

A sufficient set of experimentally implementable thermal operations for small systems

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Recent work using tools from quantum information theory has shown that **for small systems** where quantum effects become prevalent, there is not one thermodynamical second law but many. Derivations of these laws assume that an experimenter has very precise control of the system and heat bath. Here we show that these multitude of laws can be saturated using two very simple operations: changing the energy levels of the system and thermalizing over any two system energy levels. Using these two operations, one can distill the optimal amount of work from a system, as well as perform the reverse formation process. What is more, using only these two operations and one ancilla qubit in a thermal state, one can transform any state into any other state allowable by the second laws. We thus have the result that the second laws hold for fine-grained manipulation of system and bath, but can be achieved using very coarse control. This brings the full array of thermal operations towards a regime accessible by experiment, and establishes the physical relevance of these second laws, potentially opening a new direction of studies.

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I. INTRODUCTION

Thermodynamics and statistical physics are one of the most successful areas of physics, owing to its broad applicability. One can make statements which do not depend on the particulars of the dynamics, and such laws govern much of the world around us. Thermodynamics puts limitations on the efficiency of our cars' engines, determines the weather, can be used to predict many phenomena in particle accelerators, and even plays a central role in areas of fundamental physics, providing one of the only clues we have to a quantum theory of gravity through the laws of black hole thermodynamics. However, traditional thermodynamics, as derived from statistical mechanics, generally concerns itself with the average behavior of large systems, composed of many particles. Here the experimenter is only able to manipulate macroscopic quantities of the material such as its pressure and volume, and does not have access to the microscopic degrees of freedom of the system, much less the heat bath. The basic operations are limited to very crude control of the system-bath – isotherms, adiabats, isochors etc.

However, as our abilities to manipulate and control small thermodynamical systems improve, we are able to control the microscopic degrees of freedom of smaller and smaller systems [1–5]. It thus seems natural to consider the thermodynamical behavior of small, finite sized systems or heat engines composed of just a few molecules.

For a n -level system interacting with a heat bath, one can imagine an experimenter manipulating the system, who has control over each of the levels and can interact the system in any way they want with the heat bath.

From a practical point of view, needing to perform such arbitrary interactions is undesirable as they require very precise control over and be able to keep track of the entirety of the heat bath. Simpler interactions would be much more appealing. See Figure 1 for a schematic of this comparison.

However, even if one allows for such fine-grained control, the most experimentally unfeasible scenario, the second law of thermodynamics still holds (provided one computes the entropy of the system in terms of its microstates rather than using a coarse grained entropy). In fact, not only does the traditional second law hold, but additional second laws emerge **for small systems** such as the so-called *thermo-majorization* criteria [6, 7], and those given by a family of generalized free energies [8] [9]. These constrain the set of states that it is possible to transition to from a given starting state and converge to the familiar second law in the thermodynamic limit [10].

However, such precise control will be impossible to implement as it could require accurately manipulating all of the 10^{20} molecules contained in a typical heat bath. As such, it may seem that what an experimenter can achieve without such incredibly fine-grained control must be very far from what is allowed by the second laws [11, 12]. This contrasts sharply with traditional, macroscopic thermodynamics. There, those transformations allowed by the standard second law can easily be achieved by controlling macroscopic, coarse-grained parameters such as a system's volume or an external field. If the same level of control was needed macroscopically as seems necessary **for small systems**, then running a car efficiently would require control of all of the molecules in the exploding fuel and cooler. Clearly this would be an undesirable feature -

must it exist **for small systems**? The existence of a large gap between what is allowed by the most general class of operations, and what is achievable without detailed control of the heat bath, would make it hard to decide what the science of thermodynamics of microscopic systems should actually be about and how applicable the recently derived second laws are.

Surprisingly, here we show that any state transformation permitted by the additional second laws can be achieved using three simple operations. These operations, which we term *Crude Operations*, are experimentally feasible **for small systems** and do not require fine control of bath degrees of freedom to implement, only weak coupling to the bath. All allowed transformations can be implemented by applying thermalizations (assuming that the system can be thermalized), raising and lowering energy levels and rotations within energy subspaces to the system and a single thermal qubit taken from the heat bath. As a by-product, our simple operations can be viewed analogously to a universal gate set in quantum computing: they provide building blocks for the construction of more elaborate protocols.

II. THERMAL OPERATIONS

The aforementioned thermo-majorization constraints were derived [7] under the largest class of operations one is allowed to implement under thermodynamics - *Thermal Operations* [7, 13, 14]. These are presented in full detail in Section A1 of the Appendix and aim to capture all energy preserving processes that can be performed on a system in the presence of a heat bath with fixed inverse temperature β . In particular, given a system in state ρ with Hamiltonian H_S , they allow for maps of the following form:

$$\rho \mapsto \text{tr}_B \left[U_{SB} (\rho \otimes \tau_B) U_{SB}^\dagger \right], \quad (1)$$

where $\tau_B = \frac{e^{-\beta H_B}}{\text{tr}[e^{-\beta H_B}]}$ for an arbitrary bath Hamiltonian H_B and U_{SB} is an energy conserving unitary applied to system and heat bath satisfying $[U, H_S + H_B] = 0$. If a state ρ can be transformed into a state σ using a map of the above form, we will denote this by $\rho \xrightarrow{\text{TO}} \sigma$.

Energy conservation does not pose an insurmountable constraint on what is allowed since it can be enforced by incorporating a work storage device into the system to account for any energy excess or deficit. Rather, imposing energy conservation allows us to account for all sources of energy as is necessary for thermodynamics in the micro-regime. Clearly needing to apply all such unitaries to realize all possible transformations would require an enormous amount of control.

Analysis of such maps has led to: investigations into constraints on state transformations [7, 8, 10, 13, 15–17], analysis of the 3rd law of thermodynamics [18–20], the derivation of fluctuation theorems [21–24] and corrections

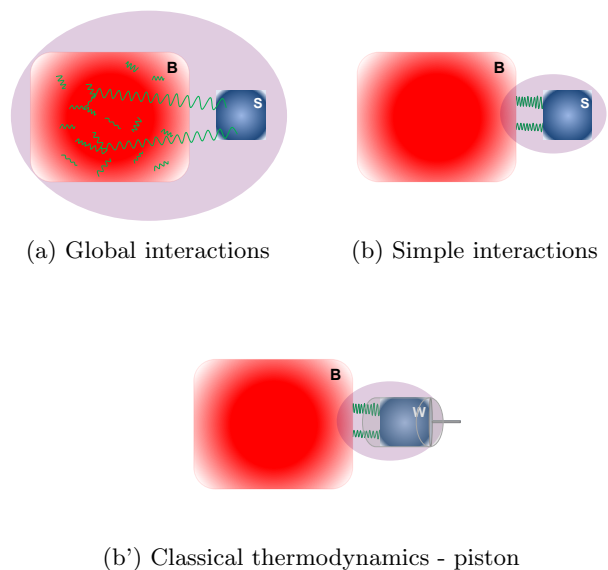


FIG. 1. *Thermal operations vs. crude operations vs. classical operations.* We consider a heat bath B together with a system S or working body W and illustrate the different levels of control an experimenter can have on the setup and interactions. Figure (a): The most detailed, experimentally unfeasible control where the experimenter keeps a record of every microstate of the system and bath (the area contained within the purple oval) and controls interactions between the system and the entirety of the heat bath (illustrated by green strings). Figure (b): The desired level of control where the experimenter keeps track of the system and a small portion of the bath (purple oval) and performs some simple interactions between these regions (green strings). Figure (b'): The previous case can be regarded as analogous to the set up in traditional thermodynamics where one has a working body W of which some parameters can be changed using simple processes such as moving a piston and weak couplings to the heat bath.

to the Carnot efficiency [25, 26]. The framework has also been recently extended to consider generalized Gibbs ensemble and scenarios where conserved charges do not commute [27–29]. Recent surveys on the field can be found in [30, 31].

It is worth noting that thermal operations can also be used to incorporate processes in which the system Hamiltonian changes. This is covered in Appendix A3. Furthermore, note that the paradigm of thermal operations is equivalent to many other methods of describing thermodynamics as the small scale. This is discussed in detail in [10, supplementary material, Section VIII] where it is shown that both time dependent system Hamiltonians and interaction Hamiltonians between system and bath can be realised within the thermal operations framework.

A. Thermo-majorization

In the absence of a source of work, a state ρ with Hamiltonian H_1 can be transformed into a state σ with Hamiltonian H_2 under thermal operations only if the familiar second law of thermodynamics holds, namely:

$$F(\rho, H_1) \geq F(\sigma, H_2), \quad (2)$$

where $F(\rho, H) = \text{tr}[H\rho] - \frac{1}{\beta}S(\rho)$ is the free energy with $S(\rho) = -\text{tr}[\rho \log \rho]$. Furthermore, in the thermodynamic limit this constraint is the only one governing state transformations.

However, the above constraint is not sufficient for determining whether a state transformation is possible under thermal operations at the nano-scale. Instead, an additional set of criteria, the thermo-majorization constraints, must be evaluated. These are best stated in terms of *thermo-majorization curves*. Given a state ρ with n -level Hamiltonian $H_S = \sum_{i=1}^n E_i |i\rangle\langle i|$, the associated thermo-majorization curve is constructed as follows:

1. Let $\eta_i = \text{tr}[\rho|i\rangle\langle i|]$, i.e. η_i denotes the occupation probability of energy level E_i .
2. β -order the probabilities and energy levels so that $\eta_i e^{\beta E_i}$ is in non-increasing order.
3. Plot the β -ordered points $\left\{ \left(\sum_{i=1}^k e^{-\beta E_i}, \sum_{i=1}^k \eta_i \right) \right\}_{k=1}^n$ together with $(0, 0)$ and connect them piece-wise linearly to form a convex curve - the thermo-majorization curve.

Given two states ρ and σ , we say that ρ *thermo-majorizes* σ if the thermo-majorization curve of ρ is never below that of σ . Examples of thermo-majorization curves are given in Figure 2.

The utility of thermo-majorization curves to determining whether a state transformation is possible under thermal operations is given by the following theorem:

Theorem 1 ([7]). *Given two states ρ and σ of an n -level system with Hamiltonian H_S in contact with a heat bath with inverse temperature β :*

1. If σ is block-diagonal in the energy eigenbasis, then $\rho \xrightarrow{TQ} \sigma$ if and only if ρ thermo-majorizes σ .
2. In general, $\rho \xrightarrow{TQ} \sigma$ only if ρ thermo-majorizes σ .

In the thermodynamic limit, the thermo-majorization criteria collapses to the condition given by Eq. (2) [10]. However, for small systems diagonal in the energy eigenbasis (or if we have access to a source of coherence [10, 17, 32, 33]) the above theorem shows that checking one constraint is not sufficient and instead it is necessary and sufficient to consider the entire thermo-majorization curve. This was shown in [23] to correspond to checking $n - 1$ inequalities.

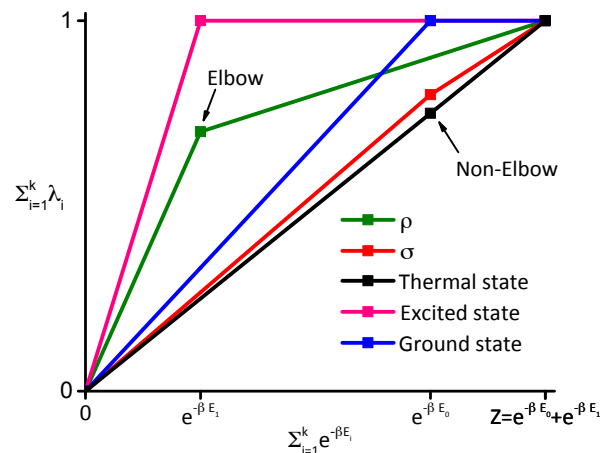


FIG. 2. *Thermo-majorization curves*. Here we plot examples of thermo-majorization curves for 2-level systems. Observe that states associated with the same Hamiltonian may have different β -orderings, as illustrated by ρ and σ here. We say that ρ *thermo-majorizes* σ as the thermo-majorization curve of ρ is never below the thermo-majorization curve of σ . The thermo-majorization curve of a Gibbs state is given by a straight line between $(0, 0)$ and $(Z, 1)$. All other states thermo-majorize it. The pure state corresponding to the highest energy level of an n -level system thermo-majorizes all other states associated with that Hamiltonian. We call a point on a curve an elbow, if the gradient of the curve changes as it passes through the point. Otherwise, it is a non-elbow.

B. Deterministic Work Values

In general, if we want a transition from ρ to σ to be possible, work may have to be added. Alternatively, if a transition can be achieved with certainty, it can be possible to extract work. Typically, there are two figures of merit that can be used to quantify the amount of work that is expended/gained: *deterministic work* and *average work*.

The deterministic work cost introduced in [7, 34, 35] is defined to be the minimum amount of work that must be added for the transformation to be possible with certainty. This amount of work is used with certainty and is suitable for characterizing the work required when manipulating a single copy of a small system.

The average work cost is a meaningful quantity when work takes a distribution of values. This can occur when we consider transforming many copies of a system and hence can build up a set of statistics or if we have a weight system to store fluctuating work values as in [24, 36]. We are interested in implementing thermal operations on single copies of a system when we do not have access to an additional weight system so do not consider the average work cost in this paper.

For further elaboration on the differences between deterministic and average work for small systems, we refer the interested reader to [7, 34, 35].

More formally, within the thermal operations frame-

work, the optimal amount of deterministic work that must be added or can be gained, the work of transition, can be quantified using (for example) the energy gap W of a 2-level system, a *wit* [7], with zero energy state $|0\rangle$ and an additional state $|1\rangle$. The associated Hamiltonian is:

$$H_W = W|1\rangle\langle 1|. \quad (3)$$

The deterministic work of transition, denoted $W_{\rho \rightarrow \sigma}$, is then defined to be the greatest value of W such that the following holds:

$$\rho \otimes |0\rangle\langle 0| \xrightarrow{\text{TO}} \sigma \otimes |1\rangle\langle 1|. \quad (4)$$

If $W_{\rho \rightarrow \sigma}$ is negative, to convert ρ into σ work has been taken from the work storage system to enable the transition to take place. On the other hand, if $W_{\rho \rightarrow \sigma}$ is positive, in converting ρ into σ it has been possible to store some extracted work in the work system.

Defining work in such a way enables the quantification of the worst-case work of a process. When $W_{\rho \rightarrow \sigma}$ is negative, it can be interpreted as the smallest amount of work that must be supplied to guarantee the transition. If it is positive, it is the largest amount of work we are guaranteed to extract in the process. As the work system is both initially and finally in a pure state, no entropy is stored in it and its energy change must be completely due to work being exchanged with the system.

III. CRUDE OPERATIONS

Implementing arbitrary transformations of the form given in Eq. (1) would require an unprecedented level of control. It is thus natural to ask whether the transformations that are possible under thermal operations can be achieved with much simpler operations. In this section we introduce three such operations which we call *crude operations*. These consist of:

1. Partial Level Thermalizations,
2. Level Transformations,
3. Subspace Rotations.

In the following subsections we describe them each in turn in more detail before concluding the section by comparing them (and in particular the level of control they require) with other classes of operations that have been introduced in the literature. In Section IV, we will show how applying protocols consisting of these operations to a system and a single thermal qubit (which is returned unchanged at the end of the protocol) is sufficient for implementing any transformation to a block-diagonal state that is possible under thermal operations.

A. Partial Level Thermalizations

The first of our three basic operations are *Partial Level Thermalizations* (PLTs). A thermalization essentially changes the state of the system into a thermal state and is usually achieved by putting the system in thermal contact with the reservoir until it equilibrates or by swapping the system with one from the reservoir. Thermalizations have no work cost or gain associated with them. A partial thermalization generalizes this, allowing one to thermalize with some probability p , implementing

$$\rho \rightarrow p\rho + (1-p)\tau_S,$$

with τ_S being the thermal state of the system at inverse temperature β . The probability p can be determined by using the ambient heat bath as a source of noise or by putting the system in contact with it for a time shorter than the equilibration time.

With Partial Level Thermalizations we go one step further and allow for the partial thermalization to act on any subset of energy levels. In order to implement them, one needs to be able to either perform the SWAP gate between any subset of energy levels of the system and of the thermal bath or to selectively put system energy levels in contact with the reservoir, for example by making use of an optical cavity or intermediate system which acts as a filter to restrict which energy levels are being addressed by the thermal contact. More precisely, a PLT is defined as:

Definition 2 (Partial Level Thermalizations). *A Partial Level Thermalization on an n level system is parametrized by $\lambda \in [0, 1]$ and a subset $\mathcal{P} \subseteq \{1, \dots, n\}$ of the system's energy levels. We denote it by $PLT_{\mathcal{P}}(\lambda)$.*

Given a diagonal state $\rho = \sum_{i=1}^n \eta_i |i\rangle\langle i|$ with associated Hamiltonian $H_S = \sum_{i=1}^n E_i |i\rangle\langle i|$ in contact with a heat bath at inverse temperature β , the action of $PLT_{\mathcal{P}}(\lambda)$ on ρ is given by:

$$\rho \xrightarrow{PLT_{\mathcal{P}}(\lambda)} \rho' \quad (5)$$

where $\rho' = \sum_{i=1}^n \eta'_i |i\rangle\langle i|$ and the η'_i are such that :

$$\eta'_i = \begin{cases} (1-\lambda)\eta_i + \frac{\lambda e^{-\beta E_i}}{\sum_{j \in \mathcal{P}} e^{-\beta E_j}} \sum_{j \in \mathcal{P}} \eta_j & \text{for } i \in \mathcal{P} \\ \eta_i & \text{for } i \notin \mathcal{P} \end{cases}. \quad (6)$$

The action of PLTs on a state is illustrated in terms of thermo-majorization curve in Figure 3. Note that if we apply PLTs to energy levels that are adjacent with respect to the β -ordering of ρ , the final states maintain this β -order.

Finally, it has recently been proposed that such operations be incorporated into Heat Bath Algorithmic Cooling protocols to enhance their performance [37]. Such protocols are important for NMR based quantum devices as they provide a route towards purifying qubits. As such, we expect the experimental implementation of PLTs to be developed.

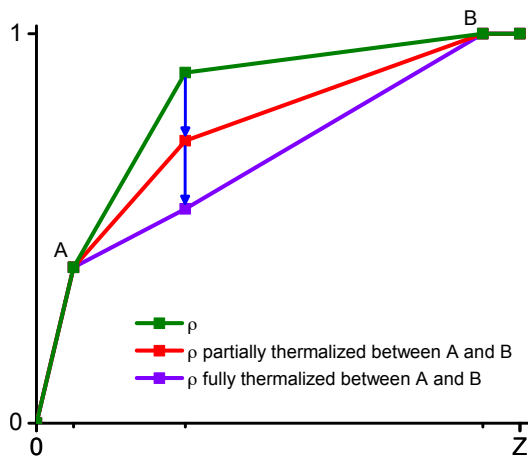


FIG. 3. *Action of Partial Level Thermalization.* Here we illustrate the action of PLTs applied to a state 4-level system with $\mathcal{P} = \{0, 1\}$. We take ρ to be block diagonal in the energy eigenbasis with occupation probabilities $p = (0.1, 0.5, 0.4, 0)$ and H_S and β to be such that $e^{-\beta E_1} = 0.4$, $e^{-\beta E_2} = 0.15$ and $e^{-\beta E_3} = e^{-\beta E_4} = 0.05$. For these choices, the affected portion of the thermo-majorization curve lies between the points A and B and we show the impact of $\text{PLT}_{\{1,2\}} (\lambda = 0.5)$ (partial thermalization) and $\text{PLT}_{\{1,2\}} (\lambda = 1)$ (full thermalization) on ρ .

B. Level Transformation

Our second type of operation are *Level Transformations* (LT), namely the raising and lowering of any subset of energy levels of the system's Hamiltonian. This type of transformation is common within thermodynamics and the work cost of implementing them is given by the change in energy of the level (when the level is populated). More formally:

Definition 3 (Level Transformations). *A level transformation on an n -level system is parametrized by a set of real numbers $\mathcal{E} = \{h_i\}_{i=1}^n$ and denoted by $LT_{\mathcal{E}}$.*

Given a state ρ with associated Hamiltonian $H_S = \sum_{i=1}^n E_i |i\rangle\langle i|$, the action of $LT_{\mathcal{E}}$ on (ρ, H_S) is:

$$(\rho, H_S) \xrightarrow{LT_{\mathcal{E}}} (\rho, H'_S), \quad (7)$$

where $H'_S = \sum_{i=1}^n (E_i + h_i) |i\rangle\langle i|$.

Their effect on thermo-majorization curves is shown in Figure 4.

The deterministic work cost of performing the Level Transformations $LT_{\mathcal{E}}$ is naturally given by $-\max h_i$ where the maximization is taken over all occupied energy levels. This captures the maximum (worst-case) amount of work that needs to be supplied to implement a transformation. **Here, we are interested in realizing those transformations that can be implemented using thermal operations at zero deterministic work cost. As such, when performing transformations using Crude Operations, we will only use LTs that do not cost work to**

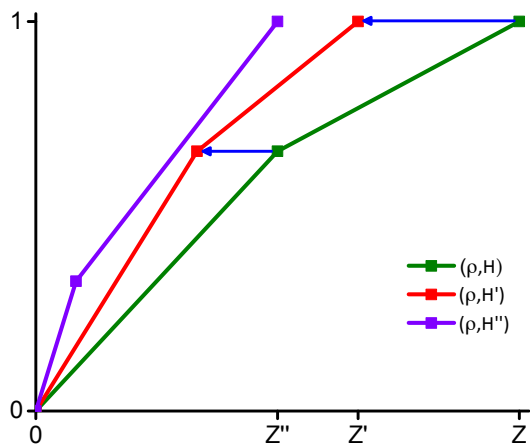


FIG. 4. *Action of Level Transformations.* Here we illustrate the action of LTs applied to a system with a thermodynamical configuration (ρ, H) . Note that LTs leave the occupation probabilities of ρ unchanged but may alter the β ordering as discussed in the Appendix.

implement i.e. $h_i \leq 0$ or $h_i \approx 0$, for all i . This will be done by combining them with Partial Level Thermalizations to form a protocol akin to an isothermal process. We will discuss this further in Section IV C 1.

C. Subspace Rotation

Finally, we will sometimes need to implement an energy conserving unitary that acts upon system only (this is in contrast to the unitaries in Eq. (1) which act on both system and heat bath). We call this operation a *Subspace Rotation* (SR) and allowing such unitaries is desirable for two reasons.

Firstly, applying such a unitary at random will cause a system to decohere into the energy eigenbasis. This may be desirable if we begin with a state which contains coherence. As decoherence is a mechanism that occurs naturally, allowing for it does not require an unreasonable amount of control.

Secondly, in the special case where the system has degenerate energy levels, one may need to implement an energy conserving unitary acting within the degenerate subspace to rotate the state to be diagonal in a specific basis.

As we will be predominantly interested in target states that are decoherent in energy, we can therefore take the initial and final states of any protocol to be diagonal in the energy eigenbasis. If necessary, we can always begin any protocol by decohering in the energy eigenbasis and then rotating the resulting state to be diagonal using an SR. At the end of the protocol we can apply a final SR to rotate within a degenerate energy subspace as required.

D. Comparison with other sets of operations

These operations are detailed with greater specificity in Section *B* of the Appendix, where it is also shown that they are a subset of Thermal Operations. Having introduced them, here we shall contrast them with operations that have appeared in other resource theoretic approaches to thermodynamics.

In [35] it was shown that full thermalizations and Level Transformations suffice for extracting the optimal amount of deterministic work from a given state under thermal operations as evaluated in [7]. Our crude operations are a larger set of operations and (as we will see in the next section) enable any allowed transformation between block-diagonal states at the optimal work cost rather than just transformations to the thermal state.

In [35] it was shown that full thermalizations and Level Transformations suffice for extracting the optimal amount of deterministic work from a given state under thermal operations as evaluated in [7]. Our crude operations are a larger set of operations and (as we will see in the next section) enable any allowed transformation between block-diagonal states at the optimal work cost rather than just transformations to the thermal state. **A protocol is also given in [35] that uses full thermalizations and LTs to perform the conversion between two block-diagonal states at the optimal average work cost/gain. However, for most transformation at least one of the LTs performed will have a work cost (as defined after Definition 3). As such, this protocol is not suitable for implementing the transformations we are interested in where we demand that no work be expended.**

A set of operations consisting of discrete unitary transformations (which allow for arbitrary unitaries to be applied to the system and arbitrary changes of Hamiltonian and hence contain Level Transformations as a special case) and discrete thermalizing transformations (which fully thermalize the system) was considered in [38]. These operations characterize processes that involve work and those that transfer heat respectively. Using these primitives, isothermal transformations between states in thermal equilibrium can be performed at optimal heat exchange. However, they do not enable all transformations that are possible under thermal operations with zero deterministic work cost.

In [15] it was shown, that for transitions between two block-diagonal states, perhaps with the expenditure or gain of **deterministic** work, it is enough to apply thermal operations on the bath and the system, and instead of the work system used in [7] just use Level Transformations. Still however, the system-bath coupling term used in [15] requires being able to implement an arbitrary Thermal Operations (see [39] and [40], where it is shown that the operations used in [15] are a subset of Thermal Operations) and thus in principle require a high degree of control.

A subset of thermal operations were considered in [36] which involved interacting with a designer heat bath

which contained an arbitrarily large number of systems in a series of states that interpolated between the input state and the target state. Such a model allowed for achieving state transformations at the optimal average work cost. Again, this would require an unfeasible amount of control in preparing the states of the heat bath. Note however, that it was shown in [41] that the interactions themselves could be taken to be thermalizations and that these are robust to experimental imperfections.

In [11], corrections to the average amount of work that can be extracted were considered under various control restrictions. In particular, bounds on the accessible set of system Hamiltonians and restrictions on the allowed interactions with the heat bath were considered. Our crude operations also consider a restricted set of heat bath interactions but we place no restriction on the allowed system Hamiltonians and consider single-shot transformations rather than the case of average work extraction.

A subclass of thermal operations were considered in [42] in which each operation could only act on two energy levels of the system at a time. While it was shown that such operations could be closely approximated in certain temperature regimes using a Jaynes-Cummings model, it was also shown that such a restricted set could not implement all of the transformations that are possible under thermal operations (a similar implication was also found in [43]). In contrast, we show that by applying PLTs that act on two energy levels (a subset of the aforementioned operations) and allowing in addition for level transformations and the use of a single thermal qubit, it is possible to reproduce the full set of transformations.

One can also consider simple operations for manipulating states that contain coherence. For example, in [44] a simple protocol for extracting the optimal amount of average work from a state with coherence that made use of non-energy preserving unitary rotations, Level Transformations and isothermal reversible processes was investigated. However, such unitary rotations are not contained within thermal operations and so we do not consider such protocols here. Furthermore, deterministic work extraction is not understood for this class of operations (see also [33] for a discussion on single-shot work extraction from coherent states using thermal operations in the presence of a reusable ancilla).

As we shall see in the next Section, crude operations allow all transformations between block-diagonal states that are possible under Thermal Operations to be implemented without the need for unreasonable levels of control **and at zero deterministic work cost.**

IV. TRANSFORMATIONS USING CRUDE OPERATIONS

In this section, our goal is to show that any transformation to a block-diagonal state that is possible under thermal operations can also be achieved by applying crude

operations to the system and a single thermal qubit. As Theorem 1 indicates that for such states $\rho \xrightarrow{TO} \sigma$ if and only if ρ thermo-majorizes σ , we hence want to show that if ρ thermo-majorizes σ , then ρ can be converted into σ using crude operations without expending any work. If this is possible we will denote it by $\rho \xrightarrow{CO} \sigma$. The converse direction (that $\rho \xrightarrow{CO} \sigma$ implies that ρ thermo-majorizes σ) also holds and this follows from the fact that such crude operations are a subset of thermal operations.

A. Trivial Hamiltonian

When the Hamiltonian of the system is trivial, $H_S = 0$ and the energy spectrum is completely degenerate, the thermo-majorization criteria collapses to a criteria known as *majorization*. If ρ has eigenvalues $\{p_i\}_{i=1}^n$ and σ has eigenvalues $\{q_i\}_{i=1}^n$, where each set is written in non-increasing order, we say ρ majorizes σ if:

$$\sum_{i=1}^k p_i \geq \sum_{i=1}^k q_i, \quad \forall k \in \{1, \dots, n\}. \quad (8)$$

If the system Hamiltonian is trivial, then ρ majorizes σ if and only if $\rho \xrightarrow{TO} \sigma$ [45].

It is well known for two probability distributions of n elements, that if p majorizes q , then p can be converted into q using at most $n - 1$ *T-transforms* [46, 47]. A T-transform T is parametrized by $r, s \in \{1, \dots, d\}$ and $\lambda \in [0, 1]$ and acts on the probability distribution p via $p \xrightarrow{T_{r,s}(\lambda)} p'$. Here

$$p'_i = \begin{cases} (1 - \lambda)p_i + \frac{\lambda}{2}(p_r + p_s) & \text{for } i \in \{r, s\} \\ p_i & \text{for } i \notin \{r, s\} \end{cases}. \quad (9)$$

This action of T-transforms is illustrated in Figure 5.

Eq. (9) is highly reminiscent of the action of PLTs defined in Eq. (6). Indeed, for trivial Hamiltonians the thermal state is just the maximally mixed one and all unitaries are energy conserving so we can assume that our states are diagonal. Hence, the action of a PLT on two energy levels is to perform a T-transform on the corresponding eigenvalues. Combining this with the aforementioned result of [46, 47] gives:

Theorem 4. *Let ρ and σ be two states of an n -level system with with trivial Hamiltonian $H_S = 0$. Then $\rho \xrightarrow{TO} \sigma$ implies that $\rho \xrightarrow{CO} \sigma$. Furthermore, this transformation can be done by applying a sequence of $n - 1$ PLTs that each act on only 2 energy levels.*

In Section F of the Appendix, we give an example of this protocol using a physical set up involving a molecule in a box (a so called Szilard engine). Furthermore, we show how to distill the optimal amount of work from a given state in this setup and also the reverse process of forming a state. These examples provide some physical insight into the crude operations we allow and the form protocols built from them take.

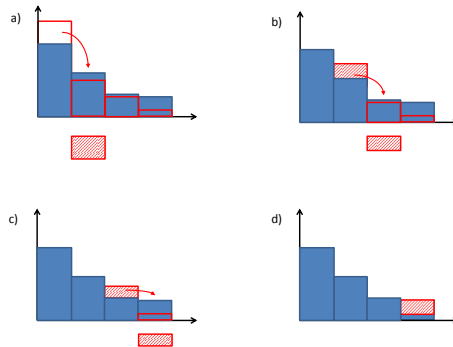


FIG. 5. The heights of each column above are given by the probability of being in a particular eigenstate. The action of *T-transforms* transforms an initial state represented by red columns into the state represented by blue columns. The probabilities (column heights) of the red state majorizes those of the blue state. We work our way from left to right, moving probability mass from the red histogram to the right until it matches the blue histogram. In each step we move some probability mass from a column of the histogram of the initial state and move it to the right until it either matches the probability required of the target state, or until the left column of the initial state matches the right column of the final state. Red dashed rectangles represent the part of the column that is added to the next one due to the action of the *T-transform*.

B. General Hamiltonian: States with the same β -order

Transformations involving trivial Hamiltonian are straightforward because in this situation there is only one β -order. In terms of thermo-majorization curves, this means that the ‘elbows’ of all states (see Figure 2 for a definition) are vertically aligned. For general Hamiltonians, the same form of alignment arises if the states that we are attempting to convert between have the same β -order. As PLTs are intuitively the generalization of T-transforms to general Hamiltonians, the result of [46] and [47] can be generalized as follows:

Theorem 5. *Let ρ and σ be two states of an n -level system with Hamiltonian H_S such that ρ and σ have the same β -order and σ is block-diagonal in the energy eigenbasis. Then $\rho \xrightarrow{TO} \sigma$ implies that $\rho \xrightarrow{CO} \sigma$. Furthermore, this transformation can be done by applying a sequence of $n - 1$ PLTs.*

Proof. The full proof is given in Theorem 17 in Appendix C but we sketch the idea here. Firstly, by making use of appropriate Subspace Rotations we can assume that ρ and σ are in fact diagonal in the energy eigenbasis. As $\rho \xrightarrow{TO} \sigma$ is equivalent to ρ thermo-majorizing σ , our task is to show that PLTs suffice for converting the thermo-majorization curve of ρ into that of σ . The protocol for achieving this is illustrated in Figure 6 which also serves as a simple, non-trivial example of how such a state

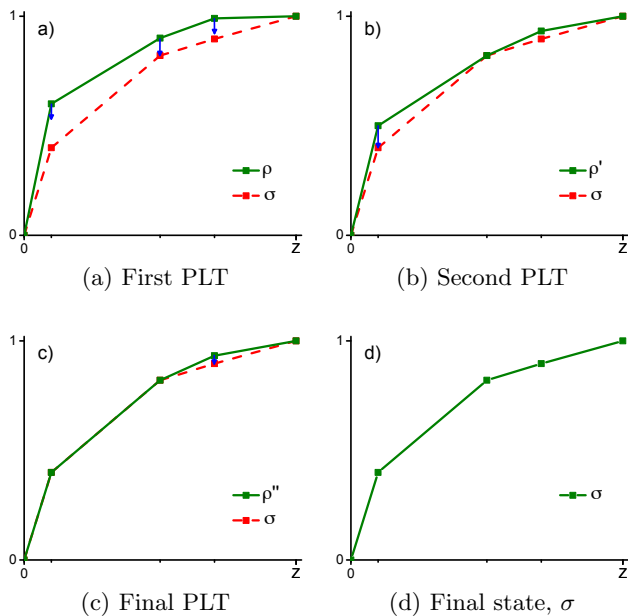


FIG. 6. *Crude Operations protocol for transforming between states with the same β -order.* If two states, ρ and σ , have the same β -order and are such that ρ thermo-majorizes σ , then ρ can be converted into σ using Partial Level Thermalizations. First a PLT is applied to ρ across the complete set of energy levels, Figure (a), lowering the thermo-majorization curve of ρ until it meets that of σ , Figure (b). Next, a second PLT is applied to those energy levels to the left of this meeting point, again lowering the curve until it meets that of σ at a second point, Figure (c). By iterating this process, ρ is transformed into σ , Figure (d).

transformation can be achieved. Roughly speaking, the aim is to use PLTs to lower the elbows on the thermo-majorization curve of ρ until one of these elbows meets an elbow of σ . This process is then repeated on subsets of energy levels that are adjacent in β -order until the curves perfectly coincide. The level of thermalization required in each PLT is detailed in the full proof in the Appendix. As the PLTs only act on adjacent energy levels, the β -order never changes during the protocol. \square

Note that T-transforms only act on two elements of a probability distribution while in the above theorem we have used PLTs that act on an arbitrary number of energy levels. It is thus natural to ask whether this result can be proved using only PLTs that act on two energy levels. In Appendix C Theorem 18, we show that the answer is yes but present only the previous statement here as the associated protocol is easier to visualize on thermo-majorization diagrams.

C. General Hamiltonian: States with different β -order

For general ρ and σ with σ block diagonal such that $\rho \xrightarrow{\text{TO}} \sigma$, the protocol described in the previous section

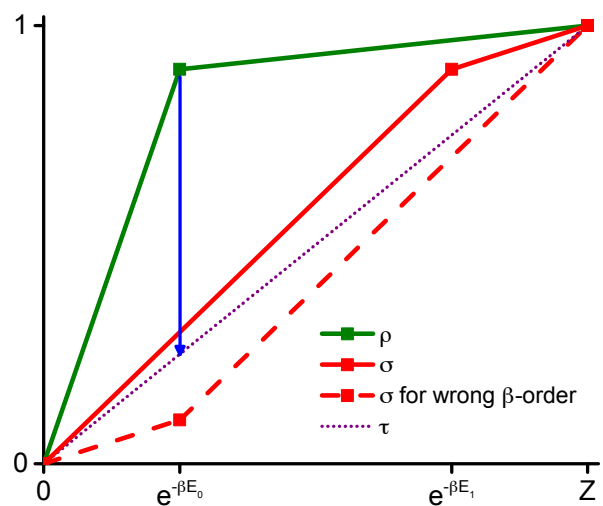


FIG. 7. *Partial Level Thermalizations are not enough.* Note that PLTs alone cannot implement all transitions possible under thermal operations. More specifically, they cannot implement changes of β -order on qubits. In the above example, we see that a PLT cannot take the initial state ρ depicted in green to σ depicted in red, because it first passes through the thermal state before it reaches σ (which we have depicted in both its β -ordered, and non- β -ordered form). Because of this, work is required to go from ρ to σ even though the optimal process does not require work.

will not work because for states with different β -orders, the associated elbows may not be perfectly aligned. The essence of why this causes a problem can be seen in Figure 7 where we consider a transformation on a 2-level system where the initial and final state have different β -orders. As we cannot write $\sigma = p\rho + (1-p)\tau_S$, a Partial Level Thermalization cannot convert ρ into σ . Furthermore, we do not believe that a combination of PLTs and LTs acting on system alone can achieve this without incurring a work cost.

However, if we are allowed to perform Crude Operations not just on the system, but on the system and a single qubit τ_A from the thermal bath, then we will show that one can transform ρ into σ even if the β -ordering is different. The effect of appending a thermal qubit on the thermo-majorization curve of ρ is to introduce n additional ‘non-elbow’ points (again, see Figure 2 for an intuitive definition). The basic idea is to use a sequence of PLTs interlaced with LTs to convert this scenario into one where the states under consideration have the same β -order. This is done by moving these new non-elbow points so that they are vertically aligned with the elbows of σ . If the sequence is performed sufficiently slowly, this can be done without expending any work.

In the next section we describe this process before giving the full description of the protocol for dealing with states with different β -orders.

1. Partial Isothermal Reversible Processes and Points Flow

We will now explain how to move non-elbow points without expending work or changing the shape of the thermo-majorization curve. First we will show how to move a non-elbow point within a line-segment of the thermo-majorization curve. Then, we will explain how to transfer a non-elbow point that is close to the end of a segment to an adjacent segment. Combining these two processes we can freely move non-elbow points along the curve. A more detailed description can be found in Section B3 of the Appendix.

PITRs: Moving non-elbows within a segment. To gain intuition for how to move a non-elbow point within a segment, let us first consider the simpler case of converting a 2-level system with Hamiltonian $H_S = \sum_{i=1}^2 E_i |i\rangle\langle i|$ that is initially in the Gibbs state $\tau_S = \frac{e^{-\beta H_S}}{Z_S}$ into a 2-level system with Hamiltonian $H'_S = \sum_{i=1}^2 E'_i |i\rangle\langle i|$ in state $\tau'_S = \frac{e^{-\beta H'_S}}{Z'_S}$. When $Z_S = Z'_S$, the thermo-majorization curves of both of these states are given by a straight line connecting $(0, 0)$ to $(Z_S, 1)$ and both curves contain a non-elbow point which w.l.o.g. we can take to be at $(e^{-\beta E_1}, e^{-\beta E_1}/Z_S)$ and $(e^{-\beta E'_1}, e^{-\beta E'_1}/Z_S)$ respectively. Thus, if we can perform $(\tau_S, H_S) \rightarrow (\tau'_S, H'_S)$ for any H'_S such that $Z'_S = Z_S$, we can effectively move the non-elbow point of τ_S to any position within the line-segment.

It was shown in [35] that performing the transformation $(\tau_S, H_S) \rightarrow (\tau'_S, H'_S)$ can be done at a deterministic work cost (i.e. without work fluctuations) of $W = -\frac{1}{\beta} \log(Z_S/Z'_S)$ by alternating Level Transformations with full thermalizations of the system. The macroscopic equivalent of this process is the isothermal expansion of gas in a container and hence this process is called an Isothermal Reversible Process.

Note that if $Z_S = Z'_S$, then no work is required to perform the above transformation. Thus in this case, by keeping Z_S constant along the course of the protocol we can convert τ_S into τ'_S for free. To do this in t steps, we need to change the two energy levels in a special way. Namely, if we change the energy level labeled by 1 by $\Delta_1 = \frac{E'_1 - E_1}{t}$, we need to alter the energy level labeled by 2 by Δ_2 such that:

$$e^{-\beta(E_1^r + \Delta_1)} + e^{-\beta(E_2^r + \Delta_2)} = Z, \quad (10)$$

where here E_1^r and E_2^r denote the energy levels of the system after the r th step of the protocol. This LT is then followed by a full thermalization and this sequence of LT followed by thermalization is repeated t times, resulting in (τ'_S, H'_S) . In the limit that t tends to infinity, it can be shown that the work cost of the transformation becomes deterministic and tends to zero [35].

A similar idea can be applied when (ρ, H_S) is such that ρ is not thermal but still contains a non-elbow. Note that a non-elbow implies that there are two energy levels E_i

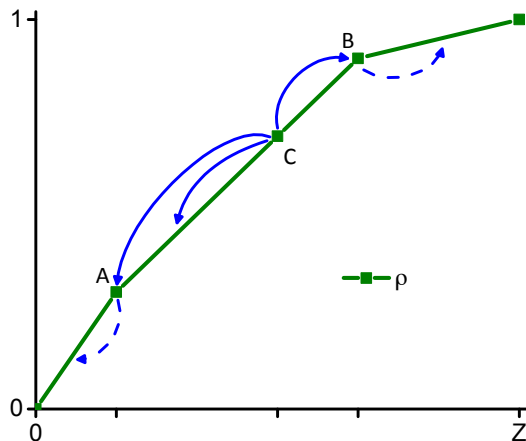


FIG. 8. *Action of Partial Isothermal Reversible Processes.* Here we illustrate the action of alternating Level Transformations and Partial Level Thermalizations (so-called, PITRs) applied to a system with state-Hamiltonian pair (ρ, H) . Using PITRs, the point at C can be moved such that it lies anywhere on the line-segment between A and B and without changing the shape of the overall thermo-majorization curve. By performing this process sufficiently slowly, this can be done with no deterministic work cost. If one moves the point C to coincide with point A (respectively B), one can then use a second PITR to move point A (B) as illustrated by the dashed arrows. Again, this does not alter the shape of the thermo-majorization curve.

and E_j that are thermal with respect to one another, i.e.:

$$\eta_i = \alpha \frac{e^{-\beta E_i}}{e^{-\beta E_i} + e^{-\beta E_j}}, \quad \eta_j = \alpha \frac{e^{-\beta E_j}}{e^{-\beta E_i} + e^{-\beta E_j}} \quad (11)$$

where $\alpha = \eta_i + \eta_j$. Moving this non-elbow along its associated line-segment can then be achieved by applying LTs to E_i and E_j in such a way that:

$$e^{-\beta(E_i^r + \Delta_i)} + e^{-\beta(E_j^r + \Delta_j)} = e^{-\beta E_i} + e^{-\beta E_j} \quad (12)$$

and then applying the Partial Level Thermalization $\text{PLT}_{\{i,j\}}(\lambda = 1)$ before repeating this sequence as necessary. In the limit that $\Delta_i \rightarrow 0$ this protocol again costs no work. We define the sequence of operations required to move a non-elbow along a line-segment (which we call a *Partial Isothermal Reversible Process* in analogy to the above) more fully in Definition 13 in the Appendix and prove that it costs no work thereafter in Lemma 14. A description of a PITR is given in terms of thermo-majorization diagrams in Figure 8.

We note here, that to the best of our knowledge, such a thermodynamical state transformation has never been performed in the lab, and we hope that our work will stimulate experimentalists to implement it.

Points flow: Transferring non-elbows between segments. Suppose that by using a PITR we have moved a non-elbow very close to the elbow defining the end of the line-segment. We shall now explain how to move it to neighboring segment.

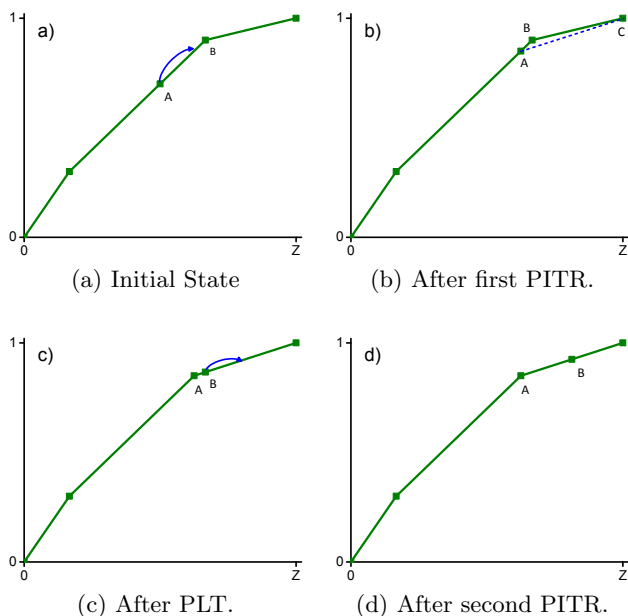


FIG. 9. *Approximate Points Flow*. Here we illustrate the protocol of Approximate Points Flow using thermo-majorization diagrams. Initially the system is as per Figure (a). Using a PITR, the non-elbow point, A , is moved towards the elbow at point B . This results in Figure (b). Next, a PLT is applied between points A and C , leading to Figure (c). Point B is now a non-elbow and can be moved using a PITR, giving Figure (d).

Let the non-elbow be associated with energy levels E_i and E_j . As it is close to the end of the line-segment, one of these energy levels (w.l.o.g., let it be E_i) must be much larger than all other energy levels of the system. Indeed, as $E_i \rightarrow \infty$ the non-elbow becomes arbitrarily close to the end of the line-segment. Let E_k denote the energy level associated with the line-segment we wish to move our non-elbow to. To do this, we apply the Partial Level Thermalization $\text{PLT}_{\{i,k\}}$ ($\lambda = 1$). This turns the non-elbow into an elbow and the elbow at the end of the line-segment into a non-elbow that we can continue to move. The whole process is depicted in Figure 9 and described more carefully in Definition 16 in Appendix B3.

Note that the process described does not leave the thermo-majorization curve completely unchanged. However, the closer the non-elbow initially was to the end of the line-segment, the lesser the change. If we wish to transfer a non-elbow without altering the thermo-majorization curve at all, we can do so if we are able to raise an energy level to infinity. This results in a simpler protocol which we describe in Appendix B3 in Definition 15 and Fig. 15.

2. The full protocol

With these concepts in place, we are now in a position to prove our main result:

Theorem 6. *Let ρ and σ be two states of an n -level system with Hamiltonian H_S such that σ is block-diagonal in the energy eigenbasis. Then $\rho \xrightarrow{\text{TQ}} \sigma$ implies that $\rho \xrightarrow{\text{CO}} \sigma$ without expending any work.*

Proof. The full proof can be found in the proof of Theorem 19 of Appendix D and we sketch it here. We again note that for σ block-diagonal, $\rho \xrightarrow{\text{TQ}} \sigma$ implies that ρ thermo-majorizes σ . Our aim is therefore to construct a sequence of Crude Operations that transform the thermo-majorization curve of ρ into that of σ . The protocol for doing this is shown in Figure 10. First a thermal qubit τ_A is appended. The thermo-majorization curve of $\rho \otimes \tau_A$ has the same shape as ρ but contains n additional non-elbows. Using the PITR and Point Flow protocols introduced in the previous section, these non-elbows can be moved so that they are vertically aligned with the elbows of $\sigma \otimes \tau_A$. Using the PLT protocol introduced in Theorem 5, this can be converted into a state which has the same thermo-majorization curve as $\sigma \otimes \tau_A$ but potentially with non-elbows in the wrong position. A final round of PITR and Point Flow protocols corrects this, leading to $\sigma \otimes \tau_A$ and upon discarding τ_A we obtain σ . \square

The protocol described above assumes that we can implement points flow exactly i.e. it is possible to raise energy levels to infinity. If this is not possible and we instead must perform them approximately, the above transformation is still possible but with a small caveat. The statement and proof of this is given in Theorem 20 in Appendix D.

D. Deterministic Work Values with Crude Operations

Crude Operations can also be used to construct protocols that achieve the optimal deterministic work values under Thermal Operations as determined through Eq. (4). Rather than using a wit to measure such a quantity, one can instead consider the work value of the Level Transformations used during the protocol. This is explored in more detail (including the ϵ -error case) in Section E of the Appendix but here we describe the optimal, zero-error protocols for both extracting work from a state and forming an incoherent state from a thermal state.

The maximum amount of work that can be deterministically extracted from a state ρ under Thermal Operations is given by [7, 35]:

$$W_{\text{distil}}(\rho) = W_{\rho \rightarrow \tau_S} = \frac{1}{\beta} \ln \left(\frac{Z_S}{\sum_{i:\eta_i > 0} e^{-\beta E_i}} \right). \quad (13)$$

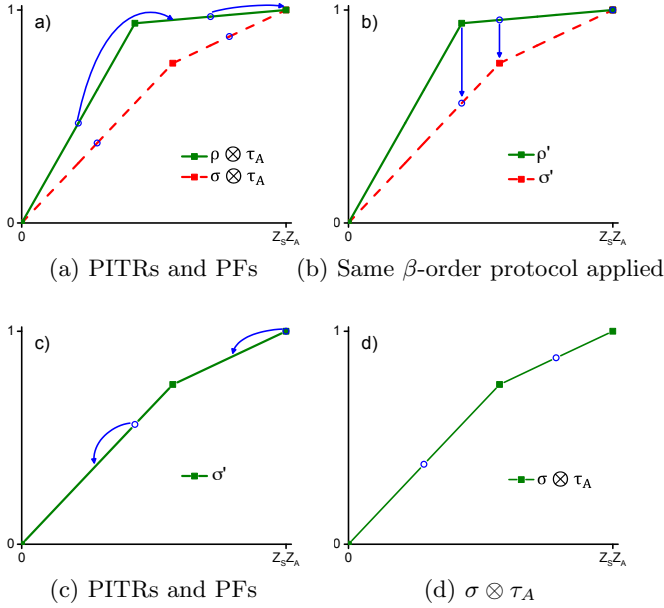


FIG. 10. *Crude Operations* protocol for transforming between states with different β -orders. If ρ thermo-majorizes σ , then it is possible to transform ρ into σ using Crude Operations. First, a thermal qubit, τ_A , with known Hamiltonian is appended, Figure (a). Using Partial Isothermal Reversible Processes the blue circles on the thermo-majorization curve of $\rho \otimes \tau_A$ can be moved to be vertically aligned with the ‘elbows’ on the curve for $\sigma \otimes \tau_A$. This forms the state ρ' , Figure (b), that has a thermo-majorization curve overlapping that of $\rho \otimes \tau_A$. Using the previously defined protocol for states with the same β -order, ρ' can be converted into σ' , Figure (c), that has a thermo-majorization curve overlapping that of $\sigma \otimes \tau_A$. A final round of PITRs converts σ' into $\sigma \otimes \tau_A$, Figure (d), and upon discarding τ_A we obtain σ .

The protocol for achieving this under Crude Operations is identical to that given in [35]. First a Level Transformation is used to raise the unoccupied energy levels (those such that $\eta_i = 0$) to infinity. As the energy levels are unoccupied, this does not cost any work. Next the state is completely thermalized. These two steps result in a thermal state τ'_S of Hamiltonian H'_S such that $Z'_S = \sum_{i:\eta_i>0} e^{-\beta E_i}$. A round of Partial Isothermal Reversible Processes can now be applied (at no work cost) to convert this to a state-Hamiltonian pair (τ_S, H''_S) where τ_S is the thermal state of both H''_S and H_S and $Z''_S = Z'_S$. Finally, the Level Transformation $LT_{\mathcal{E}}$ where $\mathcal{E} = \left\{ h_i = -\frac{1}{\beta} \ln \left(\frac{Z_S}{\sum_{i:\eta_i>0} e^{-\beta E_i}} \right) \right\}_{i=1}^n$ is performed to convert (τ_S, H''_S) into (τ_S, H_S) while extracting the amount of deterministic work given in Eq. (13).

The reverse process to work distillation is that of formation. There one starts with the thermal state τ_S and uses work to form the state ρ . For the case where ρ does not contain coherences, we can construct a process to do

this using Crude Operations achieving:

$$W_{\text{form}}(\rho) = W_{\tau_S \rightarrow \rho} = -\frac{1}{\beta} [\ln(\eta_1 e^{\beta E_1}) + \ln Z_S] \quad (14)$$

where we emphasize that the occupation probabilities and energy levels of ρ have been β -ordered. This is the optimal value achievable under Thermal Operations [7]. The protocol for achieving this runs as follows. First a round of Partial Isothermal Reversible Processes is applied to convert (τ_S, H_S) into (ρ, \tilde{H}_S) where ρ is the thermal state of \tilde{H}_S and $Z_S = \tilde{Z}_S$. As the partition function does not change, this does not cost work. Next, the Level Transformation $LT_{\mathcal{E}}$ where $\mathcal{E} = \left\{ h_i = E_i - \tilde{E}_i \right\}_{i=1}^n$ is performed to convert (ρ, \tilde{H}_S) into (ρ, H_S) . It can be shown that the deterministic work cost of this $LT - \max_i h_i$, matches the expression given in Eq. (14) (see the Appendix).

V. CONCLUSION

We have shown that Thermal Operations can be simulated by Crude Operations, a class of physical operations closer to that which can be implemented in the laboratory using current technology. This ought to bring thermodynamics of microscopic systems further into the experimental domain, and make the exploration of some of the results in the field [7, 8, 11, 23, 25, 34–36, 38, 45, 48–62] more feasible.

From a conceptual point of view, our results show that the paradigm of thermodynamics which allows for the maximum amount of control of the system and bath, is in some sense equivalent to one which allows only very crude control of the system and bath. The second laws of thermodynamics, since they are fundamental limitations on state transitions, need to be derived assuming the experimenter has as much control and technology as would be allowed by nature (i.e. Thermal Operations). Yet remarkably, the fundamental limitations captured by thermo-majorization and the generalized free energies, which are derived assuming such control, can be achieved with very little control, namely by Crude Operations. Control over bath degrees of freedom, with the exception of one qubit, is not needed.

There are additional second laws, which place constraints not only on the occupation probabilities of a state, but also place restrictions on the coherences over energy levels [8, 16, 17, 33]. **While we conjecture that Crude Operations are also sufficient for the control of quantum coherences, as we do not yet know what the necessary conditions are for coherence manipulation under thermal operations, this cannot yet be verified. Determining the allowed transformations between states that contain coherence remains an important open question in the resource theoretic approach to quantum thermodynamics. The progress that has been made on this ques-**

tion has considered thermal operations in the presence of a reusable source of coherence that typically takes the form of a high dimensional quantum system [32, 33]. However, even in these papers the extraction of deterministic work is not fully understood.

To achieve the state transformations that are possible under Thermal Operations using Crude Operations, we required the use of a single thermal qubit as an ancilla. It would be interesting to investigate the extent to which this is required. In other words, what state transformations can be performed using Crude Operations applied to the system alone?

Finally, with regards to the experimental applicability of our Crude Operations, we note that our operation of Partial Level Thermalization has been proposed to be used in protocols for Heat Bath Algorithmic Cooling [37]. There they have been used to break previous cooling bounds which are important for quantum information processing on NMR systems. Hence, it would be interesting to investigate whether allowing for the full power

of Crude Operations can be used to enhance these protocols still further. In another direction, perhaps NMR systems can provide a useful testbed for the experimental study of Crude Operations and quantum and nano-scale thermodynamics in general?

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