Linear response for intermittent maps with summable and nonsummable decay of correlations

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Abstract

We consider a family of Pomeau-Manneville type interval maps T_α, parametrized by $\alpha \in (0,1)$, with the unique absolutely continuous invariant probability measures ν_α, and rate of correlations decay $n^{1-1/\alpha}$. We show that despite the absence of a spectral gap for all $\alpha \in (0,1)$ and despite nonsummable correlations for $\alpha \geq 1/2$, the map $\alpha \mapsto \int \varphi \, d\nu_\alpha$ is continuously differentiable for $\varphi \in L^q[0,1]$ for q sufficiently large.

1 Introduction

Let $T_\alpha \colon X \to X$ be a family of transformations on a Riemannian manifold X parametrized by α and admitting unique SRB measures ν_α. Having an observable $\varphi \colon X \to \mathbb{R}$, it may be important to know how $\int \varphi \, d\nu_\alpha$ changes with α. If the map $\alpha \mapsto \int \varphi \, d\nu_\alpha$ is differentiable, then *linear response* holds.

An interesting question is, which families of maps and observables have linear response. Ruelle proved linear response in the Axiom A case [R97, R98, R09, R09.1]. It was shown in [D04, B07, M07, BS08] that spectral gap and structural stability are not necessary or sufficient conditions.

We consider a family of Pomeau-Manneville type maps with slow (polynomial) decay of correlations: $T_\alpha \colon [0,1] \to [0,1]$, given by

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } \ x \in [0,1/2] \\ 2x - 1 & \text{if } \ x \in (1/2,1] \end{cases}, \tag{1.1}$$

parametrized by $\alpha \in [0,1)$. By [LSV99], each T_α admits a unique absolutely continuous invariant probability measure ν_α, and the sharp rate of decay of correlations for Hölder observables is $n^{1-1/\alpha}$ [Y99, S02, G04, H04].

We prove linear response on the interval $\alpha \in (0,1)$, including the case when $\alpha \geq 1/2$, and correlations are not summable. This is the first time that linear response has been proved in the case of nonsummable decay of correlations. We develop a machinery which, when applied to the family T_α, yields:

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Theorem 1.1. *For any $\varphi \in C^1[0,1]$, the map $\alpha \mapsto \int \varphi \, d\nu_\alpha$ is continuously differentiable on $(0,1)$.*

Using additional structure of the family T_α, we prove a stronger result:

Theorem 1.2. *Let ρ_α be the density of ν_α. For every $\alpha, x \in (0,1) \times (0,1]$ there exists a partial derivative $\partial_\alpha \rho_\alpha(x)$. Both $\rho_\alpha(x)$ and $\partial_\alpha \rho_\alpha(x)$ are jointly continuous in α, x on $(0,1) \times (0,1]$. Moreover, for every interval $[\alpha_-, \alpha_+] \subset (0,1)$ there exists a constant K, such that for all $x \in (0,1]$ and $\alpha \in [\alpha_-, \alpha_+]$*

$$\rho_\alpha(x) \le K \, x^{-\alpha} \qquad \text{and} \qquad |\partial_\alpha \rho_\alpha(x)| \le K \, x^{-\alpha}(1 - \log x).$$

In particular, for any $q > (1-\alpha_+)^{-1}$ and observable $\varphi \in L^q[0,1]$, the map $\alpha \mapsto \int \varphi \, d\nu_\alpha$ is continuously differentiable on $[\alpha_-, \alpha_+]$.

The same problem of linear response for the family T_α has been solved independently by Baladi and Todd [BT15] using different methods. They prove that for $\alpha_+ \in (0,1)$, $q > (1-\alpha_+)^{-1}$ and $\varphi \in L^q[0,1]$, the map $\alpha \mapsto \int \varphi \, d\nu_\alpha$ is differentiable on $[0, \alpha_+)$, plus they give an explicit formula for the derivative in terms of the transfer operator corresponding to T_α. We obtain more control of the invariant measure, as in Theorem 1.2, but do not give such a formula. Instead we provide explicit formulas for ρ_α and $\partial_\alpha \rho_\alpha$ in terms of the transfer operator for the induced map (see Subsections 4.1 and 5.3), but we do not state them here because they are too technical. Whereas [BT15] were the first to treat the case $\alpha < 1/2$, we were the first to treat the case $\alpha \ge 1/2$.

In a more recent paper [BS15], Bahsoun and Saussol consider a class of dynamical systems which includes (1.1). They prove in particular that for $\beta \in (0,1)$ and $\alpha \in (0, \beta)$,

$$\lim_{\varepsilon \to 0} \sup_{x \in (0,1]} x^\beta \left| \frac{\rho_{\alpha+\varepsilon} - \rho_\alpha}{\varepsilon} - \partial_\alpha \rho_\alpha \right| = 0.$$

That is, ρ_α is differentiable as an element of a Banach space of continuous functions on $(0,1]$ with a norm $\|\varphi\| = \sup_{x \in (0,1]} x^\beta |\varphi(x)|$. We remark that for the map (1.1) this follows from Theorem 1.2.

The paper is organized as follows. In Section 2 we introduce an abstract framework, and in Section 3 we apply it to the family T_α, and prove Theorem 1.1.

Technical parts of proofs are presented separately: in Section 4 for the abstract framework, and in Section 5 for the properties of the family T_α.

Theorem 1.2 is proven in Subsection 5.3. We do not give the proof earlier in the paper, because it uses rather special technical properties of T_α.

2 Setup and Notations

Let $I \subset \mathbb{R}$ be a closed bounded interval, and $F_\alpha : I \to I$ be a family of maps, parametrized by $\alpha \in [\alpha_-, \alpha_+]$. Assume that each F_α has finitely or countably many full branches, indexed by $r \in \mathcal{R}$, the same set $\mathcal{R}$ for all α.

Technically we assume that $I = \bigcup_r [a_r, b_r]$ modulo a zero measure set (branch boundaries a_r and b_r may depend on α), and that for each r the map $F_{\alpha,r} : [a_r, b_r] \to I$ is a diffeomorphism; here $F_{\alpha,r}$ equals to F_α on (a_r, b_r), and is extended countinuously to $[a_r, b_r]$.

- We use the letter ξ for spatial variable, and notation $(\cdot)'$ for differentiation with respect to ξ, and ∂_α for differentiation with respect to α.
- Denote $G_{\alpha,r} = \left|(F_{\alpha,r}^{-1})'\right|$, defined on I. Note that $G_{\alpha,r} = \pm(F_{\alpha,r}^{-1})'$, the sign depends only on r.
- For each i let $\|h\|_{C^i} = \max(\|h\|_\infty, \|h'\|_\infty, \ldots, \|h^{(i)}\|_\infty)$ denote the C^i norm of h.
- Let m be the normalized Lebesgue measure on I, and P_α be the transfer operator for F_α with respect to m. By definition, $\int (P_\alpha u)v\, dm = \int u(v \circ F_\alpha)\, dm$ for $u \in L^1(I)$ and $v \in L^\infty(I)$. There is an explicit formula for P_α:

$$(P_\alpha h)(\xi) = \sum_r G_{\alpha,r}(\xi) h(F_{\alpha,r}^{-1}(\xi)).$$

We assume that $F_{\alpha,r}^{-1}$ and $G_{\alpha,r}$, as functions of α and ξ, have continuous second order partial derivatives for each $r \in \mathcal{R}$, and there are constants

$$0 < \sigma < 1, \qquad K_0 > 0 \qquad \text{and} \qquad \gamma_r \geq 1,\, r \in \mathcal{R}$$

such that uniformly in $\alpha \in [\alpha_-, \alpha_+]$ and $r \in \mathcal{R}$:

A1. $\|G_{\alpha,r}\|_\infty \leq \sigma$,

A2. $\|G'_{\alpha,r}/G_{\alpha,r}\|_\infty \leq K_0$,

A3. $\|G''_{\alpha,r}/G_{\alpha,r}\|_\infty \leq K_0$,

A4. $\left\|\partial_\alpha F_{\alpha,r}^{-1}\right\|_\infty \leq \gamma_r$,

A5. $\left\|(\partial_\alpha G_{\alpha,r})/G_{\alpha,r}\right\|_\infty \leq \gamma_r$,

A6. $\left\|(\partial_\alpha G'_{\alpha,r})/G_{\alpha,r}\right\|_\infty \leq \gamma_r$,

A7. $\sum_r \|G_{\alpha,r}\|_\infty \gamma_r \leq K_0$.

It is well known that under conditions A1, A2, the map F_α admits a unique absolutely continuous invariant measure (see for example [P80, Z04]), which we denote by μ_α, and its density by $h_\alpha = d\mu_\alpha/dm$.

Theorem 2.1. *$h_\alpha \in C^2(I)$ and $\partial_\alpha h_\alpha \in C^1(I)$ for each $\alpha \in [\alpha_-, \alpha_+]$. The maps*

$$\begin{aligned}[\alpha_-, \alpha_+] &\longrightarrow C^2(I) \\ \alpha &\longmapsto h_\alpha\end{aligned} \qquad \text{and} \qquad \begin{aligned}[\alpha_-, \alpha_+] &\longrightarrow C^1(I) \\ \alpha &\longmapsto \partial_\alpha h_\alpha\end{aligned}$$

are continuous.

The proof of Theorem 2.1 is postponed until Section 4.

Remark 2.2. Later in the proof of Theorem 2.1 we explicitly compute constants K_1 and K_2, such that $\|h_\alpha\|_{C^2} \leq K_1$ and $\|\partial_\alpha h_\alpha\|_{C^1} \leq K_2$. These constants depend only on K_0 and σ. Below we use K_1 and K_2 as reference bounds on $\|h_\alpha\|_{C^2}$ and $\|\partial_\alpha h_\alpha\|_{C^1}$.

Remark 2.3. Both $h_\alpha(\xi)$ and $(\partial_\alpha h_\alpha)(\xi)$ are continuous in α, and continuous in ξ uniformly in α, because $\|h_\alpha\|_{C^2} \leq K_1$. Therefore both are jointly continuous in α and ξ.

Corollary 2.4. *Assume that Φ_α is a family of observables, such that $\Phi_\alpha(F_{\alpha,r}^{-1}(\xi))$ and $\partial_\alpha[\Phi_\alpha(F_{\alpha,r}^{-1}(\xi))]$ are jointly continuous in α and ξ for each r, and*

$$\|\Phi_\alpha \circ F_{\alpha,r}^{-1}\|_\infty \leq \delta_r, \qquad \|\partial_\alpha[\Phi_\alpha \circ F_{\alpha,r}^{-1}]\|_\infty \leq \delta_r$$

for some constants $\delta_r \geq 1$, $r \in \mathcal{R}$. Assume also that $\sum_r \gamma_r \, \delta_r \, \|G_{\alpha,r}\|_\infty \leq K_3$. Then the map $\alpha \mapsto \int \Phi_\alpha \, d\mu_\alpha$ is continuously differentiable on $[\alpha_-, \alpha_+]$.

Proof. First,

$$\begin{aligned}\int \Phi_\alpha \, d\mu_\alpha &= \int h_\alpha \Phi_\alpha \, dm = \int P_\alpha(h_\alpha \Phi_\alpha) \, dm \\ &= \int \left(\sum_r (h_\alpha \circ F_{\alpha,r}^{-1})(\Phi_\alpha \circ F_{\alpha,r}^{-1}) G_{\alpha,r} \right) dm.\end{aligned}$$

By Theorem 2.1 and our assumptions, $h_\alpha \circ F_{\alpha,r}^{-1}$, $\Phi_\alpha \circ F_{\alpha,r}^{-1}$ and $G_{\alpha,r}$ are jointly continuous in α and ξ, and have jointly continuous partial derivatives by α. Since $\|h_\alpha\|_{C^2} \leq K_1$, $\|\Phi_\alpha \circ F_{\alpha,r}^{-1}\|_\infty \leq \delta_r$ and $\sum_r \delta_r \, \gamma_r \, \|G_{\alpha,r}\|_\infty \leq K_3$, the series inside the integral converges uniformly to a function which is jointly continuous in α and ξ. Therefore, $\int \Phi_\alpha \, d\mu_\alpha$ depends continuously on α.

Moreover, since $\|\partial_\alpha h_\alpha\|_{C^1} \leq K_2$, $\|\partial_\alpha F_{\alpha,r}^{-1}\|_\infty \leq \gamma_r$, $|(\Phi_\alpha \circ F_{\alpha,r}^{-1})| \leq \delta_r$, $|\partial_\alpha(\Phi_\alpha \circ F_{\alpha,r}^{-1})| \leq \delta_r$ and $|\partial_\alpha[G_{\alpha,r}]| \leq \gamma_r \, G_{\alpha,r}$, we can write

$$\begin{aligned}\left|\partial_\alpha(h_\alpha \circ F_{\alpha,r}^{-1})\right| &= \left|[\partial_\alpha h_\alpha] \circ F_{\alpha,r}^{-1} + (h'_\alpha \circ F_{\alpha,r}^{-1}) \, \partial_\alpha F_{\alpha,r}^{-1}\right| \\ &\leq K_2 + K_1 \gamma_r \qquad \text{and} \\ \left|\partial_\alpha\left[(h_\alpha \circ F_{\alpha,r}^{-1})(\Phi_\alpha \circ F_{\alpha,r}^{-1}) G_{\alpha,r}\right]\right| &\leq [(K_2 + K_1 \gamma_r) + K_1 + K_1 \gamma_r] \, \delta_r \, G_{\alpha,r} \\ &\leq (3K_1 + K_2) \, \delta_r \, \gamma_r \, G_{\alpha,r}.\end{aligned}$$

Since $\sum_r \delta_r \, \gamma_r \, \|G_{\alpha,r}\|_\infty \leq K_3$, we can write

$$\begin{aligned}\frac{d}{d\alpha} \int \Phi_\alpha \, d\mu_\alpha &= \frac{d}{d\alpha} \int \left(\sum_r (h_\alpha \circ F_{\alpha,r}^{-1})(\Phi_\alpha \circ F_{\alpha,r}^{-1}) G_{\alpha,r} \right) dm \\ &= \sum_r \int \partial_\alpha \left[(h_\alpha \circ F_{\alpha,r}^{-1})(\Phi_\alpha \circ F_{\alpha,r}^{-1}) G_{\alpha,r}\right] dm.\end{aligned}$$

The series converges uniformly, and is bounded by $K_3(3K_1 + K_2)$. The terms are continuous in α, thus so is the sum. □

3 Application to Pomeau-Manneville type maps

In this section we work with the family of maps T_α, defined by equation (1.1). Assume that $\alpha \in [\alpha_-, \alpha_+] \subset (0, 1)$. Let $\tau_\alpha(x) = \min\{k \geq 1 \colon T_\alpha^k x \in [1/2, 1]\}$ be the return time to the interval $[1/2, 1]$. Let

$$F_\alpha \colon [1/2, 1] \to [1/2, 1], \qquad x \mapsto T_\alpha^{\tau_\alpha(x)}(x)$$

be the induced map.

Branches. Let $x_0 = 1$, $x_1 = 1/2$, and define $x_k \in (0, 1/2]$ for $k \geq 1$ by setting $x_k = T_\alpha x_{k+1}$. Note that $T_\alpha^k \colon (x_{k+1}, x_k) \to (1/2, 1)$ is a diffeomorphism. Let $y_k = (1 + x_k)/2$. Then $T_\alpha^{k+1} \colon (y_{k+1}, y_k) \to (1/2, 1)$ is a diffeomorphism. It is clear that $\tau_\alpha = k + 1$ on (y_{k+1}, y_k), so the map F_α has full branches on the intervals (y_{k+1}, y_k) for $k \geq 0$.

We index branches by $r \in \mathcal{R} = \mathbb{N} \cup \{0\}$, the r-th branch being the one on (y_{r+1}, y_r). Let $F_{\alpha,r} : [y_{r+1}, y_r] \to [0, 1]$ be the continuous extension of $F_\alpha : (y_{r+1}, y_r) \to (0, 1)$.

For notational convenience we introduce a function

$$\operatorname{logg}(r) = \begin{cases} 1 & r \leq e \\ \log(r) & r > e \end{cases}.$$

Let $\varphi \in C^1[0, 1]$ be an observable; let

$$\Phi_\alpha = \sum_{k=0}^{\tau_\alpha - 1} \varphi \circ T_\alpha^k \tag{3.1}$$

be the corresponding observable for the induced system.

Theorem 3.1. *The family of maps $F_\alpha = T_\alpha^{\tau_\alpha} \colon [1/2, 1] \to [1/2, 1]$ with observables Φ_α fits into the setup of Theorem 2.1 and Corollary 2.4 with branches indexed as above, $\delta_r = K(r + 1) \|\varphi\|_{C^1}$, $\sigma = 1/2$, and $\gamma_r = K(\operatorname{logg} r)^3$, where K is a constant, depending only on α_- and α_+.*

The proof consists of verification of the assumptions of Theorem 2.1 and Corollary 2.4, and is carried out in Subsection 5.2. Here we use Theorem 3.1 to prove our main result — Theorem 1.1.

Proof of Theorem 1.1. The invariant measure ν_α for T_α is related to the invariant measure μ_α for F_α by Kac's formula:

$$\int \varphi \, d\nu_\alpha = \int \Phi_\alpha \, d\mu_\alpha \Big/ \int \tau_\alpha d\mu_\alpha,$$

where Φ_α is given by (3.1). Note that if $\varphi \equiv 1$, then $\Phi_\alpha = \tau_\alpha$. By Theorem 3.1, both integrals are continuously differentiable in α. Also, $\tau_\alpha \geq 1$, so $\int \tau_\alpha d\mu_\alpha \geq 1$. Hence $\int \varphi \, d\nu_\alpha$ is continuously differentiable in α. ☐

4 Proof of Theorem 2.1

For $h \in C^1(I)$ and $\alpha \in [\alpha_-, \alpha_+]$ define $Q_\alpha h = \partial_\alpha(P_\alpha h)$, if the derivative exists. Denote

$$\mathcal{Q}_h(\alpha) = Q_\alpha h, \qquad \mathcal{P}_h(\alpha) = P_\alpha h.$$

4.1 Outline of the proof

The proof consists of three steps:

(a) Continuity (Subsection 4.2). We show that for $i = 1, 2$ the linear operators

$$P_\alpha \colon C^i(I) \to C^i(I) \qquad \text{and} \qquad Q_\alpha \colon C^i(I) \to C^{i-1}(I)$$

are well defined, and their norms are bounded uniformly in α. Plus, they continuously depend on α in the following sense: for each $h \in C^i(I)$ the maps $\mathcal{P}_h \colon [\alpha_-, \alpha_+] \to C^i(I)$ and $\mathcal{Q}_h \colon [\alpha_-, \alpha_+] \to C^{i-1}(I)$ are continuous. Moreover, the map $\mathcal{P}_h \colon [\alpha_-, \alpha_+] \to C^{i-1}(I)$ is continuously differentiable, and its derivative is $\mathcal{Q}_h \colon [\alpha_-, \alpha_+] \to C^{i-1}(I)$.

In addition, $\int Q_\alpha h \, dm = 0$ for every $h \in C^1(I)$.

(b) Distortion bounds and coupling (Subsection 4.3).

- If h is in $C^i(I)$ for $i = 1$ or 2, and $\int h \, dm = 0$, then $\|P_\alpha^k h\|_{C^i} \to 0$ exponentially fast, uniformly in α.
- If $h \in C^2(I)$ and $\int h \, dm = 1$, then $h_\alpha = \lim_{k\to\infty} P_\alpha^k h = h + \sum_{k=0}^\infty P_\alpha^k (P_\alpha h - h)$
- The series above converges exponentially fast in C^2, and $\|h_\alpha\|_{C^2}$ is bounded on $[\alpha_-, \alpha_+]$.
- The map $\alpha \mapsto h_\alpha$ from $[\alpha_-, \alpha_+]$ to $C^2(I)$ is continuous.

(c) Computation of $\partial_\alpha h_\alpha$. Fix $\alpha \in [\alpha_-, \alpha_+]$. Start with a formula, which holds for every α, β and n:

$$P_\beta^n h_\alpha - P_\alpha^n h_\alpha = \sum_{k=0}^{n-1} P_\beta^k (P_\beta - P_\alpha) h_\alpha.$$

Since the terms in the above sum converge exponentially fast in C^2, we can take the limit $n \to \infty$. Note that $P_\alpha^n h_\alpha = h_\alpha$ and $\lim_{n\to\infty} P_\beta^n h_\alpha = h_\beta$. Hence

$$h_\beta - h_\alpha = \sum_{k=0}^{\infty} P_\beta^k (P_\beta - P_\alpha) h_\alpha.$$

For fixed α, recall that the map $\beta \mapsto P_\beta h_\alpha$ from $[\alpha_-, \alpha_+]$ to $C^1(I)$ is continuously differentiable, and its derivative is the map $\beta \mapsto Q_\beta h_\alpha$, hence

$$(P_\beta - P_\alpha) h_\alpha = (\beta - \alpha)\, Q_\alpha h_\alpha + R_\beta,$$

with $\|R_\beta\|_{C^1} = o(\beta - \alpha)$. Note that both $Q_\alpha h$ and R_β have zero mean. Next,

$$h_\beta - h_\alpha = (\beta - \alpha) \sum_{k=0}^{\infty} P_\beta^k Q_\alpha h_\alpha + \sum_{k=0}^{\infty} P_\beta^k R_\beta.$$

Both series converge exponentially fast in $C^1(I)$, uniformly in α and β, and the C^1 norm of the second one is $o(\alpha - \beta)$.

Observe that the maps

$$\begin{aligned} [\alpha_-, \alpha_+] \times C^1(I) &\longrightarrow C^1(I) \\ \alpha, h &\longmapsto P_\alpha h \end{aligned} \qquad \text{and} \qquad \begin{aligned} [\alpha_-, \alpha_+] \times C^2(I) &\longrightarrow C^1(I) \\ \alpha, h &\longmapsto Q_\alpha h \end{aligned}$$

are continuous in α, and continuous in h uniformly in α, because P_α and Q_α are linear operators, bounded uniformly in α. Thus both maps are jointly continuous in α and h. Recall that the map $\alpha \to h_\alpha$ from $[\alpha_-, \alpha_+]$ to $C^2(I)$ is continuous. Thus the map $\alpha, \beta \mapsto P_\beta^k Q_\alpha h_\alpha$ from $[\alpha_-, \alpha_+]^2$ to $C^1(I)$ is continuous.

Therefore, in $C^1(I)$ topology,

$$\lim_{\beta \to \alpha} \frac{h_\beta - h_\alpha}{\beta - \alpha} = \sum_{k=0}^{\infty} P_\alpha^k Q_\alpha h_\alpha,$$

and $\sum_{k=0}^{\infty} P_\alpha^k Q_\alpha h_\alpha$ continuously depends on α. Note that the above also implies that $\partial_\alpha h_\alpha = \sum_{k=0}^{\infty} P_\alpha^k Q_\alpha h_\alpha$ (if understood pointwise).

Therefore the map $\alpha \mapsto h_\alpha$ from $[\alpha_-, \alpha_+]$ to $C^1(I)$ is continuously differentiable, and its derivative is $\alpha \mapsto \partial_\alpha h_\alpha = \sum_{k=0}^{\infty} P_\alpha^k Q_\alpha h_\alpha$.

In the remainder of this section we make the above precise.

4.2 Continuity

Lemma 4.1. *Let $K_6 = 4K_0(1 + K_0)$. For each $i = 1, 2$ and $h \in C^i(I)$:*

a) The map $\mathcal{P}_h \colon [\alpha_-, \alpha_+] \to C^i(I)$ is continuous. Also, $\|P_\alpha h\|_{C^i} \le K_6 \|h\|_{C^i}$.

b) The map $\mathcal{Q}_h \colon [\alpha_-, \alpha_+] \to C^{i-1}(I)$ is continuous. Also, $\|Q_\alpha h\|_{C^{i-1}} \le K_6 \|h\|_{C^i}$.

c) The map $\mathcal{Q}_h \colon [\alpha_-, \alpha_+] \to C^{i-1}(I)$ is the derivative of the map $\mathcal{P}_h \colon [\alpha_-, \alpha_+] \to C^{i-1}(I)$.

d) $\int Q_\alpha h \, dm = 0$.

Proof. We do the case $i = 2$; the case $i = 1$ is similar and simpler.

a) Let $p_{\alpha,r} = (h \circ F_{\alpha,r}^{-1}) G_{\alpha,r}$. Then

$$p'_{\alpha,r} = \pm (h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^2 + (h \circ F_{\alpha,r}^{-1})\, G'_{\alpha,r}, \quad \text{and}$$
$$p''_{\alpha,r} = (h'' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^3 \pm 3 (h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r} G'_{\alpha,r} + (h \circ F_{\alpha,r}^{-1}) G''_{\alpha,r},$$

where the sign of $\pm$ depends only on r. By assumptions A1, A2 and A3, $\|p_{\alpha,r}\|_{C^2} \le (4K_0 + 1)\|h\|_{C^2} \|G_{\alpha,r}\|_\infty$.

Since $p_{\alpha,r}$, $p'_{\alpha,r}$ and $p''_{\alpha,r}$ are jointly continuous in α and ξ, we obtain that the map $\alpha \mapsto p_{\alpha,r}$ from $[\alpha_-, \alpha_+]$ to $C^2(I)$ is continuous.

By assumption A7, $\sum_r \|G_{\alpha,r}\|_\infty \le K_0$, so the map $\alpha \mapsto P_\alpha h = \sum_r p_{\alpha,r}$ is continuous from $[\alpha_-, \alpha_+]$ to C^2, and $\|P_\alpha h\|_{C^2} \le K_0(4K_0 + 1)\|h\|_{C^2}$.

b) Let $q_{\alpha,r} = \partial_\alpha [(h \circ F_{\alpha,r}^{-1}) G_{\alpha,r}]$. We use the fact that $F_{\alpha,r}^{-1}$, as a function of α and ξ, has continuous partial derivatives up to second order, to compute

$$q'_{\alpha,r} = \partial_\alpha \big[((h \circ F_{\alpha,r}^{-1}) G_{\alpha,r})' \big] = \partial_\alpha \big[\pm (h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^2 + (h \circ F_{\alpha,r}^{-1})\, G'_{\alpha,r} \big].$$

We use assumptions A1, A2, A4, A5 and A6 to estimate

$$\|q_{\alpha,r}\|_{C^1} \le \|G_{\alpha,r}\|_\infty (4 + K_0) \gamma_r \|h\|_{C^2}.$$

Since $q_{\alpha,r}$ and $q'_{\alpha,r}$ are jointly continuous in α and ξ, we obtain that the map $\alpha \mapsto q_{\alpha,r}$ from $[\alpha_-, \alpha_+]$ to $C^1(I)$ is continuous.

By assumption A7, $\sum_r \gamma_r \|G_{\alpha,r}\|_\infty \le K_0$, so the map $\alpha \mapsto Q_\alpha h = \sum_r q_{\alpha,r}$ is continuous from $[\alpha_-, \alpha_+]$ to C^1, and $\|Q_\alpha h\|_{C^1} \le K_0(4 + K_0)\|h\|_{C^2}$.

c) Note that $(Q_\alpha h)(\xi)$ and $(Q_\alpha h)'(\xi)$ are jointly continuous in α and ξ. By definition of Q_α, for every ξ and $j = 0, 1$ we can write

$$
\begin{aligned}
(P_\beta h)^{(j)}(\xi) - (P_\alpha h)^{(j)}(\xi) &= \int_\alpha^\beta (Q_t h)^{(j)}(\xi)\, dt \\
&= (\beta - \alpha)(Q_\alpha h)^{(j)}(\xi) + \int_\alpha^\beta \left[(Q_t h)^{(j)}(\xi) - (Q_\alpha h)^{(j)}(\xi)\right] dt.
\end{aligned}
$$

Fix α. Since $\lim_{t\to\alpha} \|Q_t h - Q_\alpha h\|_{C^1} = 0$, the integral on the right is $o(\beta - \alpha)$ uniformly in ξ. Therefore,

$$
\|P_\beta h - P_\alpha h - (\beta - \alpha) Q_\alpha h\|_{C^1} = o(\beta - \alpha),
$$

thus $\mathcal{Q}_h \colon [\alpha_-, \alpha_+] \to C^1(I)$ is the derivative of $\mathcal{P}_h[\alpha_-, \alpha_+] \to C^1(I)$.

d) To prove that $\int Q_\alpha h\, dm = 0$, we differentiate the identity $\int P_\alpha h\, dm = \int h\, dm$ by α:

$$
\int Q_\alpha h\, dm = \frac{d}{d\alpha} \int P_\alpha h\, dm = 0,
$$

the order of differentiation and integration can be changed because both $P_\alpha h$ and $\partial_\alpha(P_\alpha h) = Q_\alpha h$ are jointly continuous in α and ξ.

□

4.3 Distortion bounds and coupling

If $h \in C^1$ and h is positive, denote $\|h\|_L = \|h'/h\|_\infty$. If also $h \in C^2$, denote $\|h\|_P = \|h''/h\|_\infty$.

Lemma 4.2 (Distortion bounds). *If $h \in C^1$ and $h > 0$, then*

$$
\|P_\alpha h\|_L \le \sigma \|h\|_L + K_0. \tag{4.1}
$$

If also $h \in C^2$, then

$$
\|P_\alpha h\|_P \le \sigma^2 \|h\|_P + 3\sigma K_0 \|h\|_L + K_0. \tag{4.2}
$$

Proof. Recall that $P_\alpha h = \sum_r (h \circ F_{\alpha,r}^{-1}) G_{\alpha,r}$ and $(F_{\alpha,r}^{-1})' = \pm G_{\alpha,r}$, where the sign depends only on r. Inequality (4.1) follows from the following computation:

$$
\begin{aligned}
\left|\frac{(P_\alpha h)'}{P_\alpha h}\right| &= \left|\frac{\sum_r \pm (h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^2 + (h \circ F_{\alpha,r}^{-1})\, G'_{\alpha,r}}{\sum_r (h \circ F_{\alpha,r}^{-1}) G_{\alpha,r}}\right| \\
&\le \max_r \left|\frac{\pm (h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^2 + (h \circ F_{\alpha,r}^{-1})\, G'_{\alpha,r}}{(h \circ F_{\alpha,r}^{-1}) G_{\alpha,r}}\right| \le \max_r \left(\frac{|h' \circ F_{\alpha,r}^{-1}|}{h \circ F_{\alpha,r}^{-1}} G_{\alpha,r} + \frac{|G'_{\alpha,r}|}{G_{\alpha,r}} \right) \\
&\le \max_r (\|h\|_L \|G_{\alpha,r}\|_\infty + \|G_{\alpha,r}\|_L)
\end{aligned}
$$

and assumptions A1 and A2.

Next,

$$(P_\alpha h)'' = \sum_r \big[(h'' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^3 \pm 3(h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r} G'_{\alpha,r} + (h \circ F_{\alpha,r}^{-1}) G''_{\alpha,r}\big].$$

Thus

$$\begin{aligned}
\left|\frac{(P_\alpha h)''}{P_\alpha h}\right| &\le \max_r \frac{\big|(h'' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^3 \pm 3(h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r} G'_{\alpha,r} + (h \circ F_{\alpha,r}^{-1}) G''_{\alpha,r}\big|}{(h \circ F_{\alpha,r}^{-1}) G_{\alpha,r}} \\
&\le \max_r \left(\frac{|h'' \circ F_{\alpha,r}^{-1}|}{h \circ F_{\alpha,r}^{-1}} G_{\alpha,r}^2 + 3 \frac{|h' \circ F_{\alpha,r}^{-1}|}{h \circ F_{\alpha,r}^{-1}} \frac{|G'_{\alpha,r}|}{G_{\alpha,r}} G_{\alpha,r} + \frac{|G''_{\alpha,r}|}{G_{\alpha,r}} \right) \\
&\le \max_r \big(\|h\|_P \, \|G_{\alpha,r}\|_\infty^2 + 3\|h\|_L \, \|G_{\alpha,r}\|_L \, \|G_{\alpha,r}\|_\infty + \|G_{\alpha,r}\|_P \big).
\end{aligned}$$

The inequality (4.2) follows from the above and assumptions A1, A2 and A3. □

Let $K_L > 0$, $K_P > 0$ and $\theta \in (0,1)$ be constants satisfying

$$\begin{aligned}
&K_L(1 - \theta\, e^{|I| K_L}) > \sigma\, K_L + K_0, \quad \text{and} \\
&K_P(1 - \theta\, e^{|I| K_L}) > \sigma^2\, K_P + 3\sigma K_0\, K_L + K_0,
\end{aligned} \tag{4.3}$$

where $|I|$ means the length of the interval I. It is clear that such constants can be chosen, because $\sigma < 1$.

Definition 4.3. We say that a function h is *regular*, if it is positive, belongs in $C^1(I)$, and $\|h\|_L \le K_L$. If in addition $h \in C^2(I)$ and $\|h\|_P \le K_P$, we say that h is *superregular*.

Remark 4.4. It readily follows from Lemma 4.2 that if h is a regular function, then so is $P_\alpha h$. If h is superregular, then so is $P_\alpha h$.

Remark 4.5. We observe that regular functions are explicitly bounded from above and from below. If h is regular, so $\|h'/h\|_\infty < K_L$, then $h(x_1)/h(x_2) \le e^{|I| K_L}$ for all x_1, x_2. Also, there is $\hat{x} \in I$ such that $h(\hat{x}) = \int h\, dm$, hence

$$e^{-|I| K_L} \int h\, dm \le h(x) \le e^{|I| K_L} \int h\, dm \qquad \text{for all } x \in I. \tag{4.4}$$

Lemma 4.6. *Assume that h is regular. Let $g = P_\alpha h - \theta \int h\, dm$. Then g is regular. If h is superregular, then so is g.*

Proof. Since $P_\alpha h \ge e^{-|I| K_L} \int P_\alpha h\, dm = e^{-|I| K_L} \int h\, dm$ by equation (4.4),

$$g = P_\alpha h \left(1 - \frac{\theta \int h\, dm}{P_\alpha h} \right) \ge P_\alpha h \big(1 - \theta e^{|I| K_L} \big).$$

Thus $g > 0$. By Lemma 4.2 and equation (4.3)

$$\|g\|_L = \left\| \frac{g'}{g} \right\|_\infty = \left\| \frac{(P_\alpha h)'}{g} \right\|_\infty \le \left\| \frac{(P_\alpha h)'}{P_\alpha h} \right\|_\infty \frac{1}{1 - \theta e^{|I| K_L}} = \|P_\alpha h\|_L \frac{1}{1 - \theta e^{|I| K_L}} \le K_L.$$

Hence g is regular. An analogous proof works for $\|g\|_P$. □

Lemma 4.7 (Coupling Lemma). *Let f and g be two regular functions with $\int f\,dm = \int g\,dm = M$. Let $f_0 = f$ and $g_0 = g$, and define*

$$f_{n+1} = P_\alpha f_n - \theta \int f_n\,dm, \qquad g_{n+1} = P_\alpha g_n - \theta \int g_n\,dm.$$

Then for all n

$$P_\alpha^n(f - g) = f_n - g_n,$$

where f_n and g_n are regular, and $\int f_n\,dm = \int g_n\,dm = (1-\theta)^n M$.

In particular, $\|f_n\|_\infty, \|g_n\|_\infty \leq (1-\theta)^n e^{|I|K_L} M$, and

$$\|f_n'\|_\infty, \|g_n'\|_\infty \leq K_L(1-\theta)^n e^{|I|K_L} M.$$

If in addition f and g are superregular, then

$$\|f_n''\|_\infty, \|g_n''\|_\infty \leq K_P(1-\theta)^n e^{|I|K_L} M.$$

Proof. The proof of $\int f_n\,dm = \int g_n\,dm = (1-\theta)^n M$ is by induction.

By equation (4.4), $\|f\|_\infty$ and $\|g\|_\infty$ are bounded by $(1-\theta)^n e^{|I|K_L} M$. Note that if h is a regular function, then $\|h'\|_\infty \leq K_L\|h\|_\infty$, and if it is superregular, then also $\|h''\|_\infty \leq K_P\|h\|_\infty$. The bounds on $\|f'\|_\infty$, $\|g'\|_\infty$, $\|f''\|_\infty$, $\|g''\|_\infty$ follow. □

Corollary 4.8. *There is a constant K_5 such that if $h \in C^i(I)$ for $i = 1$ or 2, and h has mean zero, then*

$$\|P_\alpha^n h\|_{C^i} \leq K_5(1-\theta)^n \, \|h\|_{C^i}.$$

Proof. We can represent $h = (h+c) - c$, where $c = \|h\|_{C^i}(1 + \max(K_L^{-1}, K_P^{-1}))$. Then

$$\left\|\frac{h'}{h+c}\right\|_\infty \leq \frac{\|h\|_{C^i}}{-\|h\|_{C^i} + c} = \frac{1}{\max(K_L^{-1}, K_P^{-1})} = \min(K_L, K_P),$$

and so $h + c$ is regular. If $i = 2$, then the same identity with h'' in place of h' also holds true, so also $h + c$ is superregular.

By Lemma 4.7 applied to $f = h + c$ and $g = c$,

$$P_\alpha^n h = f_n - g_n,$$

where

$$\begin{aligned}
\|f_n\|_{C^i}, \|g_n\|_{C^i} &\leq \max(1, K_L, K_P)(1-\theta)^n e^{|I|K_L} c \\
&= (1-\theta)^n \big[\max(1, K_L, K_P) e^{|I|K_L}(1 + \max(K_L^{-1}, K_P^{-1}))\big] \|h\|_{C^i} \\
&= (1-\theta)^n \frac{K_5}{2} \|h\|_{C^i}.
\end{aligned}$$

Thus $\|P_\alpha^n h\|_{C^i} \leq (1-\theta)^n K_5 \|h\|_{C^i}$, where

$$K_5 = 2\max(1, K_L, K_P)(1 + \max(K_L^{-1}, K_P^{-1}))\, e^{|I|K_L}.$$

□

Corollary 4.9. *For any $h \in C^2(I)$ with $\int h\, dm = 1$*

$$h_\alpha = \lim_{n\to\infty} P_\alpha^n h = h + \sum_{n=0}^{\infty} P_\alpha^n (P_\alpha h - h).$$

The series converges exponentially fast in C^2. The C^2 norm of h_α is bounded by $K_1 = 1 + 2\theta^{-1} e^{|I|K_L} \max(1, K_L, K_P)$.

Proof. Let

$$f = 1 + \sum_{n=0}^{\infty} P_\alpha^n (P_\alpha 1 - 1).$$

Since 1 is a superregular function, so is $P_\alpha 1$, and by Lemma 4.7 applied to $f = P_\alpha 1$ and $g = 1$, we have that $\|P_\alpha^n(P_\alpha 1 - 1)\|_{C^2} \le 2(1-\theta)^n \, e^{|I|K_L} \max(1, K_L, K_P)$. Thus the series above converges exponentially fast in $C^2(I)$ and $\|h\|_{C^2} \le K_1$.

Since $1 + \sum_{n=0}^{N} P_\alpha^n (P_\alpha 1 - 1) = P_\alpha^{N+1} 1$, we have $f = \lim_{n\to\infty} P_\alpha^n 1$. Thus f is invariant under P_α. It is clear that $\int f\, dm = 1$. Thus $h_\alpha = f$.

By Corollary 4.8, the C^2 norm of $P_\alpha^n (h - h_\alpha) = (P_\alpha^n h) - h_\alpha$ decreases exponentially with n, thus $h_\alpha = \lim_{n\to\infty} P_\alpha^n h = h + \sum_{n=0}^{\infty} P_\alpha^n (P_\alpha h - h)$. □

Corollary 4.10. *The map $\alpha \mapsto h_\alpha$ from $[\alpha_-, \alpha_+]$ to $C^2(I)$ is continuous.*

Proof. Using Corollary 4.9, write for $N \in \mathbb{N}$:

$$h_\alpha = 1 + \sum_{n=0}^{N-1} P_\alpha^n (P_\alpha 1 - 1) + \sum_{n=N}^{\infty} P_\alpha^n (P_\alpha 1 - 1).$$

The C^2 norm of the second sum is exponentially small in N, uniformly in α. By Lemma 4.1, a map $\alpha \mapsto P_\alpha^n (P_\alpha h - h)$ from $[\alpha_-, \alpha_+]$ to $C^2(I)$ is continuous for every n. Thus the first sum depends on α continuously. Since the choice of N is arbitrary, the result follows. □

5 Proofs of Theorems 3.1 and 1.2

In this section we prove technical statements about the family of maps T_α, defined by equation (1.1). We use notations introduced in Section 3.

In Subsection 5.1 we introduce necessary notations and prove a number of technical lemmas, in Subsections 5.2 and 5.3 we use the accumulated knowledge to prove Theorems 3.1 and 1.2.

5.1 Technical Lemmas

We use notation $\mathbf{C}$ for various nonnegative constants, which only depend on α_- and α_+, and may change from line to line, and within one expression if used twice. Recall the definition of y_r from the beginning of Section 3.

It is clear that $T_\alpha(x)$, as a function of α and x, has continuous partial derivatives of all orders in $\alpha, x \in [\alpha_-, \alpha_+] \times (0, 1/2]$, and so do $F_{\alpha,r}(x)$ and $F_{\alpha,r}^{-1}(x)$ on $[\alpha_-, \alpha_+] \times [y_{r+1}, y_r]$ and $[\alpha_-, \alpha_+] \times [1/2, 1]$ respectively.

Let $E_\alpha\colon [0,1/2] \to [0,1]$, $E_\alpha x = T_\alpha x$ be the left branch of the map T_α. Note that E_α is invertible. Let $z \in [0,1]$ and write, for notational convenience, $z_r = E_\alpha^{-r}(z)$. Then $F_{\alpha,r}(z) = E_\alpha^r(T_\alpha(z)) = E_\alpha^r(2z-1)$ for $z \in [y_{r+1}, y_r]$, and for $z \in [1/2, 1]$

$$T_\alpha(F_{\alpha,r}^{-1}(z)) = 2F_{\alpha,r}^{-1}(z) - 1 = z_r. \tag{5.1}$$

By $(\cdot)'$ we denote the derivative with respect to z. Let $G_{\alpha,r}$ be defined as in Theorem 2.1. Then for $z \in [1/2,1]$

$$G_{\alpha,r}(z) = (F_{\alpha,r}^{-1})'(z) = z_r'/2. \tag{5.2}$$

We do all the analysis in terms of z_r, and the relation to $G_{\alpha,r}$ and $F_{\alpha,r}^{-1}$ is given by equations (5.1) and (5.2).

Remark 5.1. By construction, $z_0 = z$, $z_0' = 1$ and $z_0'' = 0$. Note that $z_r \le 1/2$ for $r \ge 1$. Also

$$z_r = z_{r+1}(1 + 2^\alpha z_{r+1}^\alpha), \tag{5.3}$$
$$z_r' = [1 + (\alpha+1)2^\alpha z_{r+1}^\alpha] z_{r+1}', \tag{5.4}$$
$$z_r' = \prod_{j=1}^{r} \left[1 + (\alpha+1)2^\alpha z_j^\alpha\right]^{-1}. \tag{5.5}$$

Our analysis is built around the following estimate:

Lemma 5.2. *For $r \ge 1$*

$$\frac{1}{z_0^{-\alpha} + r\,\alpha 2^\alpha} \le z_r^\alpha \le \frac{1}{z_0^{-\alpha} + r\,\alpha(1-\alpha)2^{\alpha-1}}.$$

In particular,

$$\frac{\mathbf{C}\, z_0^\alpha}{r} \le z_r^\alpha \le \frac{\mathbf{C}}{r} \qquad \textit{and} \qquad -\log z_r \le \mathbf{C}\,[\operatorname{logg} r - \log z_0].$$

Proof. Transform equation (5.3) into

$$z_{r+1}^{-\alpha} = z_r^{-\alpha} + \frac{1 - (1 + 2^\alpha z_{r+1}^\alpha)^{-\alpha}}{z_{r+1}^\alpha}.$$

Then

$$z_r^{-\alpha} = z_0^{-\alpha} + \sum_{j=1}^{r} \frac{1 - (1 + 2^\alpha z_j^\alpha)^{-\alpha}}{z_j^\alpha}. \tag{5.6}$$

For all $t \in (0,1)$ and all $\alpha \in (0,1)$

$$1 - \alpha t \le (1+t)^{-\alpha} \le 1 - \alpha t + \frac{\alpha(\alpha+1)}{2} t^2.$$

Since $z_j \in (0,1/2]$ for $j \ge 1$, using the above inequality with $t = 2^\alpha z_j^\alpha$, we obtain

$$\alpha(1-\alpha)2^{\alpha-1} \le \frac{1 - (1 + 2^\alpha z_j^\alpha)^{-\alpha}}{z_j^\alpha} \le \alpha 2^\alpha.$$

By equation (5.6),

$$r\,\alpha(1-\alpha)2^{\alpha-1} \le z_r^{-\alpha} - z_0^{-\alpha} \le r\,\alpha 2^{\alpha}.$$

for $r \ge 1$. Write

$$\frac{z_0^\alpha}{r}\frac{1}{1+\alpha 2^\alpha} \le \frac{z_0^\alpha}{r}\frac{1}{r^{-1}+z_0^\alpha\,\alpha 2^\alpha} = \frac{1}{z_0^{-\alpha}+r\,\alpha 2^\alpha} \le z_r^\alpha \le \frac{1}{z_0^{-\alpha}+r\,\alpha(1-\alpha)2^{\alpha-1}}.$$

The result follows. □

Lemma 5.3. $z_0' = 1$ *and*

$$0 \le z_r' \le \mathbf{C}\,(1+rz_0^\alpha\alpha 2^\alpha)^{-(\alpha+1)/\alpha} \le \mathbf{C}\,r^{-(\alpha+1)/\alpha}\,z_0^{-(\alpha+1)}$$

for $r \ge 1$.

Proof. By Remark 5.1, $z_0' = 1$. Let $r \ge 1$. Using the inequality

$$\frac{1}{1+t} \le \exp(-t+t^2) \quad \text{for} \quad t \ge 0$$

on equation (5.5) we obtain

$$0 \le z_r' = \prod_{j=1}^{r} \frac{1}{1+(\alpha+1)2^\alpha z_j^\alpha} \le \exp\left(-\sum_{j=1}^{r}(\alpha+1)2^\alpha z_j^\alpha + \sum_{j=1}^{r}\big((\alpha+1)2^\alpha z_j^\alpha\big)^2\right). \quad (5.7)$$

By Lemma 5.2, $(z_j^\alpha)^2 \le \mathbf{C}/j^2$, thus the second sum under the exponent is bounded by $\mathbf{C}$. Also by Lemma 5.2,

$$\begin{aligned}\sum_{j=1}^{r} z_j^\alpha \ge \sum_{j=1}^{r}\frac{1}{z_0^{-\alpha}+j\alpha 2^\alpha} &\ge \int_1^r \frac{z_0^\alpha}{1+tz_0^\alpha\alpha 2^\alpha}\,dt - \mathbf{C} \\ &= \frac{1}{\alpha 2^\alpha}\log(1+tz_0^\alpha\alpha 2^\alpha)\Big|_{t=1}^{t=r} - \mathbf{C} \\ &\ge \frac{\log(1+rz_0^\alpha\alpha 2^\alpha)}{\alpha 2^\alpha} - \mathbf{C}\end{aligned}$$

Thus

$$-(\alpha+1)2^\alpha \sum_{j=1}^{r} z_j^\alpha \le -\frac{\alpha+1}{\alpha}\log(1+rz_0^\alpha\alpha 2^\alpha) + \mathbf{C},$$

and by equation (5.7),

$$z_r' \le \mathbf{C}\,(1+rz_0^\alpha\alpha 2^\alpha)^{-(\alpha+1)/\alpha} \le \mathbf{C}\,(rz_0^\alpha\alpha 2^\alpha)^{-(\alpha+1)/\alpha} \le \mathbf{C}\,r^{-(\alpha+1)/\alpha}z_0^{-(\alpha+1)}.$$

□

Lemma 5.4. $0 \le -z_r''/z_r' \le \mathbf{C}\,z_0^{-2}/\max(r,1)$.

Proof. Differentiating both sides of the equation (5.4), we obtain

$$z_r'' = \alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-1}(z_{r+1}')^2 + (1+(\alpha+1)2^\alpha z_{r+1}^\alpha)z_{r+1}''. \tag{5.8}$$

Dividing the above by $z_r' = [1+(\alpha+1)2^\alpha z_{r+1}^\alpha]z_{r+1}'$ we get

$$\frac{z_r''}{z_r'} = \frac{\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-1} z_{r+1}'}{1+(\alpha+1)2^\alpha z_{r+1}^\alpha} + \frac{z_{r+1}''}{z_{r+1}'}.$$

Recall that $z_0''/z_0' = 0$ and $z_r' \geq 0$, thus $z_r'' \leq 0$ for all r. By Lemmas 5.2 and 5.3 we have

$$0 \leq z_r^{\alpha-1} z_r' \leq \mathbf{C}\left(\frac{z_0^\alpha}{r}\right)^{(\alpha-1)/\alpha} r^{-\frac{\alpha+1}{\alpha}}\, z_0^{-(\alpha+1)} \leq \mathbf{C}\, r^{-2}\, z_0^{-2}$$

for $r \geq 1$. Thus

$$0 \leq \frac{z_r''}{z_r'} - \frac{z_{r+1}''}{z_{r+1}'} \leq \mathbf{C}\,(r+1)^{-2}\, z_0^{-2}.$$

The result follows. ∎

Lemma 5.5. $|z_r'''/z_r'| \leq \mathbf{C}\, z_0^{-\alpha-4}/\max(r^2,1)$.

Proof. Differentiate the equation (5.8). This results in

$$\begin{aligned} z_r''' = (\alpha-1)\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-2}(z_{r+1}')^3 + 3\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-1} z_{r+1}' z_{r+1}'' \\ +(1+(\alpha+1)2^\alpha z_{r+1}^\alpha) z_{r+1}'''. \end{aligned}$$

Dividing the above by $z_r' = [1+(\alpha+1)2^\alpha z_{r+1}^\alpha]z_{r+1}'$ we get

$$\frac{z_r'''}{z_r'} = \frac{(\alpha-1)\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-2}(z_{r+1}')^2}{1+(\alpha+1)2^\alpha z_{r+1}^\alpha} + \frac{3\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-1} z_{r+1}'}{1+(\alpha+1)2^\alpha z_{r+1}^\alpha}\frac{z_{r+1}''}{z_{r+1}'} + \frac{z_{r+1}'''}{z_{r+1}'}.$$

Using Lemmas 5.2, 5.3 and 5.4 we bound the first two terms in the right hand side above by $\mathbf{C}\,(r+1)^{-3}\, z_0^{-\alpha-4}$ and $\mathbf{C}\,(r+1)^{-3}\, z_0^{-4}$ respectively. Thus

$$\left|\frac{z_r'''}{z_r'} - \frac{z_{r+1}'''}{z_{r+1}'}\right| \leq \mathbf{C}\,(r+1)^{-3}\, z_0^{-\alpha-4}.$$

Since $z_0'''/z_0' = 0$, the result follows. ∎

Lemma 5.6. $\partial_\alpha z_0 = 0$ *and for* $r \geq 1$

$$0 \leq \frac{\partial_\alpha z_r}{z_r} \leq \mathbf{C}\ \mathrm{logg}(r z_0^\alpha)\,[\mathrm{logg}\, r - \log z_0] \qquad \textit{and}$$
$$0 \leq \partial_\alpha z_r \leq \mathbf{C}\,\frac{\mathrm{logg}(r z_0^\alpha)}{r^{1/\alpha}}[\mathrm{logg}\, r - \log z_0].$$

Proof. Since $z_0 = z$ does not depend on α, $\partial_\alpha z_0 = 0$.

Differentiating the identity $z_{r+1}(1+2^\alpha z_{r+1}^\alpha) = z_r$ by α we obtain a recursive relationz

$$\partial_\alpha z_{r+1} = \frac{\partial_\alpha z_r + 2^\alpha z_{r+1}^{\alpha+1}(-\log(2z_{r+1}))}{1+(\alpha+1)2^\alpha z_{r+1}^\alpha}.$$

Since $z_{r+1} \le 1/2$ for all r, it follows that $\partial_\alpha z_r \ge 0$ for all r. It is convenient to rewrite the above, dividing by z_{r+1} and using $z_{r+1}(1 + 2^\alpha z_{r+1}^\alpha) = z_r$:

$$\frac{\partial_\alpha z_{r+1}}{z_{r+1}} = \frac{(1 + 2^\alpha z_{r+1}^\alpha)\frac{\partial_\alpha z_r}{z_r} + 2^\alpha z_{r+1}^\alpha(-\log(2z_{r+1}))}{1 + (\alpha+1)2^\alpha z_{r+1}^\alpha},$$

which implies

$$\frac{\partial_\alpha z_{r+1}}{z_{r+1}} \le \frac{\partial_\alpha z_r}{z_r} + 2^\alpha z_{r+1}^\alpha(-\log(2z_{r+1})).$$

By Lemma 5.2,

$$2^\alpha z_r^\alpha(-\log(2z_r)) \le \mathbf{C}\,\frac{\operatorname{logg} r - \log z_0}{z_0^{-\alpha} + r\alpha(1-\alpha)2^{\alpha-1}}.$$

Hence

$$\begin{aligned}\frac{\partial_\alpha z_r}{z_r} &\le \sum_{j=1}^{r} 2^\alpha z_j^\alpha(-\log(2z_j)) \le \mathbf{C}\int_1^r \frac{\operatorname{logg} t - \log z_0}{z_0^{-\alpha} + t\alpha(1-\alpha)2^{\alpha-1}}\,dt \\ &\le \mathbf{C}\,\operatorname{logg}(rz_0^\alpha)[\operatorname{logg}(rz_0^\alpha) - \log z_0].\end{aligned} \tag{5.9}$$

The first part of the lemma follows. To prove the second part, observe that by Lemma 5.2, $z_r \le \mathbf{C}\,r^{-1/\alpha}$. □

Lemma 5.7. $|(\partial_\alpha z'_r)/z'_r| \le \mathbf{C}\,[\operatorname{logg}(rz_0^\alpha)]^2\,[\operatorname{logg} r - \log z_0]$.

Proof. Note that $\partial_\alpha z'_0 = 0$, because $z_0 = z$ does not depend on α.

Differentiate equation (5.4) by α. This results in

$$\begin{aligned}\partial_\alpha z'_r = \big(2^\alpha z_{r+1}^\alpha + (\alpha+1)2^\alpha z_{r+1}^\alpha \log(2z_{r+1}) + \alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-1}\partial_\alpha z_{r+1}\big) z'_{r+1} \\ +(1 + (\alpha+1)2^\alpha z_{r+1}^\alpha)\partial_\alpha z'_{r+1}\end{aligned}$$

Dividing the above by $z'_r = [1 + (\alpha+1)2^\alpha z_{r+1}^\alpha] z'_{r+1}$ we get

$$\frac{\partial_\alpha z'_r}{z'_r} = \frac{2^\alpha z_{r+1}^\alpha + (\alpha+1)2^\alpha z_{r+1}^\alpha \log(2z_{r+1}) + \alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-1}\partial_\alpha z_{r+1}}{1 + (\alpha+1)2^\alpha z_{r+1}^\alpha} + \frac{\partial_\alpha z'_{r+1}}{z'_{r+1}}.$$

For $r \ge 1$ Lemmas 5.2 and 5.6 give $|z_r^\alpha| \le \mathbf{C}/(z_0^{-\alpha} + r\alpha(1-\alpha)2^{\alpha-1})$,

$$|z_r^\alpha \log z_r| \le \mathbf{C}\,\frac{\operatorname{logg} r - \log z_0}{z_0^{-\alpha} + r\alpha(1-\alpha)2^{\alpha-1}} \qquad \text{and}$$

$$|z_r^{\alpha-1}\partial_\alpha z_r| = \left| z_r^\alpha \frac{\partial_\alpha z_r}{z_r}\right| \le \mathbf{C}\,\frac{\operatorname{logg}(rz_0^\alpha)\,(\operatorname{logg} r - \log z_0)}{z_0^{-\alpha} + r\alpha(1-\alpha)2^{\alpha-1}}.$$

Therefore

$$\left|\frac{\partial_\alpha z'_r}{z'_r} - \frac{\partial_\alpha z'_{r+1}}{z'_{r+1}}\right| \le \mathbf{C}\,\frac{\operatorname{logg}((r+1)z_0^\alpha)\,(\operatorname{logg}(r+1) - \log z_0)}{z_0^{-\alpha} + (r+1)\alpha(1-\alpha)2^{\alpha-1}}.$$

Thus

$$\left|\frac{\partial_\alpha z'_r}{z'_r}\right| \le \mathbf{C}\int_1^r \frac{\operatorname{logg}(tz_0^\alpha)\,(\operatorname{logg} t - \log z_0)}{z_0^{-\alpha} + t\alpha(1-\alpha)2^{\alpha-1}}\,dt \le \mathbf{C}\,[\operatorname{logg}(rz_0^\alpha)]^2(\operatorname{logg} r - \log z_0).$$

□

Lemma 5.8. $|(\partial_\alpha z_r'')/z_r'| \le \mathbf{C}\, z_0^{-2}\,(1-\log z_0)$.

Proof. Differentiate both sides of the equation (5.8) by α. This gives

$$\begin{aligned}\partial_\alpha z_r'' =&[2\alpha+1+\alpha(\alpha+1)\log(2z_{r+1})]2^\alpha z_{r+1}^{\alpha-1}(z_{r+1}')^2\\ &+(\alpha-1)\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-2}(z_{r+1}')^2\partial_\alpha z_{r+1}+2\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-1}z_{r+1}'\partial_\alpha z_{r+1}'\\ &+(1+(\alpha+1)\log(2z_{r+1}))2^\alpha z_{r+1}^\alpha z_{r+1}''+\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-1}z_{r+1}''\partial_\alpha z_{r+1}\\ &+(1+(\alpha+1)2^\alpha z_{r+1}^\alpha)\partial_\alpha z_{r+1}''.\end{aligned}$$

Dividing the above by $z_r'=[1+(\alpha+1)2^\alpha z_{r+1}^\alpha]z_{r+1}'$ and using Lemma 5.2 to bound z_{r+1}, Lemma 5.3 to bound z_r', Lemma 5.4 to bound z_r''/z_r', Lemma 5.6 to bound $\partial_\alpha z_r$ and Lemma 5.7 to bound $\partial_\alpha z_r'/z_r'$, we obtain for $r\ge 0$:

$$\begin{aligned}|z_r^{\alpha-1}z_r'\log z_r| &\le \mathbf{C}\,r^{-2}\,z_0^{-2}\,(\operatorname{logg} r-\log z_0\,),\\ |z_r^{\alpha-2}z_r'\partial_\alpha z_r| &\le \mathbf{C}\,r^{-2}\,z_0^{-2}\,\operatorname{logg}(rz_0^\alpha)\,(\operatorname{logg} r-\log z_0),\\ |z_r^{\alpha-1}\partial_\alpha z_r'| &\le \mathbf{C}\,r^{-2}\,z_0^{-2}\,[\operatorname{logg}(rz_0^\alpha)]^2\,(\operatorname{logg} r-\log z_0),\\ |z_r^\alpha z_r''\log z_r/z_r'| &\le \mathbf{C}\,r^{-2}\,z_0^{-2}\,(\operatorname{logg} r-\log z_0),\\ |z_r^{\alpha-1}z_r''(\partial_\alpha z_r)/z_r'| &\le \mathbf{C}\,r^{-2}\,z_0^{-2}\,\operatorname{logg}(rz_0^\alpha)\,(\operatorname{logg} r-\log z_0).\end{aligned}$$

Hence for $r\ge 1$

$$\left|\frac{\partial_\alpha z_r''}{z_r'}-\frac{\partial_\alpha z_{r-1}''}{z_{r-1}'}\right| \le \mathbf{C}\,r^{-2}\,z_0^{-2}\,[\operatorname{logg}(rz_0^\alpha)]^2\,(\operatorname{logg} r-\log z_0).$$

Recall that $\partial_\alpha z_0''=0$. Then

$$\left|\frac{\partial_\alpha z_r''}{z_r'}\right| \le z_0^{-2}\sum_{j=1}^{r} j^{-2}\,[\operatorname{logg}(jz_0^\alpha)]^2\,[\operatorname{logg} j-\log z_0] \le \mathbf{C}\,z_0^{-2}(1-\log z_0).$$

□

5.2 Proof of Theorem 3.1

The verification of assumptions of Theorem 2.1 is as follows. Since $G_{\alpha,r}$ and $F_{\alpha,r}^{-1}$ are defined on $[1/2,1]$, we use that $z=z_0\ge 1/2$ in the bounds below. Now,

A1. By equation (5.5), $z_r'\le 1$, thus $\|G_{\alpha,r}\|_\infty \le 1/2$.

A2. By Lemma 5.4, $|z_r''/z_r'|\le \mathbf{C}$, thus $\|G_{\alpha,r}'/G_{\alpha,r}\|_\infty \le \mathbf{C}$.

A3. By Lemma 5.5, $|z_r'''/z_r'|\le \mathbf{C}$, thus $\|G_{\alpha,r}''/G_{\alpha,r}\|_\infty \le \mathbf{C}$.

A4. By Lemma 5.6, $|\partial_\alpha z_r|\le \mathbf{C}\,r^{-1/\alpha}\,(\operatorname{logg} r)^2$, and by equation (5.1) we have

$$\|\partial_\alpha F_{\alpha,r}^{-1}\|_\infty \le \mathbf{C}\,r^{-1/\alpha}\,(\operatorname{logg} r)^2 \le \mathbf{C}\,(\operatorname{logg} r)^2.$$

A5. By Lemma 5.7, $|(\partial_\alpha z_r')/z_r'|\le \mathbf{C}\,(\operatorname{logg} r)^3$, thus $\|(\partial_\alpha G_{\alpha,r})/G_{\alpha,r}\|_\infty \le \mathbf{C}\,(\operatorname{logg} r)^3$.

A6. By Lemma 5.8, $|(\partial_\alpha z_r'')/z_r'|\le \mathbf{C}$, thus $\|(\partial_\alpha G_{\alpha,r}')/G_{\alpha,r}\|_\infty \le \mathbf{C}$.

A7. By Remark 5.1, $z_0' = 1$, and by Lemma 5.3, $|z_r'| \le \mathbf{C}\, r^{-(\alpha+1)/\alpha}$ for $r \ge 1$, so

$$\sum_{r=0}^{\infty} \|G_{\alpha,r}\|_\infty (\operatorname{logg} r)^3 = \frac{1}{2}\sum_{r=0}^{\infty} \sup_z |z_r'| \cdot (\operatorname{logg} r)^3 \le \frac{1}{2} + \mathbf{C} \sum_{r=1}^{\infty} \frac{(\operatorname{logg} r)^3}{r^{1+1/\alpha}} \le \mathbf{C}.$$

To verify the assumptions of the Corollary 2.4 — we have to show in addition that

- $\sum_{r=0}^{\infty}(r+1)(\operatorname{logg} r)^3 \, \|G_{\alpha,r}\|_\infty \le \mathbf{C}$. By Lemma 5.3 and equation (5.2),

$$|G_{\alpha,r}(z)| = |z_r'|/2 \le \mathbf{C}\, r^{-(\alpha+1)/\alpha},$$

  thus

$$\sum_{r=0}^{\infty}(r+1)(\operatorname{logg} r)^3 \, \|G_{\alpha,r}\|_\infty \le \mathbf{C} \sum_{r=0}^{\infty} \frac{(\operatorname{logg} r)^3}{r^{1/\alpha}} \frac{r+1}{r} \le \mathbf{C}.$$

- $\|\partial_\alpha[\Phi_\alpha \circ F_{\alpha,r}^{-1}]\|_\infty \le \mathbf{C}\, \|\varphi\|_{C^1}\,(r+1)$ and $\|\Phi_\alpha \circ F_{\alpha,r}^{-1}\|_\infty \le \mathbf{C}\, \|\varphi\|_{C^1}\,(r+1)$. This is true because

$$\big(\Phi_\alpha \circ F_{\alpha,r}^{-1}\big)(z) = \varphi\Big(\frac{z_r+1}{2}\Big) + \sum_{j=0}^{r-1} \varphi(T_\alpha^j z_r) = \varphi\Big(\frac{z_r+1}{2}\Big) + \sum_{j=1}^{r} \varphi(z_j),$$

  and $|\partial_\alpha z_r| \le \mathbf{C}$ by Lemma 5.6.

Hence we have verified assumptions of Theorem 2.1 and Corollary 2.4 as required.

5.3 Proof of Theorem 1.2

Recall that the invariant measure of T_α is denoted by ν_α, and its density by ρ_α, while the invariant measure of the induced map F_α is denoted by μ_α, and its density by h_α.

Lemma 5.9. *$\rho_\alpha(z) = g_\alpha(z)/\int_0^1 g_\alpha(x)\,dx$ for all $z \in (0,1]$, where*

$$g_\alpha(z) = \frac{1}{2}\sum_{k=0}^{\infty} h_\alpha\Big(\frac{z_k+1}{2}\Big)\, z_k'.$$

Proof. Let φ be a nonnegative observable on $[0,1]$, and $\Phi_\alpha = \sum_{k=0}^{\tau_\alpha-1} \varphi \circ T_\alpha^k$ be the corresponding induced observable. In the beginning of Section 3 we partitioned the interval $[1/2,1]$ into intervals $[y_{r+1}, y_r]$, $r \ge 0$, where F_α has full branches and $\tau_\alpha = r+1$.

Compute

$$\begin{aligned}
\int \Phi_\alpha \, d\mu_\alpha &= \int_{1/2}^{1} \sum_{k=0}^{\tau_\alpha(y)-1} \varphi(T_\alpha^k y)\, h_\alpha(y)\, dy = \sum_{j=0}^{\infty} \int_{y_{j+1}}^{y_j} \sum_{k=0}^{j} \varphi(T_\alpha^k y)\, h_\alpha(y)\, dy \\
&= \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \int_{y_{j+1}}^{y_j} \varphi(T_\alpha^k y)\, h_\alpha(y)\, dy = \sum_{k=0}^{\infty} \int_{1/2}^{y_k} \varphi(T_\alpha^k y)\, h_\alpha(y)\, dy \\
&= \int_{1/2}^{1} \varphi(y)\, h_\alpha(y)\, dy + \frac{1}{2} \sum_{k=1}^{\infty} \int_{0}^{x_k} \varphi(T_\alpha^{k-1} x)\, h_\alpha\Big(\frac{x+1}{2}\Big)\, dx \\
&= \int_{1/2}^{1} \varphi(y)\, h_\alpha(y)\, dy + \frac{1}{2} \sum_{k=0}^{\infty} \int_{0}^{x_{k+1}} \varphi(T_\alpha^{k} x)\, h_\alpha\Big(\frac{x+1}{2}\Big)\, dx \\
&= \int_{1/2}^{1} \varphi(y)\, h_\alpha(y)\, dy + \frac{1}{2} \sum_{k=0}^{\infty} \int_{0}^{1/2} \varphi(z)\, h_\alpha\Big(\frac{z_k+1}{2}\Big)\, z_k'\, dz \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \int_{0}^{1} \varphi(z)\, h_\alpha\Big(\frac{z_k+1}{2}\Big)\, z_k'\, dz.
\end{aligned}$$

First we made a substitution $x = T_\alpha y = 2y - 1$, and then a substitution $z = T_\alpha^k x$, i.e. $x = z_k$. In the last step we used the fact that for $z \geq 1/2$

$$h_\alpha(z) = (P_\alpha h_\alpha)(z) = \sum_{k=0}^{\infty} h_\alpha\Big(\frac{z_k+1}{2}\Big) \frac{z_k'}{2}.$$

Since $\int \varphi \, d\nu_\alpha = \int \Phi_\alpha \, d\mu_\alpha / \int \tau_\alpha \, d\mu_\alpha$, the result follows. ☐

Lemma 5.10. *$g_\alpha(z)$ and $\partial_\alpha g_\alpha(z)$ are jointly continuous in α, z on $[\alpha_-, \alpha_+] \times (0, 1]$. Also, $0 \leq g_\alpha(z) \leq \mathbf{C}\, z^{-\alpha}$ and $|\partial_\alpha g_\alpha(z)| \leq \mathbf{C}\, z^{-\alpha}(1 - \log z)^3$.*

Proof. By Theorem 2.1, $\|h_\alpha\|_{C^2} \leq \mathbf{C}$ and $\|\partial_\alpha h_\alpha\|_{C^1} \leq \mathbf{C}$, and both $h_\alpha(z)$ and $\partial_\alpha h_\alpha(z)$ are jointly continuous in α and z. By Lemma 5.3, $0 \leq z_r' \leq \mathbf{C}\,(1 + r z^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha}$, hence

$$0 \leq \sum_{r=1}^{\infty} z_r' \leq \mathbf{C} \int_1^\infty (1 + t z^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha}\, dt \leq \mathbf{C} z^{-\alpha}. \tag{5.10}$$

Now,

$$0 \leq g_\alpha(z) = \frac{1}{2} \sum_{k=0}^{\infty} h_\alpha\Big(\frac{z_k+1}{2}\Big)\, z_k' \leq \mathbf{C}\, z^{-\alpha}.$$

Terms of the series are jointly continuous in α and z, and convergence is uniform away from $z = 0$, thus $g_\alpha(z)$ is also jointly continuous in α and z.

Denote $u_{\alpha,k}(z) = h_\alpha((z_k + 1)/2)\, z_k'/2$, so that $g_\alpha(z) = \sum_{k=0}^\infty u_{\alpha,k}(z)$ and compute

$$\partial_\alpha u_{\alpha,k}(z) = \bigg[(\partial_\alpha h_\alpha)\Big(\frac{z_k+1}{2}\Big) + h_\alpha\Big(\frac{z_k+1}{2}\Big)\frac{\partial_\alpha z_k}{2}\bigg]\frac{z_k'}{2} + h_\alpha\Big(\frac{z_k+1}{2}\Big)\frac{\partial_\alpha z_k'}{2}.$$

By Lemma 5.6,

$$0 \leq \partial_\alpha z_r \leq \mathbf{C}\, r^{-1/\alpha} \operatorname{logg}(r z^\alpha)\, [\operatorname{logg} r - \log z].$$

By Lemma 5.7,

$$|\partial_\alpha z_r'| \le \mathbf{C}\, z_r' \,[\operatorname{logg}(rz^\alpha)]^2\, [\operatorname{logg} r - \log z].$$

Thus $|\partial_\alpha u_{\alpha,k}(z)| \le \mathbf{C}\, z_r' \,[\operatorname{logg}(rz^\alpha)]^2\, [\operatorname{logg} r - \log z]$. Thus by Lemma 5.3,

$$\begin{aligned}\sum_{k=0}^{\infty} |\partial_\alpha u_{\alpha,k}(z)| &\le \mathbf{C} \sum_{k=0}^{\infty} z_k' \,[\operatorname{logg}(kz^\alpha)]^2\, [\operatorname{logg} k - \log z] \\ &\le \mathbf{C} \int_1^\infty (1 + tz^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha}\, [\log(tz^\alpha)]^2\, [\log t - \log z]\, dt \\ &= \mathbf{C}\, z^{-\alpha} \int_{z^\alpha}^\infty (1 + s\alpha 2^\alpha)^{-(\alpha+1)/\alpha}\, (\log s)^2 \left[\log \frac{s}{z^\alpha} - \log z\right] ds \\ &\le \mathbf{C}\, z^{-\alpha}\, (1 - \log z).\end{aligned}$$

Therefore we can write

$$(\partial_\alpha g_\alpha)(z) = \sum_{k=0}^{\infty} \partial_\alpha u_{\alpha,k}(z).$$

Away from $z = 0$, the terms of the series are jointly continuous in α and z, and series converges uniformly, so $(\partial_\alpha g_\alpha)(z)$ is jointly continuous in α and z, and $|\partial_\alpha g_\alpha(z)| \le \mathbf{C}\, z^{-\alpha}(1 - \log z)$. □

Corollary 5.11. *$\rho_\alpha(z)$ and $\partial_\alpha \rho_\alpha(z)$ are jointly continuous in α and z. Also, $0 \le g_\alpha(z) \le \mathbf{C}\, z^{-\alpha}$ and $|\partial_\alpha g_\alpha(z)| \le \mathbf{C}\, z^{-\alpha}(1 - \log z)$.*

Proof. Note that $\int_0^1 g_\alpha(z)\, dz = \int \tau_\alpha\, d\mu_\alpha \ge 1$, and

$$\frac{d}{d\alpha} \int_0^1 g_\alpha(z)\, dz = \int_0^1 (\partial_\alpha g_\alpha)(z)\, dz.$$

By Lemma 5.10, $\int_0^1 g_\alpha(z)\, dz$ is continuously differentiable in α, its derivative is bounded by $\mathbf{C}$. The result follows from Lemma 5.10 and relation, established in Lemma 5.9:

$$\rho_\alpha(z) = g_\alpha(z) \Big/ \int_0^1 g_\alpha(x)\, dx.$$

□

Corollary 5.12. *Assume that $\varphi \in L^q[0,1]$, where $q > (1 - \alpha_+)^{-1}$. Then the map $\alpha \mapsto \int \varphi(x)\rho_\alpha(x)\, dx$ is continuously differentiable on $[\alpha_-, \alpha_+]$.*

Proof. Let $p = 1/(1 - 1/q)$. Then $p < 1/\alpha_+$ and by Corollary 5.11, $\|\partial_\alpha \rho_\alpha\|_{L^p}$ is bounded uniformly in α. Since $\rho_\alpha(x)$ and $(\partial_\alpha \rho_\alpha)(x)$ are jointly continuous in α and x, we can write

$$\left| \frac{d}{d\alpha} \int \varphi(x)\rho_\alpha(x)\, dx \right| = \left| \int_0^1 \varphi(x)\, (\partial_\alpha \rho_\alpha)(x)\, dx \right| \le \|\varphi\|_{L^q}\, \|\partial_\alpha \rho_\alpha\|_{L^p}.$$

It is clear that the above is bounded on $[\alpha_-, \alpha_+]$. Continuity of $\int_0^1 \varphi(x)\, (\partial_\alpha \rho_\alpha)(x)\, dx$ follows from continuity of $(\partial_\alpha \rho_\alpha)(x)$ in α and the dominated convergence theorem. □

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