Passivity and electric circuits: a behavioral approach

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Abstract: We present the element extraction approach to electric circuit analysis and synthesis using the behavioral framework. Explicit descriptions are obtained of the behavior and the driving-point behavior of a general circuit comprising resistors, inductors, capacitors, transformers, and gyrators (an RLCTG circuit). It is shown that the internal currents and voltages are always *properly eliminable* to obtain a driving-point behavior which is the set of locally integrable (weak) solutions to a linear differential equation. We also review a recently introduced trajectory level definition of passivity, and we show that a behavior is passive if and only if it is the driving-point behavior of an RLCTG circuit.

Keywords: Passivity, Behavioral theory, Electric circuits, Network synthesis, Linear systems

1. INTRODUCTION

In this paper, we describe the element extraction approach to the analysis and synthesis of electric circuits. Several of the results are inspired by the papers [1, 2], and are presented here in the behavioral framework [7]. This framework also exposes questions which are not covered in [1, 2]. Notably, the main contributions of this paper are:

1. In [3], a new definition of passivity was proposed which differs from the definition in [2] in its treatment of uncontrollable behaviors. We will demonstrate that a system is passive in the sense of [3] if and only if it is the driving-point behavior of an electric circuit comprising resistors, inductors, capacitors, transformers and gyrators (an RLCTG circuit). This was stated in [3] without proof.
2. We show that, for *any* n-port RLCTG circuit, there exist compatible partitions of the driving-point currents and voltages as $(\mathbf{i}_1, \mathbf{i}_2)$ and $(\mathbf{v}_1, \mathbf{v}_2)$ such that the driving-point behavior has an input-state-output representation in which: (i) the state is a subset of the capacitor voltages and inductor currents; and (ii) the input comprises $\mathbf{i}_1$ and $\mathbf{v}_2$, and the output comprises $\mathbf{v}_1$ and $\mathbf{i}_2$. The currents and voltages in the circuit's resistors and the other capacitors and inductors are a linear function of this input and state.
3. We conclude that the driving-point behavior of an electric circuit is the set of (weak) solutions to a linear differential equation. In other words, the internal currents and voltages are *properly eliminable* in the sense of [6].

The paper is structured as follows. In Section 2, we describe the element extraction approach to circuit analysis. Section 3 discusses the concept of proper elimination. In Section 4, we review the material in [3] on passivity. Then, Section 5 investigates the behavior of circuits containing only resistors, transformers, and gyrators (RTG circuits), and Section 6 investigates the behavior of RLCTG circuits.

Our notation is as follows. We denote the real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, and the closed right-half plane by $\overline{\mathbb{C}}_+$. If $\lambda \in \mathbb{C}$, $\bar{\lambda}$ denotes its complex conjugate. The polynomials and rational functions in the indeterminate s with real coefficients are denoted $\mathbb{R}[s]$ and $\mathbb{R}(s)$. $\mathbb{R}^{m\times n}$ (resp., $\mathbb{R}^{m\times n}[s]$, $\mathbb{R}^{m\times n}(s)$) denotes the $m\times n$ matrices with entries from $\mathbb{R}$ (resp., $\mathbb{R}[s]$, $\mathbb{R}(s)$), and n is omitted if $n = 1$. If $H \in \mathbb{R}^{m\times n}, \mathbb{R}^{m\times n}[s]$, or $\mathbb{R}^{m\times n}(s)$, then H^T denotes its transpose, and if H is invertible then H^{-1} denotes its inverse. If $M \in \mathbb{R}^{m\times m}$, $M > 0$ ($M \geq 0$) indicates that M is positive (non-negative) definite. We denote the block column and block diagonal matrices with entries $H_1, \ldots, H_n$ by $\mathrm{col}(H_1 \ \cdots \ H_n)$ and $\mathrm{diag}(H_1 \ \cdots \ H_n)$.

The (k-vector-valued) locally integrable functions are denoted $\mathcal{L}_1^{\mathrm{loc}}\left(\mathbb{R}, \mathbb{R}^k\right)$ [7, Defns. 2.3.3, 2.3.4], and we equate any two locally integrable functions which differ only on a set of measure zero. We will consider behaviors (systems) defined in one of the following two ways: (i) as the set of weak solutions (see [7, Sec. 2.3.2]) to a differential equation

$$\mathcal{B} = \{\mathbf{w} \in \mathcal{L}_1^{\mathrm{loc}}\left(\mathbb{R}, \mathbb{R}^q\right) \mid R(\tfrac{d}{dt})\mathbf{w} = 0\}, \ R \in \mathbb{R}^{p\times q}[s]; \quad (1)$$

and (ii) as the projection of a behavior $\mathcal{B}$ as in (1) onto a subset of its components, denoted:

$$\mathcal{B}^{(T_1\mathbf{w},\ldots,T_m\mathbf{w})} = \{(T_1\mathbf{w}, \ldots, T_m\mathbf{w}) \mid \mathbf{w} \in \mathcal{B}\},$$

where $T_1, \ldots, T_m$ are real matrices which can be completed to a permutation matrix $T = \mathrm{col}(T_1 \ \cdots \ T_m \ T_{m+1})$.

2. ELECTRIC CIRCUIT ANALYSIS

Our concern in this paper is with electric circuits comprising an interconnection of resistors, inductors, capacitors, transformers, and gyrators (RLCTG circuits). Resistors, inductors, and capacitors each have a single port (pair of

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terminals) across which a voltage v can be applied and through which a current i can flow; and the behaviors of these elements satisfy

$$\{(i,v) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}) \mid v{=}iR \text{ for some } R{>}0\},$$
$$\{(i,v) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}) \mid v{=}L\tfrac{di}{dt} \text{ for some } L{>}0\},$$
$$\{(i,v) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}) \mid i{=}C\tfrac{dv}{dt} \text{ for some } C{>}0\},$$

respectively. A transformer possesses $m+n$ ports, and its behavior is determined by its *turns-ratio matrix* $T \in \mathbb{R}^{m\times n}$. Specifically, the driving-point currents $\mathbf{i} = \mathrm{col}\,(\mathbf{i}_1\ \mathbf{i}_2)$ and voltages $\mathbf{v} = \mathrm{col}\,(\mathbf{v}_1\ \mathbf{v}_2)$ (partitioned compatibly with T) satisfy

$$\left\{(\mathbf{i},\mathbf{v}){\in}\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \mid \begin{bmatrix}\mathbf{v}_1\\ \mathbf{i}_2\end{bmatrix} {=} \begin{bmatrix}0 & T^T\\ -T & 0\end{bmatrix}\begin{bmatrix}\mathbf{i}_1\\ \mathbf{v}_2\end{bmatrix}\right\}.$$

Finally, a gyrator possesses 2 ports, and its driving-point currents $\mathbf{i} = \mathrm{col}\,(i_1\ i_2)$ and voltages $\mathbf{v} = \mathrm{col}\,(v_1\ v_2)$ satisfy

$$\left\{(\mathbf{i},\mathbf{v}){\in}\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2) \mid \begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix}\right\}.$$

For each of these five elements, the inner product of the port currents and the port voltages is equal to the instantaneous power supplied to the element. An n-port RLCTG circuit is then an interconnection of resistors, inductors, capacitors, transformers, and gyrators which has n designated external ports, each associated with a current and a voltage. The vector of port currents $\mathbf{i}$ and voltages $\mathbf{v}$ are oriented so the energy transferred to the circuit in the interval from t_0 to t_1 is $\int_{t_0}^{t_1}(\mathbf{i}^T\mathbf{v})(t)dt$.

The element extraction approach to RLCTG circuit analysis views any given electric circuit N as the cascade interconnection of two circuits: N_1, in which all of the elements are removed and every single element port is replaced with an external port; and N_2, which contains each of the elements in the original circuit (disconnected from each other) [1]. If there are n external ports and m element ports in N, then N_1 is an $n{+}m$-port circuit with driving-point currents $\mathrm{col}\,(\mathbf{i}\ \mathbf{i}_2)$ and voltages $\mathrm{col}\,(\mathbf{v}\ \mathbf{v}_2)$ which satisfy Kirchhoff's current and voltage laws; and N_2 is an m-port circuit with driving-point currents $\mathbf{i}_1$ and voltages $\mathbf{v}_1$ which satisfy the individual element equations. Here, Kirchhoff's laws result in $n+m$ equations relating $\mathbf{i}, \mathbf{v}, \mathbf{i}_2$ and $\mathbf{v}_2$; [1] and the element constraints give m equations relating $\mathbf{i}_1$ and $\mathbf{v}_1$. The circuit N is obtained by connecting the final m ports of N_1 to the m ports of N_2, which results in the $2m$ interconnection equations $\mathbf{v}_2 = \mathbf{v}_1$ and $\mathbf{i}_2 = -\mathbf{i}_1$. We thus obtain $n{+}4m$ linear differential equations relating $\mathbf{i}, \mathbf{i}_1, \mathbf{i}_2, \mathbf{v}, \mathbf{v}_1$, and $\mathbf{v}_2$; where $\mathbf{i}$ and $\mathbf{v}$ correspond to the driving-point current and voltage of N; and $\mathbf{i}_1$ and $\mathbf{v}_1$ correspond to the internal currents and voltages. The full behavior of the circuit is the projection of the solutions to this equation onto the variables $(\mathbf{i}, \mathbf{i}_1, \mathbf{v}, \mathbf{v}_1)$, and the driving-point behavior is the projection onto $(\mathbf{i}, \mathbf{v})$. These are obtained by the procedure in the next section.

Note that the element extraction procedure can proceed inductively. This approach is taken in this paper. We consider six circuits: N_{1a}, the circuit N with the inductors and capacitors replaced by external ports; N_{2a}, the inductors and capacitors; N_{1b}, the circuit N_{1a} with the resistors replaced by external ports; N_{2b}, the resistors; N_{1c}, the circuit N_{1b} with the transformers and gyrators replaced by external ports; and N_{2c}, the transformers and gyrators. The procedure in the previous paragraph can then be used to obtain the driving-point behavior of N_{1b} (by letting $N_1 = N_{1c}$ and $N_2 = N_{2c}$), then N_{1a} (by letting $N_1 = N_{1b}$ and $N_2 = N_{2b}$), and finally N (by letting $N_1 = N_{1a}$ and $N_2 = N_{2a}$). This process is illustrated in Fig. 1 (see p. 6).

3. PROPER ELIMINATION

As discussed previously, the full behavior and the driving-point behavior of an electric circuit can be obtained by projecting the set of solutions $\mathbf{w}$ to an equation of the form of (1) onto subsets of its components. Accordingly, let $\hat{\mathcal{B}} = \{\mathrm{col}\,(\mathbf{w}_1\ \mathbf{w}_2) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{n_1+n_2}) \mid \hat{R}(\frac{d}{dt})\mathrm{col}\,(\mathbf{w}_1\ \mathbf{w}_2) = 0\}$. By [7, Thm. 6.2.6], there exists a unimodular U such that

$$U\hat{R} = \begin{bmatrix}R_1 & 0\\ R_2 & M_2\end{bmatrix}, \tag{2}$$

where the rightmost matrix is partitioned compatibly with $\mathrm{col}\,(\mathbf{w}_1\ \mathbf{w}_2)$, and M_2 has full row rank. Then, from [7, Thm. 2.5.4], $\hat{\mathcal{B}}$ is the set of locally integrable solutions to $R_1(\frac{d}{dt})\mathbf{w}_1 = 0$ and $M_2(\frac{d}{dt})\mathbf{w}_2 = -R_2(\frac{d}{dt})\mathbf{w}_1$. Now, let $\mathcal{B} := \{\mathbf{w}_1 \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{n_1}) \mid R_1(\frac{d}{dt})\mathbf{w}_1 = 0\}$. If, for any given $\mathbf{w}_1 \in \mathcal{B}$, there exists a $\mathbf{w}_2 \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{n_2})$ such that $M_2(\frac{d}{dt})\mathbf{w}_2 = -R_2(\frac{d}{dt})\mathbf{w}_1$, then $\mathcal{B}$ is the projection of $\hat{\mathcal{B}}$ onto $\mathbf{w}_1$; i.e., $\mathcal{B} = \hat{\mathcal{B}}^{(\mathbf{w}_1)}$. In this case, we call $\mathbf{w}_2$ *properly eliminable*. However, this is not true in general. For example, consider the system $\mathcal{B} = \{\mathrm{col}(w_{1a}\ w_{1b}) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2) \mid \exists w_2 \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ such that $w_{1a} = w_{1b}$, and $\frac{dw_{1a}}{dt} = w_2\}$, from [6, Example 2.1]. Then any trajectory in $\mathcal{B}$ must satisfy $w_{1a} = w_{1b}$, and the (weak) derivative of w_{1a} must be locally integrable. In fact, $\mathcal{B}$ cannot be represented in the form of (1) [6].

4. PASSIVITY AND CONTROLLABILITY

The driving-point behavior of each of the five electric circuit elements has the general form:

$$\mathcal{B} = \{(\mathbf{i},\mathbf{v}) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \mid P(\tfrac{d}{dt})\mathbf{i} = Q(\tfrac{d}{dt})\mathbf{v}, \text{ for some } P,Q \in \mathbb{R}^{n\times n}[s]\}. \tag{3}$$

Here, Q need not be invertible (consider a transformer). In [3], the following definition of passivity was proposed:

Definition 1. $\mathcal{B}$ in (3) is called *passive* if, for any given $(\mathbf{i},\mathbf{v}){\in}\mathcal{B}$ and $t_0{\in}\mathbb{R}$, there exists $K{\in}\mathbb{R}$ (dependent on $(\mathbf{i},\mathbf{v})$ and t_0) such that if $(\hat{\mathbf{i}},\hat{\mathbf{v}}) \in \mathcal{B}$ and $(\hat{\mathbf{i}}(t),\hat{\mathbf{v}}(t)) = (\mathbf{i}(t),\mathbf{v}(t))$ for all $t < t_0$, then $-\int_{t_0}^{t_1}(\hat{\mathbf{i}}^T\hat{\mathbf{v}})(t)dt < K$ for all $t_1 \geq t_0$.

Note that it is easily shown that the bound K is necessarily non-negative. An important contribution in [3] was to answer the question *when is an uncontrollable system passive?*. Here, $\mathcal{B}$ in (3) is called *controllable* if, for any given $(\mathbf{i}_1,\mathbf{v}_1),(\mathbf{i}_2,\mathbf{v}_2) \in \mathcal{B}$, and any given $t_0 \in \mathbb{R}$, there exists a $t_1 \in \mathbb{R}$ and an $(\mathbf{i},\mathbf{v}) \in \mathcal{B}$ such that $(\mathbf{i}(t),\mathbf{v}(t)) = (\mathbf{i}_1(t),\mathbf{v}_1(t))$ for all $t < t_0$; and $(\mathbf{i}(t),\mathbf{v}(t)) = (\mathbf{i}_2(t),\mathbf{v}_2(t))$ for all $t \geq t_1$ (see [7, Defn. 5.2.2]). In fact, if a linear system

[1] These equations can be derived from the graph G obtained by replacing each port in N_1 with an (oriented) edge. Kirchhoff's laws imply that the driving-point currents and voltages of N_1 are in the circuit and cut-set space of the graph, respectively [5]. These two vector spaces are orthogonal (so $\mathbf{i}^T\mathbf{v} + \mathbf{i}_2^T\mathbf{v}_2 = 0$), and the sum of their dimensions is equal to the number of edges in G (i.e., $n+m$).

$\mathcal{B}$ is controllable, then this property holds for all $t_1 > t_0$. Note, however, that the driving-point behavior of a general RLCTG circuit need not be controllable [4, 5].

If $\mathcal{B}$ in (3) is passive and Q is invertible, then $Q^{-1}P$ is necessarily *positive-real*. However, as emphasised in [3], the positive-real condition is not sufficient for guaranteeing that $\mathcal{B}$ is passive. This is due to the possibility of common roots (more specifically, common left divisors) of P and Q, which arise when $\mathcal{B}$ is not controllable [7, Thm. 5.2.10]. In fact, in [3] it was proved that $\mathcal{B}$ in (3) is passive if and only if (P, Q) is a *positive-real pair*, defined as follows.

Definition 2. Let $P, Q \in \mathbb{R}^{n\times n}[s]$. We call (P, Q) a *positive-real pair* if the following conditions hold:

1. $P(\lambda)Q(\bar{\lambda})^T + Q(\lambda)P(\bar{\lambda})^T \geq 0$ for all $\lambda \in \overline{\mathbb{C}}_+$.
2. $\text{rank}([P \;\; -Q](\lambda)) = n$ for all $\lambda \in \overline{\mathbb{C}}_+$.
3. If $\mathbf{p} \in \mathbb{R}^n[s]$ and $\lambda \in \mathbb{C}$ satisfy $\mathbf{p}(\lambda)^T[P \;\; -Q](\lambda) = 0$ and $\mathbf{p}(s)^T(P(s)Q(-s)^T + Q(s)P(-s)^T) = 0$, then $\mathbf{p}(\lambda) = 0$.

In this paper, we show that $\mathcal{B}$ takes the form of (3) and is passive if and only if $\mathcal{B}$ is the driving-point behavior of an RLCTG circuit. In particular, we show that the internal currents and voltages in a general RLCTG circuit are properly eliminable, a result which is not established in [2]. Another important distinction between our results and those in [2] is in the treatment of uncontrollable behaviors. In [2], a system $\mathcal{B}$ is called passive if $\int_0^T(\mathbf{i}^T\mathbf{v})(t)dt \geq 0$ for all $T \geq 0$ and all $(\mathbf{i}, \mathbf{v}) \in \mathcal{B}$ which satisfy $\mathbf{i}(t) = 0$ and $\mathbf{v}(t) = 0$ for all $t < 0$. This differs from Definition 1 when $\mathcal{B}$ is not controllable (see [3, Example 3]). In particular, a system $\mathcal{B}$ can be passive in the sense of [2], and yet it can be impossible to realise $\mathcal{B}$ as the driving-point behavior of an RLCTG circuit. In contrast, our results indicate how to design an RLCTG circuit to realise any given behavior $\mathcal{B}$ as in (3) which is passive in the sense of Definition 1.

5. BEHAVIOR OF NON-DYNAMIC PASSIVE ELECTRIC CIRCUITS

In this section, we describe the full and driving-point behavior of a general circuit comprising transformers and gyrators (a TG circuit), and a general circuit comprising resistors, transformers and gyrators (an RTG circuit).

Theorem 3. Let $\tilde{\mathcal{B}}$ be the full behavior of an n-port TG circuit, with driving-point currents and voltages $\mathbf{i}$ and $\mathbf{v}$, and m element ports with currents and voltages $\mathbf{i}_1$ and $\mathbf{v}_1$. Then $\mathbf{i}^T\mathbf{v} = 0$ for all $(\mathbf{i}, \mathbf{v}, \mathbf{i}_1, \mathbf{v}_1) \in \tilde{\mathcal{B}}$, and there exist $P, Q \in \mathbb{R}^{n\times n}$ such that $\mathcal{B} := \tilde{\mathcal{B}}^{(\mathbf{i},\mathbf{v})}$ takes the form of (3).

Proof. We follow the element extraction approach outlined in Section 2. In the terminology of that section, N_1 has driving-point currents and voltages $\text{col}(\mathbf{i} \;\; \mathbf{i}_2)$ and $\text{col}(\mathbf{v} \;\; \mathbf{v}_2)$ which satisfy $\mathbf{i}^T\mathbf{v} + \mathbf{i}_2^T\mathbf{v}_2 = 0$; N_2 comprises isolated transformers and gyrators, with driving-point currents and voltages $\mathbf{i}_1$ and $\mathbf{v}_1$; and it is easily verified from the element properties in Section 2 that $\mathbf{i}_1^T\mathbf{v}_1 = 0$. Then, given the interconnection equations $\mathbf{v}_2 = \mathbf{v}_1$ and $\mathbf{i}_2 = -\mathbf{i}_1$, we find that $\mathbf{i}^T\mathbf{v} = -\mathbf{i}_2^T\mathbf{v}_2 = \mathbf{i}_1^T\mathbf{v}_1 = 0$.

The remainder of the proof is inspired by [1]. It is easily shown that both N_1 and N_2 have a driving-point behavior of the form $\hat{\mathcal{B}} = \{(\hat{\mathbf{i}}, \hat{\mathbf{v}}) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\hat{n}}) \times \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\hat{n}}) \mid (\hat{\mathbf{v}} - \hat{\mathbf{i}}) = \hat{S}(\hat{\mathbf{v}} + \hat{\mathbf{i}})\}$ for some orthogonal matrix $\hat{S} \in \mathbb{R}^{\hat{n}\times\hat{n}}$ (i.e., $\hat{S}^T = \hat{S}^{-1}$).[2] We therefore obtain relationships of the form

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v} + \mathbf{i} \\ \mathbf{v}_2 + \mathbf{i}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v} - \mathbf{i} \\ \mathbf{v}_2 - \mathbf{i}_2 \end{bmatrix}, \quad \begin{matrix} \hat{S}(\mathbf{v}_1 + \mathbf{i}_1) = (\mathbf{v}_1 - \mathbf{i}_1), \\ \mathbf{v}_2 = \mathbf{v}_1, \text{ and } \mathbf{i}_2 = -\mathbf{i}_1, \end{matrix}$$

in which the leftmost matrix, and $\hat{S}$, are orthogonal. It is then easily verified that $\mathbf{i}_2$ and $\mathbf{v}_2$ are properly eliminable, and $\tilde{\mathcal{B}}$ is the set of locally integrable solutions to:

$$\begin{bmatrix} I & -S_{11} & 0 & -S_{12} \\ 0 & -S_{21} & I & -S_{22} \\ 0 & 0 & -\hat{S} & I \end{bmatrix} \begin{bmatrix} -I & I & 0 & 0 \\ I & I & 0 & 0 \\ 0 & 0 & I & I \\ 0 & 0 & -I & I \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{v} \\ \mathbf{i}_1 \\ \mathbf{v}_1 \end{bmatrix} = 0. \quad (4)$$

We now use the procedure in Section 3 to find $\mathcal{B} = \tilde{\mathcal{B}}^{(\mathbf{i},\mathbf{v})}$. First, let $T = \text{col}(T_1 \;\; T_2) \in \mathbb{R}^{m\times m}$ be an orthogonal matrix such that the rows of T_2 are a basis for the left null space of $I - S_{22}\hat{S}$. By orthogonality,

$$T_1^T T_1 + T_2^T T_2 = I, \;\; T_1 T_1^T = I, \;\; T_2 T_2^T = I, \quad (5)$$

$$S_{12}^T S_{12} + S_{22}^T S_{22} = I, \;\; S_{21}S_{21}^T + S_{22}S_{22}^T = I, \text{ and } \hat{S}^T = \hat{S}^{-1}. \quad (6)$$

We will show the following five conditions: (i) $T_1(I - S_{22}\hat{S})$ has full row rank; (ii) $T_2 S_{21} = 0$; (iii) $(I - S_{22}\hat{S})T_2^T = 0$; (iv) $S_{12}\hat{S}T_2^T = 0$; and (v) $T_1(I - S_{22}\hat{S}) = (I - T_1 S_{22}\hat{S}T_1^T)T_1$. We then note from conditions (i)–(v) and equation (5) that $(I - T_1 S_{22}\hat{S}T_1^T)$ is invertible and $S_{12}\hat{S}(I - T_1^T(I - T_1 S_{22}\hat{S}T_1^T)^{-1}T_1(I - S_{22}\hat{S}) = S_{12}\hat{S}(I - T_1^T T_1) = S_{12}\hat{S}T_2^T T_2 = 0$. With the notation

$$\tilde{S} := S_{11} + S_{12}\hat{S}T_1^T(I - T_1 S_{22}\hat{S}T_1^T)^{-1}T_1 S_{21},$$

$$R_1 := \begin{bmatrix} -I - \tilde{S} & I - \tilde{S} \\ 0 & 0 \end{bmatrix}, \; R_2 := \begin{bmatrix} 0 & -T_1 S_{21} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -I & I \\ I & I \end{bmatrix},$$

$$\text{and } M_2 := \begin{bmatrix} T_1(I - S_{22}\hat{S}) & 0 \\ -\hat{S} & I \end{bmatrix} \begin{bmatrix} I & I \\ -I & I \end{bmatrix},$$

then, by pre-multiplying (4) by the invertible matrix

$$\begin{bmatrix} I & S_{12}\hat{S}T_1^T(I - T_1 S_{22}\hat{S}T_1^T)^{-1}T_1 & 0 \\ 0 & T_2 & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & S_{12} \\ 0 & I & S_{22} \\ 0 & 0 & I \end{bmatrix},$$

we find that $\tilde{\mathcal{B}}$ is the set of locally integrable solutions to:

$$R_1 \begin{bmatrix} \mathbf{i} \\ \mathbf{v} \end{bmatrix} = 0, \text{ and } M_2 \begin{bmatrix} \mathbf{i}_1 \\ \mathbf{v}_1 \end{bmatrix} = -R_2 \begin{bmatrix} \mathbf{i} \\ \mathbf{v} \end{bmatrix}. \quad (7)$$

Since $T_1(I - S_{22}\hat{S})$ has full row rank, then so too does M_2. Thus, for any given locally integrable $\mathbf{i}$ and $\mathbf{v}$, there exists a locally integrable $\mathbf{i}_1$ and $\mathbf{v}_1$ satisfying the rightmost equation in (7); i.e., $\mathbf{i}_1$ and $\mathbf{v}_1$ are properly eliminable. Then, with $P := I + \tilde{S}$ and $Q := I - \tilde{S}$, it follows that the driving-point behavior $\mathcal{B} := \tilde{\mathcal{B}}^{(\mathbf{i},\mathbf{v})}$ takes the form of (3).

To complete the proof, it remains to show conditions (i)–(v). First, suppose $\mathbf{z}_1 \in \mathbb{R}^r$ satisfies $\mathbf{z}_1^T T_1(I - S_{22}\hat{S}) = 0$. Then $\mathbf{z}_1^T T_1$ is in the left null space of $(I - S_{22}\hat{S})$, and

[2] To see this, note initially that since there are $2\hat{n}$ independent linear equations relating $\hat{\mathbf{i}}$ and $\hat{\mathbf{v}}$, then there exist $A, B \in \mathbb{R}^{\hat{n}\times\hat{n}}$ such that $\hat{\mathcal{B}} = \{(\hat{\mathbf{i}}, \hat{\mathbf{v}}) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\hat{n}}) \times \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\hat{n}}) \mid A(\hat{\mathbf{v}} - \hat{\mathbf{i}}) = B(\hat{\mathbf{v}} + \hat{\mathbf{i}})\}$. Moreover, $(\hat{\mathbf{v}} + \hat{\mathbf{i}})^T(\hat{\mathbf{v}} + \hat{\mathbf{i}}) - (\hat{\mathbf{v}} - \hat{\mathbf{i}})^T(\hat{\mathbf{v}} - \hat{\mathbf{i}}) = 4\hat{\mathbf{i}}^T\hat{\mathbf{v}} = 0$, so $(\hat{\mathbf{v}} - \hat{\mathbf{i}}) = 0$ if $(\hat{\mathbf{v}} + \hat{\mathbf{i}}) = 0$, which implies that A is invertible. Thus, with the notation $\hat{S} = A^{-1}B$, we obtain $(\hat{\mathbf{v}} - \hat{\mathbf{i}}) = \hat{S}(\hat{\mathbf{v}} + \hat{\mathbf{i}})$, whence $(\hat{\mathbf{v}} + \hat{\mathbf{i}})^T(\hat{\mathbf{v}} + \hat{\mathbf{i}}) - (\hat{\mathbf{v}} - \hat{\mathbf{i}})^T(\hat{\mathbf{v}} - \hat{\mathbf{i}}) = (\hat{\mathbf{v}} + \hat{\mathbf{i}})^T(I - \hat{S}^T\hat{S})(\hat{\mathbf{v}} + \hat{\mathbf{i}})^T = 0$. Since this holds for all $(\hat{\mathbf{v}} + \hat{\mathbf{i}}) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\hat{n}})$, then $\hat{S}^T\hat{S} = I$.

so there exists $\mathbf{z}_2 \in \mathbb{R}^{m-r}$ such that $\mathbf{z}_1^T T_1 = \mathbf{z}_2^T T_2$. This implies that $\begin{bmatrix}\mathbf{z}_1^T & -\mathbf{z}_2^T\end{bmatrix} T = 0$, and so $\mathbf{z}_1 = \mathbf{z}_2 = 0$, which proves condition (i). Next, note from (6) that, if $\mathbf{z} \in \mathbb{R}^{m\times m}$ satisfies $\mathbf{z}^T(I - S_{22}\hat{S}) = 0$, then $0 = \mathbf{z}^T(I - S_{22}\hat{S}\hat{S}^T S_{22}^T)\mathbf{z} = \mathbf{z}^T(I - S_{22}S_{22}^T)\mathbf{z} = \mathbf{z}^T S_{21}S_{21}^T\mathbf{z}$, so $\mathbf{z}^T S_{21} = 0$, and $0 = S_{21}S_{21}^T\mathbf{z} = (I - S_{22}\hat{S}\hat{S}^T S_{22}^T)\mathbf{z}$. In particular, $\mathbf{z} = S_{22}\hat{S}\hat{S}^T S_{22}^T\mathbf{z} = S_{22}\hat{S}\mathbf{z}$. This implies that $0 = \mathbf{z}^T(I - \hat{S}^T S_{22}^T S_{22}\hat{S})\mathbf{z} = \mathbf{z}^T\hat{S}^T(I - S_{22}^T S_{22})\hat{S}\mathbf{z} = \mathbf{z}^T\hat{S}^T S_{12}^T S_{12}\hat{S}\mathbf{z}$, so $S_{12}\hat{S}\mathbf{z} = 0$. By letting $\mathbf{z}$ be each of the rows in T_2 in turn, we obtain conditions (ii)–(iv). Finally, condition (v) follows from (5) and condition (iii), since $T_1(I - S_{22}\hat{S}) = T_1(I - S_{22}\hat{S})(T_1^T T_1 + T_2^T T_2) = (I - T_1 S_{22}\hat{S}T_1^T)T_1$. □

Note that, if $I - S_{22}\hat{S}$ is singular, then there exists $\mathbf{z} \in \mathbb{R}^{2m}$ such that $M_2\mathbf{z} = 0$, so the internal currents and voltages are not uniquely determined by the driving-point currents and voltages. However, this is unlikely to occur in practice as the transformers and gyrators will have internal resistance, and it follows from the next theorem that the current and voltage in the resistors of an RTG circuit are determined by the driving-point current and voltage.

Theorem 4. Let $\tilde{\mathcal{B}}$ be the full behavior of an n-port RTG circuit, with internal currents and voltages $\mathbf{i}_1$ and $\mathbf{v}_1$; driving-point currents and voltages $\mathbf{i}$ and $\mathbf{v}$; and m resistors with currents and voltages $\mathbf{i}_R$ and $\mathbf{v}_R$. Then $\mathbf{i}^T\mathbf{v} = \mathbf{i}_R^T\mathbf{v}_R$ for all $(\mathbf{i}, \mathbf{v}, \mathbf{i}_1, \mathbf{v}_1) \in \tilde{\mathcal{B}}$. Furthermore, there exist: (i) compatible partitions of the driving-point currents and voltages as $(\mathbf{i}_a, \mathbf{i}_b)$ and $(\mathbf{v}_a, \mathbf{v}_b)$; (ii) a $\tilde{H} \in \mathbb{R}^{n\times n}$ with $\tilde{H} + \tilde{H}^T \geq 0$; and (iii) an $\tilde{L} \in \mathbb{R}^{2m\times n}$; such that, with $\mathbf{e} = \mathrm{col}(\mathbf{i}_a\ \mathbf{v}_b)$ and $\mathbf{r} = \mathrm{col}(\mathbf{v}_a\ \mathbf{i}_b)$, then $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r})} = \{(\mathbf{e}, \mathbf{r}) \mid \mathbf{e} \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ and $\mathbf{r} = \tilde{H}\mathbf{e}\}$, and $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r},\mathbf{i}_R,\mathbf{v}_R)} = \{(\mathbf{e}, \mathbf{r}, \mathbf{i}_R, \mathbf{v}_R) \mid (\mathbf{e}, \mathbf{r}) \in \mathcal{B}$ and $\mathrm{col}(\mathbf{i}_R\ \mathbf{v}_R) = \tilde{L}\mathbf{e}\}$.

Proof. Following Section 2, we consider the $n{+}m$-port circuit N_{1b} obtained by replacing the resistors by external ports, and the m-port circuit N_{2b} containing the resistors (disconnected from each other). Here, N_{1b} has driving-point currents and voltages $\mathrm{col}(\mathbf{i}\ \mathbf{i}_2)$ and $\mathrm{col}(\mathbf{v}\ \mathbf{v}_2)$; N_{2b} has driving-point currents and voltages $\mathbf{i}_R$ and $\mathbf{v}_R$; and interconnection results in the equations $\mathbf{v}_2 = \mathbf{v}_R$ and $\mathbf{i}_2 = -\mathbf{i}_R$. Then, from Theorem 3, $\mathbf{i}^T\mathbf{v} + \mathbf{i}_2^T\mathbf{v}_2 = 0$, so $\mathbf{i}^T\mathbf{v} = -\mathbf{i}_2^T\mathbf{v}_2 = \mathbf{i}_R^T\mathbf{v}_R$. Also, it follows from Theorem 3 that the driving-point behavior of N_{1b} takes the form of (3) and is passive. Thus, from [3, Thm. 7], there are compatible partitions of $\mathbf{i}$ and $\mathbf{v}$ as $(\mathbf{i}_a, \mathbf{i}_b)$ and $(\mathbf{v}_a, \mathbf{v}_b)$, and compatible partitions of $\mathbf{i}_2$ and $\mathbf{v}_2$ as $(\mathbf{i}_{2a}, \mathbf{i}_{2b})$ and $(\mathbf{v}_{2a}, \mathbf{v}_{2b})$, such that, with $\mathbf{e}_2 = \mathrm{col}(\mathbf{i}_{2a}\ \mathbf{v}_{2b})$, $\mathbf{r}_2 = \mathrm{col}(\mathbf{v}_{2a}\ \mathbf{i}_{2b})$, $\mathbf{e} = \mathrm{col}(\mathbf{i}_a\ \mathbf{v}_b)$, and $\mathbf{r} = \mathrm{col}(\mathbf{v}_a\ \mathbf{i}_b)$, the driving-point behavior of N_{1b} is the set of locally integrable solutions to: [3]

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}\mathbf{e} \\ \mathbf{e}_2\end{bmatrix} = \begin{bmatrix}\mathbf{r} \\ \mathbf{r}_2\end{bmatrix}. \tag{8}$$

Furthermore, from Theorem 3, then $\mathbf{i}^T\mathbf{v} + \mathbf{i}_2^T\mathbf{v}_2 = \mathbf{e}^T\mathbf{r} + \mathbf{e}_2^T\mathbf{r}_2 = 0$, whereupon it is easily shown that the leftmost matrix in (8) is skew-symmetric. By partitioning $\mathbf{i}_R$ and $\mathbf{v}_R$ compatibly with $\mathbf{i}_2$ as $(\mathbf{i}_{Ra}, \mathbf{i}_{Rb})$ and $(\mathbf{v}_{Ra}, \mathbf{v}_{Rb})$, and by defining $\mathbf{e}_R := \mathrm{col}(\mathbf{v}_{Ra}\ \mathbf{i}_{Rb})$ and $\mathbf{r}_R := \mathrm{col}(\mathbf{i}_{Ra}\ \mathbf{v}_{Rb})$, then the driving-point behavior of N_{2b} corresponds to the set of locally integrable solutions to an equation of the form $\mathbf{r}_R = \hat{H}\mathbf{e}_R$ where $\hat{H} \in \mathbb{R}^{m\times m}$ is a diagonal positive-definite matrix. Also, by letting Σ be the signature matrix partitioned compatibly with $\mathbf{e}_R$ of the form $\Sigma := \mathrm{diag}(I\ -I)$, then the interconnection equations take the form $\mathbf{e}_2 = -\Sigma\mathbf{r}_R$, $\mathbf{r}_2 = \Sigma\mathbf{e}_R$. It is then easily shown that $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r},\mathbf{e}_R,\mathbf{r}_R)}$ is the set of locally integrable solutions to:

$$\begin{bmatrix} I & -H_{11} & 0 & H_{12}\Sigma \\ 0 & -H_{21} & \Sigma & H_{22}\Sigma \\ 0 & 0 & -\hat{H} & I\end{bmatrix}\begin{bmatrix}\mathbf{r} \\ \mathbf{e} \\ \mathbf{e}_R \\ \mathbf{r}_R\end{bmatrix} = 0. \tag{9}$$

We now follow the procedure in Section 3 to obtain $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r})}$. First, note that since Σ and $\hat{H}$ are diagonal matrices, then $\Sigma\hat{H} = \hat{H}\Sigma$. Next, note that $I + H_{22}\hat{H}$ is invertible. This follows since, if $\mathbf{z} \in \mathbb{R}^m$ satisfies $\mathbf{z}^T(I + H_{22}\hat{H}) = 0$, then $0 = \mathbf{z}^T(I + H_{22}\hat{H})\hat{H}^{-1}(I + H_{22}\hat{H})^T\mathbf{z} = \mathbf{z}^T(\hat{H}^{-1} + H_{22} + H_{22}^T + H_{22}\hat{H}^{-1}H_{22}^T)\mathbf{z}$. Since H_{22} is skew-symmetric and $\hat{H}^{-1} > 0$, then this implies $\mathbf{z} = 0$. Now, let

$$\tilde{H} := H_{11} - H_{12}\hat{H}(I + H_{22}\hat{H})^{-1}H_{21},\ \tilde{R}_1 := \begin{bmatrix} I & -\tilde{H}\end{bmatrix},$$

$$\tilde{R}_2 := \begin{bmatrix}-H_{21} \\ 0\end{bmatrix},\ \text{and}\ \tilde{M}_2 := \begin{bmatrix}(I + H_{22}\hat{H})\Sigma & 0 \\ -\hat{H} & I\end{bmatrix}.$$

By pre-multiplying (9) by the invertible matrix

$$\begin{bmatrix} I & -H_{12}\hat{H}(I + H_{22}\hat{H})^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I\end{bmatrix}\begin{bmatrix} I & 0 & -H_{12}\Sigma \\ 0 & I & -H_{22}\Sigma \\ 0 & 0 & I\end{bmatrix},$$

we find that $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r},\mathbf{e}_R,\mathbf{r}_R)}$ is the set of locally integrable solutions to $\mathbf{r} = \tilde{H}\mathbf{e}$ and $\tilde{M}_2\mathrm{col}(\mathbf{e}_R\ \mathbf{r}_R) = -\tilde{R}_2\mathbf{e}$. It is easily verified that $\tilde{M}_2$ is invertible and $L := -\tilde{M}_2^{-1}\tilde{R}_2 = \mathrm{col}\begin{pmatrix}\Sigma & \hat{H}\Sigma\end{pmatrix}(I + H_{22}\hat{H})^{-1}H_{21}$, so $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r})}$ and $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r},\mathbf{i}_R,\mathbf{v}_R)}$ take the form indicated in the present theorem statement. That $\tilde{H} + \tilde{H}^T \geq 0$ follows since $\mathbf{e}^T\mathbf{r} = \frac{1}{2}\mathbf{e}^T(\tilde{H} + \tilde{H}^T)\mathbf{e} = \mathbf{i}^T\mathbf{v} \geq 0$ for all $\mathbf{e} \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$. □

6. RLCTG CIRCUIT BEHAVIORS

In Theorem 5, we show that the driving-point behavior of an RLCTG circuit has a representation:

$$\mathcal{B}_s = \{(\mathbf{u}, \mathbf{y}, \mathbf{x}) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d)$$
$$\text{such that}\ \tfrac{d\mathbf{x}}{dt} = A\mathbf{x} + B\mathbf{u}\ \text{and}\ \mathbf{y} = C\mathbf{x} + D\mathbf{u}\},$$
$$\text{with}\ A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{d\times n}, C \in \mathbb{R}^{n\times d}, D \in \mathbb{R}^{n\times n}, \tag{10}$$

in which the input $\mathbf{u}$ is a subset of the driving-point currents and voltages (with exactly one input for each port), the output $\mathbf{y}$ is the complementary subset of driving-point currents and voltages, and the state $\mathbf{x}$ is a subset of the inductor currents and capacitor voltages. The remaining inductor and capacitor currents and voltages are then uniquely determined given this input and state (and so too are the resistor currents and voltages, as can easily be shown from Theorem 4).

Theorem 5. Let $\tilde{\mathcal{B}}$ be the full behavior of an n-port RLCTG circuit, with driving-point currents and voltages $\mathbf{i}$ and $\mathbf{v}$; m_1 inductors with currents and voltages $\mathbf{i}_L$ and $\mathbf{v}_L$; and m_2 capacitors with currents and

[3] Note that such a partition need not be unique, but one way of obtaining a suitable partition is as follows. From Theorem 3, $\hat{\mathbf{i}} := \mathrm{col}\begin{pmatrix}\mathbf{i} & \mathbf{i}_2\end{pmatrix}$ and $\hat{\mathbf{v}} := \mathrm{col}\begin{pmatrix}\mathbf{v} & \mathbf{v}_2\end{pmatrix}$ satisfy $\hat{P}\hat{\mathbf{i}} = \hat{Q}\hat{\mathbf{v}}$ for some $\hat{P}, \hat{Q} \in \mathbb{R}^{\hat{n}\times\hat{n}}$. We let T be a invertible matrix such that $T\hat{Q} = \mathrm{col}\begin{pmatrix}\tilde{Q} & 0\end{pmatrix}$ where $\tilde{Q}$ has full row-rank. We then pick a set of independent columns from $\tilde{Q}$ to form a invertible matrix, and we let $\mathbf{v}_a$ and $\mathbf{v}_{2a}$ comprise the components in $\hat{\mathbf{v}}$ corresponding to these columns.

voltages $\mathbf{i}_C$ and $\mathbf{v}_C$. Then there exist: (i) compatible partitions of the driving-point currents and voltages as $(\mathbf{i}_a, \mathbf{i}_b)$ and $(\mathbf{v}_a, \mathbf{v}_b)$; (ii) compatible partitions of the inductor currents and voltages as $(\mathbf{i}_{La}, \mathbf{i}_{Lb})$ and $(\mathbf{v}_{La}, \mathbf{v}_{Lb})$; (iii) compatible partitions of the capacitor currents and voltages as $(\mathbf{i}_{Ca}, \mathbf{i}_{Cb})$ and $(\mathbf{v}_{Ca}, \mathbf{v}_{Cb})$; (iv) an input-state-output model $\mathcal{B}_s$ as in (10); and (v) a $G \in \mathbb{R}^{(2m_1+2m_2-d)\times(n+d)}$; such that, with $\mathbf{e} = \mathrm{col}\,(\mathbf{i}_a\ \mathbf{v}_b)$, $\mathbf{r} = \mathrm{col}\,(\mathbf{v}_a\ \mathbf{i}_b)$, $\mathbf{e}_{1a} = \mathrm{col}\,(\mathbf{i}_{La}\ \mathbf{v}_{Ca})$, $\mathbf{r}_{1a} = \mathrm{col}\,(\mathbf{v}_{La}\ \mathbf{i}_{Ca})$, $\mathbf{e}_{1b} = \mathrm{col}\,(\mathbf{v}_{Lb}\ \mathbf{i}_{Cb})$, and $\mathbf{r}_{1b} = \mathrm{col}\,(\mathbf{i}_{Lb}\ \mathbf{v}_{Cb})$, then $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r},\mathbf{e}_{1a},\mathbf{r}_{1a},\mathbf{e}_{1b},\mathbf{r}_{1b})} = \{(\mathbf{e},\mathbf{r},\mathbf{e}_{1a},\mathbf{r}_{1a},\mathbf{e}_{1b},\mathbf{r}_{1b}) \mid \tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r},\mathbf{e}_{1a})} = \mathcal{B}_s$ and $\mathrm{col}\,(\mathbf{r}_{1a}\ \mathbf{e}_{1b}\ \mathbf{r}_{1b}) = G\mathrm{col}\,(\mathbf{e}\ \mathbf{e}_{1a})\}$.

Proof. In this case, following Section 2, we let N_{1a} be obtained by removing all inductors and capacitors and replacing these with external ports; and N_{2a} comprises the inductors and capacitors (disconnected from each other). Then, from Theorem 4, it is easily shown that there exist partitions satisfying conditions (i)–(iii) in the present theorem statement; a matrix Σ_1 (resp., Σ_2) which, partitioned compatibly with $\mathrm{col}\,(\mathbf{i}_{La}\ \mathbf{v}_{Ca})$ (resp., $\mathrm{col}\,(\mathbf{i}_{Lb}\ \mathbf{v}_{Cb})$) takes the form $\Sigma_1 := \mathrm{diag}\,(-I\ I)$ (resp., $\Sigma_2 := \mathrm{diag}\,(-I\ I)$); and diagonal matrices Λ_1 and Λ_2 with positive diagonal entries; such that, with $\mathbf{e}, \mathbf{r}, \mathbf{e}_{1a}, \mathbf{r}_{1a}, \mathbf{e}_{1b}$ and $\mathbf{r}_{1b}$ as defined in the present theorem statement, then $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r},\mathbf{e}_{1a},\mathbf{r}_{1a},\mathbf{e}_{1b},\mathbf{r}_{1b})}$ is determined by equations of the form:

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{e}_{2a} \\ \mathbf{e}_{2b} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ \mathbf{r}_{2a} \\ \mathbf{r}_{2b} \end{bmatrix}, \tag{11}$$

$$\mathbf{r}_{1a} = \Lambda_1 \tfrac{d\mathbf{e}_{1a}}{dt},\ \ \mathbf{e}_{1b} = \Lambda_2 \tfrac{d\mathbf{r}_{1b}}{dt}, \tag{12}$$

$$\mathbf{e}_{2a}=\Sigma_1\mathbf{e}_{1a},\ \mathbf{r}_{2a}=-\Sigma_1\mathbf{r}_{1a},\ \mathbf{r}_{2b}=\Sigma_2\mathbf{r}_{1b},\ \mathbf{e}_{2b}=-\Sigma_2\mathbf{e}_{1b}. \tag{13}$$

Here, (11) corresponds to the driving-point behavior of N_{1a}; (12) corresponds to the driving-point behavior of N_{2a}; and (13) are the interconnection equations. Also, from the proof of Theorem 4, the leftmost matrix M in (11) satisfies $M + M^T \geq 0$. Next, we show that the partitions of $\mathbf{i}, \mathbf{v}, \mathbf{i}_L$ and $\mathbf{v}_C$ can be chosen such that $M_{13} = 0$, $M_{31} = 0$, and $M_{33} = 0$. To show this, we prove that if either M_{13} or M_{33} is non-zero, then we can reduce the number of entries corresponding to inductor voltages or capacitor currents on the left of (11). These correspond to the entries in $\mathbf{e}_{2b}$. This procedure must terminate in a finite number of steps with an equation of the form of (11) in which $M_{13} = 0$ and $M_{33} = 0$ (it is possible that there will be no entries remaining on the left of (11) which correspond to inductor voltages or capacitor currents). Then, since $M + M^T \geq 0$, it is easily shown that $M_{33} = 0$ implies that $M_{32} = -M_{23}^T$ and $M_{31} = -M_{13}^T = 0$. To prove that the reduction is possible, it suffices to show that: (i) if $M_{33} \neq 0$, then a subset of the entries of $\mathbf{e}_{2b}$ are a linear function of the corresponding entries in $\mathbf{r}_{2b}$ together with the entries in $\mathbf{e}, \mathbf{e}_{2a}$, and the complementary entries in $\mathbf{e}_{2b}$; and (ii) if $M_{33} = 0$ but $M_{13} \neq 0$, then one of the entries of $\mathbf{e}$ together with one of the entries of $\mathbf{e}_{2b}$ is a linear function of the corresponding entries in $\mathbf{r}$ and $\mathbf{r}_{2b}$ together with the complementary entries in $\mathbf{e}, \mathbf{e}_{2a}$ and $\mathbf{e}_{2b}$.

To show (i), we notice that if $(M_{33})_{ij}$ (the entry in the ith row and jth column of M_{33}) is non-zero, then either (a) $(M_{33})_{ii} \neq 0$; or (b) $(M_{33})_{ii} = 0$, $(M_{33})_{ji} = -(M_{33})_{ij}$, and the submatrix of M containing the entries $(M_{33})_{ii}, (M_{33})_{ij}, (M_{33})_{ji}$, and $(M_{33})_{jj}$ is invertible. This follows by considering the corresponding submatrix of $M + M^T$, which is non-negative definite since $M + M^T \geq 0$. Then, in case (a) (resp., case (b)), the ith (resp., ith and jth) entry in $\mathbf{e}_{2b}$ is a linear function of the corresponding entry (resp., entries) in $\mathbf{r}_{2b}$ together with the entries in $\mathbf{e}, \mathbf{e}_{2a}$, and the complementary entries in $\mathbf{e}_{2b}$. To see (ii), we note that if $(M_{13})_{ij}$ is non-zero and $(M_{33})_{jj} = 0$, then $(M_{31})_{ji} = -(M_{13})_{ij}$, and the submatrix of M containing the entries $(M_{11})_{ii}, (M_{13})_{ij}, (M_{31})_{ji}$, and $(M_{33})_{jj}$ is invertible . In this case, the ith entry in $\mathbf{e}$ and the jth entry in $\mathbf{e}_{2b}$ are a linear function of the corresponding entries in $\mathbf{r}$ and $\mathbf{r}_{2b}$ together with the complementary entries in $\mathbf{e}, \mathbf{e}_{2a}$ and $\mathbf{e}_{2b}$.

By letting $M_{13} = 0$, $M_{31} = 0$, $M_{33} = 0$, and $M_{32} = -M_{23}^T$ in (11), then it is easily shown that $\mathbf{e}_{2a}$, $\mathbf{e}_{2b}$, $\mathbf{r}_{2a}$ and $\mathbf{r}_{2b}$ are properly eliminable, and $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r},\mathbf{e}_{1a},\mathbf{r}_{1a},\mathbf{e}_{1b},\mathbf{r}_{1b})}$ is the set of locally integrable solutions to:

$$\begin{bmatrix} I & -M_{11} & -M_{12}\Sigma_1 & 0 & 0 & 0 \\ 0 & 0 & -\Lambda_1\frac{d}{dt} & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & -\Lambda_2\frac{d}{dt} \\ 0 & -M_{21} & -M_{22}\Sigma_1 & M_{23}\Sigma_2 & -\Sigma_1 & 0 \\ 0 & 0 & M_{23}^T\Sigma_1 & 0 & 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{e} \\ \mathbf{e}_{1a} \\ \mathbf{e}_{1b} \\ \mathbf{r}_{1a} \\ \mathbf{r}_{1b} \end{bmatrix} = 0. \tag{14}$$

Now, let

$$U_1(s) := \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & \Sigma_1 & 0 \\ 0 & 0 & I & 0 & \Lambda_2\Sigma_2 s \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

and we note that U_1 is unimodular. Thus, by pre-multiplying (14) by $U_1(\frac{d}{dt})$, we find that $\mathbf{r}_{1a}$ and $\mathbf{r}_{1b}$ are determined by the final two block rows in (14) (and are locally integrable if $\mathbf{r}, \mathbf{e}, \mathbf{e}_{1a}$, and $\mathbf{e}_{1b}$ are). Furthermore, with the notation $\hat{M}_{12} := M_{12}\Sigma_1$, $\hat{M}_{21} := \Sigma_1 M_{21}$, $\hat{M}_{22} := \Sigma_1 M_{22}\Sigma_1$, and $\hat{M}_{23} := \Sigma_1 M_{23}\Sigma_2$, then $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r},\mathbf{e}_{1a},\mathbf{e}_{1b})}$ is the set of locally integrable solutions to:

$$\begin{bmatrix} I & -M_{11} & -\hat{M}_{12} & 0 \\ 0 & -\hat{M}_{21} & -\Lambda_1\frac{d}{dt} - \hat{M}_{22} & \hat{M}_{23} \\ 0 & 0 & \Lambda_2\hat{M}_{23}^T\frac{d}{dt} & I \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{e} \\ \mathbf{e}_{1a} \\ \mathbf{e}_{1b} \end{bmatrix} = 0. \tag{15}$$

We next let $\Omega := \Lambda_1 + \hat{M}_{23}\Lambda_2\hat{M}_{23}^T$. Then $\Omega > 0$ since $\Lambda_1, \Lambda_2 > 0$. In particular, Ω is invertible. Further, we let

$$U_2 := \begin{bmatrix} I & 0 & 0 \\ 0 & -\Omega^{-1} & \Omega^{-1}\hat{M}_{23} \\ 0 & \Lambda_2\hat{M}_{23}^T\Omega^{-1} & I - \Lambda_2\hat{M}_{23}^T\Omega^{-1}\hat{M}_{23} \end{bmatrix},$$

and we note that U_2 is invertible. By pre-multiplying (15) by U_2, we find that $\mathbf{e}_{1b} = \Lambda_2\hat{M}_{23}^T\Omega^{-1}(\hat{M}_{21}\mathbf{e} + \hat{M}_{22}\mathbf{e}_{1a})$ (and is locally integrable if $\mathbf{e}$ and $\mathbf{e}_{1a}$ are), and that $\tilde{\mathcal{B}}^{(\mathbf{e},\mathbf{r},\mathbf{e}_{1a})} = \mathcal{B}_s$ where $\mathcal{B}_s$ is as in (10) with

$$A := -\Omega^{-1}\hat{M}_{22},\ B := -\Omega^{-1}\hat{M}_{21},\ C := \hat{M}_{12},\ D := M_{11}.$$

Furthermore, it follows from the preceding expressions that the present theorem statement holds, with

$$G := \begin{bmatrix} -\Lambda_1\Omega^{-1}\hat{M}_{21} & -\Lambda_1\Omega^{-1}\hat{M}_{22} \\ \Lambda_2\hat{M}_{23}^T\Omega^{-1}\hat{M}_{21} & \Lambda_2\hat{M}_{23}^T\Omega^{-1}\hat{M}_{22} \\ 0 & -\hat{M}_{23}^T \end{bmatrix}. \qquad \square$$

We finally obtain the following theorem on the driving-point behavior of RLCTG circuits.

Theorem 6. The following are equivalent:

1. $\mathcal{B}$ is the driving-point behavior of an n-port RLCTG circuit.
2. $\mathcal{B}$ is as in (3) and is passive.
3. $\mathcal{B}$ is as in (3) and (P, Q) is a positive-real pair.

Proof. The equivalence of conditions 2 and 3 is shown in [3, Thm. 7]. To see that 1 implies 2, note initially from [6, Example 3.1] that $\mathbf{x}$ is properly eliminable in (10). It is then easily shown from Theorems 3–5 that the internal currents and voltages are always properly eliminable from the full behavior of an RLCTG circuit, and the driving-point behavior necessarily takes the form of (3). Furthermore, from the expressions in the proof of Theorem 5 for A, B, C and D, we find that $\Omega > 0$ satisfies

$$\begin{bmatrix} -A^T\Omega - \Omega A & C^T - \Omega B \\ C - B^T\Omega & D + D^T \end{bmatrix} = \begin{bmatrix} \hat{M}_{22} + \hat{M}_{22}^T & \hat{M}_{12}^T + \hat{M}_{21} \\ \hat{M}_{12} + \hat{M}_{21}^T & M_{11} + M_{11}^T \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \Sigma_1 \\ I & 0 \end{bmatrix} \begin{bmatrix} M_{11} + M_{11}^T & M_{12} + M_{21}^T \\ M_{21} + M_{12}^T & M_{22} + M_{22}^T \end{bmatrix} \begin{bmatrix} 0 & I \\ \Sigma_1 & 0 \end{bmatrix} \geq 0.$$

It follows from [3, Thms. 7 and 11] that $\mathcal{B}$ is passive. Finally, to see that 2 implies 1, note that if $\mathcal{B}$ is as in (3) and is passive, then it is easily shown from results in [3, 8] that there exists: (i) an (observable) $\mathcal{B}_s$ as in (10) which satisfies

$$\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} + \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}^T \geq 0; \qquad (16)$$

and (ii) a permutation matrix $T = \operatorname{col}(T_1\ T_2)$; such that, with $\mathbf{e} = \operatorname{col}(T_1\mathbf{i}\ T_2\mathbf{v})$ and $\mathbf{r} = \operatorname{col}(T_1\mathbf{v}\ T_2\mathbf{i})$, then $\mathcal{B}^{(\mathbf{e},\mathbf{r})} = \mathcal{B}_s^{(\mathbf{u},\mathbf{y})}$. It is then easily shown that $\mathcal{B}$ can be realised as the driving-point behavior of an RLCTG circuit following the procedure described in [2]. $\square$

7. EXAMPLE

We apply Theorems 3–5 to N in Fig. 1. In this case, we obtain the following two relationships corresponding to the driving-point behaviors of N_{1c} and N_{2c}:

$$\begin{bmatrix} v-i \\ \hat{v}_a-\hat{i}_a \\ \hat{v}_b-\hat{i}_b \\ \hat{v}_c-\hat{i}_c \\ \hat{v}_d-\hat{i}_d \\ \hat{v}_e-\hat{i}_e \\ \hat{v}_f-\hat{i}_f \\ \hat{v}_g-\hat{i}_g \end{bmatrix} = \begin{bmatrix} \frac{1}{31} & \frac{10}{31} & -\frac{20}{31} & \frac{12}{31} & \frac{12}{31} & \frac{6}{31} & \frac{10}{31} & -\frac{6}{31} \\ \frac{10}{31} & \frac{7}{31} & -\frac{14}{31} & -\frac{4}{31} & -\frac{4}{31} & -\frac{2}{31} & -\frac{24}{31} & \frac{2}{31} \\ -\frac{20}{31} & -\frac{14}{31} & -\frac{3}{31} & \frac{8}{31} & \frac{8}{31} & \frac{4}{31} & -\frac{14}{31} & -\frac{4}{31} \\ \frac{12}{31} & -\frac{4}{31} & \frac{8}{31} & -\frac{11}{31} & \frac{20}{31} & \frac{10}{31} & -\frac{4}{31} & -\frac{10}{31} \\ \frac{12}{31} & -\frac{4}{31} & \frac{8}{31} & \frac{20}{31} & -\frac{11}{31} & \frac{10}{31} & -\frac{4}{31} & -\frac{10}{31} \\ \frac{6}{31} & -\frac{2}{31} & \frac{4}{31} & \frac{10}{31} & \frac{10}{31} & \frac{5}{31} & -\frac{2}{31} & \frac{26}{31} \\ \frac{10}{31} & -\frac{24}{31} & -\frac{14}{31} & -\frac{4}{31} & -\frac{4}{31} & -\frac{2}{31} & \frac{7}{31} & \frac{2}{31} \\ -\frac{6}{31} & \frac{2}{31} & -\frac{4}{31} & -\frac{10}{31} & -\frac{10}{31} & \frac{26}{31} & \frac{2}{31} & \frac{5}{31} \end{bmatrix} \begin{bmatrix} v+i \\ \hat{v}_a+\hat{i}_a \\ \hat{v}_b+\hat{i}_b \\ \hat{v}_c+\hat{i}_c \\ \hat{v}_d+\hat{i}_d \\ \hat{v}_e+\hat{i}_e \\ \hat{v}_f+\hat{i}_f \\ \hat{v}_g+\hat{i}_g \end{bmatrix},$$

$$\text{and } \begin{bmatrix} v_f-i_f \\ v_g-i_g \end{bmatrix} = \begin{bmatrix} \frac{15}{17} & -\frac{8}{17} \\ -\frac{8}{17} & -\frac{15}{17} \end{bmatrix} \begin{bmatrix} v_f+i_f \\ v_g+i_g \end{bmatrix}.$$

Next, in Theorem 3, we let $T = T_1 = I$. We then obtain the following relationship in Theorem 4 corresponding to the driving-point behavior of circuit N_{1b}:

$$\begin{bmatrix} i \\ \hat{v}_a \\ \hat{v}_b \\ \hat{v}_c \\ \hat{i}_d \\ \hat{i}_e \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -5 & 4 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & -1 & -1 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \hat{i}_a \\ \hat{i}_b \\ \hat{i}_c \\ \hat{v}_d \\ \hat{v}_e \end{bmatrix}.$$

Then, in Theorem 5, we obtain the following input-state-output representation for the driving-point behavior of N:

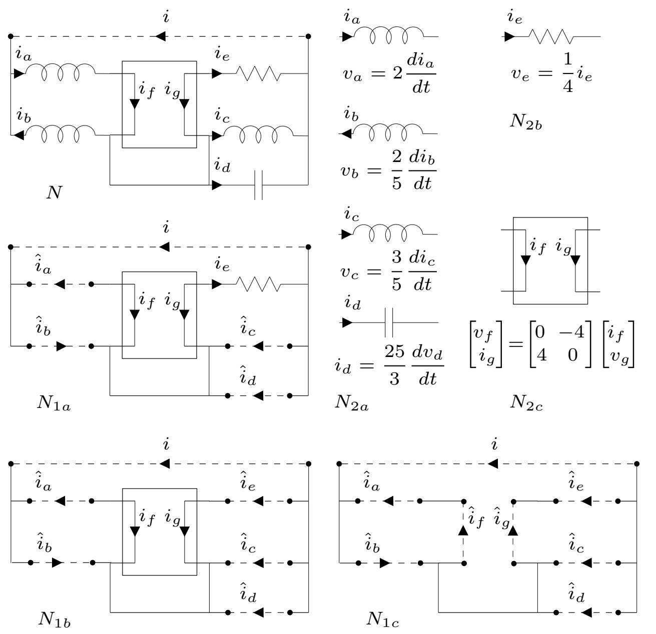

Fig. 1. Darlington synthesis for the impedance $s(s^2+s+1)/(3s^2+2s+1)$ [9, Sec. IV].

$$\frac{d}{dt}\begin{bmatrix} i_a \\ i_b \\ i_c \\ v_d \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & -\frac{5}{2} \\ 0 & 0 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & \frac{5}{3} \\ \frac{3}{5} & -\frac{3}{25} & -\frac{3}{25} & 0 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ v_d \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{5}{2} \\ 0 \\ 0 \end{bmatrix} v, \ i = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ v_d \end{bmatrix}.$$

Note that this representation is neither observable nor controllable. Finally, from Theorems 3 and 4, we find that the remaining currents and voltages in N satisfy:

$$\begin{bmatrix} v_a \\ v_b \\ v_c \\ i_d \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 5 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ v_d \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} v, \quad \begin{bmatrix} i_e \\ i_f \\ i_g \\ v_e \\ v_f \\ v_g \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ v_d \end{bmatrix}.$$

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