p-adic Deformation of motivic Chow groups

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Abstract. For a smooth projective scheme Y over $W(k)$ we consider an element in the motivic Chow group of the reduction Y_m over the truncated Witt ring $W_m(k)$ and give a "Hodge" criterion - using the crystalline cycle class in relative crystalline cohomology - for the element to lift to the continuous Chow group of the associated p-adic formal scheme $Y_\bullet$. The result extends previous work of Bloch-Esnault-Kerz on the p-adic variational Hodge conjecture to a relative setting. In the course of the proof we derive two new results on the relative de Rham-Witt complex and its Nygaard filtration, and work with a relative version of syntomic complexes to define relative motivic complexes for a smooth lifting of Y_m over the ind-scheme Spec $W_\bullet(W_m(k))$.

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Introduction

In a recent work, Bloch, Esnault and Kerz studied a p-adic analogue of Grothendieck's variational Hodge conjecture on the deformation of algebraic cycles resp. vector bundles. In the context of what is called p-adic variational Hodge Conjecture [B-E-K1], Conjecture 1.2, the above authors gave a Hodge-theoretic condition on the crystalline Chern class when a vector bundle on a smooth projective variety Y_1 over a perfect field k of char p lifts to a vector bundle on a formal lifting $Y_\bullet$ of Y_1 over the Witt vectors $W(k)$. Their method relies on a construction of a motivic pro-complex $\mathbb{Z}_{Y_\bullet}(r)$ in the derived category of pro-complexes with respect to the Nisnevich topology on Y_1, which is obtained by glueing the Suslin-Voevodsky complex on Y_1 with the syntomic complex of Fontaine-Messing on $Y_\bullet$ along the logarithmic Hodge-Witt sheaf in degree r. The continuous Chow group $\mathrm{Ch}^r_{\mathrm{cont}}(Y_\bullet)$ is defined in [B-E-K1] as the hypercohomology of the complex $\mathbb{Z}_{Y_\bullet}(r)$ and is equipped with a canonical map

$$\mathrm{Ch}^r_{\mathrm{cont}}(Y_\bullet) \longrightarrow \varprojlim_n H^{2r}(Y_1, \mathbb{Z}_{Y_n}(r)) \longrightarrow \mathrm{Ch}^r(Y_1) = H^{2r}(Y_1, \mathbb{Z}_{Y_1}(r))$$

to the usual Chow group of Y_1. The obstruction of deforming an algebraic cycle class from Y_1 to $Y_\bullet$ lies in the cohomology of a certain truncated filtered de Rham complex on Y which is already entailed in the definition of the syntomic complex. The filtered de Rham complex, denoted by $p(r)\Omega^\bullet_{Y_\bullet}$ is — as a procomplex – quasiisomorphic to a filtered version of the de Rham-Witt complex denoted by $q(r)W\Omega_{Y_1/k}$ in the étale/Nisnevich-topology [B-E-K1] Prop. 2.8. Hence the obstruction can be made visible by using the crystalline Chern classes which are induced by Gros's Chern classes [Gr] with values in the logarithmic Hodge-Witt cohomology [B-E-K1] Theorem 8.5. In another deep result Bloch-Esnault-Kerz relate the continuous Chow ring $\oplus_{r\leq d}\mathrm{Ch}^r_{\mathrm{cont}}(Y_\bullet)_\mathbb{Q}$ to continuous K-theory $K_0^{\mathrm{cont}}(Y_\bullet)_\mathbb{Q}$ [B-E-K1] Theorem 11.1. This finally enables them to give an equivalent Hodge-theoretic criterion when a vector bundle, rationally, can be lifted from Y_1 to $Y_\bullet$ [B-E-K1], Theorem 1.3.

In the present note I study a relative version of the work of Bloch-Esnault-Kerz, starting from the "motivic" Chow group $H^{2r}(Y_1,\mathbb{Z}_{Y_m}(r))$ for fixed m. The problem is to find a similar criterion when an element in the latter cohomology group (the case $m=1$ being treated in [B-E-K1]) lifts to the continuous Chow group $Ch^r_{\mathrm{cont}}(Y_\bullet)$. In such a mixed characteristic situation, especially when working with a scheme Y_m defined over the artinian local ring $W_m(k)$, it is reasonable to define the cohomological codimension r Chow group as $H^r_{\mathrm{Zar}}(Y_m,\mathcal{K}^{\mathrm{Mil}}_r)$. The graded object is automatically a ring, contravariant in Y_m (see [B-E-K2], §4 for a similar situation in char 0).

There is a canonical map

$$H^{2r}(Y_1,\mathbb{Z}_{Y_m}(r)) \xrightarrow{\pi_r} H^r(Y_m,\mathcal{K}^{\mathrm{Mil}}_r)$$

which in some cases can be shown to be an isomorphism or at least an epimorphism. Hence our problem is still related to deforming Chow groups p-adically. Whilst Bloch-Esnault-Kerz entirely work with $\mathbb{Z}_{Y_\bullet}(r)$ as a procomplex, we need to define $\mathbb{Z}_{Y_m}(r)$ at a finite level which requires some additional thoughts related to the divided Frobenius in the definition of the syntomic complex at finite level. For fixed m we consider the smooth projective scheme $Y_m = X_1$ over the ring $R = W_m(k)$ and we assume there exists a compatible system $X_n/\mathrm{Spec}\ W_n(R)$ of liftings of X_1 which is compatible with the formal lifting $Y_\bullet$ of Y_1, that is $X_{n+1}\times_{\mathrm{Spec}\ W_{n+1}(R)} \mathrm{Spec}\ W_n(R) = X_n$ and $X_n \times_{\mathrm{Spec}\ W_n(R)} \mathrm{Spec}\ W_n(k) = Y_n$. Such a system X_n defines an ind-scheme $X_\bullet$ over the ind-scheme Spec $W_\bullet(R)$ in the sense of [EGA1], Prop. 10.6.3. As multiplication by p is not injective on $W(R)$ we need an alternative definition of the relative syntomic complex $\sigma_{X_\bullet/W_\bullet(R)}(r)$, using a divided Frobenius map defined on a filtered version $N^rW_\bullet\Omega_{X_1/R}$ of the relative de Rham-Witt complex $W\Omega^\bullet_{X_1/R}$. If $m=1$, so $R=k$, then our complex $\sigma_{X_\bullet}(r)$ and the complex $\sigma_{Y_\bullet}(r)$ of Fontaine-Messing [F-M] resp. Kato [K2] are isomorphic as procomplexes. We formally define a motivic complex $\mathbb{Z}_{X_\bullet}(r)$ on X_1 in the same way as Bloch-Esnault-Kerz, by glueing $\mathbb{Z}_{X_1}(r)$ with $\sigma_{X_\bullet}(r)$ along the relative logarithmic Hodge-Witt sheaf $W_\bullet\Omega^r_{X_1/R,\log}$ in degree r and obtain a similar Hodge-theoretic condition to lifting a class in $H^{2r}(Y_1,\mathbb{Z}_{Y_m}(r))$ to $H^{2r}(X_1,\mathbb{Z}_{X_\bullet}(r))$, by using the crystalline cycle class with values in relative de Rham-Witt resp. relative crystalline cohomology.

As the ind-scheme $X_\bullet$ is assumed to be compatible with $Y_\bullet$ we can give a positive answer to our original problem (Theorem 3.6). We formulate here the main application on deforming elements in motivic Chow groups p-adically (Corollary 3.9):

THEOREM 0.1. *Let $r < p$.*

(i) Let $Y_\bullet$ be a formal smooth projective scheme over $\mathrm{Spf} W(k)$. Let $X_1 = Y_m$ for some fixed $m \in \mathbb{N}$ and assume X_1 admits a smooth lifting $X_\bullet$, over Spec $W_\bullet(W_m(k))$ compatible with $Y_\bullet$. Let $\xi \in H^{2r}(X_1, \mathbb{Z}_{X_1}(r))$.

If $c(\xi)$ is "Hodge" with respect to $X_\bullet$, i.e. $c(\xi) \in \mathrm{Image}(\mathbb{H}^{2r}(X_\bullet, \Omega^{\geq r}_{X_\bullet}) \to H^{2r}(X_1, N^r W_\bullet \Omega^\bullet_{X_1/W_m(k)}))$, then ξ lifts to an element $\hat{\xi} \in CH^r_{\mathrm{cont}}(Y_\bullet) = H^{2r}_{\mathrm{cont}}(Y_1, \mathbb{Z}_{Y_\bullet}(r))$.

(ii) Let $z \in \mathrm{image}(\pi_r)$. If its crystalline cycle class is "Hodge" with respect to $X_\bullet$, then z lifts to an element $\hat{z}$ in $\varprojlim_n H^r(Y_n, \mathcal{K}^{\mathrm{Mil}}_{Y_n,r})$.

The theorem should be compared with [B-E-K1] Theorem 8.5. In the proof we will see that the implications in (i) and (ii) do not depend on the choice of $X_\bullet$; Given two liftings $X_\bullet$, $X'_\bullet$ compatible with $Y_\bullet$, with respect to which $c(\xi)$ resp. $c(z)$ is "Hodge", the lifting property of ξ resp. z holds. In the course of the paper we need two technical results on the relative de Rham-Witt complex which play a crucial role in our construction and in the proofs.
In the relative setting the filtered de Rham complex $p(r)\Omega^\bullet_{Y_\bullet}$ mentioned earlier and used in the case $R = k$ in [B-E-K1] is replaced by the complex ($I_R := VW(R)$) denoted by $\mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)}$:

$$I_R\mathcal{O}_{X_\bullet} \xrightarrow{pd} I_R \otimes_{W(R)} \Omega^1_{X_\bullet/W_\bullet(R)} \xrightarrow{pd} \cdots \xrightarrow{pd} I_R \otimes \Omega^{r-1}_{X_\bullet/W_\bullet(R)} \xrightarrow{d} \Omega^r_{X_\bullet/W_\bullet(R)} \xrightarrow{d} \cdots$$

Then we prove Conjecture 4.1 in [L-Z2] for $r < p$

THEOREM 0.2. *Let $r < p$. The complex $\mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)}$ is in the derived category isomorphic to the complex, denoted by $N^rW_\bullet\Omega^\bullet_{X_1/R}$*

$$W_\bullet\mathcal{O}_{X_1} \xrightarrow{d} W_\bullet\Omega^1_{X_1/R} \xrightarrow{d} \cdots \xrightarrow{d} W_\bullet\Omega^{r-1}_{X_1/R} \xrightarrow{dV} W_\bullet\Omega^r_{X_1/R} \xrightarrow{d}$$

The Theorem already holds at finite level for $X_n/W_n(R)$ for any ring R on which p is nilpotent (see Theorem 1.2).
In a second technical result on the relative de Rham-Witt complex we derive an exact triangle generalizing [Il] I 5.7.2 and [B-E-K1] Corollary 4.6 in the case $R = k$.

THEOREM 0.3. *(= Theorem 1.9). Let R be artinian local with perfect residue field k and X_1 smooth over Spec R. In the derived category of procomplexes on $(X_1)_{\mathrm{et}}$ we have a short exact sequence*

$$0 \longrightarrow W_\bullet\Omega^r_{X_1/R,\log}[-r] \longrightarrow N^rW_\bullet\Omega^\bullet_{X_1/R} \xrightarrow{1-\mathrm{Fr}} W_\bullet\Omega^\bullet_{X_1/R} \longrightarrow 0.$$

Note that the complex $q(r)W_\bullet\Omega^\bullet_{X_1/k}$ appearing in [B-E-K1] Corollary 4.6 is isomorphic as procomplex to $N^rW_\bullet\Omega^\bullet_{X_1/k}$ by [L-Z2] Proposition 4.4, if $R = k$. Finally, we point out that Theorem 0.2 has been applied in the construction of higher displays ([G-L] Theorem 1.1 and [L-Z2] Conjecture 5.8).
In the equal characteristic p case, Matthew Morrow has recently studied a relative version of another arithmetic conjecture, the Crystalline Tate Conjecture (see [M1], [M2]), which is a characteristic p analogue of Grothendieck's variational Hodge conjecture.
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1 Relative syntomic complexes

Let X be a smooth scheme X over Spec R (R artinian local with perfect residue field k of characteristic $p > 0$), admitting a lifting $X_\bullet$ as ind-scheme over Spec $W_\bullet(R)$. We are going to define relative syntomic complexes $\sigma_{X_\bullet}(r)$ that will be entailed in the construction of the relative motivic complexes $\mathbb{Z}_{X_\bullet}(r)$ later on.
The definition of $\sigma_{X_\bullet}(r)$ will rely on an appropriate divided Frobenius map Fr on a filtered version of the relative de Rham-Witt complex, denoted by $N^rW_n\Omega^\bullet_{X/R}$:

$$W_{n-1}\mathcal{O}_X \xrightarrow{\mathrm{d}} W_{n-1}\Omega^1_{X/R} \xrightarrow{\mathrm{d}} \cdots \longrightarrow W_{n-1}\Omega^{r-1}_{X/R} \xrightarrow{\mathrm{d}V} W_n\Omega^r_{X/R} \xrightarrow{\mathrm{d}} W_n\Omega^{r+1}_{X/R} \xrightarrow{\mathrm{d}} \cdots$$

(compare the definition in [L-Z2], Definition 2.1). Secondly, we will need a comparison between the complex $N^rW_n\Omega^\bullet_{X/R}$ and the following 'filtered' de Rham complex on the lifting X_n, denoted by $\mathcal{F}^r\Omega^\bullet_{X_n/W_n(R)}$, where $I_R = VW_{n-1}(R)$:

$$I_R \otimes_{W_n(R)} \mathcal{O}_{X_n} \xrightarrow{p\mathrm{d}} I_R \otimes_{W_n(R)} \Omega^1_{X_n/W_n(R)} \xrightarrow{p\mathrm{d}} \cdots \xrightarrow{p\mathrm{d}} I_R \otimes_{W_n(R)} \Omega^{r-1}_{X_n/W_n(R)} \xrightarrow{\mathrm{d}} \Omega^r_{X_n/W_n(R)} \xrightarrow{\mathrm{d}} \cdots$$

We recall the following

Conjecture 1.1. *([L-Z2] Conjecture 4.1). Let R be a ring on which p is nilpotent, $X_n/W_n(R)$ smooth and $X := X_n \times_{W_n(R)} R$. There is an isomorphism in the derived category between the complexes $N^rW_n\Omega^\bullet_{X/R}$ and $\mathcal{F}^r\Omega^\bullet_{X_n/W_n(R)}$.*

We can prove the following

Theorem 1.2. *The conjecture holds if $r < p$.*

Proof. Assume first that there exists a closed embedding $X_n \hookrightarrow Z_n$ into a smooth $W_n(R)$-scheme Z_n which is a Witt lift of $Z = Z \times_{W_n(R)} R$ in the sense of [L-Z1] Definition 3.3. That is it is equipped with a map $\Delta_n : W_n(Z) \to Z_n$ fitting into a commutative diagram

$$
\begin{array}{ccccc}
W_n(X) & \longrightarrow & & & W_n(Z) \\
\uparrow {\scriptstyle w_0} & & & & \downarrow {\scriptstyle \Delta_n} \\
X & \hookrightarrow & X_n & \longrightarrow & Z_n
\end{array}
$$

Such a Witt-lift always exists locally. Let I be the ideal sheaf of X_n in $\mathcal{O}_{Z_n}$ and $\mathcal{J} = \mathcal{J}_n$ be the divided power ideal sheaf of the embedding i_n. Let $\mathcal{O}_{D_n}$ be the PD-envelope of $\mathcal{O}_{Z_n}$ with respect to $\mathcal{J}$, with underlying scheme D_n. We already know that the complex $\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^\bullet_{Z_n/W_n(R)}$ is quasiisomorphic to $\Omega^\bullet_{X_n/W_n(R)}$ ([Il], [B-O]). Let $\mathcal{J}^{[r]}$ for $r \geq 1$ be the higher divided power ideal sheaves.
To keep notation light we will write $\mathcal{O}$ for $\mathcal{O}_{D_n}$, Ω^i for $\Omega^i_{D_n}$, $I_R\mathcal{J}^{[j]}$ for $I_R \otimes_{W_n(R)} \mathcal{J}^{[j]}$ and $I_R\mathcal{J}^{[j]}\Omega^s$ for $I_R \otimes_{W_n(R)} (\mathcal{J}^{[j]} \otimes_{\mathcal{O}_{D_n}} \Omega^s_{D_n})$. Then we consider the following diagram of complexes

$$
\begin{array}{ccccccccccccccc}
I_R\mathcal{O} & & & & & & & & & & & & & & \\
 & \overset{pd}{\searrow} & & & & & & & & & & & & & \\
I_R\mathcal{J} & \longrightarrow & I_R\Omega^1 & & & & & & & & & & & & \\
 & & & \overset{pd}{\searrow} & & & & & & & & & & & \\
\vdots & & \vdots & & \ddots & & & & & & & & & & \\
 & & & & & \overset{pd}{\searrow} & & & & & & & & & \\
I_R\mathcal{J}^{[r-3]} & \xrightarrow{d} & I_R\mathcal{J}^{[r-2]}\Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & I_R\Omega^{r-3} & & & & & & & & \\
 & & & & & & & \overset{pd}{\searrow} & & & & & & & \\
I_R\mathcal{J}^{[r-2]} & \xrightarrow{d} & I_R\mathcal{J}^{[r-2]}\Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & I_R\mathcal{J}\Omega^{r-3} & \xrightarrow{d} & I_R\Omega^{r-2} & & & & & & \\
 & & & & & & & & & \overset{pd}{\searrow} & & & & & \\
I_R\mathcal{J}^{[r-1]} & \xrightarrow{d} & I_R\mathcal{J}^{[r-2]}\Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & I_R\mathcal{J}^{[2]}\Omega^{r-3} & \xrightarrow{d} & I_R\mathcal{J}\Omega^{r-2} & \xrightarrow{d} & I_R\Omega^{r-1} & & & & \\
 & & & & & & & & & & & \overset{d}{\searrow} & & & \\
\mathcal{J}^{[r]} & \xrightarrow{d} & \mathcal{J}^{[r-1]}\Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathcal{J}^{[3]}\Omega^{r-3} & \xrightarrow{d} & \mathcal{J}^{[2]}\Omega^{r-2} & \xrightarrow{d} & \mathcal{J}\Omega^{r-1} & \xrightarrow{d} & \Omega^r & \xrightarrow{d} & \cdots
\end{array}
\tag{1.3}
$$

As in the classical case for $R = k$ (see [B-E-K1] 2.8) it follows from [B-O] Theorem 7.2, applied to $X_n \hookrightarrow Z_n$ and $X_n = X_n$, that the lower horizontal sequence is quasiisomorphic to $\Omega^{\geq r}_{X_n/W_n(R)}$. All horizontal sequences are - up to the term $I_R\Omega^j$ that is placed on the diagonal - exact because all sheaves $\mathcal{J}^{[j]}$ and Ω^j are - locally - free $\mathcal{O}_{X_n}$-modules by [B-O] Prop. 3.32. Therefore the sequence $\mathcal{J}^{[s-\bullet]}\Omega^\bullet$ remains exact after $\otimes_{W_n(R)} R$ because it then coincides with the corresponding sequence for the closed embedding $X = X_n \times_{W_n(R)} R \to Z_n \times_{W_n(R)} R$. Then $I_R\mathcal{J}^{[s-\bullet]}\Omega^\bullet$ is exact as well.
It is clear that adding up the two lower horizontal sequences degree-wise yields a complex that is quasiisomorphic to

$$
\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow I_R\Omega^{r-1}_{X_n/W_n(R)} \xrightarrow{d} \Omega^r_{X_n/W_n(R)} \xrightarrow{d} \Omega^{r+1}_{X_n/W_n(R)} \longrightarrow \cdots
$$

Moreover, it is easy to see that adding up degree-wise the $k+1$ lower horizontal sequences up to the sequence starting with $I_R\mathcal{J}^{[r-k]}$ we obtain a complex that is quasiisomorphic to

$$\ldots \longrightarrow 0 \longrightarrow I_R\Omega^{r-k}_{X_n/W_n(R)} \xrightarrow{pd} \cdots \xrightarrow{pd} I_R\Omega^{r-1}_{X_n/W_n(R)} \xrightarrow{d} \Omega^r_{X_n/W_n(R)} \xrightarrow{d} \cdots \tag{1.4}$$

The quasiisomorphisms are induced by the canonical maps $\mathcal{O}_{D_n} \longrightarrow \mathcal{O}_{X_n}$, $\Omega^j_{D_n} \longrightarrow \Omega^j_{X_n}$ etc.
Define $\mathrm{Fil}^r\Omega^\bullet_{D_n/W_n(R)}$ to be the complex obtained by adding up all horizontal sequences degree-wise. Then $\mathrm{Fil}^r\Omega^\bullet_{D_n/W_n(R)}$ is quasiisomorphic to $\mathcal{F}^r\Omega^\bullet_{X_n/W_n(R)}$, the complex that is defined above before Conjecture 1.1.
Now construct a map

$$\Sigma : \mathrm{Fil}^r\Omega^\bullet_{D_n/W_n(R)} \longrightarrow N^rW_n\Omega^\bullet_{X/R} \tag{1.5}$$

The composite map $\Delta_n : \mathcal{O}_{Z_n} \to W_n(\mathcal{O}_X)$ extends to a map $\sigma : \mathcal{O}_{D_n} \to W_n(\mathcal{O}_X)$ with induced maps $\Omega^i_{D_n} \xrightarrow{\sigma} W_n\Omega^i_{X/R}$, because the image of $I \subset \mathcal{O}_{Z_n}$ is contained in $VW_{n-1}(\mathcal{O}_X)$ which is a PD-ideal in $W_n(\mathcal{O}_X)$. Let $x \in \mathcal{J}$ with image $\sigma(x) = V\eta \in VW_{n-1}(\mathcal{O}_X)$. Then $\sigma(x^n) = p^{n-1}V(\eta^n)$ hence $\sigma(\gamma_n(x)) = \frac{1}{n!}p^{n-1}V(\eta^n)$. Then for $r \leq p-1$, $j < r$ and $n > j$, the element $\sigma^{(j)} = \frac{1}{n!}p^{n-1-j}V(\eta^n)$ is well-defined. Define $F^{(j+1)}(\gamma_n(x)) =$ "$\frac{F}{p}$"$\sigma^{(j)}(\gamma_n(x)) := \frac{1}{n!}p^{n-1-j}\eta^n$ using $FV = p$. Then the map Σ is defined on entries as follows:
Consider a differential in the lower horizontal sequence

$$\mathcal{J}^{[k]}\Omega^{r-k} \xrightarrow{d} \mathcal{J}^{[k-1]}\Omega^{r-k+1}$$

For $m \geq k$ let $\gamma_m(x)\omega \in \mathcal{J}^{[k]}\Omega^{r-k}$ with $\sigma(x) = V\eta$ as above. Define $F_k(\gamma_m(x)\omega) = F^{(k)}(\gamma_m(x))F\sigma(\omega) = \frac{p^{m-1-(k-1)}}{m!}\eta^mF\sigma(\omega)$ in $W_{n-1}\Omega^{r-k}_{X/R}$. Then

$$dF_k(\gamma_m(x)\omega) = \frac{p^{m-k}}{(m-1)!}\eta^{m-1}d\eta F\sigma(\omega) + \frac{p^{m-k+1}}{m!}\eta^mFd\sigma(\omega)$$

using $dF = pFd$.
On the other hand $d(\gamma_m(x)\omega) = \gamma_{m-1}(x)dx\omega + \gamma_m(x)d\omega$ and hence

$$F_{k-1}(d\gamma_m(x)\omega) = \frac{p^{m-2-(k-2)}}{(m-1)!}\eta^{m-1}d\eta F\sigma(\omega) + \frac{p^{m-1-(k-2)}}{m!}\eta^mFd\sigma(\omega)$$

Here we have used $FdV\eta = d\eta$. We see that $dF_k(\gamma_m(x)\omega) = F_{k-1}d(\gamma_m(x)\omega)$.
Now let for $\underline{x} = (x_1,\ldots,x_\ell)$, $x_i \in \mathcal{J}$ and $m = \sum_{i=1}^\ell m_i \geq k$, $\underline{x}^{[m]} = x_1^{[m_1]}\cdots x_\ell^{[m_\ell]}$ with $x_i^{[m_i]} = \gamma_{m_i}(x_i) = \frac{x_i^{m_i}}{(m_i)!}$ (an arbitrary element in $\mathcal{J}^{[k]}$). Let $\sigma(x_i) = V(\eta_i)$. Define

$$F^{(k)}(\underline{x}^{[\underline{m}]}) = \left(\prod_{i=1}^{\ell} \frac{p^{m_i-1}}{(m_i)!}\eta_i^{m_i}\right) \cdot p^{-(k-\ell)}$$

The definition is compatible with the previous case $\ell = 1$. Again we have for $\underline{x}^{[\underline{m}]} \cdot \omega \in \mathcal{I}^k\Omega^{r-k}$ and $F_k(\underline{x}^{[\underline{m}]} \cdot \omega) := F^{(k)}(\underline{x}^{[\underline{m}]}) \cdot F\sigma(\omega)$ the equality

$$dF_k(\underline{x}^{[\underline{m}]} \cdot \omega) = F_{k-1}d(\underline{x}^{[\underline{m}]} \cdot \omega)$$

The tedious proof is omitted.
So we have a commutative diagram for $k \geq 1$

$$\begin{array}{ccc} \mathcal{I}^{[k]}\Omega^{r-k} & \xrightarrow{d} & \mathcal{I}^{[k-1]}\Omega^{r-k+1} \\ \downarrow{\scriptstyle F_k} & & \downarrow{\scriptstyle F_{k-1}} \\ W_{n-1}\Omega^{r-k}_{X/R} & \xrightarrow{d} & W_{n-1}\Omega^{r-k+1}_{X/R} \end{array} \tag{1.6.1}$$

We can extend the map F_k to a map

$$F_{k+1} : I_R\mathcal{I}^{[k]}\Omega^{\ell-k} \longrightarrow W_{n-1}\Omega^{\ell-k}_{X/R}$$

by

$$F_{k+1}(V\xi\underline{x}^{[\underline{m}]}\omega) = \xi F_k(\underline{x}^{[\underline{m}]}\omega)$$

Then

$$\begin{array}{ccc} I_R\mathcal{I}^{[k]}\Omega^{\ell-k} & \xrightarrow{d} & I_R\mathcal{I}^{[k-1]}\Omega^{\ell-k+1} \\ \downarrow{\scriptstyle F_{k+1}} & & \downarrow{\scriptstyle F_{k}} \\ W_{n-1}\Omega^{\ell-k}_{X/R} & \xrightarrow{d} & W_{n-1}\Omega^{\ell-k+1}_{X/R} \end{array} \tag{1.6.2}$$

commutes as well for $k \geq 1$. It is also clear that the diagram

$$\begin{array}{ccc} I_R\Omega^{k} & \xrightarrow{pd} & I_R\Omega^{k+1} \\ \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_1} \\ W_{n-1}\Omega^{k}_{X/R} & \xrightarrow{d} & W_{n-1}\Omega^{k+1}_{X/R} \end{array} \tag{1.6.3}$$

commutes where $F_1(V\xi\omega) = \xi F\omega$, using that $dF\omega = pFd\omega$.
In degree $r-1$ the maps d commute with dV because we have commutative diagrams

$$\begin{array}{ccc} I_R\Omega^{r-1} & \xrightarrow{d} & \Omega^{r} \\ \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle \sigma} \\ W_{n-1}\Omega^{r-1}_{X/R} & \xrightarrow{dV} & W_{n}\Omega^{r}_{X/R} \end{array} \qquad \begin{array}{ccc} \mathcal{I}\Omega^{r-1} & \xrightarrow{d} & \Omega^{r} \\ \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle \sigma} \\ W_{n-1}\Omega^{r-1}_{X/R} & \xrightarrow{dV} & W_{n}\Omega^{r}_{X/R} \end{array} \tag{1.6.4}$$

because

$$dV(F_1(V\xi\omega)) = dV(\xi F\sigma(\omega)) = d(V\xi\sigma(\omega)) = V\xi d\sigma(\omega) = V\xi\sigma(d(\omega))$$

and

$$dV(F_1(\gamma_m(x)\omega)) = dV\left(\frac{p^{m-1}}{m!}\eta^m F\sigma(\omega)\right) = d(\sigma(\gamma_m(x))\sigma(\omega)) = \sigma d(\gamma_m(x)\omega)$$

(where $\sigma(x) = V\eta$ as before).
Hence we have constructed a map

$$\Sigma : \mathrm{Fil}^r\Omega^\bullet_{D_n/W_n(R)} \longrightarrow N^rW_n\Omega^\bullet_{X/R} \tag{1.6}$$

from the complex constructed in diagram (1.3) into the Nygaard complex.
We have a diagram

$$\begin{array}{ccc} \mathcal{F}^r\Omega^\bullet_{X_n/W_n(R)} & \overset{\cong}{\longleftarrow} & \mathrm{Fil}^r\Omega_{D_n/W_n(R)} \\ & & \Big\downarrow \Sigma \\ & & N^rW_n\Omega^\bullet_{X/R} \end{array} \tag{1.7}$$

If we have two embeddings $X_n \xrightarrow{i_n} Z_n$, $X_n \xrightarrow{i'_n} Z'_n$ into Witt lifts Z_n, Z'_n with corresponding diagrams (1.3) for each embedding and corresponding complexes $\mathrm{Fil}^r\Omega^\bullet_{D_n/W_n(R)}$, $\mathrm{Fil}^r\Omega^\bullet_{D'_n/W_n(R)}$ then by considering the product embedding $X_n \xrightarrow{(i_n,i'_n)} Z_n \times Z'_n$ and the corresponding Fil^r-complex, we see that we get a canonical map

$$\mathcal{F}^r\Omega^\bullet_{X_n/W_n(R)} \longrightarrow N^rW_n\Omega^\bullet_{X/R} \tag{1.7.1}$$

in the derived category which does not depend on the choice of the embedding i_n. In order to prove Theorem 1.2 it suffices to show that the map Σ is a quasiisomorphism. This is a local question, hence we may assume that $X_n = Z_n = D_n$ are affine with Frobenius lift F. Then the assertion follows from [L-Z2] Corollary 4.3. This proves the Theorem and Conjecture 4.1 in [L-Z2] for $r < p$ assuming the existence of a global embedding into a Witt lift. If there is no embedding of X_n into a Witt lift one proceeds by simplicial methods as in [Il] II.1.1, [L-Z1] §3.2. Let $X_n(i)$, $i \in I$ be a covering of X_n, inducing a covering $X(i)$ of X, and an embedding $X_n(i) \to Y_n(i)$ which is a Witt lift of $Y(i) = Y_n(i) \times_{W_n(R)} R$. One gets simplicial schemes $X^\bullet \to X^\bullet_n \to D^\bullet_n \to Y^\bullet_n$ and quasiisomorphisms of simplicial complexes of sheaves

$$\mathcal{F}^r\Omega^\bullet_{X^\bullet_n/W_n(R)} \leftarrow \mathrm{Fil}^r\Omega^\bullet_{D^\bullet_n/W_n(R)} \to N^rW_n\Omega^\bullet_{X^\bullet/R}$$

on $X^\bullet$; let $\theta : X^\bullet \to X$ be the natural augmentation. By applying $R\theta_*$ to the quasiisomorphisms we get, by cohomological descent in Zariski/étale topology, an isomorphism (1.7.1) in $D_{\text{ét}}(X)$. □

There are well known maps of the de Rham-Witt complexes, denoted by "1" and Fr, between $N^r W_n \Omega^\bullet_{X_R}$ and $W_{n-1}\Omega^\bullet_{X/R}$:

$$
\begin{array}{ccccccccccccc}
W_{n-1}\mathcal{O}_X & \xrightarrow{d} & W_{n-1}\Omega^1_{X/R} & \xrightarrow{d} & \cdots & \xrightarrow{d} & W_{n-1}\Omega^{r-1}_{X/R} & \xrightarrow{dV} & W_n\Omega^r_{X/R} & \xrightarrow{d} & W_n\Omega^{r+1}_{X/R} & \xrightarrow{d} & \cdots \\
{\scriptstyle p^{r-1}V}\downarrow\downarrow{\scriptstyle =} & & {\scriptstyle p^{r-2}V}\downarrow\downarrow{\scriptstyle =} & & & & {\scriptstyle V}\downarrow\downarrow{\scriptstyle =} & & \downarrow\downarrow{\scriptstyle F} & & \downarrow\downarrow{\scriptstyle pF} & & \\
W_{n-1}\mathcal{O}_X & \xrightarrow{d} & W_{n-1}\Omega^1_{X/R} & \xrightarrow{d} & \cdots & \xrightarrow{d} & W_{n-1}\Omega^{r-1}_{X/R} & \xrightarrow{d} & W_{n-1}\Omega^r_{X/R} & \xrightarrow{d} & W_{n-1}\Omega^{r+1}_{X/R} & \xrightarrow{d} & \cdots
\end{array}
\tag{1.8}
$$

The diagram commutes because of $F\mathrm{d}V = d$, $\mathrm{d}F = pF\mathrm{d}$ and $V\mathrm{d} = p\mathrm{d}V$. p^iV means p^iV composed with the projection from level n to level $n-1$. The map Fr of complexes also appears in [L-Z2] in the context of (pre-)displays and plays the role of a divided Frobenius.

In the following we will consider the derived category of procomplexes $D_{\mathrm{pro,et}}(X)$ defined as follows: Let $C_{\mathrm{pro,et}}(X)$ be the category of pro-systems of unbounded complexes of sheaves on the small étale site of X. Then $D_{\mathrm{pro,et}}(X)$ is the Verdier localisation of the homotopy category of $C_{\mathrm{pro,et}}(X)$ where all objects are killed which are represented by pro-systems of complexes with levelwise vanishing cohomology sheaves (compare [B-E-K1] Definition A.4).

Theorem 1.9. *Let R be an artinian local ring with perfect residue field k, $X/\mathrm{Spec}\ R$ smooth. Then there is an exact sequence of pro-complexes in $D_{\mathrm{pro,et}}(X)$:*

$$0 \longrightarrow W_\bullet\Omega^r_{X/R,\log}[-r] \longrightarrow N^r W_\bullet \Omega^\bullet_{X/R} \overset{1-\mathrm{Fr}}{\longrightarrow} W_\bullet\Omega^\bullet_{X/R} \longrightarrow 0$$

where $W_\bullet\Omega^r_{X/R,\log}$ is, locally for $X = \mathrm{Spec}\ A$, generated by $\mathrm{d}\log[x_1]\wedge\ldots\wedge \mathrm{d}\log[x_r]$, with $x_1,\ldots,x_r \in A$, as $W_\bullet(\mathbb{F}_p)$-module.

Proof. Let $l < r$, $i \geq 0$. Consider the map

$$p^iV - \mathrm{id} : W_{n-1}\Omega^l_{X/R} \longrightarrow W_{n-1}\Omega^l_{X/R}$$

Then $(p^iV - \mathrm{id})\alpha = p^iV\alpha - \alpha$ and for given β we have $\beta = (p^iV - \mathrm{id})\alpha$ has the solution $\alpha = -\sum_{m=0}^\infty (p^iV)^m\beta$ hence $p^iV - \mathrm{id}$ is surjective. On the other hand, let $\alpha \in \mathrm{Ker}(p^iV - \mathrm{id})$. Then $\alpha = p^iV\alpha$, hence $\alpha \in (p^iV)^s W_{n-1}\Omega^l_{X/R}$ for all s, so $\alpha = 0$ and thus $1 - \mathrm{Fr}$ is an automorphism in degrees $< r$.

A formal inverse of $(1 - p^sF)$, for $s > 0$, is $\sum_{n=0}^\infty (p^sF)^n = \sum_{n=1}^\infty p^{sn}F^n$. This is an element of the Cartier-Raynaud ring because for any $u > 0$ $p^{sn} \in V^uW(R)$ for almost all n. Hence $\sum_{n\geq 0} p^{sn}F^n$ acts on the completed $W\Omega^l_{X/R}$ and provides an inverse of $1 - p^sF$ on $W\Omega^l_{X/R}$. But then $1 - p^sF$ is also surjective on the prosystem $W_\bullet\Omega^l_{X/R}$.

Since all assertions in the theorem only need to be checked locally, we may assume now that $X = \mathrm{Spec}\ B$, where B is étale over a Laurent polynomial algebra $A = R[T_1^{\pm 1}, \ldots, T_d^{\pm 1}]$. It is enough to prove the theorem when replacing B by $B \otimes_R R/\mathfrak{m}^e$ for any $e \geq 1$, where $\mathfrak{m}$ is the maximal ideal of R. For $e = 1$ this follows from [Il] I Théorème 5.7.2. We will prove the remaining assertions by inducion on e. So let B/R be such that $\mathfrak{m}^e R = 0$ and assume the theorem holds for $\bar{B} = B \otimes_R R/\mathfrak{m}^{e-1}$. To prove the injectivity of $1 - p^s F$, for $s > 0$, on the prosheaf $W_\bullet \Omega^\ell_{B/R}$ it is enough to show that

$$\ker(1 - p^s F : W_{n+1}\Omega^\ell_{B/R} \to W_n \Omega^\ell_{B/R})$$

is contained in $\mathrm{Fil}^n W_{n+1}\Omega^\ell_{B/R}$. (For $e = 1$, this is shown in [Il] I, Lemma 3.30). Consider the commutative diagram

$$\begin{array}{ccc} \ker \pi_{n+1} & \xrightarrow{1 - p^s F} & \ker \pi_n \\ \downarrow & & \downarrow \\ W_{n+1}\Omega^\ell_{B/R} & \xrightarrow{1 - p^s F} & W_n \Omega^\ell_{B/R} \\ \downarrow & & \downarrow \\ W_{n+1}\Omega^\ell_{\bar{B}/\bar{R}} & \xrightarrow{1 - p^s F} & W_n \Omega^\ell_{\bar{B}/\bar{R}} \end{array} \tag{1.9.1}$$

Let $A_n = W_n(R)[T_1^{\pm 1}, \ldots, T_d^{\pm 1}]$ and $\varphi : A_{n+1} \to A_n$ be the Frobenius, extending $F : W_{n+1}(R) \to W_n(R)$ by $T_i \to T_i^p$. The map $A_n \to W_n(A)$, $T_i \to [T_i]$ is compatible with Frobenii. As shown in [L-Z1] Prop. 3.2, φ extends to a Frobenius structure $B_{n+1} \to B_n$, where B_n is a lifting of B over $W_n(R)$, étale over A_n, equipped with a map $B_n \to W_n(B)$, again compatible with Frobenii. Let now $m \in \mathbb{N}$ be such that $p^m W_{n+1}(R) = 0$. Then étale base change for the relative de Rham-Witt complex and the proof of [L-Z1] Theorem 3.5 (applied to $A = R[T_1^{\pm 1}, \ldots, T_d^{\pm 1}]$ instead of $R[T_1, \ldots, T_d]$) gives isomorphisms of complexes

$$\begin{aligned} W_n \Omega^\bullet_{B/R} &= W_{m+n}(B) \otimes_{W_{m+n}(A), F^n} W_n \Omega^\bullet_{A/R} \\ &\cong B_{m+n} \otimes_{A_{m+n}, \varphi^n} W_n \Omega^\bullet_{A/R} \\ &= B_{m+n} \otimes_{A_{m+n}, \varphi^n} \Omega^\bullet_{A_n/W_n(R)} \oplus B_{m+n} \otimes_{A_{m+n}, \varphi^n} (W_n \Omega^\bullet_{A/R})_{frac} \\ &= (W_n \Omega^\bullet_{B/R})_{int} \oplus (W_n \Omega^\bullet_{B/R})_{frac} \end{aligned} \tag{1.9.2}$$

The decomposition into an integral and an acyclic fractional part according to weight functions with values in $\mathbb{Z}[1/p]$ is given in [L-Z1] (3.9) for polynomial algebras and in [B-M-S] Theorems 10.12 and 10.13 for Laurent polynomial algebras. From the uniqueness statement in the description of $W_n \Omega^\bullet_{A/R}$ as sums of basic Witt differentials we see that

$$\ker(W_n \Omega^\bullet_{A/R} \to W_n \Omega^\bullet_{\bar{A}/\bar{R}}) = W_n(\mathfrak{m}^{e-1})\Omega^\bullet_{A_n/W_n(R)} \oplus (W_n \Omega^\bullet_{\mathfrak{m}^{e-1}A/R})_{frac}$$

where $(W_n \Omega^\bullet_{\mathfrak{m}^{e-1}A/R})_{frac}$ consists of sums of basic Witt differentials in $(W_n \Omega^\bullet_{A/R})_{frac}$ with coefficients in $W_n(\mathfrak{m}^{e-1})$. Then $\ker \pi_n$, for $\pi_n : W_n \Omega^\ell_{B/R} \to$

$W_n\Omega^\ell_{\bar{B}/\bar{R}}$, is equal to

$$B_{m+n} \otimes_{A_{m+n},\varphi^m} W_n(\mathfrak{m}^{e-1})\Omega^\bullet_{A_n/W_n(R)} \oplus B_{m+n} \otimes_{A_{m+n},\varphi^m} (W_n\Omega^\bullet_{\mathfrak{m}^{e-1}A/R})_{frac} \tag{1.9.3}$$

Since for $\alpha \in \mathfrak{m}^{e-1}$ and $p = [p] + V\eta$ we have $p \cdot [\alpha] = [p \cdot \alpha] + V(\eta \cdot [\alpha]^p) = 0$ we see that $p \cdot x = 0$ for all $x \in W_n(\mathfrak{m}^{e-1})$ and hence $1 - p^s F : \ker \pi_{n+1} \to \ker \pi_n$ is the projection map which has kernel $\mathrm{Fil}^n W_{n+1}\Omega^\ell_{B/R} \cap \ker \pi_{n+1}$. By induction hypothesis, on the level $\bar{B}/\bar{R}$, $\ker(1 - p^s F)$ is contained in $\mathrm{Fil}^n W\Omega^\ell_{\bar{B}/\bar{R}}$. This shows that $1 - p^s F : W_\bullet\Omega^\ell_{B/R} \to W_\bullet\Omega^\ell_{B/R}$ is an isomorphism of prosheaves for $s > 0$ and hence the map $1 - \mathrm{Fr}$ in the theorem is bijective in degrees $> r$.

Now we prove the exactness of the complex of prosheaves

$$0 \to W_\bullet\Omega^r_{B/R,\log} \to W_\bullet\Omega^r_{B/R} \xrightarrow{1-\mathrm{Fr}} W_\bullet\Omega^r_{B/R} \to 0$$

in the étale topology. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \ker \pi_\bullet & \xrightarrow{1-F} & \ker \pi_\bullet & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_\bullet\Omega^r_{B/R,\log} & \longrightarrow & W_\bullet\Omega^r_{B/R} & \xrightarrow{1-F} & W_\bullet\Omega^r_{B/R} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_\bullet\Omega^r_{\bar{B}/\bar{R},\log} & \longrightarrow & W_\bullet\Omega^r_{\bar{B}/\bar{R}} & \xrightarrow{1-F} & W_\bullet\Omega^r_{\bar{B}/\bar{R}} & \longrightarrow & 0
\end{array} \tag{1.9.4}$$

By induction hypothesis, the lower sequence is exact in the étale topology. To prove the surjectivity of $1 - F$ in the étale topology it suffices to show that $\ker \pi_{n+1} \xrightarrow{1-F} \ker \pi_n$ is surjective. We use again the description (1.9.3) of $\ker \pi_n$ as a sum of an integral and a fractional part with coefficients in $W_n(\mathfrak{m}^{e-1})$, and where the fractional part is acyclic, too.

Let $x = [x_0] + V\eta \in W_{n+1}(\mathfrak{m}^{e-1})$. Then $Fx = [x_0]^p + p \cdot \eta = 0$, so $1 - F$ is the projection from level $n+1$ to level n on the integral part. In the fractional part of the decomposition (1.9.3) an element $\tilde{f} \otimes V\omega$, with $\tilde{f}$ a lift of $f \in B$ to B_{m+n+1} corresponds to $\varphi^m \tilde{f} V\omega = V(F^{m+1}\tilde{f} \cdot \omega)$ in $W_{n+1}\Omega^r_{\mathfrak{m}^{e-1}B/R}$, where we identify $\tilde{f}$ with its image in $W_{m+n+1}(B)$ and use the compatibility of φ and F under the map $B_{m+n+1} \to W_{m+n+1}(B)$. Likewise, $\tilde{f} \otimes dV\omega = \varphi^m \tilde{f} dV\omega = d(F^m \tilde{f} V\omega) = dV(F^{m+1}\tilde{f} \cdot \omega)$ because p^m annihilates $W_{n+1}(R)$ and $dF = pFd$. Since $V\omega$ has coefficients in $W_{n+1}(\mathfrak{m}^{e-1})$ we see that $F \circ V(\omega) = p \cdot \omega = 0$. So again $1 - F$ is the projection from level $n+1$ to level n on the image of V. On the other hand, $1 - F$ maps the image of dV onto the image of d. The assertion already holds in the Zariski topology. We recall here the argument in [Il] I. Prop. 3.26 which also holds for the relative de Rham-Witt complex, using the formula $FdV = d$. Let $x \in W_n\Omega^{r-1}_{B/R}$. Then

$$\begin{aligned} dx &= FdVx - dVx + FdV^2x - dV^2x + \cdots \\ &= (F-1)(dVx + \cdots + dV^n x) \end{aligned}$$

Since for $y \in W_n\Omega^{r-1}_{B/R}$

$$(F-1)(dVy) = dy - dVy$$

lies in the image of d, the assertion follows. So in particular, the image of dV in $W_n\Omega^r_{\mathfrak{m}^{e-1}B/R}$ is contained in the image of $1-F$. Hence $1-F : \ker \pi_\bullet \to \ker \pi_\bullet$ is surjective and therefore $1-F$ is surjective on the prosheaf $W_\bullet\Omega^r_{B/R}$ in the étale topology.
Now we compute the kernel of $1-F : \ker \pi_{n+1} \to \ker \pi_n$. The above considerations show that $1-F$ is the projection from level $n+1$ to level n on the integral part of $\ker \pi_{n+1} = W_{n+1}\Omega^r_{\mathfrak{m}^{e-1}B/R}$ and also on the image of V (because F vanishes there). So the kernel of $1-F$, when restricted to this integral part and the image of V, is contained in $\mathrm{Fil}^n W_{n+1}\Omega^r_{B/R} \cap \ker \pi_{n+1}$. On the other hand, the image of dV is mapped under $1-F$ onto the image of d using the formula $FdV = d$.
In the following we prove a uniqueness statement for representing elements in

$$(W_n\Omega^r_{\mathfrak{m}^{e-1}B/R})_{frac} = B_{m+n} \otimes_{A_{m+n},\varphi^m} (W_n\Omega^r_{\mathfrak{m}^{e-1}A/R})_{frac}$$

as a sum of "basic" Witt differentials. For this we recall the notion of primitive basic Witt differentials $e(1,k,\mathcal{P})$ associated to primitive weight functions $k : \{1,\ldots,d\} \to \mathbb{Z} \cup \{\infty\}$ and partitions $\mathcal{P}$ of supp k, $\mathcal{P} = I_0 \cup \cdots \cup I_r$ with $I_0 \neq \emptyset$. "Primitive" means that for at least one $i \in I_0$, $p \nmid k_i$. They are defined in [L-Z1] 2.2 and used in the uniqueness statement [L-Z1] Theorem 2.24 for polynomial algebras, where k takes values in $\mathbb{N}$. But the same statement holds for Laurent polynomial algebras as well by allowing weight functions to take values in $\mathbb{Z} \cup \{\infty\}$, where the value $k_i = k(i)$ is ∞ if the variable T_i occurs in a logarithmic differential $d\log[T_i]$. A description of the elements $e(1,k,\mathcal{P})$ in the case of Laurent polynomial algebras is given in [B-M-S], 10.4, Case 1, assuming $v(a|_{I_0}) = v(a|_{I_1}) = \cdots = v(a|_{I_{\rho_1}}) = 0$, that is $\rho_1 = 0$ using the notation in [B-M-S].
Then an element z in $(W_n\Omega^r_{\mathfrak{m}^{e-1}A/R})_{frac}$ has a unique representation

$$z = \sum_{(k',\mathcal{P}')} \sum_{j=1}^{n-1} V^j \xi'_j e(1,k',\mathcal{P}') + \sum_{(k,\mathcal{P})} \sum_{j=1}^{n-1} dV^j \xi_j e(1,k,\mathcal{P}) \tag{1.9.5}$$

where $(k',\mathcal{P}'), (k,\mathcal{P})$ are as above, $\mathcal{P}' = I'_0 \cup \cdots \cup I'_r$; $\mathcal{P} = I_0 \cup \cdots \cup I_{r-1}$, $\xi_j, \xi'_j \in W_{n-j}(\mathfrak{m}^{e-1})$. For our purposes, namely to compute the kernel of $1-F$, it is enough to consider the second sum, i.e. we will only consider exact differentials in the fractional part. In order to find elements in the kernel of

$1-F$, we need to include the case $j=0$ in the above sum, so we will consider elements

$$z=\sum_{(k,\mathcal{P})}\sum_{j=0}^{n-1} dV^j\xi_j e(1,k,\mathcal{P})$$

Since the product structure of $W_n(R)$ on $W_n(\mathfrak{m}^{e-1})$ factors through the action of k:

$$\alpha\cdot(\xi_0,\dots,\xi_{n-1})=([\alpha]\xi_0,[\alpha]^p\xi_1,\dots,[\alpha]^{p^{n-1}}\xi_{n-1})$$

we see that $\mathfrak{m}^{e-1}$, resp. $W_n(\mathfrak{m}^{e-1})$ become k-vector spaces. (Note that $I_R=VW_{n-1}(R)$ and $W_n(\mathfrak{m})$ both annihilate $W_n(\mathfrak{m}^{e-1})$.) Then the action of A_n on $(W_n\Omega^r_{\mathfrak{m}^{e-1}A/R})_{frac}$ factors through $A_k=A\otimes_R k=k[T_1^{\pm 1},\dots,T_d^{\pm 1}]$. We have an isomorphism for all $m\geq 0$ ([L-Z1], Prop. 3.2, Lemma A.9 and Corollary A.11)

$$(B_{m+n}\otimes_{A_{m+n},\varphi^m}A_n)\otimes_{A_n}A_k\cong B_n\otimes_{W_n(R)}k\cong B_k=B\otimes_R k \tag{1.9.6}$$

given by $b\otimes a\otimes 1\mapsto \bar{b}^{p^m}\cdot\bar{a}$ where $\bar{b}$, resp. $\bar{a}$ is the image of b, resp. a under the canonical map $B_{m+n}\to B_k$ resp. $A_n\to A_k$.

Let $\mathcal{M}_{<p^n}$ be the set of all primitive basic Witt differentials $e(1,k,\mathcal{P})$ with $\mathcal{P}=I_0\cup\dots\cup I_{r-1}$ such that $1\leq k_i<p^n$ or $k_i=\infty$ for all non-zero weights $k_i=k(i)$ occuring in k. Let $\{\rho_i\}_{i\in I}$ be a k-vector space basis of $\mathfrak{m}^{e-1}$. Since k is perfect $\{V^j[\rho_i]\}_{i\in I}$ is a k-vector space basis for $V^j[\mathfrak{m}^{e-1}]$ ($\subset W_n(\mathfrak{m}^{e-1})$) for all j. Then $\{V^j[\rho_i]\cdot e(1,k,\mathcal{P})\}_{i\in I,e(1,k,\mathcal{P})\in\mathcal{M}_{<p^n}}$ is a basis of the A_k-action on primitive basic Witt differentials with coefficients in $V^j[\mathfrak{m}^{e-1}]$, for all $j\in\{0,\dots,n-1\}$ via $\alpha\cdot\omega=\alpha^{p^n}\cdot\omega$ (compare Prop. 2.2 and Prop. 2.3 and its proof in [D-L-Z]; it also applies to the F-action of Laurent polynomial algebras A_k). Likewise $\{d(V^j[\rho_i]e(1,k,\mathcal{P}))\}_{i\in I,e(1,k,\mathcal{P})\in\mathcal{M}_{<p^n}}$ is a basis of the A_k-action on d(primitive basic Witt differentials with coefficients in $V^j[\mathfrak{m}^{e-1}]$) for j fixed, $j\in\{0,\dots,n-1\}$ via $\alpha d\omega=\alpha^{p^n}d\omega=d\alpha^{p^n}\omega$.

Let $\mathcal{M}_{l,n}$ be the k-vector space of primitive basic Witt differentials in degree $r-1$ with coefficients in $W_{n-l}(\mathfrak{m}^{e-1})$ and let $\mathcal{M}_{l,n}(j)$ be the subspace of $\mathcal{M}_{l,n}$ of those differentials with coefficients in $V^j[\mathfrak{m}^{e-1}]\subset W_{n-l}(\mathfrak{m}^{e-1})$, $j=0,\dots,n-l-1$. Then $\{dV^l(V^j[\rho_i]e(1,k,\mathcal{P}))\}_{i\in I,e(1,k,\mathcal{P})\in\mathcal{M}_{<p^n}}$ is a basis of the A_k-action on $dV^l(\mathcal{M}_{l,n}(j))$ via $\alpha dV^l\omega=\alpha^{p^{n-l}}dV^l\omega=dV^l\alpha^{p^n}\omega$. The isomorphism (1.9.6) shows that for all $m\geq 0$

$$B_{m+n}\otimes_{A_{m+n},\varphi^m}(W_n\Omega^r_{\mathfrak{m}^{e-1}A/R})^{exact}_{frac}\cong B_k\otimes_{A_k,F^m}(W_n\Omega^r_{\mathfrak{m}^{e-1}A/R})^{exact}_{frac} \tag{1.9.7}$$

Then $B_k\otimes_{A_k,F^{n-l}}(dV^l\mathcal{M}_{l,n})\cong dV^l(B_k^{p^n}\otimes_{A_k^{p^n}}\mathcal{M}_{l,n})$ and $\{dV^l(V^j[\rho_i]e(1,k,\mathcal{P}))\}_{i\in I,e(1,k,\mathcal{P})\in\mathcal{M}_{<p^n}}$ is a basis of the B_k-action on $B_k\otimes_{A_k,F^{n-l}}dV^l(\mathcal{M}_{l,n}(j))$ for fixed j.

Summarizing, we have isomorphisms

$$
\begin{aligned}
B_{m+n} \otimes_{A_{m+n},\varphi^m} (W_n\Omega^r_{\mathfrak{m}^{e-1}A/R})^{exact}_{frac} &\cong B_{m+n} \otimes_{A_{m+n},\varphi^m} \left(\sum_{l=0}^{n-1} dV^l(\mathcal{M}_{l,n})\right) \\
&\cong \sum_{l=0}^{n-1}(B_{m+n} \otimes_{A_{m+n},\varphi^m} dV^l(\mathcal{M}_{l,n})) \\
&\cong \sum_{l=0}^{n-1} B_k \otimes_{A_k,F^{n-l}} dV^l(\mathcal{M}_{l,n}) \\
&\cong \sum_{l=0}^{n-1} dV^l(B_k^{p^n} \otimes_{A_k^{p^n}} \mathcal{M}_{l,n}) \qquad (1.9.8)
\end{aligned}
$$

(choose $m := n - l$ for each l for the penultimate isomorphism). Then we have proven the following

LEMMA 1.10. *For $z \in B_{m+n} \otimes_{A_{m+n}} (W_n\Omega^r_{\mathfrak{m}^{e-1}A/R})^{exact}_{frac}$ we have a representation as*

$$
z = \sum_{l=0}^{n-1} dV^l \left(\sum_{e(1,k,\mathcal{P})\in\mathcal{M}_{<p^n}} \left(\sum_{j=0}^{n-l-1} \sum_{i\in I} V^j([\rho_i])[b^{p^n}_{i,l,j,k,\mathcal{P}}] \right) e(1,k,\mathcal{P}) \right)
$$

with uniquely determined elements $b_{i,l,j,k,\mathcal{P}} \in B_k$ and where $\{\rho_i\}_{i\in I}$ is a k-basis of $\mathfrak{m}^{e-1}$ as before, hence $\{V^j[\rho_i]\}_{i\in I}$ is a basis of $V^j[\mathfrak{m}^{e-1}]$ as a k-vector subspace in $W_{n-l}(\mathfrak{m}^{e-1})$.

F maps an element $z = \sum_{l=0}^{n-1} dV^l(\beta_l)$ as above to $z' = \sum_{l=1}^{n-1} dV^{l-1}(\beta_l)$, using the formula $FdV = d$ and that $Fd\beta_0$ vanishes because F annihilates $W_n(\mathfrak{m}^{e-1})$. Now we are looking at a particular summand

$$
dV^l\left(V^j([\rho_i])[b^{p^n}_{i,l,j,k,\mathcal{P}}]e(1,k,\mathcal{P})\right)
$$

It is easy to see that $b^{p^n}_{i,l,j,k,\mathcal{P}}e(1,k,\mathcal{P})$ can be written as $g_{i,l,j,k,\mathcal{P}} \cdot \omega(k,\mathcal{P})$, where $\omega(k,\mathcal{P})$ is a logarithmic differential (a product of $d\log$'s in variables $[T_1],\ldots,[T_d]$) depending only on $(k,\mathcal{P})$ and $g_{i,l,j,k,\mathcal{P}} \in B_k$ (use that $d[T]^s = \frac{[T]^s d\log[T]}{s}$ for $p \nmid s$ and $F^r d[T] = [T]^{p^r} d\log[T]$). Then

$$
V^j([\rho_i])[b^{p^n}_{i,l,j,k,\mathcal{P}}]e(1,k,\mathcal{P}) = V^j([\rho_i g^{p^j}_{i,l,j,k,\mathcal{P}}])\omega(k,\mathcal{P})
$$

Then, for fixed j and i, F maps (using $F\omega = \omega$)

$$
\sum_{l=0}^{n-1-j} dV^{l+j}[\rho_i g^{p^j}_{i,l,j,k,\mathcal{P}}]\omega(k,\mathcal{P})
$$

to

$$\sum_{l=1}^{n-1-j} dV^{l+j-1}[\rho_i g^{p^j}_{i,l,j,k,\mathcal{P}}]\omega(k,\mathcal{P}) = \sum_{l=1}^{n-1-j} dV^{l-1}(V^j[\rho_i]\cdot g_{i,l,j,k,\mathcal{P}})\omega(k,\mathcal{P})$$
$$= \sum_{l=1}^{n-1-j} dV^{l-1}(V^j[\rho_i][b^{p^n}_{i,l,j,k,\mathcal{P}}]e(1,k,\mathcal{P}))$$

Note that $dV^j[\rho_i g^{p^j}_{i,l,j,k,\mathcal{P}}]$ (the case $l=0$) vanishes under F because $d(V^{j-1}[\rho_i]\cdot [g^p_{i,l,j,k,\mathcal{P}}])=0$. So F maps

$$\sum_{l=1}^{n-1-j} dV^{l+j}[\rho_i\cdot g^{p^j}_{i,l,j,k,\mathcal{P}}]\omega(k,\mathcal{P})$$

to

$$\sum_{l=1}^{n-1-j} dV^{l+j-1}[\rho_i][g^{p^j}_{i,l,j,k,\mathcal{P}}]\cdot\omega(k,\mathcal{P})$$

Now let us first look at the case $j=0$ and consider an element

$$z = d([\alpha]\cdot[g])\cdot\omega$$

$\alpha\in\mathfrak{m}^{e-1}$, $g\in B_k$, ω a logarithmic differential satisfying $F\omega=\omega$. Then

$$\begin{aligned} z &= d([1]+[\alpha][g])\omega \\ &= d([1+\alpha g])\omega + \sum_{l=1}^{n} dV^l([x_l])\omega \quad \operatorname{mod} \operatorname{Fil}^{n+1} \end{aligned}$$

where $x_l = S_l([1],[\alpha g])$ and S_l is the polynomial defining the l-component of the sum of two Witt vectors. It is known that $S_0(\underline{X},\underline{Y}) = X_0+Y_0$, $S_1(\underline{X},\underline{Y}) = X_1+Y_1+\frac{1}{p}(X_0^p+Y_0^p-(X_0+Y_0)^p)$. We do not need to know S_n for $n\geq 2$. We see that $x_1 = S_1([1],[\alpha g]) = -\alpha g$ and get mod Fil^{n+1}

$$d([1]+[\alpha][g]) = d([1+\alpha g]) + dV([-\alpha g]) + \sum_{l=2}^{n} dV^l[x_l]$$

Now $F[\alpha]=[\alpha]^p=0$, so we get, using $FdV=d$

$$\begin{aligned} 0 &= Fd([1+\alpha g]) + d([-\alpha g]) + \sum_{l=1}^{n-1} dV^l[x_{l+1}] \\ &= d\log[1+\alpha g] + d([-\alpha g]) + \sum_{l=1}^{n-1} dV^l[x_{l+1}] \end{aligned}$$

because

$$Fd([1+\alpha g]) = [1+\alpha g]^{p-1}d([1+\alpha g]) = d\log([1+\alpha g])$$

since $[1+\alpha g]^p = 1$. Hence

$$d\log[1+\alpha g] = -d([-\alpha g]) - \sum_{l=1}^{n-1} dV^l[x_{l+1}]$$

Since $d\log[1+\alpha g]$ is invariant under F, the right hand side is invariant – modulo $\mathrm{Fil}^{n-1}W_n\Omega^r_{B/R}$ – under F as well. This implies, using Lemma 1.10, that $x_l = S_l([1],[\alpha g]) = -\alpha g$ for $l = 2$ and then by induction for all l. Returning to our element z we finally have, since $Fz = 0$ and $F\omega = \omega$,

$$d\log([1+\alpha g])\omega = (-\sum_{l=1}^{n-1} dV^l[-\alpha g] - d[-\alpha g])\omega \tag{1.11}$$

Since $(1+\alpha g)(1-\alpha g) = 1$ (because $\alpha^2 = 0$) we have

$$d\log([1+\alpha g]) = -d\log([1-\alpha g])$$

and hence (1.11) becomes

$$\begin{aligned} d\log([1+\alpha g])\omega &= \left(\sum_{l=1}^{n-1} dV^l[\alpha g] + d[\alpha g]\right)\omega \\ &= \left(\sum_{l=0}^{n-1} dV^l[\alpha g]\right)\omega \end{aligned}$$

This shows that the right hand side is a logarithmic differential η satisfying $F\eta = \eta$. We have seen that for $\rho \in \mathfrak{m}^{e-1}, g \in B_k$

$$[1] + [\rho \cdot g] = [1+\rho g] + V[-\rho g] + \sum_{j=2}^{\infty} V^j[-\rho g]$$

This implies

$$dV^l[\rho g] = dV^l([1] + [\rho g]) = dV^l[1+\rho g] + \sum_{j \geq l+1} dV^j[-\rho g]$$

or

$$dV^l[1+\rho g] = dV^l[\rho g] - \sum_{j=l+1}^{\infty} dV^j[-\rho g]$$

Replacing g by g^{p^l} yields

$$dV^l[1+\rho g^{p^l}] = dV^l[\rho g^{p^l}] - \sum_{j=l+1}^{\infty} dV^j[-\rho g^{p^l}] \tag{1.12}$$

Since $dV^{l-1}[\rho g^{p^l}] = 0$ we have

$$FdV^l[1+\rho g^{p^l}] = -\sum_{j=l}^{\infty} dV^j[-\rho g^{p^l}] = dV^{l-1}[1+\rho g^{p^l}]$$

which is invariant under F, because the infinite sum is invariant under F. Then

$$F^{l+1}dV^l[1+\rho g^{p^l}] = Fd[1+\rho g^{p^l}] = d\log[1+\rho g^{p^l}] = -\sum_{j=l}^{\infty} dV^j[-\rho g^{p^l}] \quad (1.13)$$

This shows that under the assumption $Fz = z$ modulo Fil^n

$$\sum_{l=0}^{n-1-j} dV^{l+j}[\rho_i g^{p^j}_{i,l,j,k,\mathcal{P}}]\omega(k,\mathcal{P})$$

is a logarithmic differential modulo Fil^n because $\rho_i g^{p^j}_{i,l,j,k,\mathcal{P}}$ does not depend on l. Using the uniqueness statement in Lemma 1.10. we conclude that

$$\ker(1-F|\ker\pi_\bullet) \subset W_\bullet\Omega^r_{B/R,\log}$$

This shows that

$$W_\bullet\Omega^r_{B/R,\log} = \ker(W_\bullet\Omega^r_{B/R} \xrightarrow{1-F} W_\bullet\Omega^r_{B/R})$$

and finishes the proof of Theorem 1.9. □

Now we can define relative syntomic complexes. As at the beginning of this section, let R be artinian local with perfect residue field k of char $p > 0$. Let $X/\mathrm{Spec}\, R$ be smooth, admitting a lifting $X_\bullet$ as an ind-scheme over $\mathrm{Spec}\, W_\bullet(R)$. Assume there exists a compatible system of embeddings $i_n : X_n \to Z_n$ into Witt lifts Z_n which satisfy the properties of [L-Z1] Definition 3.3. The i_n factorise through a compatible system of PD-envelopes D_n. One obtains a compatible system of quasiisomorphisms

$$\mathcal{F}^r\Omega^\bullet_{X_n/W_n(R)} \xleftarrow{\cong} \mathrm{Fil}^r\Omega^\bullet_{D_n/W_n(R)} \xrightarrow{\cong} N^rW_n\Omega^\bullet_{X/R}$$

and hence an isomorphism of procomplexes

$$\Sigma : \mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)} \to N^rW_\bullet\Omega^\bullet_{X/R} \quad (1.14)$$

in $D_{\mathrm{pro,Zar}}(X)$ resp $D_{\mathrm{pro,et}}(X)$.
To construct Σ in general, one chooses a covering $\{X(i) = \mathrm{Spec}\, A_i\}_{i\in I}$ of X such that A_i is étale over $R[T_1,\dots,T_d]$. Since $X \hookrightarrow X_n$ is a nilpotent embedding, there exists a covering $\{X_n(i) = \mathrm{Spec}\, A_{n,i}\}_{i\in I}$ of X_n such that $A_{n,i}$ is étale over $W_n(R)[T_1,\dots,T_d]$ and $A_{n,i}\times_{W_n(R)} W_{n-1}(R) = A_{n-1,i}$, in particular $A_{n,i}\times_{W_n(R)} R = A_i$. Using [L-Z1] Prop. 3.2, the $\{A_{n,i}\}_n$ form a

compatible system of Frobenius lifts, in particular of Witt lifts for all $i \in I$. For $X_n(i_1,\dots,i_s) = X_n(i_1) \cap \dots \cap X_n(i_s)$ and $Z_n(i_1,\dots,i_s) = X_n(i_1) \times_{W_n(R)} \dots \times_{W_n(R)} X_n(i_s)$, the product embeddings $X_n(i_1,\dots,i_s) \to Z_n(i_1,\dots,i_s)$ with associated PD-envelopes $D_n(i_1,\dots,i_s)$ are embeddings into Witt lifts and induce compatible morphisms of simplicial schemes $X^\bullet \to X_n^\bullet \to D_n^\bullet \to Z_n^\bullet$, hence the isomorphisms (1.7.1) are compatible and induce again an isomorphism (1.14)

$$\Sigma : \mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)} \to N^rW_\bullet\Omega^\bullet_{X/R}$$

of procomplexes in $D_{\mathrm{pro,Zar}}(X)$ resp $D_{\mathrm{pro,et}}(X)$. This completes the proof of Theorem 0.2.

In the following we always assume $r < p$. Using the composite map of $1 - \mathrm{Fr}$ with Σ:

$$\mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)} \overset{\Sigma}{\underset{\sim}{\longrightarrow}} N^rW_\bullet\Omega^\bullet_{X/R} \overset{1-\mathrm{Fr}}{\longrightarrow} W_\bullet\Omega^\bullet_{X/R}$$

we can define

$$\tilde{\sigma}_{X_\bullet}(r) = \mathrm{cone}\left(\mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)} \overset{1-\mathrm{Fr}}{\longrightarrow} W_\bullet\Omega^\bullet_{X/R}\right)[-1].$$

This complex is denoted by $\sigma^I_X(r)$ in [B-E-K1]. It plays the role of a technical variant of the syntomic complex $\sigma_{X_\bullet}(r)$ we are going to define now. Consider the composite map of associated procomplexes:

$$\Omega^{\geq r}_{X_\bullet/W_\bullet(R)} \longrightarrow \mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)} \overset{(1-\mathrm{Fr})\circ\Sigma}{\longrightarrow} W_\bullet\Omega^\bullet_{X/R}$$

which is also denoted by $1 - \mathrm{Fr}$. Here the first arrow is the canonical inclusion of complexes.

Definition 1.15.

$$\sigma_{X_\bullet}(r) = \mathrm{cone}\left(\Omega^{\geq r}_{X_\bullet/W_\bullet(R)} \overset{1-\mathrm{Fr}}{\longrightarrow} W_\bullet\Omega^\bullet_{X/R}\right)[-1]$$

is the relative syntomic complex of the ind-scheme $X_\bullet$ on $(X)_{\mathrm{et}}$ i.e. in $D_{\mathrm{pro,et}}(X)$.

Let $\mathcal{M}(r) = \mathrm{cone}(\Omega^{\geq r}_{X_\bullet/W_\bullet(R)} \to \mathcal{F}^r\Omega_{X_\bullet/W_\bullet(R)})[-1]$. Theorem 1.9 yields an exact triangle

$$\mathcal{M}(r) \longrightarrow \sigma_{X_\bullet}(r) \longrightarrow W_\bullet\Omega^r_{X/R,\log}[-r] \overset{+1}{\longrightarrow}$$

in $D_{\mathrm{pro,et}}(X)$ and we have

$$\begin{aligned} \mathcal{M}(r) &= \mathrm{cone}\left(\Omega^{\geq r}_{X_\bullet} \longrightarrow \mathcal{F}^r\Omega_{X_\bullet/W_\bullet(R)}\right)[-1] \\ &= \mathcal{F}^r\Omega^{<r}_{X_\bullet/W_\bullet(R)}[-1] \end{aligned}$$

Hence we get the following Theorem in analogy to [B-E-K1], Theorem 5.4:

Theorem 1.16 (Fundamental triangle). *There is an exact triangle in* $D_{\mathrm{pro,et}}(X)$*:*

$$\mathcal{F}^r\Omega^{<r}_{X_\bullet/W_\bullet(R)}[-1] \longrightarrow \sigma_{X_\bullet}(r) \longrightarrow W_\bullet\Omega^r_{X/R,\log}[-r] \xrightarrow{+1}$$

Apply $\tau_{\leq r}R\varepsilon_*$, where $\varepsilon : X_{\mathrm{et}} \to X_{\mathrm{Nis}}$, to this triangle and use the same argument for the Nisnevich versions of [B-E-K1] Theorem 5.4 to obtain an exact triangle in $D_{\mathrm{pro,Nis}}(X)$.

$$\mathcal{F}^r\Omega^{<r}_{X_\bullet/W_\bullet(R)}[-1] \longrightarrow \sigma_{X_\bullet,\mathrm{Nis}}(r) \longrightarrow W_\bullet\Omega^r_{X/R,\log,\mathrm{Nis}}[-r] \xrightarrow{+1}$$

where $\sigma_{X_\bullet,\mathrm{Nis}}(r) := \tau_{\leq r}R\varepsilon_*\sigma_{X_\bullet}(r)$ and $W_\bullet\Omega^r_{X/R,\log,\mathrm{Nis}} := \varepsilon_* W_\bullet\Omega^r_{X/R,\log,\mathrm{et}}$. We can also prove the analogue of Theorem 6.1 in [B-E-K1]. The statement holds in the étale and Nisnevich topology.

Theorem 1.17. *The connecting homomorphism*

$$\alpha : W_\bullet\Omega^r_{X/R,\log}[-r] \longrightarrow \mathcal{F}^r\Omega^{<r}_{X_\bullet/W_\bullet(R)}$$

resulting from the fundamental triangle is equal to the composite map

$$\beta : W_\bullet\Omega^r_{X/R,\log}[-r] \longrightarrow N^rW_\bullet\Omega^\bullet_{X/R} \xrightarrow{\sim} \mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)} \longrightarrow \mathcal{F}^r\Omega^{<r}_{X_\bullet/W_\bullet(R)}.$$

Proof. The proof is very similar to the proof of Theorem 6.1 in [B-E-K1]. From the definition of $\sigma_{X_\bullet}(r)$ we get a morphism in $D_{\mathrm{pro,et}}(X)$

$$\sigma_{X_\bullet}(r) \longrightarrow \Omega^{\geq r}_{X_\bullet/W_\bullet(R)}.$$

Define $\sigma'_{X_\bullet}(r) = \mathrm{cone}(\sigma_{X_\bullet}(r) \longrightarrow \Omega^{\geq r}_{X_\bullet/W_\bullet(R)})[-1]$. The morphism $\sigma_{X_\bullet}(r) \to W_\bullet\Omega^r_{X/R,\log}[-r]$ in the fundamental triangle induces a morphism

$$\sigma'_{X_\bullet}(r) \longrightarrow W_\bullet\Omega^r_{X/R,\log}[-r].$$

Then we have a chain of isomorphisms in $D_{\mathrm{pro}}(X)$:

$$\begin{aligned} \sigma'_{X_\bullet}(r) &\xrightarrow{\sim} \mathrm{cone}\left(\tilde\sigma_{X_\bullet}(r) \longrightarrow \mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)}\right)[-1] \\ &\xrightarrow{\sim} \mathrm{cone}\left(\mathrm{cone}\left(N^rW_\bullet\Omega^\bullet_{X/R} \xrightarrow{1-\mathrm{Fr}} W_\bullet\Omega^\bullet_{X/R}\right)[-1] \longrightarrow N^rW_\bullet\Omega^\bullet_{X/R}\right)[-1] \\ &\xleftarrow{\sim} \Sigma(r) := \mathrm{cone}\left(W_\bullet\Omega^\bullet_{X/R,\log}[-r] \longrightarrow N^rW_\bullet\Omega^\bullet_{X/R}\right)[-1] \end{aligned}$$

Then the proof of the Theorem follows from the following proposition: □

Proposition 1.18. *There is an exact triangle*

$$\mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)}[-1] \longrightarrow \sigma'_{X_\bullet}(r) \longrightarrow W_\bullet\Omega^r_{X/R,\log}[-r] \xrightarrow{+1}$$

fitting into a commutative diagram of exact triangles

$$\begin{array}{ccccccc}
N^rW_\bullet\Omega^\bullet_{X/R}[-1] & \longrightarrow & \Sigma(r) & \longrightarrow & W_\bullet\Omega^r_{X/R,\log}[-r] & \xrightarrow{+1} \\
\uparrow\cong & & \uparrow(*) & & \uparrow= & \\
\mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)}[-1] & \longrightarrow & \sigma'_{X_\bullet}(r) & \longrightarrow & W_\bullet\Omega^r_{X/R,\log}[-r] & \xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow= & \\
\mathcal{F}^r\Omega^{<r}_{X_\bullet/W_\bullet(R)}[-1] & \longrightarrow & \sigma_{X_\bullet}(r) & \longrightarrow & W_\bullet\Omega^r_{X/R,\log}[-r] & \xrightarrow{+1}
\end{array}$$

where $(*)$ *is the composite of the previous isomorphisms and the lower exact triangle is the fundamental triangle.*

The proof of the Proposition is the same as for Proposition 6.3 in [B-E-K1]. It implies Theorem 1.17.
For a smooth projective variety Y/k with lifting $Y_n/W_n(k)$ we will also work with the syntomic complex $\sigma_{Y_n}(r)$ at finite level. Our definition differs from the one in [K2] Definition 1.6. But using Proposition 4.4 in [L-Z2] it is easy to see that $\sigma_{Y_\bullet/W_\bullet(k)}(r)$ and the procomplex in [B-E-K1], Definition 4.2 are quasiisomorphic.

Proposition 1.19. *Let*

$$\mathcal{M}_n := \left[W_n\Omega^r_{Y/k,\log} + V^{n-1}\Omega^r_{Y/k} \xrightarrow{\mathrm{d}} \mathrm{Fil}^{n-1}W_n\Omega^{r+1}_{Y/k} \xrightarrow{\mathrm{d}} \mathrm{Fil}^{n-1}W_n\Omega^{r+2}_{Y/k} \xrightarrow{\mathrm{d}} \cdots\right][-r]$$

Then there is an exact triangle on (Y_{et})

$$0 \longrightarrow \mathcal{M}_n \longrightarrow N^rW_n\Omega^\bullet_{Y/k} \xrightarrow{1-\mathrm{Fr}} W_{n-1}\Omega^\bullet_{Y/k} \longrightarrow 0.$$

Proof. It follows from the proof of Theorem 1.9 that $1-\mathrm{Fr}$ is bijective in degrees $< r$ and surjective in degrees $\geq r$. Finally it follows from [B-E-K1] Lemma 4.4 and [Il] I Lemma 3.30 that in degrees $> r$ the kernel of $1-\mathrm{Fr}$ is $\mathrm{Fil}^{n-1}W_n\Omega^\bullet_{Y/k}$. Since $(1-F)\mathrm{d}V^{n-1}\Omega^{r-1}_{Y/k} = \mathrm{d}V^{n-2}\Omega^{r-1}_{Y/k} \subset W_{n-1}\Omega^r_{Y/k}$. It follows from [Il] I 5.7.2 that the kernel of $1-F$ in degree r is $W_n\Omega^r_{Y/k,\log} + V^{n-1}\Omega^r_{Y/k}$, as stated. □

Note that we have an injection $W_n\Omega^r_{Y/k,\log} \hookrightarrow \mathcal{H}^r(\mathcal{M}_n)$.

Definition 1.20. *The syntomic complex* $\sigma_{Y_n}(r)$ *is defined as follows in* $D(Y_{\mathrm{et}})$*:*

$$\sigma_{Y_n}(r) = \mathrm{cone}\left(\Omega^{\geq r}_{Y_n/W_n(k)} \longrightarrow \mathcal{F}^r\Omega^\bullet_{Y_n/W_n(k)} \xrightarrow{\sim} N^rW_n\Omega^\bullet_{Y/k} \xrightarrow{1-\mathrm{Fr}} W_{n-1}\Omega^\bullet_{Y/k}\right)[-1]$$

This is the finite level version of Definition 1.15. for $R = k$. It follows from the definitions and Proposition 1.19. that one has an exact triangle

$$\mathcal{F}^r\Omega^{<r}_{Y_n/W_n(k)}[-1] \longrightarrow \sigma_{Y_n}(r) \longrightarrow \mathcal{M}_n \xrightarrow{+1} \tag{1.21}$$

We have $\mathcal{H}^j(\sigma_{Y_n}(r)) = \mathcal{H}^j\mathcal{M}$ in degrees $> r$ and an exact sequence

$$0 \longrightarrow p\Omega^{r-1}_{Y_n}/p^2 \mathrm{d}\Omega^{r-1}_{Y_n} \longrightarrow \mathcal{H}^r(\sigma_{Y_n}(r)) \longrightarrow \mathcal{H}^r(\mathcal{M}_n) \longrightarrow 0. \tag{1.22}$$

For $\varepsilon : (Y)_{\mathrm{et}} \to (Y)_{\mathrm{Nis}}$ apply again $\tau_{\leq r} R\varepsilon_*$ to 1.23 to get the following exact triangle in $D(Y_{\mathrm{Nis}})$

$$0 \longrightarrow \mathcal{F}^r \Omega^{<r}_{Y_n/W_n(k)}[-1] \longrightarrow \sigma_{Y_n,\mathrm{Nis}}(r) \xrightarrow{\varphi} \mathcal{P}[-r] \longrightarrow 0 \tag{1.23}$$

where $\sigma_{Y_n,\mathrm{Nis}}(r) := \tau_{\leq r} R\varepsilon_* \sigma_{Y_n}(r)$ and $\mathcal{P}$ is a Nisnevich-sheaf which contains $\varepsilon_* W_n \Omega^r_{Y/k,\log} = W_n \Omega^r_{Y/k,\log,\mathrm{Nis}}$ (compare [B-E-K1] Proposition 2.4.1) as a subsheaf.

2 Relative motivic complexes

Let $\{Y_n/W_n(k)\}_n$ be a projective smooth formal scheme and let $\mathbb{Z}_{Y_1}(r)$, for $r < p$, be the Suslin-Voevodsky complex of Y_1/k [S-V]. Bloch-Esnault-Kerz have defined a motivic procomplex $\mathbb{Z}_{Y_\bullet}(r)$ in $D_{\mathrm{pro,Nis}}(Y_1)$ by

$$\mathbb{Z}_{Y_\bullet}(r) = \mathrm{cone}\left(\sigma_{Y_\bullet,\mathrm{Nis}}(r) \oplus \mathbb{Z}_{Y_1}(r) \overset{\varphi \oplus -\log}{\longrightarrow} W_\bullet \Omega^r_{Y_1,\log,\mathrm{Nis}}[-r]\right)[-1] \tag{2.1}$$

where φ is the map from the fundamental triangle (Theorem 1.16.) and log is the composite map

$$\mathbb{Z}_{Y_1}(r) \longrightarrow \mathcal{H}^r\left(\mathbb{Z}_{Y_1}(r)\right)[-r] = \mathcal{K}^{\mathrm{Mil}}_{Y_1,r}[-r] \overset{\mathrm{d}\log[\;]}{\longrightarrow} W_\bullet \Omega^r_{Y_1,\log,\mathrm{Nis}}[-r] \tag{2.2}$$

(see [B-E-K1] (7.4)).
Now we fix $m \in \mathbb{N}$ and define $X := Y_m$. Then at finite level $\mathbb{Z}_X(r)$ is defined as follows on $(X)_{\mathrm{Nis}}$

$$\mathbb{Z}_X(r) = \mathrm{cone}\left(\sigma_{X,\mathrm{Nis}}(r) \oplus \mathbb{Z}_{Y_1}(r) \overset{\varphi \oplus (-\log)}{\longrightarrow} \mathcal{P}[-r]\right)[-1] \tag{2.3}$$

where φ is the map in (1.23) and log is defined as before using the injection $W_m \Omega^r_{Y,\log,\mathrm{Nis}} \hookrightarrow \mathcal{P}$. The long exact cohomology sequence associated to 2.3 yields an exact sequence in degree r:

$$0 \longrightarrow \mathcal{H}^r(\mathbb{Z}_X(r)) \longrightarrow \mathcal{H}^r(\sigma_{X,\mathrm{Nis}}(r)) \oplus \mathcal{H}^r(\mathbb{Z}_{Y_1}(r)) \overset{\varphi \oplus (-\log)}{\longrightarrow} \mathcal{P} \longrightarrow 0. \tag{2.4}$$

The exact sequences 1.22, 1.23 and 2.4 yield the upper exact sequence in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p\Omega^{r-1}_{X/W_m(k)}/p^2 \mathrm{d}\Omega^{r-2}_{X/W_m(k)} & \longrightarrow & \mathcal{H}^r(\mathbb{Z}_X(r)) & \longrightarrow & \mathcal{H}^r(\mathbb{Z}_{Y_1}(r)) & \longrightarrow & 0 \\
 & & =\uparrow & & \uparrow & & \uparrow\cong & & \\
0 & \longrightarrow & p\Omega^{r-1}_{X/W_m(k)}/p^2 \mathrm{d}\Omega^{r-2}_{X/W_m(k)} & \longrightarrow & \mathcal{K}^{\mathrm{Mil}}_{X,r} & \longrightarrow & \mathcal{K}^{\mathrm{Mil}}_{Y_1,r} & \longrightarrow & 0
\end{array} \tag{2.5}$$

where the bottom row is the exact sequence shown in [B-E-K1], Theorem 12.3 and the middle vertical arrow is Kato's syntomic regulator map. It is a finite level version of the map $(*)$ in the commutative diagram in [B-E-K1] p. 695 and is constructed similarly as in [K2] Section 3, where Kato constructs a map (using our notation)

$$\mathcal{O}_{Y_{n+1}}^{\times} \to \mathcal{H}^1(Y_1, \mathcal{S}_n(1)_{Y_n})$$

with his definition of the syntomic complexes given in [K2] Definition 1.6. The change of level from $n+1$ to n is due to the fact that the element $p^{-1}\log\overline{(f(a)a^{-p})}$ in [K2] page 216 is only well-defined in $\mathcal{O}_{D_n}$ because multiplication by p on $\mathcal{O}_{D_{n+1}}$ factors through an injection $p : \mathcal{O}_{D_n} \to \mathcal{O}_{D_{n+1}}$. Since we work with a different definition of $\sigma_{Y_n}(r)$ using the de Rham-Witt complex the above level change is unnecessary. In the section after Prop. 2.9 below we make the symbol map explicit in the case $r = 1$. One should read this section in the case $R = k$. The element $\frac{1}{p}\log\frac{F(\tilde{a})}{\tilde{a}^p}$ that occurs there is well-defined in $W_{n-1}(\mathcal{O}_{Y_1})$, where $\tilde{a} = [\lambda](1 + V\eta)$ is in $W_n(\mathcal{O}_{Y_1})$. Hence we get a symbol map (with $X = Y_m$)

$$\mathcal{O}_X^{\times} \to \mathcal{H}^1(\sigma_{X_n,Nis}(1))$$

which induces

$$\mathcal{O}_X^{\times} \otimes \cdots \mathcal{O}_X^{\times} \to \mathcal{H}^r(\sigma_{X,\mathrm{Nis}}(r))$$

Analagous to [K2] Prop 3.2 we show that this map factors through the symbol map in the Milnor K-sheaf $\mathcal{K}_{X,r}^{\mathrm{Mil}} \to \mathcal{H}^r(\sigma_{X,\mathrm{Nis}}(r))$. Similar to [K2] Lemma 3.7.2 one sees that the composite map

$$\mathcal{K}_{X,r}^{\mathrm{Mil}} \to \mathcal{H}^r(\sigma_{X,\mathrm{Nis}}(r)) \to \mathcal{P}$$

is given by $b_1 \otimes \cdots \otimes b_r \mapsto d\log\left[\bar{b}_1\right] \wedge \cdots \wedge d\log\left[\bar{b}_r\right]$ where $\bar{b}_i$ is the reduction of b_i modulo p. Hence the composite map

$$\mathcal{K}_{X,r}^{\mathrm{Mil}} \to \mathcal{H}^r(\sigma_{X,\mathrm{Nis}}(r)) \oplus (\mathcal{K}_{Y_1,r}^{\mathrm{Mil}} = \mathcal{H}^r(\mathbb{Z}_{Y_1}(r))) \xrightarrow{\varphi\oplus(-\log)} \mathcal{P}$$

vanishes and this defines a natural map fitting into the diagram (2.5)

$$\mathcal{K}_{X,r}^{\mathrm{Mil}} \to \mathcal{H}^r(\mathbb{Z}_X(r))$$

The diagram (2.5) implies that

$$\mathcal{H}^r(\mathbb{Z}_X(r)) \cong \mathcal{K}_{X,r}^{\mathrm{Mil}}. \tag{2.6}$$

It follows from the definition that $\mathbb{Z}_X(r)$ has cohomological degree $\leq r$, because $\mathcal{H}^j(\sigma_{X,\mathrm{Nis}}(r)) = \mathcal{H}^j(\mathbb{Z}_{Y_1}(r)) = 0$ for $j > r$ and $\mathcal{H}^r(\sigma_{X,\mathrm{Nis}}(r)) \to \mathcal{P}$ is surjective. Finally it is easy to see that all the properties in [B-E-K1] Proposition 7.2 listed for the procomplex $\mathbb{Z}_{Y_\bullet}(r)$ pass over to $\mathbb{Z}_X(r)$ at finite level except the Kummer triangle Prop. 7.2 (3) which holds only for procomplexes.

In the following, let $R = W_m(k)$ and assume there exists an ind-scheme lifting $X_\bullet/\mathrm{Spec}\ W_\bullet(R)$ of $X = Y_m/R$ which is compatible with $Y_\bullet$ under the base change $R \to k$, i.e. $X_n \times_{W_n(R)} W_n(k) = Y_n$, in particular $X_m \times_{W_m(R)} W_m(k) = Y_m$.

Definition 2.7. *As object in $D_{\mathrm{pro,Nis}}(X)$ the motivic procomplex $\mathbb{Z}_{X_\bullet}(r)$ is defined for $r < p$ as follows:*

$$\mathbb{Z}_{X_\bullet}(r) = \mathrm{cone}\left(\sigma_{X_\bullet,\mathrm{Nis}}(r) \oplus \mathbb{Z}_X(r) \xrightarrow{\varphi\oplus(-\log)} W_\bullet\Omega^r_{X/R,\log,\mathrm{Nis}}[-r]\right)[-1]$$

where φ comes from the fundamental triangle (Theorem 1.16.) for the syntomic procomplex $\sigma_{X_\bullet,\mathrm{Nis}}(r)$ and $\mathbb{Z}_X(r) \xrightarrow{\log} W_\bullet\Omega^r_{X/R,\log,\mathrm{Nis}}[-r]$ is the symbol map into the relative logarithmic de Rham-Witt complex, defined as follows

$$\mathbb{Z}_X(r) \longrightarrow \mathcal{H}^r(\mathbb{Z}_X(r))[-r] = \mathcal{K}^{\mathrm{Mil}}_{X,r}[-r] \xrightarrow{\mathrm{d}\log[\ ]} W_\bullet\Omega^r_{X/R,\log,\mathrm{Nis}}[-r].$$

Here $[\ ]$ is the Teichmüller lift from $\mathcal{O}_X$ to $W_n(\mathcal{O}_X)$, the definition is analogous to [B-E-K1] (7.4).

Proposition 2.8. *The motivic procomplex $\mathbb{Z}_{X_\bullet}(r)$ has support in cohomology degrees $\leq r$. For $r \geq 1$, if the Beilinson-Soulé Conjecture is true, it has support in degrees $[1, r]$.*

Proof. Under the assumptions this holds for $\mathbb{Z}_X(r)$ by [B-E-K1] Prop. 7.2. By definition $\sigma_{X_\bullet,\mathrm{Nis}}(r)$ has support in $[1, r]$; from the definition of $\mathbb{Z}_{X_\bullet}(r)$ we get an exact sequence

$$0 \to \mathcal{H}^r(\mathbb{Z}_{X_\bullet}(r)) \to \mathcal{H}^r(\sigma_{X_\bullet,\mathrm{Nis}}(r)) \oplus \mathcal{H}^r(\mathbb{Z}_X(r)) \to W_\bullet\Omega^r_{X/R,\log,\mathrm{Nis}} \to 0$$

since $\mathcal{H}^r(\sigma_{X_\bullet,\mathrm{Nis}}(r)) \to W_\bullet\Omega^r_{X/R,\log,\mathrm{Nis}}$ is surjective by (1.16.). This proves the proposition. □

Note that the map $\mathrm{d}\log[\ ]$ is an epimorphism in the étale topology because $W_\bullet\Omega^r_{X/R,\log}$ is, by definition, locally generated by symbols. We expect that the corresponding Nisnevich sheaf $W_\bullet\Omega^r_{X/R,\log,\mathrm{Nis}} = \varepsilon_* W_\bullet\Omega^r_{X/R,\log,\mathrm{et}}$ is again generated by symbols. For $R = k$ this is shown in [B-E-K1], Prop 2.4 and [K1] Proposition 1.

Remark. It is easy to see that there is a canonical product structure

$$\mathbb{Z}_{X_\bullet}(r) \otimes^L_{\mathbb{Z}} \mathbb{Z}_{X_\bullet}(r') \longrightarrow \mathbb{Z}_{X_\bullet}(r + r')$$

compatible with the product structures on $\sigma_{X_\bullet}(r)$ and on $\mathbb{Z}_X(r)$. The argument is the same as [B-E-K1] Proposition 7.2 (5). On the other hand, property (3) in Proposition 7.2 does not seem to hold; the cone of the Kummer sequence $\mathbb{Z}_{X_\bullet}(r) \xrightarrow{p^n} \mathbb{Z}_{X_\bullet}(r)$ is likely to be much more complicated.

However, we do get the following analogy of [B-E-K1] Proposition 7.3:

Proposition 2.9 (Fundamental motivic triangle). *There is a unique commutative diagram of exact triangles*

$$\begin{array}{ccccccc}
\mathcal{F}^r\Omega^{<r}_{X_\bullet/W_\bullet(R)}[-1] & \longrightarrow & \mathbb{Z}_{X_\bullet}(r) & \longrightarrow & \mathbb{Z}_X(r) & \longrightarrow \\
\| & & \downarrow & & \downarrow \mathrm{d}\log[\ ] & \\
\mathcal{F}^r\Omega^{<r}_{X_\bullet/W_\bullet(R)}[-1] & \longrightarrow & \sigma_{X_\bullet,\mathrm{Nis}}(r) & \longrightarrow & W_\bullet\Omega^r_{X/R,\log,\mathrm{Nis}}[-r] & \longrightarrow
\end{array}$$

Proof. The right hand side square is homotopy Cartesian by definition, hence the proposition is proven in the same way as Proposition 7.3 in [B-E-K1]. □

Now we look at the special cases $r = 0, 1$:
For $r = 0$, $\sigma_{X_\bullet,\mathrm{Nis}}(r)$ is isomorphic to $W_\bullet\Omega^0_{X/R,\log,\mathrm{Nis}} = \mathbb{Z}/p^\bullet$, hence $\mathbb{Z}_{X_\bullet}(0) = \mathbb{Z}_X(0) = \mathbb{Z}$.
For $r = 1$, we construct a map $\mathcal{K}^{\mathrm{Mil}}_{X_{n,1}}[-1] = \mathcal{O}^*_{X_n}[-1] \to \sigma_{X_n}(1)$ as follows. Assume first that there exists a compatible system $X_n \hookrightarrow Z_n$ into Witt lifts Z_n with PD-envelope D_n as before and induced maps $\mathcal{O}_{D_n} \to W_n(\mathcal{O}_X)$. We have an exact sequence

$$0 \longrightarrow N \longrightarrow \mathcal{O}^*_{Z_n} \longrightarrow \mathcal{O}^*_{X_n} \longrightarrow 1$$

so $\mathcal{O}^*_{X_n}[-1]$ is isomorphic to

$$\begin{array}{ccc} N & \longrightarrow & \mathcal{O}^*_{Z_n} \\ \text{degree } 0 & & \text{degree } 1 \end{array}$$

The complex $\sigma_{X_n}(1)$ is represented by the complex

$$\mathcal{I}_{D_n} \xrightarrow{\mathrm{d}_1} \mathcal{O}_{D_n} \otimes \Omega^1_{Z_n/W_n(R)} \oplus W_{n-1}(\mathcal{O}_X) \xrightarrow{\mathrm{d}_2} \Omega^2_{D_n/W_n(R)} \oplus W_{n-1}\Omega^1_{X/R} \longrightarrow$$

where

$$\begin{array}{rcl} \mathrm{d}_1 : x & \mapsto & (\mathrm{d}x, (F_1 - 1)(x)) \\ \mathrm{d}_2 : (x, y) & \mapsto & (\mathrm{d}x, (F_1 - 1)(x) - \mathrm{d}y) \end{array}$$

and x is identified with its image under $\mathcal{I}_{D_n} \longrightarrow VW_{n-1}(\mathcal{O}_X)$ and $F_1(x = V\eta) = \frac{\text{“}F\text{”}}{p}(V\eta) = \eta$.
We define a map $(N \to \mathcal{O}^*_{Z_n}) \longrightarrow \sigma_{X_n}(1)$

$$\begin{array}{cccc} \text{in degree } 0 & : & N & \longrightarrow & \mathcal{I}_{D_n} \\ & & a & \longmapsto & \log(a) \\ \text{in degree } 1 & : & \mathcal{O}^*_{Z_n} & \longrightarrow & \mathcal{O}_{D_n} \otimes \Omega^1_{Z_n} \oplus W_{n-1}(\mathcal{O}_X) \\ & & a & \longmapsto & \left(\mathrm{d}\log a, \frac{1}{p}\log\frac{F\tilde{a}}{\tilde{a}^p}\right) \end{array}$$

Note that $\tilde{a} = [\lambda](1 + V\eta) \in W_n(\mathcal{O}_X)$ is the image of a under

$$\mathcal{O}^*_{Z_n} \longrightarrow W_n(\mathcal{O}_X)^*$$

($[\lambda]$ is the Teichmüller element of some $\lambda \in \mathcal{O}^*_X$).
Then $F(\tilde{a}) = [\lambda]^p(1 + p\eta)$ and $(\tilde{a})^p = [\lambda]^p(1 + V\eta)^p$ considered as elements in $W_{n-1}(\mathcal{O}_X)$. Then

$$\frac{F(\tilde{a})}{\tilde{a}^p} = \frac{1 + p\eta}{(1 + V\eta)^p}.$$

Because of the uniqueness of η the elements $\frac{1}{p}\log(1+p\eta)$ and $\frac{1}{p}\log(1+V\eta)^p$ are uniquely determined, hence

$$\begin{aligned}\frac{1}{p}\log\frac{F(\tilde{a})}{\tilde{a}^p} &= \frac{1}{p}\log(1+p\eta)-\frac{1}{p}\log(1+V\eta)^p\\ &= \frac{1}{p}\log(1+p\eta)-\log(1+V\eta)\end{aligned}$$

is well-defined.
This defines a map

$$\mathcal{O}^*_{X_\bullet}[-1]\longrightarrow\sigma_{X_\bullet,\mathrm{Nis}}(1)$$

of procomplexes, hence a map

$$\mathcal{O}^*_{X_\bullet}\longrightarrow\mathcal{H}^1\left(\sigma_{X_\bullet,\mathrm{Nis}}(1)\right). \tag{2.10}$$

If there is no global system of embeddings $X_n\to Z_n$ into Witt lifts Z_n one proceeds by simplicial methods as outlined before the definition of $\sigma_{X_\bullet}(r)$ (Def. 1.15.) to construct the map (2.10). We omit the details here.
There is a commutative diagram of Nisnevich sheaves

$$\begin{array}{ccc}\mathcal{O}^*_{X_\bullet} & \longrightarrow & \mathcal{O}^*_X\\ \downarrow & & \downarrow\cong\\ \mathcal{H}^1(\sigma_{X_\bullet,\mathrm{Nis}}(1)) & & \mathcal{H}^1(\mathbb{Z}_X(1))\\ \downarrow & & \downarrow\\ W_\bullet\Omega^1_{X/R,\log,\mathrm{Nis}} & = & W_\bullet\Omega^1_{X/R,\log,\mathrm{Nis}}\end{array} \tag{2.11}$$

which induces a map

$$\mathcal{O}^*_{X_\bullet}\longrightarrow\mathcal{H}^1(\mathbb{Z}_{X_\bullet}(1))$$

by the definition of $\mathbb{Z}_{X_\bullet}(1)$.

Lemma 2.12. *We have a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}0 & \longrightarrow & I_R\mathcal{O}_{X_\bullet}\cong 1+I_R\mathcal{O}_{X_\bullet} & \longrightarrow & \mathcal{H}^1(\mathbb{Z}_{X_\bullet}(1)) & \longrightarrow & \mathcal{H}^1(\mathbb{Z}_X(1)) & \longrightarrow & 0\\ & & \uparrow= & & \uparrow & & \uparrow\cong & & \\ 0 & \longrightarrow & I_R\mathcal{O}_{X_\bullet}\cong 1+I_R\mathcal{O}_{X_\bullet} & \longrightarrow & \mathcal{O}^*_{X_\bullet} & \longrightarrow & \mathcal{O}^*_X & \longrightarrow & 0\end{array}$$

where $1+V(\eta)x\mapsto\log(1+V(\eta)x)$ *is well-defined because p is nilpotent on* $\mathcal{O}_{X_n}$ *and induces the isomorphism* $1+I_R\mathcal{O}_{X_\bullet}\to I_R\mathcal{O}_{X_\bullet}$. *(Recall that* $I_R=VW_{n-1}(R)$*.)*

By assumption $X_n\times_{W_n(R)}R=X$ and so $\mathcal{O}_{X_n}/I_R\mathcal{O}_{X_n}=\mathcal{O}_X$; since I_R is nilpotent we immediately deduce that on units $\mathcal{O}^*_{X_n}/1+I_R\mathcal{O}^*_{X_n}=\mathcal{O}^*_X$, hence the lower sequence is exact. It is a slight generalisation of the p-adic logarithm isomorphism [B-E-K1] (1.3) that the log map is an isomorphism because $I_R\mathcal{O}_{X_n}$ admits a divided power structure and p is nilpotent.

The upper sequence is exact because of the fundamental motivic triangle (Proposition 2.9).
The Lemma implies that $\mathcal{O}^*_{X_\bullet}$ and $\mathcal{H}^1(\mathbb{Z}_{X_\bullet}(1))$ are isomorphic, hence

$$\mathbb{Z}_{X_\bullet}(1) \cong \mathbb{G}_{m/X_\bullet}[-1]. \tag{2.13}$$

The isomorphism 2.13 and the product structure on $\mathbb{Z}_{X_\bullet}(r)$ induce a symbol map (compare the proof of [K2], Proposition 3.2)

$$\mathcal{K}^{\mathrm{Mil}}_{X_\bullet,r} \longrightarrow \mathcal{H}^r(\mathbb{Z}_{X_\bullet}(r)). \tag{2.14}$$

But in the absence of ([B-E-K1], Theorem 12.3) which cannot be extended to a relative setting we cannot expect that 2.14 is an isomorphism.

3 p-adic deformation of motivic Chow groups

Let $X = Y_m/\mathrm{Spec}\ W_m(k)$ as before and $X_\bullet$ be a smooth projective lifting of X to Spec $W_\bullet(R)$, $R = W_m(k)$, which is compatible with $Y_\bullet$ as before. Let $r < p$.

Definition 3.1. *The continuous Chow group of $X_\bullet$ is defined as* $\mathrm{Ch}^r_{\mathrm{cont}}(X_\bullet) := H^{2r}_{\mathrm{cont}}(X, \mathbb{Z}_{X_\bullet}(r))$.

Note that we also work with continuous cohomology.
The fundamental motivic triangle (Proposition 2.9) gives rise to an exact obstruction sequence to the deformation problem lifting a class in $H^{2r}(X, \mathbb{Z}_X(r))$ to a class in $\mathrm{Ch}^r_{\mathrm{cont}}(X_\bullet)$

$$\mathrm{Ch}^r_{\mathrm{cont}}(X_\bullet) \xrightarrow{\partial} H^{2r}(X, \mathbb{Z}_X(r)) \xrightarrow{\mathrm{ob}} H^{2r}_{\mathrm{cont}}(X, \mathcal{F}^r\Omega^{<r}_{X_\bullet}). \tag{3.2}$$

Now we construct crystalline cycle classes on $H^{2r}(X, \mathbb{Z}_X(r))$. We have a canonical map

$$H^{2r}(X, \mathbb{Z}_X(r)) \longrightarrow H^r(X, \mathcal{H}^r(\mathbb{Z}_X(r)) = H^r(X, \mathcal{K}^{\mathrm{Mil}}_r) \xrightarrow{\mathrm{d}\log[\ ]} H^r(X, W\Omega^r_{X/R,\log,\mathrm{Nis}}).$$

The map of complexes (the first map in Theorem 1.9) in $C_{\mathrm{pro,et}}(X)$

$$W_\bullet\Omega^r_{X/R,\log}[-r] \longrightarrow N^rW_\bullet\Omega^\bullet_{X/R}$$

defines a map of complexes in $C_{\mathrm{pro,Nis}}(X)$

$$W_\bullet\Omega^r_{X/R,\log,\mathrm{Nis}}[-r] = \varepsilon_*W_\bullet\Omega^r_{X/R,\log}[-r] \to \varepsilon_*N^rW_\bullet\Omega^\bullet_{X/R} = N^rW_\bullet\Omega^\bullet_{X/R,\mathrm{Nis}}$$

(In the following we omit the subscript 'Nis' as all complexes and cohomology groups are taken in the Nisnevich topology) and yields the refined relative crystalline cycle class map

$$\begin{array}{ccc} H^{2r}(X, \mathbb{Z}_X(r)) & \longrightarrow & H^{2r}_{\mathrm{cont}}(X, N^rW_\bullet\Omega^r_{X/R}) \\ \xi & \longmapsto & c(\xi) \end{array} \tag{3.3}$$

Then the relative crystalline cycle class of ξ is the image $c_{\mathrm{cris}}(\xi)$ of $c(\xi)$ in $H^{2r}_{\mathrm{cont}}(X, W_\bullet\Omega^\bullet_{X/R})$. We have canonical isomorphisms (Theorem 1.2)

$$\begin{aligned} H^i_{\mathrm{cont}}\left(X, N^r W_\bullet\Omega^\bullet_{X/R}\right) &\cong H^i\left(X, \mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)}\right) \\ \text{and} \qquad H^n_{\mathrm{cont}}\left(X, W_\bullet\Omega^\bullet_{X/R}\right) &\cong \varprojlim_m H^n(X, W_m\Omega^\bullet_{X/R}) \\ &\cong H^n_{\mathrm{cris}}(X/W(R)) \\ &\cong H^n_{\mathrm{cont}}\left(X_\bullet, \Omega^\bullet_{X_\bullet/W_\bullet(R)}\right) \end{aligned} \tag{3.4}$$

where the first isomorphism follows from [L-Z1], Corollary 1.14 and the second from the main comparison theorem [L-Z1], Theorem 3.1. Note that in [B-O] §5 the crystalline site/topos and the cohomology of the crystalline structure sheaf is defined for any scheme defined over a PD-scheme S on which p is nilpotent. We apply this to the PD-scheme $S = \mathrm{Spec}\ W_n(R)$ with PD-ideal $VW_{n-1}(R)$ and consider X as an S-scheme via $X \to \mathrm{Spec}\ R \to S$. Then, by definition, $H^i_{\mathrm{cris}}(X/W(R)) = \varprojlim_n H^i_{\mathrm{cris}}(X/W_n(R))$.

DEFINITION 3.5 (Compare [B-E-K1], Definition 8.3).

(1) One says that $c(\xi)$ is Hodge with respect to the lifting $X_\bullet$ if and only if $c(\xi)$ lies in the image of $H^{2r}_{\mathrm{cont}}(X, \Omega^{\geq r}_{X_\bullet})$ in $H^{2r}_{\mathrm{cont}}(X, \mathcal{F}^n\Omega^\bullet_{X_\bullet/W_\bullet(R)}) = H^{2r}_{\mathrm{cont}}(X, N^r W_\bullet\Omega^\bullet_{X/R})$.

(2) One says that $c_{\mathrm{cris}}(\xi)$ is Hodge modulo torsion with respect to the lifting $X_\bullet$ if and only if $c_{\mathrm{cris}}(\xi) \otimes \mathbb{Q}$ lies in the image of $H^{2r}_{\mathrm{cont}}(X, \Omega^{\geq r}_{X_\bullet}) \otimes \mathbb{Q} \to H^{2r}_{\mathrm{cris}}(X/W(R)) \otimes \mathbb{Q}$.

Then we have the following

THEOREM 3.6. *Let $X_\bullet/\mathrm{Spec}\ W_\bullet(R)$ as before, let $\xi \in H^{2r}(X, \mathbb{Z}_X(r))$ and $r < p$. Then*

(1) $c(\xi)$ is Hodge with respect to the lifting $X_\bullet \iff \xi$ lies in the image of ∂ in 3.2.

(2) $c_{\mathrm{cris}}(\xi)$ is Hodge modulo torsion with respect to the lifting $X_\bullet \iff \xi \otimes \mathbb{Q}$ lies in the image of $\partial \otimes \mathbb{Q}$.

Proof. We claim that the canonical map

$$H^{2r}_{\mathrm{cont}}(X, N^r W_\bullet\Omega^\bullet_{X/R}) \longrightarrow H^{2r}_{\mathrm{cont}}\left(X, W_\bullet\Omega^\bullet_{X/R}\right)$$

induced by the map "1" (see Theorem 1.9) has kernel and cokernel killed by a power of p: Indeed, this map can be identified, via Theorem 1.2, with the map

$$H^{2r}_{\mathrm{cont}}(X, \mathcal{F}^r\Omega^\bullet_{X_\bullet/W_\bullet(R)}) \longrightarrow H^{2r}_{\mathrm{cont}}(X, \Omega^\bullet_{X_\bullet/W_\bullet(R)})$$

which is induced by the corresponding map of complexes

$$\begin{array}{ccccccccc}
I_R \otimes_{W_\bullet(R)} \mathcal{O}_{X_\bullet} & \xrightarrow{pd} & I_R \otimes_{W_\bullet(R)} \Omega^1_{X_\bullet/W_\bullet(R)} & \xrightarrow{pd} \cdots \xrightarrow{pd} & I_R \otimes_{W_\bullet(R)} \Omega^{r-1}_{X_\bullet/W_\bullet(R)} & \xrightarrow{d} & \Omega^r_{X_\bullet/W_\bullet(R)} & \xrightarrow{d} & \cdots \\
\downarrow \cdot p^{r-1} & & \downarrow \cdot p^{r-2} & & \downarrow & & \downarrow & & \\
\mathcal{O}_{X_\bullet} & \xrightarrow{d} & \Omega^1_{X_\bullet/W_\bullet(R)} & \xrightarrow{d} \cdots \xrightarrow{d} & \Omega^{r-1}_{X_\bullet/W_\bullet(R)} & \xrightarrow{d} & \Omega^r_{X_\bullet/W_\bullet(R)} & \xrightarrow{d} & \cdots
\end{array}$$

The kernel of this map of complexes is a complex of sheaves annihilated by p^{r-1}, hence its hypercohomology is killed by a power of p. The cokernel is a complex of sheaves that admits a filtration in a way that the successive quotients are complexes with entries of the form $\Omega^j_{X/R}$ or $I_R/pI_R\Omega^j_{X_\bullet/W_\bullet(R)}$. The cohomology of these sheaves is killed by a power of p since p is nilpotent on R. Hence the hypercohomology of the cokernel is killed by a power of p and therefore the map

$$H^{2r}_{\text{cont}}(N^r W_\bullet \Omega^\bullet_{X/R}) \otimes \mathbb{Q} \longrightarrow H^{2r}_{\text{cris}}(X/W(R)) \otimes \mathbb{Q}$$

is an isomorphism. Then the first part (1) implies the second part (2).
The exact sequence 3.2 can be extended to a commutative diagram with exact rows

$$\begin{array}{ccccc}
\mathrm{Ch}^r_{\text{cont}}(X_\bullet) & \xrightarrow{\partial} & H^{2r}(X, \mathbb{Z}_X(r)) & \xrightarrow{\text{ob}} & H^{2r}_{\text{cont}}\left(X, \mathcal{F}^r\Omega^{<r}_{X_\bullet/W_\bullet(R)}\right) \\
\downarrow c & & \downarrow c & & \downarrow = \\
H^{2r}\left(\Omega^{\geq r}_{X_\bullet/W_\bullet(R)}\right) & \longrightarrow & H^{2r}_{\text{cont}}\left(\mathcal{F}^r\Omega_{X_\bullet/W_\bullet(R)}\right) & \longrightarrow & H^{2r}_{\text{cont}}\left(X, \mathcal{F}^r\Omega^{<r}_{X_\bullet/W_\bullet(R)}\right)
\end{array} \tag{3.7}$$

where we have used again the isomorphisms 3.4. By Theorem 1.17. the right hand square commutes. Then the Theorem easily follows. □

Remark 3.8.

(i) *We do not need for the proof that the left vertical arrow is well-defined.*

(ii) *If the Hodge-de Rham spectral sequence of the ind-scheme $X_\bullet$ degenerates, then the map*

$$H^{2r}_{\text{cont}}\left(X, \Omega^{\geq r}_{X_\bullet}\right) \longrightarrow H^{2r}_{\text{cont}}\left(\mathcal{F}^r\Omega^\bullet_{X_\bullet/W(R)}\right)$$

is injective and hence the left vertical arrow is also well-defined.

(iii) *For $r = 1$ we are really dealing with Picard groups. As $\mathbb{Z}_{X_\bullet}(1) = \mathbb{G}_{m/X_\bullet}[-1]$ we have $H^2(X, \mathbb{Z}_{X_\bullet}(1)) = \mathrm{Pic}(X_\bullet)$. The system $\{H^0(X, \mathbb{G}_{m,X_n})\}_n$ $(= \{W_n(R)^*\}_n$ if X is connected) is obviously Mittag-Leffler, hence $\varprojlim_n{}^1 H^0(X, \mathbb{G}_{m,X_n})$ vanishes and we have an isomorphism*

$$\mathrm{Ch}^1_{\text{cont}}(X_\bullet) = H^1_{\text{cont}}(X, \mathbb{G}_{m,X_\bullet}) \cong \varprojlim_n \mathrm{Pic}(X_n)$$

Definition and Corollary 3.9. *Let $r < p$. Let $X = Y_m$, $Y_\bullet$ a formal smooth projective scheme over* Spf$W(k)$. *Let $\xi \in H^{2r}(X, \mathbb{Z}_X(r))$. We say that its refined relative crystalline cycle class $c(\xi)$ is "Hodge" if there exists a smooth, projective lifting $X_\bullet$ of X as ind-scheme over the ind-scheme Spec $W_\bullet(W_m(k))$, compatible with $Y_\bullet$, and such that $c(\xi)$ is "Hodge" with respect to $X_\bullet$. Assume $c(\xi)$ is "Hodge", then ξ deforms to a class on the formal scheme $Y_\bullet$, that is it lies in the image of the map*

$$\mathrm{Ch}^r_{\mathrm{cont}}(Y_\bullet) \longrightarrow \varprojlim_n H^{2r}(Y_n, \mathbb{Z}_{Y_n}(r)) \longrightarrow H^{2r}(X, \mathbb{Z}_X(r)).$$

Proof. By general homological algebra the first arrow is surjective (as stated in [B-E-K1], p697). For any smooth lifting $X_\bullet$ of $X = Y_m$ over Spec $W_\bullet(W_m(k))$ compatible with the formal scheme $Y_\bullet$ under the base change $W_m(k) \longrightarrow k$ there is a base change map of motivic complexes $\mathbb{Z}_{X_\bullet}(r) \longrightarrow \mathbb{Z}_{Y_\bullet}(r)$ inducing $\mathrm{Ch}^r_{\mathrm{cont}}(X_\bullet) \longrightarrow \mathrm{Ch}^r_{\mathrm{cont}}(Y_\bullet)$ through which the map

$$\delta : \mathrm{Ch}^r_{\mathrm{cont}}(X_\bullet) \longrightarrow H^{2r}(X, \mathbb{Z}_X(r))$$

factors. The Corollary follows from this and Theorem 3.6. □

Remark. Note that $H^{2r}(X, \mathbb{Z}_X(r)) \otimes \mathbb{Q} = H^{2r}(Y_1, \mathbb{Z}_{Y_1}(r)) \otimes \mathbb{Q}$, hence we do not get any new information with regard to lifting vector bundles (compare [B-E-K1], Theorem 1.3). The implication in Corollary 3.9, i.e. the lifting property of ξ does not depend on the choice of $X_\bullet$, for which $c(\xi)$ is Hodge.

For an algebraic scheme Z, it is reasonable to define the cohomological Chow group as

$$\mathrm{Ch}^p(Z) := H^p(Z, \mathcal{K}^{\mathrm{Mil}}_p).$$

The graded object $\mathrm{Ch}^*(Z)$ then has a ring structure due to the natural product structure of Milnor K-groups, it is contravariant in Z and coincides with the usual Chow group of codimension p-cycles modulo rational equivalence if Z is regular excellent over an infinite field (see [Ke]). Applying this to $X = Y_m/W_m(k)$ we define

$$\mathrm{Ch}^r(X) := H^r(X, \mathcal{K}^{\mathrm{Mil}}_{X,r}). \tag{3.10}$$

The canonical map $\mathbb{Z}_X(r) \to \mathcal{K}^{\mathrm{Mil}}_{X,r}[-r]$ defines a homomorphism.

$$\pi_r : H^{2r}(X, \mathbb{Z}_X(r)) \longrightarrow H^r(X, \mathcal{K}^{\mathrm{Mil}}_r) = \mathrm{Ch}^r(X)$$

that we already used in the construction of the crystalline cycle class. We want to give a criterion when this map is surjective or bijective.
With our definition of $\mathbb{Z}_X(r)$ it is easy to see that the fundamental motivic triangle for $\mathbb{Z}_{Y_\bullet}(r)$ holds for $\mathbb{Z}_X(r)$ as well: there is an exact sequence

$$0 \longrightarrow \mathcal{F}^r\Omega^{<r}_{X/W_m(k)}[-1] \longrightarrow \mathbb{Z}_X(r) \longrightarrow \mathbb{Z}_{Y_1}(r) \longrightarrow 0. \tag{3.11}$$

It induces the following commutative diagram, by taking hypercohomology of 3.11 and applying [B-E-K1], Theorem 12.3 to get the lower exact sequence in the diagram

$$\begin{array}{ccccccccc}
H^{2r-1}(Y_1,\mathbb{Z}_{Y_1}(r)) & \to & H^{2r-1}(X,\mathcal{F}^r\Omega^{<r}_{X/W_m(k)}) & \to & H^{2r}(X,\mathbb{Z}_X(r)) & \to & H^{2r}(Y_1,\mathbb{Z}_{Y_1}(r)) & \to & H^{2r}(X,\mathcal{F}^r\Omega^{<r}_{X/W_m(k)}) \\
\downarrow\cong & & & & & & & & \\
\mathrm{Ch}^r(Y_1,1) & & \downarrow\alpha & & \downarrow\pi_r & & \cong\downarrow\sigma & & \downarrow\beta \\
\downarrow\cong & & & & & & & & \\
H^{r-1}(Y_1,\mathcal{K}^{\mathrm{Mil}}_{Y_1,r}) & \to & H^r(X,\frac{p\Omega^{r-1}_X}{p^2 d\Omega^{r-2}_X}) & \to & H^r(X,\mathcal{K}^{\mathrm{Mil}}_{X,r}) & \to & H^r(Y_1,\mathcal{K}^{\mathrm{Mil}}_{Y,r}) & \to & H^{r+1}(X,\frac{p\Omega^{r-1}_X}{p^2 d\Omega^{r-2}_X})
\end{array} \tag{3.12}$$

The maps α, β are induced by

$$\mathcal{F}^r\Omega^{<r}_{X/W_m(k)} \longrightarrow \mathcal{H}^{r-1}\mathcal{F}^r\Omega^{<r}_{X/W_m(k)} = \frac{p\Omega^{r-1}_X}{p^2 d\Omega^{r-2}_X}.$$

The isomorphism σ is a standard map (compare [B-E-K1] 7.3). The first isomorphism in the left vertical arrow is shown in [M-V-W], Theorem 19.1, the second is explained in [M], Corollary 5.2 (b).
Let

$$\tau_{\leq r-2}\mathcal{F}^r\Omega^{<r}_{X/W_m(k)} : p\mathcal{O}_X \xrightarrow{pd} p\Omega^1_X \xrightarrow{pd} \cdots \xrightarrow{pd} p\Omega^{r-3}_X \xrightarrow{pd} \mathrm{Ker} pd(\subset p\Omega^{r-2}) \longrightarrow 0.$$

The diagram shows that if $H^{2r}(\tau_{\leq r-2}\mathcal{F}^r\Omega^{<r}_{X/W_m(k)}) = 0$ then π_r is surjective. As the cohomology of each term in the complex $\tau_{\leq r-2}\mathcal{F}^r\Omega^{<r}_{X/W_m(k)}$ vanishes in degrees $> d$ we see that $H^{2r}(\tau_{\leq r-2}\mathcal{F}^r\Omega^{<r}_{X/W_m(k)}) = 0$ for $r > \dim X - 2$ and $H^j(\tau_{\leq r-2}\mathcal{F}^r\Omega^{<r}_{X/W_m(k)}) = 0$ for $j = 2r, 2r-1$ holds for $r = d = \dim X$. In this case π_d is bijective (compare diagram 3.12) Hence we have shown

LEMMA 3.13. *Let* $\mathrm{d} = \dim X/$*Spec* $W_m(k)$. *Then*

$$\pi_{d-1} : H^{2(d-1)}\left(X, \mathbb{Z}_X(d-1)\right) \longrightarrow \mathrm{Ch}^{d-1}(X)$$

is surjective and

$$\pi_d : H^{2d}\left(X, \mathbb{Z}_X(d)\right) \xrightarrow{\sim} \mathrm{Ch}^d(X)$$

is an isomorphism.

In both cases one can give a Hodge-theoretic criterion, following 3.9, for lifting an element $z \in \mathrm{Ch}^?(X)$ $(? = d, d-1)$ to an element in the continuous Chow group $\mathrm{Ch}^?_{\mathrm{cont}}(Y_\bullet)$ by considering its (refined) crystalline cycle class in the cohomology of the relative de Rham-Witt complex. The precise formulation is clear and omitted here. Moreover, Theorem 0.1 (i) and (ii) follows from Corollary 3.9 and the above definitions.

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