A proof of Hadjicostas's conjecture

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In [1] Hadjicostas makes the following conjecture, which we prove here.

Theorem 1 *Let* $z \in \mathbf{C}$ *satisfy* $\mathrm{Re}(z) > -2$. *Then*

$$\int_0^1 \int_0^1 \frac{[-\log(xy)]^z}{1-xy}(1-x)\,dx\,dy = \Gamma(z+2)\left[\zeta(z+2) - \frac{1}{z+1}\right].$$

Proof We calculate

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{[-\log(xy)]^z}{1-xy}(1-x)\,dx\,dy &= \int_0^1 \int_0^1 \frac{[-\log(xy)]^z}{1-xy}(1-x)\,dy\,dx \\
&= \int_0^1 \int_0^x \frac{[-\log u]^z}{1-u}\frac{1-x}{x}\,du\,dx \\
&= \int_0^1 \int_u^1 \frac{[-\log u]^z}{1-u}\frac{1-x}{x}\,dx\,du \\
&= \int_0^1 \frac{[-\log u]^z}{1-u}(-\log u + u - 1)\,du \\
&= \int_0^\infty \frac{t^z}{1-e^{-t}}(t + e^{-t} - 1)e^{-t}\,dt \\
&= \int_0^\infty \left[\frac{t^{z+1}}{e^t - 1} - t^z e^{-t}\right] dt \\
&= \Gamma(z+2)\zeta(z+2) - \Gamma(z+1) \\
&= \Gamma(z+2)\left[\zeta(z+2) - \frac{1}{z+1}\right]
\end{aligned}$$

using the substitutions $y = u/x$ and $u = e^{-t}$. This calculation is valid line-by-line whenever $\mathrm{Re}(z) > -1$ and the conclusion holds by analytic continuation when $\mathrm{Re}(z) > -2$. □

As Hadjicostas points out, when $z = -1$, since $\zeta(s) = 1/(s-1) + \gamma + O(s-1)$ as $s \to 1$, we get

$$\gamma = -\int_0^1\int_0^1 \frac{1-x}{(1-xy)\log(xy)}\,dx\,dy,$$

a formula obtained by Sondow [2].

References

[1] Petros Hadjicostas, 'A conjecture-generalization of Sondow's formula', preprint, arxiv.\protect\vrule width0pt\protect\href{http://arxiv.org/abs/math/04
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[2] Jonathan Sondow, 'Double integrals for Euler's constant and ln(4/π), preprint, arxiv.\protect\vrule width0pt\protect\href{http://arxiv.org/abs/math/0211148}{m
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