ON THE PAIRWISE MAXIMA OF GENERALISED DIVISOR FUNCTIONS

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Abstract. In this paper, we prove the asymptotic growth rate of the summatory function of the pairwise maxima of the generalised divisor function $d_k(n)$, for a fixed positive integer $k \geq 2$. This result generalises previous results of Kátai, Erdős and Hall on the local behaviour of divisor function on short intervals.

1. Introduction

Let $n$ be a natural number and let $d(n)$ denote the number of divisors of $n$. Kátai, in his paper [4], studied the local behaviour of the function $d(n)$. In his paper he proved that

\begin{equation}
\sum_{n \leq x} \max \{d(n), d(n+1)\} = 2x \log x + O(x(\log x)^{1-\delta}),
\end{equation}

where $\delta$ is a suitable positive constant.

In their paper [2], Erdős and Hall determined the following asymptotic for the local maxima of $d(n)$:

**Theorem 1.1** (Erdős-Hall). If $h = o((\log x)^3 - 2\sqrt{2})$, then

\begin{equation}
\sum_{n \leq x} \max \{d(n), d(n+1), \ldots, d(n+h-1)\} = hx \log x + O(h^2x(\log x)^{2(\sqrt{2}-1)}).
\end{equation}

(1.2)

In the case $h = 2$, equation (1.2) reduces to

\begin{equation}
\sum_{n \leq x} \max \{d(n), d(n+1)\} = 2x \log x + O(x(\log x)^{2(\sqrt{2}-1)}).
\end{equation}

(1.3)

Although the authors do not state this explicitly, with slight modifications their proof of Theorem 1.1 also provides us with

\begin{equation}
\sum_{n \leq x} \max \{d(n), d(n+h)\} = 2x \log x + O(x(\log x)^{2(\sqrt{2}-1)}).
\end{equation}

(1.4)
for fixed values of $h$.

In this paper we generalise (1.4) for fixed values of $h$ and $k$ by considering the relation

$$
\sum_{n \leq x} \max\{d_k(n), d_k(n+h)\} = \sum_{n \leq x} d_k(n) + \sum_{n \leq x} d_k(n+h) - \sum_{n \leq x} \min\{d_k(n), d_k(n+h)\}
$$

$$
= 2 \sum_{n \leq x} d_k(n) - \sum_{n \leq x} \min\{d_k(n), d_k(n+h)\}
$$

$$
+ \sum_{x < n \leq x+h} d_k(n) - \sum_{n < h} d_k(n)
$$

(1.5)

$$
= 2 \sum_{n \leq x} d_k(n) + E_k(x, h).
$$

Our main result is Theorem 1.2 below, which is proved in Section 3.

**Theorem 1.2.** If $h$ and $k$ are fixed, then

$$
E_k(x, h) \ll_{h,k} x (\log x)^{(2(\sqrt{k} - 1))}
$$

as $x \to \infty$.

By using the well-known asymptotic formula for the summatory function of $d_k(n)$ [8, p. 263], Theorem 1.2 states that if $k > 4$ and $h$ a fixed number, then

$$
\sum_{n \leq x} \max\{d_k(n), d_k(n+h)\} = \frac{2}{(k-1)!} x (\log x)^{k-1} + O(x (\log x)^{k-2})
$$

and for $k \leq 4$ we have that

$$
\sum_{n \leq x} \max\{d_k(n), d_k(n+h)\} = \frac{2}{(k-1)!} x (\log x)^{k-1} + O(x (\log x)^{2(\sqrt{k} - 1)})
$$

as $x \to \infty$.

The main difficulty is that the approach of Erdős and Hall [2] breaks down for $d_k(n)$ if $k \geq 4$. Therefore new ideas are necessary to generalise their results. To overcome such intricacies we use a theorem by Nair and Tenenbaum [5] to obtain a bound on certain averages involving $d_k(n)$ which turns out to be sufficient to establish the asymptotic formula above. In Section 2 of the paper we discuss the method of Erdős and Hall and why it breaks down when we try to generalise to $d_k(n)$. In Section 3 we prove Theorem 1.2, which is the main result of this paper.

2. **The method of Erdős and Hall**

In this section we briefly describe the method of proof of (1.4) used in their paper [2], and how it must be modified to establish Theorem 1.2. Note that $d(p^\alpha) \geq d(p^{\alpha-1})$ for $\alpha \geq 1$. Since $\sqrt{d(n)}$ is multiplicative, we have
\begin{align}
\sqrt{d(n)} &= \sum_{d \mid n} f(d) \\
\end{align}

where

\begin{align}
f(p^{\alpha}) &= \sqrt{g(p^{\alpha})} - \sqrt{g(p^{\alpha-1})} \geq 0
\end{align}

for \( \alpha \geq 1 \) and \( f(1) = 1 \).

The method of Erdős and Hall begins by using the simple facts that

\begin{align}
\min\{d(n), d(n+1)\} \leq \sqrt{d(n)d(n+1)}
\end{align}

and

\begin{align}
\sum_{n \leq x} \sqrt{d(n)d(n+1)} &= \sum_{n \leq x} \sum_{d \mid n} f(d) \sum_{e \mid n+1} f(e),
\end{align}

and a crucial step of their proof establishes that there exists a constant \( C \) such that

\begin{align}
\sqrt{d(n)} = \sum_{d \mid n} f(d) \leq C \sum_{d \mid n} f(d).
\end{align}

To establish (2.5), the authors observe that

\begin{align}
\sum_{d \mid n, d \geq \sqrt{n}} f(d) \leq \frac{2}{\log n} \sum_{d \mid n, d \geq \sqrt{n}} f(d) \log d \leq \frac{2}{\log n} \sum_{d \mid n} f(d) \log d
\end{align}

for any multiplicative function \( f \) satisfying \( f(1) = 1 \), so to prove (2.5) it is sufficient to establish the existence of a \( C' < 1/2 \) such that

\begin{align}
\sum_{d \mid n} f(d) \log d \leq C' \log n \sum_{d \mid n} f(d)
\end{align}

because by (2.6) we then have

\begin{align}
\sum_{d \mid n} f(d) \leq \frac{1}{1 - 2C'} \sum_{d \mid n, d < \sqrt{n}} f(d).
\end{align}

However, we can prove that
Lemma 2.1. For a multiplicative function $f$ satisfying $f(1) = 1$, let

\begin{equation}
\sqrt{g(n)} = \sum_{d \mid n} f(d),
\end{equation}

(2.9)

then there exists a constant $C' < 1/2$ such that

\begin{equation}
\sum_{d \mid n} f(d) \log d \leq C' \log n \sum_{d \mid n} f(d)
\end{equation}

(2.10)

if and only if there exists a constant $C'' > 1/2$ such that

\begin{equation}
\sqrt{g(p^\alpha)} \leq \frac{1}{C'' \alpha} \sum_{j=0}^{\alpha-1} \sqrt{g(p^j)}
\end{equation}

(2.11)

for every $p$ and every $\alpha \geq 1$.

Proof. By logarithmic differentiation of

\begin{equation}
\sum_{d \mid n} \frac{f(d)}{d^s}
\end{equation}

(2.12)

one finds that

\begin{equation}
\frac{\sum_{d \mid n} f(d) \log d}{\sum_{d \mid n} f(d)} = \sum_{p^\alpha \mid n} \left( \frac{f(p) + 2f(p^2) + \cdots + \alpha f(p^\alpha)}{1 + f(p) + f(p^2) + \cdots + f(p^\alpha)} \right) \log p.
\end{equation}

(2.13)

From (2.13) it follows that the existence of $C'$ in (2.10) is equivalent to

\begin{equation}
\sum_{j=0}^{\alpha} j f(p^j) \leq C' \alpha \sum_{j=0}^{\alpha} f(p^j)
\end{equation}

(2.14)

for every $p$ and every $\alpha \geq 1$. By (2.2) and some elementary analysis, (2.14) reduces to (2.11). \hfill \Box

Erdős and Hall prove that (2.11) holds when $g(n) = d(n)$ so Lemma 2.1 applies. This gives a non-trivial estimate of (2.4) which implies Theorem 1.1. However, the following dilemma arises.

Corollary 2.2. The growth constraint (2.11) does not hold for $g(n) = d_k(n)$ when $k > 3$.

Proof. Since $d_k(p^j) = \left(\begin{array}{c} j+k-1 \\ k-j \end{array}\right)$, we observe that...
\[
\sqrt{\left(\frac{7}{4}\right)} > \frac{1}{2} \sum_{j=0}^{3} \sqrt{\left(\frac{3+j}{3}\right)},
\]
so (2.11) fails for \( g(n) = d_4(n) \). Similar arguments show that (2.11) fails to hold for any \( k \geq 4 \). \( \square \)

It follows from the previous corollary that Erdős and Hall approach does not apply for \( d_k(n) \) for \( k \geq 4 \). We will remedy this in the next section.

3. A proof via the theorems of Nair-Tenenbaum and Selberg-Delange

In this section Theorem 1.2 is proved by establishing a suitable bound for the l.h.s of (2.4) via Theorem 3.1 below, which is special case of a very general theorem of Nair and Tenenbaum [5] (Theorem 1 therein).

Let \( \Omega(n) \) denote the number of prime factors of \( n \) counted with multiplicity and let \( A \) and \( B \) be positive constants. Also let \( \alpha > 0 \) and \( \epsilon > 0 \) be quantities which may be taken to be arbitrarily small.

**Theorem 3.1 (Nair-Tenenbaum).** If \( F_1, F_2 \) are non-negative arithmetic functions satisfying

\[
F_1(m)F_2(n) \leq \min\{A^{\Omega(mn)}, B(\epsilon)(mn)^\epsilon\}
\]
whenever \( (m, n) = 1 \), then

\[
\sum_{x \leq n \leq x+y} F_1(n)F_2(n+h) \ll_{A,B,h,\epsilon} \frac{y}{(\log x)^2} \sum_{mn \leq x} \frac{F_1(m)F_2(n)}{mn}
\]
uniformly for \( x^\alpha \leq y \leq x \).

From (2.3) and the fact that for fixed \( h \) the sum

\[
\sum_{x < n \leq x+h} d_k(n) \ll_{h,k} \max_{n \leq x+h} d_k(n) \ll_{h,k} \frac{kC_{\log(x+h)} \log(x+h)}{\log(x+h)}
\]

\[
\ll_{h,k} x^{o(\log k)},
\]

(3.3)
it follows from (1.5) that to prove Theorem 1.2 it will be sufficient to prove the following proposition.

**Proposition 3.2.** For fixed \( h \) and \( k \) we have

\[
\sum_{n \leq x} \sqrt{d_k(n)d_k(n+h)} = O\left((\log x)^2(\sqrt{k}-1)\right)
\]

(3.4)
as \( x \to \infty \).

**Proof.** Take \( F_1(n) = F_2(n) = \sqrt{d_k(n)} \) in Theorem 3.1, so that \( F_1(m)F_2(n) = \sqrt{d_k(mn)} \) when \((m,n) = 1\). To begin, we must verify that (3.1) holds in this case, i.e. that

\[
\sqrt{d_k(n)} \leq \min\{A^{\Omega(n)}, B(\epsilon)n^\epsilon\}
\]

when \( n \) is squarefree. Since \( d_k(p) = k \) it follows that \( d_k(n) = k^{\Omega(n)} \), so we have \( A = \sqrt{k} \). Since \( \Omega(n) = O(\log n/\log \log n) \) as \( n \to \infty \) it follows that \( k^{\Omega(n)} \leq B(\epsilon)n^\epsilon \) for every \( \epsilon > 0 \), so (3.5) holds in this case.

For \( \sigma > 1 \) let

\[
D_k(s) = \sum_{m \leq x} \frac{d_k^{1/2}(m)}{n^s}.
\]

By the quantitative version of Perron’s formula—a general proof of which is given in Titchmarsh [8] (Lemma 3.12)—one now observes that for \( \delta > 0, k \geq 2, T > 0 \) and \( x \) not an integer we have

\[
\sum_{mn \leq x} \frac{F_1(m)F_2(n)}{mn} = \sum_{mn \leq x} \frac{d_k^{1/2}(m)d_k^{1/2}(n)}{mn} = \frac{1}{2\pi i} \int_{\delta - iT}^{\delta + iT} D_k^2(s+1) x^s ds
\]

\[
+ O\left( \frac{x^\delta}{T} D_k^2(\delta + 1) \right)
\]

\[
+ O\left( \frac{\log x}{T} \max_{n \leq 2x} \frac{1}{n} \sum_{d|n} d_k^{1/2}(d) \right).
\]

(3.7)

The remaining steps of the proof essentially follow the methods of Selberg [6] and Delange [1], which enable the integral on the r.h.s of (3.7) to be estimated. This proceeds by evaluating the integral along segments marginally above and below the potential branch cut \((-\infty, 0]\) and using Hankel’s integral representation of \( \Gamma(s) \).

The first step is to observe that

\[
D_k^2(s) = H_k(s)\zeta^{2k^{1/2}}(s),
\]

where \( H_k(s) \) has an absolutely convergent Euler product on compact subsets of the half plane \( \sigma > 1/2 \). As such, for fixed \( k \), \( |H_k(s)| \) is bounded above and away from zero on compact subsets of the half plane \( \sigma > 1/2 \). Moreover, due to the simple pole of \( \zeta(s) \) at \( s = 1 \), from (3.8) it is evident that \((-\infty, 0] \) is a branch cut for \( D_k^2(s+1) \) whenever \( k \) is not square.
Given $\epsilon > 0$, one takes the path of integration in (3.7) to consist of horizontal segments from $\delta - iT$ to $-\delta - iT$ and $-\delta + iT$ to $\delta + iT$, vertical segments from $-\delta - iT$ to $-\delta - i\epsilon$ and $-\delta + i\epsilon$ to $-\delta + iT$, and a truncated Hankel contour (a path from $-\delta - i\epsilon$ to $-\delta + i\epsilon$ passing around the cut along the segment $[-\delta, 0]$, but not crossing it). From (3.8), the bounds on $|H_k(s)|$ and the elementary fact that $\zeta(\sigma + iT) = O(t^{1-\sigma+\delta})$ for $\sigma \geq 0$, it is immediate that the vertical segments of the integral are

$$\left| \frac{1}{2\pi i} \int_{-\delta - i\epsilon}^{\delta + i\epsilon} H_k(s + 1) \zeta^{2k^{1/2}}(s + 1)x^s ds \right| \ll_{k, \delta} x^{-\delta} T^{4\delta k^{1/2}},$$

and that the horizontal segments of the integral are

$$\left| \frac{1}{2\pi i} \int_{-\delta - iT}^{\delta + iT} H_k(s + 1) \zeta^{2k^{1/2}}(s + 1)x^s ds \right| \ll_{k, \delta} x^{\delta} T^{4\delta k^{1/2} - 1}.$$ 

Taking $T = x^{2\delta}$ and $\delta = k^{-1/2}/8$, the r.h.s. of (3.9) is

$$x^{-\delta}(x^{2\delta})^{1/2} = x^{-\delta + 8\delta^2 k^{1/2}} = x^{-\delta + k^{-1/2}/8} = 1$$
and the r.h.s. of (3.10) is

$$x^\delta(x^{2\delta})^{1/2} - 1 = x^{-\delta + 8\delta^2 k^{1/2}} = 1,$$

so (3.9) and (3.10) are bounded as $x \to \infty$ for fixed $k$.

Moreover, with these choices for $\delta$ and $T$, the first error term on the r.h.s of (3.7) is

$$x^\delta \frac{D_k^2(\delta + 1)}{T} = x^{-\delta} D_k^2(\delta + 1) \ll_{k} x^{-\delta}$$
which is bounded as $x \to \infty$ for fixed $k$. The second error term on the r.h.s of (3.7) is

$$\log x \max_{n \leq 2x} \frac{1}{n} \sum_{d|n} d_k^{1/2}(d) \ll_{k} x^{-2\delta} \log x (k + 1)^{C \log x / \log \log x}$$

$$\ll_{k} x^{-2\delta + C \log k / \log \log x},$$

which is also bounded as $x \to \infty$ for fixed $k$.

For fixed $k$ then, it follows that

$$\sum_{mn \leq x} \frac{d_k^{1/2}(m)d_k^{1/2}(n)}{mn} = \frac{1}{2\pi i} \int_{\mathcal{H}(k, \epsilon)} D_k^2(s + 1) \frac{x^s ds}{s} + O_k(1),$$
where the path of integration $\mathcal{H}(k, \epsilon)$ is from $-k^{-1/2}/8 - i\epsilon$ to $-k^{-1/2}/8 + i\epsilon$ and not intersecting the half line $(-\infty, 0]$. Invoking (3.8) and the fact that $\zeta(s)$ has a simple pole at $s = 1$, one may expand $H_k(s + 1)$ in a power series about $s = 0$ to give

$$D_k^2(s + 1) = \sum_{n \leq 2k^{1/2}} c_n s^{n-2k^{1/2}} + O_k(1) \tag{3.16}$$

so the r.h.s of (3.15) is

$$\sum_{n \leq 2k^{1/2}} \frac{c_n}{2\pi i} \int_{\mathcal{H}(k, \epsilon)} x^s s^{n-2k^{1/2}-1} ds + O_k(1). \tag{3.17}$$

Making the change of variable $s = z/\log x$ in (3.17) then gives

$$\sum_{n \leq 2k^{1/2}} \frac{c_n (\log x)^{2k^{1/2}-n}}{2\pi i} \int_{\mathcal{H}(k, \epsilon, x)} e^z z^{n-2k^{1/2}-1} dz + O_k(1), \tag{3.18}$$

where $\mathcal{H}(k, \epsilon, x)$ indicates a path of integration from $-k^{-1/2} \log x/8 - i\epsilon \log x$ to $-k^{-1/2} \log x/8 + i\epsilon \log x$ and not intersecting the half line $(-\infty, 0]$. Taking $\epsilon = o(1/\log x)$, the path $\mathcal{H}(k, \epsilon, x)$ approaches a standard Hankel contour $\mathcal{H}$ as $x \to \infty$ therefore, using Hankel’s identity

$$\frac{1}{\Gamma(s+1)} = \frac{1}{2\pi i} \int_{\mathcal{H}} e^z z^{-s-1} dz, \tag{3.19}$$

in (3.18), from (3.7) we now have

$$\sum_{mn \leq x} \frac{d_k^{1/2}(m)d_k^{1/2}(n)}{mn} = \sum_{n \leq 2k^{1/2}} \frac{c_n (\log x)^{2k^{1/2}-n}}{\Gamma(2k^{1/2} - n + 1)} + O_k(1) \tag{3.20}$$

Thus, (3.20) and (3.2) together give

$$\sum_{x \leq n \leq x+y} d_k^{1/2}(n)d_k^{1/2}(n+h) \ll_{h,k} y(\log x)^{2(k^{1/2}-1)} \tag{3.21}$$

uniformly for $x^n \leq y \leq x$.

To complete the proof of Proposition 3.2 we take $y = x = 2^{-m-1}X$ successively in (3.21) and sum over the range $0 \leq m \leq \log_2 X$, which gives
\[
\sum_{n \leq X} \frac{d_k^{1/2}(n) d_k^{1/2}(n + h)}{X (\log X)^{2(k^{1/2} - 1)}} \ll_{h,k} \sum_{0 \leq m \leq \log_2 X} 2^{-m-1} \left( 1 - \frac{(m - 1) \log 2}{\log X} \right)^{2(k^{1/2} - 1)}
\]

as \( X \to \infty. \)  

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