ON THE PAIRWISE MAXIMA OF GENERALISED DIVISOR FUNCTIONS

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Abstract. In this paper, we prove the asymptotic growth rate of the summatory function of the pairwise maxima of the generalised divisor function $d_k(n)$, for a fixed positive integer $k \geq 2$. This result generalises previous results of Kátai, Erdős and Hall on the local behaviour of divisor function on short intervals.

1. Introduction

Let n be a natural number and let $d(n)$ denote the number of divisors of n. Kátai, in his paper [4], studied the local behaviour of the function $d(n)$. In his paper he proved that

$$\sum_{n\leq x} \max\{d(n), d(n+1)\} = 2x\log x + O(x(\log x)^{1-\delta}), \tag{1.1}$$

where δ is a suitable positive constant.

In their paper [2], Erdős and Hall determined the following asymptotic for the local maxima of $d(n)$:

Theorem 1.1 (Erdős-Hall)**.** *If $h = o((\log x)^{3-2\sqrt{2}})$, then*

$$\sum_{n\leq x} \max\{d(n), d(n+1), ..., d(n+h-1)\} = hx\log x + O(h^2 x(\log x)^{2(\sqrt{2}-1)}). \tag{1.2}$$

In the case $h = 2$, equation (1.2) reduces to

$$\sum_{n\leq x} \max\{d(n), d(n+1)\} = 2x\log x + O(x(\log x)^{2(\sqrt{2}-1)}). \tag{1.3}$$

Although the authors do not state this explicitly, with slight modifications their proof of Theorem 1.1 also provides us with

$$\sum_{n\leq x} \max\{d(n), d(n+h)\} = 2x\log x + O(x(\log x)^{2(\sqrt{2}-1)}) \tag{1.4}$$

Date: January 22, 2018.

2010 *Mathematics Subject Classification.* 11N37, 11N56.

Key words and phrases. divisor functions, multiplicative functions, pairwise maxima.

for fixed values of h.

In this paper we generalise (1.4) for fixed values of h and k by considering the relation

$$\begin{aligned}\sum_{n\le x}\max\{d_k(n),d_k(n+h)\} &= \sum_{n\le x} d_k(n)+\sum_{n\le x} d_k(n+h)-\sum_{n\le x}\min\{d_k(n),d_k(n+h)\}\\ &= 2\sum_{n\le x} d_k(n)-\sum_{n\le x}\min\{d_k(n),d_k(n+h)\}\\ &+ \sum_{x<n\le x+h} d_k(n)-\sum_{n<h} d_k(n)\\ &= 2\sum_{n\le x} d_k(n)+E_k(x,h). \qquad (1.5)\end{aligned}$$

Our main result is Theorem 1.2 below, which is proved in Section 3.

Theorem 1.2. *If h and k are fixed, then*

$$E_k(x,h)\ll_{h,k} x(\log x)^{2(\sqrt{k}-1)} \qquad (1.6)$$

as $x\to\infty$.

By using the well-known asymptotic formula for the summatory function of $d_k(n)$ [8, p. 263], Theorem 1.2 states that if $k>4$ and h a fixed number, then

$$\sum_{n\le x}\max\{d_k(n),d_k(n+h)\}=\frac{2}{(k-1)!}x(\log x)^{k-1}+O(x(\log x)^{k-2}) \qquad (1.7)$$

and for $k\le 4$ we have that

$$\sum_{n\le x}\max\{d_k(n),d_k(n+h)\}=\frac{2}{(k-1)!}x(\log x)^{k-1}+O(x(\log x)^{2(\sqrt{k}-1)}) \qquad (1.8)$$

as $x\to\infty$.

The main difficulty is that the approach of Erdős and Hall [2] breaks down for $d_k(n)$ if $k\ge 4$. Therefore new ideas are necessary to generalise their results. To overcome such intricacies we use a theorem by Nair and Tenenbaum [5] to obtain a bound on certain averages involving $d_k(n)$ which turns out to be sufficient to establish the asymptotic formula above. In Section 2 of the paper we discuss the method of Erdős and Hall and why it breaks down when we try to generalise to $d_k(n)$. In Section 3 we prove Theorem 1.2, which is the main result of this paper.

2. The method of Erdős and Hall

In this section we briefly describe the method of proof of (1.4) used in their paper [2], and how it must be modified to establish Theorem 1.2. Note that $d(p^\alpha)\ge d(p^{\alpha-1})$ for $\alpha\ge 1$. Since $\sqrt{d(n)}$ is multiplicative, we have

$$\sqrt{d(n)} = \sum_{d|n} f(d) \tag{2.1}$$

where

$$f(p^\alpha) = \sqrt{g(p^\alpha)} - \sqrt{g(p^{\alpha-1})} \geq 0 \tag{2.2}$$

for $\alpha \geq 1$ and $f(1) = 1$.

The method of Erdős and Hall begins by using the simple facts that

$$\min\{d(n), d(n+1)\} \leq \sqrt{d(n)d(n+1)} \tag{2.3}$$

and

$$\sum_{n\leq x} \sqrt{d(n)d(n+1)} = \sum_{n\leq x}\sum_{d|n} f(d) \sum_{e|n+1} f(e), \tag{2.4}$$

and a crucial step of their proof establishes that there exists a constant C such that

$$\sqrt{d(n)} = \sum_{d|n} f(d) \leq C \sum_{\substack{d|n \\ d<\sqrt{n}}} f(d). \tag{2.5}$$

To establish (2.5), the authors observe that

$$\sum_{\substack{d|n \\ d\geq\sqrt{n}}} f(d) \leq \frac{2}{\log n} \sum_{\substack{d|n \\ d\geq\sqrt{n}}} f(d)\log d \leq \frac{2}{\log n}\sum_{d|n} f(d)\log d \tag{2.6}$$

for any multiplicative function f satisfying $f(1) = 1$, so to prove (2.5) it is sufficient to establish the existence of a $C' < 1/2$ such that

$$\sum_{d|n} f(d)\log d \leq C'\log n \sum_{d|n} f(d) \tag{2.7}$$

because by (2.6) we then have

$$\sum_{d|n} f(d) \leq \frac{1}{1-2C'} \sum_{\substack{d|n \\ d<\sqrt{n}}} f(d). \tag{2.8}$$

However, we can prove that

Lemma 2.1. *For a multiplicative function f satisfying $f(1) = 1$, let*

$$\sqrt{g(n)} = \sum_{d|n} f(d), \tag{2.9}$$

then there exists a constant $C' < 1/2$ such that

$$\sum_{d|n} f(d)\log d \leq C' \log n \sum_{d|n} f(d) \tag{2.10}$$

if and only if there exists a constant $C'' > 1/2$ such that

$$\sqrt{g(p^\alpha)} \leq \frac{1}{C''\alpha} \sum_{j=0}^{\alpha-1} \sqrt{g(p^j)} \tag{2.11}$$

for every p and every $\alpha \geq 1$.

Proof. By logarithmic differentiation of

$$\sum_{d|n} \frac{f(d)}{d^s} \tag{2.12}$$

one finds that

$$\frac{\sum_{d|n} f(d)\log d}{\sum_{d|n} f(d)} = \sum_{p^\alpha \| n} \left(\frac{f(p) + 2f(p^2) + \cdots + \alpha f(p^\alpha)}{1 + f(p) + f(p^2) + \cdots + f(p^\alpha)} \right) \log p. \tag{2.13}$$

From (2.13) it follows that the existence of C' in (2.10) is equivalent to

$$\sum_{j=0}^{\alpha} j f(p^j) \leq C'\alpha \sum_{j=0}^{\alpha} f(p^j) \tag{2.14}$$

for every p and every $\alpha \geq 1$. By (2.2) and some elementary analysis, (2.14) reduces to (2.11). □

Erdős and Hall prove that (2.11) holds when $g(n) = d(n)$ so Lemma 2.1 applies. This gives a non-trivial estimate of (2.4) which implies Theorem 1.1. However, the following dilemma arises.

Corollary 2.2. *The growth constraint (2.11) does not hold for $g(n) = d_k(n)$ when $k > 3$.*

Proof. Since $d_k(p^j) = \binom{j+k-1}{j}$, we observe that

$$\sqrt{\binom{7}{4}} \quad > \quad \frac{1}{2}\sum_{j=0}^{3}\sqrt{\binom{3+j}{3}}, \tag{2.15}$$

so (2.11) fails for $g(n) = d_4(n)$. Similar arguments show that (2.11) fails to hold for any $k \geq 4$. □

It follows from the previous corollary that Erdős and Hall approach does not apply for $d_k(n)$ for $k \geq 4$. We will remedy this in the next section.

3. A proof via the theorems of Nair-Tenenbaum and Selberg-Delange

In this section Theorem 1.2 is proved by establishing a suitable bound for the l.h.s of (2.4) via Theorem 3.1 below, which is special case of a very general theorem of Nair and Tenenbaum [5] (Theorem 1 therein).

Let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity and let A and B be positive constants. Also let $\alpha > 0$ and $\epsilon > 0$ be quantities which may be taken to be arbitrarily small.

Theorem 3.1 (Nair-Tenenbaum)**.** *If F_1, F_2 are non-negative arithmetic functions satisfying*

$$F_1(m)F_2(n) \leq \min\{A^{\Omega(mn)}, B(\epsilon)(mn)^{\epsilon}\} \tag{3.1}$$

whenever $(m,n) = 1$, then

$$\sum_{x\leq n\leq x+y} F_1(n)F_2(n+h) \ll_{A,B,h,\epsilon} \frac{y}{(\log x)^2}\sum_{mn\leq x}\frac{F_1(m)F_2(n)}{mn} \tag{3.2}$$

uniformly for $x^{\alpha} \leq y \leq x$.

From (2.3) and the fact that for fixed h the sum

$$\begin{aligned} \sum_{x<n\leq x+h} d_k(n) \quad &\ll_{h,k} \quad \max_{n\leq x+h} d_k(n) \\ &\ll_{h,k} \quad k^{C\log(x+h)/\log\log(x+h)} \\ &\ll_{h,k} \quad x^{o(\log k)}, \end{aligned} \tag{3.3}$$

it follows from (1.5) that to prove Theorem 1.2 it will be sufficient to prove the following proposition.

Proposition 3.2. *For fixed h and k we have*

$$\sum_{n\leq x}\sqrt{d_k(n)d_k(n+h)} = O\left(x(\log x)^{2(\sqrt{k}-1)}\right) \tag{3.4}$$

as $x \to \infty$.

Proof. Take $F_1(n) = F_2(n) = \sqrt{d_k(n)}$ in Theorem 3.1, so that $F_1(m)F_2(n) = \sqrt{d_k(mn)}$ when $(m, n) = 1$. To begin, we must verify that (3.1) holds in this case, i.e. that

$$\sqrt{d_k(n)} \leq \min\{A^{\Omega(n)}, B(\epsilon)n^{\epsilon}\} \tag{3.5}$$

when n is squarefree. Since $d_k(p) = k$ it follows that $d_k(n) = k^{\Omega(n)}$, so we have $A = \sqrt{k}$. Since $\Omega(n) = O(\log n/\log\log n)$ as $n \to \infty$ it follows that $k^{\Omega(n)} \leq B(\epsilon)n^{\epsilon}$ for every $\epsilon > 0$, so (3.5) holds in this case.

For $\sigma > 1$ let

$$D_k(s) = \sum_1^{\infty} \frac{d_k^{1/2}(n)}{n^s}. \tag{3.6}$$

By the quantitative version of Perron's formula—a general proof of which is given in Titchmarsh [8] (Lemma 3.12)—one now observes that for $\delta > 0$, $k \geq 2$, $T > 0$ and x not an integer we have

$$\begin{aligned} \sum_{mn \leq x} \frac{F_1(m)F_2(n)}{mn} = \sum_{mn \leq x} \frac{d_k^{1/2}(m)d_k^{1/2}(n)}{mn} &= \frac{1}{2\pi i}\int_{\delta - iT}^{\delta + iT} D_k^2(s+1)\frac{x^s ds}{s} \\ &+ O\left(\frac{x^{\delta}}{T}D_k^2(\delta+1)\right) \\ &+ O\left(\frac{\log x}{T}\max_{n \leq 2x}\frac{1}{n}\sum_{d|n} d_k^{1/2}(d)\right). \end{aligned} \tag{3.7}$$

The remaining steps of the proof essentially follow the methods of Selberg [6] and Delange [1], which enable the integral on the r.h.s of (3.7) to be estimated. This proceeds by evaluating the integral along segments marginally above and below the potential branch cut $(-\infty, 0]$ and using Hankel's integral representation of $\Gamma(s)$.

The first step is to observe that

$$D_k^2(s) = H_k(s)\zeta^{2k^{1/2}}(s), \tag{3.8}$$

where $H_k(s)$ has an absolutely convergent Euler product on compact subsets of the half plane $\sigma > 1/2$. As such, for fixed k, $|H_k(s)|$ is bounded above and away from zero on compact subsets of the half plane $\sigma > 1/2$. Moreover, due to the simple pole of $\zeta(s)$ at $s = 1$, from (3.8) it is evident that $(-\infty, 0]$ is a branch cut for $D_k^2(s+1)$ whenever k is not square.

Given $\epsilon > 0$, one takes the path of integration in (3.7) to consist of horizontal segments from $\delta - iT$ to $-\delta - iT$ and $-\delta + iT$ to $\delta + iT$, vertical segments from $-\delta - iT$ to $-\delta - i\epsilon$ and $-\delta + i\epsilon$ to $-\delta + iT$, and a truncated Hankel contour (a path from $-\delta - i\epsilon$ to $-\delta + i\epsilon$ passing around the cut along the segment $[-\delta, 0]$, but not crossing it). From (3.8), the bounds on $|H_k(s)|$ and the elementary fact that $\zeta(\sigma + it) = O(t^{1-\sigma+\delta})$ for $\sigma \geq 0$, it is immediate that the vertical segments of the integral are

$$\left| \frac{1}{2\pi i} \int_{-\delta+i\epsilon}^{-\delta+iT} \frac{H_k(s+1)\zeta^{2k^{1/2}}(s+1)x^s ds}{s} \right| \ll_{k,\delta} x^{-\delta} T^{4\delta k^{1/2}}, \tag{3.9}$$

and that the horizontal segments of the integral are

$$\left| \frac{1}{2\pi i} \int_{-\delta+iT}^{\delta+iT} \frac{H_k(s+1)\zeta^{2k^{1/2}}(s+1)x^s ds}{s} \right| \ll_{k,\delta} x^{\delta} T^{4\delta k^{1/2}-1}. \tag{3.10}$$

Taking $T = x^{2\delta}$ and $\delta = k^{-1/2}/8$, the r.h.s. of (3.9) is

$$x^{-\delta}(x^{2\delta})^{4\delta k^{1/2}} = x^{-\delta+8\delta^2 k^{1/2}} = x^{-\delta+k^{-1/2}/8} = 1 \tag{3.11}$$

and the r.h.s. of (3.10) is

$$x^{\delta}(x^{2\delta})^{4\delta k^{1/2}-1} = x^{-\delta+8\delta^2 k^{1/2}} = 1, \tag{3.12}$$

so (3.9) and (3.10) are bounded as $x \to \infty$ for fixed k.

Moreover, with these choices for δ and T, the first error term on the r.h.s of (3.7) is

$$\frac{x^{\delta}}{T} D_k^2(\delta+1) = x^{-\delta} D_k^2(\delta+1) \ll_k x^{-\delta} \tag{3.13}$$

which is bounded as $x \to \infty$ for fixed k. The second error term on the r.h.s of (3.7) is

$$\begin{aligned} \frac{\log x}{T} \max_{n \leq 2x} \frac{1}{n} \sum_{d|n} d_k^{1/2}(d) \quad &\ll_k \quad x^{-2\delta} \log x (k+1)^{C \log x / \log\log x} \\ &\ll_k \quad x^{-2\delta + C \log k / \log\log x}, \end{aligned} \tag{3.14}$$

which is also bounded as $x \to \infty$ for fixed k.

For fixed k then, it follows that

$$\sum_{mn \leq x} \frac{d_k^{1/2}(m) d_k^{1/2}(n)}{mn} = \frac{1}{2\pi i} \int_{\mathcal{H}(k,\epsilon)} D_k^2(s+1) \frac{x^s ds}{s} + O_k(1), \tag{3.15}$$

where the path of integration $\mathcal{H}(k,\epsilon)$ is from $-k^{-1/2}/8 - i\epsilon$ to $-k^{-1/2}/8 + i\epsilon$ and not intersecting the half line $(-\infty, 0]$. Invoking (3.8) and the fact that $\zeta(s)$ has a simple pole at $s = 1$, one may expand $H_k(s+1)$ in a power series about $s = 0$ to give

$$D_k^2(s+1) = \sum_{n \le 2k^{1/2}} c_n s^{n-2k^{1/2}} + O_k(1) \tag{3.16}$$

so the r.h.s of (3.15) is

$$\sum_{n \le 2k^{1/2}} \frac{c_n}{2\pi i} \int_{\mathcal{H}(k,\epsilon)} x^s s^{n-2k^{1/2}-1} ds + O_k(1). \tag{3.17}$$

Making the change of variable $s = z/\log x$ in (3.17) then gives

$$\sum_{n \le 2k^{1/2}} \frac{c_n (\log x)^{2k^{1/2}-n}}{2\pi i} \int_{\mathcal{H}(k,\epsilon,x)} e^z z^{n-2k^{1/2}-1} dz + O_k(1), \tag{3.18}$$

where $\mathcal{H}(k,\epsilon,x)$ indicates a path of integration from $-k^{-1/2}\log x/8 - i\epsilon \log x$ to $-k^{-1/2}\log x/8 + i\epsilon \log x$ and not intersecting the half line $(-\infty, 0]$. Taking $\epsilon = o(1/\log x)$, the path $\mathcal{H}(k,\epsilon,x)$ approaches a standard Hankel contour $\mathcal{H}$ as $x \to \infty$ therefore, using Hankel's identity

$$\frac{1}{\Gamma(s+1)} = \frac{1}{2\pi i} \int_{\mathcal{H}} e^z z^{-s-1} dz, \tag{3.19}$$

in (3.18), from (3.7) we now have

$$\begin{aligned} \sum_{mn \le x} \frac{d_k^{1/2}(m) d_k^{1/2}(n)}{mn} &= \sum_{n \le 2k^{1/2}} \frac{c_n (\log x)^{2k^{1/2}-n}}{\Gamma(2k^{1/2} - n + 1)} + O_k(1) \\ &= O_k\left((\log x)^{2k^{1/2}}\right). \end{aligned} \tag{3.20}$$

Thus, (3.20) and (3.2) together give

$$\sum_{x \le n \le x+y} d_k^{1/2}(n) d_k^{1/2}(n+h) \ll_{h,k} y (\log x)^{2(k^{1/2}-1)} \tag{3.21}$$

uniformly for $x^\alpha \le y \le x$.

To complete the proof of Proposition 3.2 we take $y = x = 2^{-m-1}X$ successively in (3.21) and sum over the range $0 \le m \le \log_2 X$, which gives

$$\frac{\sum_{n\leq X} d_k^{1/2}(n) d_k^{1/2}(n+h)}{X(\log X)^{2(k^{1/2}-1)}} \ll_{h,k} \sum_{0\leq m\leq \log_2 X} 2^{-m-1}\left(1-\frac{(m-1)\log 2}{\log X}\right)^{2(k^{1/2}-1)}$$

$$\ll_{h,k} 1 \tag{3.22}$$

as $X \to \infty$. □

Acknowledgment: We would like to thank Professor Zeev Rudnick for suggesting that the results of Nair and Tenenbaum could be used to prove the main result of this paper. We would like to thank an anonymous referee whose comments and suggestions helped to improve the presentation of the paper. The first author is grateful to the Leverhulme Trust (RPG-2017-320) for the support through the research grant "Moments of L-functions in Function Fields and Random Matrix Theory".

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