EXPLANATORY UNIFICATION BY PROOFS IN SCHOOL MATHEMATICS

KOTARO KOMATSU, TARO FUJITA, KEITH JONES, NAOKI SUE

In a recent review of philosophical research on mathematical explanation, Mancosu (2015) explicates two major accounts of mathematical explanation within mathematics. One is Steiner's (1978) notion of *explanatory proofs*; the other is Kitcher's (1981, 1989) idea of *explanatory unification.* Steiner's model has been employed in the field of mathematics education research. When elaborating the functions of proof, mathematics education researchers have generally referred to Steiner's principle of distinguishing explanatory proofs from non-explanatory ones. Hanna (1990), for example, referred to Steiner's idea when introducing the explanatory function of proof.

In this article, we focus on the other major model, Kitcher's explanatory unification. Kitcher's model has been relatively neglected in mathematics education, and Hanna (2018) states that "Kitcher's model may not be as relevant to mathematics education as it might appear [...] because it is difficult to assign to unification per se [...] any degree of explanatory power in the cognitive sense" (p. 9). However, our premise in this article is that if Kitcher's idea of explanatory unification is regarded as being mainly related to *unification*, rather than to *explanation* [1], his idea is relevant to mathematics education, for two reasons. First, given that Kitcher proposed his model to provide insight into how scientists (including mathematicians) advance their research, introducing Kitcher's explanatory unification into school mathematics (especially in the teaching of proof and proving) is worthwhile because one of the goals of mathematics teaching is "for students to engage in an authentic way with proving as this activity is practiced in the mathematical community" (Stylianides, Stylianides & Weber, 2017, p. 247). Second, with unification being a process where one synthesises scattered facts into a coherent whole, such synthesis is not only recognised as advanced mathematical thinking (Dreyfus, 1991; Tall, 1991), but also classified as higher-order thinking in Bloom's taxonomy of educational objectives (Bloom, Engelhart, Furst, Hill, & Krathwohl, 1956); hence, the introduction of Kitcher's explanatory unification into school mathematics might offer learning opportunities for cultivating students' mathematical and higher-order thinking.

Despite the significance of Kitcher's model, it is only recently that mathematics education research has begun to pay attention to his idea of explanatory unification (Hanna, 2018; Raman-Sundström & Öhman, 2015; Stylianides, Sandefur & Watson, 2016). Furthermore, as existing studies do not illustrate explanatory unification, it is quite difficult to grasp the form of this activity in the context of school mathematics. In this article, we examine a classroom episode in a secondary school in order to address the question: What does explanatory unification by proofs look like in the context of school mathematics?

Kitcher's idea of explanatory unification

Kitcher (1989), in reviewing major accounts of scientific explanation, argues that a theory of explanation should show how scientific explanation advances human understanding and how disputes in past and present science can be comprehended and arbitrated. Kitcher takes mathematics into consideration, arguing that "mathematical knowledge is similar to other parts of scientific knowledge, and there is no basis for a methodological division between mathematics and the natural sciences" (Kitcher, 1989, p. 423). He also adds "the fact that the unification approach provides an account of explanation [...] in mathematics stands to its credit" (p. 437).

To explain the central ideas of Kitcher's explanatory unification, we begin with his illustration from the history of mathematics: the development of research on the solvability of equations. Ways to solve linear, quadratic, and some high-order equations have been known for quite some time. Indeed, the root formula for each class of equations up to and including degree four was recognised by at least the end of the 18th century. Nevertheless, rather than considering these four classes of equations separately, mathematicians desired to know systematically why these equations allow their roots to be expressed as rational functions of the coefficients. Insight into the structure underlying equations was provided partially by Lagrange's consideration of permutations of the roots of equations and ultimately by the development of Galois theory. Thanks to Galois, mathematicians acquired a fundamental criterion for judging whether the roots of equations can be expressed as rational functions of the coefficients, and this criterion can be applied to unify the four individual classes.

The above illustration captures one characteristic of Kitcher's idea of scientific explanation: Kitcher's focus is not on an individual argument for a single statement (*e.g.,* a proof for a statement about cubic equations), but rather on *clusters* of arguments for multiple statements (*e.g.,* a cluster of proofs for statements about equations up to and including degree four) [2]. Actually, he contends that "science supplies us with explanations whose worth cannot be appreciated by considering them one-by-one but only by seeing how they form part of a systematic picture of the order of nature" (p. 430). In this sense, Kitcher's account can be regarded as a model for unification of multiple statements rather than a model for explanation of a single statement such as Steiner's notion of explanatory proofs.

In summary, the following statement by Kitcher illustrates very clearly his view of explanatory unification [3]:

> Understanding the phenomena is not simply a matter of reducing the "fundamental incomprehensibilities" but of seeing connections, common patterns, in what initially appeared to be different situations. Here the switch in conception from premise-conclusion pairs to derivations proves vital. *Science advances our understanding of nature by showing us how to derive descriptions of many phenomena, using the same patterns of derivation again and again, and, in demonstrating this, it teaches us how to reduce the number of types of facts we have to accept as ultimate.* (Kitcher, 1989, p. 432)

We apply this notion of explanatory unification to the context of proving in school mathematics and define explanatory unification of statements by clusters of proofs as follows: a cluster of proofs can be regarded as providing explanatory unification of statements if: EU1) there initially appeared to be different sets of statements and proofs, EU2) one constructs the cluster of proofs through seeing a connection in these sets and showing the statements by using the same proof pattern repeatedly, EU3) the cluster of proofs can reduce the number of facts that one has to use. In what follows, we illustrate how explanatory unification can be used to deepen understanding of students' proving processes by analysing an episode from a secondary school mathematics classroom.

A classroom episode

We examine an episode from a classroom consisting of 39 eighth-grade students (13–14 years old) in a Japanese lower secondary school affiliated with a national university. The students tackled a series of tasks involving the properties of 'star polygons' over four 50-minute lessons. These lessons were originally designed as proof lessons rather than a way of examining Kitcher's idea of explanatory unification. In fact, we were not aware of his idea before designing the lessons. It was through our subsequent analysis of the lessons that we came to realise that Kitcher's model might be helpful for understanding students' thinking process.

The fourth author of this article implemented the lessons in his classroom. While he was not specifically aware of Kitcher's research at that time, he was an expert teacher with 16 years' teaching experiences across several schools and with the knowledge that the star polygon tasks would provide a very useful learning experience for students.

The mathematical capabilities of the students were above average for Japan. For data analysis, we used the video records of the lessons, the transcripts of the records, the students' worksheets, and field notes taken by the first author. We used these data to examine how the students proved the different statements that they produced. In particular, we identified two different proof clusters for the problem and found that one of those was able to unify the different statements more generally and hence satisfied our criteria for explanatory unification proposed as EU1–3 above. In the following sections, we first describe what happened in the classroom and then show how a proof produced by a student was used to prove the different statements shared in the lessons. The transcripts have been translated from the original Japanese by the authors. All students' names are pseudonyms.

Introduction of star pentagon

In the first lesson, the students tackled a task that involved finding the sum of the interior angles of a star pentagon. After individual work and small-group discussion, several solutions were shared in whole-class discussion (*e.g.,* Figures 1a–c) [4]. A student, Misaki, proved that the sum of the interior angles of a star pentagon, shown in Figure 1a, was 180° by considering a concave quadrilateral ACEP and a triangle BDP. She showed that ∠APE = $a + c + e$ using the property of concave quadrilaterals, something which had been proved in an earlier lesson, and ∠APB = $b + d$ using the property of the interior angles and exterior angles of triangles. Regarding Figure 1b, another student, Aoi, showed that ∠DFG = $a + c$ and ∠DGF = $b + e$ by considering triangles ACF and BGE, respectively, and, by focusing on triangle DFG, demonstrated the same conclusion as Misaki. After that, a third student, Riko, presented her idea on the blackboard (Figure 1c) [5].

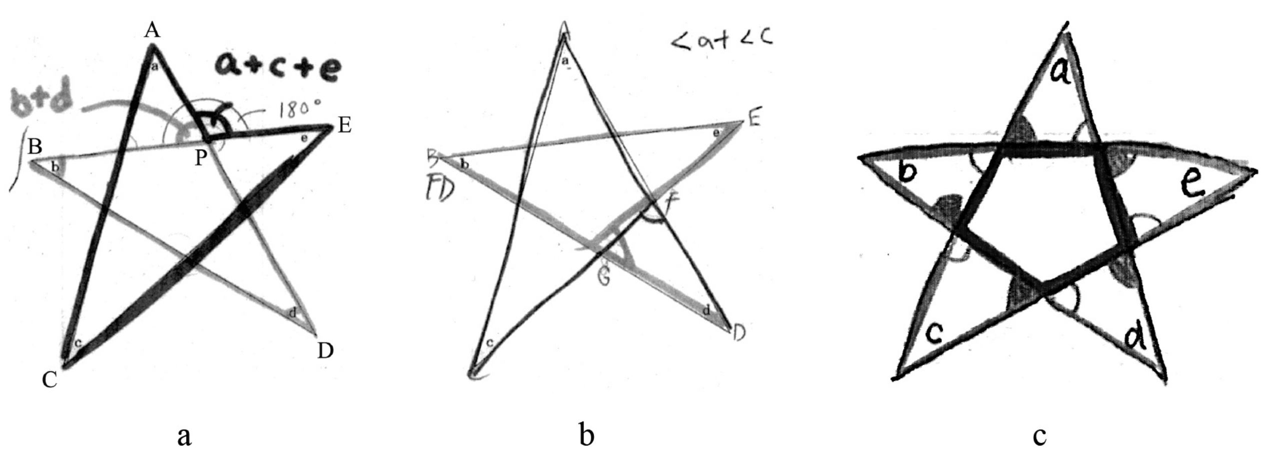


Figure 1. Star pentagons.

Riko There are five triangles including ∠a ~ ∠e. 180° × 5 = 900°. The sum of the exterior angles of the pentagon is 360° × 2 = 720°. 900° – 720° = 180°.

Riko's idea was complicated for the other students, but they gradually came to understand it through whole-class discussion facilitated by the class teacher as follows:

Teacher It was difficult to understand [*Riko's proof*]. I want other students to clarify. Can anybody understand her proof?

Shun I think, triangles including angles *a, b, c, d*, and *e* mean dividing the star pentagon into a pentagon and triangles. [...] I think she [*Riko*] considered the pointy triangles in the tops, and she thought that the sum of the interior angles of the five [*triangles*] was 900.

Teacher Riko, is it OK? [*Riko nodded.*] Next, she wrote "The sum of the exterior angles of the triangle [*The teacher misspoke here; Riko wrote 'pentagon'*] is 360° × 2 = 720°". Can anyone also understand what this means?

Takumi The sum of the exterior angles of the pentagon is 360, but this is only this part [*the black-marked angles in Figure 1c*]. In order to include the opposite one [*the white-marked angles in Figure 1c*], multiply 360 by 2. […] And the whole sum is 720°.

Teacher Lastly, why did she subtract 720 from 900? Can anyone clarify?

Shota: Well, 900° was shown initially, but this is the sum of the interior angles of the whole triangles. After that, 720° was obtained from the sum of the exterior angles of the pentagon, and these are angles subtracted from the whole triangles. […] Then, we can find the angles from *a* to *e*, and thus the answer.

As can be seen in the above exchange, Riko first obtained the sum of the interior angles of the five 'outside triangles'. However, these angles include extra angles that can be divided into the black- and white-marked angles in Figure 1c, each of which are equal to the sum of the exterior angles of the 'inside pentagon'. Hence, she calculated '180° × 5 – 360° × 2' and thus proved that the sum of the interior angles of a star pentagon is 180°.

Investigations of 'star-even polygons'

After investigating interior angles of the star pentagon in the first lesson, the teacher began the second lesson by inviting the students to work on increasing the number of the vertices of the polygon. Here, to provide a starting point for the students, a star polygon was defined as *a polygon constructed by connecting vertices with skipping the adjacent vertex.* Through drawing various star polygons using this definition, the students noticed that star polygons where the numbers of the vertices were odd (hereafter, star-odd polygons) could be drawn in one stroke, whereas star polygons where the numbers of the vertices were even (star-even polygons) could not, but could be drawn by a combination of two polygons.

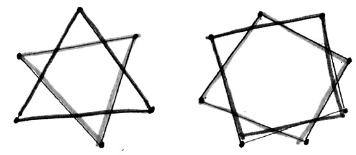


Figure 2. Star hexagon and octagon.

At this point the teacher focused the students' attention on star-even polygons (Figure 2), and the students proved that the sum of the interior angles of a star hexagon was 360° because a star hexagon consisted of two triangles (180° × 2). Similarly, they used the interior angle sum theorem of polygons, 180° × (n – 2), to prove that the sums of the interior angles of a star octagon and a star decagon were 720° (= 360° × 2) and 1080° (= 540° × 2), respectively.

Investigations of star-odd polygons

In the third lesson, the teacher used a table similar to Table 1 to summarise the results obtained in the first and second lessons. The students were invited to consider the table and they conjectured that the sums of the interior angles of a star heptagon and a star nonagon would be 540° and 900°, respectively. Then, the students individually or collaboratively attempted to prove these conjectures. In the subsequent whole-class discussion, two solutions to the star heptagon case were shared (Figures 3a and 3b).

Table 1. Summary of the first and second lessons.

The number of vertices	5	6	7	8	9	10	…
The sum of interior angles	180°	360°	?	720°	?	1080°	…

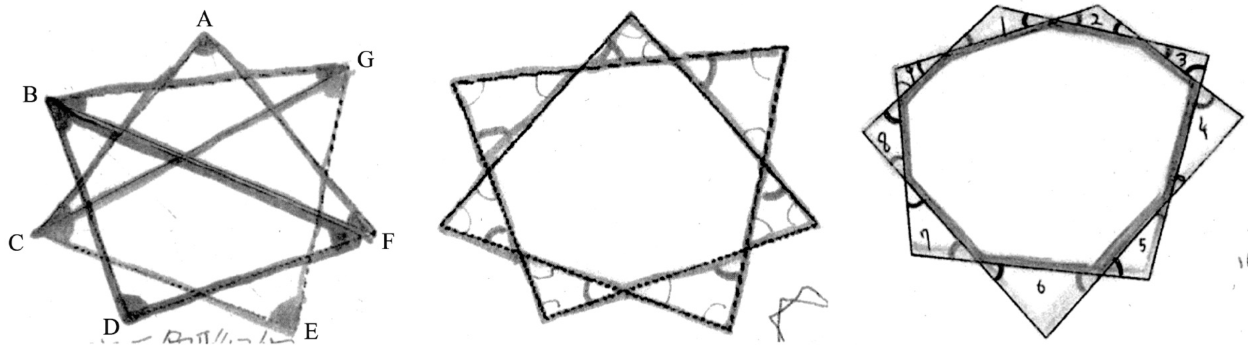
In Figure 3a, student Kenta divided the star heptagon into star pentagon ABCFG and triangles BDF and CEG and calculated '180° + 180° × 2 = 540°'. Regarding Figure 3b, Kaito described his idea as follows:

Kaito I applied […] the same method that Riko used for a star pentagon. Seven triangles including all vertices [...] and all of the sums of the interior angles. Then, I used the property of the sum of exterior angles in order to remove angles that were not the angles of the vertices. […] I got 1260° as the sum of the interior angles of the seven [*outside triangles*], and 720° is extra angles, so the answer is 540°.

Teacher Well, Kaito, what do you mean by the property of exterior angles?

Kaito The sum of the exterior angles of any polygon is 360°. [...] The triangles include extra angles, and all of them are the exterior angles. There are two sets of the exterior angles, so I multiplied 360° by 2.

As can be seen, Kaito proved that the sum of the interior



a b c

Figure 3. Star heptagons and nonagon.

angles of a star heptagon is 540° by adopting Riko's approach to the star pentagon in the first lesson (Figure 1c). Kaito focused on the sum of the interior angles of the seven 'outside triangles' and found that the extra angles consisted of two sets of the exterior angles of the 'inside heptagon'. Then, he concluded his answer by calculating '180° × 7 (outside triangles) – 360° × 2 (the two sets of exterior angles)'.

Another student, Miwa, suggested that the same approach could be applied to the case of star nonagons and proved that the sum of the interior angles of a star nonagon is 900° by calculating '180° × 9 (outside triangles) – 360° × 2 (the two sets of exterior angles)' (Figure 3c).

Star-even polygons revisited for unification

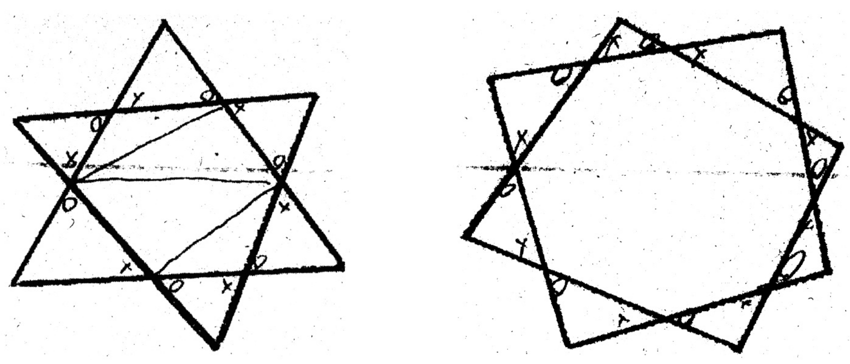
As the students' approach to star-even polygons (in the second lesson) was different to their approach to star-odd polygons (in the third lesson), the teacher began the fourth lesson by encouraging the students to look back at their investigations regarding star-even polygons. He posed the question of whether Riko's approach (Figure 1c) could be applied to the case of star-even polygons. The students individually or collaboratively worked on this task by taking star hexagons and octagons as examples. They then shared their ideas in a whole-class discussion.

Student Daiki, for instance, expressed his idea regarding the star octagon case as follows (Figure 4b):

Daiki A star octagon consists of eight stars, oops, eight triangles and a single octagon. [...] Because the interior angles of a triangle add to 180° and there are eight, 1440°. Because the sum of the exterior angles of the decagon [*He misspoke, meaning 'octagon'*] is 360 times 2 and 720, and because the angles, except for the top vertices of the triangles, are extra sides and angles, I subtracted the sum of the exterior angles from 1440, and got 720°.

In his account, Daiki employed Riko's approach shared in the first and third lessons and proved that the sum of the interior angles of a star octagon is 720°. Thus, with the encouragement of the teacher, the students found that the approach to star-odd polygons was also applicable to star-even polygons and thereby unified their approaches to these two types of star polygons.

After this, student Masaru reflected on what they had done since the first lesson and commented that there was a general pattern; when the number of the vertices of a star polygon increases by one, the sum of the interior angles increases by 180°. The teacher responded to this observation by inviting other students to explain why this pattern was true. After individual work and small-group discussion, student Kenta wrote his ideas on the blackboard to share with the class:



a b

Figure 4. Star hexagon and octagon revisited.

> The number of triangles of a star polygon is equal to the number of vertices. The sum [*of the interior angles of a star polygon*] can be found by 'the sum of the interior angles of the triangles – two sets of the sum of the exterior angles'. Because the two sets of the sum of the exterior angles are always constant, [*the sum of the interior angles of a star polygon*] increases by the interior angles of the triangles.

This shows genuine cognitive progress whereby the student explained *why* the general pattern—the sum of the interior angles of a star polygon increases by 180° if the number of the vertices increases by one—is true, based on the increase in the number of outside triangles and the constancy of the sum of exterior angles in any polygons. Subsequently another student, Aoi, agreed with Kenta's idea and shared her algebraic formula $y = 180x - 720 = 180(x - 4)$ (where y is the sum of the interior angles of a star polygon and x is the number of the vertices of the star polygon).

Discussion

In the classroom episode described in this article, the students constructed various proofs for different statements; this included the sum of the interior angles of a star pentagon is 180°, the sum of the interior angles of a star hexagon is 360°, and so on. These students' proofs can be divided into two clusters in terms of the sequence of the lessons. The first cluster of proofs consists of those constructed up to the third lesson, in which the students performed analysis according to two cases, whether the number of the vertices of a star polygon was even (Figure 2) or odd (Figures 3b–c). In this analysis star-odd polygons and star-even polygons were considered to be different cases and hence not unified. A unification was subsequently achieved by the second cluster of proofs, namely the proofs constructed in the first, second and fourth lessons, in which the students employed Riko's original approach to the star pentagon (Figure 1c) for not only star-odd polygons (Figures 3b–c), but also star-even polygons (Figures 4a–b).

The second cluster of proofs derived from Riko's approach can be regarded as providing explanatory unification of statements about star polygons according to our definition shown in the earlier section. Until the third lesson, the students considered that there were two different sets of statements and proofs; one set of statements and proofs was related to star-even polygons, while the other set was related to star-odd polygons. These two sets were different for the students as they performed the aforementioned case analysis (EU1). In the fourth lesson (involving the second proof cluster), however, the students saw a connection in these sets: every star polygon could be regarded as consisting of an inside polygon and outside triangles. They thereby proved the statements about both star-odd and star-even polygons by employing Riko's approach repeatedly (EU2). Furthermore, three theorems were used until the third lesson: the interior angle sum theorem of polygons in star-even polygons, the interior angle sum theorem of triangles in star-odd polygons, and the exterior angle sum theorem of polygons in star-odd polygons. By using only the second and last theorems, the second proof cluster successfully reduced the number of facts that the students needed to use (EU3) [6]. All of these attributes regarding the second proof cluster are consistent with Kitcher's notion of explanatory unification [7].

Mancosu (2015) regards Steiner's model of explanatory proofs and Kitcher's model of explanatory unification as two major accounts of mathematical explanation within mathematics. A relationship between these two models can be seen in the classroom episode described in this article. The students created the general statement that the sum of the interior angles of a star polygon is $180n - 720$ (where n is the number of the vertices of the star polygon) through the unification of different proofs. They explained why this statement, derived from the unification, is true by using one of the characterising properties (Steiner, 1978) of star polygons: a star polygon can be regarded as consisting of the 'inside polygon' and 'outside triangles'. The students explained why the sum of the interior angles of a star polygon increases by 180° if the number of the vertices increases by one, by referring to the increase in the number of outside triangles and the constancy of the sum of exterior angles in any polygons (Kenta's explanation in the fourth lesson). They provided these explanations by producing the second cluster of proofs through explanatory unification of statements regarding individual star polygons (*e.g.,* pentagons, hexagons, etc.), and encapsulating this cluster into a single comprehensive proof dealing with star polygons in general. In summary, the relationship between explanation and unification observed in this classroom episode is that the students devised the explanatory proof of the general statement through the unification of the individual statements/proofs.

As the students (*e.g.,* Kenta and Aoi) devised and proved a general formula for the sums of the interior angles of star polygons, explanatory unification can be regarded as representing a specific type of generalisation. However, not all generalisation leads to explanatory unification: it is possible to produce generalisations without experiencing different sets of statements and proofs and without reducing the number of facts used (for more on the wider role of generalisation, see Mason, 2002, and Rivera, 2013). Some forms of generalisation do not meet the conditions of EU1–3, and such a process is not related to explanatory unification.

In the introduction to this article, we referred to the advantages of explanatory unification in terms of authentic mathematical practice as well as mathematical and higher-order thinking. The classroom episode of this article shows another benefit of this activity, namely fostering mathematical understanding. According to the recursive theory of mathematical understanding by Pirie and Kieren (1994), *folding back* plays a vital role in deepening understanding:

> When faced with a problem or question at any level, which is not immediately solvable, one needs to *fold back* to an inner level in order to extend one's current, inadequate understanding. This returned-to, inner level activity, however, is not identical to the original inner level actions; it is now informed and shaped by outer level interests and understandings. Continuing with our metaphor of folding, we can say that one now has a 'thicker' understanding at the returned-to level. This inner level action is part of a recursive reconstruction of

knowledge, necessary to further build outer level knowing. Different students will move in different ways and at different speeds through the levels, folding back again and again to enable them to build broader, but also more sophisticated or deeper understanding. (Pirie & Kieren, 1994, p. 69)

This folding-back process can be observed in the classroom episode in this article. In the fourth lesson, the students returned to the case of star-even polygons. This reflection was not identical to their original consideration in the second lesson, but was shaped by a different interest that encouraged the students to examine whether their approach to star-odd polygons was applicable to star-even polygons. The students thereby recursively reconstructed their knowledge of star-even polygons and unified star-even and star-odd polygons. This unification enabled the students to reach a more sophisticated level where they not only verified *that* the statement—the sum of the interior angles of a star polygon increases by 180° if the number of the vertices increases by one—was true, but also understood *why* this statement was true (Hanna, 1995). Furthermore, the students invented the algebraic formula $y = 180(x - 4)$ [8]. Thus, explanatory unification can be an effective activity for fostering student mathematical understanding.

Although we did not design the lessons according to Kitcher's idea of explanatory unification, the episode described in this article implies that explanatory unification activity could not be achieved without the intentional design of task sequences based on teachers' knowledge of the subject matter. For example, after the star pentagon task, one possibility for a subsequent task might be one investigating other star polygons without dividing them into even and odd cases. However, the teacher in this episode initially focused the students' attention on star-even polygons; this decision was derived from his anticipation that the students would notice that star polygons can be divided into two types according to the possibility of one-stroke drawing, and from his knowledge that it is more straightforward to examine star-even than star-odd polygons. After the star-even polygon task and then the star-odd polygon task, the teacher set up another task revisiting star-even polygons, based on his additional knowledge that the students' approach to star-odd polygons can be applied to star-even polygons. As a result, this task sequence enabled the students to engage in explanatory unification. The curriculum the students had learnt seems to be another factor supporting the achievement of this activity; the students had sufficient knowledge of the subject matter related to the tasks, such as knowledge about the exterior angles of a polygon. Thus, purposeful design of task sequences (and curriculum) is essential for engaging students in explanatory unification.

Conclusions

We believe it is worthwhile to introduce explanatory unification activity into school mathematics and to consider how to better support students in this activity. We do not think that explanatory unification can be achieved in the case of all tasks; purposeful design of task sequences is necessary. In this sense, we agree with Sierpinska (2004) who considers "the design, analysis and empirical testing of mathematical tasks, whether for the purposes of research or teaching, as one of the most important responsibilities of mathematics education" (p. 10). Thus, future research should address task design for explanatory unification. Examples of such tasks would be those related to finding a formula for $1 + 2 + 3 + \ldots + n$ and to the property of the sums of consecutive natural numbers (*e.g.,* $3 + 4 + 5$ and $8 + 9 + 10 + 11$), because, like the classroom episode in this article, some students may tackle these tasks by case analysis according to whether the number of natural numbers is odd or even. More general direction for task design for explanatory unification is implied in the episode in this article. The students' activity consisted of (1) obtaining several statements, (2) employing proof ideas to prove the statements, and (3) looking back at their own investigations and realising that one of the proof ideas was enough to prove the statements. Another possibility for (3) is that one may devise a fresh proof idea that can be employed for all the statements. Task sequences eliciting this process would enable students to experience explanatory unification by proofs.

Notes

[1] Kitcher himself considers explanation in terms of unification, as he states that "to explain is to fit the phenomena into a unified picture insofar as we can" (Kitcher, 1989, p. 500).

[2] Such argument clusters are often provided by building theories (e.g., Galois theory and Newton's gravitational theory).

[3] Kitcher introduces several technicalities to elaborate his theory of explanatory unification (*e.g.,* general argument patterns consisting of schematic arguments, filling instructions, and classifications). However, we do not deal with them in this article because this quote provides a sufficient characterisation of explanatory unification for our purposes.

[4] All the diagrams shown in this article are taken from the students' worksheets. We added the labels in Figures 1a and 3a for clarification.

[5] The transcripts have been translated from Japanese and edited for clarity. Editorial omissions are shown by […].

[6] Logically speaking, it is sufficient to accept only the interior angle sum theorem of triangles as a basis, because this theorem can be used for deriving the other two theorems. However, the students had learnt this logical relationship long before, and they were not conscious of it in the lessons described in this article.

[7] We do not intend to argue that the second proof cluster gives the absolute explanatory unification. There are other arguments that can deal with more general star polygons (which are defined as polygons constructed by connecting the vertices with skipping the next k vertices).

[8] This invention of the algebraic formula was achieved by the second cluster of proofs and thus is relevant to the discovery function of proofs (de Villiers, 1990; for a recent review of the discovery function, see Komatsu, Tsujiyama, & Sakamaki, 2014).

Acknowledgements

We wish to express our thanks to Isao Ohira for providing us with an inspiration for the lessons about star polygons and to the anonymous reviewers for their helpful comments on earlier versions of this article. This study is supported by the Japan Society for the Promotion of Science (Nos. 15H05402, 16H02068, and 26282039).

References

Bloom, B.S., Engelhart, M.D., Furst, E.J., Hill, W.H. & Krathwohl, D.R. (1956) *Taxonomy of Educational Objectives, Handbook 1: The Cognitive Domain.* New York: David McKay Company.

De Villiers, M. (1990) The role and function of proof in mathematics. *Pythagoras* **24**, 17–24.

Dreyfus, T. (1991) Advanced mathematical thinking processes. In Tall, D. (Ed.) *Advanced Mathematical Thinking,* pp. 25–41. Dordrecht: Kluwer.

Hanna, G. (1990) Some pedagogical aspects of proof. *Interchange* **21**(1),

6-13.
Hanna, G. (1995) Challenges to the importance of proof. *For the Learning of Mathematics,* **15**(3), 42-49.
Hanna, G. (2018) Reflections on proof as explanation. In Stylianides, A.J. & Harel, G. (Eds.) *Advances in Mathematics Education Research on Proof and Proving: An International Perspective (ICME-13 Monographs)* pp. 3-18. New York: Springer.
Kitcher, P. (1981) Explanatory unification. *Philosophy of Science* **48**(4), 507-531.
Kitcher, P. (1989) Explanatory unification and the causal structure of the world. In Kitcher, P. & Salmon, W.C. (Eds.) *Scientific Explanation,* pp. 410-505. Minneapolis, MN: University of Minnesota Press.
Komatsu, K., Tsujiyama, Y. & Sakamaki, A. (2014) Rethinking the discovery function of proof within the context of proofs and refutations. *International Journal of Mathematical Education in Science and Technology* **45**(7), 1053-1067.
Mancosu, P. (2015) Explanation in mathematics. In Zalta, E.N. (Ed.) *The Stanford Encyclopedia of Philosophy.* https://plato.stanford.edu/entries/mathematics-explanation
Mason, J. (2002) Generalisation and algebra: exploiting children's powers. In Haggarty, L. (Ed.) *Aspects of Teaching Secondary Mathematics: Perspectives on Practice,* pp. 105-120. London: Routledge Falmer.
Pirie, S. & Kieren, T. (1994) Growth in mathematical understanding: how can we characterise it and how can we represent it? *Educational Studies in Mathematics* **26**(2), 165-190.
Raman-Sundström, M. & Öhman, L.D. (2015) Mathematical fit: a first approximation. In Krainer, K. & Vondrová, N. (Eds.) *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education,* pp. 185-191. Prague: Charles University.
Rivera, F. (2013) *Teaching and Learning Patterns in School Mathematics: Psychological and Pedagogical Considerations.* Dordrecht: Springer.
Sierpinska A. (2004) Research in mathematics education through a keyhole: task problematization. *For the Learning of Mathematics* **24**(2), 7-15.
Steiner, M. (1978) Mathematical explanation. *Philosophical Studies* **34**(2), 135-151.
Stylianides, G.J., Sandefur, J. & Watson, A. (2016) Conditions for proving by mathematical induction to be explanatory. *Journal of Mathematical Behavior* **43**, 20-34.
Stylianides, G.J., Stylianides, A.J. & Weber, K. (2017) Research on the teaching and learning of proof: taking stock and moving forward. In Cai, J. (Ed.) *Compendium for Research in Mathematics Education,* pp. 237-266. Reston, VA: National Council of Teachers of Mathematics.
Tall, D. (1991) The psychology of advanced mathematical thinking. In Tall, D. (Ed.) *Advanced Mathematical Thinking,* pp. 3-21. Dordrecht: Kluwer.

This photo was provided by David Wagner of the University of New Brunswick.