ON ÉTALE FUNDAMENTAL GROUPS OF FORMAL FIBRES OF p-ADIC CURVES

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Abstract. We investigate a certain class of (geometric) finite (Galois) coverings of formal fibres of p-adic curves and the corresponding quotient of the (geometric) étale fundamental group. A key result in our investigation is that these (Galois) coverings can be compactified to finite (Galois) coverings of proper p-adic curves. We also prove that the maximal prime-to-p quotient of the geometric étale fundamental group of a (geometrically connected) formal fibre of a p-adic curve is (pro-)prime-to-p *free of finite computable rank.*

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§0. Introduction/Main Results. A classical result in the theory of étale fundamental groups is the description of the structure of the geometric étale fundamental group of an affine, smooth, and geometrically connected curve over a field of characteristic 0 (cf. [Grothendieck], Exposé XIII, Corollaire 2.12). In this paper we investigate the structure of a certain quotient of the geometric étale fundamental group of a formal fibre of a p-adic curve.

Let R be a complete discrete valuation ring, $K = \mathrm{Fr}(R)$ its quotient field, and k its residue field which we assume to be algebraically closed of characteristic $p \geq 0$. Let X be a proper, flat, and normal formal R-curve whose special fibre X_k is reduced and consists of $n \geq 1$ distinct irreducible components $\{P_i\}_{i=1}^n$ which intersect at a (closed) point $x \in X_k(k)$, and x is the unique singular point of X_k. Write $\widetilde{P}_i \to P_i$ for the morphism of normalisation. We assume $\widetilde{P}_i = \mathbb{P}^1_k$ is a projective line, the morphism $\widetilde{P}_i \to P_i$ is a homeomorphism, and if x_i is the (unique) pre-image of x in $\widetilde{P}_i$ then $x_i \in \widetilde{P}_i(k)$ is the zero point of $\widetilde{P}_i$. In particular, the configuration of the irreducible components of X_k is tree-like. The formal curve X has a formal covering $X = B \cup D_1 \cup \ldots \cup D_n$ where $B \subset X$ is a formal sub-scheme with

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special fibre $B_k = X_k \setminus \{\infty_i\}_{i=1}^n$ (∞_i is the image in P_i of the infinity point of $\widetilde{P}_i$, $1 \leq i \leq n$), $D_i = \operatorname{Spf}\langle \frac{1}{T_i} \rangle$ is an R-formal closed unit disc with special fibre $D_{i,k} = P_i \setminus \{x\}$ and generic fibre $D_{i,K} = \operatorname{Sp} K\langle \frac{1}{T_i} \rangle$ which is a closed unit K-rigid disc centred at the point $\infty_i \in D_{i,K}(K)$ (which specialises in $\infty_i \in D_{i,k}$), $1 \leq i \leq n$. Write $\mathcal{F} \overset{\text{def}}{=} \mathcal{F}_x = \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ for the formal germ of X at x and $\mathcal{F}_K \overset{\text{def}}{=} \mathcal{F}_{x,K} = \operatorname{Spec}(\hat{\mathcal{O}}_{X,x} \otimes_R K)$ for the formal fibre of the generic fibre X_K of the algebraisation of X at x (cf. 1.2 for more details, as well as Remark 3.1 which asserts that any formal germ of a formal R-curve at a closed point admits a compactification as above).

Let $S \subset \mathcal{F}_K$ be a (possibly empty) finite set of closed points. Write $\pi_1(\mathcal{F}_K \setminus S)^{\text{geo}}$ for the geometric étale fundamental group of $\mathcal{F}_K \setminus S$ (in the sense of Grothendieck, cf. 1.3 for more details), and consider the quotient $\pi_1(\mathcal{F}_K \setminus S)^{\text{geo}} \twoheadrightarrow \widehat{\pi}_1(\mathcal{F}_K \setminus S)^{\text{geo}}$ which classifies finite coverings $\mathcal{Y}' \to \mathcal{F}' \overset{\text{def}}{=} \mathcal{F} \times_R R'$, where R'/R is a finite extension, $\mathcal{Y}'$ is normal and geometrically connected, which are étale above $\mathcal{F}_{K'} \setminus S_{K'}$ ($K' \overset{\text{def}}{=} \operatorname{Fr} R'$ and $S_{K'} \overset{\text{def}}{=} S \times_K K'$) and étale above the generic points of $\mathcal{F}_k$ (cf. loc. cit.). Similarly, write $\pi_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n))^{\text{geo}}$ for the geometric étale fundamental group of the affine curve $X_K \setminus (S \cup \{\infty_i\}_{i=1}^n)$ and consider the quotient $\pi_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n))^{\text{geo}} \twoheadrightarrow \widehat{\pi}_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n); \{\infty_i\}_{i=1}^n)^{\text{geo}}$ which classifies finite coverings $Y' \to X_{R'}$ which are étale above $X_{K'} \setminus (S_{K'} \cup \{\infty_i\}_{i=1}^n)$, possibly ramified above the points $\{\infty_i\}_{i=1}^n$ with ramification indices prime-to-p, and which are étale above the generic points of X_k (here R', K' and $S_{K'}$ are as above). We also write $\widehat{\pi}_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n))^{\text{geo},p} \overset{\text{def}}{=} \widehat{\pi}_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n); \{\infty_i\}_{i=1}^n)^{\text{geo},p}$ (resp. $\widehat{\pi}_1(\mathcal{F}_K \setminus S)^{\text{geo},p}$) for the maximal pro-$p$ quotient of $\widehat{\pi}_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n); \{\infty_i\}_{i=1}^n)^{\text{geo}}$ (resp. $\widehat{\pi}_1(\mathcal{F}_K \setminus S)^{\text{geo}}$). Our first main result is the following (cf. Theorem 3.2).

Theorem 1. *The (scheme) morphism $\mathcal{F}_K \to X_K$ induces a continuous homomorphism $\widehat{\pi}_1(\mathcal{F}_K \setminus S)^{\text{geo}} \to \widehat{\pi}_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n); \{\infty_i\}_{i=1}^n)^{\text{geo}}$ (resp. $\widehat{\pi}_1(\mathcal{F}_K \setminus S)^{\text{geo},p} \to \widehat{\pi}_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n); \{\infty_i\}_{i=1}^n)^{\text{geo},p}$) which makes $\widehat{\pi}_1(\mathcal{F}_K \setminus S)^{\text{geo}}$ (resp. $\widehat{\pi}_1(\mathcal{F}_K \setminus S)^{\text{geo},p}$) into a semi-direct factor (cf. Definition 1.1.4 and Lemma 1.1.5) of $\widehat{\pi}_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n); \{\infty_i\}_{i=1}^n)^{\text{geo}}$ (resp. $\widehat{\pi}_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n); \{\infty_i\}_{i=1}^n)^{\text{geo},p}$). In particular, the above homomorphisms are injective.*

In the course of proving Theorem 1 (cf. proof of Theorem 3.2) we prove the following.

Theorem 2. *Let $f : \mathcal{Y} \to \mathcal{F}$ be a finite (Galois) covering with $\mathcal{Y}$ normal and geometrically connected, which is étale above $\mathcal{F}_K \setminus S$ and above the generic points of $\mathcal{F}_k$. Then there exists, after possibly a finite extension of K, a finite (Galois) covering $\tilde{f} : Y \to X$ of formal schemes with Y normal and geometrically connected, which is étale above $X_K \setminus (S \cup \{\infty_i\}_{i=1}^n)$ and above the generic points of X_k, is possibly ramified above the points $\{\infty_i\}_{i=1}^n$ with ramification indices prime-to-p, and which induces by pull back via the (scheme) morphism $\mathcal{F} \to X$ the covering f.*

Let $g_x \overset{\text{def}}{=} \operatorname{genus}(X_K)$, which is also called the genus of the formal fibre $\mathcal{F}_K$. Write $\pi_1(\mathcal{F}_K \setminus S, \eta)^{\text{geo},p'}$ (resp. $\pi_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n), \eta)^{\text{geo},p'}$) for the maximal prime-to-p quotient of the geometric étale fundamental group $\pi_1(\mathcal{F}_K \setminus S, \eta)^{\text{geo}}$ (resp. $\pi_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n), \eta)^{\text{geo}}$). Our second main result is the following (cf. Theorem 3.4).

Theorem 3. *Let $S(\overline{K})=\{y_1,\dots,y_m\}$ of cardinality $m\geq 0$. Then the continuous homomorphism $\pi_1(\mathcal{F}_K\setminus S,\eta)^{\mathrm{geo},p'}\to\pi_1(X_K\setminus(S\cup\{\infty_i\}_{i=1}^n),\eta)^{\mathrm{geo},p'}$ (induced by the (scheme) morphism $\mathcal{F}_K\to X_K$) is an isomorphism. In particular, $\pi_1(\mathcal{F}_K\setminus S,\eta)^{\mathrm{geo},p'}$ is (pro-)prime-to-p free of rank $2g_x+n+m-1$ and can be generated by $2g_x+n+m$ generators $\{a_1,\dots,a_g,b_1,\cdots,b_g,\sigma_1,\dots,\sigma_n,\tau_1,\dots,\tau_m\}$ subject to the unique relation $\prod_{i=1}^g[a_i,b_i]\prod_{j=1}^n\sigma_j\prod_{t=1}^m\tau_t=1$, where σ_j (resp τ_t) is a generator of inertia at ∞_i (resp. y_t).*

Next, we outline the content of the paper. In §1 we collect some well-known background material. In §2 we investigate a certain quotient of the absolute Galois group of a formal boundary of a formal germ of a p-adic curve and prove Proposition 2.5 which is used in the proof of Theorem 1. In §3 we prove Theorems 1 and 3.

Notations. In this paper K is a complete discrete valuation field, R its valuation ring, π a uniformising parameter, and $k\overset{\mathrm{def}}{=}R/\pi R$ the residue field which we assume to be algebraically closed of characteristic $p\geq 0$.

We refer to [Raynaud], 3, for the terminology we will use concerning K-rigid analytic spaces, R-formal schemes, as well as the link between formal and rigid geometry. For an R-(formal) scheme X we will denote by $X_K\overset{\mathrm{def}}{=}X\times_R K$ (resp. $X_k\overset{\mathrm{def}}{=}X\times_R k$) the generic (resp. special) fibre of X (the generic fibre is understood in the rigid analytic sense in the case where X is a formal scheme). Moreover, if $X=\mathrm{Spf}\,A$ is an affine formal R-scheme of finite type we denote by $X_K\overset{\mathrm{def}}{=}\mathrm{Sp}(A\otimes_R K)$ the associated K-rigid affinoid space and will also denote, when there is no risk of confusion, by X_K the affine scheme $X_K\overset{\mathrm{def}}{=}\mathrm{Spec}(A\otimes_R K)$.

A formal (resp. algebraic) R-curve is an R-formal scheme of finite type (resp. R-scheme of finite type) flat, separated, and whose special fibre is equi-dimensional of dimension 1. For a K-scheme (resp. K-rigid analytic space) X and L/K a field extension (resp. a finite extension) we write $X_L\overset{\mathrm{def}}{=}X\times_K L$ which is an L-scheme (resp. an L-rigid analytic space). If X is a proper and normal formal R-curve we also denote, when there is no risk of confusion, by X the algebraisation of X which is an algebraic R-curve and by X_K the proper normal and algebraic K-curve associated to the rigid K-curve X_K via the rigid GAGA functor.

For a profinite group H and a prime integer ℓ we denote by H^ℓ the maximal pro-ℓ quotient of H, and $H^{\ell'}$ the maximal prime-to-ℓ quotient of H.

§**1 Background.** In this section we collect some background material used in this paper.

1.1. Let $p>1$ be a prime integer. We recall some well-known facts on profinite pro-p groups. First, we recall the following characterisations of free pro-p groups.

Proposition 1.1.1. *Let G be a profinite pro-p group. Then the following properties are equivalent.*
(i) G is a free pro-p group.
(ii) The p-cohomological dimension of G satisfies $\mathrm{cd}_p(G)\leq 1$.
In particular, a closed subgroup of a free pro-p group is free.

Proof. Well-known (cf. [Serre], and [Ribes-Zalesskii], Theorem 7.7.4). □

Next, we recall the notion of a *direct factor* of a free pro-p group (cf. [Garuti], 1, the discussion preceding Proposition 1.8, and [Saïdi], §1).

Definition/Lemma 1.1.2 (Direct factors of free pro-p groups). *Let F be a free pro-p group, $H \subseteq F$ a closed subgroup, and $\iota : H \to F$ the natural homomorphism. We say that H is a direct factor of F if there exists a continuous homomorphism $s : F \to H$ such that $s \circ \iota = \mathrm{id}_H$. There exists then a (non unique) closed subgroup N of F such that F is isomorphic to the free direct product $H \star N$. We will refer to such a subgroup N as a supplement of H.*

Proof. See [Saïdi] Lemma 1.1.2. $\square$

One has the following cohomological characterisation of direct factors of free pro-p groups.

Proposition 1.1.3. *Let H be a pro-p group, F a free pro-p group, and $\sigma : H \to F$ a continuous homomorphism. Assume that the map induced by σ on cohomology*

$$h^1(\sigma) : H^1(F, \mathbb{Z}/p\mathbb{Z}) \to H^1(H, \mathbb{Z}/p\mathbb{Z})$$

is surjective, where $\mathbb{Z}/p\mathbb{Z}$ is considered as a trivial discrete module. Then σ induces an isomorphism $H \xrightarrow{\sim} \sigma(H)$ and $\sigma(H)$ is a direct factor of F. In particular, H is pro-p free. We say that σ makes H into a direct factor of F.

Proof. cf. [Garuti], Proposition 1.8 and Proposition 1.1.1 above. $\square$

Next, we consider the notion of a *semi-direct factor* of a profinite group.

Definition 1.1.4 (Semi-direct factors of profinite groups). Let G be a profinite group, $H \subseteq G$ a closed subgroup, and $\iota : H \to G$ the natural homomorphism. We say that H is a *semi-direct factor* of G if there exists a continuous homomorphism $s : G \to H$ such that $s \circ \iota = \mathrm{id}_H$ (s is necessarily surjective).

Lemma 1.1.5. *Let $\tau : H \to G$ be a continuous homomorphism between profinite groups. Write $H = \varprojlim_{j \in J} H_j$ as the projective limit of the inverse system $\{H_j, \phi_{j'j}, J\}$ of finite quotients H_j of H with index set J. Suppose there exists, $\forall j \in J$, a surjective homomorphism $\psi_j : G \twoheadrightarrow H_j$ such that $\tau \circ \psi_j : H \twoheadrightarrow H_j$ is the natural map and $\psi_j = \phi_{j'j} \circ \psi_{j'}$ whenever this makes sense. Then τ induces an isomorphism $H \xrightarrow{\sim} \tau(H)$ and $\tau(H)$ is a semi-direct factor of G. We say that τ makes H into a semi-direct factor of G.*

Proof. Indeed, the $\{\psi_j\}_{j \in J}$ give rise to a continuous (necessarily surjective) homomorphism $\psi : G \to H$ which is a right inverse of τ. $\square$

1.2. Formal Patching. Next, we explain the procedure which allows to construct (Galois) coverings of curves in the setting of formal geometry by patching coverings of formal (affine, non-proper) curves with coverings of formal germs at closed points of the special fibre along the boundaries of these formal germs.

1.2.1. Let X be a proper, normal, formal R-curve with X_k reduced. For $x \in X$ a closed point let $\mathcal{F}_x \overset{\text{def}}{=} \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ be the formal completion of X at x which we will refer to as the *formal germ* of X at x. Thus, $\hat{\mathcal{O}}_{X,x}$ is the completion of the local ring of the algebraisation of X at x. Write $\mathcal{F}_{x,K} \overset{\text{def}}{=} \operatorname{Spec}(\hat{\mathcal{O}}_{X,x} \otimes_R K)$. We will refer to $\mathcal{F}_{x,K}$ as the *formal fibre* of X_K at x. Let $\{\mathcal{P}_i\}_{i=1}^n$ be the minimal prime ideals of $\hat{\mathcal{O}}_{X,x}$ which contain π; they correspond to the branches $\{\eta_i\}_{i=1}^n$ of the completion of X_k at x (i.e., closed points of the normalisation of X_k above x), and

$\mathcal{X}_i = \mathcal{X}_{x,i} \overset{\text{def}}{=} \operatorname{Spf} \hat{\mathcal{O}}_{x,\mathcal{P}_i}$ the formal completion of the localisation of $\mathcal{F}_x$ at $\mathcal{P}_i$. The local ring $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$ is a complete discrete valuation ring with uniformiser π. We refer to $\{\mathcal{X}_i\}_{i=1}^n$ as the set of *boundaries* of the formal germ $\mathcal{F}_x$. We have a canonical morphism $\mathcal{X}_i \to \mathcal{F}_x$ of formal schemes, $1 \le i \le n$.

Let Z be a finite set of closed points of X and $U \subset X$ a formal sub-scheme of X whose special fibre is $U_k \overset{\text{def}}{=} X_k \setminus Z$.

Definition 1.2.2. We use the notations above. A (G-)covering patching data for the pair (X, Z) consists of the following.
(i) A finite (Galois) covering $V \to U$ of formal schemes (with Galois group G).
(ii) For each point $x \in Z$, a finite (Galois) covering $\mathcal{Y}_x \to \mathcal{F}_x$ of formal schemes (with Galois group G).

The above data (i) and (ii) must satisfy the following compatibility condition.
(iii) If $\{\mathcal{X}_i\}_{i=1}^n$ are the boundaries of the formal germ at the point x, then for $1 \le i \le n$ is given a (G-equivariant) $\mathcal{X}_i$-isomorphism

$$\mathcal{Y}_x \times_{\mathcal{F}_x} \mathcal{X}_i \overset{\sim}{\to} V \times_U \mathcal{X}_i.$$

Property (iii) should hold for each $x \in Z$. (Note that there are natural morphisms $\mathcal{X}_i \to U$ of formal schemes, $1 \le i \le n$.)

The following is the main patching result that we will use in this paper (cf. [Pries], Theorem 3.4, [Harbater], Theorem 3.2.8).

Proposition 1.2.3. *We use the notations above. Given a (G-)covering patching data as in Definition 1.2.2 there exists a unique, up to isomorphism, (Galois) covering $Y \to X$ of formal schemes (with Galois group G) which induces the above (G-)covering in Definition 1.2.2(i) when restricted to U, and induces the above (G-)covering in Definition 1.2.2(ii) when pulled-back to $\mathcal{F}_x$ for each point $x \in Z$.*

1.2.4. With the same notations as above, let $x \in X$ be a closed point and $\widetilde{X}_k$ the normalisation of X_k. There is a one-to-one correspondence between the set of points of $\widetilde{X}_k$ above x and the set of boundaries of the formal germ of X at the point x. Let x_i be the point of $\widetilde{X}_k$ above x which corresponds to the boundary $\mathcal{X}_i$, $1 \le i \le n$. Then the completion of $\widetilde{X}_k$ at x_i is isomorphic to the spectrum of a ring of formal power series $k[[t_i]]$ over k where t_i is a local parameter at x_i. The complete local ring $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$ is a discrete valuation ring with uniformiser π and residue field isomorphic to $k((t_i))$. Fix an isomorphism $k((t_i)) \overset{\sim}{\to} \hat{\mathcal{O}}_{x,\mathcal{P}_i}/\pi$. Let $T_i \in \hat{\mathcal{O}}_{x,\mathcal{P}_i}$ be an element which lifts (the image in $\hat{\mathcal{O}}_{x,\mathcal{P}_i}/\pi$ under the above isomorphism of) t_i; we shall refer to such an element T_i as a parameter of $\hat{\mathcal{O}}_{x,\mathcal{P}_i}$, or of the boundary $\mathcal{X}_i$. Then there exists an isomorphism $R[[T_i]]\{T_i^{-1}\} \overset{\sim}{\to} \hat{\mathcal{O}}_{x,\mathcal{P}_i}$, where

$$R[[T]]\{T^{-1}\} \overset{\text{def}}{=} \left\{ \sum_{i=-\infty}^{\infty} a_i T^i, \ \lim_{i \to -\infty} |a_i| = 0 \right\}$$

and $| \ |$ is a normalised absolute value of R (cf. [Bourbaki], §2, 5).

1.3. Let X be a normal and geometrically connected flat R-scheme (resp. R-formal affine scheme) whose special fibre is equidimensional of dimension 1, $F \subset X_K$ a finite set of closed points, and η a geometric point of X above its generic point.

Then η determines an algebraic closure $\overline{K}$ of K and we have an exact sequence of arithmetic fundamental groups

$$1 \to \pi_1(X_K \setminus F, \eta)^{\text{geo}} \to \pi_1(X_K \setminus F, \eta) \to \mathrm{Gal}(\overline{K}/K) \to 1,$$

where $\pi_1(X_K \setminus F, \eta)^{\text{geo}} \overset{\text{def}}{=} \mathrm{Ker}\left(\pi_1(X_K \setminus F, \eta) \twoheadrightarrow \mathrm{Gal}(\overline{K}/K)\right)$ is the geometric fundamental group of X_K with generic point η. (In case $X = \mathrm{Spf}\, A$ is formal affine we define $\pi_1(X_K, \eta) \overset{\text{def}}{=} \pi_1(\mathrm{Spec}\, A_K, \eta)$ and similarly we define $\pi_1(X_K \setminus F, \eta)$, cf. [Saïdi], 2.1.)

Definition 1.3.1. Let $S, T \subset X_K$ be (possibly empty) finite sets of closed points (which we also view as reduced closed sub-schemes of X_K). Assume that the special fibre X_k of X is reduced. Let $I \overset{\text{def}}{=} I_{X_k,T} \subset \pi_1(X_K \setminus (S \cup T), \eta)^{\text{geo}}$ be the subgroup normally generated by the inertia subgroups above the generic points of X_k and the pro-p Sylow subgroups of the inertia groups above all points in T. We define

$$\widehat{\pi}_1(X_K \setminus (S \cup T); T, \eta)^{\text{geo}} \overset{\text{def}}{=} \pi_1(X_K \setminus (S \cup T), \eta)^{\text{geo}}/I$$

and refer to it as the geometric étale fundamental group of $X_K \setminus (S \cup T)$; with base point η, *generically étale above* X_k and *tamely ramified* above T. In case $T = \emptyset$ and $U_K \overset{\text{def}}{=} X_K \setminus S$ we simply write $\widehat{\pi}_1(U_K, \eta)^{\text{geo}} \overset{\text{def}}{=} \widehat{\pi}_1(X_K \setminus S; \emptyset, \eta)^{\text{geo}}$.

Note that the definition of $\widehat{\pi}_1(X_K \setminus (S \cup T); T, \eta)^{\text{geo}}$ depends on the model X of X_K (the model X of X_K will be fixed in later discussions in this paper). The profinite group $\widehat{\pi}_1(X_K \setminus (S \cup T); T, \eta)^{\text{geo}}$ classifies finite covers $f : Y_L \to X_L \overset{\text{def}}{=} X \times_K L$ where L/K is a finite extension with valuation ring R_L, which are étale above $X_L \setminus (S \cup T)_L$ (here $(S \cup T)_L \overset{\text{def}}{=} (S \cup T) \times_K L$) and possibly ramified with ramification indices prime-to-p above the points in $T_L \overset{\text{def}}{=} T \times_K L$, Y_L is geometrically connected, and such that f extends after possibly a finite extension of L to a finite cover $\tilde{f} : Y \to X_{R_L} \overset{\text{def}}{=} X \times_R R_L$ with Y normal and $\tilde{f}$ is étale above the generic points of X_k. Note that if X is a smooth R-formal affine scheme as above which is an R-formal curve then $\widehat{\pi}_1(X_K, \eta)^{\text{geo}}$ is isomorphic to the geometric étale fundamental group of the affine scheme X_k as follows from the theorems of liftings of étale coverings (cf. [Grothendieck], Exposé I, Corollaire 8.4) and the theorem of purity of Zarizski-Nagata (cf. loc. cit. Exposé X, Théorème de pureté 3.1). Note also that $\widehat{\pi}_1(X_K \setminus (S \cup T); T, \eta)^{\text{geo},p'} = \pi_1(X_K \setminus (S \cup T), \eta)^{\text{geo},p'}$, as follows easily from Abhyankar's lemma (cf. loc. cit. Exposé X, Lemme 3.6).

§2. Geometric Galois groups of formal boundaries of formal germs of p-adic formal curves. In this section we investigate the structure of a certain quotient of the geometric Galois group of a formal boundary of a formal germ of a formal R-curve. The results in this section will be used in §3.

Let $D \overset{\text{def}}{=} \mathrm{Spf}\, R\langle \frac{1}{T} \rangle$ be the formal standard R-closed unit disc and $D_K \overset{\text{def}}{=} \mathrm{Sp}\, K\langle \frac{1}{T} \rangle$ its generic fibre which is the standard rigid K-closed unit disc centred at ∞. Write $\mathcal{X} = \mathrm{Spf}\, R[[T]]\{T^{-1}\}$ and $\mathcal{X}_K \overset{\text{def}}{=} \mathrm{Spec}(R[[T]]\{T^{-1}\} \otimes_R K)$. We have natural morphisms $\mathcal{X} \to D$ of formal R-schemes, and $\mathcal{X}_K \to D_K$ of K-schemes (cf. Notations). Let η be a geometric point of $\mathcal{X}_K$ with value in its generic point which determines a generic point of D_K; which we denote also η, as well as algebraic closures $\overline{K}$ of K, $\overline{k}$ of k, and $\overline{M}$ of $M \overset{\text{def}}{=} \text{Fr}(R[[T]]\{T^{-1}\})$. We have an exact sequence of Galois groups

$$1 \to \text{Gal}(\overline{M}/\overline{K}.M) \to \text{Gal}(\overline{M}/M) \to \text{Gal}(\overline{K}/K) \to 1.$$

Let $I \overset{\text{def}}{=} I_{(\mathcal{X}_k)} \subset \text{Gal}(\overline{M}/\overline{K}.M)$ be the subgroup normally generated by the inertia subgroups above the generic point of $\mathcal{X}_k$. Write $\Delta \overset{\text{def}}{=} \text{Gal}(\overline{M}/\overline{K}.M)/I$ and $\Gamma \overset{\text{def}}{=} \Delta^{p'}$. We have an exact sequence

$$1 \to P \to \Delta \to \Gamma \to 1,$$

where $P \overset{\text{def}}{=} \text{Ker}(\Delta \twoheadrightarrow \Gamma)$.

Lemma 2.1. *With the notations above, P is the unique pro-p Sylow subgroup of Δ, P is pro-p free, and Γ is canonically isomorphic to the maximal prime-to-p quotient $\hat{\mathbb{Z}}(1)^{p'}$ of the Tate twist $\hat{\mathbb{Z}}(1)$.*

Proof. Indeed, it follows from the various Definitions that Δ is isomorphic to the absolute Galois group of $\overline{k}((t))$ which is known to be an extension of $\hat{\mathbb{Z}}(1)^{p'}$ by a free pro-p group. $\square$

Lemma 2.2. *Assume $p > 0$. Then the pro-p group $\widehat{\pi}_1(D_K, \eta)^{\text{geo},p}$ is free.*

Proof. Indeed, it follows from the various Definitions that $\widehat{\pi}_1(D_K, \eta)^{\text{geo},p}$ is isomorphic to the maximal pro-p quotient of the geometric fundamental group of $D_k = \mathbb{A}^1_k$ which is pro-p free (cf. [Serre1], Proposition 1). $\square$

Proposition 2.3. *Assume $p > 0$. Then the homomorphism $\Delta \to \widehat{\pi}_1(D_K, \eta)^{\text{geo}}$ induced by the morphism $\mathcal{X}_K \to D_K$ induces a homomorphism $\Delta^p \to \widehat{\pi}_1(D_K, \eta)^{\text{geo},p}$ which makes Δ^p into a direct factor of $\widehat{\pi}_1(D_K, \eta)^{\text{geo},p}$. Moreover, Δ^p is a free pro-p group of infinite rank.*

Proof. We show that the map $\psi : H^1(\widehat{\pi}_1(D_K, \eta)^{\text{geo}}, \mathbb{Z}/p\mathbb{Z}) \to H^1(\Delta, \mathbb{Z}/p\mathbb{Z})$ induced by the homomorphism $\Delta \to \widehat{\pi}_1(D_K, \eta)^{\text{geo}}$ on cohomology is surjective (cf. Proposition 1.1.3). Let $\tilde{f} : \Delta \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ be a surjective homomorphism and $f : \mathcal{Y} \to \mathcal{X}$ the corresponding Galois cover (which we can assume, without loss of generality, defined over K) with $\mathcal{Y}$ normal, geometrically connected, and f is étale above the generic point of $\mathcal{X}_k$ (hence f is étale above $\mathcal{X}$). Thus, $f_k : \mathcal{Y}_k \to \mathcal{X}_k = \text{Spec}\, k((t))$ is an étale $\mathbb{Z}/p\mathbb{Z}$-torsor. By Artin-Schreier theory the torsor f_k can be approximated by a Galois cover $g_k : Y_k \to \mathbb{P}^1_k$ of degree p which is étale outside the point $t = 0$ and whose completion above this point is isomorphic to f_k. The étale $\mathbb{Z}/p\mathbb{Z}$-torsor $g_k^{-1}(\mathbb{A}^1_k) \to \mathbb{A}^1_k \overset{\text{def}}{=} \mathbb{P}^1_k \setminus \{t = 0\} = \text{Spec}\, k[\frac{1}{t}]$ lifts (uniquely up to isomorphism) to an étale $\mathbb{Z}/p\mathbb{Z}$-torsor $g : Z_K \to D_K$ by the theorems of liftings of étale covers (cf. [Grothendieck], Exposé I, Corollaire 8.4) which gives rise to a class in $H^1(\widehat{\pi}_1(D_K, \eta)^{\text{geo}}, \mathbb{Z}/p\mathbb{Z})$ that is easily verified to map to the class of f in $H^1(\Delta, \mathbb{Z}/p\mathbb{Z})$. Moreover, Δ^p has infinite rank as it is isomorphic to the maximal pro-p quotient of the absolute Galois group of $\overline{k}((t))$ which is known to be free of infinite rank. $\square$

Write $\widetilde{\Gamma} \overset{\text{def}}{=} \widehat{\pi}_1(D_K \setminus \{\infty\}; \{\infty\}, \eta)^{\text{geo},p'} = \pi_1(D_K \setminus \{\infty\}, \eta)^{\text{geo},p'}$ (cf. 1.3) for the maximal prime-to-p quotient of $\widehat{\pi}_1(D_K \setminus \{\infty\}; \{\infty\}, \eta)^{\text{geo}}$.

Lemma 2.4. *The morphism $\mathcal{X}_K \to D_K$ induces a canonical homomorphism $\Gamma \to \widetilde{\Gamma}$ which is an isomorphism. In particular, $\widetilde{\Gamma}$ is (canonically) isomorphic to $\hat{\mathbb{Z}}(1)^{p'}$.*

Proof. Follows easily from the fact that a Galois covering $Y_K \to D_K$ of order prime-to-p with Y_K geometrically connected, ramified only above ∞ is, possibly after a finite extension of K and for a suitable choice of the parameter T of D_K, generically a μ_n-torsor given generically by the equation $S^n = T$ for some positive integer n prime-to-p. $\square$

Consider the following exact sequence

$$1 \to \mathcal{H} \to \widehat{\pi}_1(D_K \setminus \{\infty\}; \{\infty\}, \eta)^{\mathrm{geo}} \to \widetilde{\Gamma} \to 1,$$

where $\mathcal{H} \overset{\mathrm{def}}{=} \mathrm{Ker}(\widehat{\pi}_1(D_K \setminus \{\infty\}; \{\infty\}, \eta)^{\mathrm{geo}} \twoheadrightarrow \widetilde{\Gamma})$. Further, let $\widetilde{P} \overset{\mathrm{def}}{=} \mathcal{H}^p$ be the maximal pro-p quotient of $\mathcal{H}$. By pushing out the above sequence by the (characteristic) quotient $\mathcal{H} \twoheadrightarrow \widetilde{P}$ we obtain an exact sequence

$$1 \to \widetilde{P} \to \widetilde{\Delta} \to \widetilde{\Gamma} \to 1.$$

Proposition 2.5. *The morphism $\mathcal{X}_K \to D_K$ induces a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \widetilde{P} & \longrightarrow & \widetilde{\Delta} & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & 1
\end{array}$$

where the right vertical homomorphism $\Gamma \to \widetilde{\Gamma}$ is an isomorphism (cf. Lemma 2.4) and the middle vertical homomorphism $\Delta \to \widetilde{\Delta}$ makes Δ into a semi-direct factor of $\widetilde{\Delta}$ (cf. Lemma 1.1.5).

Proof. Let $\Delta \twoheadrightarrow G$ be a finite quotient which sits in an exact sequence $1 \to Q \to G \to \Gamma_n \to 1$ where Γ_n is the unique quotient of Γ of cardinality n; for some integer n prime-to-p, with Q a p-group (cf. Lemma 2.1). We will show there exists a surjective homomorphism $\widetilde{\Delta} \twoheadrightarrow G$ whose composition with $\Delta \to \widetilde{\Delta}$ is the above homomorphism. We can assume, without loss of generality, that the corresponding Galois covering $f : \mathcal{Y} \to \mathcal{X}$ with group G is defined over K, $\mathcal{Y}$ is normal and connected, and f is étale. This covering factorises as $\mathcal{Y} \to \mathcal{X}' \to \mathcal{X}$ where $\mathcal{X}' \to \mathcal{X}$ is Galois with group $\Gamma_n \xrightarrow{\sim} \mu_n$ and $\mathcal{Y} \to \mathcal{X}'$ is Galois with group Q. After possibly a finite extension of K the μ_n-torsor $\mathcal{X}' \to \mathcal{X}$ extends to a generically μ_n-torsor $D' \to D$ defined generically by an equation $S^n = T$, for a suitable choice of the parameter T of D, which is (totally) ramified only above ∞, $D' = \mathrm{Spf}\, R\langle \frac{1}{S} \rangle$ is a closed formal unit disc centred at the unique point; which we denote also ∞, above $\infty \in D$ and $\mathcal{X}' = \mathrm{Spf}\, R[[S]]\{S^{-1}\}$ (cf. Lemma 2.4 and the isomorphism $\Gamma \xrightarrow{\sim} \widetilde{\Gamma}$ therein).

For the rest of the proof we assume $p > 0$. By Proposition 2.3, applied to $\mathcal{X}' \to D'$, there exists (after possibly a finite extension of K) an étale Galois covering $Y \to D'$ with group Q, Y is normal and geometrically connected, and such that we have a commutative diagram of cartesian squares

$$\begin{array}{ccccc}
\mathcal{Y} & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & D' & \longrightarrow & D.
\end{array}$$

Next, we borrow some ideas from [Garuti] (preuve du Théorème 2.13). We claim one can choose the above (geometric) covering $Y \to D'$ such that the finite composite covering $Y \to D$ is Galois with group G. Indeed, consider the quotient $\Delta \twoheadrightarrow \Delta_{\mathcal{X}'}$ (resp. $\widetilde{\Delta} \twoheadrightarrow \widetilde{\Delta}_{D'}$) of Δ (resp. $\widetilde{\Delta}$) which sits in the following exact sequence $1 \to P_{\mathcal{X}'} \to \Delta_{\mathcal{X}'} \to \Gamma_n \to 1$ where $P_{\mathcal{X}'} \overset{\text{def}}{=} \widehat{\pi}_1(\mathcal{X}', \eta)^{\text{geo},p}$ (resp. $1 \to \widetilde{P}_{D'} \to \widetilde{\Delta}_{D'} \to \widetilde{\Gamma}_n \to 1$ where $\widetilde{P}_{D'} \overset{\text{def}}{=} \widehat{\pi}_1(D', \eta)^{\text{geo},p}$). We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_{\mathcal{X}'} & \longrightarrow & \Delta_{\mathcal{X}'} & \longrightarrow & \Gamma_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \widetilde{P}_{D'} & \longrightarrow & \widetilde{\Delta}_{D'} & \longrightarrow & \widetilde{\Gamma}_n & \longrightarrow & 1 \end{array}$$

where the right vertical map is an isomorphism (cf. Lemma 2.4). The choice of a splitting of the upper sequence in the above diagram (which splits since $P_{\mathcal{X}'}$ is pro-p and Γ_n is cyclic (pro-)prime-to-p) induces an action of Γ_n on $\widetilde{P}_{D'}$ and $P_{\mathcal{X}'}$ is a direct factor of $\widetilde{P}_{D'}$ (cf. Proposition 2.3) which is stable by this action of Γ_n. Further, $P_{\mathcal{X}'}$ has a supplement E in $\widetilde{P}_{D'}$ which is invariant under the action of Γ_n by [Garuti], Corollaire 1.11. The existence of this supplement E implies that one can choose $Y \to D'$ as above such that the finite composite covering $Y \to D$ is Galois with group G. More precisely, if the Galois covering $\mathcal{Y} \to \mathcal{X}'$ corresponds to the surjective homomorphism $\rho : P_{\mathcal{X}'} \twoheadrightarrow Q$ (which is stable by Γ_n since $\mathcal{Y} \to \mathcal{X}$ is Galois) then we consider the Galois covering $Y \to D'$ corresponding to the surjective homomorphism $\widetilde{P}_{D'} = P_{\mathcal{X}'} \star E \twoheadrightarrow Q$ which is induced by ρ and the trivial homomorphism $E \to Q$, which is stable by Γ_n.

The above construction can be performed in a functorial way with respect to the various finite quotients of Δ. More precisely, let $\{\phi_j : \Delta \twoheadrightarrow G_j\}_{j \in J}$ be a cofinal system of finite quotients of Δ where G_j sits in an exact sequence $1 \to Q_j \to G_j \to \Gamma_{n_j} \to 1$, for some integer n_j prime-to-p, and Q_j a p-group. Assume we have a factorisation $\Delta \twoheadrightarrow G_{j'} \twoheadrightarrow G_j$ for $j', j \in J$. Thus, n_j divides $n_{j'}$, and we can assume without loss of generality (after replacing the group extension G_j by its pull-back via $\Gamma_{n'_j} \twoheadrightarrow \Gamma_{n_j}$) that $n \overset{\text{def}}{=} n_j = n_{j'}$. With the above notations we then have surjective homomorphisms $\rho_{j'} : P_{\mathcal{X}'} \twoheadrightarrow Q_{j'}$, $\rho_j : P_{\mathcal{X}'} \twoheadrightarrow Q_j$ (which are stable by Γ_n), and ρ_j factorises through $\rho_{j'}$. Then we consider the Galois covering(s) $Y_{j'} \to D'$ (resp. $Y_j \to D'$) corresponding to the surjective homomorphism(s) $\psi_{j'} : \widetilde{P}_{D'} = P_{\mathcal{X}'} \star E \twoheadrightarrow Q$ (resp. $\psi_j : \widetilde{P}_{D'} = P_{\mathcal{X}'} \star E \twoheadrightarrow Q$) which are induced by $\rho_{j'}$ (resp. ρ_j) and the trivial homomorphism $E \to Q$, which are stable by Γ_n and ψ_j factorises through $\psi_{j'}$. We deduce from this construction the existence of a surjective continuous homomorphism $\widetilde{\Delta} \twoheadrightarrow \Delta$ which is a right inverse to the natural homomorphism $\Delta \to \widetilde{\Delta}$ (cf. Lemma 1.1.5). $\square$

§3 Geometric fundamental groups of formal fibres of p-adic curves. In this section we investigate the structure of $\widehat{\pi}_1$ of a formal fibre of a K-curve. Let X be a proper, normal, formal R-curve whose special fibre X_k is reduced and consists of $n \geq 1$ distinct irreducible components $\{P_i\}_{i=1}^n$ which intersect at a (closed) point $x \in X_k(k)$, and x is the unique singular point of X_k. Write $\widetilde{P}_i \to P_i$ for the morphism of normalisation. We assume $\widetilde{P}_i = \mathbb{P}^1_k$ is a projective line, the morphism $\widetilde{P}_i \to P_i$ is a homeomorphism, and if x_i is the (unique) pre-image of x

in $\widetilde{P}_i$ then $x_i \in \widetilde{P}_i(k)$ is the zero point of $\widetilde{P}_i$. In particular, the configuration of the irreducible components of X_k is tree-like. The formal curve X has a formal covering $X = B \cup D_1 \cup \ldots \cup D_n$ where $B \subset X$ is a formal sub-scheme with special fibre $B_k = X_k \setminus \{\infty_i\}_{i=1}^n$, ∞_i is the image in P_i of the infinity point of $\widetilde{P}_i$, $D_i = \operatorname{Spf} R\langle \frac{1}{T_i} \rangle$ is an R-formal closed unit disc with special fibre $D_{i,k} = P_i \setminus \{x\}$ and generic fibre $D_{i,K} = \operatorname{Sp} K\langle \frac{1}{T_i} \rangle$ which is a closed unit K-rigid disc centred at the point $\infty_i \in D_{i,K}(K)$ which specialises into the infinity point $\infty_i \in P_i$, $1 \leq i \leq n$. Write $\mathcal{F} \overset{\text{def}}{=} \mathcal{F}_x = \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ for the formal germ of X at x and $\mathcal{F}_K \overset{\text{def}}{=} \mathcal{F}_{x,K} = \operatorname{Spec}(\hat{\mathcal{O}}_{X,x} \otimes_R K)$ for the formal fibre of X_K at x (cf. 1.2.1). For $1 \leq i \leq n$, let $\mathcal{X}_i$ be the formal boundary of $\mathcal{F}$ corresponding to the point x_i above. The completion of the normalisation X_k^{nor} of X_k at x_i is isomorphic to the spectrum of a ring of formal power series $k[[t_i]]$ in one variable over k, and $\mathcal{X}_i \xrightarrow{\sim} \operatorname{Spf} R[[T_i]]\{T_i^{-1}\}$ (cf. 1.2.4).

Remark 3.1. Let $\widetilde{Y}$ be a proper and normal formal R-curve with $\widetilde{Y}_k$ reduced and $y \in \widetilde{Y}(k)$ a closed point. Write $\mathcal{G} \overset{\text{def}}{=} \mathcal{G}_y = \operatorname{Spf} \hat{\mathcal{O}}_{\widetilde{Y},y}$ for the formal germ of $\widetilde{Y}$ at y and $\mathcal{G}_K \overset{\text{def}}{=} \mathcal{G}_{y,K} = \operatorname{Spec}(\hat{\mathcal{O}}_{\widetilde{Y},y} \otimes_R K)$ for the formal fibre of $\widetilde{Y}_K$ at y (cf. 1.2.1). Let $\{\mathcal{Y}_i\}_{i=1}^n$ be the set of formal boundaries of $\mathcal{G}$, and $y_i \in (\widetilde{Y}_k)^{\text{nor}}(k)$ the point of the normalisation $(\widetilde{Y}_k)^{\text{nor}}$ of $\widetilde{Y}_k$ above y which corresponds to the boundary $\mathcal{Y}_i$, $1 \leq i \leq n$. The completion of $(\widetilde{Y}_k)^{\text{nor}}$ at y_i is isomorphic to the spectrum of a ring of formal power series $k[[s_i]]$ in one variable over k and $\mathcal{Y}_i \xrightarrow{\sim} \operatorname{Spf} R[[S_i]]\{S_i^{-1}\}$ (cf. 1.2.4). One can construct a compactification of $\mathcal{G}$ (as in the above discussion where $\mathcal{G} = \mathcal{F}$) which is a formal and proper R-curve $Y \overset{\text{def}}{=} Y_y$ obtained by patching an R-formal closed unit disc $Y_i = \operatorname{Spf} R\langle \frac{1}{S_i} \rangle$ with $\mathcal{G}$ along the boundary $\mathcal{Y}_i$, for $1 \leq i \leq n$. The resulting formal R-curve Y has a special fibre Y_k consisting of n distinct reduced irreducible components $\{Q_i\}_{i=1}^n$ which intersect at the (closed) point y, and y is the unique singular point of Y_k. Moreover, if we write $\widetilde{Q}_i \to Q_i$ for the morphism of normalisation then $\widetilde{Q}_i = \mathbb{P}^1_k$ is a projective line and the morphism $\widetilde{Q}_i \to Q_i$ is a homeomorphism. By construction the formal germ (resp. formal fibre) of Y (resp. of Y_K) at the closed point y is isomorphic to $\mathcal{G}$ (resp. $\mathcal{G}_K$). (cf. [Bosch-Lütkebohmert], Definition 4.4, for a rigid analytic construction of the generic fibre $Y_K^{\text{rig}} \overset{\text{def}}{=} Y_K$ of the above compactification Y endowed with a formal covering corresponding to the above formal model Y of Y_K, as well as [Bosch], Theorem 5.8, for the invariance of the formal germ at y under this construction.)

Let η be a geometric point of $\mathcal{F}_K$ with value in its generic point which induces a geometric point η of X_K via the natural (scheme theoretic) morphism $\mathcal{F}_K \to X_K$ (cf. Notations) and determines an algebraic closure $\overline{K}$ of K. Let $S \subset \mathcal{F}_K$ be a (possibly empty) finite set of closed points. We have an exact sequence of arithmetic fundamental groups

$$1 \to \pi_1(\mathcal{F}_K \setminus S, \eta)^{\text{geo}} \to \pi_1(\mathcal{F}_K \setminus S, \eta) \to \operatorname{Gal}(\overline{K}/K) \to 1,$$

where $\pi_1(\mathcal{F}_K \setminus S, \eta)^{\text{geo}} \overset{\text{def}}{=} \operatorname{Ker}\big(\pi_1(\mathcal{F}_K \setminus S, \eta) \twoheadrightarrow \operatorname{Gal}(\overline{K}/K)\big)$. Write $\widehat{\pi}_1(\mathcal{F}_K \setminus S, \eta)^{\text{geo}}$ for the quotient of $\pi_1(\mathcal{F}_K \setminus S, \eta)^{\text{geo}}$ defined in 1.3.1. Thus, $\widehat{\pi}_1(\mathcal{F}_K \setminus S, \eta)^{\text{geo}}$ classifies finite (geometric) coverings $\mathcal{Y} \to \mathcal{F}$ (which we assume without loss of generality defined over K) with $\mathcal{Y}$ normal and geometrically connected, which are étale above

$\mathcal{F}_K\setminus S$ and above the generic points of $\mathcal{F}_k$. Write $U_K \overset{\text{def}}{=} X_K\setminus(S\cup\{\infty_i\}_{i=1}^n)$ which is an affine curve and $\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo}} \overset{\text{def}}{=} \widehat{\pi}_1(X_K\setminus(S\cup\{\infty_i\}_{i=1}^n);\{\infty_i\}_{i=1}^n,\eta)^{\text{geo}}$ for the quotient of $\pi_1(U_K,\eta)^{\text{geo}}$ defined in 1.3.1. Thus, $\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo}}$ classifies finite (geometric) coverings $Y\to X$ (which we assume without loss of generality defined over K) with Y normal and geometrically connected, which are étale above $X_K\setminus(S\cup\{\infty_i\}_{i=1}^n)$, are possibly ramified above the points $\{\infty_i\}_{i=1}^n$ with ramification indices prime-to-p, and are étale above the generic points of X_k. One of our main results is the following.

Theorem 3.2. *The (scheme) morphism $\mathcal{F}_K\to X_K$ induces a continuous homomorphism $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo}}\to\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo}}$ (resp. $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo},p}\to\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo},p}$) which makes $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo}}$ (resp. $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo},p}$) into a semi-direct factor of $\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo}}$ (resp. $\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo},p}$).*

Proof. We prove the first assertion by showing the criterion in Lemma 1.1.5 is satisfied. Let $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo}}\twoheadrightarrow G$ be a finite quotient (which we can assume without loss of generality) corresponding to a finite Galois covering $f:\mathcal{Y}\to\mathcal{F}$ with group G, with $\mathcal{Y}$ normal and geometrically connected, which is étale above $\mathcal{F}_K\setminus S$ and above the generic points of $\mathcal{F}_k$. We will show the existence of a surjective homomorphism $\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo}}\twoheadrightarrow G$ whose composite with $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo}}\to\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo}}$ is the above homomorphism. For $1\le i\le n$, let $f_i:\mathcal{Y}_i=\cup_{j=1}^{n_i}\mathcal{Y}_{i,j}\to\mathcal{X}_i$ be the pull-back of f to $\mathcal{X}_i$ via the natural morphism $\mathcal{X}_i\to\mathcal{F}$; $\{\mathcal{Y}_{i,j}\}_{j=1}^{n_i}$ are the connected components of $\mathcal{Y}_i$ and the morphism $f_{i,j}:\mathcal{Y}_{i,j}\to\mathcal{X}_i$ induced by f_i is Galois with group G_j a subgroup of G. Thus, G_j is a quotient of $\widehat{\pi}_1(\mathcal{X}_i,\eta_i)$ (η_i is a suitable base point of $\mathcal{X}_i$). Fix $1\le j_0\le n_j$, then $f_i\overset{\sim}{\to}\operatorname{Ind}_{G_{j_0}}^G f_{i,j_0}$ is an induced cover (cf. [Raynaud], 4.1). By Proposition 2.5 there exists (after possibly a finite extension of K) a finite Galois covering $\tilde{f}_{i,j_0}:Y_{i,j_0}\to D_i$ with group G_{j_0}, where Y_{i,j_0} is normal and geometrically connected, whose pull-back to $\mathcal{X}_i$ via the natural morphism $\mathcal{X}_i\to D_i$ is isomorphic to f_{i,j_0}. Further, the morphism $\tilde{f}_{i,j_0}$, is ramified above $D_{i,K}$ possibly only above ∞_i with ramification index prime-to-p, and $\tilde{f}_{i,j_0}$ is étale above the generic point of $D_{i,k}$. Let $\tilde{f}_i:Y_i\overset{\text{def}}{=}\operatorname{Ind}_{G_{j_0}}^G Y_{i,j_0}\to D_i$ be the induced cover (cf. loc. cit.), for $1\le i\le n$. By Proposition 1.2.3 one can patch the covering f with the coverings $\{\tilde{f}_i\}_{i=1}^n$ to construct a finite Galois covering $\tilde{f}:Y\to X$ between formal R-curves with group G, Y is normal and geometrically connected (since $\mathcal{Y}_K$ is), which gives rise (via the formal GAGA functor) to a surjective homomorphism $\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo}}\twoheadrightarrow G$ as required. Moreover, one verifies easily that the above construction can be performed in a functorial way with respect to the various quotients of $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo}}$ (in the sense of lemma 1.1.5) using Proposition 2.5, so that one deduces the existence of a continuous homomorphism $\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo}}\to\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo}}$ which is right inverse to $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo}}\to\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo}}$. The proof of the second assertion is entirely similar using similar arguments. □

Proposition 3.3. *The (scheme) morphism $\mathcal{F}_K\to X_K$ induces a continuous homomorphism $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo},p'}\to\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo},p'}$ which makes $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\text{geo},p'}$ into a semi-direct factor of $\widehat{\pi}_1(U_K;\{\infty_i\}_{i=1}^n,\eta)^{\text{geo},p'}$.*

Proof. The proof follows by using similar arguments to the ones used in the proof of Theorem 3.2. More precisely, with the notations in the proof of Theorem 3.2

the morphism $\mathcal{Y}_{i,j} \to \mathcal{X}_i$ in this case is Galois with group μ_n, where n is an integer prime-to-p, and extends (uniquely, possibly after a finite extension of K) to a cyclic Galois covering $Y_{i,j} \to D_i$ of degree n ramified only above ∞_i (cf. Lemma 2.1 and Lemma 2.4).

In [Saïdi1] we defined the *genus* g_x of the closed point x of X, whose definition depends only on the local (étale) structure of X_k at x, and which equals the genus of the proper, connected, and smooth K-curve X_K constructed above (cf. loc. cit. Lemma 3.3.1 and the discussion before it). (The genus g_x of x is also called the genus of the formal fibre $\mathcal{F}_K$.)

Theorem 3.4. *Let $S(\overline{K}) = \{y_1, \ldots, y_m\}$ of cardinality $m \geq 0$. Then the continuous homomorphism $\pi_1(\mathcal{F}_K \setminus S, \eta)^{\mathrm{geo},p'} \to \pi_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n), \eta)^{\mathrm{geo},p'}$ (cf. Proposition 3.3) is an isomorphism. In particular, $\pi_1(\mathcal{F}_K \setminus S, \eta)^{\mathrm{geo},p'}$ is (pro-)prime-to-p free of rank $2g_x + n + m - 1$ and can be generated by $2g_x + n + m$ generators $\{a_1, \ldots, a_g, b_1, \ldots, b_g, \sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m\}$ subject to the unique relation $\prod_{i=1}^g [a_i, b_i] \prod_{j=1}^n \sigma_j \prod_{t=1}^m \tau_t = 1$, where σ_j (resp τ_t) is a generator of inertia at ∞_i (resp. y_t).*

Proof. The homomorphism $\pi_1(\mathcal{F}_K \setminus S, \eta)^{\mathrm{geo},p'} \to \pi_1(X_K \setminus (\{\infty_i\}_{i=1}^n \cup S), \eta)^{\mathrm{geo},p'}$ is injective as follows from Proposition 3.3 (note that $\widehat{\pi}_1 = \pi_1$ in this case). We show it is surjective. To this end it suffices to show that given a finite Galois covering $f : Y \to X$ with group G of cardinality prime-to-p, with Y normal and geometrically connected, which is étale above $X_K \setminus (S \cup \{\infty_i\}_{i=1}^n)$, and $\tilde{f} : \mathcal{Y}_K \to \mathcal{F}_K$ its restriction to $\mathcal{F}_K$, then $\mathcal{Y}_K$ is geometrically connected. Equivalently, we need to show (possibly after passing to a finite extension of K) that $f^{-1}(x)$ consists of a single closed point. (The set of connected components of $\mathcal{Y}_K$ is in one-to-one correspondence with the set $f^{-1}(x)$.) We can assume, without loss of generality, that Y_k is reduced (cf. Lemme d'Abhyankar, [Grothendieck], Exposé X, Lemme 3.6). Let $y \in f^{-1}(x)$ and $D_y \subset G$ its decomposition group. Let Y_i be an irreducible component of Y_k above P_i passing through y, $\widetilde{Y}_i \to Y_i$ the morphism of normalisation, and $\widetilde{Y}_i \to \widetilde{P}_i$ the natural morphism which is Galois with group $D_{Y_i} \subset G$ the decomposition group of Y_i. The morphism $\widetilde{Y}_i \to \widetilde{P}_i$ is étale outside $\{x_i, \infty_i\}$ by Zariski's purity Theorem. Hence $D_{Y_i} = \mu_n$ is cyclic of order n, for some integer n prime-to-p, and the above morphism $\widetilde{Y}_i \to \widetilde{P}_i$ is totally ramified above ∞_i and x_i as follows from the structure of $\pi_1(\mathbb{P}^1_{\bar{k}} \setminus \{0, \infty\})^{p'}$. In particular, $D_{Y_i} \subset D_y$. Moreover, Y_k is regular outside $f^{-1}(x)$ (cf. [Raynaud], Lemma 6.3.2). We can associate a graph Γ to Y_k whose vertices are the irreducible components of Y_k and edges are the closed points of Y_k above x, two vertices Y_i and $Y_{i'}$ passing by a closed point y above x are linked by the edge y. Assume that $f^{-1}(x)$ has cardinality > 1 and let $\{y, y'\} \subseteq f^{-1}(x)$ be two distinct points. Then no irreducible component of Y_k passes through both y and y' (cf. the above fact that $\widetilde{Y}_i \to \widetilde{P}_i$ is totally ramified above x_i). More precisely, if Y_i is an irreducible component of Y_k then Y_i passes through a unique point y of Y_k above x. From this (and the above facts) it follows easily that the connected components of Γ are in one-to-one correspondence with the elements of $f^{-1}(x)$ and Γ is disconnected which contradicts the fact that Y_k is connected. Thus, $f^{-1}(x)$ has cardinality 1 necessarily as required. The last assertion follows form the well-known structure of $\pi_1(X_K \setminus (S \cup \{\infty_i\}_{i=1}^n), \eta)^{\mathrm{geo},p'}$ (cf. [Grothendieck], Exposé XIII, Corollaire 2.12). $\square$

Examples 3.5. Suppose K is of mixed characteristics with $\mathrm{char}(k)=p>0$. Let $\mathcal{F}=\mathrm{Spf}\, R[[T]]$ (resp. $\mathcal{F}=\mathrm{Spf}\, R[[T_1,T_2]]/(T_1T_2-\pi^e)$) be the formal open unit disc (resp. formal open annulus of thickness $e\geq 1$) and $S=\{y_1,\ldots,y_m\}\subset \mathcal{F}(K)$ a set of $m\geq 0$ distinct K-rational points (in the second case $e>1$ necessarily if $m\neq 0$). In this case $\mathcal{F}$ has a compactification $X=\mathbb{P}^1_R$ the R-projective line with parameter T and $\mathcal{F}$ is the formal germ at $T=0$ (resp. a compactification X which is a formal model of the projective line $\mathbb{P}^1_K$ consisting of two formal closed unit discs D_1 and D_2 centred at ∞_1 and ∞_2; respectively, which are patched with $\mathcal{F}$ along its two boundaries. The special fibre X_k consists of two projective lines which intersect at the double point x and $\mathcal{F}$ is the formal germ at x). The results of §3 in this case read as follows. First, the homomorphism $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\mathrm{geo}}\to\widehat{\pi}_1(\mathbb{P}^1_K\setminus(T\cup\{\infty\});\{\infty\},\eta)^{\mathrm{geo}}$ (resp. $\widehat{\pi}_1(\mathcal{F}_K\setminus S,\eta)^{\mathrm{geo}}\to\widehat{\pi}_1(\mathbb{P}^1_K\setminus(T\cup\{\infty_1,\infty_2\});\{\infty_1,\infty_2\},\eta)^{\mathrm{geo}}$) makes $\pi_1(\mathcal{F}_K\setminus S,\eta)^{\mathrm{geo}}$ into a semi-direct factor of $\widehat{\pi}_1(\mathbb{P}^1_K\setminus(T\cup\{\infty\});\{\infty\},\eta)^{\mathrm{geo}}$ (resp. $\widehat{\pi}_1(\mathbb{P}^1_K\setminus(T\cup\{\infty_1,\infty_2\});\{\infty_1,\infty_2\},\eta)^{\mathrm{geo}}$) and the maximal prime-to-p quotient $\pi_1(\mathcal{F}_K\setminus S,\eta)^{\mathrm{geo},p'}$ is free of rank m (resp. $m+1$).

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