [S_1 \ S_2]$ be nonsingular; and partition $T := S^{-1}$ compatibly with $S$ as $t = (T_1 \ T_2)$. Then

\[
\begin{pmatrix}
T_1 & 0 \\
T_2 & 1
\end{pmatrix}
A [S_1 \ S_2] =
\begin{pmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{pmatrix}
C [S_1 \ S_2] = \begin{pmatrix}
\tilde{C}_1 & 0
\end{pmatrix},
\]

where $(\tilde{C}_1, A_{11})$ is observable: the observer staircase form [Polderman and Willems 1998 Corollary 5.3.14].

D3 Let $B_s$ be as in (1.2); let $V_o$ and $S$ be as in note D2; let $B := B^{[0, \infty)}$; and let $t_0 \in \mathbb{R}$. If $(u, y, x) \in B_s$, $(u, y) \in B$, and $(\hat{u}(t), y(t)) = (u(t), y(t))$ for all $t < t_0$, then there exists $(\hat{u}, y, x) \in B_s$ with $x(t_0) = x(t_0)$. This follows from the following two observations, which are easily shown from the variation of the constants formula: (i) if $(u, y) \in B$ and $z \in \mathbb{R}^d$, then there exists $(\hat{u}, y, x) \in B_s$ with $T_2x(t_0) = z$; and (ii) if $(u, y, x), (\bar{u}, \bar{y}, \bar{x}) \in B_s$, and $(\hat{u}(t), y(t)) = (\bar{u}(t), \bar{y}(t))$ for all $t < t_0$, then $T_2x(t_0) = T_2\bar{x}(t_0)$.

Also, with $A_{11}$ and $\tilde{C}_1$ as in note D2, and $\hat{B}_1 := T_1B$, then it follows from the variation of the constants formula that $B = \hat{B}_s(\hat{u}, y, \hat{x})$, with

\[
\hat{B}_s = \{(u, y, \hat{x}) \in L^1_{loc}(\mathbb{R}, \mathbb{R}^n) \times L^1_{loc}(\mathbb{R}, \mathbb{R}^n) \times L^1_{loc}(\mathbb{R}, \mathbb{R}^d) \mid \text{such that} \quad \frac{df}{dt} = \hat{A}_{11}\hat{x} + B_1u \text{ and } y = \hat{C}_1\hat{x} + D_1u\}.
\]
D4 Let $\mathcal{B} = \{(u, x) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \times L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d) | \frac{dx}{dt} = Ax + Bu\}$ and let $V \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R})$ and let $V_x := [B \ AB \cdots A^{d-1}B]$. If $\mathcal{B}$ is controllable, then we also call the pair $(A, B)$ controllable. From [Polderman and Willems (1998), Section 5.2.1], the following are equivalent: (i) $(A, B)$ is controllable; (ii) $[\lambda I - A B]$ has full row rank for all $\lambda \in \mathbb{C}$; and (iii) $\text{rank}(V_x) = d$. Now, let $\mathcal{B}_s$ be as in (1.2); let $\mathcal{B} = \mathcal{B}(u^n)$; and let $(C, A)$ be observable. Then $\mathcal{B}$ is controllable (resp., stabilizable) if and only if $[\lambda I - A B]$ has full row rank for all $\lambda \in \mathbb{C}$ (resp., $\lambda \in \hat{\mathbb{C}}_+$). The proof is similar to [Hughes (2016a), proof of Theorem 5.2].

References


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