On the dynamics and control of mechanical properties of hierarchical auxetics

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ABSTRACT

In this work, we investigate the deformation mechanism of auxetic hierarchical rotating square systems through a dynamics approach. We show how their deformation behaviour and, hence their mechanical properties, can be manipulated solely by altering the resistance to rotational motion of the hinges within the system. This provides enhanced tunability without necessarily changing the geometry of the system, a phenomenon which is not typically observed in other non-hierarchical unimode auxetic systems.

Introduction

The design of multifunctional materials entails a clear understanding of their underlying physics. Mechanical metamaterials which exhibit a negative Poisson's ratios (auxetics)[1–12] have recently been the focus of many studies due to the various enhanced properties they exhibit, the relative ease with which they may be produced, and the potential applications where they may be used. For example, it has been shown that auxeticity can be achieved through the introduction of cuts and perforations[13–15], molding[16] and 3D printing[17] techniques which are well established within the field of engineering. This also permits the production of custom-designed systems even at industrial scales at a relative low cost. Of particular interest are auxetics incorporating elements of hierarchy[18–21], with the hierarchical rotating square structure proposed by Cho *et al.*[22] and Gatt *et al.*[23] being one of the most prominent examples of these systems. It is composed of rotating 'square units' which are made up of an array of smaller rotating units themselves, a process which in theory may be repeated indefinitely. The inclusion of hierarchy in the classic rotating square system[27] has been shown to result in increased versatility, expandability and enhanced tunability of phononic wave propagation[22,23,32,33].

Despite all this, no attempts have yet been made to study the dynamic behaviour of the mechanical properties of these hierarchical systems, meaning that some of the more fundamental properties have hitherto not been explored. For example, the current methods of analysis give no information on the rate of deformation meaning that it is impossible to estimate the time required for a system to move from one conformation to another, or the path the system takes as a function of time during this deformation.

In view of the above, the deformation behaviour of a hierarchical structure based on the rigid rotating units model (see Fig. 1) was investigated through a set of dynamics equations which take into consideration the rotational motion of the units in the respective hierarchical levels. Each structure considered in this work consists of squares (or square-like units) connected through hinges having the same properties (i.e. the same resistance to rotational motion). Using this dynamics approach, the mechanical versatility of these hierarchical structures was studied for systems having the same initial geometry but with each system having hinges that offer different resistance to rotational motion (when compared to other systems) albeit each hinge having the same properties within the same system. All this contrasts with previous works on these systems[22,23,32,33] which have focused mainly on the geometric versatility and implementation of these frameworks as perforated systems.

Model

In this paper, a model specifically designed to describe the dynamical behaviour and predict the mechanical properties of the hierarchical system shown in Fig. 1(b) will be presented and discussed. This may be described as a finite two-level heirarchical system having four square-like units in the upper level (corresponding to Level 1 of the structure) each of which is made from

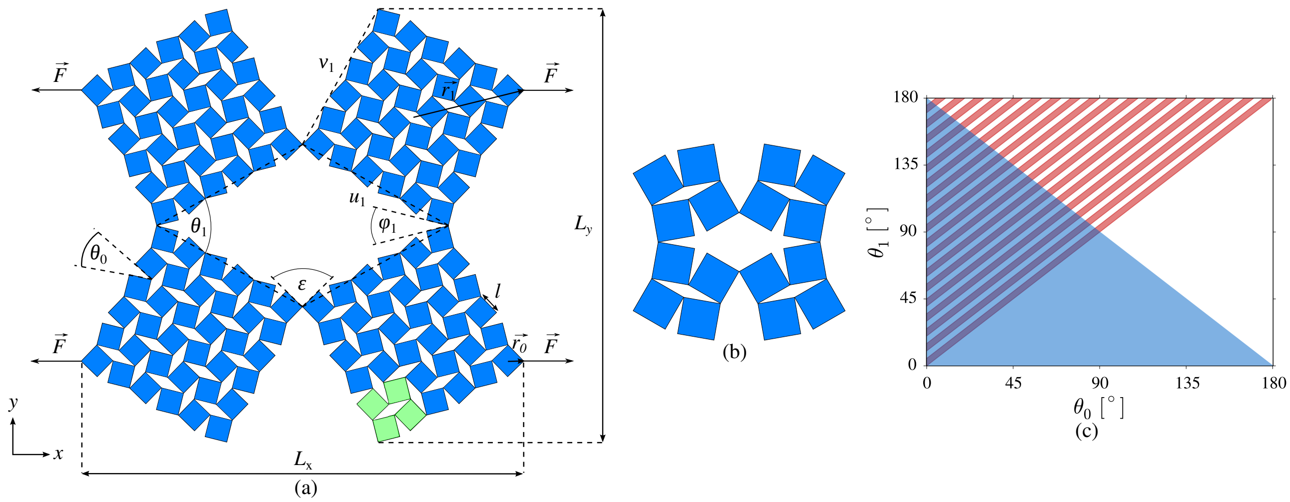

Figure 1. The panels in this figure present (a) the two-level hierarchical auxetic system with four square-like units corresponding to the level 1 of the structure, where each unit consists of $N_0 \times N_0$ (in the provided example $N_0 = 3$) Level 0 repeat units (bright green), (b) an example of the structure corresponding to N_0 = 1 and (c) the permissable angles for θ_0 and θ_1, which conditions ensure that the squares do not overlap whith each other and the system retains the same connectivity. This is attained when conditions $\theta_1 > \theta_0$ and $\varepsilon = \pi - \theta_1 - \theta_0 > 0$ are satisfied.

four other sqaures (the Level 0) having a linear dimension of l. This sytem is a particular case of a more general two-level heirarchical system where each of the Level 1 building blocks consists of $2N_{0,x} \times 2N_{0,y}$, where $N_{0,x}$ and $N_{0,y}$ stand for the number of Level 0 units in the two orthogonal directions associated with a Level 1 building block, see Fig. 1(a). Note that if $N_{0,x} = N_{0,y} = N_0$, the Level 0 would approximate the shape of a square (as is the case in the present work) whilst if $N_{0,x} \neq N_{0,y}$, then the Level 0 would assume the shape of a rectangle.

Variables θ_0 and θ_1 correspond to angles between the adjacent units of the zeroth and first level respectively. These quantities can in turn be used in order to determine the linear dimensions (L_x and L_y) of the discussed system (see Supplementary Information). It is also very important to note that the hierarchical systems considered in this study cannot be constructed for any combination of θ_0 and θ_1. The possible values of these angles are given in Fig. 1(c).

This model operates under the assumption that the Level 0 rotating square units of the system are completely rigid and cannot distort or change shape in any way during deformation. Furthermore, the Level 0 squares and Level 1 building blocks will be assumed to be connected together through hinges which permit relative rotation of two connected units. The symmetry of the system, as well as its geometric constraints result in a system which only has few degrees of freedom. In particular, under the condition of uniaxial on-axis loading, all of the units constituting the i-th level of the system are geometrically constrained to rotate by the same angle, while the 0-th level squares remain rigid, i.e. they do not distort or change shape in any way. Under these conditions, for a given value of l (size of the length of a single square), the geometry of the whole system can be described through just two independent variables, angles θ_0 and θ_1. This means that it is sufficient to investigate the rotation of individual units in both levels in order to obtain a complete picture of the deformation mechanism of the considered structure, i.e. the dynamics of the system may be fully described through a set of equations which are the rotational analogues of Newton's equation of motion, which equations may be solved numerically. In other words, the deformation through time, of the whole system may be divined by solving the equations governing the changes in the angles θ_i at the given time, an approach which allows for perfect rigidity of the 0-th level rotating units and is highly efficient from a computational time point of view, as the number of equations which must be solved at the given time corresponds to the number of levels within the system and is nearly independent of the number of units constituting a given level. This stems from the fact that the number of rigid units corresponding to the i-th level is only a number parameter in the equation describing the dynamics of the angle θ_i, as will be discussed later in this paper.

In order to induce a deformation in the systems discussed in this work, a force $\overrightarrow{F}$ is applied on each of the leftmost and rightmost vertices of the system, as shown in Fig. 1(a) (or topmost and bottommost, depending on the direction of loading).

This model assumes that a resistance to rotation is solely due to a friction where all hinges in the system are identical to each other irrespective of their position within the system. In this case, the resistance to rotation may be quantified in terms of a friction torque $f = f_i$ (i=0,1) resulting from the friction caused by the rotational motion of the hinge and that the value of f remains constant regardless of changes in angular velocity and angle of aperture of the subunit. Under such assimptions, for a given level to start deforming, the resultant friction torque associated with this level has to be overcomed by the torque corresponding to the force applied to the system. The deformation of the system depends on the collective magnitude of torques associated with the Level 0 squares (S_0) and the Level 1 subunits (S_1), as follows:

$$S_0 = 16N_0^2|\overrightarrow{\tau_0}| \qquad \text{and} \qquad S_1 = 4|\overrightarrow{\tau_1}| \tag{1}$$

where, $\overrightarrow{\tau_0}$ stands for the torque associated with each of the Level 0 squares and analogically, $\overrightarrow{\tau_1}$ corresponds to the torque associated with the Level 1 building block. In this case, the value of S_i depends on the angular acceleration $\frac{d^2\theta_i}{dt^2}$ (which is induced by the force $\overrightarrow{F}$) and the moment of inertia I_i corresponding to the i-th level, hence it can be defined as $S_i = I_i\frac{d^2\theta_i}{dt^2}$. The deformation of the system can then be established through the rotation of the units found in each level, by means of the rotational analog of Newton's equation of motion:

$$I_i\frac{d^2\theta_i}{dt^2} = 8r_iF\sin\left(\sphericalangle\overrightarrow{r_i},\overrightarrow{F}\right) \mp n_if \tag{2}$$

where, n_i, is the number of hinges corresponding to Level i of the system, r_i, is the distance between the vertex where the force is applied and the centre of mass of the Level i building block.

In the case of this study, a two-level system is being considered and therefore $i = 0, 1$. Here, I_0 corresponds to the rotation of individual squares with respect to their own centres and I_1 is associated with the rotation of the Level 1 building blocks with respect to their centres of mass. The information concerning the way how these quantities were calculated can be found in the Methods section.

The sign in front of the term n_if in Eq. 2, is set in a way so that the resultant friction torque always opposes the rotational motion of units corresponding to the i-th level. Furthermore, the factor 8 in Eq. 2 is associated with the number of forces being applied to the system and the resulting reaction forces as described in[26].

At this point, it is important to highlight the fact that although the model proposed above is based on a specific type of a hinge, the derived model can be generalised for any type of potential V governing the hinging process of the rigid subunits. The potential V would then depend on the way how the hinge is constructed and the type of the material used. Thus, for an arbitrary potential V dependent on the extent of the angle θ_i associated with a given level, the term n_if in Eq. 2 may be generalised as $\frac{\partial V}{\partial\theta_i}$. For example, if the hinging process of the system is governed through a harmonic potential, which was the approach adopted in eqalier work[23], the Newton's equation of motion (Eq. 2) can be rewritten in the following manner:

$$I_1\frac{d^2\theta_1}{dt^2} = 8r_1F\sin\left(\sphericalangle\overrightarrow{r_1},\overrightarrow{F}\right) \mp 4K_h\left[(\theta_1-\theta_0)-(\theta_{1,eq}-\theta_{0,eq})\right] \tag{3}$$

$$I_0\frac{d^2\theta_0}{dt^2} = 8r_0F\sin\left(\sphericalangle\overrightarrow{r_0},\overrightarrow{F}\right) \mp 4K_h\left[(\theta_1-\theta_0)-(\theta_{1,eq}-\theta_{0,eq})\right] \mp (n_i-4)\,K_h\left(\theta_0-\theta_{0,eq}\right) \tag{4}$$

where, K_h and $\theta_{i,eq}$ (i=0,1) stand for the stiffness constant and the equilibrium angle corresponding to the given level of the system. Moreover, the signs depend on whether the angles θ_0 and θ_1 are greater than $\theta_{0,eq}$ and $\theta_{1,eq}$ respectively.

Results & Discussion

Numerical solutions of the model presented above suggest that irrespective of the hinge being used, a tensile force results in the system deforming through a relative rotation of the consitutent units where, for a given initial structure, the actual manner of deformation, and hence the Poisson's ratio, is dependent on the resistance to motion offered by the hinges. This is the first time that a change in the Poisson's ratio is being reported solely due to a change in the resitance associated with the hinges of the hierarchical system (with all the hinges in the system offering the same resistance to rotation), rather than a change in the geometry or a change in the resistance to rotation of the hinges present in different levels of a particular hierarchical structure. This is clearly shown by the results plotted in Fig. 2 for systems with the same initial geometry, set to $l = 0.05\,\text{m}$, $N_0 = 1$, $\theta_0 = 10^\circ$ and $\theta_1 = 20^\circ$, having hinges which offer different resitance to motion. Here it is important to note that the changes in the mechanical properties of the hierarchical systems occur while the system is still deforming via the rotating mechanism. This is completely different from the effect observed in other auxetic systems such as hexagonal re-entrant honeycombs[29–31], where the mechanical properties depend on the interplay between the three main deformation mechanisms present in these systems,

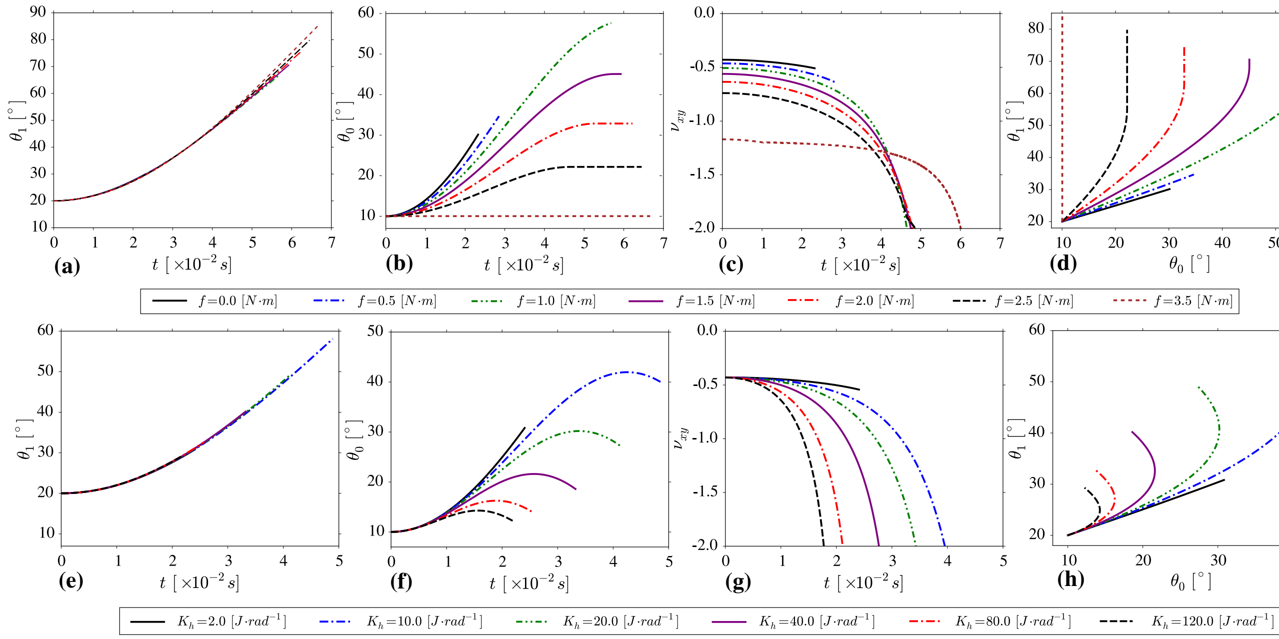

Figure 2. Plots showing the variation in (a) θ_1, (b) θ_0 and (c) Poisson's ratio ν_{xy} as a function of time t in the loading direction for systems with f values ranging from 0N·m to 3.5N·m and (d) the relation of θ_1 to θ_0 for a deforming structure. Both solutions correspond to the same value of $N_0 = 1$.

i.e. the stretching, hinging and flexing mechanisms. For such re-entrant systems one could alter the mechanical properties by changing the ratio of their respective stiffness constants. However in our case, there is no such interplay of mechanisms and the change in mechanical properties was obtained solely as a result of the relative rotations of Level 0 and Level 1 units.

Fig. 2(a)-(d) show the reults for systems where the resistance to motion is governed through a friction torque (f) associated with every hinge within the system. These different structures have the same initial geometry but different values of f ranging from 0 N·m to 3.5 N·m. A detailed analysis of these results indicates that in the case of the hierarchical systems having relatively high values of f, Level 1 opens to a greater extent than Level 0 of the structure. This can be concluded from the fact that angle θ_1 opens to a greater extent than angle θ_0 as shown in Fig. 2 and Table 1. This is in accordance with the numerical and experimental work conducted by Gatt *et al.*[23] and Tang *et al.*[32] concerning the deformation of hierarchical perforated materials. However this deformation behaviour is not observed for all systems with the equivalent initial geometric configuration. In fact, for systems having relatively low values of f, the opposite behaviour is observed, with the Level 0 squares opening to a greater extent than the larger Level 1 units. Therefore, these results suggest that the deformation behaviour depends on the magnitude of the f coefficient. It is also important to note that although θ_1 always increases, for relatively low values of f, referring to Fig. 1(a) and Fig. 3, angle ϕ_1 becomes smaller with time whilst for relatively large values of f this angle becomes increasingly

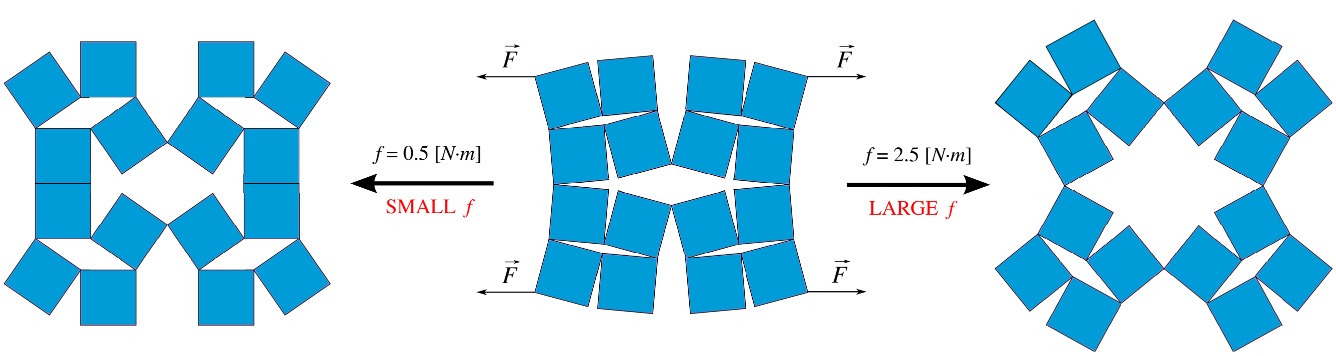

Figure 3. Diagrams showing the final state of the deformation of two systems, where the hinging process is governed by friction, with f values of 0.5N·m and 2.5N·m. Animations presenting the entire deformation range of these systems are provided in the Supplementary Information: ANIM1.gif and ANIM2.gif respectively.

larger.

f [N·m]	0	0.5	1.0	1.5	2.0	2.5	3.5
$\theta_{0,final} - \theta_{0,initial}$ [°]	20.134	24.688	47.659	35.075	22.869	12.176	0.033
$\theta_{1,final} - \theta_{1,initial}$ [°]	10.133	14.689	46.793	50.491	55.351	59.802	65.184

Table 1. Values concerning the difference between the final and initial value of the angle θ_i for a particular value of f.

This difference in the deformation mechanisms upon altering the f coefficient may be explained if one considers the number of hinges present within each level of the system. The deformation of the rotating units in the respective levels is governed by the ratio of the resultant friction torque in the zeroth and first level of system. For the hierarchical structure considered above, the number of Level 0 and Level 1 hinges, n_0 and n_1, is equal to 20 and 4 respectively. Based on Eq. 2, the resultant friction torque, which the system has to overcome in order to expand, can be expressed by means of the terms $n_0 f$ and $n_1 f$ for the Level 0 and Level 1 of the structure respectively. Hence, in the considered case of $N_0 = 1$, the corresponding friction torque of the zeroth level of the structure is five times greater than that of the first level. Thus for relatively large values of f, for example $f = 2.5\,\mathrm{N{\cdot}m}$, the magnitude of the resultant friction torque corresponding to Level 0 is relatively large when compared to that of the torque associated with the applied force. At the same time, Level 1 units are not as affected by the value of f (see Fig. 2(b)) due to the fact that in this case, the friction torque associated with only 4 hinges has to be overcomed. This results in a greater deformation of the Level 1 units. However, in cases where the f constant assumes a relatively small value (such as in the case of $f = 0.5\,\mathrm{N{\cdot}m}$), the resultant friction torque becomes insignificant in comparison to the applied force $\left(\text{i.e.}\, 8Fr_i \sin\left(\sphericalangle \overrightarrow{F}, \overrightarrow{r_i}\right) \gg n_i f\right)$ and the distance r_i becomes the governing factor for the deformation of the system. This means that the Level 0 units deform to a greater extent in the case of a relatively small value of f. The effect which the value of f has on the deformation of the system is clearly shown in Fig. 2(d) where the final configuration of the system corresponding to both of the discussed values of f is presented.

The above results indicate that one may control the deformation behaviour of the system simply by changing the friction coefficient, f. This also suggests that for any two-level hierarchical rotating squares geometry (assuming that both levels open at the same time), there is a specific value of f where the rate of angle opening of θ_0 and θ_1 is equal. Such a treshold value of f can be denoted as f_T. It can be obtained by means of Eq. 2, where by assuming that $\omega_1 = \omega_0$, the value of f_T can be found once the condition $\frac{d^2\theta_0}{dt^2} = \frac{d^2\theta_1}{dt^2}$ is satisfied, i.e. when:

$$f_T = \frac{8F\left[r_1 I_0 \sin\left(\sphericalangle \overrightarrow{F}, \overrightarrow{r_1}\right) - r_0 I_1 \sin\left(\sphericalangle \overrightarrow{F}, \overrightarrow{r_0}\right)\right]}{N_1 I_0 - N_0 I_1} \tag{5}$$

The value f_T, can be determined only for a given time since it varies as the angles change.

Moreover, as shown in Fig. 2(c), a change in f results in a change in the Poisson's ratio of the hierarchical structure under consideration. This system tends to become more auxetic as the value of f increases (in the case of the geometric parameters used). This stems from the fact that the geometry of Level 1 units (defined by u_1 and v_1 in Fig. 1(a)) can be described as rectangles rather than as squares, with $u_1 = 0.1083\,\mathrm{m}$ and $v_1 = 0.0996\,\mathrm{m}$. For relatively large values of f, θ_0 changes to a small extent, meaning that the dimensions of the Level 1 units remain roughly constant throughout the deformation of the hierarchical structure. It is well known that the Poisson's ratios of the rigid rectangles model is dependent on the dimensions of the rectangles and the angle between them. In fact, referring to Fig. 4, it can be shown that for the hierarchical system having $f = 3.5\,\mathrm{N{\cdot}m}$, the Poisson's ratio follows the same profile as the rigid rotating rectanlges model proposed by Grima *et al.*[28]. This result also proves the suitability of this dynamics method to model systems based on rigid rotating units.

On the other hand, for relatively small values of f, (for example $f = 0.5\,\mathrm{N{\cdot}m}$) θ_0 changes to a large extent, meaning that the dimensions of the Level 1 units change throughout the deformation of the hierarchical structure. This means that in this case, the Poisson's ratio of the hierarchical system will not follow that of rotating rectangles model since the rectangles are deforming themselves.

At this point it is important to note that the results presented above relate only to a hinge governed by the friction torque, but the findings that the deformation pathway and mechnaical properties are affected by the properties of the hinges is a general result. As an example, a similar set of results was produced for structures in which the hinging process is governed by a harmonic potential rather than friction. The values of the stiffness constant K_h (associated with the harmonic potential) for these structures were set in the range between $2\,\mathrm{J{\cdot}rad^{-1}}$ and $120\,\mathrm{J{\cdot}rad^{-1}}$, while the same geometric parameters considered for the hinges governed by friction were used. From the results obtained, see Fig. 2(e)-(h), one can note that even though the deformation patterns are slightly different than it was the case for Fig. 2(a)-(d) (which stems from a different nature of the the hinging process), both of the systems lead to the same conclusions as analogical trends can be observed in both sets of figures.

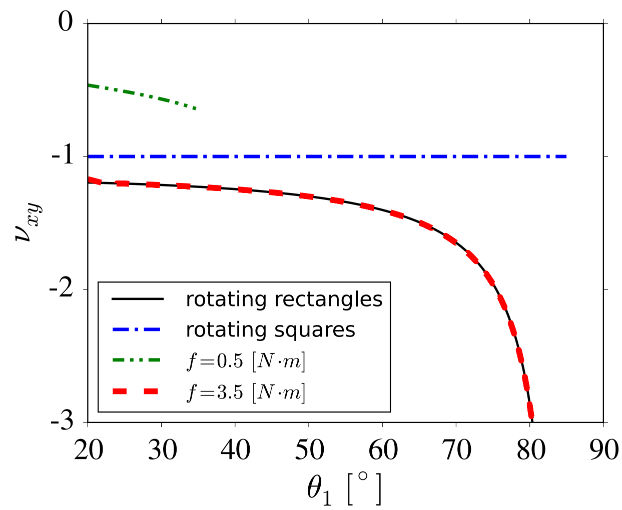

Figure 4. A plot showing a comparison of the Poisson's ratios obtained from the numerical solutions presented here for $f = 0.5\,\text{N}\cdot\text{m}$ and $f = 3.5\,\text{N}\cdot\text{m}$ with those calculated from analytical models for uni-level rotating rigid rectangle and square systems.

Based on Fig. 2(f), one can note that Level 0 starts closing during the process of deformation. This result is associated with the fact that the resultant restoring force (for this level) is greater than in the case of Level 1 (number of hinges is greater for Level 0 than it is the case for Level 1), hence it is more prone to exceed the torque corresponding to the external force. Another interesting result concerns the fact that all of the lines corresponding to the Poisson's ratio start at the same point which was not the case for the friction-based hinges discussed in this paper. This effect can be explained by the fact that in the case of the harmonic potential, at the time equal to zero, the term corresponding to the restoring torque (see Eq. 3 and Eq. 4) assumes the value of zero as the hinges are in their equilibrium states.

Also, although the results discussed here are specific to a particular geometry, the same trends in deforamtion and Poisson's ratios are expected to occur for other initial geometric conformations of the hierarchical system, as shown in the Supplementary Information. Furthermore, this result is expected to be valid for larger values of N_0. However, one may presume that as N_0 increases, Level 0 becomes increasingly rigid meaning that the Poisson's ratio would increasingly depend on the deformation of the Level 1 units. Moreover, as $N_0 \to \infty$, the geometry of Level 1 units resemble more closely that of a square and thus the Poisson's ratio of the system would approach -1 as expected for the rotating squares model[27].

All this is very significant, since the work presented here shows that it is possible to alter the deformation mechanism, and hence the mechanical properties, of a hierarchical rotating rigid unit system simply by changing the stiffness of the hinges in an even manner, i.e. in this case, simply by changing the magnitude of the friction torque of its hinges despite each hinge having the same f value. This property is not observed in currently known non-hierarchical rotating rigid unit systems and adds another element of versatility to this class of auxetic structures, which has already been shown through previous studies to possess the potential to exhibit a considerable range of mechanical properties through geometric variation alone[22,23,32]. This means that if, for example, one were to build a hierarchical system where the Level 0 squares are connected together through 'smart/intelligent' hinges with tuneable friction coefficients, one could achieve a considerable range of negative Poisson's ratio without altering the initial geometry of the system. Moreover, the examples discussed here only provide a glimpse of the true potential of these systems. A greater degree of versatility is envisaged if other geometries besides the rotating square motif are employed and if the number of hierarchical levels in the system is increased. This increased versatility could make these systems ideal for a number of niche applications such as smart filters, where the friction coefficients of 'intelligent' hinges may be customized according to the required pore sizes. This way, one filter may be used to filter a range of substances with different parameters, hence reducing material costs. Also, such a filter would be much easier to clean than a normal filter due to the adjustable pore size.

Conclusions

In conclusion, through a dynamics approach, a model was designed to predict the Poisson's ratio and quantify the relative rotations of the units at each hierarchical level for a hierarchical rotating rigid unit systems. It was shown that unlike unilevel systems, deformation behaviour and the Poisson's ratios (including the auxetic potential) of such hierarchical structures may be altered solely by changing the relative resistance to the rotational motion of the hinges of the systems. This contrasts sharply with the behaviour of other auxetic systems where, unless the geometry of the system is altered, changes in the mechanical properties can only be attained through changes in the interplay of different deformation mechanisms. This is very significant as it suggests that if one were to construct such a system through the use of hinges with tuneable friction torques, one may be able to control with ease the relative deformations of the various hierarchical levels and hence the overall macroscopic and mechanical properties of the system. It is hoped that this work will lead to further interest in the field of hierarchical auxetic systems and, in the future, even lead to the production of 'smart' hierarchical auxetic systems with tuneable deformation behaviour.

Methods

Calculation of the moment of inertia corresponding to the Level 0 and Level 1 of the system

The moment of inertia I_0 associated with the Level 0 of the structure can be considered as the sum of moments of inertia of all of the Level 0 squares rotating with respect to their centres. Assuming that the rotatong rigid units are made of a material having a surface density ρ, the moment of inertia of a single square can be expressed as $I_{sq} = \frac{1}{6}\rho a^4$. In such a case, I_0 can be calculated by means of the following expression: $I_0 = 16N_0^2 I_{sq}$.

The moment of inertia I_1, can be defined as four times the moment of inertia corresponding to the Level 1 building block ($I_{1,BB}$) rotating with respect to its centre of mass (there are 4 Level 1 building blocks). The moment of inertia $I_{1,BB}$ depends on the contribution from each of the Level 0 squares, constituting the Level 1 building block and the distance from the centre of mass of the Level 1 building block to the centre of the particular square within the considered Level 1 unit d_m. Thus, I_1 can be expressed as follows:

$$I_1 = 4I_{1,BB} = 4\left(\sum_{m}^{4N^2}\left(I_{sq} + \rho a^2 d_m^2\right)\right) \tag{6}$$

Numerical solutions

In order to investigate the deformation behaviour of the discussed system, the Newton's equation of motion proposed in Eq. 2 was written in the form of the following equations:

$$\frac{d\theta_{i,k}}{dt} = \omega_{i,k} \tag{7}$$

$$\frac{d\omega_{i,k}}{dt} = \left(8r_{i,k}F\sin\left(\sphericalangle\overrightarrow{r_{i,k}},\overrightarrow{F}\right) \mp n_i f\right) \Big/ I_{i,k} \tag{8}$$

where, $i = 0,1$ and k is a number of a given time step. Equations 7 and 8 were solved numerically by means of the classical fourth-order Runge-Kutta numerical algorithm[36], which is one of the most common methods of solving ordinary differential equations. The initial conditions concerning the value of $\theta_{i,0}$ and $\omega_{i,0}$ are provided in the Parameters subsection.

At each time step k the following conditions have to satisifed:

$$\theta_{1,k} > \theta_{0,k} \qquad \text{and} \qquad \varepsilon_k = \pi - \theta_{1,k} - \theta_{0,k} > 0 \qquad \text{and} \qquad L_{x,k+1} > L_{x,k} \tag{9}$$

Calculation of the Poisson's ratio

In general, for loading in the x direction, the Poisson's ratio can be expressed as follows:

$$\nu_{xy} = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} \tag{10}$$

where ε_{xx} and ε_{yy} are the strains in the x and y directions respectively. In case of this work, the value of ν_{xy} at the given time was established by means of the following formula:

$$\nu_{xy,k+1} = -\frac{L_{y,k+1} - L_{y,k}}{L_{y,k}} \Big/ \frac{L_{x,k+1} - L_{x,k}}{L_{x,k}} \tag{11}$$

Parameters

In order to investigate the deformation behaviour of the discussed system for loading in the x-direction, the structure consisting of $1 \times 1(N_0 \times N_0)$ Level 0 building blocks was used. The constants characterising this system were set as follows: $F = 500\,\mathrm{N}$, $l = 0.05\,\mathrm{m}$, $\rho = 3000\,\mathrm{kg{\cdot}m^{-2}}$ (density of material making up the Level 0 subunits), ω_1 and $\omega_0 = 0\,\mathrm{rad{\cdot}s^{-1}}$ (the initial angular velocity of the rotating units in the respective first and zeroth levels), $\Delta t = 10^{-7}\mathrm{s}$. In addition, the initial geometric parameters of the system were set as: $\theta_1 = 20^\circ$, $\theta_0 = 10^\circ$, which in case of $N_0 = 1$ leads to, $L_x = 0.248\,\mathrm{m}$ and $L_y = 0.234\,\mathrm{m}$. Furthermore, in order to show how does the deformation of the system change upon chaning the value of f, f was set to be equal to $\{0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.5\}\,\mathrm{N{\cdot}m}$ respectively.

In the case of the structure constructed by means of hinges governed by the harmonic potential, all of the parameters with exception for the K_h variable remain the same. In the considered cases K_h assumes the values of $\{2, 10, 20, 40, 80, 120\} J\,rad^{-1}$.

References

1. Wojciechowski, K.W., & Branka, A.C. Negative Poisson's ratio in a two-dimensional "isotropic" solid. *Phys. Rev. A* **40**, 7222 (1989)
2. Evans, K.E., Nkansah M.A., & Hutchinson I.J. Molecular network design. Nature **353**, 124 (1991)
3. Baughman, R.H., Shacklette, J.M., Zakhidov, A.A., & Stafstrom, S. Nagative Poisson's ratios as a common feature of cubic matals. *Nature* **392**, 362-364 (1998)
4. Alderson, A., & Evans, K.E. Molecular Origin of Auxetic Behaviour in Tetrahedral Framework Silicates. *Phys. Rev. Lett.* **89**, 225503 (2002)
5. Wojciechowski, K.W. Remarks on "Piosson ratio beyond the limits of the elasticity theory" *J. Phys. Soc. Jpn.* **72**, 1819-1820 (2003)
6. Sparavigna, A. Phonons in conventional and auxetic honeycomb lattices *Phys. Rev. B* **76**, 134302 (2007)
7. Spadoni, A., Ruzzene, M., Gonella, S., & Scarpa F. Phononic properties of hexagonal chiral lattices *Wave Motion* **46**, 435-450 (2009)
8. Chen, L., Liu, C., Wang, J., Zhang, W., Hu, C., & Fan, S. Auxetic materials with large negative Poisson's ratios based on highly oriented carbon nanotube structures *Appl. Phys. Lett.* **94**, 253111 (2009)
9. Li, D., Jaglinski, T., Stone, D.S., & Lakes, R.S. Temperature insensitive negative Poisson's ratios in isotropic alloys near a morphotropic phase boundary *Appl. Phys. Lett.* **101**, 251903 (2012)
10. Lv, C., Krishnaraju, D., Konjevod, G., Yu, H., & Jiang, H. Origami based Mechanical Metamaterials *Sci. Rep.* **4**, 5979 (2014)
11. Shin, D., Urzhumov, Y., Lim, D., Kim, K., & Smith D.R. A versatile smart transformation optics device with auxetic elasto-electromagnetic metamaterials *Sci. Rep.* **4**, 4084 (2014)
12. Zhang, L.C. *et al.* Tinselenidene: a Two-dimensional Auxetic Material with Ultralow Lattice Thermal Conductivity and Ultrahigh Hole Mobility *Sci. Rep.* **6**, 19830 (2016)
13. Bertoldi, K., Reis, P.M., Willshaw, S., & Mullin, T. Negative Poisson's Ratio Behaviour Induced by an Elastic Instability *Adv. Mater.* **22** 3, 361-366 (2010)
14. Florijn, B., Coulais, C., & van Hecke, M. Programmable Mechanical Metamaterials *Phys. Rev. Lett.* **113**, 175503 (2014)
15. Mizzi, L., Azzopardi, K.M., Attard, D., Grima, J.N., & Gatt, R. Auxetic metamaterials exhibiting giant negative Poisson's ratios *Phys. Status Solidi Rapid Res. Lett.* **9** 7, 425-430 (2015)
16. Cicala, G. *et al.* Hexachiral truss-core with twisted hemp yarns: Out-of-plane shear properties *Compos. Struct.* **94**, 3556-3562 (2012)
17. Buckmann, T. *et al.* Tailored 3D Mechanical Metamaterials Made by Dip-in Direct-Laser-Writing Optical Lithography *Adv. Mater.* **24**, 2710-2714 (2012)
18. Lakes, R. Materials with structural hierarchy *Nature* **361**, 511-515 (1993)
19. Sun, Y. & Pugno, N.M.In plane stiffness of multifunctional hierarchical honeycombs with negative Poisson's ratio sub-structures *Compos. Struct.* **106**, 681-689 (2013)
20. Oftadeh, R., Haghpanah, B., Vella, D., Boudaoud, A., & Vaziri, A. Optimal Fractal-Like Hierarchical Honeycombs *Phys. Rev. Lett.* 113, 104301 (2014)

21. Mousanezhad, D., Babaee, S., Ebrahimi, H., Ghosh, R., Hamouda, A. S., Bertoldi, K., & Vaziri, A. Hierarchical honeycomb auxetic metamaterials *Sci. Rep.* **5**, 18306 (2015)

22. Cho, Y. *et al.* Engineering the shape and structure of materials by fractal cut *Proc. Natl. Acad. Sci.* **111**, 17390-17395 (2014)

23. Gatt, R. *et al.* Hierarchical Auxetic Mechanical Metamaterials *Sci. Rep.* **5**, 8395 (2015)

24. Grima, J.N., Caruana-Gauci, R., Dudek, M.R., Wojciechowski, K.W., & Gatt, R. Smart metamaterials with tunable auxetic and other properties *Smart Mater. Struct.* **22** 8, 084016 (2013)

25. Dudek, M.R., Wojciechowski, K.W., Grima, J.N., Caruana-Gauci R., & Dudek, K.K. Colossal magnetocaloric effect in magneto-auxetic systems *Smart Mater. Struct.* **24** 8, 085027 (2015)

26. Attard, D., Manicar, E., Gatt, R., & Grima, J.N. On the properties of auxetic rotating stretching squares *Phys. Status Solidi B* 246, 2045-2054 (2009)

27. Grima, J.N., & Evans, K.E. Auxetic behaviour from rotating squares *J. Mater. Sci. Lett.* **19** 17, 1563-1565 (2000)

28. Grima, J.N., Alderson, A., & Evans, K.E. Auxetic behaviour from rotating rigid units *Phys. Status Solidi B* **242** 3, 561-575 (2005)

29. Gibson, L.J., Ashby, M.F., Schajer G.S., & Robertson, C.I. The mechanics of Two-Dimensional Cellular Materials *Proc. R. Soc. A* **382** 1782, 25-42 (1982)

30. Evans, K.E., Alderson A., & Christian, F.R. Auxetic two-dimensional polymer networks. An example of tailoring geometry for specific mechanical properties *J. Chem. Soc., Faraday Trans.* **91**, 2671-2680 (1995)

31. Masters I.G., & Evans, K.E. Models for the elastic deformation of honeycombs *Comp. Struct.* **35** 4, 403-422 (1996)

32. Tang, Y. *et al.* Design of hierarchically cut hinges for highly stretchable and reconfigurable metamaterials with enhanced strength *Adv. Mater.* **27** 44, 7181-7190 (2015)

33. Kunin, V., Yang, S., Cho, Y., Deymier, P., Srolovitz, D.J. Static and dynamic elastic properties of fractal-cut materials *Extreme Mechanics Letters* **6**, 103-114 (2016)

34. Dudek, K.K., Attard, D., Caruana-Gauci, R., Wojciechowski, K.W., & Grima, J.N. *et al.* Unimode metamaterials exhibiting negative linear compressibility and negative thermal expansion *Smart Mater. Struct.* **25**, 025009 (2016)

35. Grima, J.N., Zammit, V., Gatt, R., Alderson, A., & Evans, K.E. Auxetic behaviour from rotating semi-rigid units *Phys. Status Solidi B* **244** 3, 866-882 (2007)

36. Burden, R.L., & Faires, J.D. Numerical Analysis 220 (PWS Publishers 1985)