



Properties of the Conditionally Filtered Equations: Conservation, Normal Modes, and Variational Formulation

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Conditionally filtered equations have recently been proposed as a basis for modelling the atmospheric boundary layer and convection. Conditional filtering decomposes the fluid into a number of categories or components, such as convective updrafts and the background environment, and derives governing equations for the dynamics of each component. Because of the novelty and unfamiliarity of these equations, it is important to establish some of their physical and mathematical properties, and to examine whether their solutions might behave in counter-intuitive or even unphysical ways. It is also important to understand the properties of the equations in order to develop suitable numerical solution methods. The conditionally filtered equations are shown to have conservation laws for mass, entropy, momentum or axial angular momentum, energy, and potential vorticity. The normal modes of the conditionally filtered equations include the usual acoustic, inertio-gravity, and Rossby modes of the standard compressible Euler equations. In addition, they possess modes with different perturbations in the different fluid components that resemble gravity modes and inertial modes but with zero pressure perturbation. These modes make no contribution to the total filter-scale fluid motion, and their amplitude diminishes as the filter scale diminishes. Finally, it is shown that the conditionally filtered equations have a natural variational formulation, which can be used as a basis for systematically deriving consistent approximations.

Key Words: Approximate equations, Conditional average, Convection, Hamilton's principle

Received ...

1. Introduction

Conditionally filtered equations have recently been proposed as a basis for mathematical and numerical modelling of the atmospheric boundary layer and convection (Thuburn *et al.* 2018). Conditional filtering itself is an extension of coarse-graining ideas that are commonly used in large-eddy turbulence modelling, and that enable one to write down equations of motion valid

19 for a particular scale of motion, with the subgrid-scale terms then appearing on the right-hand
20 side and in need of parameterization – see Leonard (1975), Frisch (1995) and Aluie *et al.* (2018)
21 for a range of examples. The conditionally filtered equations extend this idea so that prognostic
22 equations can be constructed for particular fluid types as well as particular scales, for example for
23 different ‘components’ of the fluid, such as convective updrafts, downdrafts, and the background
24 environment. These prognostic equations may then be solved in a numerical model, even when the
25 individual convective updrafts and downdrafts are too small-scale to be resolved.

26 The conditionally filtered equations provide a natural way of representing qualitatively quite
27 different types of small-scale physical process within the same mathematical framework. For
28 example, local turbulent fluxes might be represented by right-hand side subgrid terms as an eddy
29 diffusion, while fluxes associated with coherent structures such as deep boundary layer thermals
30 or convective updrafts might be represented by one of the fluid components whose dynamics is
31 explicitly represented by the left-hand side terms. (See (1)-(5) and figure 1 below.) By making certain
32 approximations to the conditionally filtered equations and certain choices for the parameterized
33 terms, they can be shown to reduce to a typical mass flux convection scheme, or to a typical eddy
34 diffusion scheme, coupled to resolved-scale dynamics. Thus, the conditionally filtered equations could
35 provide a useful and self-consistent basis for improving the coupling of different parameterization
36 schemes with each other and with the resolved dynamics, or for building unified parameterization
37 schemes that can smoothly transition between different regimes, for example between a dry
38 convective boundary layer and shallow convection. A particular motivation for us is the possibility
39 of extending the dynamical core of a weather or climate model to solve the left-hand sides of
40 the conditionally filtered equations for all fluid components, thus explicitly capturing some of the
41 dynamics of convection. Ultimately we wish to explore the potential of this approach to improve
42 some of the well-known modelling problems in convection-dynamics coupling, including memory
43 of the dynamical state of convection, the propagation of convective systems to neighboring grid
44 columns, and the horizontal location of compensating subsidence. These motivations are discussed
45 in more detail by Thuburn *et al.* (2018).

46 Similar ideas, leading to prognostic equations for multiple fluid components, may be found in the
47 work of Yano *et al.* (2010), Yano (2012), and in the prognostic cloud scheme of Randall and Fowler
48 (1999). The conditionally filtered approach, however, is more systematic and leads to consistent
49 prognostic equations for all the dynamical variables as well as thermodynamic variables and
50 component volume fractions. Similar equation sets are also used for modelling multi-phase flows
51 in engineering applications (e.g. Drew 1983; Abgrall and Karni 2001). The conditionally filtered
52 compressible Euler equations are given in section 2 below.

53 The right-hand sides of the conditionally filtered equations represent a range of important,
54 subgrid-scale physical processes such as local turbulent fluxes and entrainment and detrainment.
55 The eventual applications envisaged for the conditionally filtered equations will depend critically
56 on the choices made to parameterize these terms. The focus of the present paper, however, is on the
57 left-hand sides, which represent a modified form of the resolved-scale dynamics of the compressible
58 Euler equations. Complex models, built from multiple components which are themselves complex,
59 can behave in unexpected and unphysical ways if the individual components are not sufficiently
60 well understood and well behaved (see e.g. Gross *et al.* 2017, for some examples). This motivates
61 us to analyse and document some of the physical and mathematical properties of the conditionally
62 filtered equations when their right-hand sides are zero. We consider this an important preliminary
63 before attempting to increase the complexity of the system by coupling to parameterized right-hand
64 side terms. It is also important to understand the properties of the equations in order to develop
65 suitable numerical solution methods. This paper examines their conservation properties and normal
66 modes, and presents a variational formulation.

67 Conservation properties are fundamental properties of a physical system, and respecting relevant
 68 conservation properties is widely regarded as essential in any mathematical model. Budgets
 69 of conserved quantities can help to understand physical mechanisms (e.g. Hoskins *et al.* 1985;
 70 Peixoto and Oort 1992; Pauluis and Held 2002), and respecting conservation properties in numerical
 71 models can help to ensure their stability and accuracy (e.g. Thuburn 2008, and references therein).
 72 Section 3 discusses conservation of mass, entropy, momentum, energy, and potential vorticity for
 73 the conditionally filtered equations.

74 The conditionally filtered equations have a rather unusual structure, with separate density,
 75 entropy, and velocity fields for each fluid component, but a single common pressure field (section 2).
 76 This raises the question of what types of motion the equations might support; these might be
 77 counter-intuitive or even unphysical. One way to address this question is to examine the normal
 78 modes of the linearized equations (e.g. Gill 1982; Vallis 2017). This is done for the conditionally
 79 filtered equations in section 4. Normal modes can also give useful insight for the development of
 80 numerical solution methods, including choice of grid staggering to best capture mode structures
 81 (e.g. Arakawa and Lamb 1977; Thuburn *et al.* 2002), identification of modes that might be most
 82 challenging for a numerical method, identification of computational modes, and understanding the
 83 structure of the Helmholtz problem that arises for implicit time integration schemes. They are also
 84 valuable as test cases for numerical models (e.g. Baldauf and Brdar 2013; Shamir and Paldor 2016).

85 A variational formulation of fluid dynamical equations can be useful in several ways. The
 86 conservation properties of the system can be related to certain symmetries of the Lagrangian (e.g.
 87 Salmon 1998). Approximate versions of the governing equations, for example hydrostatic, pseudo-
 88 incompressible, or Boussinesq can be derived in a systematic way by approximating the Lagrangian
 89 and the conservation properties will be preserved by the approximation provided the corresponding
 90 symmetries are preserved (e.g. Cotter and Holm 2014; Dubos and Voitus 2014; Staniforth 2014;
 91 Tort and Dubos 2014). Such approximate versions of the governing equations might be useful for
 92 more idealized modelling or as the basis for simple theoretical models. Section 5 confirms that the
 93 conditionally filtered compressible Euler equations can be obtained from a variational formulation.

94 2. Governing equations

95 As in the derivation of the coarse-grained equations used in Large-Eddy Simulation (LES),
 96 conditional filtering makes use of an Eulerian spatial filter that retains only the flow variations
 97 on scales larger than some filter scale. But in addition to the filter it also employs a set of quasi-
 98 Lagrangian labels I_i , $i = 1, \dots, n$; at any point in the fluid exactly one of the I_i is equal to 1 and
 99 the rest are equal to 0. In the proposed application it is envisaged that the labels might be used
 100 to pick out coherent structures in the flow, such as convective updrafts and downdrafts and their
 101 environment. This quasi-Lagrangian labelling of fluid parcels is intended to capture, in mathematical
 102 form, some of the intuitive ideas behind the way we think about coherent structures such as cumulus
 103 clouds. For example, we typically think of an air parcel as retaining its identity as a cloud parcel
 104 over some time period until physical processes such mixing and evaporation change its physical
 105 properties, at which point it may be relabelled as an environment parcel.

106 To proceed, the fluid dynamical equations are multiplied by each of the I_i before applying the
 107 spatial filter. This then leads to a set of equations of motion for each fluid component i . When
 108 the starting equations are the dry non-hydrostatic compressible Euler equations, the resulting
 109 conditionally filtered equations are the following (Thuburn *et al.* 2018):

$$\sum_{i=1}^n \sigma_i = 1, \quad (1)$$

110

$$\frac{\partial}{\partial t}(\sigma_i \rho_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) = \sum_{j \neq i} (\mathcal{M}_{ij} - \mathcal{M}_{ji}), \quad (2)$$

111

$$\frac{\partial}{\partial t}(\sigma_i \rho_i \eta_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i \eta_i) = \sum_{j \neq i} (\mathcal{M}_{ij} \hat{\eta}_{ij} - \mathcal{M}_{ji} \hat{\eta}_{ji}) - \nabla \cdot \mathbf{F}_{\text{SF}}^{\eta_i}, \quad (3)$$

$$\begin{aligned} \frac{\partial}{\partial t}(\sigma_i \rho_i \mathbf{u}_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i \mathbf{u}_i) + \sigma_i \nabla \bar{p} + \sigma_i \rho_i \nabla \Phi \\ = \sum_{j \neq i} (\mathcal{M}_{ij} \hat{\mathbf{u}}_{ij} - \mathcal{M}_{ji} \hat{\mathbf{u}}_{ji}) - \nabla \cdot \mathbf{F}_{\text{SF}}^{\mathbf{u}_i} - \mathbf{b}_i - \sum_j \mathbf{d}_{ij}, \end{aligned} \quad (4)$$

112

$$\bar{p} - P(\rho_i, \eta_i) = P_{\text{SF}}^i. \quad (5)$$

113 Here σ_i , ρ_i , η_i , and \mathbf{u}_i are the volume fraction, density, specific entropy, and velocity, respectively, of
 114 the i^{th} fluid component on the filter scale, \bar{p} is the filter-scale pressure, and Φ is the geopotential. See
 115 figure 1 for a schematic illustration of the meaning of the conditionally filtered fields. Equation (1)
 116 expresses the fact that the volume fractions must sum to one, (2) expresses mass conservation, (3)
 117 entropy conservation, and (4) momentum conservation, while (5) is a generic form for the equation
 118 of state relating pressure to entropy and density. For simplicity the Coriolis terms associated with
 119 planetary rotation have been neglected here. However, it is straightforward to re-introduce them
 120 and we do so for the purpose of section 4 below.

121 The right-hand sides of the above equations allow for the possibility that fluid parcels may be
 122 relabelled as the flow evolves; this could represent processes such as entrainment and detrainment
 123 of fluid between convective updrafts and their environment. Thus, for example, \mathcal{M}_{ij} is the rate per
 124 unit volume at which mass is relabelled from type j to type i , and $\hat{\eta}_{ij}$ and $\hat{\mathbf{u}}_{ij}$ are representative
 125 values of specific entropy and velocity for that relabelled fluid. If the fluid labels I_i were exactly
 126 materially conserved then the relabelling terms \mathcal{M}_{ij} would vanish. Note also that the time over
 127 which a parcel keeps a recognizable identity is much longer than a model timestep – the lifetimes of
 128 small individual clouds is of order several minutes but in a model approaching cloud resolution the
 129 timestep is measured in seconds. In a climate model the timestep might be of order tens of minutes,
 130 but the cloud populations at that resolution last of order hours. Relabelling, and its relation to
 131 physical processes such as evaporation and mixing, is further discussed by Thuburn *et al.* (2018).

132 As in the equations of LES, subfilter-scale variability contributes to the filter-scale behaviour.
 133 Here $\mathbf{F}_{\text{SF}}^{\eta_i}$ is a subfilter-scale flux of entropy, $\mathbf{F}_{\text{SF}}^{\mathbf{u}_i}$ is a subfilter-scale momentum flux tensor, and
 134 P_{SF}^i accounts for variations in pressure between the fluid components as well as effects of filtering a
 135 nonlinear equation of state. The right-hand sides cannot be derived from the equations of motion;
 136 rather, they must be parameterized, as must terms representing similar processes in, for example,
 137 a mass flux scheme.

138 Note that the same filter-scale pressure \bar{p} appears in the pressure gradient term on the left-
 139 hand side of the momentum equation (4) for every i . This is a similar assumption to that made
 140 in conventional parcel arguments, where it is assumed that the parcel takes on the pressure of the
 141 environment (e.g. Bohren and Albrecht 1998). The assumption may be justified by noting that (in
 142 most convective circumstances) the acoustic adjustment time—the time required for an acoustic
 143 wave to propagate the width of a cloud and so remove unbalanced pressure fluctuations—is short
 144 compared to the timescales of interest. Thus, acoustic oscillations will very quickly equilibrate the
 145 pressure between components, and by making the equal pressure assumption we are supposing this
 146 adjustment to take place instantaneously. A consequence of the assumption is that the equations
 147 do not support those acoustic modes for which fluid component i has a different pressure from
 148 fluid component $j \neq i$ (see also section 4). These acoustic modes would in any case have very
 149 small amplitude, and explicitly resolving them would present unnecessary difficulties for numerical
 150 solution methods with no gain in accuracy. In the Boussinesq and anelastic approximations acoustic
 151 modes are eliminated ab initio because the speed of sound is taken to be infinite. The acoustic

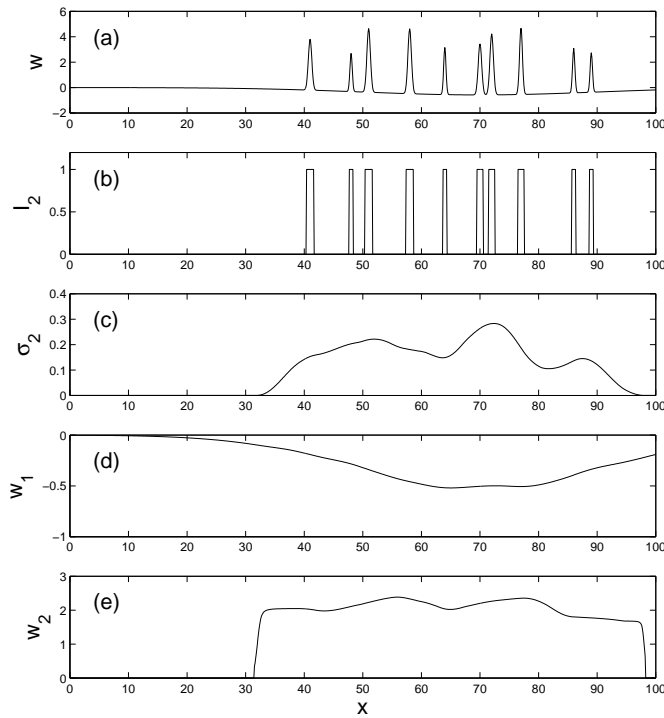


Figure 1. Schematic one-dimensional illustration of the idea of conditional filtering. (a) Hypothetical unfiltered vertical velocity field w as a function of horizontal coordinate x , showing a number of strong updrafts embedded in a region of weak descent. (b) Label I_2 picking out the updraft regions; in this example $n = 2$, with $I_1 = 1 - I_2$. (c) Volume fraction of updraft fluid on the filter scale. In this example the filter has a cosine-squared kernel of full width 19 units. (d) w_1 : the conditionally filtered value of w in the non-updraft fluid. (e) w_2 : the conditionally filtered value of w in the updraft fluid.

152 adjustment between different fluid components then occurs instantaneously, and the assumption of
 153 the same filter-scale pressure is a very natural one.

154 In a convecting fluid the pressure gradient is not, in fact, homogeneous on the scale of the
 155 convective updrafts, and rising thermals experience a significant drag due to pressure variations
 156 on the scale of the thermal (e.g. Romps and Charn 2015). These pressure variations do not
 157 represent acoustic waves; they are present in Boussinesq and anelastic numerical simulations. In
 158 the conditionally filtered equations, the fact that the net pressure gradient experienced by fluid i
 159 departs from $\nabla \bar{p}$ is accounted for by the terms $-\mathbf{b}_i - \sum_j \mathbf{d}_{ij}$ on the right-hand side. In particular,
 160 \mathbf{d}_{ij} is minus the pressure drag exerted by fluid j on fluid i . These terms have the properties that

$$\sum_i \mathbf{b}_i = 0 \quad (6)$$

161 and

$$\mathbf{d}_{ij} = -\mathbf{d}_{ji}. \quad (7)$$

162 These terms are not predicted by the conditionally filtered equations and so, in general must be
 163 parameterized, just as the analogous terms are parameterized in typical mass flux convection
 164 schemes (e.g. de Roode *et al.* 2012, and references therein). In this paper we will mainly be
 165 concerned with the left-hand sides of the conditionally filtered equations, so we will often neglect
 166 these terms along with the other right-hand side terms.

167 It may be useful to note how the conditionally filtered equations (1)–(5) are related to the
 168 usual filtered single-fluid equations. The conditionally filtered equations reduce to the usual filtered

169 single-fluid equations simply by setting the number of fluid components n to 1; in that case the
 170 fluid relabelling terms \mathcal{M}_{ij} and the terms representing pressure forces between fluid components \mathbf{b}_i
 171 and \mathbf{d}_{ij} all vanish, and $\sigma_1 \equiv 1$. Thuburn *et al.* (2018) also show that the usual filtered single-fluid
 172 equations are obtained by summing the conditionally filtered equations over all fluid components i .
 173 Note also that, although the left hand sides of (2)-(4) are written here in Eulerian flux form, this is
 174 not a requirement; it is straightforward to convert them to Lagrangian form, as we do, for example,
 175 in (18) and (24) below.

176 In the absence of the right-hand sides, equations (1)–(5) form a closed system (see below). All of
 177 the right-hand side terms, on the other hand, must be modelled or parameterized by making some
 178 additional assumptions. The present paper focuses mainly on the properties of the equations in the
 179 absence of the parameterized terms, whereupon they reduce to

$$\sum_{i=1}^n \sigma_i = 1, \quad (8)$$

$$\frac{\partial}{\partial t}(\sigma_i \rho_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) = 0, \quad (9)$$

$$\frac{\partial}{\partial t}(\sigma_i \rho_i \eta_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i \eta_i) = 0, \quad (10)$$

$$\frac{\partial}{\partial t}(\sigma_i \rho_i \mathbf{u}_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i \mathbf{u}_i) + \sigma_i \nabla \bar{p} + \sigma_i \rho_i \nabla \Phi = 0, \quad (11)$$

$$\bar{p} - P(\rho_i, \eta_i) = 0, \quad (12)$$

184 where (8) is the same as (1) but is included for completeness.

185 In the case of a single fluid component $n = 1$, equations (8)-(12) reduce to the usual non-
 186 hydrostatic compressible Euler equations. For $n > 1$ the equations for different i are coupled by
 187 the common pressure gradient term $\nabla \bar{p}$ and the requirement (8) (these two points are related—see
 188 section 5). Also, for $n > 1$ it is not immediately obvious that (8)-(12) form a closed system. It can be
 189 confirmed, simply by counting, that the number of equations is equal to the number of unknowns.
 190 Appendix A outlines how the given equations imply the time evolution of σ_i , ρ_i , and \bar{p} and how
 191 they allow \bar{p} to be diagnosed. For the linearized version of these equations, the fact that a dispersion
 192 relation can be derived (section 4) provides further confirmation that they form a closed system.

193 A potentially useful variant of the conditionally filtered equations, mentioned by Thuburn *et al.*
 194 (2018), is one in which all fluid components are constrained to have identical horizontal velocity:
 195 $\mathbf{v}_i = \bar{\mathbf{v}}^{*\dagger}$, where $\mathbf{u}_i = (\mathbf{v}_i, w_i)$. In this variant the horizontal components of the inter-fluid pressure
 196 forces $\mathbf{b}_i + \sum_j \mathbf{d}_{ij}$ are assumed to be just what is required to maintain the equality of the \mathbf{v}_i .
 197 The ansatz that the horizontal velocities are all the same is an additional physical assumption that
 198 may be useful in some circumstances, but it is not demanded by the mathematical structure of the
 199 equations. Making this assumption does not change the vertical part of (4), namely

$$\begin{aligned} & \frac{\partial}{\partial t}(\sigma_i \rho_i w_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i w_i) + \sigma_i \bar{p}_z + \sigma_i \rho_i \Phi_z = \\ & \sum_{j \neq i} (\mathcal{M}_{ij} \hat{w}_{ij} - \mathcal{M}_{ji} \hat{w}_{ji}) - \nabla \cdot \mathbf{F}_{\text{SF}}^{w_i} - b_i^{(z)} - \sum_j d_{ij}^{(z)}, \end{aligned} \quad (13)$$

200 where subscript z indicates a vertical derivative and superscript (z) indicates a vertical component.
 201 However, the horizontal part is replaced by the sum over all fluid components

$$\frac{\partial}{\partial t}(\bar{\rho} \bar{\mathbf{v}}^*) + \nabla \cdot (\bar{\rho} \bar{\mathbf{u}}^* \bar{\mathbf{v}}^*) + \nabla_H \bar{p} + \bar{\rho} \nabla_H \Phi = -\nabla \cdot \mathbf{F}_{\text{SF}}^{\bar{\mathbf{v}}^*}, \quad (14)$$

[†]The notation \bar{X} to indicate a filtered value of X and \bar{X}^* to indicate a density-weighted filtered value, so that $\bar{\rho} \bar{X}^* = \overline{\rho X}$, is retained for consistency with Thuburn *et al.* (2018).

202 where ∇_H is the horizontal part of the gradient,

$$\bar{\rho} = \sum_i \sigma_i \rho_i \quad (15)$$

203 is the total filter-scale density, and $\bar{\mathbf{u}}^*$ is the density-weighted filter-scale velocity given by

$$\bar{\rho} \bar{\mathbf{u}}^* = \sum_i \sigma_i \rho_i \mathbf{u}_i. \quad (16)$$

204 The \mathbf{b} and \mathbf{d} terms have cancelled in (14) because of (6) and (7), while the relabelling terms also
 205 cancel when summed over i and j . Appendix B summarizes how the main results of the paper carry
 206 over to this equal- \mathbf{v}_i variant.

207 3. Conservation properties

208 This section examines the conservation properties of the conditionally filtered equations. We focus on
 209 the compressible Euler equations, but similar derivations may be carried out for other, approximate,
 210 governing equation sets such as hydrostatic, pseudo-incompressible, or Boussinesq equations.

211 3.1. Mass

212 Equation (9) is manifestly in the form of a conservation law for the mass of the i^{th} fluid component. If
 213 there is no mass flux across domain boundaries then it implies that the mass of each fluid component
 214 is individually conserved, and hence that their sum, the total fluid mass, is also conserved.

215 If relabelling terms are re-introduced, (9) becomes (2). Then the mass of each fluid component is
 216 no longer conserved. However, summing (2) over i and noting that the relabelling terms then cancel
 217 (because they are relabelling terms) gives

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot (\bar{\rho} \bar{\mathbf{u}}^*) = 0, \quad (17)$$

218 with $\bar{\rho}$ and $\bar{\mathbf{u}}^*$ given by (15) and (16). Thus the total fluid mass is conserved even when relabelling
 219 terms are included.

220 3.2. Entropy

221 Equation (10) is manifestly in the form of a conservation law for the entropy of the i^{th} fluid
 222 component. If there is no entropy flux across domain boundaries then it implies that the entropy of
 223 each fluid component is individually conserved, and hence that their sum, the total fluid entropy,
 224 is also conserved.

225 Subtracting η_i times (9) from (10) gives

$$\frac{D_i \eta_i}{Dt} \equiv \frac{\partial \eta_i}{\partial t} + \mathbf{u}_i \cdot \nabla \eta_i = 0. \quad (18)$$

226 This shows that η_i is materially conserved following fluid parcels that move with velocity \mathbf{u}_i .

227 If the subfilter-scale flux term $\nabla \cdot \mathbf{F}_{\text{SF}}^{\eta_i}$ is included in (10) then the equation is still in the form of
 228 a flux form conservation law, so the entropy of each fluid component is still conserved in an integral
 229 sense, though it is no longer materially conserved. If, in addition, the relabelling terms are included
 230 to give (3) then the entropy of each fluid component is no longer conserved. However, summing (3)
 231 over i and noting the cancelling of the relabelling terms shows that

$$\frac{\partial}{\partial t} (\bar{\rho} \bar{\eta}^*) + \nabla \cdot \mathbf{F}^\eta = 0, \quad (19)$$

232 where the total filter-scale specific entropy of the fluid $\bar{\eta}^*$ is given by

$$\bar{\rho}\bar{\eta}^* = \sum_i \sigma_i \rho_i \eta_i, \quad (20)$$

233 and the total entropy flux is given by

$$\mathbf{F}^\eta = \sum_i (\sigma_i \rho_i \mathbf{u}_i \eta_i + \mathbf{F}_{\text{SF}}^{\eta_i}). \quad (21)$$

234 Analogous results would hold for any function of entropy, for example potential temperature θ ,
 235 and also for any materially conserved scalar such as specific humidity in the absence of precipitation
 236 or a chemical tracer mixing ratio.

237 Note that the derivation of (3) neglects sources of entropy due to diabatic heating and also due
 238 to mixing and other irreversible processes. In realistic flows such sources are often not negligible
 239 (e.g. Pauluis and Held 2002; Raymond 2013) so a comprehensive model would need to take them
 240 into account.

241 3.3. Momentum

242 The geopotential gradient $\nabla\Phi$ provides an external force and hence an external source of momentum.
 243 Even if this term is ignored for the moment, (11) does not conserve the momentum of the i^{th} fluid
 244 component because the $\sigma_i \nabla\bar{p}$ term is not in conservation form. However, the $\sigma_i \nabla\bar{p}$ term does
 245 represent a conservative exchange of momentum between different fluid components, as do the \mathbf{b}_i ,
 246 \mathbf{d}_{ij} , and relabelling terms. This can be seen by summing (4) over i and using (6) and (7) to obtain

$$\frac{\partial}{\partial t} (\bar{\rho} \bar{\mathbf{u}}^*) + \nabla \cdot \mathbf{F}^{\mathbf{u}} + \bar{\rho} \nabla \Phi = 0, \quad (22)$$

247 where $\bar{\rho} \bar{\mathbf{u}}^*$ is given by (16) and

$$\mathbf{F}^{\mathbf{u}} = \bar{\rho} I + \sum_i (\sigma_i \rho_i \mathbf{u}_i \mathbf{u}_i + \mathbf{F}_{\text{SF}}^{\mathbf{u}_i}) \quad (23)$$

248 is the total momentum flux tensor, with I the identity matrix. Thus, the total fluid momentum is
 249 conserved except for the effect of the external force.

250 If the Coriolis terms are re-introduced for a rotating planet then the relevant conserved quantity
 251 is the axial angular momentum. The axial angular momentum of the i^{th} fluid is not conserved, but
 252 it is straightforward to verify that the $\nabla\bar{p}$, \mathbf{b}_i , \mathbf{d}_{ij} , and relabelling terms all describe conservative
 253 transfers between fluid components and the total axial angular momentum is conserved.

254 3.4. Energy

255 In this subsection we ignore the subfilter-scale flux terms and the relabelling terms; in general they
 256 do not conserve the energy of the filter-scale flow. For the moment the \mathbf{b}_i and \mathbf{d}_{ij} terms are retained.

257 Subtracting \mathbf{u}_i times (9) from (4) and neglecting $\mathbf{F}_{\text{SF}}^{\mathbf{u}_i}$ and \mathcal{M}_{ij} gives the advective form of the
 258 momentum equation

$$\sigma_i \rho_i \frac{D_i \mathbf{u}_i}{Dt} + \sigma_i \nabla\bar{p} + \sigma_i \rho_i \nabla\Phi = -\mathbf{b}_i - \sum_j \mathbf{d}_{ij}. \quad (24)$$

259 Taking the dot product of \mathbf{u}_i with (24) gives

$$\sigma_i \rho_i \frac{D_i}{Dt} \left(\frac{1}{2} |\mathbf{u}_i|^2 + \Phi \right) + \sigma_i \mathbf{u}_i \cdot \nabla\bar{p} = -\mathbf{u}_i \cdot \left(\mathbf{b}_i + \sum_j \mathbf{d}_{ij} \right). \quad (25)$$

260 Next, defining $e_i(\rho_i, \eta_i)$ to be the specific internal energy of fluid component i ,

$$\frac{D_i e_i}{Dt} = \frac{\partial e_i}{\partial \rho_i} \bigg|_{\eta_i} \frac{D_i \rho_i}{Dt} + \frac{\partial e_i}{\partial \eta_i} \bigg|_{\rho_i} \frac{D_i \eta_i}{Dt}. \quad (26)$$

261 Noting that

$$\frac{\partial e_i}{\partial \rho_i} = \frac{\bar{p}}{\rho_i^2}, \quad (27)$$

262 and using (9) to obtain the material derivative of ρ_i

$$\sigma_i \frac{D_i \rho_i}{Dt} + \rho_i \frac{\partial \sigma_i}{\partial t} + \rho_i \nabla \cdot (\sigma_i \mathbf{u}_i) = 0 \quad (28)$$

263 and (18) for the material derivative of η_i , (26) becomes

$$\sigma_i \rho_i \frac{D_i e_i}{Dt} = -\bar{p} \left(\frac{\partial \sigma_i}{\partial t} + \nabla \cdot (\sigma_i \mathbf{u}_i) \right). \quad (29)$$

264 Adding this result to (25) gives

$$\begin{aligned} \sigma_i \rho_i \frac{D_i}{Dt} \varepsilon_i + \nabla \cdot (\sigma_i \mathbf{u}_i \bar{p}) + \bar{p} \frac{\partial \sigma_i}{\partial t} \\ = -\mathbf{u}_i \cdot \left(\mathbf{b}_i + \sum_j \mathbf{d}_{ij} \right), \end{aligned} \quad (30)$$

265 where

$$\varepsilon_i = \frac{1}{2} |\mathbf{u}_i|^2 + \Phi + e_i \quad (31)$$

266 is the total filter-scale energy per unit mass of the i^{th} fluid component. Finally, adding ε_i times (9)
267 to (30) gives

$$\begin{aligned} \frac{\partial}{\partial t} (\sigma_i \rho_i \varepsilon_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i \varepsilon_i + \sigma_i \mathbf{u}_i \bar{p}) + \bar{p} \frac{\partial \sigma_i}{\partial t} \\ = -\mathbf{u}_i \cdot \left(\mathbf{b}_i + \sum_j \mathbf{d}_{ij} \right). \end{aligned} \quad (32)$$

268 The quantity $\sigma_i \rho_i \varepsilon_i$ is the contribution from the i^{th} fluid component to the total filter-scale energy
269 density. In general it is not conserved. The term $\bar{p} \partial \sigma_i / \partial t$ represents a conservative exchange of
270 energy between fluid components, since $\sum_i \sigma_i = 1$ implies $\sum_i \bar{p} \partial \sigma_i / \partial t = 0$. The terms $\mathbf{b}_i + \sum_j \mathbf{d}_{ij}$
271 will typically tend to reduce differences between the \mathbf{u}_i ; they thus represent a sink of filter-scale
272 energy and a transfer to subfilter scales. If the $\mathbf{b}_i + \sum_j \mathbf{d}_{ij}$ terms can be ignored then summing
273 (32) over i shows that the total filter-scale energy is conserved:

$$\frac{\partial}{\partial t} \left(\sum_i \sigma_i \rho_i \varepsilon_i \right) + \nabla \cdot \left(\sum_i (\sigma_i \rho_i \mathbf{u}_i \varepsilon_i + \sigma_i \mathbf{u}_i \bar{p}) \right) = 0. \quad (33)$$

274 3.5. Potential vorticity

275 Using standard vector calculus identities, the advective form of the momentum equation (24) may
276 be written in so-called vector invariant form

$$\frac{\partial \mathbf{u}_i}{\partial t} + \boldsymbol{\zeta}_i \times \mathbf{u}_i + \frac{1}{\rho_i} \nabla \bar{p} + \nabla \left(\Phi + \frac{1}{2} |\mathbf{u}_i|^2 \right) = -\frac{1}{\sigma_i \rho_i} \left(\mathbf{b}_i + \sum_j \mathbf{d}_{ij} \right). \quad (34)$$

277 For now suppose the right-hand side can be neglected. Taking the curl and using further vector
278 calculus identities gives the vorticity equation for the i^{th} fluid component

$$\frac{D_i \boldsymbol{\zeta}_i}{Dt} + \boldsymbol{\zeta}_i \nabla \cdot \mathbf{u}_i - \boldsymbol{\zeta}_i \cdot \nabla \mathbf{u}_i + \nabla \times \left(\frac{1}{\rho_i} \nabla \bar{p} \right) = 0. \quad (35)$$

279 Rewriting (9) in the form

$$\frac{D_i}{Dt} (\sigma_i \rho_i) + \sigma_i \rho_i \nabla \cdot \mathbf{u}_i = 0 \quad (36)$$

280 allows the velocity divergence term to be eliminated:

$$\sigma_i \rho_i \frac{D_i}{Dt} \left(\frac{\zeta_i}{\sigma_i \rho_i} \right) - \zeta_i \cdot \nabla \mathbf{u}_i + \nabla \times \left(\frac{1}{\rho_i} \nabla \bar{p} \right) = 0. \quad (37)$$

281 Now consider a scalar λ that is materially conserved following the velocity field \mathbf{u}_i , i.e. $D_i \lambda / Dt =$
282 0. Taking the gradient, expanding, and rearranging gives

$$\frac{D_i}{Dt} \nabla \lambda + (\nabla \mathbf{u}_i) \cdot \nabla \lambda = 0. \quad (38)$$

283 If we construct the quantity

$$\Pi_i = \frac{\zeta_i \cdot \nabla \lambda}{\sigma_i \rho_i} \quad (39)$$

284 and use the product rule to evaluate its material derivative we obtain

$$\frac{D_i \Pi_i}{Dt} - \frac{1}{\sigma_i \rho_i^3} \nabla \rho_i \times \nabla \bar{p} \cdot \nabla \lambda = 0. \quad (40)$$

285 If λ is chosen to be the specific entropy η_i , or any function of the specific entropy such as the
286 potential temperature (Π_i is then the potential vorticity of the i^{th} fluid), then λ can be expressed
287 as a function of ρ_i and \bar{p} , $\nabla \lambda$ at every point is a linear combination of $\nabla \rho_i$ and $\nabla \bar{p}$, and so the
288 scalar triple product term in (40) vanishes, leaving

$$\frac{D_i \Pi_i}{Dt} = 0. \quad (41)$$

289 Thus the potential vorticity of the i^{th} fluid component is materially conserved following \mathbf{u}_i . This
290 derivation closely parallels the standard textbook derivation of potential vorticity conservation for
291 a single component fluid (e.g. Vallis 2017). A notable difference is the appearance of σ_i as well as
292 ρ_i in the denominator of (39).

293 If the $\mathbf{b}_i + \sum_j \mathbf{d}_{ij}$ terms can not be neglected then they may be carried through the derivation
294 to appear as source terms in (41). The potential vorticity of the i^{th} fluid component is then no
295 longer materially conserved.

296 Haynes and McIntyre (1987) showed that potential vorticity satisfies a flux form conservation law
297 even in the presence of diabatic heating and frictional forces. They also proved the *impermeability*
298 *theorem*, that there is no net flux of potential vorticity across an isentropic surface. These results are
299 purely kinematic (Bretherton and Schär 1993; Vallis 2017); they do not depend on the governing
300 dynamical equations, only on the definition of potential vorticity and the fact that the vorticity is
301 the curl of a vector and hence divergence free. It comes as no surprise, then, that the conservation
302 law and impermeability theorem generalize straightforwardly to the conditionally filtered equations,
303 as follows.

304 The conservation law is obtained from (39), setting $\lambda = \eta_i$ and using $\nabla \cdot \zeta_i = 0$,

$$\sigma_i \rho_i \Pi_i = \nabla \cdot (\eta_i \zeta_i). \quad (42)$$

305 Taking the time derivative then gives

$$\frac{\partial}{\partial t} (\sigma_i \rho_i \Pi_i) + \nabla \cdot \mathcal{F}_i = 0 \quad (43)$$

306 where

$$\mathcal{F}_i = - \frac{\partial}{\partial t} (\eta_i \zeta_i). \quad (44)$$

307 The time derivative in the expression for \mathcal{F}_i can be removed using the prognostic equations for η_i
 308 and ζ_i (including diabatic heating and friction if present). Note also that the flux is not unique;
 309 any divergence free vector may be added to \mathcal{F}_i leaving the conservation law intact.

310 Next consider the integral of potential vorticity within a volume bounded by an isentropic surface,
 311 i.e. a surface of constant η_i . For example, this surface might envelope the Earth.

$$\int \sigma_i \rho_i \Pi_i dV = \int \nabla \cdot (\eta_i \zeta_i) dV = \int_{\partial} \eta_i \zeta_i \cdot d\mathbf{A}, \quad (45)$$

312 where the last integral is the area integral of the outward normal component of $\eta_i \zeta_i$ over the
 313 boundary of the original volume. Since η_i is constant on this boundary it can be brought outside
 314 the integral:

$$\int \sigma_i \rho_i \Pi_i dV = \eta_i \int_{\partial} \zeta_i \cdot d\mathbf{A} = \eta_i \int \nabla \cdot \zeta_i dV = 0. \quad (46)$$

315 Similarly, the integral of potential vorticity within a volume bounded by a pair of isentropic surfaces
 316 must also vanish.

317 Finally, consider the integral of potential vorticity within a volume that is bounded in part by
 318 an isentropic surface A on which $\eta_i = \eta_i^{(A)} = \text{const}$ and in part by a surface B, such as the ground,
 319 on which $\eta_i = \eta_i^{(B)}$ may vary in space and time. The boundary integral in (45) may be split into
 320 two contributions:

$$\begin{aligned} \int \sigma_i \rho_i \Pi_i dV &= \eta_i^{(A)} \int_{\partial_A} \zeta_i \cdot d\mathbf{A} + \int_{\partial_B} \eta_i^{(B)} \zeta_i \cdot d\mathbf{A} \\ &= \eta_i^{(A)} \int_{\partial_A + \partial_B} \zeta_i \cdot d\mathbf{A} + \int_{\partial_B} (\eta_i^{(B)} - \eta_i^{(A)}) \zeta_i \cdot d\mathbf{A} \\ &= \int_{\partial_B} (\eta_i^{(B)} - \eta_i^{(A)}) \zeta_i \cdot d\mathbf{A}. \end{aligned} \quad (47)$$

321 Thus the integral of potential vorticity within the volume, and therefore its rate of change, depends
 322 only on contributions from surface B; there is no contribution from surface A.

323 In summary, for the conditionally filtered equations, the potential vorticity of each fluid
 324 component i satisfies a flux form conservation law and the impermeability theorem.

325 4. Normal modes

326 In this section we focus mainly on the case of two fluid components. The case of more fluid
 327 components is discussed briefly at the end. To analyse the normal modes, all of the right-hand
 328 side terms in (2)-(5) are neglected, so the starting point is (8)-(12). For simplicity, planar geometry
 329 is assumed and the equations are written in Cartesian coordinates. However, Coriolis terms are
 330 re-introduced with a linear dependence of the Coriolis parameter on the northward coordinate y ,
 331 i.e. we use a β -plane, because the Coriolis terms and β -effect are crucial to the dynamics of the
 332 normal modes.

333 Small perturbations to a basic state are considered. The basic state (indicated by superscript (r))
 334 is at rest and in hydrostatic balance, and the basic state thermodynamic quantities $\rho^{(r)}, \eta^{(r)}, p^{(r)}$ are
 335 identical for the two fluid components, though their volume fractions $\sigma_1^{(r)}, \sigma_2^{(r)}$ might be different.
 336 Basic state quantities are functions only of the vertical coordinate z .

337 Equations (8)-(12) are linearized about the basic state and wavelike solutions proportional to
 338 $\exp\{i(kx + ly - \omega t)\}$ are sought, where k, l are the horizontal components of the wave vector and
 339 ω is the frequency. The β -effect is included, while still permitting such wavelike solutions, following
 340 the approximation made by Thuburn and Woollings (2005). Including the β -effect is useful for
 341 identifying the Rossby modes and distinguishing them from any zero-frequency modes.

342 The resulting linearized equations are

$$\sum_{i=1}^2 \sigma_i = 0, \quad (48)$$

$$-i\omega \left(\sigma_i^{(r)} \rho_i + \sigma_i \rho^{(r)} \right) + \sigma_i^{(r)} \rho^{(r)} (iku_i + ilv_i) + \left(\sigma_i^{(r)} \rho^{(r)} w_i \right)_z = 0, \quad (49)$$

$$-i\omega \eta_i + w_i \eta_z^{(r)} = 0, \quad (50)$$

$$-i\tilde{\omega} u_i - f v_i + ik \frac{p}{\rho^{(r)}} = 0, \quad (51)$$

$$-i\tilde{\omega} v_i + f u_i + il \frac{p}{\rho^{(r)}} = 0, \quad (52)$$

$$-i\omega w_i + \frac{1}{\rho^{(r)}} p_z + g \frac{\rho_i}{\rho^{(r)}} = 0, \quad (53)$$

$$\frac{1}{c^2} \frac{p}{\rho^{(r)}} = \frac{\rho_i}{\rho^{(r)}} + Q \eta_i. \quad (54)$$

Here, σ_i , ρ_i , η_i , and p are now the perturbations to volume fraction, density, specific entropy, and pressure, respectively, and u_i , v_i , w_i are the velocity perturbation components. f is a mean Coriolis parameter and β is the northward gradient of the Coriolis parameter, both taken as constant. The gravitational acceleration g is minus the vertical component of $\nabla\Phi = (0, 0, -g)$. Subscript z indicates a vertical derivative. The modified frequency $\tilde{\omega}$ is given by

$$\tilde{\omega} = \omega + \frac{k\beta}{K^2} \quad (55)$$

where $K^2 = k^2 + l^2$. The linearized equation of state (54) has been obtained by writing $\rho = \rho(p, \eta)$ and considering small perturbations to the reference state; the quantity Q is given by

$$Q = - \left. \frac{\partial \ln \rho}{\partial \eta} \right|_p^{(r)} \quad (56)$$

while

$$c^2 = \left. \frac{\partial p}{\partial \rho} \right|_\eta^{(r)} \quad (57)$$

is the sound speed squared in the reference state.

An equation for a single unknown perturbation field, in this case p , is derived by systematically eliminating the other unknowns. First use (50) to eliminate η_i from (54):

$$i\omega \left(\frac{1}{c^2} \frac{p}{\rho^{(r)}} - \frac{\rho_i}{\rho^{(r)}} \right) = \frac{N^2}{g} w_i, \quad (58)$$

where the buoyancy frequency squared N^2 for a general equation of state is given by

$$\frac{N^2}{g} = - \frac{\rho_z^{(r)}}{\rho^{(r)}} - \frac{g}{c^2} = Q \eta_z^{(r)} \quad (59)$$

(e.g. IOC *et al.* 2010; Thuburn 2017).

Using (58) to eliminate $\rho_i/\rho^{(r)}$ from (53) gives

$$\left(\omega^2 - N^2 \right) \rho^{(r)} w_i + i\omega \left(p_z + \frac{g}{c^2} p \right) = 0. \quad (60)$$

Also, combining (51) and (52) gives

$$\rho^{(r)} u_i = \frac{k\tilde{\omega} + ilf}{\tilde{\omega}^2 - f^2} p, \quad (61)$$

364

$$\rho^{(r)} v_i = \frac{l\tilde{\omega} - ikf}{\tilde{\omega}^2 - f^2} p. \quad (62)$$

 365 Summing (49) over i and using (48) to eliminate σ_i gives

$$\sum_{i=1}^2 \left\{ -i\omega\sigma_i^{(r)}\rho_i + \sigma_i^{(r)}\rho^{(r)}(iku_i + ilv_i) + \left(\sigma_i^{(r)}\rho^{(r)}w_i\right)_z \right\} = 0. \quad (63)$$

 366 Using (58) to eliminate ρ_i gives

$$\sum_{i=1}^2 \left\{ -i\omega\sigma_i^{(r)}\frac{p}{c^2} + \sigma_i^{(r)}\rho^{(r)}(iku_i + ilv_i) + \left(\sigma_i^{(r)}\rho^{(r)}w_i\right)_z + \frac{N^2}{g}\sigma_i^{(r)}\rho^{(r)}w_i \right\} = 0, \quad (64)$$

 367 Now u_i , v_i and w_i may be eliminated using (61), (62) and (60), giving an equation in the single
 368 unknown p :

$$\sum_{i=1}^2 \left\{ -i\omega\sigma_i^{(r)}\frac{p}{c^2} + \sigma_i^{(r)}\left(\frac{iK^2\tilde{\omega}}{\tilde{\omega}^2 - f^2}\right)p - \left(\sigma_i^{(r)}\frac{i\omega}{\omega^2 - N^2}\left(p_z + \frac{g}{c^2}\right)\right)_z - \sigma_i^{(r)}\frac{N^2}{g}\frac{i\omega}{\omega^2 - N^2}\left(p_z + \frac{g}{c^2}\right) \right\} = 0. \quad (65)$$

 369 This equation can be simplified by noting that $\sum_i \sigma_i^{(r)} = 1$, to obtain

$$\left(\frac{\omega}{c^2} - \frac{K^2\tilde{\omega}}{\tilde{\omega}^2 - f^2}\right)p + \left(\frac{d}{dz} + \frac{N^2}{g}\right)\left(\frac{\omega}{\omega^2 - N^2}\right)\left(\frac{d}{dz} + \frac{g}{c^2}\right)p = 0. \quad (66)$$

 370 For a general equation of state and for arbitrary basic state profiles, (66) could be solved
 371 numerically. Normal modes can be obtained analytically if a perfect gas equation of state is assumed,
 372 the basic state is assumed to be isothermal, and g is taken to be constant. In that case c^2 and N^2
 373 are constant, and so is the density scale height H , which is given by

$$\frac{1}{H} = \frac{N^2}{g} + \frac{g}{c^2}. \quad (67)$$

 374 Then (66) is a constant coefficient equation for p . The solutions have a simpler structure when
 375 expressed in terms of a rescaled variable

$$q = p \exp(z/2H); \quad (68)$$

376 (66) then reduces to

$$\left(\frac{\omega}{c^2} - \frac{K^2\tilde{\omega}}{\tilde{\omega}^2 - f^2}\right)q + \left(\frac{\omega}{\omega^2 - N^2}\right)\left(\frac{d^2}{dz^2} - \Gamma^2\right)q = 0, \quad (69)$$

 377 where the inverse length scale Γ is given by

$$\Gamma = \frac{1}{2}\left(\frac{g}{c^2} - \frac{N^2}{g}\right). \quad (70)$$

378 *4.1. Single-fluid-equivalent modes*

379 Equation (69) has solutions for q proportional to $\exp(imz)$ with real vertical wavenumber m . These
 380 are internal mode solutions. The allowed values of m are determined by the lower and upper
 381 boundary conditions, for example $w_i = 0$ at $z = 0$ and at some domain height D , though we will not
 382 dwell on this detail here. Substituting such solutions into (69) and cancelling q gives the dispersion
 383 relation, relating ω to k , l and m :

$$\frac{\omega}{c^2} - \frac{K^2 \tilde{\omega}}{\tilde{\omega}^2 - f^2} - \omega \frac{m^2 + \Gamma^2}{\omega^2 - N^2} = 0. \quad (71)$$

384 It is clear that (71) is a quintic equation for ω , and it is easily confirmed that it is identical to
 385 the dispersion relation obtained by Thuburn and Woollings (2005) for the single-fluid-component
 386 compressible Euler equations. The five roots for ω for any given (k, l, m) correspond to five branches
 387 of normal modes: eastward and westward propagating acoustic modes, eastward and westward
 388 propagating inertio-gravity modes, and westward propagating Rossby modes.

389 To examine the structure of these normal modes, note that (60)-(62) imply $(u_1, v_1, w_1) =$
 390 (u_2, v_2, w_2) . (It has been assumed here that $\omega^2 \neq N^2$ and $\tilde{\omega}^2 \neq f^2$, but it can be confirmed that
 391 such values of ω are not solutions of (71) except for very special and unrealistic parameter values.)
 392 It then follows from (50) and (54) that $\rho_1 = \rho_2$ and $\eta_1 = \eta_2$, while (49) implies that that σ_1 and
 393 σ_2 are determined simply by vertical advection of the background values $\sigma_1^{(r)}$ and $\sigma_2^{(r)}$. Thus, these
 394 normal modes have identical perturbations in the two fluid components. Their structure, as well
 395 as their frequency, is exactly that of the normal modes for the single-fluid-component compressible
 396 Euler equations. In other words, the single-fluid normal modes are a subset of the two-fluid normal
 397 modes.

398 The single-fluid compressible Euler equations also support external modes, with zero vertical
 399 velocity and entropy perturbation (assuming a rigid lid upper boundary condition) and exponential
 400 profiles of the other perturbation variables. Seeking such modes in the two-fluid case, only the
 401 first line is retained on the left-hand sides of (64), (65), (66), and (69), and the dispersion relation
 402 becomes

$$\frac{\omega}{c^2} - \frac{K^2 \tilde{\omega}}{\tilde{\omega}^2 - f^2} = 0. \quad (72)$$

403 This is a cubic equation for ω , giving three branches of normal modes: eastward and westward
 404 external acoustic modes and westward external Rossby modes. Again, the frequencies are identical
 405 to those in the single-fluid case, and the mode structures are identical in the two fluid components,
 406 so again the single-fluid normal modes are a subset of the two-fluid normal modes.

407 In order to obtain (71) from (69) it was assumed that q was non-zero in order to cancel q . Another
 408 way to satisfy (69) is for q to be identically zero. There are then three ways to obtain non-trivial
 409 solutions.

 410 *4.2. Two-fluid gravity modes*

411 To have zero p but non-zero vertical velocity, (60) implies that $\omega^2 = N^2$. Equations (61) and (62)
 412 then imply that $u_i = v_i = 0$; the motion is purely vertical. From (58) and (54), the entropy and
 413 density perturbations are related to the vertical velocity perturbation by

$$\pm i N Q \eta_i = \mp i N \frac{\rho_i}{\rho^{(r)}} = \frac{N^2}{g} w_i. \quad (73)$$

414 Equation (64) reduces to

$$\left(\frac{d}{dz} + \frac{N^2}{g} \right) \left(\sum_{i=1}^2 \sigma_i^{(r)} \rho^{(r)} w_i \right) = 0. \quad (74)$$

415 The bottom boundary condition implies $\sum_{i=1}^2 \sigma_i^{(r)} \rho^{(r)} w_i = 0$ there, and then (74) implies
 416 $\sum_{i=1}^2 \sigma_i^{(r)} \rho^{(r)} w_i = 0$ at all heights. Thus

$$\sigma_1^{(r)} w_1 = -\sigma_2^{(r)} w_2; \quad (75)$$

417 the vertical mass fluxes by the two fluid components are equal and opposite. Finally, substituting
 418 from (73) for ρ_i in (49) gives the volume fraction perturbations in terms of w_i :

$$\pm i N \sigma_i = \frac{1}{\rho^{(r)}} \left(\sigma_i^{(r)} \rho^{(r)} w_i \right)_z + \frac{N^2}{g} w_i. \quad (76)$$

419 The essential dynamics of these motions involves the coupling of vertical velocity with buoyancy
 420 perturbations, and their structure and frequency are reminiscent of deep internal gravity waves.
 421 This justifies our classification of them as two-fluid gravity modes. At the same time, there are some
 422 important differences from fully-resolved gravity modes of the single-fluid equations. For example,
 423 because the pressure and horizontal velocity perturbations vanish there is no horizontal coupling.

424 The frequency of these motions is independent of their vertical structure. Therefore, there
 425 is no unique way to define a set of vertical normal modes. A convenient choice is $w_1 \propto$
 426 $(\sigma_2^{(r)} / \sigma_1^{(r)} \rho^{(r)})^{1/2} \exp\{imz\}$, etc. This choice ensures that the modes for different m are indeed
 427 mutually orthogonal (i.e. normal) with respect to the energy of the linearized equations

$$E_{\text{lin}} = \sum_i \left\{ \frac{\sigma_i^{(r)} \rho^{(r)}}{2} \left(|\mathbf{u}_i|^2 + \frac{g^2 Q^2}{N^2} |\eta_i|^2 + \frac{|p|^2}{\rho^{(r)2} c^2} \right) \right\}, \quad (77)$$

428 and it allows us to discuss the vertical wavenumber m . (The expression (77) reduces to that given
 429 by e.g. Phillips 1990; Thuburn *et al.* 2002, in the case of a single fluid component.)

430 For any given (k, l, m) there are two possible frequencies, $\omega = \pm N$, giving two branches to the
 431 dispersion relation. Although the structures and frequencies of these modes resemble those of deep
 432 internal gravity modes in some respects, these features hold for all m , including large m , so the
 433 mode structures do not, in fact, have to be deep.

434 Finally, since the frequency of these modes is independent of k, l and m their group velocity is
 435 identically zero. They propagate neither horizontally nor vertically.

436 4.3. Two-fluid inertial modes

437 To have zero p with non-zero horizontal velocity, (61) and (62) imply that $\tilde{\omega}^2 = f^2$, i.e. $\omega =$
 438 $\pm f - k\beta/K^2$. (60) then implies that $w_i = 0$, and hence $\rho_i = 0$ and $\eta_i = 0$. Either (51) or (52)
 439 shows that

$$v_i = \mp i u_i, \quad (78)$$

440 from which it follows that the horizontal divergence

$$\delta_i = i k u_i + i l v_i = i k u_i \pm l u_i \quad (79)$$

441 is in quadrature with the vertical component of vorticity

$$\zeta_i = i k v_i - i l u_i = \pm k u_i - i l u_i = \mp i \delta_i. \quad (80)$$

442 Equation (49) implies that

$$\sum_i \sigma_i \delta_i = 0, \quad (81)$$

443 so the net mass flux convergence vanishes everywhere. Combining with (79) then shows that the
 444 net mass flux $\sum_i \sigma_i^{(r)} \mathbf{u}_i$ vanishes everywhere. The essential dynamics of these motions involves the
 445 coupling between u and v via the Coriolis term, and they have structure and frequency resembling

446 inertial modes, but with compensating horizontal mass fluxes in the two fluid components rather
 447 than in layers at nearby heights. Hence we classify them as two-fluid inertial modes. As for the
 448 two-fluid gravity modes, the pressure perturbation vanishes, but now there is the possibility for
 449 horizontal coupling through horizontal advection.

450 As for the two-fluid gravity modes, the frequency of these motions is independent of their vertical
 451 structure. Again there is no unique way to define a set of vertical normal mode structures, but a
 452 convenient choice is $u_1 \propto (\sigma_2^{(r)}/\sigma_1^{(r)}\rho^{(r)})^{1/2} \exp\{imz\}$, etc., so that the modes for different m are
 453 orthogonal with respect to (77).

454 For any given (k, l, m) there are two branches of these normal modes, with frequencies $\omega =$
 455 $\pm f - \beta k/K^2$. The frequency is independent of m , so their vertical group velocity is zero. Their
 456 horizontal group velocity is small but non-zero, similar to that of barotropic Rossby waves.

457 4.4. Relabelling modes

458 One further branch of modes is possible, in which $u_i, v_i, w_i, \rho_i, \eta_i$, and p all vanish. The frequency
 459 is zero, but the volume fraction perturbations are non-zero and satisfy

$$\sigma_1 = -\sigma_2. \quad (82)$$

460 These represent modes in which some fluid has been relabelled, but the physical state of the system
 461 is identical to the basic state. The energy perturbation (77) vanishes for these modes.

462 4.5. Normal modes for $n > 2$ fluid components

463 The normal modes for the two-fluid case discussed above generalize in a straightforward way to
 464 the case of any number n of fluid components. The derivation of (71) carries through exactly as
 465 before, so we have the same branches of single-fluid-equivalent modes: eastward and westward
 466 propagating acoustic modes, eastward and westward propagating inertio-gravity modes, and
 467 westward propagating Rossby modes. As before, ρ_i, η_i and \mathbf{u}_i are all independent of i .

468 The two branches of two-fluid gravity modes become $2(n-1)$ branches of multi-fluid gravity
 469 modes. They all have u_i and v_i identically zero, and satisfy $\sum_i \sigma_i^{(r)} \rho^{(r)} w_i = 0$. One way to confirm
 470 the number of branches is to note that, for $\omega = \pm N$ (hence the factor 2), and for any vertical profile
 471 $w_1(z)$, there are $n-1$ linearly independent modes with $\sigma_1^{(r)} w_1 + \sigma_i^{(r)} w_i = 0$ and $w_j = 0$ when $j \neq i$,
 472 for $i = 2, \dots, n$. If an orthogonal set of modes is needed then this can be obtained (non-uniquely)
 473 by writing the vertical velocity of the j^{th} mode as

$$\sigma_i^{(r)} \rho^{(r)1/2} w_i^{(j)} = a_i^{(j)} f^{(m)}(z) \quad (83)$$

474 where

$$f^{(m)}(z) = \left(\sum_i \frac{1}{\sigma_i^{(r)}} \right)^{-1/2} \exp(imz) \quad (84)$$

475 and

$$a_i^{(j)} = \exp\left(\frac{2\pi i}{n} i j\right). \quad (85)$$

476 As before, all of these modes have zero group velocity.

477 In an analogous way, the two branches of two-fluid inertial modes become $2(n-1)$ branches of
 478 multi-fluid inertial modes. They all have ρ_i, η_i and w_i identically zero, and satisfy $\sum_i \sigma_i^{(r)} \mathbf{u}_i = 0$.
 479 An orthogonal set of modes is obtained by defining the u_i for the j^{th} mode as

$$\sigma_i^{(r)} \rho^{(r)1/2} u_i^{(j)} = a_i^{(j)} f^{(m)}(z), \quad (86)$$

480 etc., with $f^{(m)}(z)$ and $a_i^{(j)}$ as above. As before, all these modes have zero vertical group velocity.

481 Finally the branch of two-fluid relabelling modes becomes $n-1$ branches of multi-fluid relabelling
 482 modes.

483 4.6. Normal modes in the Boussinesq equations

To interpret the normal modes it is helpful to consider the two-fluid Boussinesq equations. These equations eliminate acoustic modes ab initio, and if we further restrict attention to the f -plane we more transparently expose the physically new modes of the system. Allowing density to vary only in terms associated with gravity, the Boussinesq versions of equations (8)-(12) are given by, now including the Coriolis term,

$$\sum_{i=1}^n \sigma_i = 1, \quad \text{Volume fractions must sum to unity} \quad (87)$$

$$\frac{\partial \sigma_i}{\partial t} + \nabla \cdot (\sigma_i \mathbf{u}_i) = 0, \quad \text{Mass or volume conservation} \quad (88)$$

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i + \nabla \bar{\phi} + f \mathbf{k} \times \mathbf{u}_i - \mathbf{k} b_i = 0, \quad \text{Momentum equation} \quad (89)$$

$$\frac{\partial b_i}{\partial t} + \mathbf{u}_i \cdot \nabla b_i = 0, \quad \text{Buoyancy conservation} \quad (90)$$

484 where \mathbf{k} is the unit vector in the vertical, $\bar{\phi}$ is the deviation of the kinematic filter-scale pressure
485 (\bar{p}/ρ_0) from a hydrostatic reference state, and b_i is the buoyancy of the i -th component.

If we take f to be a constant and linearize these equations around a state of rest, of given basic-state volume fractions $\sigma_i^{(r)}$, and constant stratification, N^2 , we obtain,

$$\sum_i \sigma_i = 0 \quad \text{perturbation volume fractions sum to zero,} \quad (91)$$

$$-i\omega \sigma_i + \sigma_i^{(r)} (iku_i + ilv_i + imw_i) = 0 \quad \text{mass conservation,} \quad (92)$$

$$-i\omega u_i - fv_i + ik\phi = 0 \quad u \text{ momentum,} \quad (93)$$

$$-i\omega v_i + fv_i + il\phi = 0 \quad v \text{ momentum,} \quad (94)$$

$$-i\omega w_i + im\phi - b_i = 0 \quad w \text{ momentum,} \quad (95)$$

$$-i\omega b_i + N^2 w_i = 0 \quad \text{buoyancy.} \quad (96)$$

486 The variables σ_i , u_i , v_i , w_i , b_i , ϕ are now all perturbation quantities. Eliminating b_i from the
487 buoyancy and vertical momentum equations gives

$$(\omega^2 - N^2) = \omega m \phi. \quad (97)$$

488 The two horizontal momentum equations may be written as

$$u_i = \frac{k\omega + ilf}{\omega^2 - f^2} \phi, \quad v_i = \frac{l\omega - ikf}{\omega^2 - f^2} \phi. \quad (98)$$

489 If the pressure perturbation ϕ is non-zero then these equations reduce to

$$\omega^2 = \frac{m^2 f^2 + N^2 (k^2 + l^2)}{(k^2 + l^2 + m^2)}. \quad (99)$$

490 This is just the standard dispersion relation for an inertia-gravity wave (e.g., Vallis 2017, chapter
491 7). If the pressure perturbation is zero then, using (97) and (98), we find the *additional* two-fluid
492 modes,

$$\omega^2 = N^2, \quad u_i = v_i = 0, \quad (100)$$

493 and

$$\omega^2 = f^2, \quad w_i = 0. \quad (101)$$

494 These modes are gravity wave modes and inertial modes respectively. They are similar to their one-
495 fluid counterparts, but they obey an additional constraint that arises from (91) and (92), namely

496

$$\sum_i \sigma_i^{(r)} (iku_i + ilv_i + imw_i) = 0. \quad (102)$$

497 The gravity and inertial waves must then obey, respectively,

$$\omega^2 = N^2, \quad \sigma_1^{(r)} w_1 = -\sigma_2^{(r)} w_2, \quad (103)$$

498 and

$$\omega^2 = f^2, \quad \sigma_1^{(r)} (ku_1 + lv_1) = -\sigma_2^{(r)} (ku_2 + lv_2). \quad (104)$$

499 The two-fluid gravity-wave mode, (103), is of particular note. Since $\sigma_i^{(r)}$ is positive for all i , the mode
 500 represents ascending motion by one fluid and descending motion by the other fluid. (The mode is of
 501 course present in the fully compressible equations but in the Boussinesq derivation it is seen most
 502 plainly.) It is not an unphysical mode, since the conditionally filtered equations represent motion
 503 on a large scale. Rather, within that large-scale there is an oscillation consisting of ascent of one
 504 fluid component and descent of the other. It may be the most important new mode introduced by
 505 the conditional filtering, since subfilter-scale buoyancy-driven motions such as cumulus convection
 506 will project strongly onto this mode.

507 4.7. Behaviour in the limit of short filter scale

508 One of the motivations for the introduction of the conditionally filtered equations was the desire to
 509 formulate a mathematical framework that could represent cumulus convection both in the unresolved
 510 case, where the scale of convection is much smaller than the filter scale, and in the resolved case,
 511 where the scale of convection is greater than the filter scale, with the potential to be able to work
 512 also for intermediate cases in the so-called ‘grey zone’. Since the usual single-fluid equations are
 513 able to represent convection in the resolved case, a desirable property of the conditionally filtered
 514 equations is that their behaviour should smoothly reduce to that of the single-fluid equations as the
 515 filter scale is reduced. Among other things, this will require the parameterized relabelling terms and
 516 inter-fluid pressure forces to behave appropriately in the limit of short filter scale. Here we focus on
 517 the behaviour of the normal modes.

518 In the limit of short filter scale it is desirable that the flow field \mathbf{u} should be represented more
 519 and more completely by the mean filter-scale field $\bar{\mathbf{u}}^*$. That is, \mathbf{u} should project more and more
 520 onto the single-fluid-equivalent normal modes, with the multi-fluid normal modes as well as the
 521 subfilter-scale contributions becoming less significant.

522 Let us examine this behaviour in the simplest possible scenario. Consider a field w that is a
 523 function only of x , such as that illustrated in figure 1, and suppose that there are two Lagrangian
 524 labels I_1 and I_2 , which pick out updrafts and environment, respectively. For simplicity take the
 525 density to be constant, so that it can be ignored in the rest of this subsection. Using the normal
 526 mode structures for gravity modes discussed above, at each point x the filter-scale vertical mass
 527 fluxes $\sigma_1^{(r)} w_1$ and $\sigma_2^{(r)} w_2$ can be projected onto the single-fluid-equivalent and two-fluid modes:

$$\begin{pmatrix} \sigma_1^{(r)} w_1 \\ \sigma_2^{(r)} w_2 \end{pmatrix} = A_1 \begin{pmatrix} \sigma_1^{(r)} \\ \sigma_2^{(r)} \end{pmatrix} + A_2 \begin{pmatrix} -(\sigma_1^{(r)} \sigma_2^{(r)})^{1/2} \\ (\sigma_1^{(r)} \sigma_2^{(r)})^{1/2} \end{pmatrix}, \quad (105)$$

528 where A_1 is the amplitude of the single-fluid-equivalent mode and A_2 is the amplitude of the
 529 two-fluid mode. Solving for A_1 and A_2 gives

$$A_1 = \sigma_1^{(r)} w_1 + \sigma_2^{(r)} w_2 \quad (106)$$

530 and

$$A_2 = (\sigma_1^{(r)} \sigma_2^{(r)})^{1/2} (w_2 - w_1) = (\sigma_1^{(r)} / \sigma_2^{(r)})^{1/2} \sigma_2^{(r)} w_2 - (\sigma_2^{(r)} / \sigma_1^{(r)})^{1/2} \sigma_1^{(r)} w_1. \quad (107)$$

Table 1. Summary of the normal modes of the conditionally filtered equations with n fluid components.

Family	Physical character	Number of branches		
		Full equations	Equal- \mathbf{v}_i variant	Boussinesq
Single-fluid-equivalent	Acoustic	2	2	0
	Inertio-gravity	2	2	2
	Rossby	1	1	1
Multi-fluid gravity	Deep gravity waves	$2(n-1)$	$2(n-1)$	$2(n-1)$
Multi-fluid inertial	Inertial waves	$2(n-1)$	0	$2(n-1)$
Multi-fluid relabelling		$(n-1)$	$(n-1)$	$(n-1)$
Total number of branches		$5n$	$3n+2$	$5n-2$

531 We consider the behaviour of this decomposition as the filter scale approaches zero, holding the
 532 field $w(x)$ and the labels $I_1(x)$ and $I_2(x)$ fixed. The amplitude of the single-fluid-equivalent mode A_1
 533 is just the total mass-weighted filter-scale velocity \bar{w}^* . It will approach w in the limit of short filter
 534 scale, as it would in the usual unconditionally filtered equations. The amplitude of the two-fluid
 535 mode A_2 , on the other hand, depends on $\sigma_1^{(r)}$ and $\sigma_2^{(r)}$. Since $\sigma_i^{(r)} = \bar{I}_i$, $\sigma_i^{(r)}$ will approach I_i as the
 536 filter scale diminishes. Thus, at almost every point in the domain either $\sigma_1^{(r)}$ or $\sigma_2^{(r)}$ will approach
 537 zero. (There will be a finite number of exceptions at those points where the I_i switch between zero
 538 and one.) If the filter kernel is non-zero only over a finite range, which shrinks with the filter scale,
 539 then $\sigma_1^{(r)}$ will become equal to zero when there are no points with $I_1 = 1$ within range of the filter,
 540 and similarly for $\sigma_2^{(r)}$. In this way, the amplitude A_2 will approach zero, or actually become zero,
 541 at almost every point in the fluid as the filter scale tends to zero.

542 Thus, as desired, the representation of the complete flow field by the conditionally filtered
 543 equations converges to its representation by the single-fluid or unconditionally filtered equations,
 544 and the contribution from multi-fluid modes tends to zero, as the filter scale tends to zero.

545 4.8. Summary of normal modes

546 Table 1 summarizes the normal modes of the conditionally filtered equations. It is notable that
 547 the only acoustic modes are the single-fluid-equivalent acoustic modes. This is expected since all
 548 fluid components have the same pressure, and, as mentioned in the Introduction, this is a desirable
 549 feature of the conditionally filtered equations. The table also shows that the effect of constraining
 550 the horizontal velocities of different fluid components to be equal is to remove the multi-fluid inertial
 551 modes; the other modes are unchanged. See Appendix B for a brief discussion.

552 5. Variational formulation

553 Hamilton's principle expresses the equations of motion as the stationarity of the action

$$\delta\mathcal{L} = 0 \quad (108)$$

554 under arbitrary small variations of some state variables \mathbf{X} , where the action \mathcal{L} is the integral of the
 555 Lagrangian density $L(\mathbf{X})$ over space and time:

$$\mathcal{L} = \iint dt d\mathbf{x} L(\mathbf{X}). \quad (109)$$

556 The Lagrangian density is essentially the kinetic energy density minus the potential and internal
 557 energy density, but there are different flavours of the idea depending on whether an Eulerian or
 558 Lagrangian description of the fluid motion is of interest, and whether constraints such as

559 conservation of mass are imposed through restricting the allowed perturbations $\delta\mathbf{X}$ or through
 560 Lagrange multipliers in (108) (e.g. Salmon 1998).

561 In this section we focus on an Eulerian description of the fluid motion, following Salmon (1998)
 562 (chapter 7, section 8). As we would for a single fluid, we impose conservation of mass, and
 563 also material conservation of entropy and material conservation of another Lagrangian label, via
 564 Lagrange multipliers, for each fluid component i . (See Salmon 1998, for a discussion of how the
 565 extra Lagrangian label relates to Lin's constraint). Equality of the pressures in the different fluid
 566 components and the requirement for the volume fractions to sum to unity are also imposed through
 567 Lagrange multipliers. Hence the appropriate expression for \mathcal{L} is

$$\begin{aligned} \mathcal{L} = \iint dt d\mathbf{x} \sum_i \left\{ \sigma_i \rho_i \frac{1}{2} |\mathbf{u}_i|^2 - \sigma_i \rho_i \Phi \right. \\ \left. - \sigma_i \rho_i e_i(\rho_i, \eta_i) - \sigma_i \rho_i \frac{D_i \phi_i}{Dt} \right. \\ \left. - \sigma_i \rho_i A_i \frac{D_i \eta_i}{Dt} - \sigma_i \rho_i C_i \frac{D_i \lambda_i}{Dt} \right\} \\ - \sum_{i \neq 1} \nu_i (p_1 - p_i) - \mu \left(\sum_i \sigma_i - 1 \right). \end{aligned} \quad (110)$$

568 Here ϕ_i is the Lagrange multiplier associated with conservation of mass of the i^{th} fluid, A_i is the
 569 Lagrange multiplier associated with material conservation of η_i , λ_i is a Lagrangian label for the
 570 i^{th} fluid and C_i is the Lagrange multiplier associated with material conservation of λ_i , the ν_i are
 571 a set of Lagrange multipliers associated with the equality of the pressure in the different fluid
 572 components, and μ is the Lagrange multiplier associated with the volume fractions summing to
 573 unity. The pressure p_i is related to the internal energy density e_i by

$$p_i = \rho_i^2 \left. \frac{\partial e_i}{\partial \rho_i} \right|_{\eta_i}. \quad (111)$$

574 Hamilton's principle now states that $\delta\mathcal{L} = 0$ for arbitrary, independent, small variations of σ_i , ρ_i ,
 575 η_i , \mathbf{u}_i , λ_i , ϕ_i , A_i , C_i , ν_i , and μ . Boundary conditions (in space and time) are assumed to be such
 576 that any boundary terms arising through integration by parts vanish.

577 For variations in μ , $\delta\mathcal{L} = 0$ implies

$$\sum_i \sigma_i - 1 = 0, \quad (112)$$

578 in agreement with (8). For variations in ν_i , $\delta\mathcal{L} = 0$ implies

$$p_i = p_1; \quad (113)$$

579 thus the pressures in all the fluid components take the same value, which we can call \bar{p} for consistency
 580 with the earlier notation. For variations in A_i and C_i , $\delta\mathcal{L} = 0$ implies

$$\frac{D_i \eta_i}{Dt} = 0, \quad (114)$$

581 consistent with (18), and

$$\frac{D_i \lambda_i}{Dt} = 0. \quad (115)$$

582 For variations in ϕ_i , $\delta\mathcal{L} = 0$ implies, after integration by parts

$$\frac{\partial}{\partial t} (\sigma_i \rho_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) = 0, \quad (116)$$

583 consistent with (9).

584 For variations in σ_i and in ρ_i , $\delta\mathcal{L} = 0$ implies

$$\rho_i \left(\frac{1}{2} |\mathbf{u}_i|^2 - \Phi - e_i - \frac{D_i \phi_i}{Dt} \right) - \mu = 0 \quad (117)$$

585 and

$$\frac{1}{2} |\mathbf{u}_i|^2 - \Phi - e_i - \frac{\bar{p}}{\rho_i} - \frac{D_i \phi_i}{Dt} = 0, \quad (118)$$

586 where (114) and (115) have been used to eliminate terms involving $D_i \eta_i / Dt$ and $D_i \lambda_i / Dt$, and
587 (113) has been used to write $p_i = p_1 = \bar{p}$.

588 Taking (117) minus ρ_i times (118) shows that

$$\mu = \bar{p}. \quad (119)$$

589 Thus \bar{p} is the Lagrange multiplier corresponding to the requirement for the volume fractions to
590 sum to unity. This is reflected in the fact that the volume fractions summing to unity is crucial
591 for determining \bar{p} ; see (140). This result is analogous to the well-known interpretation of pressure
592 as the Lagrange multiplier corresponding to the incompressibility condition for an incompressible
593 fluid.

594 For variations in η_i and λ_i , $\delta\mathcal{L} = 0$ implies, after integration by parts,

$$-\sigma_i \rho_i T_i + \frac{\partial}{\partial t} (\sigma_i \rho_i A_i) + \nabla \cdot (\sigma_i \rho_i A_i \mathbf{u}_i) = 0, \quad (120)$$

595 and

$$\frac{\partial}{\partial t} (\sigma_i \rho_i C_i) + \nabla \cdot (\sigma_i \rho_i C_i \mathbf{u}_i) = 0, \quad (121)$$

596 where

$$T_i = \left. \frac{\partial e_i}{\partial \eta_i} \right|_{\rho_i} \quad (122)$$

597 is the temperature of the i^{th} fluid. Finally, for variations in \mathbf{u}_i , $\delta\mathcal{L} = 0$ implies

$$\mathbf{u}_i - \nabla \phi_i - A_i \nabla \eta_i - C_i \nabla \lambda_i = 0. \quad (123)$$

598 To obtain the equations of motion we systematically eliminate the remaining Lagrange multipliers
599 and the materially conserved scalars λ_i . Taking (120) minus A_i times (116) gives

$$\frac{D_i A_i}{Dt} = T_i, \quad (124)$$

600 while (121) minus C_i times (116) gives

$$\frac{D_i C_i}{Dt} = 0. \quad (125)$$

601 Taking $\partial/\partial t$ of (123) gives

$$\begin{aligned} \frac{\partial \mathbf{u}_i}{\partial t} - \nabla \frac{\partial \phi_i}{\partial t} - \frac{\partial A_i}{\partial t} \nabla \eta_i - A_i \nabla \frac{\partial \eta_i}{\partial t} \\ - \frac{\partial C_i}{\partial t} \nabla \lambda_i - C_i \nabla \frac{\partial \lambda_i}{\partial t} = 0. \end{aligned} \quad (126)$$

602 Taking $\mathbf{u}_i \cdot (123)$, subtracting (118), and using (114) and (115) gives

$$\frac{1}{2} |\mathbf{u}_i|^2 + \Phi + e_i + \frac{\bar{p}}{\rho_i} + \frac{\partial \phi_i}{\partial t} + A_i \frac{\partial \eta_i}{\partial t} + C_i \frac{\partial \lambda_i}{\partial t} = 0. \quad (127)$$

603 Then taking (126) plus the gradient of (127) and using (114), (115), (124), and (125) gives

$$\begin{aligned} \frac{\partial \mathbf{u}_i}{\partial t} + \nabla \left(\frac{1}{2} |\mathbf{u}_i|^2 + \Phi + e_i + \frac{\bar{p}}{\rho_i} \right) \\ = (\mathbf{u}_i \cdot \nabla \eta_i) \nabla A_i + (T_i - \mathbf{u}_i \cdot \nabla A_i) \nabla \eta_i \\ + (\mathbf{u}_i \cdot \nabla \lambda_i) \nabla C_i - (\mathbf{u}_i \cdot \nabla C_i) \nabla \lambda_i. \end{aligned} \quad (128)$$

604 Taking the curl of (123) gives

$$\zeta_{\mathbf{i}} = \nabla \times \mathbf{u}_i = \nabla A_i \times \nabla \eta_i + \nabla C_i \times \nabla \lambda_i, \quad (129)$$

605 so that

$$\begin{aligned} \zeta_{\mathbf{i}} \times \mathbf{u}_i &= (\mathbf{u}_i \cdot \nabla A_i) \nabla \eta_i - (\mathbf{u}_i \cdot \nabla \eta_i) \nabla A_i \\ &+ (\mathbf{u}_i \cdot \nabla C_i) \nabla \lambda_i - (\mathbf{u}_i \cdot \nabla \lambda_i) \nabla C_i. \end{aligned} \quad (130)$$

606 Hence, (128) simplifies to

$$\frac{\partial \mathbf{u}_i}{\partial t} + \zeta_{\mathbf{i}} \times \mathbf{u}_i + \nabla \left(\frac{1}{2} |\mathbf{u}_i|^2 + \Phi + e_i + \frac{\bar{p}}{\rho_i} \right) = T_i \nabla \eta_i. \quad (131)$$

607 Finally, noting that

$$\nabla \left(e_i + \frac{\bar{p}}{\rho_i} \right) = T_i \nabla \eta_i + \frac{1}{\rho_i} \nabla \bar{p}, \quad (132)$$

608 (131) reduces to

$$\frac{\partial \mathbf{u}_i}{\partial t} + \zeta_{\mathbf{i}} \times \mathbf{u}_i + \frac{1}{\rho_i} \nabla \bar{p} + \nabla \left(\frac{1}{2} |\mathbf{u}_i|^2 + \Phi \right) = 0, \quad (133)$$

609 which agrees with (34) in the absence of its right-hand side.

610 Equations (112), (116), (114), and (133) derived from the variational method thus agree with the
611 conditionally filtered equations (8), (9), (18), and (34).

612 6. Summary and discussion

613 We have documented the conservation properties, normal modes, and a variational formulation
614 of the conditionally filtered equations. The results confirm that these equations have a natural
615 mathematical structure, respecting key physical properties, lending them some credibility for their
616 use in modelling atmospheric flows. In particular the normal mode results, with real frequency ω
617 provided $N^2 > 0$, imply that the equations are free from spurious unphysical instabilities, at least
618 for small perturbations to a simple basic state. Furthermore, the modes themselves have a sensible
619 physical interpretation. The usual Rossby, inertia-gravity and acoustic modes exist and have the
620 same frequency and structure as in the single-fluid case. In addition, we have identified inertia and
621 gravity modes with zero pressure perturbation in which the fluid components move separately, and
622 in general in opposite vertical and horizontal directions. This is precisely a property one might wish
623 for when modelling subgrid-scale convection, in which some of the subgrid-scale fluid ascends while
624 some of it descends. Furthermore, the amplitude of these modes goes to zero as the filter scale
625 diminishes, which is an attractive property when considering how the fluid system might behave as
626 the model resolution increases.

627 The availability of a variational formulation implies that a variety of standard approximations,
628 such as hydrostatic or pseudo-incompressible, should be applicable to the conditionally averaged
629 equations, leading to simpler equation sets that might be appropriate for some applications, both
630 theoretical and numerical. We have already begun to experiment with hydrostatic and Boussinesq
631 versions of the conditionally filtered equations. It is even possible to make different approximations
632 in different fluid components, for example making one component hydrostatic (though some thought
633 must then be given to the relabelling terms if strict energy consistency is required). However, one
634 would of course normally wish for the fluid component that represents convecting fluid to be treated
635 non-hydrostatically. The results may also be of use in developing and testing numerical methods for
636 the solution of the conditionally filtered equations. For example, numerical methods should respect
637 the conservation properties of the continuous equations, at least to within numerical truncation
638 error. The normal modes derived here provide known, exact, stable, linear solutions that a numerical
639 method should be able to reproduce.

640 Finally, the results also give some early indications of the suitability of the conditionally filtered
 641 equations for modelling cumulus convection, the application for which they were originally proposed.
 642 The multi-fluid gravity modes show that the conditionally filtered equations can capture the essential
 643 dynamics of vertical buoyancy-driven motion of one fluid component relative to another, which will
 644 be required in order to model convective updrafts and downdrafts. Of course the subfilter-scale
 645 terms, inter-fluid pressure terms, and relabelling terms, that is the right-hand sides of (2)–(4) that
 646 would need to be parameterized, are also of leading order importance for such flows (e.g. Siebesma *et al.*
 647 2007; de Rooy *et al.* 2013; Romps and Charn 2015). On the other hand, the vanishing group velocity
 648 of the multi-fluid gravity modes suggests that the conditionally filtered equations would not help
 649 to capture convectively generated gravity waves (e.g. Lane and Moncrieff 2010) unless those waves
 650 project onto the single-fluid-equivalent gravity modes. It is also conceivable that, away from the
 651 region of convection, the two-fluid gravity modes and inertial modes might have undesirable
 652 behaviour. For example, their dispersion properties might lead to behaviour analogous to that
 653 of some numerical computational modes. If this turns out to be the case then some measures to
 654 suppress them might be needed. The analysis presented here should, at least, help to identify such
 655 problems.

656 Acknowledgements

657 We are grateful to two anonymous reviewers for their constructive comments on an earlier version
 658 of this paper. This work was funded by the Natural Environment Research Council under grant
 659 NE/N013123/1 as part of the ParaCon programme.

660 Appendix A. Prognostic equations for σ_i , ρ_i and \bar{p} , and a diagnostic equation for \bar{p}

661 It is not immediately obvious how equations (8)–(12) imply the time evolution of all variables. As
 662 well as the fundamental question of whether the system of equations is closed, this is relevant to
 663 the design of numerical methods for the solution of the conditionally averaged equations.

664 First note that (9) can be expanded as

$$\sigma_i \frac{\partial \rho_i}{\partial t} + \rho_i \frac{\partial \sigma_i}{\partial t} + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) = 0. \quad (134)$$

665 The time derivative of (12) can be written

$$\frac{1}{c_i^2 \rho_i} \frac{\partial \bar{p}}{\partial t} = \frac{1}{\rho_i} \frac{\partial \rho_i}{\partial t} + Q_i \frac{\partial \eta_i}{\partial t}, \quad (135)$$

666 where $c_i^2 = \partial P / \partial \rho_i |_{\eta_i}$ is the sound speed squared in the i^{th} fluid, and $Q_i = -\partial \ln \rho_i / \partial \eta_i |_P$
 667 (compare (54)).

668 Multiplying by σ_i and substituting from (134) and (18) gives

$$\frac{\sigma_i}{c_i^2 \rho_i} \frac{\partial \bar{p}}{\partial t} = -\frac{\partial \sigma_i}{\partial t} - \frac{1}{\rho_i} \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) - \sigma_i Q_i \mathbf{u}_i \cdot \nabla \eta_i. \quad (136)$$

669 Summing over i and using (8) gives an equation for the rate of change of \bar{p} in terms of known
 670 quantities:

$$\left(\sum_i \frac{\sigma_i}{c_i^2 \rho_i} \right) \frac{\partial \bar{p}}{\partial t} = -\sum_i \frac{1}{\rho_i} \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) - \sum_i \sigma_i Q_i \mathbf{u}_i \cdot \nabla \eta_i. \quad (137)$$

671 Having obtained $\partial \bar{p} / \partial t$, $\partial \rho_i / \partial t$ follows from (135), and $\partial \sigma_i / \partial t$ follows from (134).

672 Alternatively, a diagnostic equation for \bar{p} in terms of the predicted quantities $\sigma_i \rho_i$ and η_i can be
 673 derived as follows. The equation of state can be rearranged to make ρ_i the subject:

$$\rho_i = R(\bar{p}, \eta_i). \quad (138)$$

674 Then

$$\sigma_i = \frac{(\sigma_i \rho_i)}{R(\bar{p}, \eta_i)}, \quad (139)$$

675 and summing over i gives

$$\sum_i \frac{(\sigma_i \rho_i)}{R(\bar{p}, \eta_i)} = 1. \quad (140)$$

676 Thus we have a single equation for the single unknown \bar{p} .

677 In the special case of a perfect gas equation of state, and predicting potential temperature θ_i
678 instead of entropy η_i , (140) simplifies to

$$\left(\frac{\bar{p}}{p_0}\right)^{(1-R_d/c_p)} = \frac{R_d}{p_0} \sum_i \sigma_i \rho_i \theta_i, \quad (141)$$

679 where p_0 is a constant reference pressure, R_d is the gas constant, and c_p is the specific heat capacity
680 at constant pressure (Thuburn *et al.* 2018).

681 Appendix B. Properties of the equal- \mathbf{v}_i variant

682 This appendix briefly examines how the results discussed in the main body of the paper carry over,
683 or are modified, for the variant of the conditionally filtered equations in which all fluid components
684 have the same horizontal velocity.

685 Conservation properties

686 The equal- \mathbf{v}_i variant effectively assumes that the $-\mathbf{b}_i - \sum_j \mathbf{d}_{ij}$ terms on the right-hand side of
687 (4) are exactly what is needed to maintain equality of the \mathbf{v}_i . Since (9), (10), and (18) do not
688 involve $-\mathbf{b}_i - \sum_j \mathbf{d}_{ij}$, the conservation laws for mass and entropy, and the material conservation of
689 entropy, remain the same as for the full equations. Equation (14) for the evolution of $\bar{\mathbf{v}}^*$ is obtained
690 by summing (4) over i (neglecting $\mathbf{F}_{SF}^{\mathbf{u}_i}$ and \mathcal{M}_{ij}) and using (6) and (7), and is entirely consistent
691 with (22). Therefore, the equal- \mathbf{v}_i variant also conserves momentum.

692 The derivation of the energy equation (32) does not depend on any assumption about the
693 $\mathbf{b}_i + \sum_j \mathbf{d}_{ij}$ terms, so it holds for the equal- \mathbf{v}_i variant too. Because the $\mathbf{b}_i + \sum_j \mathbf{d}_{ij}$ terms can
694 no longer be assumed zero, we can no longer make the step to (33). However, the contributions to
695 the total energy change from the horizontal components of $\mathbf{b}_i + \sum_j \mathbf{d}_{ij}$ sum to zero, leaving

$$\begin{aligned} \frac{\partial}{\partial t} \left(\sum_i \sigma_i \rho_i \varepsilon_i \right) + \nabla \cdot \left(\sum_i (\sigma_i \rho_i \mathbf{u}_i \varepsilon_i + \sigma_i \mathbf{u}_i \bar{p}) \right) \\ = - \sum_i w_i \left(b_i^{(z)} + \sum_j d_{ij}^{(z)} \right); \end{aligned} \quad (142)$$

696 only the vertical components (indicated by superscript (z)) contribute to the change in total energy.

697 Material conservation of potential vorticity (41) does depend on the vanishing of the $\mathbf{b}_i + \sum_j \mathbf{d}_{ij}$
698 terms and therefore no longer holds for the equal- \mathbf{v}_i variant. The impermeability theorem, however,
699 involves no assumptions about the forcing terms and continues to hold.

700 Normal modes

701 The single-fluid-equivalent modes, multi-fluid gravity modes, and the relabelling modes found in
702 section 4 all have identical \mathbf{v}_i for all i . Therefore they continue to exist, with exactly the same
703 frequency and structure, in the equal- \mathbf{v}_i variant. The multi-fluid inertial modes, on the other hand,
704 must satisfy $\sum_i \sigma_i^{(r)} \mathbf{v}_i = 0$. This could only hold with equal \mathbf{v}_i if $\mathbf{v}_i = 0$ for all i , but then there
705 would be no disturbance at all. Thus, the multi-fluid inertial modes do not exist in the equal- \mathbf{v}_i
706 variant. These rather general arguments are confirmed by detailed calculation analogous to that in
707 section 4.

708 *Variational formulation*

709 We have not, so far, been able to discover a suitable variational formulation of the equal- \mathbf{v}_i variant of
710 the conditionally filtered equations. There appear to be considerable technical subtleties associated
711 with the equal- \mathbf{v}_i constraint.

712 **References**

- 713 Abgrall R, Karni S. 2001. Computations of compressible multifluids. *J. Comput. Phys.* **169**: 594–623.
- 714 Aluie H, Hecht M, Vallis GK. 2018. Mapping the energy cascade in the North Atlantic ocean: The coarse-
715 graining approach. *J. Phys. Oceanogr.* **48**: 225–244.
- 716 Arakawa A, Lamb VR. 1977. Computational design of the basic dynamical processes of the UCLA general
717 circulation model. In: *General Circulation Models of the Atmosphere, Methods in Computational Physics*,
718 vol. 17, Chang J (ed). Academic Press, pp. 173–265.
- 719 Baldauf M, Brdar S. 2013. An analytic solution for linear gravity waves in a channel as a test for numerical
720 models using the non-hydrostatic, compressible Euler equations. *Quart. J. Roy. Meteor. Soc.* **139**: 1977–
721 1989.
- 722 Bohren CF, Albrecht BA. 1998. *Atmospheric Thermodynamics*. Oxford University Press.
- 723 Bretherton CS, Schär C. 1993. Flux of potential vorticity substance: A simple derivation and a uniqueness
724 property. *J. Atmos. Sci.* **50**: 1834–1836.
- 725 Cotter CJ, Holm DD. 2014. Variational formulation of sound-proof models. *Quart. J. Roy. Meteor. Soc.*
726 **140**: 1966–1973.
- 727 de Roode SR, Siebesma AP, Jonker HJJ, de Voogd Y. 2012. Parameterization of the vertical velocity equation
728 for shallow cumulus clouds. *J. Atmos. Sci.* **140**: 2424–2436.
- 729 de Rooy WC, Bechtold P, Fröhlich K, Hohenegger C, Jonker H, Mironov D, Siebesma AP, Teixeira J, Yano
730 JI. 2013. Entrainment and detrainment in cumulus convection: an overview. *Quart. J. Roy. Meteor. Soc.*
731 **139**: 1–19.
- 732 Drew DA. 1983. Mathematical modeling of two-phase flow. *Annu. Rev. Fluid Mech.* **15**: 261–291.
- 733 Dubos T, Voitus F. 2014. A semihydrostatic theory of gravity-dominated compressible flow. *J. Atmos. Sci.*
734 **71**: 4621–4638.
- 735 Frisch U. 1995. *Turbulence: The legacy of A. N. Kolmogorov*. Cambridge University Press: Cambridge, U.K.
- 736 Gill AE. 1982. *Atmosphere-Ocean Dynamics*. Academic Press: New York.
- 737 Gross M, Wan H, Rasch PJ, Caldwell PM, Klocke DLWD, Jablonowski C, Thatcher DR, Wood N,
738 Beare MCB, Willett M, Lemarié F, Blayo E, Malardel S, Termonia P, Gassmann A, Lauritzen PH,
739 Johansen H, Zarzycki CM, Sakaguchi K, Leung R. 2017. Recent progress and review of issues related
740 to Physics Dynamics Coupling in geophysical models. *Submitted to Monthly Weather Review* URL
741 <https://arxiv.org/abs/1605.06480v2>.
- 742 Haynes PH, McIntyre ME. 1987. On the evolution of vorticity and potential vorticity in the presence of
743 diabatic heating and frictional or other sources. *J. Atmos. Sci.* **44**: 828–841.
- 744 Hoskins BJ, McIntyre ME, Robinson AW. 1985. On the use and significance of isentropic potential vorticity
745 maps. *Quart. J. Roy. Meteor. Soc.* **111**: 877–946.
- 746 IOC, SCOR, IAPSO. 2010. The international thermodynamic equation of seawater–2010: Calculation and
747 use of thermodynamic properties. Intergovernmental Oceanographic Commission, Manuals and Guides
748 No. 56, URL <http://www.TEOS-10.org>.
- 749 Lane TP, Moncrieff MW. 2010. Characterization of momentum transport associated with organized moist
750 convection and gravity waves. *J. Atmos. Sci.* **67**: 3208–3225.
- 751 Leonard A. 1975. Energy cascade in large-eddy simulations of turbulent fluid flows. *Adv. Geophys.* **18**(i):
752 237–248.
- 753 Pauluis P, Held IM. 2002. Entropy budget of an atmosphere in radiative-convective equilibrium. Part I:
754 Maximum work and frictional dissipation. *J. Atmos. Sci.* **59**: 125–139.
- 755 Peixoto JP, Oort AH. 1992. *Physics of Climate*. American Institute of Physics: New York.
- 756 Phillips NA. 1990. Dispersion processes in large-scale weather prediction. World Meteorological Organization
757 Report No. 56, URL https://library.wmo.int/pmb_ged/wmo_700_en.pdf.
- 758 Randall DA, Fowler LD. 1999. Eauliq: The Next Generation. Atmospheric Science Paper No. 673, Colorado
759 State University.
- 760 Raymond DJ. 2013. Sources and sinks of entropy in the atmosphere. *JAMES* **5**: 755–763.

- 761 Romps DM, Charn AB. 2015. Sticky thermals: Evidence for a dominant balance between buoyancy and drag
762 in cloud updrafts. *J. Atmos. Sci.* **72**: 2890–2901.
- 763 Salmon R. 1998. *Lectures on Geophysical Fluid Dynamics*. Oxford University Press: New York.
- 764 Shamir O, Paldor N. 2016. A quantitative test case for global-scale dynamical cores based on analytic wave
765 solutions of the shallow-water equations. *Quart. J. Roy. Meteor. Soc.* **142**: 2705–2714.
- 766 Siebesma AP, Soares PMM, Teixeira J. 2007. A combined eddy-diffusivity mass-flux approach for the
767 convective boundary layer. *J. Atmos. Sci.* **64**: 1230–1248.
- 768 Staniforth A. 2014. Deriving consistent approximate models of the global atmosphere using Hamilton’s
769 principle. *Quart. J. Roy. Meteor. Soc.* **140**: 2383–2387.
- 770 Thuburn J. 2008. Some conservation issues for the dynamical cores of NWP and climate models. *J. Comput.*
771 *Phys.* **227**: 3715–3730.
- 772 Thuburn J. 2017. Use of the Gibbs thermodynamic potential to express the equation of state in atmospheric
773 models. *Quart. J. Roy. Meteor. Soc.* **143**: 1185–1196.
- 774 Thuburn J, Weller H, Weller HG, Vallis GK, Beare RJ, Whittall M. 2018. A framework for convection and
775 boundary layer parameterization derived from conditional filtering. To appear in *J. Atmos. Sci.*
- 776 Thuburn J, Wood N, Staniforth A. 2002. Normal modes of deep atmospheres. I: Spherical geometry. *Quart.*
777 *J. Roy. Meteor. Soc.* **128**: 1771–1792.
- 778 Thuburn J, Woollings TJ. 2005. Vertical discretizations for compressible Euler equation atmospheric models
779 giving optimal representation of normal modes. *J. Comput. Phys.* **203**: 386–404.
- 780 Tort M, Dubos T. 2014. Usual approximations to the equations of atmospheric motion: A variational
781 perspective. *J. Atmos. Sci.* **71**: 2452–2466.
- 782 Vallis GK. 2017. *Atmospheric and Oceanic Fluid Dynamics*. Cambridge University Press, 2nd edn.
- 783 Yano JI. 2012. Mass-flux subgrid-scale parameterization in analogy with multi-component flows: a
784 formulation towards scale independence. *Geosci. Model Dev.* **5**: 1425–1440.
- 785 Yano JI, Bénard P, Couvreur F, Lahellec A. 2010. NAM-SCA: A nonhydrostatic anelastic model with
786 segmentally constant approximation. *Mon. Wea. Rev.* **138**: 1957–1974.