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Questions of minimality in RLC circuit synthesis

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Abstract— **It is well known that the impedance of a passive circuit is necessarily positive-real [1]. The Bott-Duffin procedure shows that any positive-real function has an RLC realisation, possibly with the number of reactive elements (inductors and capacitors) greatly exceeding the McMillan degree. In [2] it was shown, for series-parallel circuits, that the Bott-Duffin procedure is minimal in the number of reactive elements (six) for the biquadratic minimum function. For general circuits, the best available result is the Reza-Pantell-Fialkow-Gerst simplification, published simultaneously in the 1954 papers [3–5], which reduces the number of reactive elements to five for the general biquadratic minimum function. In this extended abstract, we present an additional class of equivalent circuits which have not appeared previously in the literature. In the accompanying talk, we will show the remarkable result that the Reza-Pantell-Fialkow-Gerst simplification produces circuits which contain the least possible number of reactive elements for the realisation of certain biquadratic minimum functions.**

Key words. Circuit synthesis, mechanical control, passivity, realisation, inerter

AMS subject classification. 70Q05, 93C05, 94C99

I. Introduction

This extended abstract describes results obtained in [6] relating to the realisation of positive-real functions with the least number of reactive elements (inductors and capacitors). Given the analogy between electrical circuits and mechanical networks, which was established by the recent invention of the inerter, this question is of immediate relevance to the effective design of passive mechanical controllers for a broad range of applications [7].

As is well known, the impedance (and the admittance) of a circuit which contains only resistors, inductors, and capacitors (an RLC circuit) is necessarily PR. In [8], Bott and Duffin provided the first explicit construction for the realisation of a given PR function. As will be shown in this extended abstract, the application of various circuit transformations leads to several equivalent circuits to those of Bott and Duffin. These include the Reza-Pantell-Fialkow-Gerst simplification, which, to the present day, provides the simplest circuit realisation available for a given PR function among all general RLC circuit synthesis procedures. Here, we present additional equivalent circuits which contain the same number of reactive elements (and the same number of resistors) as those obtained by the Reza-Pantell-Fialkow-Gerst procedure. This disproves the claim made in [9] that the circuits obtained by the Reza-Pantell-Fialkow-Gerst simplification are "the only general seven-element realisations of the biquadratic minimum function".

In a forthcoming paper [2], it is shown that the procedure of Bott and Duffin produces circuits which contain the least possible number of reactive elements among all *series-parallel* circuits which realise a biquadratic minimum function. In [6], it is shown that the circuits produced by the Reza-Pantell-Fialkow-Gerst simplification, and the alternatives presented here, contain the least possible number of reactive elements among all RLC circuits which realise a biquadratic minimum function, with the exception of some special cases which are described explicitly. This is surprising since the number of reactive elements used by these procedures for the realisation of a biquadratic minimum function (five) considerably exceeds the McMillan degree of a biquadratic minimum function (two). In contrast, it is known that there is a large class of impedance functions of McMillan degree two which can be realised by circuits containing two reactive elements [10], [11]. Indeed, the McMillan degree of a general PR function is the only known lower bound on the number of reactive elements required to realise that function.

The Bott-Duffin procedure, and the Reza-Pantell-Fialkow-Gerst simplification, are inductive procedures, in which the McMillan degree of the function to be realised is reduced at each stage. The procedure relies on a preliminary procedure at each stage, called the Foster preamble. In Section II, we provide a brief description of the Foster preamble. In Section III, we describe the Bott-Duffin procedure itself, then in Section IV we introduce the aforementioned simplifications to this procedure. Finally, in Section V, we discuss the minimality of these identified simplifications to Bott-Duffin for the realisation of certain biquadratic minimum functions.

II. The Foster Preamble

The Foster preamble is a circuit synthesis procedure which either provides an RLC circuit realisation for a given PR function, or provides a partial realisation and reduces the given PR function to a minimum function. A PR function, and a minimum function, are defined as follows:

Definition 1 (PR): $F(s)$ is PR if it is real-rational, analytic in the open right half plane, and satisfies $\Re(F(s_0)) \geq 0$ for all s_0 with $\Re(s_0) > 0$.

Definition 2 (minimum function): $F(s)$ is a minimum function if it is PR, not identically zero, has no poles or zeros on the extended imaginary axis, and satisfies $\Re(F(j\omega_0)) = 0$ for at least one strictly positive value of ω_0 (this implies $\Im(F(j\omega_0)) \neq 0$).

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The Foster preamble depends on the properties of PR functions stated in the following two theorems:

Theorem 3 ([1], Theorem III, Coroll. 1): If $F(s)$ is PR and R_1 is less than or equal to the minimum value of $\Re(F(s))$ on the extended imaginary axis, then $F_r(s) = F(s) - R_1$ is PR.

Theorem 4 ([1], Theorem IV): If $F(s)$ is PR and has poles at $s = j\omega_r$ for $\omega_r > 0$ with residue $k_r/2$ ($r = 1, 2, \ldots, n$), in addition to a pole at $s = 0$ with residue k_0, and a pole at $s = \infty$ with residue k_∞, then

$$F(s) = \frac{k_0}{s} + k_\infty s + \sum_{r=1}^{n} \frac{k_r s}{s^2 + \omega_r^2} + F_r(s), \tag{1}$$

where each term is PR.

In Theorem 3, $F(s)$ is the impedance of a circuit with impedance $F_r(s)$ in series with a resistor with resistance R_1. Moreover, each term in the sum in (1) is the impedance of a parallel connection of an inductor and a capacitor, and the first two terms on the right hand side of equation (1) are the impedance of a capacitor and an inductor respectively. Hence, in Theorem 4, $F(s)$ is the impedance of a circuit with impedance $F_r(s)$ in series with a circuit comprising reactive elements. Similar partial circuit realisations can be obtained for which $F(s)$ is equal to the admittance of the circuit. Evidently, the inductive application of these two procedures will result in a function $F_r(s)$ which is either identically zero or is a minimum function. Moreover, the McMillan degree of $F_r(s)$ cannot exceed that of $F(s)$.

III. The Bott-Duffin Procedure

The paper by Bott and Duffin [8] provides a circuit realisation for a given minimum function $F(s)$ which comprises reactive elements and two circuits whose impedances have McMillan degree at least two fewer than $F(s)$. When used inductively together with the Foster preamble, this procedure provides an RLC circuit for the realisation of any given PR function. The procedure relies on a generalisation of Richard's transformation:

$$R(s) := \frac{\mu F(s) - sF(\mu)}{\mu F(\mu) - sF(s)},$$

which, for $\mu > 0$, transforms a PR function $F(s)$ into a PR function $R(s)$ whose McMillan degree does not exceed that of $F(s)$. For a minimum function $F(s)$ with $\Re(F(j\omega_0)) = 0$ and $\Im(F(j\omega_0)) > 0$, Bott and Duffin demonstrate the existence of a particular value of $\mu > 0$ which satisfies $\mu F(j\omega_0) - j\omega_0 F(\mu) = 0$. Moreover, for this choice of μ, $1/R(s)$ has poles at $s = \pm j\omega_0$. Then, let

$$\frac{1}{F_r(s)} := \frac{1}{R(s)} - \frac{2\alpha s}{s^2 + \omega_0^2},$$

where α is the residue of the pole of $1/R(s)$ at $s = j\omega_0$, and it follows that

$$\begin{aligned} F(s) &= \left(\frac{R(s)}{F(\mu)} + \frac{\mu}{F(\mu)s}\right)^{-1} + \left(\frac{1}{F(\mu)R(s)} + \frac{s}{F(\mu)\mu}\right)^{-1} \\ &= F(\mu)\frac{s^3 + F_r(s)(2\alpha + \mu)s^2 + \omega_0^2 s + F_r(s)\mu\omega_0^2}{F_r(s)s^3 + \mu s^2 + F_r(s)(2\alpha\mu + \omega_0^2)s + \mu\omega_0^2}, \end{aligned} \tag{2}$$

which is the impedance of the circuit on the top left of Fig. 1.1. Moreover, the function $F_r(s)$ is PR, and its McMillan degree is at least two fewer than that of $F(s)$.

For the case when $\Im(F(j\omega_0)) < 0$, a dual argument to the above demonstrates the existence of $\nu, \beta > 0$, and a PR function $\tilde{F}_r(s)$ with McMillan degree at least two fewer than $F(s)$, such that

$$F(s) = F(\nu)\frac{\tilde{F}_r(s)s^3 + \nu s^2 + \tilde{F}_r(s)(\omega_0^2 + 2\beta\nu)s + \nu\omega_0^2}{s^3 + \tilde{F}_r(s)(2\beta + \nu)s^2 + \omega_0^2 s + \tilde{F}_r(s)\nu\omega_0^2}. \tag{3}$$

In this case, $F(s)$ is the impedance of the circuit on the top right of Fig. 1.1.

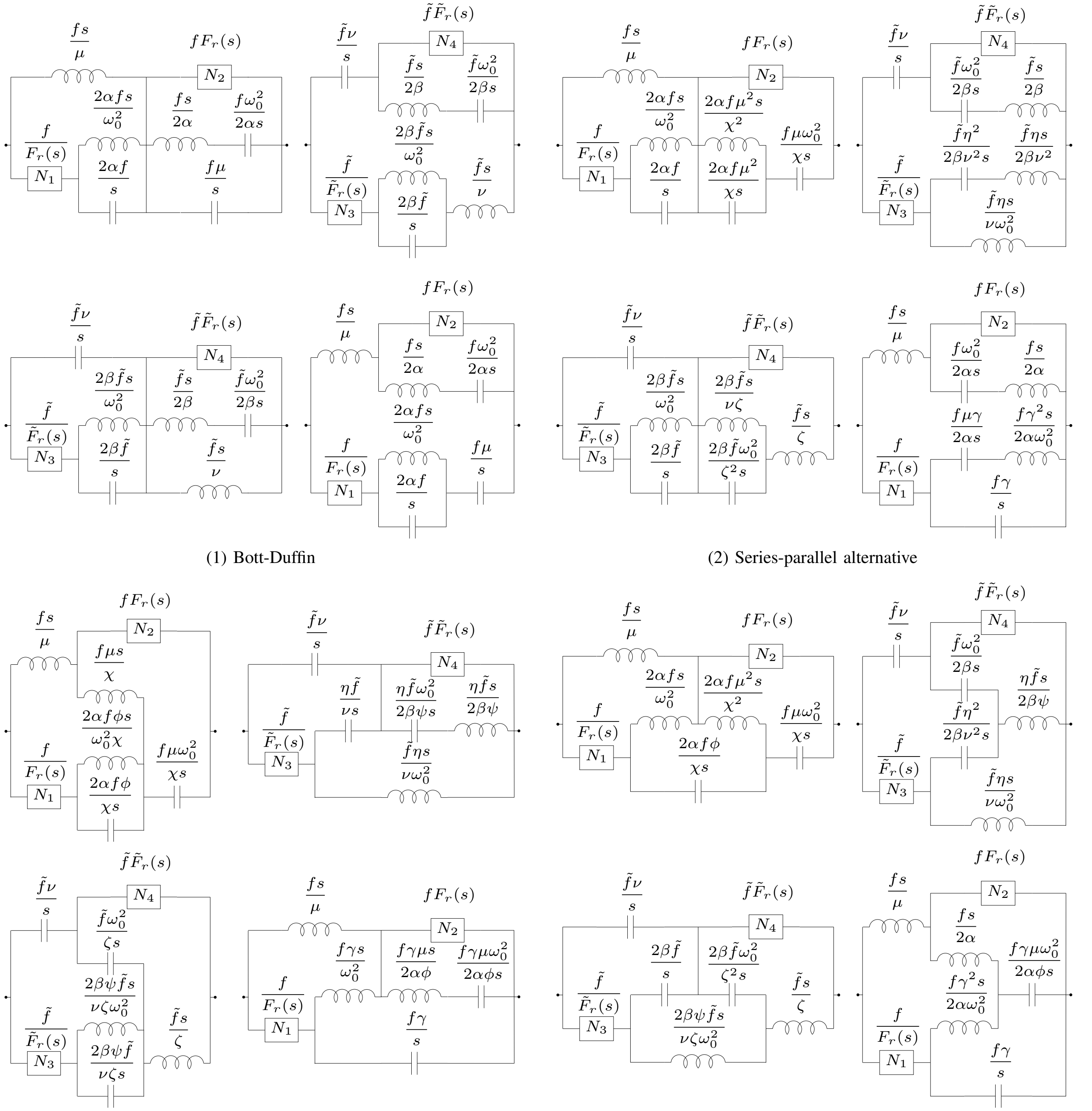
By applying the Foster preamble and the Bott-Duffin procedure to the remainder function $F_r(s)$, or $\tilde{F}_r(s)$, and proceeding inductively, an RLC circuit can be obtained to realise any given PR function. The number of reactive elements in any circuit thus obtained is significantly greater than the McMillan degree of the function being realised.

IV. Simplifications to Bott-Duffin

In Figs. 1.1 to 1.4, we present the circuits obtained by the procedure of Bott and Duffin (Fig. 1.1), in addition to several equivalent circuits (Figs. 1.2 to 1.4). These include the circuits obtained by the Reza-Pantell-Fialkow-Gerst simplification in Fig. 1.3. The circuits in Fig. 1.2 are introduced for the first time in the forthcoming paper [2]. The circuits in Fig. 1.4 appear for the first time in [6]. These circuits contain the same number of reactive elements, and the same number of resistors, as the circuits obtained from the Reza-Pantell-Fialkow-Gerst simplification.

It may be verified by direct calculation that the impedances of the circuits on the top left, and on the bottom right, of Figs. 1.1 to 1.4 are each equal to $F(s)$ in equation (2). Moreover, the impedances of the circuits on the top right, and on the bottom left, of these figures are each equal to $F(s)$ in equation (3). In the case of a biquadratic minimum function, the impedances $fF_r(s)$, $f/F_r(s)$, $\tilde{f}\tilde{F}_r(s)$ and $\tilde{f}/\tilde{F}_r(s)$ may each be realised by a resistor, and the circuits obtained in this case contain exactly five reactive elements and two resistors. It was these circuits which were initially identified in our examination of those circuits which contain five or fewer reactive elements and which realise a biquadratic minimum function. It was subsequently recognised that these circuits could be generalised to the circuits shown in Fig. 1.4, in order to provide a realisation for any given minimum function.

The circuits in Fig. 1.4 may be obtained by the application of circuit transformations to the Bott-Duffin circuits, as may all the other circuits in Fig. 1. As an example of one of these circuit transformations, in the circuit on the top left of Fig. 1.2, it may be verified that there can never be any current through the vertical wire which is shared by the two parallel connections of an inductor and a capacitor. Consequently, the impedance of that circuit is unchanged by the removal of this wire. This leaves two capacitors in series which may be combined, thus obtaining the circuit on the top left of Fig. 1.4.



(1) Bott-Duffin (2) Series-parallel alternative

(3) Reza-Pantell-Fialkow-Gerst simplification (4) New simplification

Fig. 1. Here we show the circuits generated in a single inductive step in the realisation of a minimum function $F(s)$. In this figure, $f = F(\mu)$, $\chi = \omega_0^2 + 2\alpha\mu$, $\gamma = \mu + 2\alpha$, $\phi = \chi + \mu^2$, $\tilde{f} = F(\nu)$, $\eta = \omega_0^2 + 2\beta\nu$, $\zeta = \nu + 2\beta$, $\psi = \eta + \nu^2$, and $F_r(s)$, μ, α, $\tilde{F}_r(s)$, ν, β are as described in Section III. If $\Im(F(j\omega_0)) > 0$ then $F(s)$ is realised by the circuits on the top left and bottom right of Figs. 1.1 to 1.4. If $\Im(F(j\omega_0)) < 0$, then $F(s)$ is realised by the circuits on the top right and bottom left of Figs. 1.1 to 1.4.

V. Minimality of the Simplifications to Bott-Duffin

In the case of a biquadratic minimum function, the circuits in Figs. 1.3 and 1.4 each contain five reactive elements. As shown in [6], this is the least possible number of reactive elements for the realisation of almost all biquadratic minimum functions. In other words, there are certain biquadratic minimum functions which cannot be realised by any circuit which contains four or fewer reactive elements, irrespective of the number of resistors possessed by that circuit. The argument we will employ differs considerably from that in [2], in which the minimality of the Bott-Duffin procedure for series-parallel circuit realisations of biquadratic minimum functions is considered, since the results in [2] rely on the special form of the impedance function of a series-parallel circuit. Instead, the argument focuses on the permissible currents and voltages within those circuits which realise minimum functions. Using the well known result that the instantaneous power supplied to a circuit must equal the sum of the instantaneous powers dissipated in the resistors in the circuit, it may be shown that if the driving-point current through the circuit and the currents through the elements within the circuit are all varying sinusoidally at the minimum frequency then there can be no current through the resistors in the circuit. Hence, there must be regions within the circuit comprising elements with no current flow through them. By describing the circuit as an interconnection of such regions together with elements through which current does flow (which are necessarily reactive), it is possible to identify those biquadratic minimum functions which can be realised by circuits containing fewer than five reactive elements. We will describe these biquadratic minimum functions explicitly, and thus demonstrate the minimality (in number of reactive elements) of the circuits in Figs. 1.3 and 1.4 for the realisation of almost all biquadratic minimum functions. Finally, as shown in [6], these circuits also contain the minimum total number of elements for the realisation of almost all biquadratic minimum functions.

REFERENCES

[1] O. Brune, "Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency," *J. Math. Phys.*, vol. 10, pp. 191–236, 1931.

[2] T. H. Hughes and M. C. Smith, "On the minimality and uniqueness of the Bott-Duffin realisation procedure," *To appear in IEEE Trans. on Automatic Control*, 2014.

[3] F. M. Reza, "Synthesis without ideal transformers," *J. Appl. Phys.*, vol. 25, pp. 807–808, March 1954.

[4] R. H. Pantell, "A new method of driving point impedance synthesis," *Proc. IRE (Correspondence)*, vol. 42, p. 861, 1954.

[5] A. Fialkow and I. Gerst, "Impedance synthesis without mutual coupling," *Quart. Appl. Math.*, vol. 12, pp. 420–422, 1955.

[6] T. H. Hughes, "On the synthesis of passive networks without transformers," Ph.D. dissertation, University of Cambridge, 2014.

[7] M. C. Smith, "Synthesis of mechanical networks: the inerter," *IEEE Trans. on Automatic Control*, vol. 47, no. 10, pp. 1648–1662, 2002.

[8] R. Bott and R. J. Duffin, "Impedance synthesis without use of transformers," *J. Appl. Phys.*, vol. 20, p. 816, 1949.

[9] S. Seshu, "Minimal realizations of the biquadratic minimum function," *IRE Trans. Circuit Theory*, vol. 6, pp. 345–350, 1959.

[10] J. Z. Jiang and M. C. Smith, "Regular positive-real functions and five-element network synthesis for electrical and mechanical networks," *IEEE Trans. on Automatic Control*, vol. 56, no. 6, pp. 1275–1290, June 2011.

[11] M. Reichert, "Die Kanonisch und übertragerfrei realisierbaren Zweipolfunktionen zweiten Grades (transformerless and canonic realisation of biquadratic immittance functions)," *Arch. Elek. Übertragung*, vol. 23, pp. 201–208, Apr. 1969.