Perceived Ambiguity, Ambiguity Attitude and Strategic Ambiguity in Games

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Submitted for the degree of

Doctor of Philosophy in Economics

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October 2018

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Submitted by Lorenz Hartmann to the University of Exeter as a thesis for the degree of Doctor of Philosophy in Economics in October 2018.

I certify that all material in this thesis which is not my own work has been identified and that any material that has previously been submitted and approved for the award of a degree by this or any other University has been acknowledged.

Signature: ....

Abstract

This thesis contributes to the theoretical work on decision and game theory when decision makers or players perceive ambiguity. The first article introduces a new axiomatic framework for ambiguity aversion and provides axiomatic characterizations for important preference classes that thus far had lacked characterizations. The second article introduces a new axiom called Weak Monotonicity which is shown to play a crucial role in the multiple prior model. It is shown that for many important preference classes, the assumption of monotonic preferences is a consequence of the other axioms and does not have to be assumed. The third article introduces an intuitive definition of perceived ambiguity in the multiple prior model. It is shown that the approach allows an application to games where players perceive strategic ambiguity. A very general equilibrium existence result is given. The modelling capabilities of the approach are highlighted through the analysis of examples. The fourth article applies the model from the previous article to a specific class of games with a lattice-structure. We perform comparative statics on perceived ambiguity and ambiguity attitude. We show that more optimism does not necessarily lead to higher equilibria when players have α-Maxmin preferences. We present necessary and sufficient conditions on the structure of the prior sets for this comparative statics result to hold.

The introductory chapter provides the basis of the four articles in this thesis. An overview of axiomatic decision theory, decision-making under ambiguity and ambiguous games is given. It introduces and discusses the most relevant results from the literature.

Acknowledgements

I thank Rebecca for her love and support. I thank Florian Kauffeldt for numerous extensive, instructive and at times heated debates on axiomatic decision theory in our shared office in Heidelberg. I thank Jean Baccelli and Christopher Kops for their expertise during very insightful discussions. I thank my great friend and "long-distance housemate" Ricardo Oliveira for making my time in Exeter so enjoyable. I thank and congratulate Qizhi Wang for his surprisingly good sense of humour (and his expertise in game theory). I thank Haytem Troug and Lorenz Adams for their friendship and support throughout the four years of our study at Exeter University.

I am extremely thankful for the 1+3 ESRC scholarship granted to me by the University of Exeter. Without this financial support I would not have been able to complete, or even begin, this project.

I thank my supervisor Dieter Balkenborg for his time and incontestable expertise whenever the mathematics became complicated. I thank Jürgen Eichberger for making the 1-year visit to his department possible. His constructive critique has been invaluable and was absolutely crucial for this thesis.

Most of all I thank my supervisor David Kelsey for the extensive support throughout my PhD. Without David's guidance as well as trust in me in the first place I would not have started, or finished, this project.

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1 Axiomatic Decision Theory, Decision-Making under Ambiguity and its Application to Games

This chapter provides the basis for the four articles of this thesis. We start with the famous axiomatic representation of expected utility theory by Von Neumann and Morgenstern as well as the insights of De Finetti on subjective probabilities. We then introduce the subjective expected utility theories of Savage as well as Anscombe and Aumann. This is followed by the introduction to two of the main approaches of decision-making under ambiguity: Choquet Expected Utility theory and the Multiple Prior approach. We discuss the concepts of perceived ambiguity and ambiguity attitude as well as the separation of these two concepts. Eventually we illustrate how these theories can be used to model strategic interaction when players perceive ambiguity about the strategic behaviour of other players. We focus on the approach of Eichberger and Kelsey (2014) and illustrate our own contribution to model ambiguous beliefs of players in the Multiple Prior model.

Throughout the chapter we, for convinience and consistency, occasionally deviate from the original notation as well as state simplified versions of the theorems. Our main objective is to provide the reader with the necessary background for the articles presented in this thesis.

1.1 Foundations of Decision Theory

This section provides an overview of some of the groundbreaking achievements in decision theory.

Von Neumann and Morgenstern

Consider the following game. A fair coin is repeatedly tossed. The game ends when Heads comes up for the first time. The payout is the following: if the coin is tossed n-times you receive 2^n in monetary terms. How much would you pay to play this game?

Nicolas Bernoulli states this game in a letter to Pierre Raymond De Montmort in 1713 (see De Montmort (1713)). At that time, expected value maximization was considered the rational approach to decision making under risk. Given the choice between two risky lotteries, the decision maker (DM) should always choose the lottery with the higher expected value. Bernoulli points out that the game has an infinite expected value. Thus an expected value maximizer is willing to pay any finite amount to play this game. This is obviously ridiculous. Bernoulli's game[1] thus shows that people are not expected value maximizers.

Nicolas' cousin Daniel Bernoulli shows in Bernoulli (2011) that peoples' preferences in this game can be describes as maximization of an expectation of a function which maps monetary payouts to real numbers. This approach can lead to a finite expectation of Bernoulli's game.[2] Such a function if referred to as a *utility function*. Daniel Bernoulli thus introduces *expected utility maximization*, a concept that has been and still is hugely important in decision theory.

Daniel Bernoulli's suggestion that peoples preferences in risky choice situations can be described by expected utility maximization triggered the hugely influential contribution by Von Neumann and Morgenstern (1944). They introduce a set of axioms[3] on preferences over risky lotteries which they show to be equivalent to expected utility maximization.

[1] The game is often referred to as the St. Petersburg Paradox. The word *paradox* is frequently used in decision-theory when theories of allegedly rational choice clash with observed choice behaviour or intuition. Personally we are of the opinion that the term paradox is used wrongly. A deviation from a theory is not paradoxical, it merely highlights the limitations of the model or arguably the irrationality of peoples' preferences.

[2] For instance with this function being of some logarithmic type.

[3] In decision theory, axioms can be understood as assumptions about preferences.

In the theory of Von Neumann and Morgenstern the objects of choice are objective lotteries, i.e. lotteries where the probabilities of alternatives are known. Examples are bets on the outcome of a roulette wheel or a die with known probabilities. Their theory stays silent on decisions in which probabilities are not known to the DM such as bets on horse races or a die with unknown probabilities.

Von Neumann and Morgenstern assume a set of consequences X which is not constrained to have any particular topological structure. They denote by L the set of finite-support lotteries over X. Preferences over lotteries are modelled by a binary relation $\succsim$ on L, i.e. $\succsim \subseteq L \times L$.

Two lotteries can be mixed, resulting in a new lottery. For two lotteries $P, Q \in L$ and some $\alpha \in [0,1]$, the compound lottery $\alpha P + (1-\alpha)Q \in L$ is defined by

$$(\alpha P + (1-\alpha)Q)(x) = \alpha P(x) + (1-\alpha)Q(x)$$

for all $x \in X$. This construction of compount lotteries induces a mixture space over L, a concept that is crucial for the framework of Anscombe and Aumann (1963) which is introduced later and provides the conceptual basis of the papers in this thesis.

Von Neumann and Morgenstern provide the following axioms.

Axiom 1.1 (Weak Order). For all $P, Q, R \in L$

1. $P \succsim Q$ or $Q \succsim P$.

2. If $P \succsim Q$ and $Q \succsim R$, then $P \succsim R$.

Axiom 1.2 (Continuity). For all $P, Q, R \in L$ with $P \succ Q \succ R$ there exist $\alpha, \beta \in (0,1)$ such that

$$\alpha P + (1-\alpha)R \succ Q \succ \beta P + (1-\beta)R.$$

Axiom 1.3 (Independence). For every $P, Q, R \in L$ and $\alpha \in (0,1)$

$$P \succsim Q \iff \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R.$$

Weak Order consists of two assumptions: Completeness and Transitivity. Continuity is a technical assumption.[4] Independence states that a preference of one

[4] Note that this axiom cannot be refuted by a finite number of choices.

lottery over another is not reversed when both lotteries are mixed with a third lottery. It is the most interesting and controversial axiom.[5]

Von Neumann and Morgenstern show that these three axioms characterize expected utility maximization.

Theorem 1.1 (Von Neumann and Morgenstern). *Let $\succsim$ be a preference relation on L, the set of finite-support lotteries over some set of consequences X. Then the following are equivalent:*

1. *$\succsim$ satisfies Weak Order, Continuity and Independence.*
2. *There exists a function $u : X \to \mathbb{R}$ such that for all $P, Q \in L$*

$$P \succsim Q \iff \sum_{x \in X} P(x)u(x) \geq \sum_{x \in X} Q(x)u(x).$$

Furthermore, u is unique up to positive affine transformations.

From a normative viewpoint the Von Neumann and Morgenstern axioms are compelling in the sense that a violation of these axioms can never be advisable. If this is accepted, the theorem suggests that expected utility maximization is the correct way to make decisions under risk. From a descriptive viewpoint the issue is more debatable. Allais (1953) introduces a choice-problem in which observed behaviour is typically inconsistent with expected utility maximization.[6]

De Finetti

In the framework of Von Neumann and Morgenstern, choices are made over objective lotteries: the probabilities of consequences are known. However, in hardly any decision that we make in the real world are probabilites of consequences known to us. De Finetti (1937) studies preferences over monetary bets in which no probabilistic information is available.[7] Through the DM's willingness to bet, De Finetti aims to derive subjective probabilities. His aim is thus to derive probabilities from observable choice behaviour.

[5] We discuss this axiom and its limitations for constructing a descriptive model of decision making in section 2.

[6] The choice-problem of Allais (1953) inspired a very interesting and important area of research. It is however a very different direction from this thesis. We therefore only mention its existence.

[7] Ramsey (1931) independently suggests a very similar approach.

De Finetti's framework consists of a finite state space $S = (s_1, \dots, s_{|S|})$ and prizes in monetary terms. A bet is a function from S to $\mathbb{R}^{|S|}$. The set of bets is $X = \mathbb{R}^{|S|}$. A bet $x \in X$ can be written as $(x_1, \dots, x_{|S|})$, where $x(s_i) = x_i$. Preferences are modelled by a binary relation $\succsim$ over X.

De Finetti provides the following axioms.

Axiom 1.4 (Weak Order). For all $x, y, z \in X$

1. $x \succsim y$ or $y \succsim x$.

2. If $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Axiom 1.5 (Continuity). For every $x \in X$ the sets

$$\{y|y \succ x\}, \{y|x \succ y\}$$

are open.

Axiom 1.6 (Additivity). For all $x, y, z \in X$

$$x \succsim y \iff x + z \succsim y + z.$$

Axiom 1.7 (Monotonicity). For all $x, y \in X$, if $x_i \geq y_i$ for all $i \in \{1, \dots, n\}$, then $x \succsim y$.

Axiom 1.8 (Non-Degeneracy). There exist $x, y \in X$ such that $x \succ y$.

De Finetti shows that these axioms are equivalent to the existence of a subjective probability distribution over the state space according to which the DM is an expected value maximizer.

Theorem 1.2 (De Finetti)**.** *Let S be a state space and $\succsim$ a preference relation over the set of bets $X = \mathbb{R}^{|S|}$. Then the following are equivalent:*

1. *$\succsim$ satisfies Weak Order, Continuity, Additivity, Monotonicity and Non-Degeneracy.*

2. *There exists a probability distribution P over S such that*

$$x \succsim y \iff \sum_{s \in S} P(s)x(s) \geq \sum_{s \in S} P(s)y(s).$$

Moreover, P is unique.

The crucial and most debatable axiom is Additivity. The framework is criticized as Additivity implies that DM's are expected value maximizers and therefore risk-neutral. It thus ignores the insights of Bernoulli.[8]

Savage

Recall the different approaches taken by Von Neumann and Morgenstern and De Finetti. Von Neumann and Morgenstern take objective probabilities as primitive and use it to measure the utility function. De Finetti can be interpreted as taking utilities as given and using it to measure subjective probabilities. Thus von Neumann and Morgenstern assume linearity in probabilities, De Finetti assumes linearity in utilities. Savage (1954) combines these ideas.

He assumes an abstract framework with no mathematical machinery. Neither probabilities, nor utilities are taken as primitive. Rather they are measured at the same time. Savage introduces seven axioms and shows their equivalence to subjective expected utility maximization. This means that a preference relation satisfies the seven axioms if and only if it can be represented by a unique probability distribution over the state space and a utility function over consequences according to which the DM maximizes expected utility.

Savage's framework consists of an exogenous state space S and a set of consequences X. Acts are mappings from S to X and $\mathcal{F} = \{f : S \to X\}$ is the set of acts. Preferences are modelled by a binary relation $\succsim$ over $\mathcal{F}$. A state in Savage's framework "resolves all uncertainty" in the sense that if the DM has chosen an act $f \in \mathcal{F}$ and is informed that state $s \in S$ has ocurred she knows that the consequence is $x = f(s)$.[9]

It is remarkable that Savage does not assume anything else. The state space is not restricted by any kind of measurability of events requirement. The set of consequences and the set of acts do not require any topological structure. We do not have a mixture-space as we do in Von Neumann and Morgenstern or Anscombe and Aumann (1963), see next subsection.

We do not state all of Savage's axioms but restrict attention to the axiom P_2, typically referred to as the *Sure-Thing Principle*. It is the most crucial and most

[8] One way around this is to interpret the monetary payouts as utilities. The additivity axiom would then be about adding utils.

[9] A consequence of this is that the state space is typically very large if one aims to model choice-problems in the Savage framework.

frequently criticized axiom and will become important in the second section when ambiguity takes the stage.

Axiom (Savage's P_2). For all acts $f, g, f', g' \in \mathcal{F}$ and all events $E \subseteq S$ such that it holds that $f(s) = f'(s), g(s) = g'(s)$ for all $s \in E$ and $f(s) = g(s), f'(s) = g'(s)$ for all $s \notin E$ it holds that

$$f \succsim g \iff f' \succsim g'.$$

The Sure-Thing Principle requires that the preference between two acts does not depend on the states of the world where both acts have identical consequences. When comparing two acts, it suffices to consider the states of the world in which these acts yield different outcomes. It is thus a separability axiom.

Savage shows that a DM's preferences satisfy his 7 axioms if and only if there exists a utility function u over X, unique up to positive affine transformations, and a unique subjective probability distribution P over S, such that the DM's preferences can be modelled as maximization of expected utility according the these entities.[10,11]

Theorem 1.3 (Savage)**.** *Let S be a state space, X a set of consequences and $\succsim$ a preference relation over acts $\mathcal{F} = \{f : S \to X\}$. Then the following are equivalent:*

1. *$\succsim$ satisfies the axioms $P_1 - P_7$.*
2. *There exists a non-atomic finitely-additive probability distribution Pon S and a non-constant, bounded function $u : X \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$*

$$f \succsim g \iff \int u(f)\ dP \geq \int u(g)\ dP,$$

where $\int u(f)\ dP$ denotes the expected utility of f, given u and P.

Moreover, P is unique and u is unique up to positive affine transformations.

A huge amount is to be and has been said, discussed, praised and criticized about Savage. We refrain from adding to this discussion at this point, but return to

[10] The probability distribution has the properties of being non-atomic and finitely additive. Non-atomic means that for every event $A \subseteq S$ and every $\alpha \in [0, 1]$ there exists an event $E' \subseteq E$ such that $P(E') = \alpha P(E)$. Finitely additive means that for all $E, E' \subseteq S$ for which $E \cap E' = \emptyset$ it holds that $P(E) + P(E') = P(E \cup E')$.

[11] Savage's axioms P_6 rules out the possibility that the state space is finite. The model of Anscombe and Aumann (1963), introduced in the next subsection, does not suffer from this limitation.

Savage and the Sure-Thing Principle in the next section when we discuss the Ellsberg Thought Experiment.

Anscombe and Aumann

The model of Anscombe and Aumann (1963) provides the basis for the models on decision-making under ambiguity that are introduced and discussed in this thesis. Just like Savage, Anscombe and Aumann provide a set of axioms that are equivalent to subjective expected utility maximization. However, their framework has more structure.

Just like Savage, their framework consists of a state space S and a set of consequences X. The difference is that acts do not map from S into X but from S into L, the set of finite-support lotteries over X which we already encountered in Von Neumann and Morgenstern (1944). The set of acts is thus $\mathcal{F} = \{f : S \to L\}$. An act which assigns the same lottery to every state is called a constant act. With a slight abuse of notation the set of constant acts can be associated with the set of lotteries L. Preferences are modelled as usual by a binary relation $\succsim$ over $\mathcal{F}$.

The interpretation of this framework is that the DM faces two sources of uncertainty: a *horse lottery* and a *roulette lottery*. When choosing amongst acts the DM does not necessarily have information on objective probabilites over the state space: she faces a *horse lottery* over the state space. For every state the DM faces a *roulette lottery* over the set of consequences: she knows the probabilities over the consequences. She thus faces subjective uncertainty about which state will occur and once the true state has been determined she faces objective uncertainty about consequences.

The disadvantage of this framework compared to Savage's are the structural assumptions that make it less general. The first advantage is that it allows the state space to be finite. The second and more crucial advantage is that the mixture-space on L, which we already know from Von Neumann and Morgenstern, allows the construction of a mixture-space on $\mathcal{F}$. Mixtures are performed pointwise: for $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act $\alpha f + (1 - \alpha)g$ is defined by

$$(\alpha f + (1 - \alpha)g)(s) = \alpha f(s) + (1 - \alpha)g(s) \qquad \forall s \in S.$$

The fact that we have this mixture-space over $\mathcal{F}$ is crucial for our first chapter in which we define different levels of ambiguity aversion via different levels of preference for mixing amongst acts. Since the Savage-framework does not have a mixture-space, this cannot be done there.[12]

Anscombe and Aumann consider the following axioms.

Axiom (AA1: Weak Order). For all $f, g, h \in \mathcal{F}$

1. $f \succsim g$ or $g \succsim f$.
2. If $f \succsim g$ and $g \succsim h$, then $f \succsim h$.

Axiom (AA2: Continuity). For all $f, g, h \in \mathcal{F}$ with $f \succ g \succ h$ there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h.$$

Axiom (AA3: Independence). For every $f, g, h \in \mathcal{F}$ and $\alpha \in (0, 1)$

$$f \succsim g \iff \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

Axiom (AA4: Monotonicity). For all $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.

Axiom (AA5: Non-Degeneracy). There exist $x, y \in X$ such that $x \succ y$.

Recall that the axioms of Von Neumann and Morgenstern are defined for preferences over L. The first three axioms of Anscombe and Aumann are the same as the ones from Von Neumann and Morgenstern, however defined on $\mathcal{F}$ instead of L. Axioms 4 and 5 are basically the last two axioms from De Finetti, adapted to the framework.

Anscombe and Aumann show that their five axioms are equivalent to subjective expected utility maximization: preferences satisfy their axioms if and only if they are representable by a unique probability distribution over the state space and a utility function over consequences, unique up to positive affine transformations, according to which the DM maximizes expected utility. The following theorem states their result for a finite state space.[13,14]

[12] In Ghirardato et al. (2003) the authors construct a mixture-space in the Savage framework. However they need some typological assumptions on the set of consequences.

[13] In the following we denote by the term $\int u(f)\, dP$ the double integration that is needed in this framework: the inner expected utility calculation and the outer subjective expected utility calculation. Since S is assumed to be finite we have that $\int u(f)\, dP = \sum_{s\in S} P(s) \sum_{x\in X} f(s)(x)u(x)$.

[14] In Savage's model this assumption of a finite state space is not possible as the axioms imply

Theorem 1.4 (Anscombe and Aumann). *Let S be a finite state space, L the set of finite-support lotteries over a set of consequences X and $\succsim$ a preference relation on $\mathcal{F} = \{f : S \to L\}$. Then the following are equivalent:*

1. *$\succsim$ satisfies the axioms AA1 - AA5.*
2. *There exists a probability distribution P on S and a non-constant function $u : X \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$*

$$f \succsim g \iff \int u(f)\ dP \geq \int u(g)\ dP.$$

Furthermore, P is unique and u is unique up to positive affine transformations.

1.2 Decision Making under Ambiguity

The theories of Savage as well as Anscombe and Aumann are very appealing. The axioms make sense from a normative perspective. Furthermore the axiomatic systems teach us what exactly we are assuming by modelling DM's as subjective expected utility maximizers. However, peoples' preferences systematically violate these theories even in simple choice problems. The most prominent example for this is the 1-urn thought experiment of Daniel Ellsberg (1961).[15]

Example 1.1 (The Ellsbergs 1-Urn Thought Experiment). Assume that a DM is confronted with an urn containing 90 balls. She receives the information that exactly 30 of these balls are red and that the other 60 balls are either yellow or black, but the exact number is not given. The DM is asked to choose amongst different bets before a ball is randomly drawn from the urn. Two different bets are offered.

In the **first bet** she can either choose to bet on a red ball being drawn or alternatively on a yellow ball being drawn. If she bets correctly she receives a payout of 100, otherwise she receives nothing.

In the **second bet** she can choose to bet on a red or black ball being drawn or alternatively on a yellow or black ball being drawn. Again she receives 100 if she bets correctly, otherwise nothing.

an infinite state space.

[15] In the literature Ellsberg's thought experiments are often referred to as paradoxes. As already mentioned we do not agree with how the word "paradox" is used in decision theory. We therefore refer to the "Ellsberg Thought Experiment" throughout this chapter.

These choice-problems can be modelled in the Savage framework. The state space S has three elements: red (R), yellow (Y) and black (B). The relevant consequences are 100 and 0. There are four acts: f_1 is the bet on R, f_2 is the bet on Y, g_1 is the bet on $R \cup B$ and g_2 is the bet on $Y \cup B$. This is illustrated in Table 1.1.

	R	Y	B
f_1	100	0	0
f_2	0	100	0
g_1	100	0	100
g_2	0	100	100

Table 1.1: The Ellsberg Thought Experiment

Ellsberg realizes that the typical preferences are $f_1 \succ f_2$ and $g_2 \succ g_1$.[16] People tend to prefer the acts for which the probability of winning is known. They dislike the acts for which the probability of winning is unknown. These preferences are incompatible with SEU theory. There does not exist a probability distribution which is compatible with this choice behaviour. To see why, assume for contradiction that the DM's preferences can be represented by the probability distribution[17] $P : \mathcal{P}(S) \to [0,1]$ and a utility function $u : X \to \mathbb{R}$. Assume that $u(100) > u(0)$.[18]

Since the DM strictly prefers f_1 to f_2, the bet f_1 results in a higher subjective expected utility than f_2, i.e.

$$P(R)u(100) + (1 - P(R))u(0) > P(Y)u(100) + (1 - P(Y))u(0),$$

which implies

$$(P(R) - P(Y))(u(100) - u(0)) > 0.$$

Since $u(100) > u(0)$ it follows that

$$P(R) > P(Y).$$

[16] Ellsberg did not conduct experiments in the lab, but asked economists about their preferences. His findings were later replicated in the lab.

[17] $\mathcal{P}(S)$ denotes the powerset of S.

[18] Note that this assumption is not needed to obtain a contradiction, but is easily justified. The assumption $u(100) < u(0)$ would suffice as well.

Furthermore the DM strictly prefers g_2 to g_1. It follows that

$$P(R \cup B)u(100) + (1 - P(R \cup B))u(0) < P(Y \cup B)u(100) + (1 - P(Y \cup B))u(0),$$

which implies

$$(P(R \cup B) - P(Y \cup B))(u(100) - u(0)) < 0.$$

Again with $u(100) > u(0)$ it follows that

$$P(R \cup B) < P(Y \cup B).$$

Combined this leads to the contradiction

$$1 = P(R) + P(Y \cup B) > P(Y) + P(R \cup B) = 1.$$

This implies, through the theorems of Savage as well as Anscombe and Aumann, that the axiomatic systems for SEU are violated by the typical preferences in the Ellsberg Thought Experiment. In the following we demonstate how the preferences violate the Sure-Thing Principle (STP) as well as the Independence axiom.

The violation of STP is straight-forward. Consider the event $E = B$. For all $s \notin E$ we have that $f_1(s) = g_1(s)$ and $f_2(s) = g_2(s)$ as well as for all $s \in E$ we have $f_1(s) = f_2(s)$ and $g_1(s) = g_2(s)$. Thus the STP implies that

$$f_1 \succsim f_2 \iff g_1 \succsim g_2,$$

in contradiction to the typical preferences.

To see that the Independence axiom is violated consider the acts $h_1 = 100_B0$ and $h_2 = 0$.[19] Note that $\frac{1}{2}f_1 + \frac{1}{2}h_1 = \frac{1}{2}g_1 + \frac{1}{2}h_2$ and $\frac{1}{2}f_2 + \frac{1}{2}h_1 = \frac{1}{2}g_2 + \frac{1}{2}h_2$. Due to the Independence axiom $f_1 \succ f_2$ implies $\frac{1}{2}f_1 + \frac{1}{2}h_1 \succ \frac{1}{2}f_2 + \frac{1}{2}h_1$ and $g_2 \succ g_1$ implies $\frac{1}{2}g_2 + \frac{1}{2}h_2 \succ \frac{1}{2}g_1 + \frac{1}{2}h_2$, a contradiction.

The question that springs to mind now is: why does the SEU theory fail so systematically? Or put differently: what makes people violate the allegedly convincing axioms so consistently?

In all four acts of the Ellsberg Thought Experiment the DM faces uncertainty

[19] The notation $h_1 = 100_B0$ means that the act h_1 results in consequence 100 on the state B and consequence 0 on the states R and Y.

in the sense that the consequence is not known ex-ante. However the type of uncertainty is very different. The acts f_1 and g_2 are risky acts as the probabilities of consequences are known. For the acts f_2 and g_1 the probabilities of consequences are unknown: these acts are *ambiguous*. This distinction between risk and ambiguity was propagated by Knight (1921).[20] The typical preferences in the Ellsberg Thought Experiment suggest that people tend to dislike betting on ambiguous events: they are *ambiguity averse*.[21]

The Ellsberg Thought Experiment and its lessons from it unfold the necessity to construct alternative approaches to model preferences in ambiguous choice-problems. In the following, two approaches are presented that can accommodate phenomena observed in choice under ambiguity. The first one is *Choquet Expected Utility* theory, the second one is the *Multiple Prior* approach. Both the Savage and the Anscombe and Aumann framework have been used in the literature as the basis of these approaches. We focus on the ones that use the Anscombe-Aumann framework as they provide the basis for the papers of this thesis.

1.2.1 The Choquet Expected Utility Model

Itzhak Gilboa points out in Gilboa (2009) that "P_2 [the Sure-Thing Principle] implies, among other things, that the decision maker should be indifferent between likelihood judgements that are well-reasoned and those that are arbitrary."[22]

With this in mind consider the following scenario which goes back to David Schmeidler.[23] There are two coins. The first coin is known to be fair, i.e. Heads and Tails are known to occur with equal probabilities There is no information on the second coin. Assume that the DM is forced to assign a probability to the second coin coming up Heads. As there is no information, ignorance on probabilities of Heads and Tails is symmetric. The DM thus assigns probability $\frac{1}{2}$ to Heads. The coins now have been assigned the same probability distributions, however Schmeidler suggests

[20] What we refer to as *Ambiguity* is also referred to *Uncertainty* or *Knightian Uncertainty*. Throughout this thesis we use the term *Uncertainty* in its generic sense. *Risk* refers to uncertainty with known probabilities, *Ambiguity* refers to uncertainty with possibly unknown probabilities.

[21] Of course not all people have the typical preferences. Some exhibit *ambiguity loving* preferences. Some do not violate the SEU axioms at all. See subsection 2.3 for a discussion on ambiguity attitude and how to model different attitudes towards ambiguity.

[22] Gilboa (2009), page 188.

[23] The Ellsberg 2-urn Thought Experiment is very similar.

that they still feel very different.

This different feel of the probability judgements in Schmeidler's Thought Experiment and Gilboa's explanation of the implications of the STP suggests that this axiom has a hard time in choice-problems in which ambiguity plays a role. Indeed we have already seen that the typical preferences in the Ellsberg Thought Experiment violate the STP.

Schmeidler's approach to cope with this is to provide an axiomatic system which does not imply beliefs to be representable by a probability distribution. In his model, beliefs are not additive, but are represented by not necessarily additive set functions called *capacities*.

Capacities

Definition 1.1 (Capacity)**.** *Let S be a set and $\mathcal{A}$ a σ-algebra on S.*[24] *A function $\nu : \mathcal{A} \to [0,1]$ is called capacity if the following holds:*

1. $\nu(\emptyset) = 0$,
2. $\nu(S) = 1$,
3. $E \subseteq E' \Rightarrow \nu(E) \leq \nu(E') \qquad \forall E, E' \subseteq S$.

A capacity is a normalized and monotonic, but not necessarily additive, set function. A probability distribution is therefore a special case of a capacity.

The concept of capacities can be applied to choice-problems. We can apply them to achieve a representation of beliefs which takes into account the ambiguity that the DM faces. For instance we can construct the following capacities ν_1 and ν_2 to represent beliefs over the two coins in Schmeidler's Thought Experiment. The state space is $S = \{H, T\}$ and the σ-algebra is $\mathcal{A} = \mathcal{P}(S) = \{\emptyset, H, T, S\}$ for both coins. Define $\nu_1(\emptyset) = \nu_2(\emptyset) = 0$, $\nu_1(S) = \nu_2(S) = 1$, $\nu_1(H) = \nu_1(T) = \frac{1}{2}$ and $\nu_2(H) = \nu_2(T) = \frac{2}{5}$. The DM's information about the first coin is reflected in ν_1, which is the uniform probability distribution over S. The DM does not have information on probabilities about coin 2. This is reflected by the values $\frac{2}{5}$ for both H and T, an assignment which is allowed in this framework as additivity is not required for capacities. The fact that $\nu_2(H) = \nu_2(T)$ reflects the symmetric ignorance about Heads and Tails.

[24] For finite S, which is what we assume in all our articles, $\mathcal{A}$ is the powerset of S.

The interpretation of the capacity ν_2 may be that it represents a lower bound of the DM's belief about the probability of events, i.e. the DM might believe that Heads will come up with probability at least $\frac{2}{5}$. Capacities are thus one approach to represent beliefs of a DM who perceives ambiguity.

The Choquet Integral

When beliefs are represented by a capacity, the evaluation of acts can be carried out with the *Choquet Integral* which is introduced in Choquet (1954).

Definition 1.2 (Choquet Integral)**.** *Let S be a set and $\mathcal{A}$ a σ-algebra on S. Let $X : S \to \mathbb{R}$ be a $\mathcal{A}$-measurable function and $\nu : \mathcal{A} \to [0,1]$ a capacity. The Choquet Integral of X with respect to ν is defined as*

$$\int_S X d\nu := \int_{-\infty}^{0} (\nu(\{s \in S | X(s) \geq x\}) - 1)dx + \int_{0}^{\infty} \nu(\{s \in S | X(s) \geq x\})dx, \quad (1.1)$$

where the integrals on the right side of the equation are Riemann integrals.

With the Choquet Integral, an expectation of a real-valued function X given a capacity ν can be calculated. When ν is additive, the Choquet integral reduces to the Riemann integral and is the normal expected value of X, given ν.

We apply the Choquet Integral in the framework of Anscombe and Aumann. This means that acts map from states into objective lotteries. The real-valued functions that we consider are mappings from the state space into expected utilities. Furthermore we assume a finite state space.

Under these assumptions consider an act $f \in \mathcal{F}$. There exist lotteries $l_1, \dots l_n \in L$ and a partition $(E_1, \dots, E_n)$ of S such that on E_i the act f results in lottery l_i for $i \in \{1, \dots, n\}$. Consider a utility function $u : L \to \mathbb{R}$. With slight abuse of notation we denote by $u(l_i)$ the expected utility of lottery l_i, given u. Without loss of generality we can assume that $u(l_1) \geq \dots \geq u(l_n)$. By defining $\bigcup_{j=1}^{0} = \emptyset$ the Choquet Integral in (2.2) reduces to

$$\int u(f)\ d\nu = \sum_{i=1}^{n} u(l_i) \left[\nu\left(\bigcup_{j=1}^{i} E_j \right) - \nu\left(\bigcup_{j=1}^{i-1} E_j \right) \right].$$

Throughout the thesis we use the simplified notation $\int u(f)\ d\nu$ for the Choquet expected utility (CEU) of the act f, given utility function u and capacity ν. The state space is always clear from the context and is thus omitted in the notation.

Solving Ellsberg with CEU

One strength of the Choquet Expected Utility approach is that we can easily model the typical preferences in the Ellsberg Thought Experiment. This can be done by assigning weights to the ambiguous events $Y, B, R \cup Y$ and $R \cup B$ that are smaller than the weight assigned by the uniform distribution over the state space. Assume that for $\delta \in (0, \frac{1}{3}]$ the beliefs of the DM are represented by the following capacity $\nu : \mathcal{P}(S) \rightarrow [0, 1]$:

$$\nu(E) = \begin{cases} 0, & E = \emptyset \\ 1/3, & E = R \\ 1/3 - \delta, & E \in \{Y, B\} \\ 2/3, & E = Y \cup B \\ 2/3 - \delta, & E \in \{R \cup Y, R \cup B\} \\ 1, & E = S \end{cases}.$$

The DM has the information that the probability of R is $\frac{1}{3}$ and the probability of $Y \cup B$ is $\frac{2}{3}$. About the events $Y, B, R \cup Y$ and $R \cup B$ the DM perceives the ambiguity δ.[25] If $\delta = \frac{1}{3}$, the decision maker perceives the maximum amount of ambiguity. If $\delta = 0$, ν is additive which coincides with no perceived ambiguity. We have already shown that this case cannot explain the typical preferences.

With the Choquet Integral we can now calculate the expectation of the acts f_1, f_2, g_1 and g_2 with respect to the capacity ν. Without loss of generality we assume

[25] The parameter δ can be interpreted as a measure of the amount of ambiguity that the DM perceives. The topic of perceived ambiguity is exceptionally important throughout this thesis. We introduce and discuss this topic in subsection 2.3. For the current example the same conclusion holds for all considered values of δ in the allowed range.

that $u(0) = 0$ and $u(100) = 100$.

$$\begin{aligned}
\int u(f_1)\ d\nu &= u(100)[\nu(R) - \nu(\emptyset)] + u(0)[\nu(S) - \nu(R)] \\
&= 100\nu(R) \\
&> 100\nu(Y) \\
&= u(100)[\nu(Y) - \nu(\emptyset)] + u(0)[\nu(S) - \nu(Y)] \\
&= \int u(f_2)\ d\nu.
\end{aligned}$$

$$\begin{aligned}
\int u(g_2)\ d\nu &= u(100)[\nu(Y \cup B) - \nu(\emptyset)] + u(0)[\nu(S) - \nu(Y \cup B)] \\
&= 100\nu(Y \cup B) \\
&> 100\nu(R \cup B) \\
&= u(100)[\nu(R \cup B) - \nu(\emptyset)] + u(0)[\nu(S) - \nu(R \cup B)] \\
&= \int u(g_1)\ d\nu.
\end{aligned}$$

Given ν, the Choquet expected utility is larger for f_1 than for f_2 as well as larger for g_2 than for g_1. We have thus shown that the CEU approach can model the typical preferences in the Ellsberg Thought Experiment.

Schmeidler's Axiomatization of CEU

Schmeidler (1989) axiomatizes CEU preferences in the Anscombe-Aumann framework.[26] The crucial axiom is *Comonotonic Independence*. It is a weakening of the Independence axiom of Anscombe and Aumann. The intuition behind this axiom can be explained through the Ellsberg Thought Experiment. One reason why DMs violate the Independence axiom may be that the *mix* of the unambiguous f_1 and the ambiguous f_2 with the act $h_1 = 100_B0$ shifts the ambiguity from one act to the other: $\frac{1}{2}f_1 + \frac{1}{2}h_1$ is ambiguous, $\frac{1}{2}f_2 + \frac{1}{2}h_1$ is unambiguous. The act h_1 is thus a better *hedge against ambiguity* for f_2 than it is for f_1. We have an asymmetric effect on ambiguity due to the mix. Schmeidler's idea is that the independence axiom must

[26] Sarin and Wakker (1992) axiomatize CEU preferences in the Savage framework. Throughout the thesis we rely on the Anscombe-Aumann framework and thus in the following only introduce Schmeidler's contribution.

be preserved only for cases in which such asymmetry is impossible, as in such a case the mix effects ambiguity in the same way. Such asymmetry cannot occur when two acts, and a third act with which it is mixed, order states in the same way in terms of expected utility. Schmeidler calls such acts *comonotonic acts*.

Definition 1.3. *Two acts $f, g \in \mathcal{F}$ are comonotonic if there exist no $s, s' \in S$ such that*

$$f(s) \succ g(s) \qquad \textit{and} \qquad g(s') \succ f(s').$$

Schmeidler weakens the Independence axiom to pairwise comonotonic acts.

Axiom 1.9 (Comonotonic Independence)*.* For all pairwise comonotonic acts $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$ it holds that

$$f \succsim g \iff \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

Schmeidler shows that the Anscombe-Aumann axioms with Independence replaced by Comonotonic Independence are equivalent to preferences being representable within the CEU framework.

Theorem 1.5 (Schmeidler)**.** *Let S be a finite state space, L the set of finite-support lotteries over a set of consequences X and $\succsim$ a preference relation on $\mathcal{F} = \{f : S \to L\}$. Then the following are equivalent:*

1. *$\succsim$ satisfies the axioms AA1, AA2, Comonotonic Independence, AA4 and AA5.*
2. *There exists a capacity $\nu : \mathcal{P}(S) \to [0,1]$ and a function $u : L \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$*
$$f \succsim g \iff \int u(f)\ d\nu \geq \int u(g)\ d\nu.$$

Furthermore, ν is unique and u is unique up to positive affine transformations.

To get some intuition on this result, note that Comonotonic Independence implies that the representation functional is additive for comonotonic acts. Furthermore since constant acts are comonotonic to all acts and every act is comonotonic to itself we have linearity of the functional for such acts. All these properties are easily checked to be true for the Choquet Integral: for a capacity ν, comonotonic acts f, g, constant act l and $\alpha \in [0,1]$ we have that

$$\int u(\alpha f + (1-\alpha)g + l)\ d\nu = \alpha \int u(f)\ d\nu + (1-\alpha) \int u(g)\ d\nu + u(l).$$

The reason the CEU model can explain the typical preferences in the Ellsberg Thought Experiment is that Comonotonic Independence does not restrict preferences as much as the standard Independence axiom or the Sure-Thing Principle does. The acts f_1 and f_2 as well as g_1 and g_2 are not comonotonic, thus Comonotonic Independence does not per se rule out preferences that are observed in the Ellsberg Thought Experiment. Of course, the modelling capabilities of the CEU approach do not end here.

Capacity Subclasses

In the following we introduce some if the most important capacity subclasses, including the ones that are relevant for this thesis. The important ideas behind and results on them are explained.

A capacity ν is called *convex* if

$$\nu(E) + \nu(E^{'}) \leq \nu(E \cup E^{'}) + \nu(E \cap E^{'}) \qquad \forall E, E^{'} \subseteq S. \tag{1.2}$$

Schmeidler shows that, under the standard CEU axioms, a capacity is convex if and only if the DM always has a preference for mixing amongst acts, an axiom he calls Uncertainty Aversion.[27] Thus he suggests that convex capacities reflect ambiguity aversion.

Axiom 1.10 (Uncertainty Aversion). For all $f, g \in \mathcal{F}$ such that $f \sim g$ and $\alpha \in (0,1)$

$$\alpha f + (1-\alpha) g \succsim f.$$

A capacity ν is called *concave* if in equation (1.2), $\leq$ is replaced by $\geq$. Of course, this capacity class is axiomatized by the axiom in which $\succsim$ is replaced by $\precsim$ in Uncertainty Aversion.

A capacity ν is called *neo-additive* if $\nu = \delta(1-\alpha) + (1-\delta)\pi$, where π is a probability distribution on S and $\alpha, \delta \in [0,1]$. We denote them by $\nu_{\pi,\delta,\alpha}$. A neo-additive capacity for which $\alpha = 1$ is called *simple*.

Neo-additive capacities are introduced and axiomatized in Chateauneuf et al. (2007). They can be viewed as a δ-mixture of the additive capacity π and the capacity that puts weight $1-\alpha$ on all events (except $\emptyset$ and S). The latter only

[27] Schmeidler calls "Uncertainty" what we call "Ambiguity".

distinguishes between whether an event is impossible, possible or certain. The interpretation is that the DM has additive beliefs π over S but she may not be absolutely confident in π. The parameter $1-\delta$ represents this confidence, δ represents the degree of perceived ambiguity. The larger δ is, the less confidence the DM has in her beliefs. If $\delta = 0$, then $\nu = \pi$ corresponding to absolute confidence in beliefs. If $\delta = 1$, the DM has no confidence in her beliefs. The parameter α can be interpreted as the DM's ambiguity attitude. The larger α, the more pessimistic is the DM. Thus α can be interpreted as the degree of pessimism.[28]

Chateauneuf et al. (2007) show that when $\nu = \nu_{\pi,\delta,\alpha}$ is a neo-additive capacity and u is a utility function, the Choquet integral of an act f is

$$\int u(f)d\nu_{\pi,\delta,\alpha} = (1-\delta)\int u(f)\ d\pi + \delta(1-\alpha)\sup_{s\in S} u(f(s)) + \delta\alpha \inf_{s\in S} u(f(s)). \quad (1.3)$$

Thus the Choquet integral is the mix of the evaluation at the probability estimate π as well as the best and the worst case scenario for the act f. Neo-additive capacities can therefore be used to represent beliefs of decision makers that are both ambiguity averse and ambiguity loving. Furthermore they provide a clear separation of perceived ambiguity and ambiguity attitude through the parameters δ and α.

JP-capacities were introduced by Jaffray and Philippe (1997). These capacities take the form $\nu = \alpha\mu + (1-\alpha)\overline{\mu}$, where μ is a convex capacity, $\overline{\mu}$ is its dual[29] and $\alpha \in [0,1]$. The convex μ represents perceived ambiguity and α represents the ambiguity attitude. By choosing $\mu = (1-\delta)\pi$ we get a neo-additive capacity. Thus neo-additive capacities are a subclass of JP-capacities.

The core of a capacity ν is the set of probability distributions over S that pointwise dominate ν:[30]

$$Core(\nu) = \{P \in \Delta(S) | P(E) \geq \nu(E)\ \forall E \in \mathcal{P}(S)\}.$$

The *core* of a capacity may be empty. The capacities that have a non-empty core are called *balanced*. A capacity is *exact* if its values are equal to the lower envelope of the core, i.e. if $\nu(E) = \min_{P \in Core(\nu)} P(E)$ for all $E \in \mathcal{P}(S)$. It is a well-known fact that convex capacities are exact and exact capacities are balanced but that

28 Note that in the original paper Chateauneuf et al. (2007), the authors use α for the degree of optimism. We deviate from this as most of the literature uses α as the degree of pessimism.
29 The dual capacity is defined by $\bar{\mu}(E) = 1 - \mu(E^c)$ for all $E \subseteq S$.
30 The set $\Delta(S)$ is the set of probability distribution over S.

the reverse implications do not hold. Balanced, exact and convex capacities are extremely important in the first article of this thesis. We provide an axiomatization of exact capacities, an open problem in decision-theory ever since these set functions were introduced in Schmeidler (1972).[31] Chateauneuf and Tallon (2002) characterize balanced capacities. However their characterization is not an axiomatization as it is not stated purely in terms of preferences. Inspired by their result we propose the axiom 1 - Ambiguity Aversion which we show axiomatizes balanced capacities.

1.2.2 The Multiple Prior Approach

The Ellsberg Thought Experiment shows that DMs systematically violate the SEU models, i.e. their preferences cannot be described by a unique probability distribution over the state space. We have already seen that we can explain the typical preferences by replacing Independence by Comonotonic Independence which leads to a relaxation of the additivity assumption and to the CEU model. A different approach is to allow beliefs to be represented not by a single prior as in SEU, but by a set of priors: the *Multiple Prior* (MP) approach. In Siniscalchi (2006) the MP approach is motivated as follows:

> "The decision maker may wish to consider multiple possible probabilistic descriptions of the underlying uncertainty."[32]

The aim is to represent the ambiguity that the DM perceives by a set of priors over the state space. Let S be a state space and $\Delta(S)$ the set of probability distributions over S. A *prior set* is a subset of $\Delta(S)$ which is typically assumed to be non-empty, convex and compact.[33] The probability distributions contained in the prior set can be interpreted as the priors that the DM cannot rule out. In this sense a prior set can reflect the *perceived ambiguity* of the DM.[34]

[31] We call the relevant axiom 2 - Ambiguity Aversion.

[32] Siniscalchi (2006), page 3.

[33] Convexity is for convenience as it is behaviourally equivalent if one assumed a prior set or its convex hull. Compactness is assumed to guarantee a well-defined minimum for acts, given the prior set.

[34] This interpretation poses some problems as a preference relation may have multiple prior set representations, see later discussion and Siniscalchi (2006) as well as its online appendix.

Maxmin Expected Utility

The most prominent class of MP models is the *Maxmin Expected Utility* (MEU) model. The DM evaluates acts at the worst case scenario of some prior set. The evaluation of an act f, given a prior set $\mathcal{C}$ and utility function u is

$$\min_{P \in \mathcal{C}} \int u(f)\, dP.$$

This is illustrated in Figure 1.1 for the state space $S = \{s_1, s_2, s_3\}$. The MEU approach can be interpreted as modelling preferences of a DM who cannot rule out the priors in $\mathcal{C}$ and has a pessimistic attitude towards ambiguity.

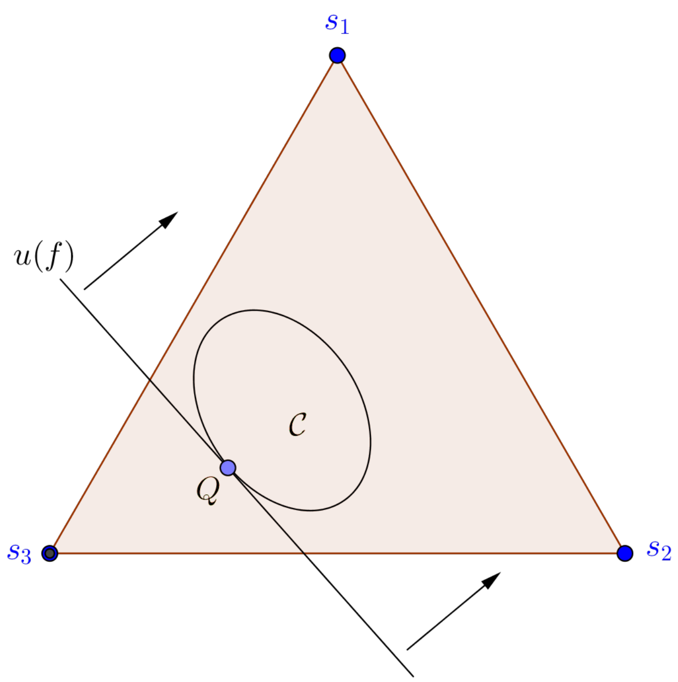


Figure 1.1: The line $u(f)$ is the lowest indifference curve of f. It is evaluated at Q. Arrows show the direction of increase in expected utility.

Solving Ellsberg with MEU

The pessimism in the MEU model allows us to model the typical preferences of the Ellsberg Thought Experiment. For some $\delta \in [0, \frac{1}{3})$, consider the prior set $\mathcal{C} =$

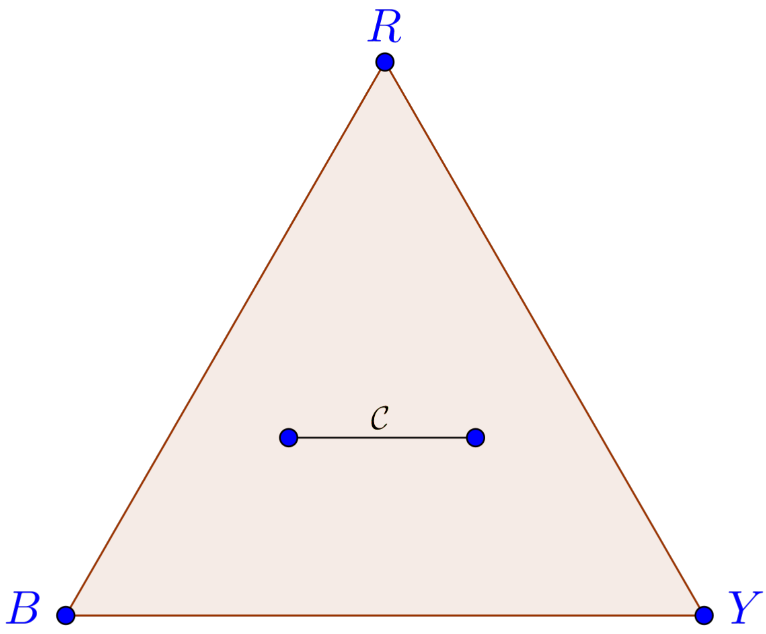


Figure 1.2: The prior set $\mathcal{C}$ reflects the perceived ambiguity of the DM in the Ellsberg Thought Experiment. It has length 2δ.

$\{P \in \Delta(S) | P(R) = \frac{1}{3}, \frac{1}{3} - \delta \leq P(B) \leq \frac{1}{3} + \delta\}$. The set is depicted in Figure 1.2.[35] The prior set reflects the information about the urn. It contains only probability distributions that put weight $\frac{1}{3}$ on red. The number δ can be interpreted as the perceived ambiguity. The MEU-evaluations of the four acts can now be calculated.

$$\min_{P\in\mathcal{C}} \int u(f_1)\ dP = \frac{1}{3}100 > (\frac{1}{3} - \delta)100 = \min_{P\in\mathcal{C}} \int u(f_2)\ dP$$

and

$$\min_{P\in\mathcal{C}} \int u(g_2)\ dP = \frac{2}{3}100 > (\frac{2}{3} - \delta)100 = \min_{P\in\mathcal{C}} \int u(g_1)\ dP.$$

The Maxmin Expected Utility of f_1 is therefore greater than for f_2 as well as greater for g_2 than for g_1.

[35] For the same value of δ, the set $\mathcal{C}$ and the capacity ν from above represent the same preferences. This is because ν is convex and $\mathcal{C}$ is the core of ν, see later discussion.

Gilboa and Schmeidler's Axiomatization of MEU

Gilboa and Schmeidler (1989) axiomatize MEU preferences in the Anscombe-Aumann framework. They introduce the following axiom, which is a weakening of Comonotonic Independence. It requires independence only when the act with which is mixed is constant.

Axiom 1.11 (Certainty Independence). For all acts $f, g \in \mathcal{F}$, constant acts $l \in L$ and $\alpha \in (0,1)$

$$f \succsim g \iff \alpha f + (1-\alpha)l \succsim \alpha g + (1-\alpha)l.$$

The intuition behind this axiom is that it is not possible to hedge against ambiguity when mixing with a constant act, thus preferences should not be affected by such a mix. The axiom stays silent about mixtures amongst pairwise comonotonic acts and is thus much weaker than Comonotonic Independence.

Gilboa and Schmeidler (1989) furthermore assume the Uncertainty Aversion axiom.

Theorem 1.6 (Gilboa and Schmeidler)**.** *Let S be a finite state space, L the set of finite-support lotteries over a set of consequences X and $\succsim$ a preference relation on $\mathcal{F} = \{f : S \to L\}$. Then the following are equivalent:*

1. *$\succsim$ satisfies the axioms AA1, AA2, Certainty Independence, AA4, AA5 and Uncertainty Aversion.*
2. *There exists a non-empty, compact and convex set $\mathcal{C} \subseteq \Delta(S)$ and a non-constant function $u : X \to \mathbb{R}$ such that for all $f, g \in \mathcal{C}$*

$$f \succsim g \iff \min_{P \in \mathcal{C}} \int u(f)\ dP \geq \min_{P \in \mathcal{C}} \int u(g)\ dP.$$

Furthermore, $\mathcal{C}$ is unique and u is unique up to positive affine transformations.

Uncertainty Aversion is responsible for the min-functional. The axiom is thus responsible for the pessimistic attitude towards ambiguity.[36] The CEU model does not have this restriction. However the MEU model has more degrees of freedom

[36] We show in our second article that this statement is only true in the presence of the other MEU axioms, most importantly Certainty Independence. We provide an example of a preference relation that satisfies Uncertainty Aversion but that cannot be represented by a prior set in combination with the min-functional.

than CEU regarding the limitations due to its independence axiom as Comonotonic Independence constrains beliefs much more than Certainty Independence does. For instance an MEU preference relation that can be represented by a ball-shaped prior set does not have a CEU representation. The CEU and MEU models however have an overlap: convex capacities. Schmeidler shows that a CEU DM with convex capacity ν has the same preferences as an MEU DM with prior set $\mathcal{C} = Core(\nu)$.

α-MEU Preferences

A more general model than MEU is α-MEU, introduced by Ghirardato and Marinacci (2002). The idea is to extend MEU to allow also optimistic attitudes towards ambiguity. Preferences are represented by a non-empty, convex and compact prior set $\mathcal{C}$, a utility function u and a parameter $\alpha \in [0, 1]$. The evaluation of an act is the α-mix of the worst and the best case scenario, given the prior set and the utility function. The evaluation of an act f is thus

$$\alpha \min_{P \in \mathcal{C}} \int u(f)\ dP + (1 - \alpha) \max_{P \in \mathcal{C}} \int u(f)\ dP.$$

The α parameter reflects the ambiguity attitude of the DM. The case $\alpha = 1$ coincides with pure pessimism and thus corresponds to MEU. The case $\alpha = 0$ coincides with pure optimism and corresponds to Maxmax Expected Utility (MMEU). For values strictly between 0 and 1, the DM exhibits both pessimistic and optimistic attitudes towards ambiguity.

The α-MEU approach is intuitively very appealing as an extension of MEU but poses a few problems. Thus far there does not exist a satisfactory axiomatization of α-MEU preferences. In Ghirardato et al. (2004) an axiomatization is provided, but Eichberger et al. (2011) show that in a finite state space the axioms imply $\alpha \in \{0, 1\}$, i.e. either MEU or MMEU. Furthermore the α-MEU axiom of Ghirardato et al. (2004) is not purely in terms of preferences. In the third article of this thesis we provide an equivalent version to their α-MEU axiom which is purely in terms of preferences. However our axiom does not solve the problem highlighted in Eichberger et al. (2011).

Another problem of the α-MEU model is that the distinction between perceived ambiguity represented through the prior set $\mathcal{C}$ and ambiguity attitude α is not given. Siniscalchi (2006) shows that there can be more than one representation of the same preferences, i.e. for some preference relation $\succsim$ there may exist $\mathcal{C}, \alpha$ and $\mathcal{C}', \alpha'$ which both represent $\succsim$.[37] Thus the α-MEU model does not achieve a clear separation of perceived ambiguity and ambiguity attitude.

Invariant Biseparable Preferences

Ghirardato et al. (2004) introduce a class of MP preferences which they call *invariant biseparable* (IB) preferences, a class that contains CEU and α-MEU preferences. Crucially, they suggest a solution to the earlier highlighted problem of non-uniqueness of the prior set for these preferences. They assume the standard Anscombe-Aumann axioms except Independence, which they replace by Certainty Independence.[38] They show that the axioms guarantee the existence of a smallest prior set which represents an IB preference relation. They interpret this set as the perceived ambiguity of the preference relation. Furthermore they prove the existence of a function a which assigns an ambiguity attitude between 0 and 1 to every act.[39] The evaluation of an act f is thus[40]

$$a(f) \min_{P \in \mathcal{C}} \int u(f)\ dP + (1 - a(f)) \max_{P \in \mathcal{C}} \int u(f)\ dP.$$

Their set of priors is characterized via the *unambiguous preference relation* in the style of Bewley (2002), introduced by Nehring (2001): For $\succsim$ the unambiguous preference relation $\succsim^*$ is constructed such that for $f, g \in \mathcal{F}$

$$f \succsim^* g \iff \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h, \qquad \forall h \in \mathcal{F}.$$

The act f is unambiguously preferred to g if no hedge can reverse the preference of f over g. The preference $\succsim^*$ is an incomplete preference relation on $\mathcal{F}$. Ghirardato et al. (2004) show that if $\succsim$ satisfies the axioms AA1, AA2, Certainty Independence, AA4 and AA5, there exists a *unique* prior set $\mathcal{C}$ such that

$$f \succsim^* g \iff \int u(f)\ dP \geq \int u(g)\ dP \qquad \forall P \in \mathcal{C}. \tag{1.4}$$

[37] A simple example is $\mathcal{C} = B_\epsilon(P) \subseteq \Delta(S)$, i.e. a ball of radius ϵ around some $P \in \Delta(S)$, $\alpha = \frac{3}{4}$ and $\mathcal{C}' = B_{\frac{\epsilon}{2}}, \alpha' = 1$.

[38] Or put differently: the MEU axioms of Gilboa and Schmeidler (1989) without Uncertainty Aversion.

[39] If we assume Uncertainty Aversion we are back to the Gilboa and Schmeidler (1989). Here the ambiguity attitude function is constant 1, i.e. pure pessimism.

[40] This is a simplified version. We omit some characteristics of the ambiguity attitude function here. The next pages for details.

The act f is unambiguously preferred to g if and only if f results in a higher expected utility than g for every $P \in \mathcal{C}$. The set $\mathcal{C}$ is not only the smallest prior set that can represent $\succsim$ but also the Clarke-Differential at 0.[41] Throughout this thesis we refer to this prior set $\mathcal{C}$ as the *GMM prior set* of $\succsim$.

Ghirardato et al. (2004) furthermore show that, under their axioms, ambiguity attitude is constant for acts that perceive "similar ambiguity" given the GMM prior set $\mathcal{C}$. The acts f and g perceive similar ambiguity (denoted by $f \asymp g$, this definition is also derived from the unambiguous preference relation) if they order the elements of $\mathcal{C}$ in the same way, i.e.

$$f \asymp g \iff \left(\int u(f)\ dP \geq \int u(f)\ dQ \iff \int u(g)\ dP \geq \int u(g)\ dQ\ \forall P, Q \in \mathcal{C} \right).$$

They denote by $[f]$ the equivalence class of $\asymp$ that contains f. The acts in $[x]$ are called crisp acts. Thus an act is crisp if

$$\int u(f)\ dP = \int u(f)\ dQ \qquad \forall P, Q \in \mathcal{C}.$$

Of course all constant acts are crisp. We can now state the representation result of Ghirardato et al. (2004).

Theorem 1.7 (Ghirardato et al.)**.** *Let $\succsim$ be a preference relation on $\mathcal{F}$ that satisfies AA1, AA2, Certainty Independence, AA4 and AA5. Then there exists a non-empty, convex, compact prior set $\mathcal{C}$, a non-constant affine function $u : L \to \mathbb{R}$ and a function $a : \mathcal{F}_{/\asymp} \to [0, 1]$ such that $\succsim$ is represented by the functional $I : \mathcal{F} \to \mathbb{R}$ defined by*

$$I(f) = a([f]) \min_{P \in \mathcal{C}} \int u(f)\ dP + (1 - a([f])) \max_{P \in \mathcal{C}} \int u(f)\ dP, \tag{1.5}$$

and u and $\mathcal{C}$ represent $\succsim^*$ in the sense of (1.4). Moreover $\mathcal{C}$ is unique, u is unique up to positive affine transformations and the function a restricted to $\mathcal{F}_{/\asymp} \backslash [x]$ is unique.

Via this construction of the preference functional, Ghirardato et al. (2004) claim to have achieved a separation of perceived ambiguity and ambiguity attitude. Note however that there may still be (and will almost always be) multiple representations of the same preferences.

[41] Clarke-differentiability is a generalization of the Gateaux-differentiability of functionals. The set of priors is obtained by this "appropriately generalized notion of derivative of a preference functional", see Ghirardato et al. (2004).

Furthermore note that Theorem 1.7 is an *only if* statement, thus it is possible that $\succsim$ can be represented by a functional I which has the properties in the theorem but which does not satisfy the stated axioms. For instance the ambiguity attitude function may be very *steep* such that preferences violate the Monotonicity axiom.[42]

To sum up this section on CEU and MP approaches, we want to highlight where the axiomatic differences between these two approaches lie. The crucial difference is the different version of the Independence axiom implying the different characteristics of the representation functional. Where CEU assumes Comonotonic Independence, the classic MP approaches assume Certainty Independence.[43] For instance preferences representable by a convex capacity are characterized by the same axioms as MEU preferences except that we have Comonotonic Indepedence instead of Certainty Independence. JP-capacities and the α-MEU model are the natural extensions of convex capacities and the MEU model to allow also optimistic attitudes towards ambiguity.

1.2.3 Perceived Ambiguity, Ambiguity Attitude and their Separation

The question of what perceived ambiguity and ambiguity attitude is behaviourally and axiomatically and how these two concepts can be separated is a much debated topic in the literature on decision-making under ambiguity. Several approaches have already been introduced here. In this subsection we take a detailed look at the approaches on this topic which are relevant for this thesis.

Two Definitions of Ambiguity Aversion and our Hierarchy of Ambiguity Aversion

Schmeidler's definition of ambiguity aversion has been already introduced. He defines it via the Uncertainty Aversion axiom. In the CEU framework this implies convex capacities. In the MP approach it implies MEU.

A different approach is taken by Ghirardato and Marinacci (2002). In the spirit of Yaari (1987), they define ambiguity aversion via a benchmark for ambiguity neutrality as well as a comparative notion of ambiguity aversion. Combined this allows

[42] See the second article for an example of a MP preference that violates Monotonicity.

[43] See the second article for an axiomatization of MP preferences without the Certainty Independence axiom.

an absolute notion of ambiguity aversion. Their benchmark for ambiguity neutrality is subjective expected utility.[44] Their comparative notion states that if for a preference relation some act f is preferred to a constant act l, then a less ambiguity averse preference relation prefers f to l as well. That is $\succsim_1$ is more ambiguity averse than $\succsim_2$ if for all acts f and constant acts l

$$f \succsim_1 l \implies f \succsim_2 l. \tag{1.6}$$

Kelsey and Nandeibam (1996) independently suggests the same comparative notion of ambiguity aversion for CEU preferences. The absolute notion of Ghirardato and Marinacci is thus the following.

Definition 1.4 (Ghirardato and Marinacci). *A preference relation $\succsim$ reveals ambiguity aversion if there exists a subjective expected utility preference $\succsim_{SEU}$ such that for all acts $f \in \mathcal{F}$ and all constant acts $l \in L$:*

$$f \succsim l \implies f \succsim_{SEU} l.$$

The intuition is clear: a DM is ambiguity averse if she is more ambiguity averse than some ambiguity neutral (SEU) DM. Restricted to the CEU framework, a preference relation $\succsim$ is ambiguity averse if and only if the corresponding capacity ν has a non-empty core, i.e. is balanced. Every element of $P \in Core(\nu)$ induces an SEU preference $\succsim_{SEU}$ such that $\succsim$ is more ambiguity averse than $\succsim_{SEU}$.

Balanced capacities are characterized in the Anscombe-Aumann framework by Chateauneuf and Tallon (2002) via their "Sure Expected Utility Diversification". As already highlighted, this characterization is not purely in terms of preferences and therefore not a proper axiom. We present in our first article of this thesis an axiom on mixing preferences that we call *1 - Ambiguity Aversion.* It states that if the mix of indifferent acts constitutes a constant act, then this constant act is preferred. We show this axiom to be equivalent to Sure Expected Utility Diversification. We thus provide an axiomatization of balanced capacities. We therefore also axiomatize the definition of ambiguity aversion by Ghirardato and Marinacci (2002) in the

[44] Epstein (1999) introduces a similar approach but takes probabilistic sophistication (Machina and Schmeidler (1992)) as the benchmark for ambiguity neutrality.

CEU framework.[45] Crucially this implies that the concept of ambiguity aversion by Ghirardato and Marinacci (2002) is equivalent to an axiom on mixing preferences, just like Schmeidler's, but weaker. This crucial insight leads to our *Hierarchy of Ambiguity Aversion*, which we introduce in our first article.

The axiom 1 - Ambiguity Aversion is the weakest axiom of the Hierarchy. Our axiom *2 - Ambiguity Aversion* is stronger and requires a preference for mixing when the mix constitutes a binary act.[46] We show that this axiom characterizes exact capacities, an open problem in decision-theory. The hierarchy proceeds in this fashion until it reaches $|S|$ - *Ambiguity Aversion* which we show to be equivalent to Schmeidler's Uncertainty Aversion and thus to his definition of ambiguity aversion. We therefore introduce a new conceptual framework with different levels of preference for mixing amongst acts which has the above two popular definitions of ambiguity aversion as its extreme cases.

Perceived Ambiguity

In the Multiple Prior model, one would like to interpret prior sets as the perceived ambiguity of the DM. We have already highlighted the problem that MP preferences have multiple representations. If there are many representations, then which prior set is the *correct* representation of perceived ambiguity? Only after answering this question one can aim at separating perceived ambiguity from ambiguity attitude.

Ghirardato et al. (2004) provide a uniqueness result via their unambiguous preference relation that induces a prior set which they interpret as the perceived ambiguity. They suggest a comparative notion of perceived ambiguity for their invariant biseparable preferences: $\succsim_1$ perceives more ambiguity than $\succsim_2$ if for all $f, g \in \mathcal{F}$

$$f \succsim_2^* g \implies f \succsim_1^* g.$$

Ghirardato et al. show that this is equivalent to the utility functions being positive affinely related and $\mathcal{C}_2 \subseteq \mathcal{C}_1$, where $\mathcal{C}_i$ is the GMM prior set of $\succsim_i, i = \{1, 2\}$. At first this makes intuitive sense: when one prior set is a subset of another then it reflects less ambiguity. In the third article we criticize their approach. We propose that perceived ambiguity is location independent, i.e. whether one prior set reflects more ambiguity than another should not depend on their location within the prob-

[45] Actually we axiomatize their definition of ambiguity aversion in a larger framework than CEU.
[46] A binary act maps to at most two different lotteries.

ability simplex. We claim that we hereby provide a more satisfactory definition of comparative perceived ambiguity. As we show in the article, our approach allows an application to games as well as a very general equilibrium existence result, since we can exogenously fix degrees of perceived ambiguity for players.

Separation of Perceived Ambiguity and Ambiguity Attitude: Exact Capacities

We have introduced several definitions and comparative notions for both perceived ambiguity and ambiguity attitude. Separating these two concepts is tricky. To illustrate that this separation still allows a fruitful debate consider the following absurdity: restricted to exact capacities, the *comparative notion of ambiguity aversion* by Ghirardato and Marinacci (2002) as well as Kelsey and Nandeibam (1996) and the *comparative notion of perceived ambiguity* by Ghirardato et al. (2004) *are identical.* To illustrate this, assume that $\succsim_1$ and $\succsim_2$ are representable by the exact capacities ν_1 and ν_2, respectively. Assume that $\succsim_1$ is more ambiguity averse than $\succsim_2$ according to the definition of Ghirardato and Marinacci as well as Kelsey and Nandeibam.

Kelsey and Nandeibam (1996) shows that this is equivalent to

$$\nu_1(E) \leq \nu_2(E) \text{ for all } E \in \mathcal{P}(S).$$

It is easily shown that this in turn is equivalent to

$$Core(\nu_2) \subseteq Core(\nu_1)$$

and that this is equivalent to

$$\mathcal{C}_2 \subseteq \mathcal{C}_1,$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the GMM prior sets of the preferences $\succsim_1$ and $\succsim_2$, respectively.[47] This in turn, according to Ghirardato et al., is equivalent to $\succsim_1$ perceiving more ambiguity than $\succsim_2$.

This illustrates the curious fact that for exact capacities, the notion of comparative ambiguity aversion by Ghirardato and Marinacci (2002) as well as Kelsey and Nandeibam (1996) is exactly the notion of comparative perceived ambiguity by Ghir-

[47] Note that the GMM set of priors is not equal to the core, it is always a superset as well as equal to the core if and only if the capacity is convex.

ardato et al. (2004). In our opinion this suggests that the questions about the correct definition of perceived ambiguity and ambiguity aversion is still very controversial.

In this thesis we suggest that, in a to be specified way, a larger set of priors reflects more perceived ambiguity than a small set of priors, see our third article. We furthermore suggest that comparative ambiguity aversion is captured by our *Hierarchy of Ambiguity Aversion*, hereby following the spirit of Schmeidler's intuition of defining ambiguity aversion via a preference for mixing amongst acts, see our first article.

Separation of Perceived Ambiguity and Ambiguity Attitude: JP-Capacities

JP-capacities and especially their subclass neo-additive capacities allow a nice separation of perceived ambiguity and ambiguity attitude. These classes of preferences do not suffer from having multiple representations within their class.[48] The perceived ambiguity is uniquely represented by the core of the convex part of the capacity and ambiguity attitude is represented by the α-parameter. Eichberger and Kelsey (2014) successfully use this fact in application to games as we illustrate in the fourth article. The clear separation of perceived ambiguity and ambiguity attitude makes it possible to perform comparative statics in one factor whilst holding the other constant. This way the influence of one factor can be analyzed without the other one interfering.

[48] For instance a JP-capacity ν cannot be represented by both some convex capacity μ and $\alpha \in [0,1]$ as well as some different μ', α'.

1.3 Ambiguous Games

The CEU and MP models introduced thus far can be utilised to model strategic interaction under ambiguity. Capacities or prior sets can for instance represent ambiguous beliefs of players about the strategic behaviour of the other players. The crucial part is to construct an equilibrium concept. In a Nash equilibrium, players choose optimal strategies given their beliefs. These beliefs are probability distributions over the pure strategy set of the other players and are consistent in the sense that the support of the beliefs only contain optimal responses of the other players. One approach for ambiguous games is to extend this idea of consistency to ambiguous beliefs by defining a convincing support notion for capacities or prior sets. Several approaches have been proposed. In the following we introduce the theories of the papers that are relevant for our contribution, most importantly Eichberger and Kelsey (2014). In the third and fourth article of this thesis we add to the literature on ambiguous games. We introduce a concept of perceived ambiguity in the MP approach which allows an application to normal-form games. We prove a very general equilibrium existence result and illustrate a broad range of modelling capabilities. Here we only sketch our contribution as it is discussed in detail in the third article.

Framework, Notation and Motivation for Ambiguous Games

We study normal-form games. A normal-form game $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ consists of a finite set of players N, finite pure strategy sets S_i and payoff function u_i for player i. The set of pure strategy combinations is denoted by S and S_{-i} is the set of strategy combinations of player i's opponents. Player i has payoff function $u_i : S \to \mathbb{R}$. The set $\Delta(S_{-i})$ denotes the set of probability distributions over S_{-i}.

		Player 2 L	Player 2 R
Player 1	U	$100, 1$	$0, 0$
Player 1	D	$99, 1$	$99, 0$

Figure 1.3: Is the Nash Equilibrium a good prediction?

We illustrate the ideas and theories by means of the game in Figure 1.3. The game has the strategy combination (U, L) as its unique Nash Equilibrium. However strategy D is very tempting for player 1 as it avoids getting 0 when R is played and is hardly worse then U when L is played. It is thus imaginable that the Nash Equilibrium provides a bad prediction of the outcome and that we observe (D, L) instead.

We illustrate how models with ambiguous beliefs provide the flexibility to have the strategy combination (D, L) as an equilibrium under ambiguity. Such models thus have the potential to model behaviour patterns that are closer to real-life behaviour in games where the Nash Equilibrium fails.

1.3.1 Eichberger and Kelsey (2014)

Eichberger and Kelsey (2014) represent beliefs of players about the strategic choice of their opponents by JP-capacities. As already highlighted this capacity class has the nice characteristic of a clean separation of perceived ambiguity from ambiguity attitude. The perceived ambiguity of a JP-capacity $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ is represented by the convex capacity μ and its core. The ambiguity attitude is represented by the parameter α. This separation makes comparative statics exercises in perceived ambiguity and ambiguity attitude possible.[49]

To define an equilibrium concept, Eichberger and Kelsey (2014) need a support notion for JP-capacities. This allows the definition of Equilibrium under Ambiguity. In equilibrium, the support of the beliefs of the players only contain optimal strategies for the other players, given their beliefs. Through this approach the idea of consistency of Nash Equilibrium is generalized to ambiguous games.

Eichberger and Kelsey (2014) define the support of a JP-capacity $\nu = \alpha\mu + (1 - \alpha)\bar{\mu} : \mathcal{P}(S) \to [0, 1]$ as

$$supp(\nu) = \bigcap_{P \in Core(\mu)} supp(P),$$

where as usual $supp(P) = \{s \in S | P(s) > 0\}$ for $P \in \Delta(S)$. This support notion for prior sets, in this case $Core(\mu)$, goes back to Ryan (2002).

In equilibrium the players maximize Choquet Expected Utility given their beliefs. The following equilibrium concept captures this.

[49] We introduce their approach for performing comparative statics in detail in our fourth article.

Definition 1.5 (Equilibrium under Ambiguity: Eichberger and Kelsey)**.** *Let* $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ *be a normal-form game and consider JP-capacities* $\hat{\nu}_i : \mathcal{P}(S_{-i}) \to [0,1]$ *for* $i \in \{1, \dots, N\}$*. Then* $\hat{\nu} = \langle \hat{\nu}_1, \dots, \hat{\nu}_N \rangle$ *is an equilibrium in beliefs under ambiguity (EUA) if for all* $i \in \{1, \dots, N\}$

$$\emptyset \neq supp(\hat{\nu}_i) \subseteq \times_{j \neq i} \arg\max_{s_j \in S_j} \int u_j(s_j, s_{-j}) \; d\hat{\nu}_j,$$

with $\int u_j(s_j, s_{-j}) \; d\hat{\nu}_j$ *being the Choquet integral of the strategy* s_j *given the capacity* $\hat{\nu}_j$*.*

If $supp(\hat{\nu}_i)$ contains just one element $\hat{s}_i$ for all $i \in \{1, \dots, N\}$ then $\hat{s} = (\hat{s}_1, \dots, \hat{s}_N)$ is called singleton equilibrium in beliefs under ambiguity.

Eichberger and Kelsey apply this model to games with positive externalities and increasing differences. We illustrate their results in detail on our fourth article.

1.3.2 Our Multiple Prior Approach

In our third article, we represent beliefs of players by MP preferences: a prior set $\mathcal{C}_i$ over $\Delta(S_{-i})$ as well as an ambiguity attitude $\alpha_i \in [0,1]$ over the players' own strategies S_i.[50] For player i with belief $\mathcal{C}_i \subseteq \Delta(S_{-i})$ and ambiguity attitude function α_i the evaluation of a strategy $s_i \in S_i$ is therefore

$$V_i(s_i | \mathcal{C}_i, \alpha_i) = \alpha_i \min_{P \in \mathcal{C}_i} \int u(s_i, s_{-i}) \; dP + (1 - \alpha_i) \max_{P \in \mathcal{C}_i} \int u(s_i, s_{-i}) \; dP.$$

To define an equilibrium notion, we adapt the support and equilibrium notion of Eichberger and Kelsey (2014). It is the natural extension of their approach to the MP model and also uses the support notion of Ryan (2002).

Definition 1.6. *Let* $\mathcal{C}$ *be a prior set on* $\Delta(S)$*. The support of* $\mathcal{C}$ *is defined by*

$$supp(\mathcal{C}) = \bigcap_{P \in \mathcal{C}} supp(P).$$

The support of a prior set consists of the strategies that receive positive weight by all elements of the prior set. In equilibrium, the support of the prior set is non-empty

[50] For a simpler illustration we reduce attention to α-MEU preferences in this subsection. In the third article we assume a much larger class of preferences where ambiguity attitude is not necessarily constant.

and only contains the opponents' best responses given their beliefs.

Definition 1.7 (Equilibrium under Ambiguity). *Let* $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N\rangle$ *be a normal-form game. The tuple* $(\mathcal{C}_i, \alpha_i)_{i=1}^N$ *is an Equilibrium under Ambiguity if for all* $1 \leq i \leq N$*,*

$$\emptyset \neq supp(\mathcal{C}_i) \subseteq \times_{j \neq i} \arg\max_{s_j \in S_j} \left[V(s_j | \mathcal{C}_j, \alpha_j) \right].$$

If $supp(\mathcal{C}_i)$ contains just a single element $\hat{s}_i \in S_i$ for all $i \in \{1, \ldots, N\}$ we refer to the equilibrium as a singleton equilibrium and $\hat{s} = (\hat{s}_i, \ldots, \hat{s}_N)$ as its strategy profile.

In the third article we introduce a measure of perceived ambiguity in the MP model. The key and desired implication of this measure is that two prior sets reflect the same perceived ambiguity if and only if they only differ in location. This is illustrated in Figure 1.4. This definition allows us to exogenously fix perceived ambiguity for the players without loosing dynamics.

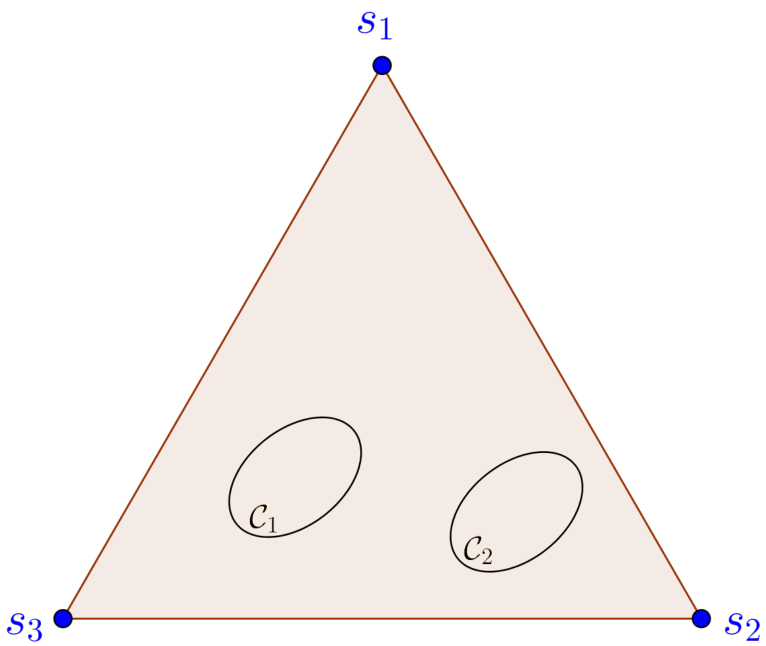


Figure 1.4: The prior sets $\mathcal{C}_1$ and $\mathcal{C}_2$ differ only in location and thus reflect the same ambiguity.

We prove equilibrium existence for normal-form games. The class of preferences considered is a superclass of IB preferences and therefore also of CEU and α-MEU. Thus our equilibrium existence result holds for these preference classes as well.

1.3.3 Back to the Example

Reconsider the normal-form game from Figure 1.3. We have argued that strategy D for player 1 is likely to occur, even though the strategy combination (U, L) is the unique Nash equilibrium. We show that both of the above approaches can easily induce the strategy combination (D, L) as an equilibrium under ambiguity.

Eichberger and Kelsey (2014) applied to the Game

Assume that we can represent the beliefs of players by the neo-additive capacities $\nu_1 = \nu_{\pi_1,\delta_1,\alpha_1}$ and $\nu_2 = \nu_{\pi_2,\delta_2,\alpha_2}$.[51] Since L strictly dominates R, player 2 will always play L, regardless of the beliefs about what player 1 does. We can thus restrict attention to player 1.

Player 1 believes that player 2 plays the optimal strategy L, i.e. $\pi(L) = 1$. But she is not completely confident in this belief. Her confidence is reflected by $(1 - \delta_1)$, her degree of perceived ambiguity by δ_1. Furthermore she has degree of pessimism α_1. Thus the belief of player 1 can be represented by the capacity $\nu_1 : \mathcal{P}(\{L, R\}) \to [0, 1]$ with

$$\nu_1(E) = \begin{cases} 0, & E = \emptyset \\ (1-\alpha_1)\delta_1, & E = R \\ (1-\alpha_1)\delta_1 + 1 - \delta_1, & E = L \\ 1, & E = S \end{cases}.$$

The Choquet expected utilities of the two strategies U and D, given ν_1 can be calculated with equation (1.3):

$$\int U \, d\nu_1 = (1-\alpha_1)\delta_1 100 + \alpha_1\delta_1 0 + (1-\delta_1)100$$
$$\int D \, d\nu_1 = 99.$$

Thus $D \succ U$ if and only if $\alpha_1\delta_1 > \frac{1}{100}$. Thus for $\alpha_1\delta_1 > \frac{1}{100}$, the pair of capacities (ν_1, ν_2) with $\nu_2 = \nu_{\pi_2,\delta_2,\alpha_2}$ such that $\pi_2(D) = 1$ constitutes an Equilibrium under

[51] Recall that neo-additive capacities are a subclass of JP-capacities. Thus we are within the framework of Eichberger and Kelsey (2014). We choose neo-additive capacities for this example because of their intuitive interpretation.

Ambiguity. The strategy combination (D, L) is a singleton equilibrium. Indeed it is straightforward to see that for values of δ_1 and α_1 such that $\alpha_1\delta_1 > \frac{1}{100}$, the above is the unique equilibrium under ambiguity.

This result makes sense intuitively. When the degrees of perceived ambiguity δ_1 as well as pessimism α_1 are sufficiently large, player 1 does not choose U, but D instead to avoid the possibility of the bad outcome 0. Mathematically this is possible since when $\alpha_1\delta_1$ gets larger, the Choquet integral puts more weight on the bad outcome R when U is evaluated. This holds even when R is not in the support of ν_1.

The example shows how the model of Eichberger and Kelsey (2014) can cope with behaviour patterns that are intuitive but clash with the Nash Equilibrium concept.

The Multiple Prior Approach applied to the Game

In a game with two players and two strategies the set S_{-i} consists of two elements. This implies that prior sets are intervals. A prior set in the game is thus of the kind $\mathcal{C}_i = Conv(P, Q | P, Q \in \Delta(S_{-i}))$. We can represent perceived ambiguity by the length of this interval, i.e. by a parameter $\delta_i \in [0, 1]$.[52]

The evaluation of strategy U, given a prior set $\mathcal{C}_1$ and ambiguity attitude α_1 is

$$\alpha_1 \min_{P \in \mathcal{C}_1} \int U \, dP + (1 - \alpha_1) \max_{P \in \mathcal{C}_1} \int U \, dP.$$

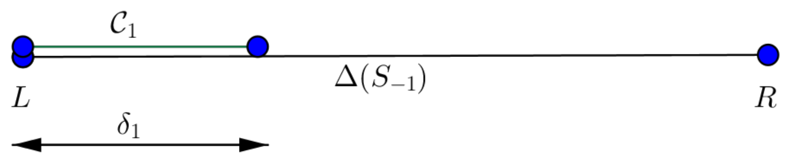


Figure 1.5: The equilibrium belief of player 1 with perceived ambiguity δ_1.

For the prior set $\mathcal{C}_1$ with interval length δ_1 depicted in Figure 4.7, the evaluation of U is

$$\alpha_1(\delta_1 0 + (1 - \delta_1)100) + (1 - \alpha_1)100.$$

The evaluation of D is again 99. Thus again $D \succ U$ if and only if $\alpha_1\delta_1 > \frac{1}{100}$. Consider any prior set $\mathcal{C}_2$ such that $supp(\mathcal{C}_2) = \{D\}$ and some ambiguity attitude

[52] To exogenously fix perceived ambiguity would mean to fix an interval-length. Our result in the third article implies equilibrium existence for any exogenously fixed interval length.

$\alpha_2 \in [0,1]$. Note that $supp(\mathcal{C}_1) = \{L\}$. Now for $\alpha_1\delta_1 > \frac{1}{100}$ the tuple $(\mathcal{C}_i, \alpha_i)_{i=1}^2$ constitutes an Equilibrium under Ambiguity. This illustrates how our MP approach can model deviations from the Nash Equilibrium prediction.

2 A Hierarchy of Ambiguity Aversion and the Axiomatization of Balanced and Exact Capacities

Abstract

This article introduces a new conceptual framework of *ambiguity aversion*. Higher levels of ambiguity aversion are axiomatically characterized by a more pronounced *preference for mixing* amongst acts. The weakest level of this hierarchy corresponds to a preference for mixing, conditional on this mix eliminating all ambiguity. We show that this axiomatically characterizes the definition of ambiguity aversion by Ghirardato and Marinacci (2002). The strongest level of the hierarchy corresponds to an unconditional preference for mixing and matches the definition of ambiguity aversion by Schmeidler (1989). We illustrate how preferences can exhibit mixing preferences that lie strictly in between these two approaches.

We show that every level of the hierarchy is characterized by a specific geometric property concerning the set of measures that dominate the preference relation. By-products of our approach are the axiomatizations of balanced and exact capacities, thus far open problems in decision theory.

Keywords: Ambiguity Aversion, Choquet Expected Utility, Balanced Capacities, Exact Capacities, Multiple Priors

2.1 Introduction

Ambiguity aversion is the aversion towards unknown risk. Ellsberg (1961) and others have demonstrated that ambiguity aversion occurs systematically in human decision-making. This sparked several approaches for defining ambiguity aversion in terms of preferences.

Schmeidler (1989) defines ambiguity aversion through preference for mixing amongst acts.[1] The intuition is that mixing smooths out utility distributions across states and thus provides a hedge against ambiguity. An ambiguity averse decision maker (DM) is thus better off. Ghirardato and Marinacci (2002) propose an alternative definition. A DM is ambiguity averse if she is, in the spirit of Yaari (1987), more ambiguity averse then some subjective expected utility (SEU)[2] DM: whenever the DM prefers an act to a constant, the SEU DM does so as well. This article shows that their definition is characterized by a preference for mixtures which eliminate all ambiguity, i.e. a preference for mixtures that constitute a constant act. An ambiguity averse DM is thus only guaranteed to prefer perfect hedges.

Schmeidler's definition is strong as it postulates a preference for mixing regardless of what act the mix constitutes. In contrast, the definition of Ghirardato and Marinacci is weak as it postulates a preference for mixing only when the mix eliminates *all* ambiguity.

These two approaches are the extreme cases of what we refer to as the *Hierarchy of Ambiguity Aversion*. We illustrate how DM's can exhibit levels of ambiguity aversion that lie strictly in between those approaches. Every level of the hierarchy is characterized through an axiom on mixing preferences. As we go up the hierarchy the corresponding axiom increases in strength.

An Example

To build intuition consider the following example. There are three states of the world s_1, s_2, s_3 and three acts f, g, h. The consequences l_1, l_2, l_3 are lotteries over some set of prizes.

Assume that the DM is indifferent between f, g and h. The $\frac{1}{2}$-mix between the acts f and g results in the lottery $\frac{1}{2}l_1 + \frac{1}{2}l_2$ in every state and thus eliminates all ambiguity. It is a perfect hedge against ambiguity. The $\frac{1}{2}$-mix between f and h reduces ambiguity to two different lotteries: $\frac{1}{2}l_1 + \frac{1}{2}l_3$ and $\frac{1}{2}l_1 + \frac{1}{2}l_2$. It reduces ambiguity but does not eliminate it. The $\frac{1}{2}$-mix between g and h results in three different lotteries: $\frac{1}{2}l_2 + \frac{1}{2}l_3$, l_2 and l_1.

[1] Acts are mappings from the state space into finite-support lotteries on some set of prizes. Mixtures are performed pointwise. See section 2.2 for the details.

[2] Ghirardato and Marinacci (2002) suggest that SEU decision makers are ambiguity neutral.

	s_1	s_2	s_3
f	l_1	l_1	l_2
g	l_2	l_2	l_1
h	l_3	l_2	l_1

It is plausible that an ambiguity averse DM prefers the $\frac{1}{2}$-mix between the acts f and g. However, one may ask whether ambiguity aversion necessarily leads to a preference for the $\frac{1}{2}$-mix between f and h as well as between g and h.

Consider three types of our decision maker: DM 1, DM 2 and DM 3. Assume that DM 1 prefers all three mixes. DM 2 prefers the mix between f and g as well as between f and h. DM 3 only prefers the mix between f and g. We suggest that DM 1 exhibits a stronger level of ambiguity aversion than DM 2, who exhibits a stronger level of ambiguity aversion than DM 3. The rationale for this is the following: DM 3 is willing to mix if this eliminates all ambiguity, but she is not willing to mix otherwise. DM 2 is willing to mix in the cases where ambiguity is reduced to one or two different lotteries. Thus DM 2 is willing to mix when ambiguity is reduced by a lesser extent and thus exhibits a higher level of ambiguity aversion then DM 3. DM 1 is always willing to mix, regardless of how much the ambiguity is reduced by, and thus exhibits a higher level of ambiguity aversion than DM 2.

The Hierarchy of Ambiguity Aversion

In the above example DM 1 is ambiguity averse in the spirit of Schmeidler (1989). DM 3 is ambiguity averse in the spirit of Ghirardato and Marinacci (2002). DM 2 exhibits a level of ambiguity aversion that lies strictly in between the two.

We characterize different levels of ambiguity aversion in the spirit of this example. Let S be a finite state space. A DM who prefers all mixtures that reduce ambiguity to at most k different lotteries, $k \in \{1, \ldots, |S|\}$, is called *k - ambiguity averse*. The higher is k, the more pronounced is the preference for mixing and thus the ambiguity aversion. This axiomatic structure is the *Hierarchy of Ambiguity Aversion*.

Outline

The article is organized as follows. After introducing the framework and notation in Section 2.2, we introduce our Hierarchy of Ambiguity Aversion in Section 2.3 and show that the extreme cases are equivalent to the well-known approaches by Schmeidler (1989) and Ghirardato and Marinacci (2002). Section 2.4 discusses the special case of Choquet Expected Utility preferences. Section 2.5 provides examples. Section 2.6 concludes. All proofs are in the Appendix.

2.2 Framework and Notation

We assume the classic framework of Anscombe and Aumann (1963). Consider a finite state space S. The powerset of S, $\mathcal{P}(S)$ is the set of events.

We study preference relations $\succsim$ on the set of acts $\mathcal{F} = \{f : S \to L\}$, where L is the set of finite-support lotteries over some set of prizes X. The asymmetric and symmetric components of $\succsim$ are denoted by $\succ$ and $\sim$, respectively. With the usual abuse of notation, L also denotes the set of constant acts. Mixtures of acts are performed pointwise: for $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$ we denote by $\lambda f + (1-\lambda)g$ the act which results in $\lambda f(s) + (1-\lambda)g(s) \in L$ for all $s \in S$.

An act is called n - *act* if it maps to at most n different lotteries, i.e. it holds that $|\{l \in L | \exists s \in S : f(s) = l\}| \leq n$. The set of n - acts is denoted by $\mathcal{F}_n$. Thus $\mathcal{F}_1 = L$ is the set of constant acts and $\mathcal{F}_2$ is the set of binary acts. Binary acts map to at most two different lotteries. They can be written as $l_E l'$ with $l, l' \in L$ and $E \subseteq S$, i.e. the act results in the lottery l on E and lottery l' on E^c, the complement of E.

Let $u : L \to \mathbb{R}$ be a non-constant affine function (*utility function*). An act is called n - *expected utility act* if it maps to at most n different expected outcomes, i.e. it holds that $|\{a \in \mathbb{R} | \exists s \in S : u(f(s)) = a\}| \leq n$, where $u(f(s))$ is the expected utility of the act f in state s. The set of n - expected utility acts is denoted by $\mathcal{F}_n^u$. Note that for every utility function u it holds that $\mathcal{F}_n \subseteq \mathcal{F}_n^u$ but the reverse does not hold.

We denote by B_0 the set of real-valued functions on S. For $f \in \mathcal{F}$ and utility function $u : L \to \mathbb{R}$, the function $u(f)$ is the element of B_0 defined by $u(f)(s) = u(f(s))$ for all $s \in S$. A functional $I : B_0 \to \mathbb{R}$ and utility function $u : L \to \mathbb{R}$

represent a preference relation $\succsim$ if for all $f, g \in \mathcal{F}$

$$f \succsim g \iff I(u(f)) \geq I(u(g)).$$

The function $I \circ u : \mathcal{F} \to \mathbb{R}$ is called a *representational functional* of $\succsim$. A functional $I : B_0 \to \mathbb{R}$ is *monotonic* if $I(\phi) \geq I(\psi)$ for all $\phi, \psi \in B_0$ for which it holds that $\phi(s) \geq \psi(s)$ for all $s \in S$. A functional $I : B_0 \to \mathbb{R}$ is *constant-linear* if $I(a\phi + b) = aI(\phi) + b$ for all $\phi \in B_0, a \geq 0$ and $b \in \mathbb{R}$.

To prepare for later discussion we introduce the concept of capacities and the Choquet Integral that will be of specific interest later on. A *capacity* is a normalized and monotonic set-function on S. The *Choquet Integral* (Choquet (1954)) of a function $Y \in B_0$ with respect to a capacity $\nu : \mathcal{P}(S) \to [0, 1]$ is defined by

$$\int_{-\infty}^{0} (\nu(\{s \in S | Y(s) \geq x\}) - 1)dx + \int_{0}^{\infty} \nu(\{s \in S | Y(s) \geq x\})dx,$$

where the integrals on the right side of the equation are Riemann integrals.

The set of probability distributions over S is denoted by $\Delta(S)$. The *core of a capacity* $\nu : \mathcal{P}(S) \to [0, 1]$ is the set of probability distributions over S that pointwise dominate ν:

$$Core(\nu) = \{P \in \Delta(S) | P(E) \geq \nu(E),\ \forall E \in \mathcal{P}(S)\}.$$

For $E \in \mathcal{P}(S)$ and capacity ν we define

$$H_\nu(E) = \{P \in \Delta(S) | P(E) = \nu(E)\},$$

the set of probability distributions over S that put the same weight on E as the capacity does. There are three capacity classes that are particularly relevant for this paper: balanced, exact and convex capacities. A capacity ν is *balanced* if its core is non-empty. A capacity ν is *exact* if its capacity values are equal to the lower envelope of the core, i.e. when $\nu(E) = \min_{P \in Core(\nu)} P(E)$ for all $E \in \mathcal{P}(S)$. A capacity ν is *convex* if $\nu(E_1) + \nu(E_2) \leq \nu(E_1 \cup E_2) + \nu(E_1 \cap E_2)$ for all $E_1, E_2 \in \mathcal{P}(S)$. It is a well-known fact that convex capacities are exact and exact capacities are balanced but that the reverse implications do not hold. A preference relation is a *Subjective Expected Utility* (SEU) preference if it can be represented by a utility function and an additive capacity.

Axioms and the Set of Dominating Measures $\mathcal{D}_{\succsim}$

Throughout this article we consider preferences that satisfy the following axioms.

Axiom 2.1 (Weak Order). 1. For all $f, g \in \mathcal{F}$ either $f \succsim g$ or $g \succsim f$.
2. For all $f, g, h \in \mathcal{F}$ if $f \succsim g \succsim h$ then $f \succsim h$.

Axiom 2.2 (Certainty Independence). For all $f, g \in \mathcal{F}$, $l \in L$ and $\lambda \in (0, 1]$

$$f \succsim g \iff \lambda f + (1-\lambda)l \succsim \lambda g + (1-\lambda)l.$$

Axiom 2.3 (Archimedean). For all $f, g, h \in \mathcal{F}$ if $f \succ g \succ h$ then there exist $\lambda, \mu \in (0, 1)$ such that $\lambda f + (1-\lambda)h \succ g \succ \mu f + (1-\mu)h$.

Axiom 2.4 (Monotonicity). For all $f, g \in \mathcal{F}$ if $f(s) \succsim g(s)$ for all $s \in S$ then $f \succsim g$.

Axiom 2.5 (Non-Degeneracy). There are $f, g \in \mathcal{F}$ such that $f \succ g$.

Ghirardato et al. (2004) refer to preferences satisfying these five axioms as *invariant biseparable*. They show that a preference relation $\succsim$ satisfies these five axioms if and only if there exists a nonconstant affine $u : L \to \mathbb{R}$ and a monotonic, constant-linear functional $I : B_0 \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$

$$f \succsim g \iff I(u(f)) \geq I(u(g)),$$

i.e. $I \circ u$ represents $\succsim$. This result makes possible the following definition. For an act $f \in \mathcal{F}$ define

$$\mathcal{H}_{I \circ u}(f) = \left\{ P \in \Delta(S) | \sum_{s \in S} u(f(s))P(s) \geq I(u(f)) \right\}.$$

The set $\mathcal{H}_{I \circ u}(f)$ consists of the probability distributions over S that result in an evaluation of f that is at least as good as the evaluation of f with the representation functional. Furthermore define

$$H_{I \circ u}(f) = \left\{ P \in \Delta(S) | \sum_{s \in S} u(f(s))P(s) = I(u(f)) \right\}.$$

The set $H_{I\circ u}(f)$ consists of the probability distributions over S that result in an evaluation of f that is exactly as good as the evaluation of f with the representation functional.

These sets are guaranteed to be non-empty for all $f \in \mathcal{F}$ due to Proposition 7 in Ghirardato et al. (2004) which states that

$$\max_{P\in\Delta(S)} \sum_{s\in S} u(f(s))P(s) \geq I(u(f)) \geq \min_{P\in\Delta(S)} \sum_{s\in S} u(f(s))P(s) text for all f \in \mathcal{F}.$$

Consider the set

$$\mathcal{D}_{\succsim} = \left\{ P \in \Delta(S) | \sum_{s\in S} u(f(s))P(s) \geq I(u(f)) \quad \forall f \in \mathcal{F} \right\},$$

a set that is introduced in Ghirardato and Marinacci (2002).[3] It is the set of SEU measures inducing preferences which assign weakly higher expected utility to all acts. Or put differently: $\mathcal{D}_{\succsim} = \bigcap_{f\in\mathcal{F}} \mathcal{H}_{I\circ u}(f)$. The set $\mathcal{D}_{\succsim}$ is convex as well as compact and may be empty. An important special case is when $\succsim$ can be represented by the Choquet Integral with respect to some capacity ν. It then holds that $\mathcal{D}_{\succsim} = Core(\nu)$.[4] To see that this is the case consider a capacity $\nu : \mathcal{P}(S) \to [0, 1]$. For a binary act $f = l_E l' \in \mathcal{F}_2$ with $l \succsim l'$ it holds that $H_{I\circ u}(f) = \{P \in \Delta(S)|P(E) = \nu(E)\} = H_\nu(E)$ due to the way the Choquet Integral is defined. This implies

$$Core(\nu) = \bigcap_{f\in\mathcal{F}_2} \mathcal{H}_{I\circ u}(f).$$

With this insight we can state an alternative definition of exact capacities which becomes convenient in Section 2.4. A capacity $\nu : \mathcal{P}(S) \to [0, 1]$ is exact if

$$H_{I\circ u}(f) \cap Core(\nu) \neq \emptyset \text{ for all } f \in \mathcal{F}_2.$$

[3] Note that this is not the set $\mathcal{C}$ derived from the above five axioms in Ghirardato et al. (2004), i.e. the Clarke-Differential at 0. One can show that $\mathcal{D} \subseteq \mathcal{C}$ with equality if and only if the preferences are of the Maxmin expected utility type.

[4] We therefore view $\mathcal{D}_{\succsim}$ as a generalization of the core concept.

2.3 The Hierarchy

For $k \in \{1, \dots, |S|\}$ consider the following axiom.

Axiom 2.6 (k - Ambiguity Aversion). If $f_1, \dots, f_n \in \mathcal{F}$, $\alpha_1, \dots, \alpha_n \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$, $\sum_{i=1}^{n} \alpha_i f_i = f \in \mathcal{F}_k$, then $f_1 \sim \dots \sim f_n$ implies $f \succsim f_1$.

The axiom states a preference for mixing amongst acts if the mixture constitutes a k - act. A DM satisfying this axiom wants to mix amongst acts if this mix results in an act which maps to at most k different lotteries. The strength of the axiom increases with k: as k becomes larger the preference for mixing amongst acts increases.

In this section we show that the case $k = 1$ corresponds to the definition of ambiguity aversion by Ghirardato and Marinacci (2002) and that the case $k = |S|$ corresponds to the definition of ambiguity aversion by Schmeidler (1989). This provides the justification for our interpretation of Axiom 2.6 as an axiomatic *Hierarchy of Ambiguity Aversion*.

2.3.1 Definition of Ambiguity Aversion by Ghirardato and Marinacci (2002)

Ghirardato and Marinacci (2002) define comparative ambiguity aversion in the spirit of Yaari (1987): a preference relation $\succsim$ is more ambiguity averse than $\succsim'$ if for all $f \in \mathcal{F}$ and $l \in L$ it holds that

$$f \succsim l \implies f \succsim' l.$$

The intuition is that if a DM prefers some act to a constant act, then a less ambiguity averse DM does so as well. Ghirardato and Marinacci suggest that SEU preferences are ambiguity neutral.[5]

This allows the following absolute notion of ambiguity aversion, the intuition being that a preference relation reveals ambiguity aversion if it is more ambiguity averse then some ambiguity neutral preference relation.

[5] Epstein (1999) introduces a similar approach but takes probabilistic sophistication (Machina and Schmeidler (1992)) as the benchmark for ambiguity neutrality.

Definition 2.1 (GM - Ambiguity Aversion)**.** *A preference relation $\succsim$ is ambiguity averse if there exists an SEU preference $\succsim_{SEU}$ such that for all $f \in \mathcal{F}$ and $l \in L$ it holds that*

$$f \succsim l \implies f \succsim_{SEU} l.$$

The following theorem shows that this definition is characterized by the axiom 1 - Ambiguity Aversion.

Theorem 2.1. *Under the five standard axioms, a preference relation satisfies GM - Ambiguity Aversion if and only if it satisfies the axiom 1 - Ambiguity Aversion.*

2.3.2 Definition of Ambiguity Aversion by Schmeidler (1989)

Schmeidler (1989) introduces the following axiom.[6]

Axiom 2.7 (Schmeidler - Ambiguity Aversion). For all $f, g \in \mathcal{F}$ with $f \sim g$ and $\alpha \in [0, 1]$ it holds that $\alpha f + (1 - \alpha)g \succsim g$.

The axiom states that the DM always has a preference for mixing. Schmeidler suggests that this axiom characterizes ambiguity aversion as mixing smooths out utility distributions across states and thus reduced ambiguity.

Definition 2.2 (Schmeidler - Ambiguity Aversion)**.** *Under the five standard axioms, a preference relation $\succsim$ is ambiguity averse if it satisfies the axiom Schmeidler - Ambiguity Aversion.*

The following theorem shows that our axiom $|S|$ - Ambiguity Aversion is equivalent to this definition.

Theorem 2.2. *A preference relation satisfies Schmeidler - Ambiguity Aversion if and only if it satisfies the axiom $|S|$ - Ambiguity Aversion.*

Note that this result is model free, i.e. Theorem 2.2 does not rely on preferences to satisfy the five standard axioms, which Theorem 2.1 does.[7]

[6] Schmeidler calls the axiom "Uncertainty Aversion". Recall that in this thesis, the term "uncertainty" is used in its generic sense, comprising risk and ambiguity. We therefore deviate from Schmeidler's terminology.

[7] We conjecture that there is a model-free version of Theorem 2.2. See discussion at the end of the main text of this article.

2.3.3 The Hierarchy

The two preceding subsections illustrate that two of the most popular definitions of ambiguity aversion are characterized by the extreme cases $k = 1$ and $k = |S|$ of our axiomatic hierarchy introduced through Axiom 2.6. This suggests that the intermediate cases $k \in \{2, \ldots, |S|-1\}$ of Axiom 2.6 correspond to levels of ambiguity aversion in between these popular definitions. The following definition thus suggests itself.

Definition 2.3 (k - Ambiguity Aversion)**.** *Under the five standard axioms, a preference relation exhibits k - Ambiguity Aversion if it satisfies the axiom k - Ambiguity Aversion.*

Thus a DM exhibits level k ambiguity aversion if her preferences satisfy the axiom k - Ambiguity Aversion.

2.3.4 The Relationship between k - Ambiguity Averse Preferences and the set $\mathcal{D}_{\succsim}$

Consider some preference relation $\succsim$. Recall that $\mathcal{D}_{\succsim}$ corresponds to the set of probability measures that induce preferences which assign weakly higher expected utility to all acts. If for some act $f \in \mathcal{F}$ there is a *gap* between $H_{I \circ u}(f)$ and $\mathcal{D}_{\succsim}$, i.e $H_{I \circ u}(f) \cap \mathcal{D}_{\succsim} = \emptyset$, then f is evaluated in a pessimistic way in the sense that $I(u(f)) < \min_{P \in \mathcal{D}_{\succsim}} \sum_{s \in S} u(f(s))P(s)$. The act f is evaluated more pessimistic then any SEU DM, whose preferences can be represented by some $P \in \mathcal{D}_{\succsim}$, does. Conversely, if for some $f \in \mathcal{F}$ there is no *gap* between $H_{I \circ u}(f)$ and $\mathcal{D}_{\succsim}$, i.e. $H_{I \circ u}(f) \cap \mathcal{D}_{\succsim} \neq \emptyset$, then f is not evaluated in such a pessimistic way. We can find an SEU DM, whose preferences can be represented by some $P \in \mathcal{D}_{\succsim}$, who evaluates f in the same way.

The following theorem shows that there is an intuitive relationship between the axiomatic Hierarchy of Ambiguity Aversion and the abovementioned *gaps* between $H_{I \circ u}(f)$ and $\mathcal{D}_{\succsim}$. A preference relation is level k - ambiguity averse if and only if there are no *gaps* for all k - acts.[8]

Theorem 2.3. *Let $\succsim$ be a preference relation on $\mathcal{F}$ satisfying the five standard axioms. Then the following are equivalent:*

[8] The reason for the truth of this result is that there is a *gap* for some $f \in \mathcal{F}$ if and only if f is not necessarily preferred when it constitutes the mix of some indifferent acts. This is the crucial step in the proof of Theorem 2.3.

1. $\succsim$ satisfies k - Ambiguity Aversion.

2. $H_{I\circ u}(f) \cap \mathcal{D}_{\succsim} \neq \emptyset$ for all $f \in \mathcal{F}_k$.

The case $k = 1$ is a consequence of Theorem 2.1 and Theorem 12 from Ghirardato and Marinacci (2002).[9] The case $k = |S|$ is a consequence of the classic Maxmin Expected Utility representation result from Gilboa and Schmeidler (1989) in combination with Theorem 2.2.

2.4 A Special Case: the Choquet Expected Utility Model

This section analyses the consequences that the Hierarchy of Ambiguity Aversion has for the Choquet Expected Utility (CEU) model. We show that the case $k = 1$ characterizes balanced capacities[10] and that the case $k = 2$ characterizes exact capacities. Both of these preference classes have thus far lacked an axiomatization. The case $k = |S|$ characterizes convex capacities, a direct consequence of Theorem 2.2.

Furthermore we show that within the CEU model, for $k = 3, \ldots, |S|$, the axioms k - Ambiguity Aversion are not independent, i.e. they all characterize the same class of capacities: convex capacities. At first this result may be surprising as k - Ambiguity Aversion increases in strength with k. It turns out however that the crucial axiom of the CEU framework, *Comonotonic Independence*, is sufficiently strong to prevent a distinction between these levels of ambiguity aversion. The axiom Certainty Independence however is weak enough to allow a proper distinction between levels of the hierarchy, see in particular Example 2.2 below.

2.4.1 Choquet Expected Utility

The CEU model was introduced and axiomatized by Schmeidler (1989). The axioms of the model are the five axioms from above with Certainty Independence replaced

[9] Note that $H_{I\circ u}(f) \cap \mathcal{D}_{\succsim} \neq \emptyset$ for all $f \in \mathcal{F}_1$ is equivalent to the non-emptyness of $\mathcal{D}_{\succsim}$.

[10] Chateauneuf and Tallon (2002) characterize balanced capacities with their Sure Expected Utility Diversification. However this characterization is not a proper axiomatization as it is not purely in terms of preferences over acts. Their result was however a huge, if not the biggest, inspiration for this article.

by the axiom Comonotonic Independence. It requires independence only for acts that are pairwise comonotonic.[11]

Axiom 2.8 (Comonotonic Independence). For all pairwise comonotonic acts $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$

$$f \succsim g \iff \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

Schmeidler states the following representation result.

Theorem 2.4 (Schmeidler (1989)). *Let $\succsim$ be a preference relation on $\mathcal{F}$. The following are equivalent:*

1. *$\succsim$ satisfies the axioms Weak Order, Monotonicity, Comonotonic Independence, Archimedean and Non-Degeneracy.*

2. *There exists a capacity $\nu : \mathcal{P}(S) \to [0,1]$ and an affine function $u : L \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$*

$$f \succsim g \iff \int u(f)\ d\nu \geq \int u(g)\ d\nu,$$

where $\int u(f)\ d\nu$ is the Choquet Expected Utility of the act f, given u and ν.

Furthermore, ν is unique and u is unique up to positive affine transformations.

We have the following corollary of Theorem 2.3 which characterizes balanced and exact capacities. It follows directly from the fact that $Core(\nu) = \mathcal{D}_{\succsim}$ when $\succsim$ is represented by the capacity ν.

Corollary 2.1. *Let $\succsim$ be a preference relation on $\mathcal{F}$ satisfying the CEU axioms from Theorem 2.4. Let ν be the corresponding capacity. Then for $k \in \{1, 2, |S|\}$ the following are equivalent:*

1. *$\succsim$ satisfies k - Ambiguity Aversion.*

2. *$H_{I \circ u}(f) \cap Core(\nu) \neq \emptyset$ for all $f \in \mathcal{F}_k$.*

[11] Two acts $f, g \in \mathcal{F}$ are comonotonic if there exist no $s, s' \in S$ such that $f(s) \succ g(s)$ and $g(s') \succ f(s')$.

Corollary 2.1 implies that 1 - Ambiguity Aversion axiomatizes balanced capacities,[12] that 2 - Ambiguity Aversion axiomatizes exact capacities and that $|S|$ - Ambiguity Aversion axiomatizes convex capacities.

The following theorem shows that these three cases correspond to the only levels of the hierarchy that the CEU model can distinguish amongst. It implies in particular that a CEU preference satisfying 3 - Ambiguity Aversion is automatically represented by a convex capacity. Thus within the CEU framework, for $k \geq 3$, the k - Ambiguity Aversion axiom is equivalent to Schmeidler - Ambiguity Aversion.

Theorem 2.5. *Under the CEU axioms from Theorem 2.4, k - Ambiguity Aversion characterizes convex capacities for all $k \geq 3$.*

2.5 Examples

This section contains 4 examples. The first example demonstrates that preferences can be 2 - ambiguity averse without being 3 - ambiguity averse. The second example demonstrates that preferences can be 3 - ambiguity averse without being 4 - ambiguity averse. The other two examples live within the CEU framework. The third example introduces a preference relation that is 1 - ambiguity averse but not 2 - ambiguity averse, i.e. it is represented by a capacity that is balanced but not exact. The last example introduces a preference relation that is 2 - ambiguity averse but not 3 - ambiguity averse, i.e. it is represented by a capacity that is exact but not convex.

Throughout the examples we assume for simplicity that acts map to utilities, i.e. to the real numbers. In the first two examples the following notation is convenient: for any act $f \in \mathcal{F}$ there exists a (non-unique) ordering of the state space $s_f^1, s_f^2, \dots, s_f^{|S|}$ such that $f(s_f^1) \geq f(s_f^2) \geq \dots \geq f(s_f^{|S|})$. For a partition $E_1, \dots, E_n$ of the state space S we use the notation $x_{1E_1} x_2 \dots x_{n-1E_{n-1}} x_n$ to denote the act that results in x_i on E_i, $i \in \{1, \dots, n\}$.

Ghirardato et al. (2004) show that invariant biseparable preferences, i.e. preferences that satisfy the five standard axioms, can be represented by a prior set $\mathcal{C} \subseteq \Delta(S)$, which is also the Clarke-Differential at 0, as well as an ambiguity attitude function $a : \mathcal{F} \to [0,1]$.[13] We make use of this result for the first two examples

[12] Recall that $H_{I\circ u}(f) \cap Core(\nu) \neq \emptyset$ for all $f \in \mathcal{F}_1$ is equivalent to the non-emptyness of $Core(\nu)$.

[13] See introductory chapter for the details.

and introduce the preference relations via this representation.

Example 2.1. Consider the state space $S = \{s_1, s_2, s_3\}$. Consider the preference relation $\succsim$ represented by the following polyhedral prior set $\mathcal{C}$ and ambiguity attitude for non-constant acts a:

$$\mathcal{C} = Conv\left\{\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},0,\frac{1}{2}\right),\left(0,\frac{1}{2},\frac{1}{2}\right)\right\},$$
$$a(f) = 1-\frac{1}{2}\left(\frac{f(s_f^1)-f(s_f^2)}{f(s_f^1)-f(s_f^3)}\right)^2.$$

Note that $\mathcal{C}$ is the Clarke-Differential at 0. The set of dominating measures is

$$\mathcal{D}_{\succsim} = \mathcal{H}_I(1_{s_1}0)\cap\mathcal{H}_I(1_{s_2}0)\cap\mathcal{H}_I(1_{s_3}0) = Conv\left\{\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right),\left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)\right\}.$$

The sets $\mathcal{C}$ and $\mathcal{D}$ are illustrated in Figure 2.1.

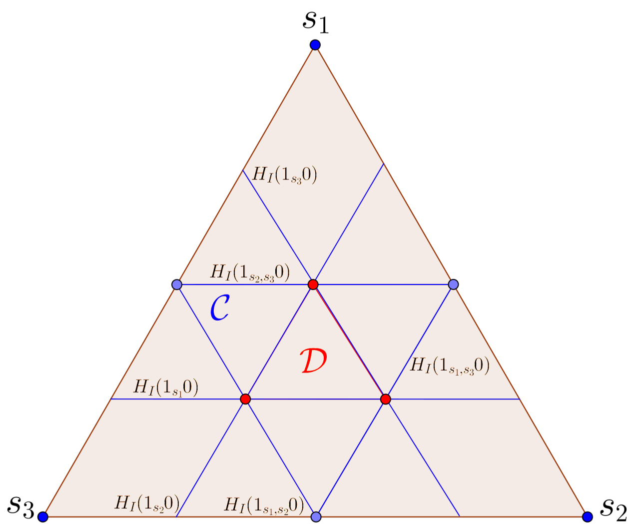


Figure 2.1: The sets $\mathcal{C}$ and $\mathcal{D}$ from Example 2.1.

Observe that for all $f \in \mathcal{F}_2$ it holds that $H_I(f)\cap\mathcal{D}_{\succsim} \neq \emptyset$. Theorem 2.3 thus implies that 2 - Ambiguity Aversion holds for $\succsim$. The preferences do not however satisfy 3 - Ambiguity Aversion. Consider the acts $f = 3_{s_1}1_{s_2}0 \in \mathcal{F}_3$ and $f_1 = 4_{s_1}0$, $f_2 = 2_{s_1,s_2}0 \in \mathcal{F}_2$. We have $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$. It holds that $I(f_1) = I(f_2) = 1$ and $I(f) = \frac{5}{6}$. Therefore 3 - Ambiguity Aversion fails.

The next example is particularly important. It illustrates that preference relations exist that satisfy 3 - Ambiguity Aversion but not 4 - Ambiguity Aversion. Recall Theorem 2.5 which states that this is not possible within the CEU framework. The example thus shows that the axioms 3 - Ambiguity Aversion and 4 - Ambiguity Aversion are indeed independent of each other, given the five standard axioms.[14]

Example 2.2. Consider the state space $S = \{s_1, s_2, s_3, s_4\}$. Consider the preference relation $\succsim$ represented by the following polyhedral prior set $\mathcal{C}$ and ambiguity attitude for non-constant acts a:

$$\mathcal{C} = Conv\Bigg\{ \left(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right), \left(\frac{3}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{4}\right), \left(\frac{3}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}\right), \left(\frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{1}{4}\right), \left(\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, \frac{1}{4}\right) \left(\frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{1}{8}\right), \left(\frac{1}{4}, \frac{1}{8}, \frac{3}{8}, \frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{3}{8}\right), \left(\frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}\right), \left(\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8}\right) \Bigg\}.$$

$$a(f) = \frac{1}{2} + \frac{3}{2}\frac{\min\{f(s_f^1) - f(s_f^2), f(s_f^2) - f(s_f^3), f(s_f^3) - f(s_f^4)\}}{f(s_f^1) - f(s_f^4)}.$$

Note that again $\mathcal{C}$ is the Clarke-Differential at 0. It holds that $a(f) = \frac{1}{2}$ for all $f \in \mathcal{F}_3$ and $a(f) > \frac{1}{2}$ for all $f \notin \mathcal{F}_3$. All $f \in \mathcal{F}_3$ are thus evaluated at the uniform distribution $P_{Unif} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \in \Delta(S)$, i.e. $I(f) = \int f \, dP_{Unif}$ for all $f \in \mathcal{F}_3$. This implies that $\mathcal{D}_{\succsim} = \{P_{Unif}\}$, i.e. the set $\mathcal{D}_{\succsim}$ consists just of the uniform distribution over S.

Theorem 2.3 implies that $\succsim$ satisfies 3 - Ambiguity Aversion. It does not however satisfy 4 - Ambiguity Aversion. Consider the acts $f = 4_{s_1}3_{s_2}2_{s_3}1 \in \mathcal{F}_4$ and $f_1 = 4_{s_1,s_2}1, f_2 = 4_{s_1}3_{s_2,s_3}0, f_3 = 4_{s_1}2 \in \mathcal{F}_3$. It holds that $f = \frac{1}{3}f_1 + \frac{1}{3}f_2 + \frac{1}{3}f_3$.

Given the prior set $\mathcal{C}$ and ambiguity attitude a we can determine that $a(f) = 1$ and $\arg\min_{P \in \mathcal{C}} \int f \, dP = (\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8})$. It follows that $I(f) = \frac{17}{8}$. Furthermore $I(f_i) = \frac{5}{2}$ for $i \in \{1, 2, 3\}$. Thus $f_1 \succ f$ and therefore 4 - Ambiguity Aversion fails.

The next two examples are on preferences that can be represented within the CEU framework. Recall that for a preference relation $\succsim$ represented by capacity ν it holds that $\mathcal{D}_{\succsim} = Core(\nu)$.

[14] Our conjecture is that it is possible to construct examples for preferences that satisfy k - Ambiguity Aversion but not $k+1$ - Ambiguity Aversion in general. For $k = 4$ we need at least 5 states which makes the analysis very complicated.

Example 2.3. Consider the state space $S = \{s_1, s_2, s_3\}$. Consider the preference relation $\succsim$ represented by the capacity ν. The core of ν is illustrated in Figure 2.2.

$$\nu(E) = \begin{cases} 1, & \text{for } E = S \\ 0, & \text{for } E = \emptyset \\ \frac{1}{6}, & \text{for } |E| = 1 \\ \frac{1}{3}, & \text{for } E = \{s_1, s_2\}; E = \{s_2, s_3\} \\ \frac{1}{4}, & \text{for } E = \{s_1, s_3\} \end{cases}$$

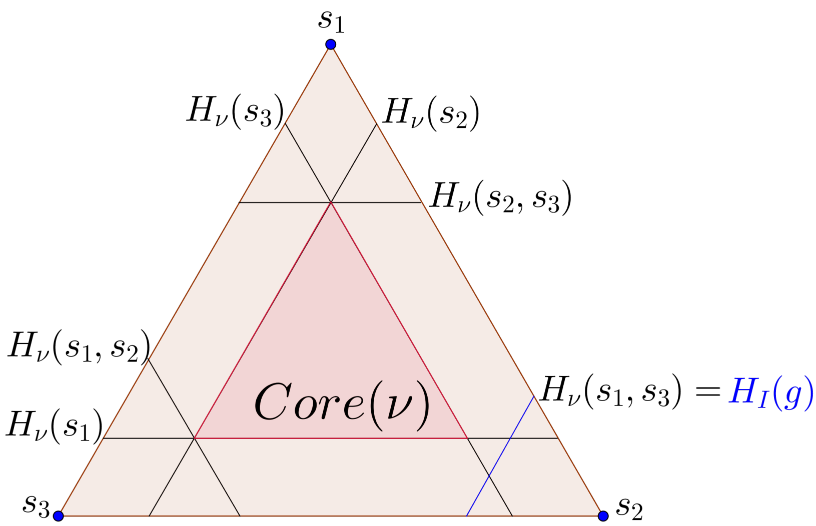


Figure 2.2: The capacity ν is balanced, but not exact.

The capacity ν is balanced as the core is non-empty, but it is not exact. Non-exactness can be determined through the standard definition by observing that $\nu(\{s_1, s_3\}) < \min_{P \in Core(\nu)} P(\{s_1, s_3\})$. Alternatively by observing that for the binary act $g = 1_{s_1,s_3}0$ it holds that $\int g\, d\nu = \nu(\{s_1, s_3\}) = \frac{1}{4} < \frac{1}{3} = \min_{P \in Core(\nu)} \int g\, dP$, i.e. $H_I(g) \cap Core(\nu) = \emptyset$.

Theorem 2.3 thus implies that $\succsim$ violates 2 - Ambiguity Aversion. To illustrate this consider the binary acts $g = 1_{s_1,s_3}0$, $f_1 = 2_{s_1}0$ and $f_2 = 2_{s_3}0$. Obviously $f_1 \sim f_2$ and $\frac{1}{2}f_1 + \frac{1}{2}f_2 = f$. However $f_1 \succ g$, hence $\succsim$ violates 2 - Ambiguity Aversion.

Example 2.4. Consider the state space $S = \{s_1, s_2, s_3, s_4\}$. Consider the preference relation $\succsim$ represented by the capacity μ. The edges of the core of μ are illustrated in Figure 2.3.

$$\mu(E) = \begin{cases} 1, & \text{for } E = S \\ \frac{1}{10}, & \text{for } E = \{s_1, s_2\}; E = \{s_1, s_3\}; E = \{s_1, s_2, s_3\}; \\ & E = \{s_1, s_2, s_4\}; E = \{s_1, s_3, s_4\}; E = \{s_2, s_3, s_4\} \\ 0, & \text{for } E \text{ otherwise} \end{cases}.$$

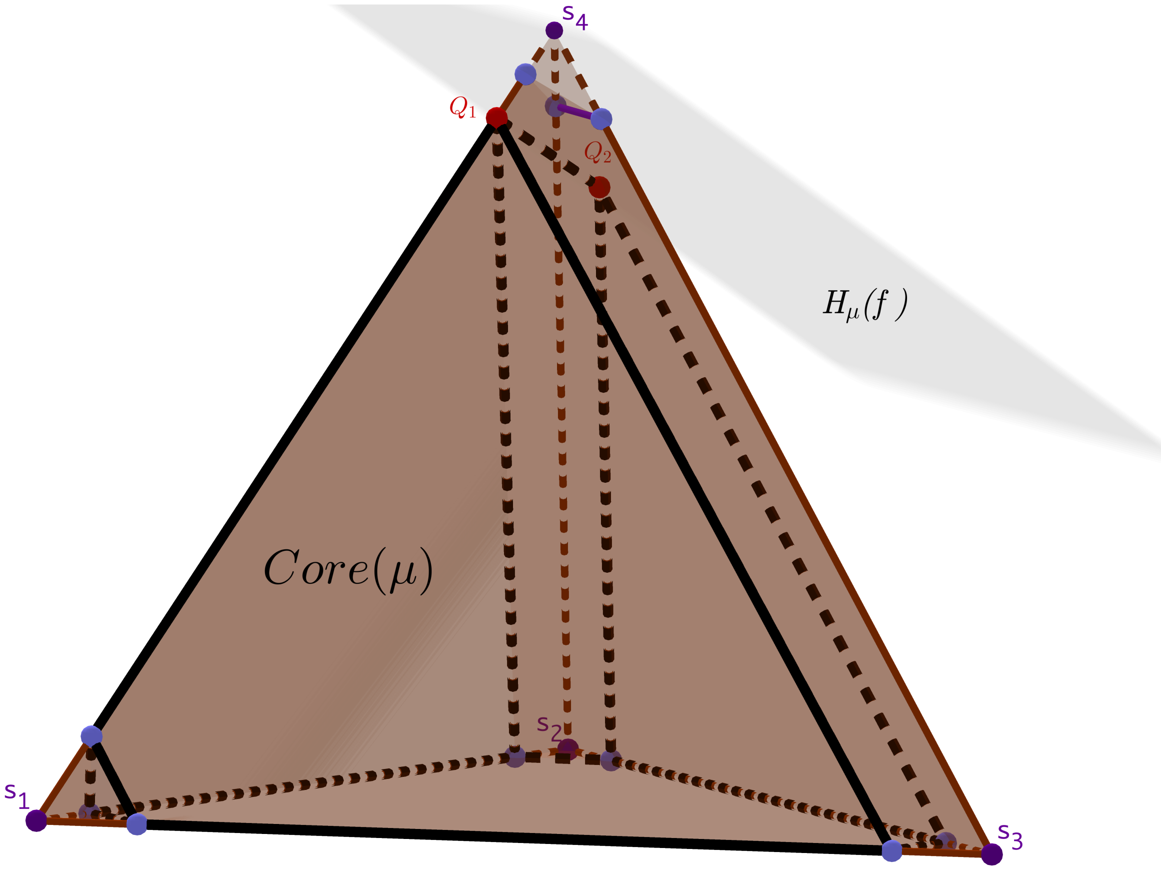


Figure 2.3: The capacity μ is exact, therefore balanced, but not convex.

The capacity μ is exact but not convex. To illustrate that μ is exact via the standard definition consider the distribution $Q_1 = (\frac{1}{10}, 0, 0, \frac{9}{10}) \in \Delta(S)$, pictured in Figure 2.3. The distribution Q_1 is an element of $Core(\mu)$ and also of $H_\mu(E)$

for $E \in \big\{\{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_2, s_3\}\big\}$.[15] Thus $H_\mu(E) \cap Core(\mu) \neq \emptyset$ for all $E \subseteq S$ which shows that μ is exact. Non-convexity follows from Corollary 2.1 and Theorem 2.5 if we find an act $f \in \mathcal{F}_3$ such that $H_\mu(f) \cap Core(\mu) = \emptyset$. Consider the act $f = 2_{s_1} 1_{s_2,s_3} 0 \in \mathcal{F}_3$. It holds that

$$\int f \, d\mu = 2\mu(\{s_1\}) + \mu(\{s_1, s_2, s_3\}) = \frac{1}{10} < \frac{2}{10} = \int f \, dQ_1 = \min_{P \in Core(\mu)} \int f \, dP.$$

Figure 2.3 illustrates that $H_\mu(f) \cap Core(\mu) = \emptyset$.

To illustrate that $\succsim$ does not satisfy 3 - Ambiguity Aversion consider the acts $f_1 = 2_{s_1,s_2}0$ and $f_2 = 2_{s_1,s_3}0$. It holds that $f_1 \sim f_2$ as well as $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$. It holds that $f_1 \succ f$, therefore the 3 - Ambiguity Aversion axiom fails.

2.6 Conclusion

This article shows that two of the most prominent definitions of ambiguity aversion are the extreme cases of a conceptual framework that we call Hierarchy of Ambiguity Aversion. We show that preferences can exhibit levels of ambiguity aversion that lie strictly in between the two known ones. We show that every level of the hierarchy can be characterized by a geometric property concerning the set of dominating measures $\mathcal{D}_{\succsim}$. We show that within the CEU framework the hierarchy has only three levels, regardless of the cardinality of the state space. A consequence of our work is the axiomatization of balanced and exact capacities, important preference classes that thus far had lacked an axiomatization.

We view the insights of this article as a starting point for future research. We conjecture that model-free versions of Theorems 2.1 and 2.3 can be proved, i.e. that we can drop the assumption that preferences satisfy the five standard axioms. If completely model-free versions of these theorems do not exist, we should aim to find the weakest axiomatic foundation that allow these results. One may for instance relax Certainty Independence to Risk Independence or Monotonicity to Weak Monotonicity, see the next article for details on these axioms.

Another promising extension of this work is to generalize the analysis to a purely subjective framework (Savage (1954)). We do not have a mixture-space in Savage,

[15] These are the crucial events. With the help of Figure 2.3 it can be checked that also the hyperplanes of all other events intersect $Core(\mu)$.

thus the approach would be quite different. We suspect however that a similar kind of hierarchy can be defined.

2.7 Appendix

First we show that we do not lose any generality by assuming that acts are mappings into utilities when discussing the axiom k - Ambiguity Aversion. Consider the following definition.

Definition 2.4 (k - "Expected" Utility Ambiguity Aversion)**.**
A preference $\succsim$ satisfies k - "Expected" Utility Ambiguity Aversion if $f_1 \sim \cdots \sim f_n, \sum_{i=1}^{n} \alpha_i f_i = f \in \mathcal{F}_k^u, \alpha_1, \ldots, \alpha_n \geq 0, \sum_{i=1}^{n} \alpha_i = 1 \implies f \succsim f_1$.

The $k = 1$ version is introduced in Chateauneuf and Tallon (2002) under the name *sure "expected" utility diversification.* They show that in the CEU framework it characterizes balanced capacities. However this does not provide a proper axiomatization as it is not purely in terms of preferences over acts. The following lemma shows that we can drop the expected utility part of the definition.

Lemma 2.1. *Under the five standard axioms, k - "Expected" Utility Ambiguity Aversion is equivalent to the axiom k - Ambiguity Aversion.*

Proof. The non-trivial direction is that k - Ambiguity Aversion implies k - "Expected" Utility Ambiguity Aversion. Assume that k - Ambiguity Aversion holds. Assume that $g_1 \sim \cdots \sim g_n, \sum_{i=1}^{n} \alpha_i g_i = g \in \mathcal{F}_k^u$. We need to show that $g \succsim g_1$.

There exist $x_*, x^* \in X$ with $x^* \succ x_*$ such that $x^* \succsim g_i(s_j) \succsim x_*$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, |S|\}$. Define the acts $f_1, \ldots, f_n$ in the following way: $f_i(s_j) = \beta_{ij} x_* + (1 - \beta_{ij}) x^*$ such that $f_i(s_j) \sim g_i(s_j)$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, |S|\}$. Due to the Archimedean axiom (continuity) these β-values always exist, are unique and are between 0 and 1. Furthermore

$$\beta x_* + (1 - \beta) x^* \succ \beta' x_* + (1 - \beta') x^* \iff \beta < \beta', \tag{2.1}$$

thus the function which maps β to $\beta x_* + (1-\beta) x^*$ is injective. Monotonicity implies $f_i \sim g_i$ for all $i \in \{1, \ldots, n\}$. Define the act f such that $f(s) = \sum_{i=1}^{n} \alpha_i f_i(s)$. Certainty Independence implies independence for constant acts, thus due to the construction of $f, f_1, \ldots, f_n$ we achieve $f(s) \sim g(s)$ for all $s \in S$. Monotonicity implies $f \sim g$.

Assume $g(s) \sim g(s')$. This implies $f(s) = \sum_{i=1}^{n} \alpha_i f_i(s) \sim \sum_{i=1}^{n} \alpha_i f_i(s') = f(s')$. Observation (2.1) implies $f(s) = f(s')$. Thus $f \in \mathcal{F}_k$. The k - Ambiguity Aversion axiom implies that $f \succsim f_1$. Thus $g \sim f \succsim f_1 \sim g_1$. This finishes the proof.

□

Proof of Theorem 2.1

For two acts $f, g \in \mathcal{F}$ such that $f(s) \sim g(s)$ for all $s \in S$ the Monotonicity axiom implies $f \sim g$. This fact was already used twice in the proof of Lemma 2.1. It implies that for an act f we can replace the resulting lotteries in every state, $f(s)$, by its expected utility, $u(f(s))$, without losing any generality. We thus assume from now on that acts are mappings from the state space into the real numbers. We denote by $\mathbb{I}_E$ the indicator function of the event E.

Consider the following more general version of Theorem 2.1.

Theorem 2.6. *Let $\succsim$ be a preference relation satisfying the five standard axioms. The following are equivalent:*

1. *$\succsim$ is GM - ambiguity averse*
2. *$\mathcal{D}_{\succsim} \neq \emptyset$*
3. *$\succsim$ satisfies 1 - Ambiguity Aversion*
4. $\sup\left\{\sum_{i=1}^{n} \lambda_i I(f_i) | \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, f_i \in \mathcal{F}, \sum_{i=1}^{n} \lambda_i f_i \leq \mathbb{I}_S\right\} = 1$, *where I is the normalized functional that represents $\succsim$, i.e. $I(\mathbb{I}_S) = 1$, $I(0) = 0$.*

Theorem 2.1 is the equivalence of 1. and 3. The equivalence of 2. and 4. is a generalization of a result of Kannai (1992) in which he characterizes balanced games, i.e. games with a non-empty core. Statement 1. $\Longleftrightarrow$ 2. is shown in Ghirardato and Marinacci (2002). The crucial step if 3. $\Longrightarrow$ 2., for which we need the following Lemma.

Lemma 2.2. *Let $\succsim$ be a preference relation satisfying the five standard axioms. The following are equivalent:*

1. *$\mathcal{D}_{\succsim} \neq \emptyset$.*
2. *$\bigcap_{i=1}^{n} \mathcal{H}_I(f_i) \neq \emptyset$ for all finite subsets of acts $f_1, \ldots, f_n \in \mathcal{F}$.*
3. *$\bigcap_{i=1}^{n} \mathcal{H}_I(f_i) \neq \emptyset$ for all finite subsets of acts $f_1, \ldots, f_n \in \mathcal{F}$ such that $\sum_{i=1}^{n} \alpha_i f_i = \mathbb{I}_S, \sum_{i=1}^{n} \alpha_i = 1, \alpha_1, \ldots, \alpha_n \geq 0$.*

4. $\bigcap_{i=1}^{n} \mathcal{H}_I(f_i) \neq \emptyset$ *for all finite subsets of acts* $f_1 \sim \cdots \sim f_n \in \mathcal{F}$ *such that* $\sum_{i=1}^{n} \alpha_i f_i = \mathbb{I}_S, \sum_{i=1}^{n} \alpha_i = 1, \alpha_1, \ldots, \alpha_n \geq 0.$

Proof. 1. $\implies$ 2., 2. $\implies$ 3. and 3. $\implies$ 4. are straightforward.

2. $\implies$ 1.:

Note that the set of acts $\mathcal{F}_{\mathbb{Q}} = \{f : S \to \mathbb{Q}\}$ is dense in $\mathcal{F}$. Furthermore $\mathcal{F}_{\mathbb{Q}}$ is countable.

2. implies that for all $f_1, \ldots, f_n \in \mathcal{F}$ there exists some $P \in \Delta(S)$ such that $I(f_i) \leq \int f_i \, dP$ for all $i \in \{1, \ldots, n\}$. Continuity of I implies that for all sequences $(f_i)_{i \in \mathbb{N}}$, $f_i \in \mathcal{F}_{\mathbb{Q}}$ there exists some $P \in \Delta(S)$ such that $I(f) \leq \int f_i \, dP$ for all $i \in \mathbb{N}$. Countability of $\mathcal{F}_{\mathbb{Q}}$ and continuity of I imply that $\mathcal{D}_{\succsim} \neq \emptyset$.

3. $\implies$ 2.:

Assume that 2. fails. Then there are acts $f_1, \ldots, f_n \in \mathcal{F}$ such that $\bigcap_{i=1}^{n} \mathcal{H}_I(f_i) = \emptyset$. There exists an act f_{n+1} and some $\alpha > 0$ such that $\sum_{i=1}^{n+1} \frac{1}{n+1} \frac{f_i}{\alpha} = \mathbb{I}_S$. Constant-linearity of the preference functional I implies that $\mathcal{H}_I(af) = \mathcal{H}_I(f)$ for all $a > 0$ and $f \in \mathcal{F}$. Therefore $\bigcap_{i=1}^{n+1} \mathcal{H}_I(\frac{f_i}{\alpha}) \subseteq \bigcap_{i=1}^{n} \mathcal{H}_I(f_i) = \emptyset$. Thus 3. fails.

4. $\implies$ 3.:

Assume that 3. fails. Then there exist acts $f_1, \ldots, f_n \in \mathcal{F}$ such that $\sum_{i=1}^{n} \alpha_i f_i = \mathbb{I}_S, \sum_{i=1}^{n} \alpha_i = 1, \alpha_1, \ldots, \alpha_n \geq 0$ and $\bigcap_{i=1}^{n} \mathcal{H}_I(f_i) = \emptyset$. We can find constants $x_1, \ldots, x_n$ such that $f_1 + x_1 \sim \cdots \sim f_n + x_n$. Now $\sum_{i=1}^{n} \alpha_i (f_i + x_i) = (1 + \sum_{i=1}^{n} x_i)\mathbb{I}_S$. Define $f' = \frac{f_i + x_i}{1 + \sum_{i=1}^{n} x_i}$. It holds that $f'_1 \sim \cdots \sim f'_n$ and $\sum_{i=1}^{n} \alpha_i f'_i = \mathbb{I}_S$. Constant-linearity of I implies that $\mathcal{H}_I(f'_i) = \mathcal{H}_I(f_i)$, thus $\bigcap_{i=1}^{n} \mathcal{H}_I(f'_i) = \bigcap_{i=1}^{n} \mathcal{H}_I(f_i) = \emptyset$, which shows that 4. fails.

□

Proof of Theorem 2.6. 3. $\implies$ 2.:

Assume that 1 - Ambiguity Aversion holds. Then for $f_1 \sim \cdots \sim f_n$, $\sum_{i=1}^{n} \alpha_i f_i = x$, $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_1, \ldots, \alpha_n \geq 0$ it holds that $x \succsim f_1$. This implies the existence of some $P \in \Delta(S)$ such that $\sum_S P(s) f_i(s) \leq \sum_S P(s) x(s) = I(x)$ for all $i \in \{1, \ldots, n\}$. This is because if such a P does not exist then $x \succsim f_i$ fails for at least one $i \in \{1, \ldots, n\}$ and thus for all f_i. This implies that 5. from Lemma 2.2 holds. Lemma 2.2 thus implies that $\mathcal{D}_{\succsim} \neq \emptyset$.

<u>2. $\implies$ 3.</u>: Consider some $P \in \mathcal{D}_{\succsim}$. Consider $f_1 \sim \cdots \sim f_n$, $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_1, \ldots, \alpha_n \geq 0$, $\sum_{i=1}^{n} \alpha_i f_i = x$. It holds that

$$\sum_{i=1}^{n} \alpha_i I(f_i) \leq \sum_{i=1}^{n} \alpha_i \int f_i \, dP = \int \sum_{i=1}^{n} \alpha_i f_i \, dP = \int x \, dP = x.$$

Thus $x \succsim f_1$, so 1 - Ambiguity Aversion holds.

<u>4. $\implies$ 3.</u>: Assume that 1 - Ambiguity Aversion fails. Then there exist $f_1 \sim \cdots \sim f_n$, $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_1, \ldots, \alpha_n \geq 0$, $\sum_{i=1}^{n} \alpha_i f_i = x$ and $\sum_{i=1}^{n} \alpha_i I(f_i) > x$. It holds that $\sum_{i=1}^{n} \alpha_i \frac{f_i}{x} > \mathbb{I}_S$. Define $f_i' = \frac{f_i}{x}$ for $i \in \{1, \ldots, n\}$. Constant-linearity of I implies that $f_1' \sim \cdots \sim f_n'$. It thus holds that $\sum_{i=1}^{n} \alpha_i I(f_i') > 1$. Thus 4. fails.

<u>3. $\implies$ 4.</u>: Assume that 4. fails, so we can find $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_1, \ldots, \lambda_n \geq 0$, $\sum_{i=1}^{n} \lambda_i f_i = \mathbb{I}_S$ and $\sum_{i=1}^{n} \lambda_i I(f_i) > 1$.

Define $f_i' = \lambda_i f_i$ for $i \in \{1, \ldots, n\}$, which implies $\sum_{i=1}^{n} f_i' = \mathbb{I}_S$. Constant-linearity of I implies that there exist $x_1, \ldots, x_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} x_i = 0$ and $f_1' + x_1 \sim \cdots \sim f_n' + x_n$. Define $f_i^\dagger = n(f_i' + x_i)$ for $i \in \{1, \ldots, n\}$. This implies that $\sum_{i=1}^{n} \frac{1}{n} f_i^\dagger = \mathbb{I}_S$. Constant-linearity implies that $\sum_{i=1}^{n} \lambda_i I(f_i^\dagger) = \sum_{i=1}^{n} \lambda_i I(f_i) > 1$. It follows that $f_1^\dagger \succsim \mathbb{I}_S$, therefore 1 - Ambiguity Aversion fails.

□

Proof of Theorem 2.2

Proof of Theorem 2.2. The nontrivial direction is that Schmeidler - Ambiguity Aversion implies $|S|$ - Ambiguity Aversion. We prove it by induction over n, the amount of indifferent acts. The base case is $n = 2$ which holds by assumption. Assume that it holds for $n - 1$. We need to show that it then also holds for n.

Assume that $f_1 \sim \cdots \sim f_n$ with $\alpha_i \geq 0$ for all $i \in \{1, \dots, n\}$, $\sum_{i=1}^{n} \alpha_i = 1$ and $\sum_{i=1}^{n} \alpha_i f_i = f$. Consider the act

$$f' = \frac{\sum_{i=2}^{n} \alpha_i f_i}{1 - \alpha_1}.$$

Note that it holds that $\frac{\sum_{i=2}^{n} \alpha_i}{1-\alpha_1} = 1$. So the act f' is a convex mix of the $n-1$ acts $f_2, \dots, f_n$. By the induction hypothesis we have $f' \succsim f_2$ which implies that $f' \succsim f_1$.

It holds that $f = \alpha_1 f_1 + (1-\alpha_1) f'$. Therefore by Schmeidler - Ambiguity Aversion we have that

$$f \succsim f_1.$$

This finishes the induction step. □

Proof of Theorem 2.3

Recall that we assume a finite state space S. Throughout the proofs of this section we again assume that acts map from the state space S into utilities, i.e. the real numbers. Without loss of generality we assume throughout that I is the normalized representation functional of the preference relation $\succsim$, i.e. $I(0) = 0$ and $I(\mathbb{I}_S) = 1$.

Let $\mathcal{F}_{\geq 0} = \{f \in \mathcal{F} | f(s) \geq 0 \ \forall s \in S\}$ be the set of acts that map to the non-negative reals. Due to the constant-linearity of the preference functional I, $H_I(f + x) = H_I(f)$ for all $f \in \mathcal{F}, x \in \mathbb{R}$. This implies that

$$H_I(f) \cap \mathcal{D}_{\succsim} \neq \emptyset \ \forall \ f \in \mathcal{F}_k \iff H_I(f) \cap \mathcal{D}_{\succsim} \neq \emptyset \ \forall \ f \in \mathcal{F}_k \cap \mathcal{F}_{\geq 0}.$$

It thus suffices to consider acts in $\mathcal{F}_{\geq 0}$ to prove Theorem 2.3.

Definition 2.5. *For a preference relation $\succsim$ with $\mathcal{D}_{\succsim} \neq \emptyset$ define the following functions $\hat{V} : \mathcal{F}_{\geq 0} \to \mathbb{R}$ and $\tilde{V} : \mathcal{F}_{\geq 0} \to \mathbb{R}$:*

$$\hat{V}(f) := \inf \left\{ \sum_S P(s) f(s) | P \in \mathcal{D}_{\succsim} \right\}.$$

$$\tilde{V}(f) :=$$

$$\sup \left\{ \sum a_i I(f_i) - a | (a_i, f_i) \text{ finite sequence in } \mathbb{R}_{\geq 0} \times \mathcal{F}_{\geq 0}, a \in \mathbb{N}, \sum a_i f_i - a \mathbb{I}_S \leq f \right\}.$$

The following Lemma establishes the remarkable observation that if $\mathcal{D}_{\succsim} \neq \emptyset$, $\hat{V}$ and $\tilde{V}$ are identical. This result is crucial for the proof of our main theorem. The proof is inspired by Schmeidler (1972) where a similar approach is used.

Lemma 2.3. *Let $\succsim$ be a preference relation satisfying the five standard axioms. Assume that $\mathcal{D}_{\succsim} \neq \emptyset$. Then for all $f \in \mathcal{F}_{\geq 0}$*

$$\hat{V}(f) = \tilde{V}(f).$$

To prove the Lemma a separating hyperplane theorem is used.

Theorem 2.7 (Separating Hyperplane Theorem (Dunford and Schwartz (1958)))**.** *In a linear topological space, any two disjoint convex sets, one of which has an interior point, can be separated by a non-zero continuous linear functional.*

Proof of Lemma 2.3. $\underline{\hat{V} \geq \tilde{V}}$: Consider an act $f \in \mathcal{F}_{\geq 0}$ and an arbitrary $P \in \mathcal{D}_{\succsim}$. Assume that

$$\sum_{i=1}^{n} a_i f_i - a\mathbb{I}_S \leq f.$$

It follows that

$$\begin{aligned}
\sum_S P(s)f(s) &\geq \sum_S P(s)\left(\sum_{i=1}^{n} a_i f_i(s) - a\right) \\
&= \sum_{i=1}^{n} a_i \sum_S P(s)f_i(s) - a \\
&\geq \sum_{i=1}^{n} a_i I(f_i) - a.
\end{aligned}$$

It follows that $\sum\limits_S P(s)f(s) \geq \tilde{V}(f)$. Since P was arbitrary and $\mathcal{D}_{\succsim}$ is compact we have that $\hat{V}(f) \geq \tilde{V}(f)$.

$\underline{\hat{V} \leq \tilde{V}}$:

The function $\tilde{V}$ is obviously monotonic. Schmeidler (1972) shows that $\tilde{V}$ is also homogenous and superlinear, i.e.

$$\begin{aligned}
\tilde{V}(\alpha f) &= \alpha\tilde{V}(f) && \forall \alpha \in \mathbb{R}_+ \quad \forall f \in \mathcal{F}_{\geq 0} \\
\tilde{V}(f+g) &\geq \tilde{V}(f) + \tilde{V}(g) && \forall f, g \in \mathcal{F}_{\geq 0}.
\end{aligned}$$

Consider an act $h \in \mathcal{F}_{\geq 0}$. The idea is to construct a set function $F \in \Delta(S)$ which is an element of $\mathcal{D}_{\succsim}$ such that $F(h) = \tilde{V}(h)$, which implies $\tilde{V}(h) \geq \hat{V}(h)$.

Case 1: $\tilde{V}(h) \neq 0$. Consider the following sets

$$A = \{f \in \mathcal{F}_{\geq 0} | \tilde{V}(f) \geq 1\}, \qquad B = \{f \in \mathcal{F}_{\geq 0} | f \leq \mathbb{I}_S\},$$
$$C = \left\{ f \in \mathcal{F}_{\geq 0} | f \leq \frac{h}{\tilde{V}(h)} \right\}.$$

The set A is convex due to the superlinearity of $\tilde{V}$ and the set $D := Co(B \cup C)$ is convex by construction. The set $int(A)$ is convex and has an interior point. To apply Theorem 3 for the sets $int(A)$ and D we need to show that $int(A) \cap D = \emptyset$. It suffices to show that $\tilde{V}(f) \leq 1$ for all $f \in D$. Due to the monotonicity of $\tilde{V}$ it suffices to show that $\tilde{V}\left(r\mathbb{I}_S + (1-r)\frac{h}{\tilde{V}(h)}\right) \leq 1$ for $0 \leq r \leq 1$. Due to the homogeneity of $\tilde{V}$ it suffices to show that

$$\tilde{V}\left(\mathbb{I}_S + th\right) = 1 + t\tilde{V}\left(h\right) \tag{2.2}$$

for $t > 0$. The $\geq$ direction follows from the superlinearity of $\tilde{V}$. For the $\leq$ direction let

$$\sum a_i f_i - a\mathbb{I}_S \leq \mathbb{I}_S + th.$$

Rearrange to get

$$\sum a_i f_i - (1+a)\mathbb{I}_S \leq th.$$

From the definition of $\tilde{V}$ we get

$$\sum a_i I(f_i) - (1+a) \leq t\tilde{V}\left(h\right),$$

implying

$$\sum a_i I(f_i) - a \leq 1 + t\tilde{V}\left(h\right).$$

It therefore holds that

$$\tilde{V}\left(\mathbb{I}_S + th\right) \leq 1 + t\tilde{V}\left(h\right).$$

The requirements for Theorem 2.7 for the sets $int(A)$ and D are therefore fulfilled and we can conclude the existence of a linear functional $F : \mathcal{F}_{\geq 0} \to \mathbb{R}$ such that

$$F(f) \leq F(\mathbb{I}_S) = 1 = F\left(\frac{h}{\tilde{V}(h)}\right) \leq F(g)$$

for all $f \in D$ and $g \in A$. The equalities are due to the fact that both $\mathbb{I}_S$ and $\frac{h}{\tilde{V}(h)}$ are in $A \cap D$. Since F is linear and $F(\mathbb{I}_S) = 1$ we have that $F \in \Delta(S)$. The second equality and the linearity of F furthermore imply that $F(h) = \tilde{V}(h)$.

We need to show that $\tilde{V}(f) \leq F(f)$ for all $f \in \mathcal{F}_{\geq 0}$, which implies that $F \geq \tilde{V}$ and so $F \in \mathcal{D}_{\succsim}$.

Fix some $f \in \mathcal{F}_{\geq 0}$. The linearity of F implies that $F(f) \geq 0$. If $\tilde{V}(f) = 0$ then $F(f) \geq \tilde{V}(f)$. So assume that $\tilde{V}(f) > 0$. There exists an $r > 0$ such that $\tilde{V}(rf) = 1$. This implies that $rf \in A$ and therefore $F(f) \geq 1$. Homogeneity of $\tilde{V}$ and F then implies the required $\tilde{V}(f) \leq F(f)$. Therefore

$$\tilde{V}(h) = F(h) \geq \inf\left\{\sum_S P(s)h(s) | P \in \mathcal{D}_{\succsim}\right\} = \hat{V}(h).$$

<u>Case 2:</u> $\tilde{V}(h) = 0$. Define A and B as above and

$$C = \bigcup_{n=1}^{\infty} \{f \in \mathcal{F}_{\geq 0} | f \leq nh\}.$$

To show $int(A) \cap D = \emptyset$ we show with the same approach as above that for $0 \leq r \leq 1$

$$\begin{aligned} \tilde{V}(r\mathbb{I}_S + (1-r)nh) &= r + (1-r)nf\left(\sum_{i=1}^{k} F_i^*\right) \\ &= r \leq 1 \qquad\qquad \forall\ 0 \leq r \leq 1. \end{aligned}$$

So again we can use Theorem 3 to get the separating function F as above with $F(\mathbb{I}_S) = 1$, which implies $F \in \Delta(S)$. We need to show $F(h) = 0$. Assume that $F(h) > 0$. Then there exists an n such that $F(nh) > 1$. So $nh \in A$ as well as $nh \in D$ which contradicts the separating property of F. Therefore $F(h) = 0$. To show $F(f) \geq \tilde{V}(f)$ for all $f \in \mathcal{F}_{\geq 0}$ the above approach is used. So $F \in \mathcal{D}_{\succsim}$ which implies $\tilde{V}(h) \geq \hat{V}(h)$. This finishes the proof.

□

We can now prove Theorem 2.3.

Proof of Theorem 2.3. Fix some $k \in \{1, \dots, |S|\}$.

<u>"1. $\implies$ 2."</u>: Assume that 2. fails. Then there exists an $f \in \mathcal{F}_k \cap \mathcal{F}_{\geq 0}$ such that $I(f) < \tilde{V}(f)$. It follows that there exist $(a_i, f_i)_{i=1}^m \in \mathbb{R}_+^m \times \mathcal{F}_{\geq 0}^m, a \in \mathbb{N}$ with

$$\sum_{i=1}^{m} a_i f_i - a\mathbb{I}_S = f \tag{2.3}$$

such that

$$I(f) < \sum_{i=1}^{m} a_i I(f_i) - a. \tag{2.4}$$

We get strict equality in (2.3) since we can always add other terms of the form $a_i f_i$.

Define $g_i = a_i f_i$ for $i \in \{1, \dots m\}$ and $g_{m+1} = -a$. By (2.3) it holds that

$$f = \sum_{i=1}^{m+1} g_i.$$

Due to the constant-linearity of the preference functional I we can find constant acts $b_1, \dots, b_{m+1} \in \mathbb{R}$ with $\sum_{i=1}^{m+1} b_i = 0$ such that for the acts $g_i' := g_i + b_i, i \in \{1, \dots, m+1\}$ it holds that

$$g_1' \sim \dots \sim g_{m+1}'.$$

Obviously $f = \sum_{i=1}^{m+1} g_i'$. For $g_i^\dagger := (m+1)g_i'$ it holds that

$$g_1^\dagger \sim \dots \sim g_{m+1}^\dagger \tag{2.5}$$

and

$$f = \sum_{i=1}^{m+1} \frac{1}{m+1} g_i^\dagger. \tag{2.6}$$

Now

$$\begin{aligned} I(f) &\overset{(2.4)}{<} \sum_{i=1}^{m} a_i I(f_i) - a \\ &= \sum_{i=1}^{m+1} I(g_i) \\ &= \sum_{i=1}^{m+1} I(g_i') \\ &= \sum_{i=1}^{m+1} \frac{1}{m+1} I(g_i^\dagger) \\ &= I(g_1^\dagger). \end{aligned}$$

Therefore $g_1^\dagger \succ f$ and this in combination with (2.5) and (2.6) contradicts the axiom k - Expected Utility Ambiguity Aversion. Lemma 2.1 implies that we get a contradiction to k - Ambiguity Aversion.

<u>"2. $\implies$ 1.":</u> Consider some act $h \in \mathcal{F}_k$ and acts $f_1 \sim \cdots \sim f_m$ with $\sum_{i=1}^{m} \alpha_i f_i = h$, $\sum_{i=1}^{m} \alpha_i = 1$ and $\alpha_i \geq 0$.

Consider some $Q \in \arg\min_{P \in \mathcal{D}_{\succsim}} \int h \, dP$. It holds that

$$\begin{aligned} I(f_1) &= \sum_{i=1}^{m} \alpha_i I(f_i) \\ &\leq \sum_{i=1}^{m} \alpha_i \min_{P \in \mathcal{D}_{\succsim}} \sum_{S} P(s) f_i(s) \\ &\leq \sum_{i=1}^{m} \alpha_i \sum_{S} Q(s) f_i(s) \\ &= \sum_{S} \sum_{i=1}^{m} \alpha_i Q(s) f_i(s) \\ &= \sum_{S} Q(s) h(s) \\ &= I(h), \end{aligned}$$

where the second inequality holds since $Q \in \mathcal{D}_{\succsim}$. We have therefore shown that $h \succsim f_1$, so the axiom k - Ambiguity Aversion holds.

□

Proof of Theorem 2.5. We need to show that when the capacity is not convex the axiom 3 - Ambiguity Aversion is violated. This implies that for $k \in \{3, \dots, |S|\}$ the axioms characterize the same class of capacities: convex capacities.

The case where $Core(\nu) = \emptyset$ is trivial. Assume that $Core(\nu) \neq \emptyset$. Assume that the capacity ν is not convex. Then there exist events $E, F \in \mathcal{P}(S)$ such that

$$\nu(E) + \nu(F) > \nu(E \cup F) + \nu(E \cap F).$$

Consider the act $f = 2_{E \cap F} 1_{E \cup F \setminus (E \cap F)} 0 \in \mathcal{F}_3$. The Choquet expected utility of f is

$$\begin{aligned} \int f \; d\nu &= 2\nu(E \cap F) + 1(\nu(E \cup F) - \nu(E \cap F)) \\ &= \nu(E \cap F) + \nu(E \cup F) \\ &< \nu(E) + \nu(F) \\ &\leq \min_{P \in Core(\nu)} P(E) + P(F) \\ &= \min_{P \in Core(\nu)} P(E \cup F) + P(E \cap F) \\ &\leq \min_{P \in Core(\nu)} \int f \; dP. \end{aligned}$$

Thus we have found an $f \in \mathcal{F}_3$ such that $\mathcal{H}_I(f) \cap Core(\nu) = \emptyset$. Corollary 2.1 implies that 3 - Ambiguity Aversion fails.

□

3 Weak Monotonicity and Multiple Priors

Abstract

This article introduces a new axiom called Weak Monotonicity. It is shown that Weak Monotonicity is, given some standard axioms, necessary and sufficient for a preference relation to be representable within the multiple prior (MP) model. It is furthermore shown that in the popular axiomatizations of Subjective Expected Utility by Anscombe and Aumann (1963) and Maxmin Expected Utility by Gilboa and Schmeidler (1989), the standard Monotonicity axiom can be replaced by Weak Monotonicity without changing the result. We suggest that these characterizations are attractive as axioms can more easily be traced back from the representation functional. We illustrate by example how non-monotonic preferences as well as preferences that do not satisfy Certainty Independence can be modelled within the MP model. We illustrate that convex preferences do not necessarily imply the min-functional that is characteristic for the Maxmin Expected Utility model.

Keywords: Multiple Priors, Monotonicity, Weak Monotonicity, Maxmin Expected Utility, Subjective Expected Utility

3.1 Introduction

Basically all influencial papers on axiomatic decision theory assume some kind of monotonicity of preferences: if an act is preferred to another in every state, then it is preferred overall. Such monotonicity of preferences also implies a separation of beliefs and tastes.

We propose an axiom that we call *Weak Monotonicity*: if the worst case scenario of an act is preferred to the best case scenario of another, then it is preferred overall. As the name suggests this is a weak version of Monotonicity.[1] It does not imply a separation of beliefs and tastes. We show that in the classic representations of Subjective Expected Utility (SEU) by Anscombe and Aumann (1963) and of Maxmin Expected Utility (MEU) by Gilboa and Schmeidler (1989), Monotonicity can be replaced by Weak Monotonicity without changing the result. Monotonicity is a consequence of the other axioms (Lemma 3.2 and its corollaries) and thus in addition implies that the separation of beliefs and tastes does not need to be assumed, rather it is also a consequence of the other axioms.

Furthermore we show that, under very standard and weak axioms, Weak Monotonicity is necessary and sufficient for the existence of a multiple prior (MP) representation of preferences (Lemma 3.1). In light of this result we view our version of the axiomatizations of SEU and MEU worthy of consideration. The reason being that Weak Monotonicity can be directly *traced back* from the respective representation functional as it is responsible for the MP representation in the first place.[2]

The insight of the role that Weak Monotonicity has in the MP model allows modelling of preferences that cannot be modelled in any of the existing frameworks that we are aware of. We illustrate by example that non-monotonic preferences can be modelled by a MP-functional (Example 3.2). We illustrate how preferences can be convex without being MEU, i.e. they cannot be represented by a prior set in combination with the min-functional (Example 3.3).

Outline

Section 3.2 introduces the framework, the MP model and the relevant axioms. Section 3.3 discusses Weak Monotonicity and illustrates its important role in the MP model. Section 3.4 discusses the consequences for the important preference classes SEU and MEU. Section 3.5 presents examples of MP preferences that cannot be modelled within the known frameworks. All proofs are in the Appendix.

[1] Weaker in the sense that it restricts preferences less.
[2] Note that SEU preferences are merely a special class of MP preferences.

3.2 Preliminaries

We assume the classic framework of Anscombe and Aumann (1963). Consider a finite state space S. The powerset of S, $\mathcal{P}(S)$ is the set of events and $\Delta(S)$ is the set of probability distributions over S. We study preference relations $\succsim$ on the set of acts $\mathcal{F} = \{f : S \to L\}$, where L is the set of finite-support lotteries over some set of prizes X. The asymmetric and symmetric components of $\succsim$ are denoted by $\succ$ and $\sim$, respectively. With the usual abuse of notation, L also denotes the set of constant acts. Mixtures of acts are performed pointwise: for $f, g \in \mathcal{F}$ and $\lambda \in [0, 1]$ we denote by $\lambda f + (1 - \lambda)g$ the act which results in $\lambda f(s) + (1 - \lambda)g(s) \in L$ for all $s \in S$. A function $u : L \to \mathbb{R}$ that is nonconstant and affine is called (Von Neumann Morgenstern) *utility function*. We define $\mathcal{F}_1^u$ as the set of constant expected utility acts, i.e.

$$\mathcal{F}_1^u = \{f \in \mathcal{F} | u(f(s)) = u(f(s')) \; \forall s, s' \in S\}.$$

Multiple Prior Preferences

Consider a preference relation $\succsim$. Assume that there exists

1. a utility function $u : L \to \mathbb{R}$
2. a non-empty, convex and compact set of priors $\mathcal{C} \subseteq \Delta(S)$
3. a function $a : \mathcal{F} \to [0, 1]$, continuous on $\mathcal{F} \backslash \mathcal{F}_1^u$

such that for all $f, g \in \mathcal{F}$

$$f \succsim g \iff I(f) \geq I(g),$$

where $I(f) = a(f) \min_{P \in \mathcal{C}} \int u(f) \; dP + (1 - a(f)) \max_{P \in \mathcal{C}} \int u(f) \; dP$. We then say that $\succsim$ has a *MP representation*, or $\succsim$ is a MP preference.

The utility function u of a MP preference is unique up to positive affine transformations. The prior set $\mathcal{C}$ and the function a are not unique in general.[3]

A MP preference that has a representation in which the function $a : \mathcal{F} \to [0, 1]$ is constant 1 is called a *Maxmin Expected Utility* (MEU) preference. MEU preferences are axiomatized in Gilboa and Schmeidler (1989). A MP preference that

[3] If we want to interpret $\mathcal{C}$ as perceived ambiguity and a as ambiguity attitude this is problematic as we then do not achieve a separation of these two concepts. However this separation is not a topic of this article.

has a representation in which the prior set $\mathcal{C}$ is a singleton is called a *Subjective Expected Utility* (SEU) preference. SEU preferences are axiomatized in Anscombe and Aumann (1963).

Axioms

The following three axioms are well-known.

Axiom 3.1 (Weak Order). 1. For all $f, g \in \mathcal{F}$ either $f \succsim g$ or $g \succsim f$.
2. For all $f, g, h \in \mathcal{F}$ if $f \succsim g \succsim h$ then $f \succsim h$.

Axiom 3.2 (Archimedean). For all $f, g, h \in \mathcal{F}$ if $f \succ g \succ h$ then there exist $\lambda, \mu \in (0,1)$ such that $\lambda f + (1-\lambda)h \succ g \succ \mu f + (1-\mu)h$.

Axiom 3.3 (Non-Degeneracy). There are $f, g \in \mathcal{F}$ such that $f \succ g$.

The following axioms are independence axioms, increasing in strength. Risk Independence means independence for constant acts and is introduced in Cerreia-Vioglio et al. (2011). Certainty Independence is introduced in Gilboa and Schmeidler (1989) and means independence when acts are mixed with constant acts. Independence itself is introduced in Anscombe and Aumann (1963).

Axiom 3.4 (Risk Independence). For all $l_1, l_2, l_3 \in L$, $\alpha \in [0,1]$

$$l_1 \succsim l_2 \iff \alpha l_1 + (1-\alpha)l_3 \succsim \alpha l_2 + (1-\alpha)l_3.$$

Axiom 3.5 (Certainty Independence). For all $f, g \in \mathcal{F}$, $l \in L$ and $\alpha \in (0,1]$

$$f \succsim g \iff \alpha f + (1-\alpha)l \succsim \alpha g + (1-\alpha)l.$$

Axiom 3.6 (Independence). For acts $f, g, h \in \mathcal{F}$ and $\alpha \in [0,1]$

$$f \succsim g \iff \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

The following axiom is introduced in Schmeidler (1989). It states a preference for mixing, i.e. convex preferences.

Axiom 3.7 (Uncertainty Aversion). For $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$ if $f \succsim g$ then $\alpha f + (1-\alpha)g \succsim g$.

The following axioms are the main topic of this article.

Axiom 3.8 (Monotonicity). For $f, g \in \mathcal{F}$ if $f(s) \succsim g(s)$ for all $s \in S$ then $f \succsim g$.

Axiom 3.9 (Weak Monotonicity). For all $f, g \in \mathcal{F}$ if $f(s) \succsim g(s')$ for all $s, s' \in S$ then $f \succsim g$.

Given Weak Order, Weak Monotonicity has an equivalent version which at times is easier to interpret.

Axiom 3.10 (Weak Monotonicity'). For all $f \in \mathcal{F}$ and $l \in L$

1. If $f(s) \succsim l$ for all $s \in S$ then $f \succsim l$.

2. If $l \succsim f(s)$ for all $s \in S$ then $l \succsim f$.

Weak Monotonicity' states that if some act is preferred (disliked) to some constant in every state, then this act is preferred (disliked) to that constant overall. The axiom does not separate beliefs and tastes as Monotonicity does. This is because it does not rule out that for some $f \in \mathcal{F}$, $l_1, l_2 \in L$ and $E \subseteq S$ it holds that $l_1 \succ l_2$ and $f_E l_2 \succ f_E l_1$.[4]

3.3 Weak Monotonicity

The following lemma shows that, under the axioms Weak Order, Archimedean, Non-Degeneracy and Risk-Independence,[5] Weak Monotonicity is necessary and sufficient for preferences to have a MP representation. It illustrates the crucial role that this axiom plays for models in which preferences have a MP representation.

Lemma 3.1. *Let $\succsim$ be a preference relation on $\mathcal{F}$ that satisfies Weak Order, Archimedean, Non-Degeneracy and Risk-Independence. The following are equivalent:*

1. $\succsim$ satisfies Weak Monotonicity.

2. $\succsim$ has a MP representation.

[4] $g_E f$ is the act that results in $g(s)$ for $s \in E$ and $f(s)$ for $s \notin E$.

[5] These axioms guarantee the existence of a nonconstant and affine utility function, i.e. the DM's preferences over constant acts are of the expected utility type.

3.4 Implication for MEU and SEU Preferences

The following lemma shows that under the axioms of Lemma 3.1 and under the assumption that th set of prizes X is unbounded, Uncertainty Aversion implies Monotonicity.[6]

Lemma 3.2. *Assume that X is unbounded. Let $\succsim$ be a preference relation that satisfies Weak Order, Archimedean, Non-Degeneracy, Risk-Independence, Weak Monotonicity and Uncertainty Aversion. Then $\succsim$ satisfies Monotonicity.*

Gilboa and Schmeidler (1989) show that a preference relation satisfies Weak Order, Archimedean, Non-Degeneracy, Monotonicity, Certainty Independence and Uncertainty Aversion if and only if the preferences have an MEU representation. The following corollary of Lemma 3.2 shows that we can replace Monotonicity by Weak Monotonicity without changing this result.[7]

Corollary 3.1. *Assume that X is unbounded. Let $\succsim$ be preference relation on $\mathcal{F}$. Then the following are equivalent:*

1. *$\succsim$ satisfies Weak Order, Archimedean, Non-Degeneracy, Certainty Independence, Weak Monotonicity and Uncertainty Aversion.*

2. *$\succsim$ has an MEU representation.*

In light of Lemma 3.1 we suggest that the representation of MEU in Corollary 3.1 has an advantage over the original. Firstly, it provides a weaker set of axioms which is desirable. More importantly, the axiom Weak Monotonicity can be *traced back* from the preference functional more easily than the axiom Monotonicity. This is because it is clear why an MEU functional satisfies Weak Monotonicity: Lemma 3.1 shows that the axiom is necessary for the MP representation in the first place. Monotonicity of preferences is then implied by the existence of a MP representation in combination with Uncertainty Aversion. We therefore suggests that Monotonicity should not be an assumption. Rather monotonicity of preferences should be viewed as a consequence of the other axioms.

Another consequence of Lemma 3.2 is that the same conclusion can be drawn for the original axiomatization of SEU by Anscombe and Aumann (1963).

[6] Note that the unboundedness condition could also be stated axiomatically via the Unboundedness axiom in Maccheroni et al. (2006).
[7] We conjecture that the unboundedness condition can be dropped, see conclusion.

Corollary 3.2. *Assume that X is unbounded. Let $\succsim$ be preference relation on $\mathcal{F}$. Then the following are equivalent:*

1. *$\succsim$ satisfies Weak Order, Archimedean, Non-Degeneracy, Independence and Weak Monotonicity.*

2. *$\succsim$ has an SEU representation.*

Corollary 3.2 shows that, as for MEU preferences, Monotonicity can be replaced by Weak Monotonicity without changing the result. Again monotonicity of preferences is a consequence of the other axioms. Corollary 3.2 thus also shows that the usual separation of beliefs and tastes does not need to be assumed. It is a consequence.

3.5 Examples

This section discusses three examples of MP preferences that cannot be modelled within the frameworks known to us. The preferences in the examples all violate Certainty Independence, one of them (Example 2) violates Monotonicity and one of them (Example 3) cannot be represented by the min-functional even though it satisfies Uncertainty Aversion. Throughout the examples the state space is $S = \{s_1, s_2\}$. To simplify analysis we assume that acts map to utilities.

Example 1: A MP preference that violates Certainty Independence

Consider the preference relation $\succsim$ represented by the following prior set $\mathcal{C} \subseteq \Delta(S)$ and continuous function $a : \mathcal{F} \to [0, 1]$. A representative indifference curve is illustrated in Figure 3.1.

$$\mathcal{C} = \left\{ P \in \Delta(S) | \frac{1}{4} \leq P(s_1) \leq \frac{3}{4} \right\},\ a(f) = \begin{cases} \frac{1}{2} & \text{if } |f(s_1) - f(s_2)| \leq 1 \\ 1 & \text{if } |f(s_1) - f(s_2)| \geq 2 \\ \frac{3}{2} - \frac{1}{f(s_1) - f(s_2)} & \text{if } f(s_1) - f(s_2) \in [1, 2] \\ \frac{3}{2} - \frac{1}{f(s_2) - f(s_1)} & \text{if } f(s_2) - f(s_1) \in [1, 2] \end{cases}.$$

The prior set $\mathcal{C}$ is the convex hull of the probability distributions $P_1 = (\frac{1}{4}, \frac{3}{4})$ and $P_2 = (\frac{3}{4}, \frac{1}{4})$.

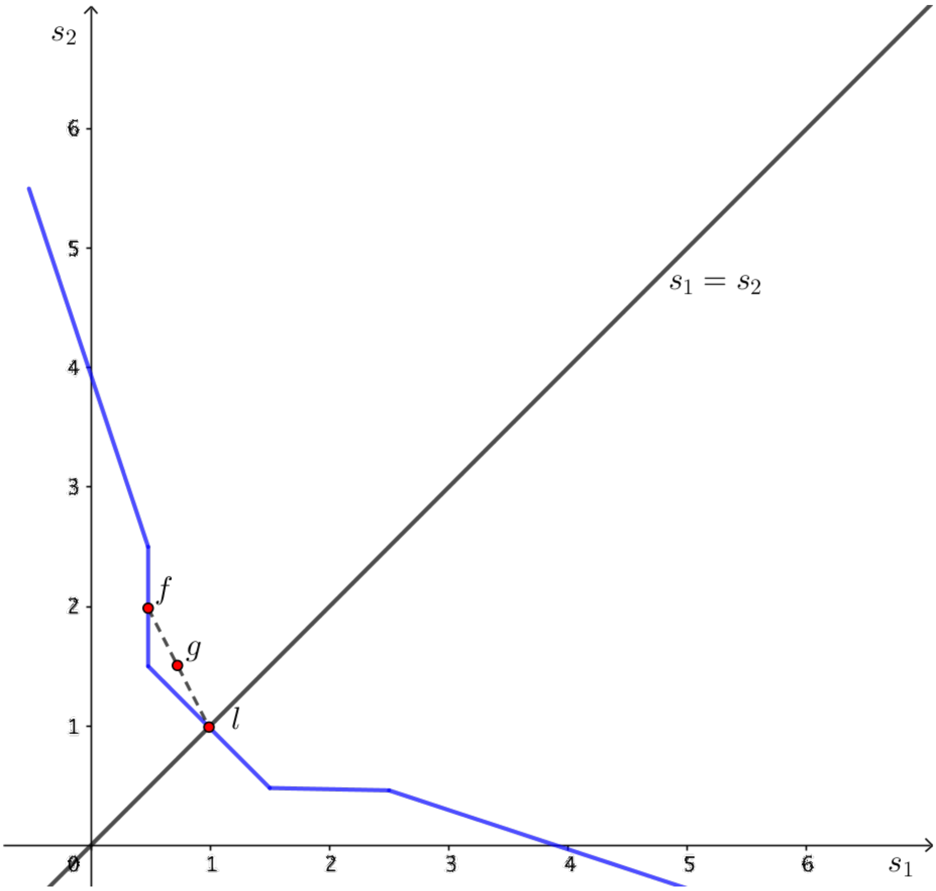


Figure 3.1: Preferences violate Certainty Independence.

To see that Certainty Independence is violated consider the acts $f = \frac{1}{2}{}_{s_1}2$ and $l = 1$. From the prior set $\mathcal{C}$ and the function a we can check that $f \sim l$. Consider the act $g = \frac{1}{2}f + \frac{1}{2}l = \frac{3}{4}{}_{s_1}\frac{3}{2}$. It can be checked that $I(g) = \frac{9}{8} > \frac{3}{4} = I(f)$, so $g \succ f$ which is a violation of Certainty Independence. This is illustrated in Figure 3.1.

Example 2: A MP preference that violates Certainty Independence as well as Monotonicity

Consider the preference relation $\succsim$ represented by the following prior set $\mathcal{C} \subseteq \Delta(S)$ and continuous function $a : \mathcal{F} \to [0, 1]$. A representative indifference curve is illustrated in Figure 3.2.

$$\mathcal{C} = \left\{ P \in \Delta(S) \middle| \frac{1}{4} \leq P(s_1) \leq \frac{3}{4} \right\},\ a(f) = \begin{cases} \frac{1}{2} & \text{if } |f(s_1) - f(s_2)| \leq 1 \\ 1 & \text{if } |f(s_1) - f(s_2)| \geq \frac{4}{3} \\ \frac{5}{2} - \frac{2}{f(s_1) - f(s_2)} & \text{if } f(s_1) - f(s_2) \in [1, \frac{4}{3}] \\ \frac{5}{2} - \frac{2}{f(s_2) - f(s_1)} & \text{if } f(s_2) - f(s_1) \in [1, \frac{4}{3}] \end{cases}.$$

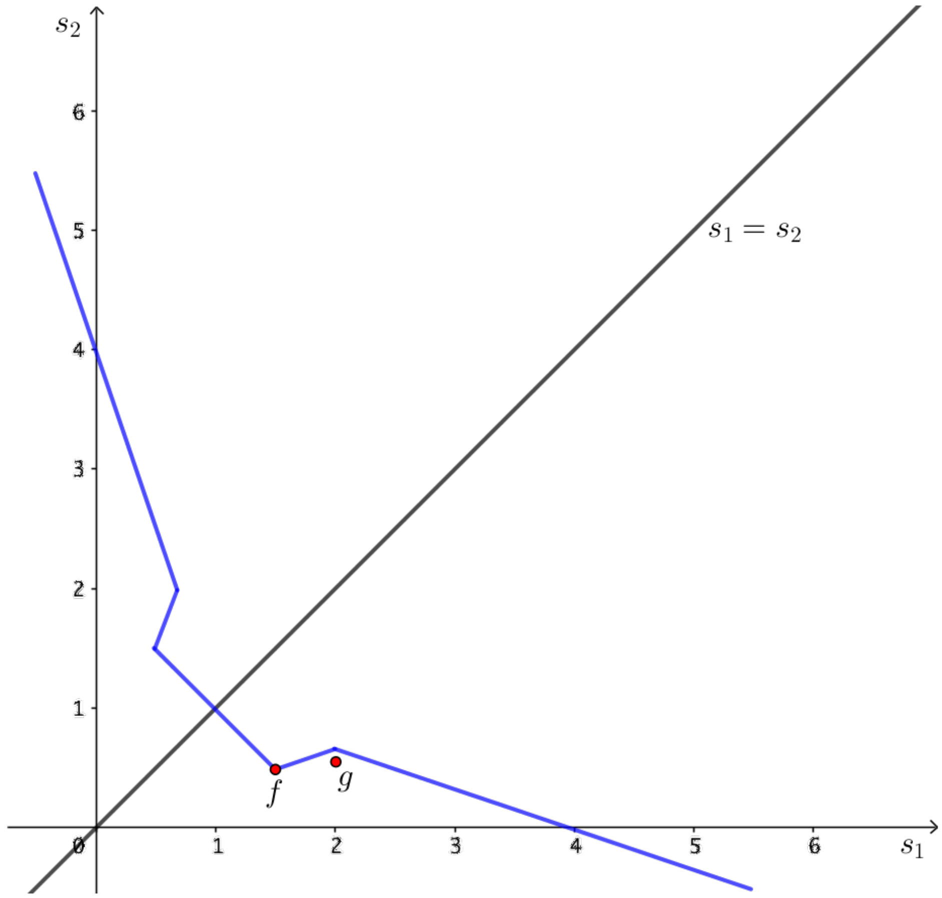


Figure 3.2: Non-monotonic preferences.

The function a is *steeper* as in Example 3.1, leading to non-monotonic preferences. To see that preferences are not monotonic consider the act $f = \frac{3}{2}{}_{s_1}\frac{1}{2}$. It is strictly worse than the act $g = 2_{s_1}\frac{3}{5}$ in every state, but nonetheless $I(f) = 1 > \frac{3}{5} = I(g)$, so $f \succ g$. This is illustrated in Figure 3.2.

Example 3: A MP preference that violates Certainty Independence and satisfies Uncertainty Aversion but cannot be represented by the min-functional

Consider the preference relation $\succsim$ represented by the following prior set $\mathcal{C} \subseteq \Delta(S)$ and continuous function $a : \mathcal{F} \to [0,1]$. A representative indifference curve is illustrated in Figure 3.3.

$$\mathcal{C} = \left\{ P \in \Delta(S) | \frac{1}{4} \leq P(s_1) \leq \frac{3}{4} \right\},\ a(f) = \begin{cases} \frac{1}{2} & \text{if } |f(s_1) - f(s_2)| \leq 1 \\ \frac{5}{6} - \frac{1}{3(f(s_1)-f(s_2))} & \text{if } f(s_1) - f(s_2) \geq 1 \\ \frac{5}{6} - \frac{1}{3(f(s_2)-f(s_1))} & \text{if } f(s_2) - f(s_1) \geq 1 \end{cases}.$$

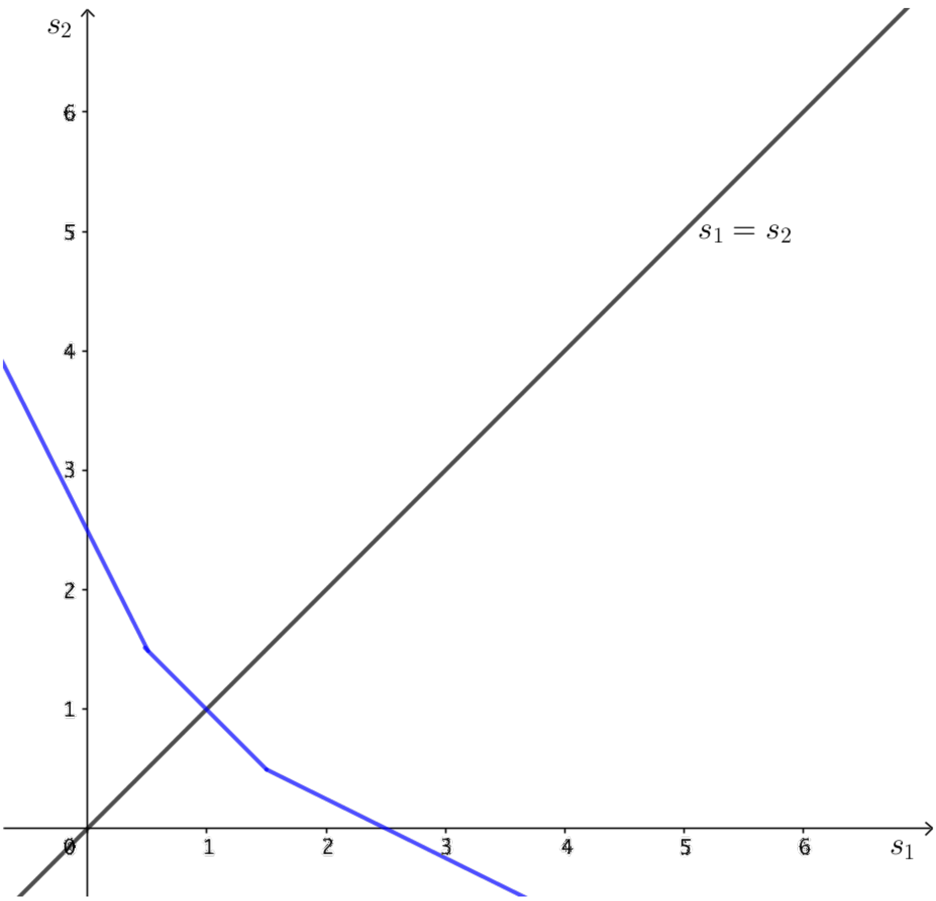


Figure 3.3: Convex Preferences that cannot be represented by the min-functional.

From the figure it can be seen that preferences are convex, i.e. they satisfy

Uncertainty Aversion. What is interesting is that these preferences are not MEU, i.e. these preferences cannot be represented by a prior set in combination with the min-functional. The reason for this is that Uncertainty Aversion implies an MEU representation only in the presence of Certainty Independence, which is violated here.[8]

3.6 Conclusion

This article introduces the new axiom Weak Monotonicity. We show that it plays a crucial role when preferences can be represented within the multiple prior model. We furthermore show that in some of the most influencial results in decision theory monotonicity must not be assumed. Rather, under Weak Monotonicity it is a consequence of the other axioms. These results are merely a starting point for future research. The consequences of Weak Monotonicity for other preference classes should be examined in more detail.

We conjecture that the results from this article have an analogues consequence in the purely subjective framework of Savage (1954). In this framework, the axiom P_3 implies monotonic preferences. We believe that it should be possible to find an axiom in the style of Weak Monotonicity which, in combination with the other axioms, implies P_3.

Another important question is whether the assumption of unboundedness of the set of prizes is really necessary. The proof of Lemma 3.2 relies on it. We conjecture that the assumption is not necessary which would make the result stronger.

[8] Consider for instance the acts $f = \frac{1}{4}{}_{s_1}2$ and $l = 1$ which lie on the same indifference curve. The act $g = \frac{1}{2}f + \frac{1}{2}l = \frac{5}{8}{}_{s_1}\frac{3}{2}$ is strictly preferred to f and l. This violates Certainty Independence.

3.7 Appendix

Proof of Lemma 3.1. Weak Order and Non-Degeneracy imply that there exists a nonconstant functional $I : \mathcal{F} \to \mathbb{R}$ which represents $\succsim$. Note that, reduced to the constant acts L, the axioms Weak Order, Archimedean, Non-Degeneracy and Risk-Independence constitute the von Neumann Morgenstern axioms which implies the existence of the affine utility function $u : L \to \mathbb{R}$ and its uniqueness up to positive affine transformations.

<u>1. $\Longrightarrow$ 2.</u>: We need to show that Weak Monotonicity is sufficient for the preference relation to have a multiple prior representation. First note that if a preference relation $\succsim$ is represented by $(\mathcal{C}, a, u)$ then there exists a continuous function a' such that $\succsim$ is also represented by $(\Delta(S), a', u)$. Thus we can reduce attention to the prior set $\Delta(S)$.

Consider some $f \in \mathcal{F}$. Weak Monotonicity implies that

$$f(\arg\max_{s\in S} u(f(s))) \succsim f \succsim f(\arg\min_{s\in S} u(f(s))).$$

This implies that

$$\begin{aligned} I(f) &\in [I(f(\arg\min_{s\in S} u(f(s)))), I(f(\arg\max_{s\in S} u(f(s))))] \\ &= [\min_{s\in S} u(f(s)), \max_{s\in S} u(f(s))] \\ &= [\min_{P\in\Delta(S)} \int u(f)dP, \max_{P\in\Delta(S)} \int u(f)dP]. \end{aligned}$$

It follows that there exists an $a \in [0,1]$ such that

$$I(f) = a \min_{P\in\Delta(S)} \int u(f)dP + (1-a) \max_{P\in\Delta(S)} \int u(f)dP.$$

This implies the existence of the ambiguity attitude function $a : \mathcal{F} \to [0,1]$. Continuity of a on $\mathcal{F}\backslash\mathcal{F}_1^u$ is implied by the Archimedean axiom.

Thus we have shown the existence of some $\mathcal{C}, a$ and u with the required properties that represent $\succsim$.

<u>2. $\Longrightarrow$ 1.</u>: Assume that $\succsim$ is represented by $(\mathcal{C}, a, u)$. We need to show that Weak Monotonicity holds. Assume that for $f, g \in \mathcal{F}$ it holds that $f(s) \succsim g(s')$ for

all $s, s' \in S$. This implies $\min_{s\in S} u(f(s)) \geq \max_{s\in S} u(g(s))$. It follows that

$$\int u(f)\ dP \geq \int u(g)\ dQ \qquad \forall P, Q \in \Delta(S).$$

Thus for any $a(f), a(g) \in [0,1]$ we have

$$\begin{aligned} & a(f) \min_{P\in\Delta(S)} \int f\ dP + (1-a(f)) \max_{P\in\Delta(S)} \int f\ dP \\ \geq\ & a(g) \min_{P\in\Delta(S)} \int g\ dP + (1-a(g)) \max_{P\in\Delta(S)} \int g\ dP. \end{aligned}$$

Thus $I(f) \geq I(g)$ and therefore $f \succsim g$, so Weak Monotonicity holds.

□

Proof of Lemma 3.2. Assume that Monotonicity fails. Then there exist some $f, g \in \mathcal{F}$ with $f(s) \succsim g(s)$ for all $s \in S$ such that $g \succ f$. The Archimedean axiom implies that there exists an act f' such that $f'(s) \succ g(s)$ for all $s \in S$ and $g \succ f'$.

For all $\alpha \in (0,1)$ there exists an act $h_\alpha \in \mathcal{F}$ such that $(\alpha g + (1-\alpha)h_\alpha)(s) \sim f'(s)$ for all $s \in S$. This can be done since we assume unboundedness of the set of prizes. It follows that $\alpha g + (1-\alpha)h_\alpha \sim f'$.

For α sufficiently large it holds that $h_\alpha(s) \succ g(s')$ for all $s, s' \in S$. Weak Monotonicity thus implies that $h_\alpha \succsim g$.

Uncertainty Aversion now implies that $f' \sim \alpha g + (1-\alpha)h_\alpha \succsim \min\{h_\alpha, g\} = g$ which contradicts $g \succ f'$.

□

4 A Definition of Perceived Ambiguity and its Application to Games

Abstract

This article introduces a multiple prior model in which perceived ambiguity is location invariant, i.e. two prior sets reveal the same ambiguity if and only if they differ only in location within the probability simplex over the state space. We pin down exactly what our definition of perceived ambiguity implies in terms of preferences and show that our approach allows a straightforward application to games. Our equilibrium existence result for normal-form games generalizes many results from the existing literature. We illustrate the modelling capabilities of our approach through several examples.

Keywords: Multiple Priors, Perceived Ambiguity, Ambiguity Attitude, Ambiguous Games.

4.1 Introduction

Ellsberg (1961) and others have demonstrated that, when facing ambiguity, i.e. when exact probabilities of events are unknown, decision makers systematically violate the Subjective Expected Utility (SEU) theory of Savage (1954). Several approaches have been suggested for a more realistic modelling of human decision making under ambiguity. A particularly successful and intuitive approach is to relax the postulation that preferences can be represented by a unique subjective probability distribution over the state space, as the axiomatic approaches by Savage (1954) and Machina and

Schmeidler (1992) imply. Preferences are represented instead by a set of probability distributions over the state space, the so called multiple prior (MP) approach. The objective is that this set of priors reflects the perceived ambiguity of the decision maker (DM) about the probabilities of occurrence of states. A large literature successfully demonstrates the potential of the MP approach, see for instance Siniscalchi (2006) and Gilboa and Schmeidler (1989).

The pivotal question addressed in this article is how to measure and compare perceived ambiguities in the MP model. We provide a definition of perceived ambiguity which is very intuitive and generalizes numerous measures suggested in the existing literature. The key property of this definition is location-invariance: the ambiguity reflected by a prior set is independent of the location of this prior set within the set of probabilities over states. A consequence is that prior sets that differ only in location reflect the same ambiguity, even though they may induce very different preferences.

We illustrate the implication that the location-invariance of perceived ambiguity has on preferences, hereby focusing on Invariant Biseparable (IB) preferences introduced in Ghirardato et al. (2004).

We show that our MP model can be applied to normal-form games to model players that perceive strategic ambiguity. The location-invariance of perceived ambiguity allows a very general equilibrium existence result: for any exogenously fixed perceived ambiguity and ambiguity attitude for each player there exists an Equilibrium under Ambiguity. Since the class of preferences considered is very rich, this result generalizes many results from the existing literature. We illustrate the modelling capabilities based on several examples. We show that our model can explain intuitive deviations from the Nash Equilibrium solution concept and that it can rationalize cooperation in the Prisoners Dilemma.

4.1.1 Motivation: Perceived Ambiguity is Location - Invariant

Consider the state space $S = \{s_1, s_2, s_3\}$ and the prior sets $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ in the probability simplex $\Delta(S)$ in Figure 4.1. Imagine that these prior sets reflect the ambiguity perceived by three DM's. It is undebatable that $\mathcal{C}_1$ reflects more ambiguity than $\mathcal{C}_2$ since $\mathcal{C}_2$ is a subset of $\mathcal{C}_1$. What about $\mathcal{C}_3$? Can we compare this prior set with the other two in terms of the perceived ambiguity that it reflects? We claim that we can. The prior set $\mathcal{C}_3$ differs from $\mathcal{C}_2$ only in location and thus these prior

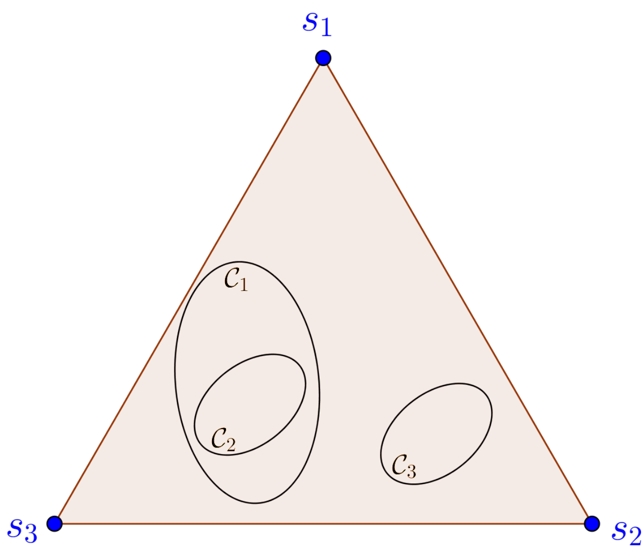


Figure 4.1: $\mathcal{C}_1$ reflects more ambiguity than $\mathcal{C}_2$. What about $\mathcal{C}_3$?

sets reflect the same ambiguity. This in turn implies that $\mathcal{C}_1$ reflects more ambiguity than $\mathcal{C}_3$.

To justify this claim consider two subjective expected utility (SEU) preferences represented by singleton prior sets P_1 and P_2. Clearly these prior sets reflect the same ambiguity (as they reflect no ambiguity at all) even though they are not subsets of each other. Consider now as prior sets ϵ-balls around some P_1 and P_2 with $\epsilon > 0$ small. This tiny change from P to $B_\epsilon(P)$ changes the perceived ambiguity in exactly the same way for both prior sets. In our opinion it would thus be inconsistent to not proclaim that $B_\epsilon(P_1)$ and $B_\epsilon(P_2)$ reflect the same ambiguity. Once this location-invariance for ϵ prior sets is accepted it suggests itself that this logic of thought has to be driven further. It leads us to the conclusion that reflected ambiguity does not depend on the location of the prior set. Going back to Figure 4.1 this implies that $\mathcal{C}_2$ and $\mathcal{C}_3$ reflect the same ambiguity as they only differ in location within $\Delta(S)$. This then implies that $\mathcal{C}_1$ reflects more ambiguity than $\mathcal{C}_2$ as well as $\mathcal{C}_3$.

In this article we introduce a definition of perceived ambiguity which implies this location-invariance. Thus in our model two DM's perceive the same ambiguity if and only if their prior sets only differ in location. The comparative notion of perceived ambiguity then becomes: DM 1 perceives more ambiguity than DM 2 if and only if there exists a translation of DM 2's prior set[1] such that this translated prior set is

[1] A translation is a geometric transformation that moves every point of a figure or a space by

a subset of DM 1's prior set. Figure 4.2 illustrates this in terms of our example.

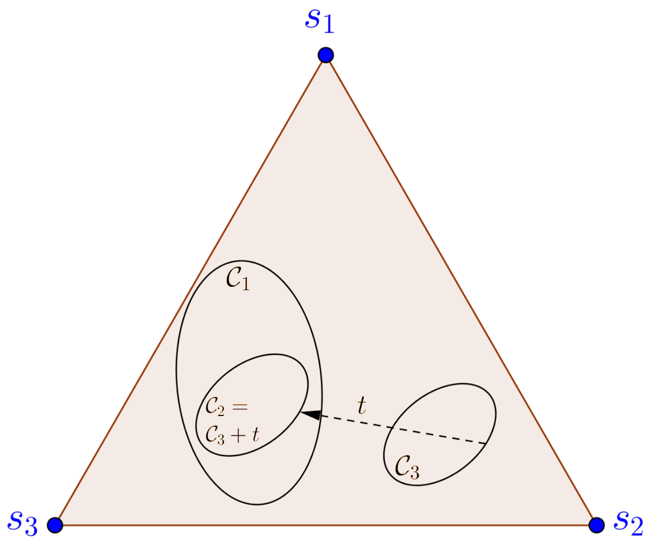


Figure 4.2: $\mathcal{C}_1$ reflects more ambiguity than $\mathcal{C}_2$ and $\mathcal{C}_3$. $\mathcal{C}_2$ and $\mathcal{C}_3$ reflect the same ambiguity.

4.1.2 Outline of the Paper

We introduce the framework in section 4.2. In section 4.3 we formally introduce our definition of perceived ambiguity. In section 4.4 we analyze what our definition of perceived ambiguity implies for IB preferences. In section 4.5 we provide comparative notions in terms of preferences for both perceived ambiguity and ambiguity attitude. We also provide an axiomatization of α-MEU preferences that is derived purely from preferences over acts. Throughout the section we focus in particular on the relationship to the model of Ghirardato et al. (2004) to which our model is related. In section 4.6 we apply our model to games. We define Equilibrium under Ambiguity in the spirit of Eichberger and Kelsey (2014) and prove equilibrium existence for normal-form games. All proofs are in the Appendix A. In the Appendix B we show how our definition of perceived ambiguity is related to existing measures, for instance from Marinacci (2000) and Chateauneuf et al. (2007).

the same amount in a given direction. A translation of a prior set is thus a change in location of this prior set.

4.2 Notation and Preliminaries

We assume a framework in the style of Anscombe and Aumann (1963). Our setting consists of a finite state space S and a set of prizes X. The powerset $\mathcal{P}(S)$ of S is the set of events. We denote by L the set of finite support lotteries over X. By $\mathcal{F}$ we denote the set of functions $f : S \to L$, called acts. As is common in the literature we abuse notation and denote by l the act such that $l(s) = l$ for all $s \in S$. Thus L also denotes the set of constant acts and X the set of constant acts that result in a degenerate lottery.

This setup allows the usual mixture-space over $\mathcal{F}$. For $f, g \in \mathcal{F}, \lambda \in [0,1]$ we denote by $\lambda f + (1-\lambda)g$ the act which results in $\lambda f(s) + (1-\lambda)g(s) \in L$ for all $s \in S$.

We assume X to be a 1-dimensional vector space, for instance the real numbers. For $f \in \mathcal{F}$ and $x \in X$ we denote by $f + x$ the act which results in $f(s) + x$ for all $s \in S$, i.e. state s results in the lottery $f(s)$ with prize x unconditionally added.[2]

We model a DM's preferences over $\mathcal{F}$ by a binary relation $\succsim$, where $\succ$ and $\sim$ respectively denote the asymmetric and symmetric components of $\succsim$. A functional $I : \mathcal{F} \to \mathbb{R}$ represents $\succsim$ if $I(f) \geq I(g) \iff f \succsim g$. A functional I is constant-linear if for all $f \in \mathcal{F}, l \in L$ and $\alpha \in [0,1]$ it holds that $I(\alpha f + (1-\alpha)l) = \alpha I(f) + (1-\alpha)I(l)$. This implies that $I(f + x) = I(f) + I(x)$.

We denote by $B_0(S)$ the set of real-valued functions on S. For an affine utility function $u : L \to \mathbb{R}$ and $f \in \mathcal{F}$ we denote by $u(f)$ the element of $B_0(S)$ defined by $u(f)(s) = u(f(s))$ for all $s \in S$. We tacitly assume throughout that the considered utility functions are nonconstant and affine, i.e. of the Von Neumann and Morgenstern type. We call two utility functions cardinally equivalent if they are positive affinely related. We define $u(\mathcal{F}) = \{u(f) \in B_0(S) | f \in \mathcal{F}\}$ and $u(X) = \{u(x) \in B_0(S) | x \in X\}$. For a utility function u we denote by $\mathcal{F}_1^u = \{f \in \mathcal{F} | u(f(s)) = u(f(s')) \quad \forall s, s' \in S\}$ the constant - expectation acts. The set $\langle f \rangle_u$ contains the acts that are positive affinely related to f in utility terms, i.e. $g \in \langle f \rangle_u$ if and only if $au(g) + b = u(f)$ for some $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$. We suppress the subscript u when convenient.

A probability over S is a normalized, monotonic and additive set-function on

[2] For instance if $f(s)$ is the lottery that results in prize y with probability α and z with probability $1-\alpha$, then for $x \in X$, $f(s) + x$ is the lottery that results in prize $y + x$ with probability α and $z + x$ with probability $1 - \alpha$.

$\mathcal{P}(S)$. We denote by $\Delta(S)$ the set of probability distributions on S. A non-empty, compact and convex set $\mathcal{C} \subseteq \Delta(S)$ is called a prior set. We denote by $\mathfrak{C}$ the set of prior sets. An ambiguity attitude is a function $a : \mathcal{F} \to [0,1]$. For a prior set $\mathcal{C}$, ambiguity attitude function a and utility function u we say that $(\mathcal{C}, a, u)$ represents the preference relation $\succsim$ if $I_{\mathcal{C},a,u}$ represents $\succsim$, where

$$I_{\mathcal{C},a,u}(f) = a(f) \min_{P \in \mathcal{C}} \int u(f)\ dP + (1 - a(f)) \max_{P \in \mathcal{C}} \int u(f)\ dP.$$

Here $\int u(f)\ dP$ denotes the expected value of the function $u(f) : S \to \mathbb{R}$, given the probability distribution P.

A translation is a geometric transformation that moves every point of a figure or a space by the same amount in a given direction. We are interested in translations of prior sets within $\Delta(S)$. For this we define the following vector-subspace $T = \{t \in B_0(S) | \sum_S t(s) = 0\}$. We say that two prior sets $\mathcal{C}$ and $\mathcal{C}'$ are translations if there exists a $t \in T$ such that $\mathcal{C}' = \mathcal{C} + t$, denoted by $\mathcal{C} \approx \mathcal{C}'$.[3] We denote by $[\mathcal{C}]$ the equivalence class of $\approx$ that contains $\mathcal{C}$ and by $\mathfrak{C}_{/[\approx]}$ the quotient of $\mathfrak{C}$ with respect to $\approx$.

4.3 A Measure for Perceived Ambiguity

In this section we introduce our definition of perceived ambiguity for prior sets and illustrate its consequence: the location-invariance of perceived ambiguity. Consider the state space $S = \{s_1, s_2, s_3\}$ and the prior set $\mathcal{C}$ in Figure 4.3. Consider the indifference curves of the act f depicted in Figure 4.3 for some utility function $u : L \to \mathbb{R}$.[4]

The probability distributions P and Q in Figure 4.3 represent the worst and best scenarios for the act f, given prior set $\mathcal{C}$ and utility function u.[5] We suggest that the difference between the indifference curves through these extreme scenarios measures the perceived ambiguity for the act f. That is perceived ambiguity is characterized by the function $\delta_{\mathcal{C},u} : \mathcal{F} \to [0,1]$ defined by

[3] Recall that prior sets are subsets of $\Delta(S)$. Translations of prior sets *beyond the border* of $\Delta(S)$ are not translations in our sense.

[4] The indifference curve of f through some $P \in \Delta(S)$ is given by $\{Q \in \Delta(S) | \int u(f) dQ = \int u(f) dP\}$. Recall that we assume u to be affine, which implies that the indifference curves are straight and parallel.

[5] Note that in general these extreme scenarios will not be unique.

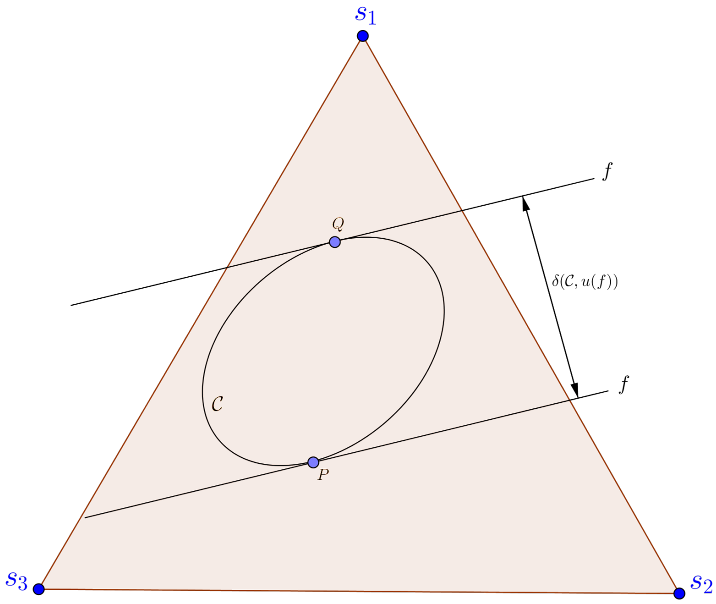


Figure 4.3: Worst and best scenario for the act f given prior set $\mathcal{C}$ and utility function u.

$$\delta_{\mathcal{C},u}(f) = \begin{cases} \frac{\max_{P\in\mathcal{C}} \int u(f)\ dP - \min_{P\in\mathcal{C}} \int u(f)\ dP}{\max_{P\in\Delta(S)} \int u(f)\ dP - \min_{P\in\Delta(S)} \int u(f)\ dP}, & f \notin \mathcal{F}_1^u \\ 0, & f \in \mathcal{F}_1^u. \end{cases} \tag{4.1}$$

For $f \notin \mathcal{F}_1^u$, $\delta_{\mathcal{C},\delta}(f)$ is the difference in evaluation of the best and worst case scenario. The denominator normalizes the measures.

Definition 4.1 (Perceived Ambiguity of Prior Sets)**.** *Consider a prior set $\mathcal{C}$ and utility function u. The perceived ambiguity reflected by $\mathcal{C}$ is characterized by $\delta_{\mathcal{C},u}$. Given $\mathcal{C}$ and u, an act f is called unambiguous if $\delta_{\mathcal{C},u}(f) = 0$.*

An act $f \in \mathcal{F}_1^u$ is always unambiguous. Whether an act $f \notin \mathcal{F}_1^u$ is unambiguous depends on the prior set $\mathcal{C}$ and the utility function u: it is unambiguous if the difference in evaluation between the best and the worst case scenario is zero. This in particular implies the desired consequence that all acts are unambiguous when the prior set is a singleton, i.e. when preferences are SEU.

The following comparative notion of perceived ambiguity for prior sets suggests itself.

Definition 4.2 (Comparative Perceived Ambiguity of Prior Sets)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be prior sets and u a utility function. Then $\mathcal{C}$ reflects more ambiguity than $\mathcal{C}'$ if $\delta_{\mathcal{C},u}(f) \geq \delta_{\mathcal{C}',u}(f)$ for all $f \in \mathcal{F}$. The prior sets $\mathcal{C}$ and $\mathcal{C}'$ reflect the same ambiguity if and only if $\delta_{\mathcal{C},u} \equiv \delta_{\mathcal{C}',u}$.*

The proposed measure has several desirable consequences that make it worthy of consideration. Firstly it is simple and makes intuitive sense. The further the extreme scenarios are apart for an act, the more ambiguity is perceived for this act. Secondly the measure generalizes and unifies all the proposed measures suggested in the literature that we are aware of.[6] Thirdly, as already highlighted, Definition 4.1 implies for SEU preferences that all acts are unambiguous which is clearly desirable. It thus implies that any two SEU preferences represent the same perceived ambiguity. Fourthly the comparative notion in Definition 4.2 is independent of the utility function (see Observation 4.1) as well as any kind of ambiguity attitude. Thus perceived ambiguity indeed only depends on the prior set and not on risk or ambiguity attitudes which is desirable. Finally Definition 4.1 implies the location-invariance of perceived ambiguity that we aspire. The following Observation provides some properties of the function $\delta_{\mathcal{C},u}$ leading up to this result.

Observation 4.1. *Let $\mathcal{C}$ be a prior set and u a utility function. Then*

1. *$\delta_{\mathcal{C},u}$ is translation-invariant, i.e. $\delta_{\mathcal{C},u} \equiv \delta_{\mathcal{C}+t,u}$ for all $t \in T$ for which $\mathcal{C} + t \subseteq \Delta(S)$.*
2. *$\delta_{\mathcal{C},u}$ is constant on affinely related acts, i.e. $f \in \langle g\rangle_u \cup \langle -g\rangle_u$ implies $\delta_{\mathcal{C},u}(f) = \delta_{\mathcal{C},u}(g)$.*
3. *The comparative notion in Definition 4.2 is independent of the utility function, i.e. for two utility functions u_1 and u_2 we have*

$$\delta_{\mathcal{C},u_1}(f) \geq \delta_{\mathcal{C}',u_1}(f) \quad \forall f \in \mathcal{F} \iff \delta_{\mathcal{C},u_2}(f) \geq \delta_{\mathcal{C}',u_2}(f) \quad \forall f \in \mathcal{F}.$$

Parts 1. and 3. of Observation 4.1 imply that Definition 4.1 captures exactly the location invariance of perceived ambiguity that we want to achieve.

[6] We address this in Appendix B.

Observation 4.2. *Let $\mathcal{C}$ and $\mathcal{C}'$ be prior sets. Then $\mathcal{C}$ reflects more ambiguity than $\mathcal{C}'$ if and only if there exists a translation $t \in T$ such that $\mathcal{C}' + t \subseteq \mathcal{C}$. Furthermore $\mathcal{C}$ and $\mathcal{C}'$ reflect the same ambiguity if and only if $\mathcal{C}$ and $\mathcal{C}'$ are translations of each other.*

The equivalence class $[\mathcal{C}]$, which contains all the translations of $\mathcal{C}$ within $\Delta(S)$, thus contains exactly those prior sets that reflect the same ambiguity according to Definition 4.1. We illustrate in section 4.6 how this allows exogenous fixation of perceived strategic ambiguity for players in normal-form games, allowing a very general equilibrium existence results.

4.4 Location-Invariance and Preferences

Section 4.3 introduces a definition of perceived ambiguity for prior sets which leads to the desired location-invariance property. What does this location-invariance imply for preferences? When does one preference relation reflect more ambiguity than another? When is one preference relation more ambiguity averse than another? To answer these questions we need to answer the following:

> How are two preference relations related, if they can be represented by the same ambiguity attitude and utility functions as well as prior sets that only differ in location?

We focus on Invariant Biseparable (IB) preferences, introduced in Ghirardato et al. (2004).[7] The reason we focus on IB preferences is that it is a large class of MP preferences containing important preference subclasses such as CEU and α-MEU. Furthermore IB preferences induce a preference functional that is constant-linear, which is necessary to answer the above question as will become clear. Another reason is that Ghirardato et al. (2004) suggest notions of comparative perceived ambiguity and comparative ambiguity attitude to which our notion can be compared easily.

4.4.1 The Effect of a Translation on Preferences

Consider the four priors in Figure 4.4. The translation $t \in T$ is the difference between P and P' as well as between Q and Q'. Consider a utility function u. For

[7] See the introductory chapter for details.

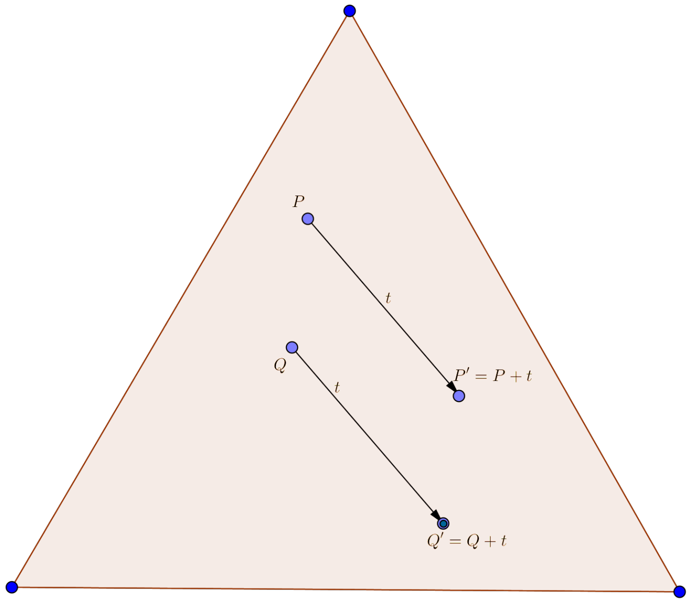


Figure 4.4: P and Q translated by t.

all $f \in \mathcal{F}$ we then have

$$\begin{aligned}
\int u(f)\ dP' - \int u(f)\ dP &= \sum_{s\in S} u(f(s))P'(s) - \sum_{s\in S} u(f(s))P(s) \\
&= \sum_{s\in S} u(f(s))(P'(s) - P(s)) \\
&= \sum_{s\in S} u(f(s))t(s) \\
&= \sum_{s\in S} u(f(s))(Q'(s) - Q(s)) \\
&= \sum_{s\in S} u(f(s))Q'(s) - \sum_{s\in S} u(f(s))Q(s) \\
&= \int u(f)\ dQ' - \int u(f)\ dQ. \qquad (4.2)
\end{aligned}$$

That is for a given translation t the resulting difference in evaluation of acts is location independent. The difference in evaluation only depends on t and on the utility function u.

Now consider two IB preference relations $\succsim_1$ and $\succsim_2$ that are represented by the same ambiguity attitude function a and utility function u as well as GMM prior sets[8]

[8] Recall that the GMM prior set is the unique prior set induced by the unambiguous pref-

$\mathcal{C}_1$ and $\mathcal{C}_2$ that are translations of each other, see Figure 4.5. Let their representation functionals be I_1 and I_2, respectively. By applying Equation (4.2) twice we observe that for every $f \in \mathcal{F}$

$$\begin{aligned} I_2(f) - I_1(f) &= a([f]) \min_{P \in \mathcal{C}_2} \int u(f)\ dP + (1 - a([f])) \max_{P \in \mathcal{C}_2} \int u(f)\ dP \\ &\quad - a([f]) \min_{P \in \mathcal{C}_1} \int u(f)\ dP + (1 - a([f])) \max_{P \in \mathcal{C}_1} \int u(f)\ dP \\ &= \max_{P \in \mathcal{C}_2} \int u(f)\ dP - \max_{P \in \mathcal{C}_1} \int u(f)\ dP \\ &= \int u(f)\ dP_2 - \int u(f)\ dP_1 \qquad \forall P_1, P_2 \in \Delta(S) \text{ such that } P_2 = P_1 + t. \end{aligned} \tag{4.3}$$

This shows that the change in evaluation of act f going from $\succsim_1$ to $\succsim_2$ only depends on the translation t and the utility function u, this property being inherited from Equation (4.2). It does not depend on the ambiguity attitude function a, the prior sets $\mathcal{C}_1$ and $\mathcal{C}_2$ or their location within $\Delta(S)$.

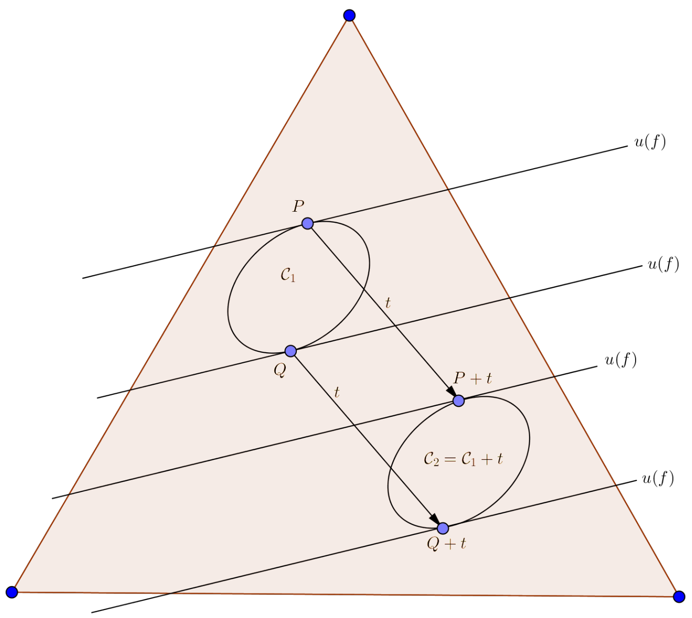


Figure 4.5: $\mathcal{C}_1$ and $\mathcal{C}_2$ differ only in location: $\mathcal{C}_2 = \mathcal{C}_1 + t$.

For an IB preference $\succsim$ and act $f \in \mathcal{F}$ define $x_f^{\succsim}$ as some certainty prize of f,

erence relation and is interpreted by the authors as reflecting the perceived ambiguity, see introductory chapter.

given preferences $\succsim$, i.e. $f \sim x_f^{\succsim}$. This prize exists since X is a 1-dimensional vector space and due to the Archimedean axiom.[9] It holds that

$$\begin{aligned} I_2(f) - I_1(f) &= I_2(x_f^{\succsim_2}) - I_1(x_f^{\succsim_1}) \\ &= u(x_f^{\succsim_2}) - u(x_f^{\succsim_1}). \end{aligned}$$

For $f \in \mathcal{F}$, $t \in T$ and utility function u define

$$x_u(f,t) = \{x \in X | u(x) = u(x_f^{\succsim_1}) - u(x_f^{\succsim_2})\}.$$

The prize $x_u(f,t)$ exists.[10] It can be interpreted as the prize that has to be added to the act f as *compensation* when preferences change from $\succsim_1$ to $\succsim_2$, i.e. when, ceteris paribus, the prior set changes location by the translation t.

It holds that $u(x_u(l,t)) = 0$ for all $l \in L$ and thus also $u(x_u(x,t)) = 0$ for all $x \in X$. Furthermore $x_u(f,0) = x_u(g,0)$ for all $f, g \in \mathcal{F}$.[11]

Now assume that for two acts f and g we have $f \succsim_1 g$. What does this imply for $\succsim_2$? Going from $\succsim_1$ to $\succsim_2$ the evaluations of f and g change from $I_1(f)$ to $I_2(f)$ and from $I_1(g)$ to $I_2(g)$. Given the insight that $x_u(\cdot, t)$ is the compensation due to translation t we can conclude that $f + x_u(f,t) \succsim_2 g + x_u(g,t)$.[12] By adding the unconditional prizes $x_u(f,t)$ and $x_u(g,t)$ to the acts f and g we thus eliminate the effect that the translation t has on the evaluation of these acts.[13]

We state this result formally in the following theorem, hereby answering the question stated at the beginning of this section.

[9] Note that $x_f^{\succsim}$ is not necessarily unique. We abuse notation slightly and denote by $x_f^{\succsim}$ an arbitrary element of the set.

[10] Again it is not necessarily unique. Again $x_u(f,t)$ denotes an arbitrary element of the set.

[11] Note that all these insights are unaffected by a replacement of the utility function by a cardinally equivalent one. Most crucially, the set $x_u(f,t)$ stays the same.

[12] Recall that the act $f + x$ is defined as the act f with the prize x unconditionally added to the resulting lotteries in every state.

[13] Note that this last step is where we need the constant-linearity of the preference functional, implying $a(f) = a(f + x)$. Without this property the set $x_u(f,t)$ would in addition depend on the ambiguity attitude. In principle this could be done, for instance by relaxing Certainty Independence to Risk Independence.

Theorem 4.1. *Let $\succsim_1$ and $\succsim_2$ be IB preference relations on $\mathcal{F}$ with the same ambiguity attitude function, utility functions that are cardinally equivalent and GMM prior sets $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Then for $t \in T$ the following are equivalent:*

1. $\mathcal{C}_1 + t = \mathcal{C}_2$.
2. $f \succsim_1 g \iff f + x_u(f,t) \succsim_2 g + x_u(g,t)$.

From Theorem 4.1 we can deduce the following for the unambiguous preference relation. Note that Corollary 4.1 does not require identical ambiguity attitudes.

Corollary 4.1. *Let $\succsim_1$ and $\succsim_2$ be IB preferences on $\mathcal{F}$ with utility functions that are cardinally equivalent and GMM prior sets $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Then for $t \in T$ the following are equivalent:*

1. $\mathcal{C}_1 + t = \mathcal{C}_2$.
2. $f \succsim_1^* g \iff f + x_u(f,t) \succsim_2^* g + x_u(g,t)$.

4.5 Comparative Perceived Ambiguity, Comparative Ambiguity Attitude and α-MEU Preferences

This section builds on the insights of the previous section. It introduces comparative notions for perceived ambiguity and ambiguity attitude for preferences that respect the location-invariance of perceived ambiguity.

4.5.1 Comparative Perceived Ambiguity

Ghirardato et al. (2004) suggest a comparative notion in which $\succsim_1$ perceives more ambiguity than $\succsim_2$ if for all $f, g \in \mathcal{F}$

$$f \succsim_1^* g \implies f \succsim_2^* g. \tag{4.4}$$

If a DM unambiguously prefers f to g, then another DM who perceives less ambiguity does so as well. Ghirardato et al. show that this is the case if and only if the utility functions are cardinally equivalent and $\mathcal{C}_1 \supseteq \mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the GMM prior sets of $\succsim_1$ and $\succsim_2$, respectively.

In section 4.3 we suggest that the comparative notion should respect location-invariance of perceived ambiguity. The following definition achieves this, as shown in Lemma 4.1.

Definition 4.3 (Comparative Perceived Ambiguity of Preferences)**.** *Let* $\succsim_1$ *and* $\succsim_2$ *be IB preference relations on* $\mathcal{F}$*. Then* $\succsim_1$ *reflects more ambiguity than* $\succsim_2$ *if there exists a translation* $t \in T$ *such that for all* $f, g \in \mathcal{F}$

$$f \succsim_1^* g \implies f + x_u(f,t) \succsim_2^* g + x_u(g,t). \tag{4.5}$$

Note the close relationship between (4.4) and (4.5). In section 4.3 we deduced that by adding the unconditional prizes $x_u(f,t)$ and $x_u(g,t)$ to the acts f and g we eliminate the effect that the translation t has on the evaluation of these acts. The case $t = 0$ gives exactly the Ghirardato et al. (2004) notion since $x_u(f,0) = x_u(g,0)$ for all acts $f, g \in \mathcal{F}$ and preferences are constant-linear. The following lemma shows that Definition 4.3 reflects the desired location-invariance.

Lemma 4.1. *Let* $\succsim_1$ *and* $\succsim_2$ *be two IB preference relations on* $\mathcal{F}$*. Then the following are equivalent:*

1. $\succsim_1$ *reflects more ambiguity than* $\succsim_2$*.*
2. $\succsim_1$ *and* $\succsim_2$ *have utility functions that are cardinally equivalent and there exists a* $t \in T$ *such that* $\mathcal{C}_1 + t \supseteq \mathcal{C}_2$*.*

4.5.2 Comparative Ambiguity Attitude

We suggest two different notions of comparative ambiguity aversion. The first one is in the spirit of Ghirardato and Marinacci (2002). In order to compare ambiguity attitudes the DMs are required to perceive the same ambiguity. In the second notion this requirement can be dropped, i.e. it is independent of perceived ambiguity being identical which we believe is superior.

The First Notion of Comparative Ambiguity Attitude

Ghirardato and Marinacci (2002) suggest a comparative notion in which $\succsim_1$ is more ambiguity averse than $\succsim_2$ if for all $l \in L$ and $f \in \mathcal{F}$

$$f \succsim_1 l \implies f \succsim_2 l.$$

If one DM prefers the act f to a constant act l, then a less ambiguity averse DM does so as well. Ghirardato et al. adopt this notion for their model and show that if for two IB preference relations the GMM prior sets are identical and utility functions cardinally equivalent, then $\succsim_1$ is more ambiguity averse than $\succsim_2$ if and only if $a_1([f]) \geq a_2([f])$ for all $f \in \mathcal{F}\backslash[x]$.

The following definition is in this spirit but respects the location-invariance of perceived ambiguity as shown in Lemma 4.2.

Definition 4.4 (Comparative Ambiguity Aversion of Preferences Type I). *Let $\succsim_1$ and $\succsim_2$ be IB preferences with utility functions that are cardinally equivalent. Then $\succsim_1$ is more ambiguity averse of type I than $\succsim_2$ if there exists a $t \in T$ such that for all $f \in \mathcal{F}$ and all $l \in L$*

$$f \succsim_1 l \implies f + x_u(f,t) \succsim_2 l.$$

Recall that $u(x_u(l,t)) = 0$ which is why it does not show up in this definition. Again the case $t = 0$ gives exactly the definition of Ghirardato and Marinacci (2002).

Lemma 4.2. *Let $\succsim_1$ and $\succsim_2$ be IB preferences with utility functions that are cardinally equivalent. Assume that there exists a $t \in T$ such that $\mathcal{C}_2 + t = \mathcal{C}_1$. Let a_1 and a_2 be the ambiguity attitude functions. Then the following are equivalent:*

1. *$\succsim_1$ is more ambiguity averse of type I than $\succsim_2$.*
2. *$a_1([f]) \geq a_2([f])$ for all $f \in \mathcal{F}\backslash[x]$.*

The Second Notion of Comparative Ambiguity Attitude

If we have a separation of perceived ambiguity and ambiguity attitude, which Ghirardato et al. (2004) claim to have found, we should aim to define comparative ambiguity attitude without requiring identical perceived ambiguity. Why? Because of the separation! The ambiguity attitude is clearly characterized by a function and thus a comparative notion should only depend on this characterization. In the following we present a definition which achieves this.

Assume that $\succsim$ is an IB preference relation on $\mathcal{F}$. Define

$$\begin{aligned}\overline{x}(f) &= \inf_{x\in X}\{x \succsim^* f\} &&= \inf_{x\in X}\{\lambda x + (1-\lambda)h \succsim \lambda f + (1-\lambda)h \quad \forall \lambda \in [0,1], h \in \mathcal{F}\},\\ \underline{x}(f) &= \sup_{x\in X}\{f \succsim^* x\} &&= \sup_{x\in X}\{\lambda f + (1-\lambda)h \succsim \lambda x + (1-\lambda)h \quad \forall \lambda \in [0,1], h \in \mathcal{F}\}.\end{aligned}$$

Thus $\overline{x}(f)$ is the infimum of the prizes that are unambiguously preferred to f and $\underline{x}(f)$ is the supremum of the prizes to which f is unambiguously preferred. Since X is a 1-dimensional vector-space, these sets are non-empty.[14] Note that $\overline{x}(f) \sim^* f$ does not hold. This is the case if and only if f is crisp in which case it also holds that $\overline{x}(f) \sim \underline{x}(f)$.[15] With $\mathcal{C}$ being the GMM prior set of $\succsim$ it is easy to see that

$$\begin{aligned} u(\overline{x}(f)) &= \max_{P\in\mathcal{C}} \int u(f)dP,\\ u(\underline{x}(f)) &= \min_{P\in\mathcal{C}} \int u(f)dP. \end{aligned}$$

Thus the DM is indifferent between $\overline{x}(f)$ and the best scenario for f within $\mathcal{C}$ as well as indifferent between $\underline{x}(f)$ and the worst scenario for f within $\mathcal{C}$.

We can now state our second definition of comparative ambiguity attitude. If a DM prefers the act f to the α-mixture of $\overline{x}(f)$ and $\underline{x}(f)$, then a less ambiguity averse DM will also prefer f to this mixture. Note however that $\overline{x}(f)$ and $\underline{x}(f)$ may differ for the two DM's.[16]

Definition 4.5 (Comparative Ambiguity Aversion of Preferences Type II)**.** *Let $\succsim_1$ and $\succsim_2$ be IB preferences with utility functions that are cardinally equivalent. $\succsim_1$ is more ambiguity averse of type II than $\succsim_2$ if for all $f \in \mathcal{F}$ and $\alpha \in [0,1]$*

$$f \succsim_1 \alpha \underline{x}_1(f) + (1-\alpha)\overline{x}_1(f) \implies f \succsim_2 \alpha \underline{x}_2(f) + (1-\alpha)\overline{x}_2(f).$$

This definition allows an analogue to Lemma 4.2 without the requirement $\mathcal{C}_2 + t = \mathcal{C}_1$.

[14] Once more we abuse notation and refer to $\overline{x}(f)$ and $\underline{x}(f)$ as arbitrary elements of these sets.
[15] See the introductory chapter for the definition of crispness of an act.
[16] The set $\overline{x}(f)$ and $\underline{x}(f)$ are the same for both DM's and all acts f if and only if the utility functions are cardinally equivalent and the prior sets are identical. The latter however is exactly what we want to relax.

Lemma 4.3. *Assume that $\succsim_1$ and $\succsim_2$ are IB preferences with utility functions that are positive affinely related. Let a_1 and a_2 be the ambiguity attitude functions. Then the following are equivalent:*

1. *$\succsim_1$ is more ambiguity averse of type II than $\succsim_2$.*

2. *$a_1([f]) \geq a_2([f])$ for all $f \in \mathcal{F} \backslash [x]$.*

We thus have a definition of comparative ambiguity attitude which does not need to assume that the DMs have the same perceived ambiguity. We think that this second notion is superior to the first. If we have a separation of perceived ambiguity and ambiguity attitude then we should aim at defining the comparative notion of the one independent of the other.

4.5.3 Excursus: α-MEU

With the entities $\overline{x}(f)$ and $\underline{x}(f)$ we can also axiomatize α-MEU, i.e. IB preferences with a constant ambiguity attitude $\alpha \in [0,1]$. Ghirardato et al. (2004) also provide such an axiomatization. It has the drawback that it is not derived purely from preferences over acts. The following axiom however is just in terms of the preference relation.[17]

Axiom (α-MEU). There exists an $\alpha \in [0,1]$ such that $f \sim \alpha \underline{x}(f) + (1-\alpha)\overline{x}(f)$ for all $f \in \mathcal{F}$.

Theorem 4.2. *Let $\succsim$ be a IB preference relation on $\mathcal{F}$. Then the following are equivalent:*

1. *$\succsim$ satisfies the axiom α-MEU.*

2. *$a([f]) = \alpha$ for all $f \in \mathcal{F} \backslash [x]$, i.e. the ambiguity attitude function is constant.*

[17] Eichberger et al. (2011) critizise the axiomatization of α-MEU preferences by Ghirardato et al. (2004). They show that in a finite state space the prior set will never be equal to the set from (Ghirardato et al. (2004)), i.e. the Clarke differential at 0, when $\alpha \in (0,1)$. Thus they show inconsistency of the axioms. We cannot solve this problem with our axiomatization, thus this criticism applies to our axiomatization as well.

4.6 Application to Games

This section applies the insights thus far to strategic interaction. We utilize our MP approach to represent players' ambiguous beliefs about the strategy choice of the other players. We introduce an equilibrium notion, our approach being closely related to Eichberger and Kelsey (2014). We prove equilibrium existence in normal-form games for any exogenously fixed perceived ambiguities via the equivalence class $[\cdot]$ and exogenously fixed ambiguity attitudes for all players.[18] Several examples illustrate the modelling capabilities of the approach.

4.6.1 Normal-Form Games and Strategic Ambiguity

A normal-form game $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ consists of a finite set of players N, finite pure strategy sets S_i and payoff function u_i for player i. The set of pure strategy combinations is denoted by S and S_{-i} is the set of pure-strategy combinations of player i's opponents. Player i has payoff function $u_i : S \to \mathbb{R}$.

Players perceive ambiguity about the strategic choice of the other players. Beliefs are represented by a prior set over the pure strategy combinations of the opponents as well as an ambiguity attitude function over the players' own strategies. For player i with belief $\mathcal{C}_i \subseteq S_{-i}$ and ambiguity attitude function $a_i : S_i \to [0, 1]$ the evaluation of a strategy $s_i \in S_i$ is therefore

$$V_i(s_i|\mathcal{C}_i, a_i) = a_i(s_i) \min_{P \in \mathcal{C}_i} \int u(s_i, s_{-i})\ dP + (1 - a_i(s_i)) \max_{P \in \mathcal{C}_i} \int u(s_i, s_{-i})\ dP.$$

Note that we do not restrict the ambiguity attitude in any way.[19] The following analysis and equilibrium result holds in this surpisingly general framework.

4.6.2 Equilibrium under Ambiguity and Equilibrium Existence

In equilibrium, players choose optimal pure strategies, given their beliefs. Mixed strategies are not an object of choice. Furthermore beliefs must be consistent in the sense that the support of the belief only contains best responses of the opponents, given their beliefs. The notion of support of a prior set pins down this consist-

[18] Recall that $[\cdot]$ contains exactly the prior sets that reflect the same ambiguity according to Definition 4.1, i.e. prior sets that differ only in location.

[19] Thus preferences are not even constant-linear.

ency. Following Eichberger and Kelsey (2014), we use the following support notion introduced by Ryan (2002).

Definition 4.6 (Support notion: Ryan (2002)). *For some finite state space S let $\mathcal{C}$ be a prior set on $\Delta(S)$. The support of $\mathcal{C}$ is defined by*

$$supp(\mathcal{C}) = \bigcap_{P \in \mathcal{C}} supp(P),$$

where for a probability distribution $P \in \Delta(S)$ the support is defined as is usual by

$$supp(P) = \{s \in S | P(s) > 0\}.$$

In a game context, a strategy combination s_{-i} is in the support of some prior set $\mathcal{C}_i$ if all elements of $\mathcal{C}_i$ assign strictly positive weight to s_{-i}. Thus the elements of $supp(\mathcal{C}_i)$ are the opponents' strategy combinations which player i cannot rule out. The consistency requirement in equilibrium is that these strategy combinations that the player cannot rule out have to be optimal for the other players.

The strategy combinations in the support are guaranteed to always receive positive weight by the evaluation functional. Elements that are not in the support may however also receive positive weight. It is this property of the model which distinguishes it mathematically from Nash Equilibrium and allows us to model non-Nash behaviour.

We now formally introduce our equilibrium notion.

Definition 4.7 (Equilibrium under Ambiguity). *Let $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ be a normal-form game. The tuple $(\mathcal{C}_i, a_i)_{i=1}^N$ is an Equilibrium under Ambiguity if for all $1 \leq i \leq N$*

$$\emptyset \neq supp(\mathcal{C}_i) \subseteq \times_{j \neq i} R_j(\mathcal{C}_j, a_j),$$

where $R_j(\mathcal{C}_j, a_j) = \arg\max_{s_j \in S_j} [V(s_j | \mathcal{C}_j, a_j)]$ is the best response correspondence.

If $supp(\mathcal{C}_i)$ contains just a single element $\hat{s}_i \in S_i$ for all $i \in \{1, \ldots, N\}$ we refer to the equilibrium as a singleton equilibrium and $\hat{s} = (\hat{s}_i, \ldots, \hat{s}_N)$ as its strategy profile.

If there are just two players and the prior sets are singletons then Definition 5.6 coincides with the standard Nash Equilibrium. With three or more players it is however possible in an Equilibrium under Ambiguity that prior sets are singletons but this does not constitute a Nash Equilibrium. For instance players 1 and 2 may have different beliefs about what player 3 will do.

Recall that mixed strategies are not an object of choice. An equilibrium in which some support contains multiple strategy combinations cannot be interpreted as an equilibrium in which players randomize. Instead the equilibrium notion should be interpreted as an *equilibrium in beliefs*.

The following theorem shows that for any exogenously chosen perceived ambiguities $[\mathcal{C}_1], \ldots, [\mathcal{C}_N]$ and ambiguity attitudes $a_1, \ldots, a_N$ there exists an Equilibrium under Ambiguity. To rule out uninteresting cases we assume that $[\mathcal{C}_i]$ does not only contain the element $\mathcal{C}_i$, or equivalent that $supp(\mathcal{C}_i) \neq \emptyset$.

Theorem 4.3 (Equilibrium Existence)**.** *Let $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ be a normal-form game. Then for any exogenously given n-tuples of perceived ambiguities $([\mathcal{C}_i])_{i=1}^N$ such that $supp(\mathcal{C}_i) \neq \emptyset$ for all $i \in \{1, \ldots, N\}$ and ambiguity attitudes $(a_i)_{i=1}^N$ there exist $(\mathcal{C}_i^*)_{i=1}^N$ with $\mathcal{C}_i^* \in [\mathcal{C}_i]$ for all $i \in \{1, \ldots, N\}$ such that $(\mathcal{C}_i^*, a_i)_{i=1}^N$ is an Equilibrium under Ambiguity.*

Due to the weak assumptions that we make about the preferences of players, Theorem 4.3 is an extremely general existence result. We can conclude equilibrium existence for subclasses like IB preferences, α-MEU preferences, neo-additive capacities or MEU preferences. The theorem also generalizes several equilibrium existence results from the literature, for instance Eichberger and Kelsey (2014) and Marinacci (2000).

4.6.3 Examples

In this subsection we study some 2-player games with which the modelling capabilities of the approach introduced in this section can be illustrated. The intuitive appeal of the resulting equilibria is highlighted.

Example 1 illustrates that our approach allows modelling of intuitive outcomes that cannot be explained by the Nash Equilibrium solution concept. Example 2 illustrated the flexibility of our model and provides comparative statics in perceived ambiguity and ambiguity attitude. Example 3 is the Rock-Paper-Scissors game. We show that for sufficient perceived ambiguity the game has a singleton Equilibrium under Ambiguity. Example 4 is the Prisoners Dilemma. We show that by allowing non-constant ambiguity attitude functions we can explain cooperation.

In a game with two players and two strategies the set S_{-i} consists of two elements. This implies that prior sets are intervals. A prior set in the game is thus of the kind $\mathcal{C}_i = Conv(P, Q | P, Q \in \Delta(S_{-i}))$. We can represent perceived ambiguity by the length of this interval, i.e. by a parameter $\delta_i \in [0, 1]$. To exogenously fix perceived ambiguity means to fix an interval-length, i.e. an exogenously fixed perceived ambiguity $[\mathcal{C}]$ consists of the prior sets with the same interval length as $\mathcal{C}$.

Example 1: Modelling intuitive non-Nash behaviour

		Player 2	
		L	R
Player 1	U	$(100, 1)$	$(0, 0)$
	D	$(99, 1)$	$(99, 0)$

Figure 4.6: Is the Nash Equilibrium realistic?

Consider the 2-player game in Figure 4.6. The unique Nash equilibrium is the strategy combination (U, L). It is however quite intuitive that player 1 plays D, in which case the strategy combination (D, L) results. We show in the following that for sufficient perceived ambiguity as well as sufficient pessimism, the strategy combination (D, L) will indeed constitute an Equilibrium under Ambiguity.

The evaluation of strategy U, given a prior set $\mathcal{C}_1 \subseteq \Delta(\{L, R\})$ and ambiguity attitude $a_1 : \{U, D\} \to [0, 1]$ is

$$a_1(U) \min_{P \in \mathcal{C}} \int U \; dP + (1 - a_1(U)) \max_{P \in \mathcal{C}} \int U \; dP.$$

For the prior set $\mathcal{C}_1$ with interval length δ_1 depicted in Figure 4.7 the evaluation of U is

$$a_1(U)(\delta_1 0 + (1 - \delta_1)100) + (1 - a_1(U))100.$$

The evaluation of D is 99. Thus, given the prior set $\mathcal{C}_1$, $D \succ U$ if and only if $a_1(U)\delta_1 > \frac{1}{100}$. Consider any prior set $\mathcal{C}_2$ such that $supp(\mathcal{C}_2) = \{D\}$ and any ambiguity attitude $a_2 : \{L, R\} \to [0, 1]$. Note that $supp(\mathcal{C}_1) = \{L\}$. Now for $a_1(U)\delta_1 > \frac{1}{100}$ the tuple $(\mathcal{C}_i, a_i)_{i=1}^2$ constitutes an Equilibrium under Ambiguity. This is because D is optimal as well as the unique element of $supp(\mathcal{C}_2)$ and L is optimal as well as the unique element of $supp(\mathcal{C}_1)$.

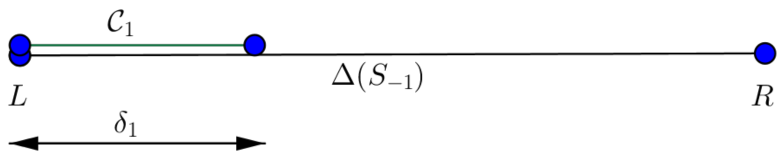


Figure 4.7: The equilibrium belief of player 1 with perceived ambiguity δ_1.

We have thus shown that (D, L) is the unique equilibrium strategy profile when player 1 perceives sufficient ambiguity and is sufficiently ambiguity averse, i.e. when $a_1(U)\delta_1 > \frac{1}{100}$. This result is very much in line with intuition and cannot be explained within the Nash-framework.

Example 2: A Game with a safe strategy for both Players

		Player 2 L	Player 2 R
Player 1	U	$(2, 1)$	$(0, 0)$
	D	$(1, 1)$	$(1, 2)$

Figure 4.8: A Game with Safe Strategies.

Consider the game in Figure 4.8. It has two pure strategy Nash Equilibria: (U, L) and (D, R). Both players have a safe strategy that guarantees a payout of 1 and a risky strategy that either results on 0 or 2. We show that, depending on the levels of perceived ambiguity and ambiguity attitude, every strategy combination can constitute a singleton Equilibrium under Ambiguity in a way that is in our opinion very much in line with intuition. Throughout the analysis δ_i denotes the exogenously fixed perceived ambiguity and a_i the exogenously fixed ambiguity attitude of player i, $i \in \{1, 2\}$.[20]

The strategy combination (D, L) constitutes a singleton Equilibrium under Ambiguity when $\frac{1}{2} \leq a_1\delta_1$ and $\frac{1}{2} \leq a_2\delta_2$. This is very intuitive. When the players are pessimistic and perceive sufficient ambiguity, they will choose the safe strategy even when they believe that the other player will play the safe strategy as well.

[20] We assume constant ambiguity attitude. No generality is lost in this case since the ambiguity attitude plays no role in the evaluation of the safe strategy that guarantees payout 1.

It might at first look curious that (U, R) can constitute a singleton Equilibrium under Ambiguity. This is the case if $\frac{1}{2} \leq (1 - a_1)\delta_1$ and $\frac{1}{2} \leq (1 - a_2)\delta_2$. When the players are sufficiently optimistic and perceive sufficient ambiguity, they play the risky strategy even when they believe that the other player will do so as well.

The strategy combination (D, R) constitutes a singleton Equilibrium under Ambiguity when $\frac{1}{2} \geq (1 - a_1)\delta_1$ and $\frac{1}{2} \geq a_2\delta_2$. This can be understood by looking at the two above cases: considering the strategy combination (D, L), player 2 will deviate to R when it is not the case that $\frac{1}{2} \leq a_2\delta_2$. Considering the strategy combination (U, R), player 1 will deviate to D when it is not the case that $\frac{1}{2} \leq (1 - a_1)\delta_1$.

Due to symmetry, (U, L) constitutes a singleton Equilibrium under Ambiguity if $\frac{1}{2} \geq a_1\delta_1$ and $\frac{1}{2} \geq (1 - a_2)\delta_2$.

Example 3: Rock-Paper-Scissors

In the previous examples the players had two strategies each. All the results could have been achieved by restricting attention to neo-additive capacities.[21]

In the following example the players have three strategies which allows modelling of behaviour that cannot be achieved with neo-additive capacities or the CEU framework in general.

Player 1 \ Player 2	R_2	P_2	S_2
R_1	$(0, 0)$	$(-1, 1)$	$(1, -1)$
P_1	$(1, -1)$	$(0, 0)$	$(-1, 1)$
S_1	$(-1, 1)$	$(1, -1)$	$(0, 0)$

Figure 4.9: Rock-Paper-Scissors

The unique Nash Equilibrium consists of the uniform mixing over the three pure strategies for both players. Our equilibrium notion allows a rich set of equilibria in this game.

First assume that both players' perceived ambiguity is represented by

$$[\mathcal{C}_i] = \left\{ \mathcal{C} \subseteq \Delta(S_{-i}) | \mathcal{C} = B_{\epsilon_i}(P), P \in \Delta(S_{-i}), \epsilon_i \in \left(0, \frac{1}{3}\right) \right\},$$

[21] This is because when the state space has just two elements, preferences with a constant ambiguity attitudes have a representation as a neo-additive capacity.

i.e. the set of balls within the simplex of radius ϵ_i.[22] Assume that ambiguity attitude a_i is constant for both players, i.e. $a_i \equiv \alpha_i$ for $i \in \{1,2\}$. Then there is the unique Equilibrium under Ambiguity

$$\left(B_{\epsilon_i}\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \alpha_i\right)_{i=1}^{2}.$$

In equilibrium the perceived ambiguity is represented by the circular prior set centered around the uniform distribution. It is easy to check that this is indeed an Equilibrium under Ambiguity. All three strategies result in the same evaluation and the support of the prior sets contain all three strategies. Note that this result allows the players to have different ϵ and ambiguity attitudes. The equilibrium is illustrated for the case $\epsilon_1 = \frac{1}{6}$ and $\epsilon_2 = \frac{1}{12}$ in Figure 4.10.

The equilibrium can be interpreted in the following way: the players believe that the opponent will uniformly mix amongst pure strategies, however the players perceive ambiguity about this belief. This ambiguity is represented by ϵ.

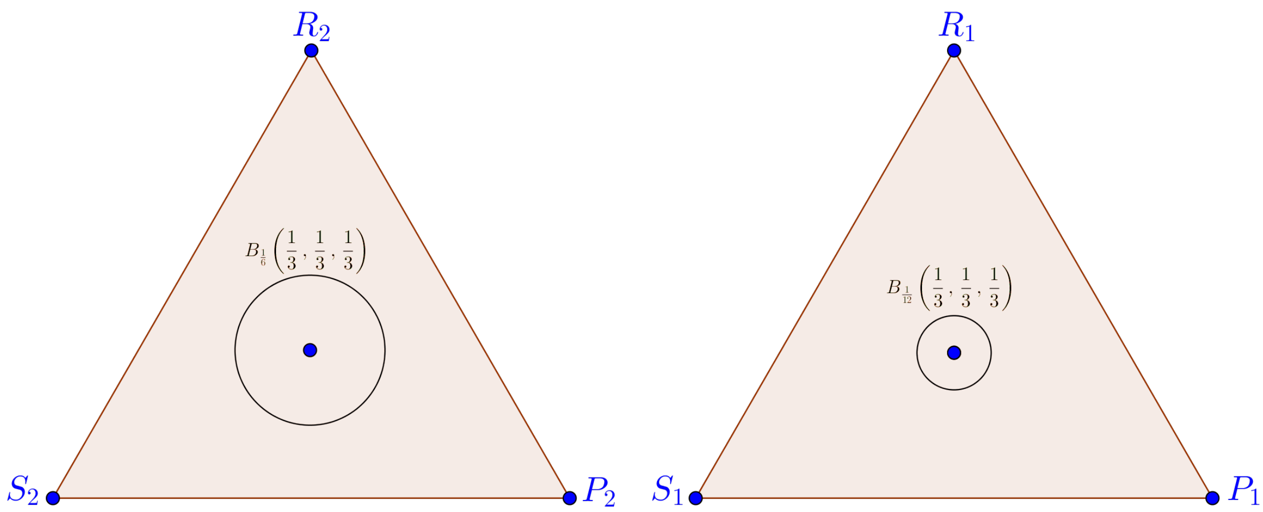


Figure 4.10: The unique equilibrium for $\epsilon_1 = \frac{1}{6}$ and $\epsilon_2 = \frac{1}{12}$ and arbitrary α_1 and α_2.

Our framework is capable of modelling other equilibria in Rock-Paper-Scissors, even singleton ones. Lets assume that player 2 is a subjective expected utility

[22] This radius must not be bigger than $\frac{1}{3}$ as $[\mathcal{C}_i]$ would be empty. The case $\epsilon_i = \frac{1}{3}$ corresponds to the case where $[\mathcal{C}_i]$ consists of a single element with an empty support, which we want to rule out.

maximizer, i.e. perceived ambiguity is represented by the set of singletons. Player 1 perceives ambiguity about whether player 2 will play P_2 or S_2. She does not perceived any ambiguity about the strategy R_2. This belief can be represented by

$$[\mathcal{C}_1] = \{\mathcal{C} \subseteq \Delta(S_2) | \mathcal{C} = Conv(P, Q), P(R_2) = Q(R_2), P(P_2) - Q(P_2) = \delta \in (0, 1)\}.$$

The parameter δ measures the ambiguity that player 1 perceives about whether P_2 or S_2 is played. In the following we consider two cases: player 1 is a complete pessimist, i.e. has Maxmin Expected Utility preferences and player 1 is a complete optimist, i.e. has Maxmax Expected Utility preferences.

The equilibria of the game depend on the parameter δ. When this parameter is sufficiently small, the equilibrium will be close to the Nash Equilibrium. However when δ is large we get equilibria that are very different from the Nash Equilibrium, we can even get singleton equilibria. The reason and intuition why a singleton equilibrium is possible here is the following: assume that player 1 perceives a lot of ambiguity, i.e. δ is close to 1. When evaluating strategy R_1, the player assigns a lot of weight on the worst case scenario P_2 due to ambiguity and pessimism. When evaluating strategy P_1, the player assigns a lot of weight on the worst case scenario S_2. When evaluating S_1 however, the player assigns less weight on the worst case scenario R_2 as most of the weight goes to the scenario S_2. This is because, given a large δ, there is simply no room for the prior set to get close to the R_2 corner of the simplex, see Figure 4.11a. Exogenously fixing perceived ambiguity the way we do in our model allows this kind of phenomena. Thus for δ sufficiently large, player 1 plays S_1 even when he believes that player 2 plays R_2. Player 2, being an SEU decision maker, believes that player 1 plays S_1. She responds optimally and thus plays R_2. Therefore the strategy profile (S_1, R_2) indeed constitutes an Equilibrium under Ambiguity for δ sufficiently large.[23] By fixing perceived ambiguity via the δ-parameter, we prevent that the prior set can get close to the R_2 corner of the simplex. For large δ this strategy thus receives very little weight, even when the support contains just R_2.

The following two tables illustrate equilibria of the game for different values of δ for the pessimism and optimism case.

[23] Whether this makes sense is surely debatable. There is room for critique on this result and the fact that these phenomenon are possible in our model. Note that exactly this phenomenon is also possible in the model of Eichberger and Kelsey (2014) as the belief of player 1 is the core of a JP-capacity, which is the preference class that they consider.

δ	$\mathcal{C}_1^*$	$supp(\mathcal{C}_1^*)$	$\mathcal{C}_2^*$	$supp(\mathcal{C}_2^*)$
$\delta \in [\frac{2}{3}, 1)$	$Conv\,\{(1-\delta, \delta, 0); (1-\delta, 0, \delta)\}$	$\{R_2\}$	$(0, 0, 1)$	$\{S_1\}$
$\delta \in [\frac{1}{2}, \frac{2}{3})$	$Conv\,\{(\frac{1}{3}, \delta, \frac{2}{3}-\delta); (\frac{1}{3}, 0, \frac{2}{3})\}$	$\{R_2, S_2\}$	$(0, \frac{1}{3}, \frac{2}{3})$	$\{P_1, S_1\}$
$\delta \in [0, \frac{1}{2})$	$Conv\,\{(\frac{1}{3}, \frac{1-2\delta}{3}, \frac{1+2\delta}{3}); (\frac{1}{3}, \frac{1+\delta}{3}, \frac{1-\delta}{3})\}$	$\{R_2, P_2, S_2\}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\{R_1, P_1, S_1\}$

Table 4.1: Equilibria under Ambiguity with a pessimistic player 1.

δ	$\mathcal{C}_1^*$	$supp(\mathcal{C}_1^*)$	$\mathcal{C}_2^*$	$supp(\mathcal{C}_2^*)$
$\delta \in [\frac{1}{2}, 1)$	$Conv\,\{(\frac{2-2\delta}{3}, \frac{1+2\delta}{3}, 0); (\frac{2-2\delta}{3}, \frac{1-\delta}{3}, \delta)\}$	$\{R_2, P_2\}$	$(\frac{2}{3}, 0, \frac{1}{3})$	$\{R_1, S_1\}$
$\delta \in [0, \frac{1}{2})$	$Conv\,\{(\frac{1}{3}, \frac{1+2\delta}{3}, \frac{1-2\delta}{3}); (\frac{1}{3}, \frac{1-\delta}{3}, \frac{1+\delta}{3})\}$	$\{R_2, P_2, S_2\}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\{R_1, P_1, S_1\}$

Table 4.2: Equilibria under Ambiguity with an optimistic player 1.

Figure 4.11 and Figure 4.12 illustrate these equilibria for the cases $\delta = \frac{5}{6}, \frac{7}{12}, \frac{1}{4}$ and a pessimistic player 1 as well as $\delta = \frac{3}{4}, \frac{1}{4}$ and an optimistic player 1.

Example 4: Cooperation in the Prisoners Dilemma

Consider the Prisoners Dilemma in Figure 4.13.

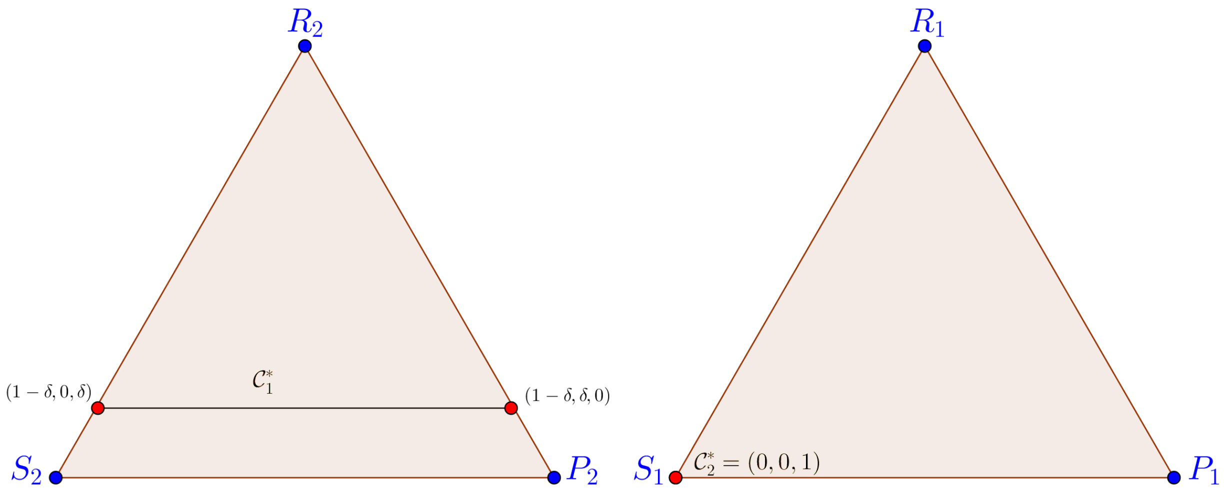
Assume that both players perceive ambiguity $[\mathcal{C}] = \{\mathcal{C} \subseteq \Delta(\{Coop, NoCoop\}) | \mathcal{C} = Conv(P, Q), P(Coop) - P(NoCoop) = \frac{3}{4}\}$ and have ambiguity attitudes $a(NoCoop) = 1, a(Coop) = 0$. Thus the players perceive ambiguity represented by an interval in $\Delta(\{Coop, NoCoop\})$ of length $\frac{3}{4}$. The players are optimistic when evaluating $Coop$ and pessimistic when evaluating $NoCoop$.

Under these conditions there are three Equilibria under Ambiguity, two of them singletons. The first Equilibrium is given by

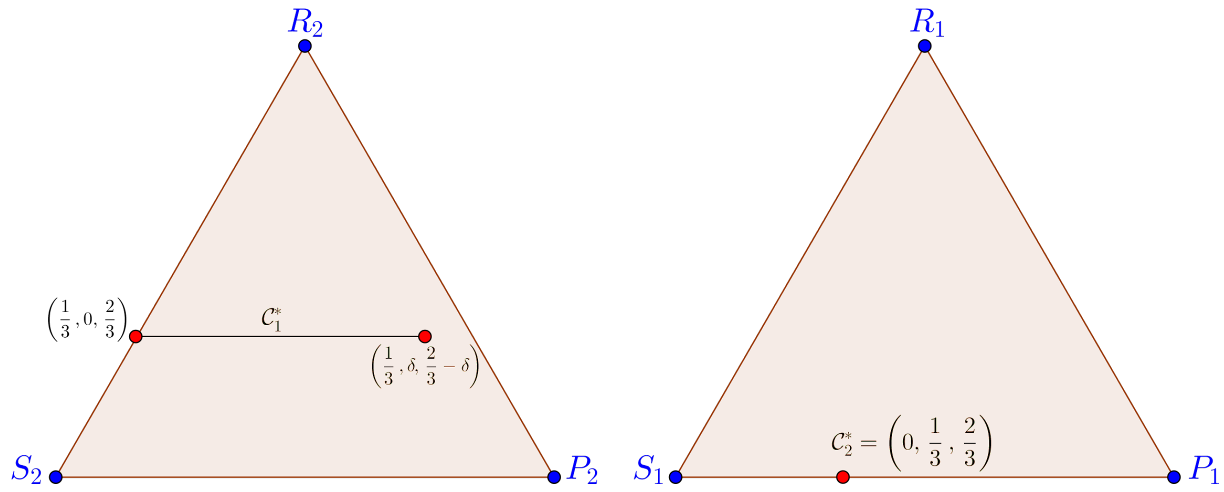
$$(\mathcal{C}^*, a)_{i=1}^2,$$

such that $\mathcal{C}^* = [0, \frac{3}{4}] \subseteq \Delta(\{Coop, NoCoop\})$. It holds that $\mathcal{C}^* \in [\mathcal{C}]$ and $supp(\mathcal{C}^*) = \{Coop\}$. To check that this is indeed an Equilibrium under Ambiguity is suffices to check that

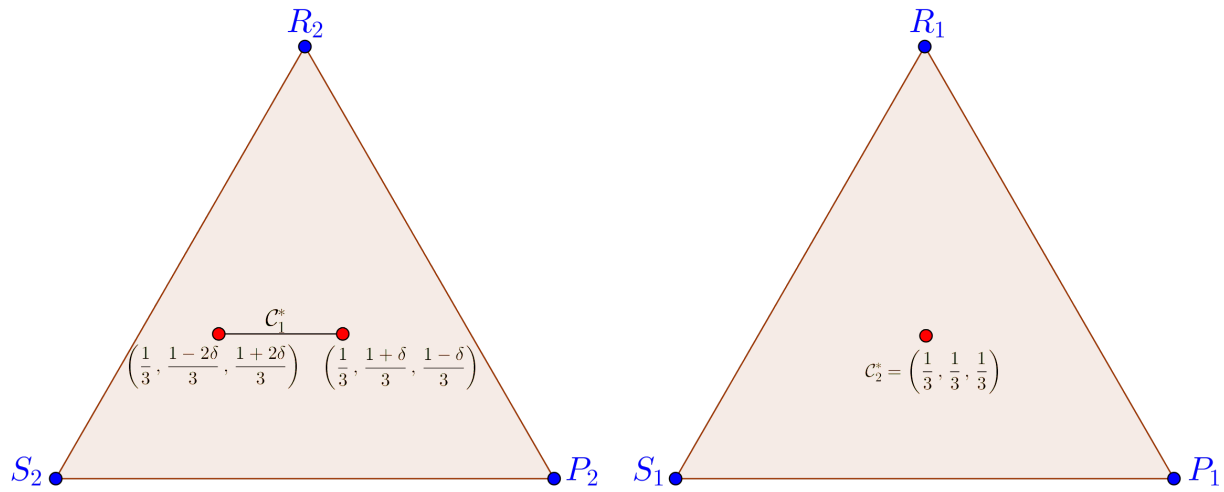
$$V(Coop|\mathcal{C}^*, a) = \max_{P \in \mathcal{C}^*} \int u(Coop, Coop)\; dP = 3,$$
$$V(NoCoop|\mathcal{C}^*, a) = \frac{1}{4} \max_{P \in \mathcal{C}^*} \int u(Coop, Coop)\; dP + \frac{3}{4} \min_{P \in \mathcal{C}^*} \int u(NoCoop, Coop)\; dP = \frac{5}{2}.$$



(a) $\delta = \frac{5}{6}$

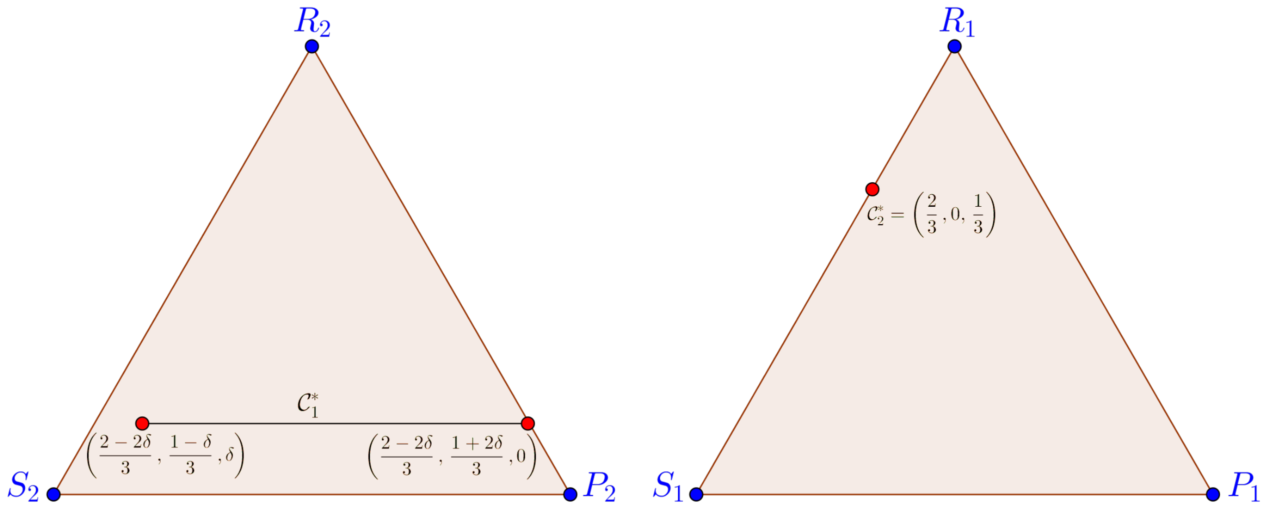


(b) $\delta = \frac{7}{12}$

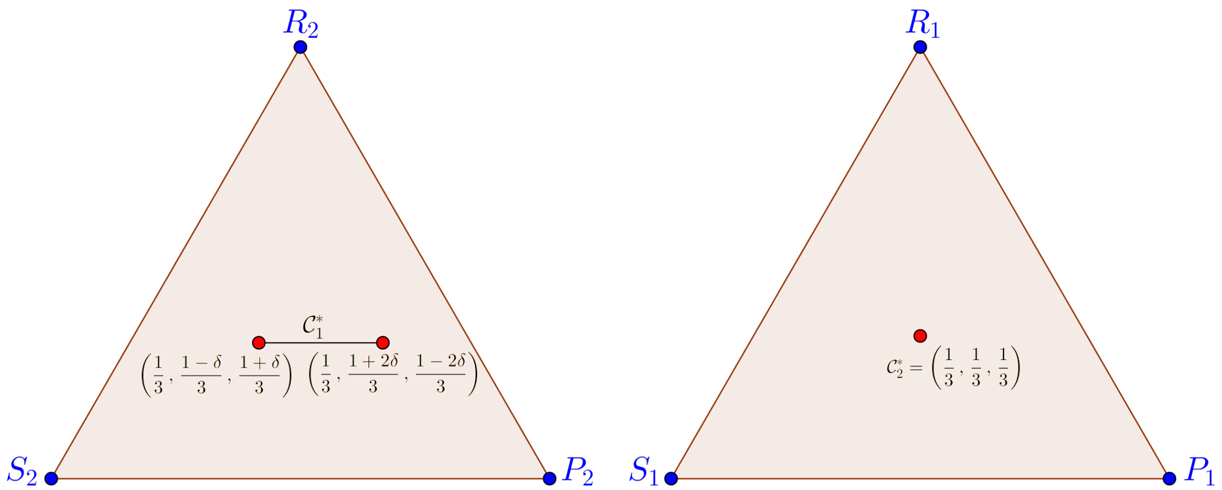


(c) $\delta = \frac{1}{4}$

Figure 4.11: Equilibrium for pessimistic player 1 and $\delta = \frac{5}{6}, \frac{7}{12}, \frac{1}{4}$.



(a) $\delta = \frac{3}{4}$



(b) $\delta = \frac{1}{4}$

Figure 4.12: Equilibrium for optimistic player 1 and $\delta = \frac{3}{4}, \frac{1}{4}$.

		Player 2	
		$Coop$	$NoCoop$
Player 1	$Coop$	$(3,3)$	$(1,4)$
	$NoCoop$	$(4,1)$	$(2,2)$

Figure 4.13: A Prisoners Dilemma.

The second Equilibrium is given by

$$(\mathcal{C}^\dagger, a)_{i=1}^2,$$

such that $\mathcal{C}^\dagger = [\frac{1}{4}, 1] \subseteq \Delta(\{Coop, NoCoop\})$. It holds that $\mathcal{C}^\dagger \in [\mathcal{C}]$ and $supp(\mathcal{C}^\dagger) = \{NoCoop\}$. We have that $V(Coop|\mathcal{C}^\dagger, a) = \frac{3}{2} < 2 = V(NoCoop|\mathcal{C}^\dagger, a)$, thus this is an Equilibrium under Ambiguity.

There is a third equilibrium in which the support of the prior sets are not singletons: $\mathcal{C}^\ddagger = [\frac{1}{8}, \frac{7}{8}] \in [\mathcal{C}]$. Given these beliefs, the players are indifferent between the two strategies.

All equilibrium prior sets are illustrated in Figure 4.14.

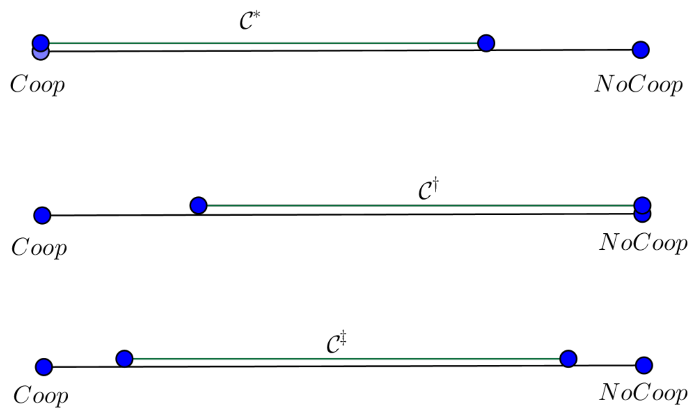


Figure 4.14: The prior sets of the three Equilibria.

In the first equilibrium the players cooperate. The reason our model allows cooperation in the prisoners dilemma is very simple: ambiguity attitude is not restricted to being constant and thus allows non-monotonic preferences.[24] The assumed

[24] See the article on Weak Monotonicity on how to model non-monotonic preferences in the multiple prior model.

preference structure therefore does not rule out that strictly dominated strategies are part of an equilibrium. We achieve this result because we artificially choose ambiguity attitudes such that the players are pessimistic when evaluating $NoCoop$ and optimistic when evaluating $Coop$. Whether this is realistic is a question that is surely debatable. Personally we do not believe that ambiguous beliefs in combination with extreme ambiguity attitude is a suitable approach to explain cooperation in the prisoners dilemma. The example does however illustrate the potential of our approach in modelling a large variety of phenomena in strategic interaction. By adding assumptions we can filter out unrealistic equilibria. By restricting attention to monotonic preferences we can for instance eliminate cooperation in the Prisoners Dilemma.

4.7 Conclusion

This article introduces a new definition of perceived ambiguity in the multiple prior model which implies location-invariance of perceived ambiguity. The definition generalizes and unifies the existing definitions of perceived ambiguity that we are aware of. We illustrate what our approach implies for preferences and suggest comparative notions for perceived ambiguity and ambiguity attitude. We also provide an axiomatization of α-MEU preferences. We show that our approach can be used to model players that perceive strategic ambiguity in normal-form games. Our equilibrium existence proof generalizes many results from the literature.

The next step is to analyse whether and how our model can be used for economic applications. What are for instance the implications for Bertrand/Cournot Oligopoly interactions. Furthermore the models' flexibility should be tested against empirical results. An interesting extension would be to study how the model can be applied to dynamic interaction.

4.8 Appendix A: Proofs of Theorems in the Main Text

Proofs of Section 4.4

Proof of Theorem 4.1. 1. $\implies$ 2.: First note that $u(x_u(h,t)) = I_1(h) - I_2(h)$ for all $h \in \mathcal{F}$.

Consider the acts f and g and assume that $f \succsim_1 g$. This is equivalent to $I_1(f) \geq I_1(g)$. Since $\mathcal{C}_1 + t = \mathcal{C}_2$, $u(x_u(f,t)) = I_1(f) - I_2(f)$ and $u(x_u(g,t)) = I_1(g) - I_2(g)$ this is equivalent to

$$I_2(f) + u(x_u(f,t)) \geq I_2(g) + u(x_u(g,t)).$$

The constant linearity of the functionals implies that this is equivalent to

$$I_2(f + x_u(f,t)) \geq I_2(g + x_u(g,t)),$$

which is equivalent to $f + x_u(f,t) \succsim_2 g + x_u(g,t)$.

2. $\implies$ 1.: Assume that $\mathcal{C}_1 + t \neq \mathcal{C}_2$. Then there exists an $f \in \mathcal{C}$ such that $I_1(f) - I_2(f) \neq u(x_u(f,t))$. Consider $x_f^{\succsim_1} \in X$, i.e. $f \sim_1 x_f^{\succsim_1}$ which implies $u(x_f^{\succsim_1}) = I_1(f)$. It holds that

$$I_2(f) + u(x_u(f,t)) \neq u(xx_f^{\succsim_1}) + u(x_u(x_f^{\succsim_1}, t)).$$

Recall that $u(x_u(x_f^{\succsim_1}, t)) = 0$ which together with the constant linearity of I_2 implies

$$I_2(f + x_u(f,t)) \neq I_2(x_f^{\succsim_1}).$$

This implies $f + x_u(f,t) \nsim_2 x_f^{\succsim_1}$. Therefore 2. fails. □

Proof of Corollary 4.1. 1. $\implies$ 2.: Assume that $\mathcal{C}_1 + t = \mathcal{C}_2$. Consider the acts f and g and assume that $f \succsim_1^* g$. This is equivalent to

$$\int u(f)dP \geq \int u(g)dP \qquad \forall P \in \mathcal{C}_1.$$

Since $\mathcal{C}_2 = \{Q \in \Delta(S) | Q = P + t, P \in \mathcal{C}_1\}$ and due to the way $x_u(\cdot, t)$ is defined we have $u(x_u(h,t)) = \int u(h) \ d(P+t) - \int u(h) \quad dP$ for all $h \in \mathcal{F}$ and for all

$P, P - t \in \Delta(S)$ we have that for any $P, P + t \in \Delta(S)$

$$\int u(f) + u(x_u(f,t))\ dP \geq \int u(g) + u(x_u(g,t))\ dP, \qquad \forall P \in \mathcal{C}_2.$$

The constant linearity of u implies than this is equivalent to $f + x_u(f,t) \succsim_2^* g + x_u(g,t)$.

2. $\implies$ 1.: Assume that $\mathcal{C}_1 + t \neq \mathcal{C}_2$. One of the following most hold: Either these exists $P \in (\mathcal{C}_1 + t)\backslash\mathcal{C}_2$ or there exists $P \in \mathcal{C}_2\backslash(\mathcal{C}_1 + t)$ or both. Assume that the first holds (considering the second is nearly identical and omitted). The separating hyperplane theorem for the separation of two disjoint convex sets implies that there exists an $f \in \mathcal{F}$ such that $\int u(f)\ dP > 0 > \max_{Q \in \mathcal{C}_2} \int u(f)\ dQ$. Intuition: The zero-indifference curve $\{P \in \Delta(S) | \int u(f) dP = 0\}$ of f is this hyperplane. Consider $x \in X$ such that $u(x) = 0$. Then $x \succsim_2^* f$. However $\int u(f)\ dP > u(x)$ and since $P \in \mathcal{C}_1 + t$ we have that $x \not\succsim_1^* f + x_u(f,t)$.

□

Proofs of Section 4.5

Proof of Lemma 4.1. 2. $\implies$ 1.: Assume that 1. fails, i.e. that there exist $f, g \in \mathcal{F}$ such that $f \succsim_1^* g$ and $f + x_u(f,t) \not\succsim_2^* g + x_u(g,t)$. The latter implies that there exists a $P \in \mathcal{C}_2$ such that

$$\int u(f) + u(x_u(f,t)) dP < \int u(g) + u(x_u(g,t)) dP.$$

If $\mathcal{C}_1 + t \supseteq \mathcal{C}_2$ holds this implies $P \in \mathcal{C}_1 + t$ which implies that $f \not\succsim_1^* g$. Thus 2. fails.

1. $\implies$ 2.: Assume that $\succsim_1$ reflects more ambiguity than $\succsim_2$, i.e. there exists a $t \in T$ such that $f \succsim_1^* g \implies f + x_u(f,t) \succsim_2^* g + x_u(g,t)$. Ghirardato et al. (2004) show in Corollary B.3 that u_1 and u_2 are positive affinely related if and only if $u_1(l_1) \geq u_1(l_2) \implies u_2(l_1) \geq u_2(l_2)$ for all $l_1, l_2 \in L$. Recall that $u(x_u(l,t)) = 0$ for all $l \in L$. It holds that

$$\begin{aligned} u_1(l_1) \geq u_1(l_2) &\implies l_1 \succsim_1 l_2 \implies l_1 \succsim_1^* l_2 \implies l_1 + x_u(l_1,t) \succsim_2^* l_2 + x_u(l_2,t) \\ &\implies u_2(l_1 + x_u(l_1,t)) \geq u_2(l_2 + x_u(l_2,t)) \implies u_2(l_1) \geq u_2(l_2). \end{aligned}$$

Thus we have shown that the utility functions are positive affinely related. Assume that $\mathcal{C}_1 + t \not\supseteq \mathcal{C}_2$. This implies that there exists a $P \in \mathcal{C}_2 \backslash (\mathcal{C}_1 + t)$. Therefore with a similar construction as in Corollary 4.1 we get a contradiction to 1.

□

Proof Lemma 4.2. 1. $\Longrightarrow$ 2.: Assume that $a_1(f) < a_2(f)$ for some $f \in \mathcal{F}$. Theorem 4.1 implies that $I_1(f) < I_2(f) + u(x_u(f,t))$. Consider $l \in L$ such that $f \sim_1 l$. The above inequality implies that $f + x_u(f,t) \succ_2 l$. Thus 1. fails.

2. $\Longrightarrow$ 1.: Assume that for some $f \in \mathcal{F}$ and $l \in L$ we have $f \succsim_1 l$ and $f + x_u(f,t) \not\precsim_2 l$. Thus $I_1(f) \geq u(l)$ and $I_2(f + x_u(f,t)) < u(l)$. This implies that $I_1(f) > I_2(f + x_u(f,t)) = I_2(f) + u(x_u(f,t))$. Through the way $x_u(f,t)$ is defined we conclude that $a_2(f) > a_1(f)$. Thus 2. fails.

□

Proof of Lemma 4.3. Assume that 1. holds or equivalently for all $f \in \mathcal{F}\backslash[x], x \in X$ and $\alpha \in [0,1]$

$$I_1(f) \geq \alpha u(\underline{x}_1(f)) + (1-\alpha)u(\overline{x}_1(f)) \implies I_2(f) \geq \alpha u(\underline{x}_2(f)) + (1-\alpha)u(\overline{x}_2(f)).$$

Recall that $u(\underline{x}(f)) = \min_{P \in \mathcal{C}} \int u(f)\ dP$ and $u(\overline{x}(f)) = \max_{P \in \mathcal{C}} \int u(f)\ dP$. This implies that the above is equivalent to

$$I_1(f) \geq I_{\mathcal{C}_1,\alpha,u} \implies I_2(f) \geq I_{\mathcal{C}_2,\alpha,u}.$$

This is equivalent to

$$a_1([f]) \leq \alpha \implies a_2([f]) \leq \alpha.$$

Since this holds for all $f \in \mathcal{F}$ this is equivalent to $a_1([f]) \geq a_2([f])$.

□

Proof of Theorem 4.2. Assume that 1. holds and consider some $f \in \mathcal{F}\backslash[x]$. This is equivalent to

$$I(f) = \alpha u(\underline{x}(f)) + (1-\alpha)u(\overline{x}(f)).$$

This is equivalent to

$$I(f) = \alpha \min_{P \in \mathcal{C}} \int u(f)\ dP + (1-\alpha) \max_{P \in \mathcal{C}} \int u(f)\ dP.$$

This is equivalent to $a([f]) = \alpha$. □

Proofs of Section 4.6

Before we can prove Theorem 4.3 we need a preliminary result, which is interesting in its own respect.

An Isomorphism from Σ_{-i} to $[\mathcal{C}_i]$

Consider a prior set $\mathcal{C}_i \subseteq \Delta(S_{-i})$ and assume that $supp(\mathcal{C}_i) \neq \emptyset$. We first show that there exists an isomorphism from Σ_{-i} to $[\mathcal{C}_i]$, where Σ_{-i} is the set of mixed strategy combinations of player i's opponents.

The first step is to show that for every pure-strategy combination $s^*_{-i} \in S_{-i}$ there exists a unique prior set $\mathcal{C}_i^{s^*_{-i}} \in [\mathcal{C}_i]$ such that $supp(\mathcal{C}_i) = \{s^*_{-i}\}$. The intuition is that $\mathcal{C}_i^{s^*_{-i}}$ is the prior set in $[\mathcal{C}_i]$ "in the s^*_{-i}" corner of the simplex. See Figure 4.15 for illustration.

For $s_{-i} \in S_{-i}$ consider the set $\mathcal{C}^{s_{-i}}_{[\mathcal{C}_i]} = \{\mathcal{C} \in [\mathcal{C}_i] | s_{-i} \notin supp(\mathcal{C})\}$, i.e. the prior sets in $[\mathcal{C}_i]$ that do not have s_{-i} in their support. Consider the set $\bigcap_{s_{-i} \in S_{-i} \setminus \{s^*_{-i}\}} \mathcal{C}^{s_{-i}}_{[\mathcal{C}_i]}$. For an element of this set it holds that the support equals s^*_{-i}. We need to show that this set contains exactly one element.

<u>Uniqueness</u>: Assume that $\mathcal{C}_{i1}$ and $\mathcal{C}_{i2}$ are elements of $\bigcap_{s_{-i} \in S_{-i} \setminus \{s^*_{-i}\}} \mathcal{C}^{s_{-i}}_{[\mathcal{C}_i]}$. Then $supp(\mathcal{C}_{i1}) = supp(\mathcal{C}_{i1}) = \{s^*_{-i}\}$. Since $\mathcal{C}_{i1}$ and $\mathcal{C}_{i2}$ are elements of $[\mathcal{C}_i]$ there exists a $t \in T$ such that $\mathcal{C}_{i2} + t = \mathcal{C}_{i1}$.

Assume that $t(s_{-i}) < 0$ for $s_{-i} \in S_{-i} \setminus \{s^*_{-i}\}$. Consider some $P \in \mathcal{C}_{i2}$ such that $P(s_{-i}) = 0$. This P exists since $\mathcal{C}_{i2} \in [\mathcal{C}^{s_{-i}}_{[\mathcal{C}_i]}]$ and the way the support of a prior set of defined. Thus it holds that $P + t \in \mathcal{C}_{i1}$ and $P + t \notin \Delta(S_{-i})$ which is a contradiction to $\mathcal{C}_{i1} \subseteq \Delta(S_{-i})$. Since $t \in T$ is holds that $\sum_{s_{-i} \in S_{-i}} t(s_{-i}) = 0$ and we can conclude that $t \equiv 0$ and therefore $\mathcal{C}_{i1} = \mathcal{C}_{i2}$.

<u>Existence</u>: Consider an arbitrary $\mathcal{C} \in [\mathcal{C}_i]$. Define

$$t(s_{-i}) = \begin{cases} -\min_{P \in \mathcal{C}} P(s_{-i}) & , s_{-i} \neq s^*_{-i} \\ \sum_{s_{-i} \in S_{-i} \setminus \{s^*_i\}} \min_{P \in \mathcal{C}} P(s'_{-i}) & , s_{-i} = s^*_{-i} \end{cases}$$

The minima are well-defined as prior sets are compact. Thus t is well-defined and an element of T. Consider the prior set $\mathcal{C}' = \mathcal{C} + t \in [\mathcal{C}_i]$. It holds that $\min_{P \in \mathcal{C}'} P(s_{-i}) = \min_{P \in \mathcal{C}} P(s_{-i}) + t(s_{-i}) = 0$ for all $s_{-i} \in S_{-i} \backslash \{s^*_{-i}\}$, implying that $supp(\mathcal{C}') \subseteq \{s^*_{-i}\}$. As $supp(\mathcal{C}') \neq \emptyset$ we can conclude that $supp(\mathcal{C}') = \{s^*_{-i}\}$. Therefore $\mathcal{C}' \in \mathcal{C}^{s^*_{-i}}_{[\mathcal{C}_i]}$.

This shows that $\bigcap_{s_{-i} \in S_{-i} \backslash \{s^*_{-i}\}} \mathcal{C}^{s_{-i}}_{[\mathcal{C}_i]}$ contains exactly one element. We call this prior set $\mathcal{C}^{s^*_{-i}}_{[\mathcal{C}_i]}$.

For $[\mathcal{C}_i]$ define for any $s_{-i} \in S_{-i}$

$$t_{[\mathcal{C}_i]} = \min_{P \in \mathcal{C}^{s_{-i}}_{[\mathcal{C}_i]}} P(s_{-i}) \in (0, 1].$$

The number $t_{[\mathcal{C}_i]}$ does not depend on s_{-i} and can be interpreted as a measure for the *space* that the prior sets in $[\mathcal{C}_i]$ have to move around in $\Delta(S_{-i})$.

Next note that for $s^*_{-i} \in S_{-i}$,

$$[\mathcal{C}_i] = \{\mathcal{C} \subseteq \Delta(S_{-i}) | \mathcal{C} = \mathcal{C}^{s^*_{-i}}_{[\mathcal{C}_i]} + t, t(s_{-i}) \geq 0 \ \forall \ s_{-i} \neq s^*_{-i}, \sum_{s_{-i} \in S_{-i} \backslash \{s^*_{-i}\}} \leq t_{[\mathcal{C}_i]}, t \in T\}.$$

For $\sigma_{-i} \in \Sigma_{-i}$ we define $\mathcal{C}^{\sigma_{-i}}_{[\mathcal{C}_i]} = \mathcal{C}^{s^*_{-i}}_{[\mathcal{C}_i]} + t^{s^*_{-i}, \sigma_{-i}}_{[\mathcal{C}_i]}$ with $t^{s^*_{-i}, \sigma_{-i}}_{[\mathcal{C}_i]} = (\sigma_{-i} - \mathbb{1}_{s^*_{-i}}) t_{[\mathcal{C}_i]}$.

The prior set $\mathcal{C}^{\sigma_{-i}}_{[\mathcal{C}_i]}$ is the "σ_{-i} mix" of the sets $\mathcal{C}^{s_{-i}}_{[\mathcal{C}_i]}$, $s_{-i} \in S_{-i}$. Thus we have found an isomorphism from Σ_{-i} to $[\mathcal{C}_i]$, where

$$\sigma_{-i} \mapsto \mathcal{C}^{\sigma_{-i}}_{[\mathcal{C}_i]}.$$

Figure 4.15 illustrates the work done for the case $S_{-i} = \{s_{-i1}, s_{-i2}, s_{-i3}\}$. The isomorphism maps the strategy s_{-i1} to the set $\mathcal{C}^{s_{-i1}}_{[\mathcal{C}_i]}$. It maps the mixed strategy $\frac{1}{2}s_{-i1} + \frac{1}{2}s_{-i3}$ to the set $\mathcal{C}^{\frac{1}{2}s_{-i1} + \frac{1}{2}s_{-i3}}_{[\mathcal{C}_i]}$.

Note that due to this construction we have that

- $supp(\sigma_{-i}) = supp(\mathcal{C}^{\sigma_{-i}}_{[\mathcal{C}_i]})$ for all $\sigma_{-i} \in \Sigma_{-i}$.
- For every prior set $\mathcal{C}_i$ we have

$$[\mathcal{C}_i] = \{\mathcal{C}^{\sigma_{-i}}_{[\mathcal{C}_i]} | \sigma_{-i} \in \Sigma_{-i}\}.$$

- Let $a_i : S_i \to [0, 1]$ be an ambiguity attitude function. Then the evaluation

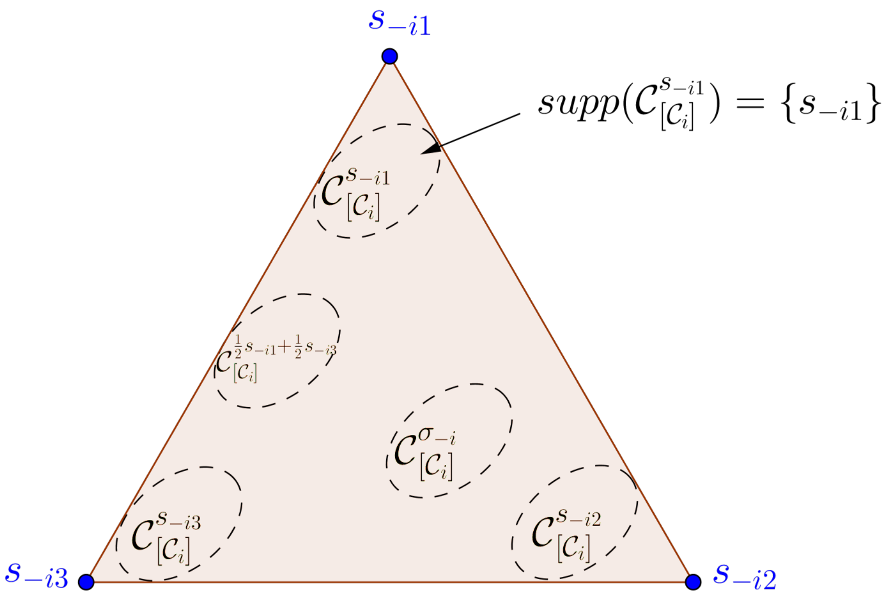


Figure 4.15: The sets $\mathcal{C}_{[\mathcal{C}_i]}^{\sigma_{-i}}$.

functional V is linear in σ_{-i}, i.e. for $s_i \in S_i$

$$V(s_i|\mathcal{C}_{[\mathcal{C}_i]}^{\sigma_{-i}}, a_i) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) V(s_i|\mathcal{C}_{[\mathcal{C}_i]}^{s_{-i}}, a_i).$$

The Distorted Game Γ^{dist}

For a normal-form game Γ and any exogenous perceived ambiguities and ambiguity attitudes we can define what we call the distorted game of Γ.

Definition 4.8. *Let $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ be a normal-form game. For exogenous perceived ambiguities $([\mathcal{C}_i])_{i=1}^N$ and ambiguity attitudes $(a_i)_{i=1}^N$ define the distorted game $\Gamma^{dist} = \langle N; S_i, u_i^{dist} : 1 \leq i \leq N \rangle$ by*

$$u_i^{dist}(s_i, s_{-i}) = a_i(s_i) \min_{P \in \mathcal{C}_{[\mathcal{C}_i]}^{s_{-i}}} \int u_i(s_i) \; dP + (1 - a_i(s_i)) \max_{P \in \mathcal{C}_{[\mathcal{C}_i]}^{s_{-i}}} \int u_i(s_i) \; dP.$$

The game Γ^{dist} is well-defined. It is a normal-form game with exactly the same players and strategy sets as Γ. The payoff functions will differ unless the players do not perceive any ambiguity, i.e. the $[\mathcal{C}_i]$'s are not the singleton sets. The game Γ^{dist} is therefore guaranteed to have a mixed Nash Equilibrium. The following Lemma is the crucial step for proving Theorem 4.3. It shows that the Nash Equilibria of Γ^{dist} induces an Equilibria under Ambiguity of Γ.

Lemma 4.4. *Let* $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ *be a normal-form game and* $([\mathcal{C}_i])_{i=1}^N$ *and* $(a_i)_{i=1}^N$ *exogenous perceived ambiguities and ambiguity attitudes. Then for every strategy combination* $(\sigma_i)_{i=1}^N$ *it holds that*

$$(\sigma_i)_{i=1}^N \text{ is a Nash Equilibrium of } \Gamma^{dist}$$
$$\Downarrow$$
$$(\mathcal{C}_{[\mathcal{C}_i]}^{\sigma_{-i}}, a_i)_{i=1}^N \text{ is an Equilibrium under Ambiguity of } \Gamma.$$

Proof. Assume that $\sigma = (\sigma_i)_{i=1}^N$ is a Nash Equilibrium of Γ^{dist}. Then

$$\emptyset \neq supp(\sigma_{-i}) \subseteq \times_{j \neq i} \arg\max_{s_j \in S_j} u_j^{dist}(s_j, \sigma_{-j}).$$

Since $supp(\sigma_{-i}) = supp(\mathcal{C}_{[\mathcal{C}_i]}^{\sigma_{-i}})$ and by the definition of u_j^{dist} this is equivalent to

$$\emptyset \neq supp(\mathcal{C}_{[\mathcal{C}_i]}^{\sigma_{-i}}) \subseteq \times_{j \neq i} \arg\max_{s_j \in S_j} V(s_j | \mathcal{C}_{[\mathcal{C}_j]}^{\sigma_{-i}}, a_j).$$

This implies that $(\mathcal{C}_i^{\sigma_{-i}}, a_i)_{i=1}^N$ is an Equilibrium under Ambiguity.

□

Note that the reverse direction does not hold in Lemma 4.4 since in a Nash Equilibrium beliefs of two players about a third player must be identical. Thus Γ may have an Equilibrium under Ambiguity that does not induce a Nash Equilibrium in Γ^{dist}.

Proof of Theorem 4.3. The proof follows directly from Lemma 4.4. The game Γ^{dist} always has a Nash Equilibrium which induces an Equilibrium under Ambiguity in Γ. □

The following two games are the distorted games from the Rock-Paper-Scissors example from the main text, where player 1 perceives ambiguity $[\mathcal{C}_1] = \{\mathcal{C} \subseteq \Delta(S_2) | \mathcal{C} = Conv(P, Q), P(R_2) = Q(R_2), P(P_2) - Q(P_2) = \delta \in (0, 1)\}$ and is a complete pessimist/optimist. Player 2 does not perceive any ambiguity, i.e. she is an SEU DM.

Player 2

		R_2	P_2	S_2
Player 1	R_1	$(-\delta, 0)$	$(-1, 1)$	$(1-2\delta, -1)$
	P_1	$(1-2\delta, -1)$	$(-\delta, 0)$	$(-1, 1)$
	S_1	$(-1+\delta, 1)$	$(1-\delta, -1)$	$(0, 0)$

Figure 4.16: The pessimism distorted Rock-Paper-Scissors game

Player 2

		R_2	P_2	S_2
Player 1	R_1	$(\delta, 0)$	$(-1+2\delta, 1)$	$(1, -1)$
	P_1	$(1-\delta, -1)$	$(0, 0)$	$(-1+\delta, 1)$
	S_1	$(-1+2\delta, 1)$	$(1, -1)$	$(\delta, 0)$

Figure 4.17: The optimism distorted Rock-Paper-Scissors game

4.9 Appendix B: Implications of our Definition of Perceived Ambiguity on Important Preference Classes

We show in this appendix that our definition of perceived ambiguity generalizes several of the existing definitions of perceived ambiguity for specific preference subclasses. We thus unify these definitions. The existing definitions of perceived ambiguity in the literature are used mainly to study the influence that ambiguous beliefs have on behaviour in games. This is needed if one wants to exogenously fix perceived ambiguity or ambiguity attitude in order to perform equilibrium existence results or perform comparative static exercises. Throughout this section the term $\delta_{\mathcal{C}}$ denotes the function introduced in Definition 4.1.[25]

4.9.1 Marinacci (2000): Convex Capacities

Recall that a capacity is a normalized and monotone set-function on $\mathcal{P}(S)$. The core of a capacity ν is defined by

$$Core(\nu) = \{P \in \Delta(S) | P(E) \geq \nu(E) \text{ for all } E \in \mathcal{P}(S)\}.$$

[25] For simplicity we assume in this Appendix that payouts are in utilities.

A capacity ν is convex if for all $E_1, E_2 \in \mathcal{P}(S)$

$$\nu(E_1) + \nu(E_2) \leq \nu(E_1 \cup E_2) + \nu(E_1) \cap E_2.$$

Convex capacities can be represented by the core of the capacity (which is also the GMM set) and ambiguity attitude function $a \equiv 1$, i.e. pure pessimism. Marinacci (2000) introduces a measure of ambiguity levels for convex capacities. For a convex capacity ν and an event $A \subseteq S$ the number

$$1 - \nu(A) - \nu(A^c) \tag{4.6}$$

measures the perceived ambiguity of A (and A^c). Marinacci (2000) introduces the function

$$\begin{aligned} \Psi_\nu :& \mathcal{P}(S) \to [0,1], \\ \Psi_\nu(A) =& 1 - \nu(A) - \nu(A^c). \end{aligned}$$

He then defines $\Lambda(S, \Psi_\nu)$ to be the set of all convex capacities over S which have the same Ψ-function, i.e. $\Lambda(S, \Psi_\nu)$ contains all the convex capacities for which the perceived ambiguity via (4.1) is the same for all events. The following Lemma shows that two convex capacities that perceive the same ambiguity according to our definition have the same Ψ. Or put differently: if the cores of two convex capacities are translations of each other then this implies that the capacities have the same Ψ-function.

Lemma 4.5. *Let $\succsim$ and $\succsim'$ be representable by convex capacities ν and ν', respectively. Then*

$$\delta_{Core(\nu)} = \delta_{Core(\nu')} \implies \Psi_\nu = \Psi_{\nu'}.$$

The reverse does not hold. This shows that our definition of perceived ambiguity restricted to convex capacities is finer than the one from Marinacci (2000), illustrated in Example 4.1. The reason for this is that the Ψ-function only fixes perceived ambiguity for binary acts (bets). It stays silent however about more complex acts which our definition *does* address.

Example 4.1. Consider the following two convex capacities over the state space

$S = \{A, B, C\}$.

$$\nu(E) = \begin{cases} 0, & \text{for } E = \emptyset \\ \frac{1}{6}, & \text{for } |E| = 1 \\ \frac{1}{3}, & \text{for } |E| = 2 \\ 1, & \text{for } E = S \end{cases}$$

$$\nu'(E) = \begin{cases} 0, & \text{for } E = \emptyset \\ \frac{1}{6}, & \text{for } E = A, B \\ \frac{1}{8}, & \text{for } E = C \\ \frac{1}{3}, & \text{for } E = A \cup C, B \cup C \\ \frac{3}{8}, & \text{for } E = A \cup B \\ 1, & \text{for } E = S \end{cases}$$

It is easily checked that $\Psi_\nu = \Psi_{\nu'}$. However the cores depicted in Figure 4.18 are not translations of each other. It therefore follows from Observation 4.2 that $\delta_{Core(\nu)} \neq \delta_{Core(\nu')}$.

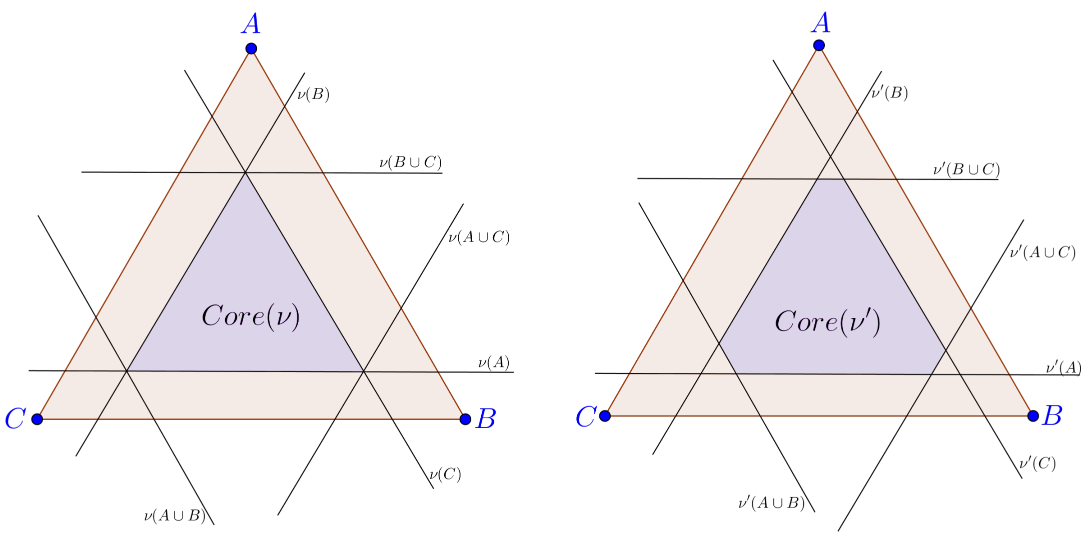


Figure 4.18: The cores of ν and ν'.

4.9.2 Eichberger and Kelsey (2014): JP-capacities

Eichberger and Kelsey (2014) assume that players beliefs can be represented by

JP-capacities, introduced by Jaffray and Philippe (1997). These capacities take the form $\nu = \alpha\mu + (1-\alpha)\overline{\mu}$, where μ is a convex capacity, $\overline{\mu}$ is its dual[26] and $\alpha \in [0,1]$. The core of μ represents perceived ambiguity and α represents ambiguity attitude. It is therefore a special case of α-MEU preferences.

Eichberger and Kelsey (2014) define the maximal and minimal degree of perceived ambiguity for a convex capacity μ by

$$\lambda(\mu) = \max_{\emptyset \neq A \neq S} \{1 - \mu(A) - \mu(A^c)\}$$
$$\gamma(\mu) = \min_{\emptyset \neq A \neq S} \{1 - \mu(A) - \mu(A^c)\}.$$

Again our definition is a generalization.

Lemma 4.6. *Let $\succsim$ and $\succsim'$ be representable by the JP-capacities $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ and $\nu' = \alpha'\mu' + (1-\alpha')\bar{\mu}'$, respectively. Then*

$$\delta_{Core(\mu)} = \delta_{Core(\mu')} \implies \lambda(\mu) = \lambda(\mu') \text{ and } \gamma(\mu) = \gamma(\mu').$$

4.9.3 Chateauneuf et al. (2007): Neo-additive capacities

Chateauneuf et al. (2007) introduce and axiomatize neo-additive capacities. These capacities allow both optimistic and pessimistic attitudes towards ambiguity. A neo-additive capacity ν is characterized by an additive distribution π over S and two parameters $\delta, \alpha \in [0,1]$ such that $\nu = (1-\delta)\pi + \delta\alpha$.

The authors interpret δ as the degree of perceived ambiguity[27] and α as the degree of optimism.[28] They show that neo-additive capacities are a subclass of JP-capacities. Their definition of perceived ambiguity is equivalent to ours restricted to neo-additive capacities.

Lemma 4.7. *Let $\succsim$ and $\succsim'$ be representable by the neo-additive capacities $\nu = (1-\delta)\pi + \delta\alpha$ and $\nu' = (1-\delta')\pi' + \delta'\alpha'$, respectively. Then*

$$\delta_{Core((1-\delta)\pi)} = \delta_{Core((1-\delta)\pi)} \iff \delta = \delta'.$$

[26] Recall that the dual capacity is defined by $\bar{\mu}(E) = 1 - \mu(E^c)$ for all $E \subseteq S$.
[27] This notation is where our δ-notation was inspired from.
[28] A special case are the so called simple capacities which are neo-additive with $\alpha = 1$, i.e. pure pessimism. Eichberger and Kelsey (2000) represent beliefs of players by simple capacities and also interpret δ as the degree of perceived ambiguity.

4.9.4 Dominiak and Eichberger (2016): Belief Functions

A belief function Φ^γ takes the form

$$\Phi^\gamma(E) = \sum_{A \subseteq E} \gamma(A) \qquad \forall E \subseteq S,$$

where γ is a probability distribution over $\mathcal{P}(S)$, also called the Mbius transformation of the capacity Φ^γ. Belief functions are a special case of convex capacities.

For a belief function Φ^γ, Dominiak and Eichberger (2016) suggest as a measure for perceived ambiguity the function $\delta = \gamma_{|\{E||E|\geq 2\}}$. It is the part of the Mbius transformation restricted to events with at least two elements. We show that two belief functions have the same δ if and only if they perceive the same ambiguity according to our definition.

Lemma 4.8. *Let $\succsim$ and $\succsim'$ be representable by the belief functions Φ^γ and $\Phi^{\gamma'}$. Then*

$$\delta_{Core(\Phi^\gamma)} = \delta_{Core(\Phi^{\gamma'})} \iff \delta_\gamma = \delta_{\gamma'}.$$

Proofs of Appendix B

Proof of Lemma 4.5. For a binary act $f = a_E b$ we have that

$$\delta(Core(\nu), f) = 1 - \nu(E) - \nu(E^c),$$

which is all we need for the proof. □

Proof of Lemma 4.6. Follows directly from Lemma 4.5 □

Proof of Lemma 4.7. Assume that the cores of the capacities $(1-\delta)\pi$ and $(1-\delta')\pi$ are translations. Due to Lemma 4.5 this is equivalent to $1-(1-\delta)\pi(E^c)-(1-\delta)\pi(E^c) = 1-(1-\delta')\pi(E^c)-(1-\delta')\pi(E^c)$. This is equivalent to $\delta = \delta'$. □

Proof of Lemma 4.8. $\Rightarrow$ follows from Lemma 4.5 since belief functions are convex capacities. $\Leftarrow$ is obvious. □

5 Optimism and Pessimism in Games with α-MEU Preferences

Abstract

Using the theory presented in the previous article, we extend the theory of Eichberger and Kelsey (2014) to α-MEU preferences. The authors study a class of games which are characterized by a lattice-structure over the pure-strategy space. We replicate their equilibrium existence result as well as the equilibrium uniqueness result for sufficient ambiguity. We show that their comparative statics result in ambiguity attitude breaks down: more optimism does not lead to higher equilibria when players have α-MEU preferences. We provide a necessary and sufficient condition on the perceived ambiguities of the players for this comparative statics result to hold.

Keywords: Ambiguous Games, α-MEU Preferences, Perceived Ambiguity, Ambiguity Attitude, Comparative Statics.

5.1 Motivation

The theory and results of the previous article allow a wide range of analysis and applications to games. We focus here on a special case of the framework introduced there. We consider the class of normal-form games with positive externalities and increasing differences, introduces and extensively analysed in Eichberger and Kelsey (2014). In these games the pure strategy space S has a lattice structure and S_i is ordered. This structure allows comparative statics exercises in perceived ambiguity and ambiguity attitude.

As Eichberger and Kelsey (2014), we assume that players perceive ambiguity about the strategic behaviour of the other players. Whereas Eichberger and Kelsey

(2014) represent beliefs by JP-capacities, we assume the larger class of α-MEU beliefs. As in the previous article we adapt the support notion and equilibrium concept of Eichberger and Kelsey (2014). Due to the larger class of beliefs our model is a real generalization of their model.

The goal of this article is to analyse and highlight the similarities and differences in modelling capabilities between the two approaches. We analyse to what extent the results from Eichberger and Kelsey (2014) can be generalized. We show that all but one result can be generalized. The result which cannot is on comparative statics in ambiguity attitude: when players have α-MEU preferences, more optimism does not necessarily lead to higher equilibria. We introduce a necessary and sufficient condition on the prior sets for the comparative statics result to hold. For this we introduce a generalized concept of first order stochastic dominance for lattice-structures that we call *lattice stochastic dominance*.

Outline

In section 5.2 we introduce the games studied as well as α-MEU preferences and lattice stochastic dominance. In section 5.3 we introduce in detail the model and results from Eichberger and Kelsey (2014). Section 5.4 contains our model and the results. Section 5.5 concludes. All proofs are in the Appendix.

5.2 Preliminaries

5.2.1 The Games studied: Positive Externalities and Increasing Differences

We study finite normal-form games $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$. Such a game consists of a finite set of players $P_1, \dots, P_N$, finite pure strategy set S_i and payoff function u_i for $P_i, i \in \{1, \dots, N\}$. The set of pure strategy combinations is denoted by S and S_{-i} is the set of pure strategy combinations of P_i's opponents. The payoff function of P_i, u_i, is a mapping from S to $\mathbb{R}$. The set $\Delta(S_{-i})$ denotes the set of probability distributions over S_{-i}. For $s_{-i} \in S_{-i}$, $P_{s_{-i}}$ denotes the degenerate lottery that results in s_{-i} with probability 1. Thus $\Delta(S_{-i})$ is the convex hull of the degenerate lotteries $P_{s_{-i}}$.

We assume that the sets S_i are ordered. This induces a partial ordering over S:

For $s = (s_1, \dots, s_N)$ and $s' = (s'_1, \dots, s'_N)$ we define $s \geq s' \iff s_i \geq s'_i$ for all $i \in \{1, \dots, N\}$. This implies a partial ordering over S_{-i} for $i \in \{1, \dots, N\}$. This implies that the sets S and S_{-i} are bounded lattices: for two arbitrary elements of the set there exists a smallest element of the set which is weakly larger than both as well as a largest element which is weakly smaller than both.[1] For a two-player game the lattice structure of S_{-i} reduces to a complete order.

Given this lattice structure we assume that payouts exhibit positive externalities and increasing differences. A normal-form game $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ exhibits positive externalities if for all $s_i \in S_i$ it holds that $u_i(s_i, s_{-i})$ is increasing in s_{-i} for all $i \in \{1, \dots, N\}$. A normal-form game exhibits increasing differences if for all $s_i, s'_i \in S_i$ with $s_i > s'_i$ it holds that $u(s_i, s_{-i}) - u(s'_i, s_{-i})$ is increasing in s_{-i} for $i \in \{1, \dots, N\}$.

The interpretation of these payout assumptions is that, due to positive externalities, a player is better off when an opponent plays a higher strategy. Increasing differences creates an incentive to choose a higher strategy when an opponent chooses a higher strategy. Eichberger and Kelsey (2014) highlight that many economic situations are of this kind, for instance Bertrand Oligopoly with linear demand and constant marginal cost.

The term $\Gamma_{peid}(N, S_i; 1 \geq i \geq N)$ denotes the set of all normal-form games with players $1, \dots, N$, pure strategy sets $S_1, \dots, S_N$ and payouts that exhibit positive externalities and increasing differences.

We denote by $\overline{s}_{-i} \in S_{-i}$ the strategy combination in S_{-i} in which the opponents of P_i play their highest strategy. We denote by $\underline{s}_{-i} \in S_{-i}$ the strategy combination in S_{-i} in which the opponents of P_i play their lowest strategy. Two strategies $s_i, s'_i \in S_i$ are positive affinely related if there exist $a > 0, b \in \mathbb{R}$ such that $u_i(s_i, s_{-i}) = au_i(s'_i, s_{-i}) + b$ for all $s_{-i} \in S_{-i}$. Note that $s_i > s'_i$ implies $a \geq 1$.

5.2.2 α-MEU Preferences

We assume that players perceive ambiguity about their opponents strategic choice. For P_i this ambiguity is represented by a non-empty, convex compact set of priors $\mathcal{C}_i \subseteq \Delta(S_{-i})$. In addition we assume that the players have a constant ambiguity attitude $\alpha_i \in [0, 1]$. Thus the players have α-MEU preferences as introduced in

[1] For instance for $s = (s_1, \dots, s_N), s' = (s'_1, \dots, s'_N) \in S$ the smallest element that is larger than s and s' is $(\max\{s_1, s'_1\}, \dots, \max\{s_N, s'_N\})$.

Ghirardato and Marinacci (2002).

For P_i with prior set $\mathcal{C}_i \subseteq \Delta(S_{-i})$ and ambiguity attitude $\alpha_i \in [0,1]$ the evaluation of a strategy $s_i \in S_i$ is therefore

$$V_i(s_i|\mathcal{C}_i, \alpha_i) = \alpha_i \min_{P \in \mathcal{C}_i} \int u(s_i, s_{-i})\ dP + (1-\alpha_i) \max_{P \in \mathcal{C}_i} \int u(s_i, s_{-i})\ dP, \qquad (5.1)$$

where $\int u_i(s_i, s_{-i})\ dP$ denotes the expected utility of the strategy combination (s_i, s_{-i}) at probability distribution $P \in \Delta(S_{-i})$.

This class of α-MEU preferences is larger than the class of preferences considered in Eichberger and Kelsey (2014) since every JP capacity[2] has an α-MEU representation but the reverse is not the case.[3]

5.2.3 Extreme Points and Lattice Stochastic Domination

Throughout the paper the best and worst case scenarios of a strategy given a prior set are crucial. For a prior set $\mathcal{C}_i \subseteq \Delta(S_{-i})$ and strategy $s_i \in S_i$ we define

$$M_{s_i}(\mathcal{C}_i) = \arg\max_{P \in \mathcal{C}_i} \int u_i(s_i, s_{-i})\ dP, \qquad m_{s_i}(\mathcal{C}_i) = \arg\min_{P \in \mathcal{C}_i} \int u_i(s_i, s_{-i})\ dP.$$

$M_{s_i}(\mathcal{C}_i)$ and $m_{s_i}(\mathcal{C}_i)$ are the sets of probability distributions in $\mathcal{C}_i$ at which the expected utility of the strategy s_i is maximized and minimized, respectively. These sets are well-defined and non-empty since prior sets are compact. Note that these sets may contain more than one element. We abuse notation and write $\int u_i(s_i, s_{-i})\ dM_{s_i}(\mathcal{C}_i)$ and $\int u_i(s_i, s_{-i})\ dm_{s_i}(\mathcal{C}_i)$ for the highest and lowest expected utility given strategy s_i and prior set $\mathcal{C}_i$. Note that $\int u_i(s_i, s_{-i})\ dM_{s_i}(\mathcal{C}_i)$ is the expected utility of the strategy s_i for a player with perceived ambiguity $\mathcal{C}_i$ who is a complete optimist, i.e. for $\alpha_i = 0$. The term $\int u_i(s_i, s_{-i})\ dm_f(\mathcal{C}_i)$ is the expected utility of a complete pessimist, i.e. for $\alpha_i = 1$. As α_i increases from 0 to 1, the weight on $\int u_i(s_i, s_{-i})\ dM_{s_i}(\mathcal{C}_i)$ is shifted to $\int u_i(s_i, s_{-i})\ dm_{s_i}(\mathcal{C}_i)$. Thus (5.1) can be rewritten as

$$V(s_i|\mathcal{C}_i, \alpha_i) = \alpha_i \int u_i(s_i, s_{-i})\ dM_{s_i}(\mathcal{C}_i) + (1-\alpha_i) \int u_i(s_i, s_{-i})\ dm_{s_i}(\mathcal{C}_i).$$

In the following we extend the concept of first-order stochastic dominance to

[2] See section 5.3 where we introduce their model.
[3] Take as prior set the core of the convex part of the JP-capacity.

lattice structures. For two probability distributions $P, Q \in \Delta(S_{-i})$ we say that P *lattice-stochastically dominates* Q if for all $s'_{-i} \in S_{-i}$ it holds that

$$P(\{s_{-i} | s_{-i} \geq s'_{-i}\}) \geq Q(\{s_{-i} | s_{-i} \geq s'_{-i}\}).$$

The probability distribution P lattice stochastically dominates Q if for all s'_{-i} it assigns a weakly higher probability to the set of strategy combinations that are at least as high as s'_{-i}.

For two players this reduces to the standard first-order stochastic dominance since in this case S_{-i} is ordered. An important feature of this definition in combination with the games studied in this paper is the following: P lattice stochastically dominates Q if and only if $\int f \, dP \geq \int f \, dQ$ for all acts $f : S_{-i} \to \mathbb{R}$ with positive externalities. The concept of lattice stochastic dominance is crucial for our comparative statics result on ambiguity attitude.

5.3 Eichberger and Kelsey (2014)

The majority of our results are inspired by Eichberger and Kelsey (2014). In this section we introduce their model and results. In the next section we analyze which of their results hold in our more general framework.

5.3.1 JP - Capacities and a Measure for Perceived Ambiguity

Eichberger and Kelsey (2014) represent beliefs by JP-capacities, introduced by Jaffray and Philippe (1997). Recall that a capacity is a normalized and monotonic mapping from the powerset of a state space $\mathcal{P}(S)$ to $[0, 1]$. A capacity $\mu : \mathcal{P}(S) \to [0, 1]$ is convex if $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$ for all $A, B \subseteq S$.

Definition 5.1 (Jaffray and Philippe (1997))**.** *A capacity $\nu : \mathcal{P}(S) \to [0, 1]$ is a JP-capacity if there exists a convex capacity μ and an $\alpha \in [0, 1]$ such that $\nu = \alpha\mu + (1 - \alpha)\bar{\mu}$, where $\bar{\mu}$ denotes the dual capacity of μ, i.e. $\bar{\mu}(A) = 1 - \mu(A^c)$ for all $A \subseteq S$.*

JP-capacities have some nice characteristics, most importantly they cleanly separate perceived ambiguity from ambiguity attitude. The perceived ambiguity of a JP-capacity ν is represented by the convex capacity μ and its core. The ambiguity attitude is represented by the parameter α. The case $\alpha = 1$ represents pure pessimism and corresponds to Maxmin Expected Utility (MEU) preferences. The case $\alpha = 0$ represents pure optimism and corresponds to Maxmax Expected Utility (MMEU) preferences. For $\alpha \in (0, 1)$ the player's preferences exhibit both pessimistic and optimistic features.

Thus in representing beliefs by JP capacities, the authors assume a class of preferences that allow a clear separation of perceived ambiguity and ambiguity attitude. This makes comparative statics exercises in perceived ambiguity and ambiguity attitude possible.

To perform comparative statics the authors introduce upper and lower bounds for the perceived ambiguity of a JP-capacity. They adapt a notion from Dow and Werlang (1994) for convex capacities to define maximal and minimal degrees of perceived ambiguity.

Definition 5.2 (Eichberger and Kelsey (2014))**.** *Let $\mu : \mathcal{P}(S) \to [0, 1]$ be a convex capacity. The maximal degree of ambiguity of μ is given by $\lambda(\mu) = \max\{\bar{\mu}(A) - \mu(A) | \emptyset \subsetneq A \subsetneq S\}$. The minimal degree of ambiguity is given by $\gamma(\mu) = \min\{\bar{\mu}(A) - \mu(A) | \emptyset \subsetneq A \subsetneq S\}$.*

5.3.2 Support and Equilibrium in Beliefs under Ambiguity

Recall the support notion for prior sets by Ryan (2002).

Definition 5.3 (Support notion: Ryan (2002))**.** *Let $\mathcal{C}$ be a prior set on $\Delta(S)$. The support of $\mathcal{C}$ is defined by*

$$supp(\mathcal{C}) = \bigcap_{P \in \mathcal{C}} supp(P).$$

where for a probability distribution $P \in \Delta(S_{-i})$ the support is defined as is usual by

$$supp(P) = \{s_{-i} \in S_{-i} | P(s_{-i}) > 0\}.$$

The support of a prior set consists of the states that receive positive weight by all elements of the prior set. Eichberger and Kelsey (2014) define the support of a JP-capacity in this spirit with the prior set being the core of the convex part of the

JP-capacity. The support of a JP-capacity $\nu = \alpha\mu + (1-\alpha)\bar{\mu} : \mathcal{P}(S) \to [0,1]$ is thus

$$supp(\nu) = \bigcap_{P \in Core(\mu)} supp(P).$$

In equilibrium the players maximize Choquet Expected Utility (CEU) given their beliefs which are represented by JP-capacities.[4] Beliefs are reasonable in the sense that the players believe that their opponents play best responses.

Definition 5.4 (Eichberger and Kelsey (2014))**.** *Let* $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ *be a normal-form game and consider JP-capacities* $\hat{\nu}_i : \mathcal{P}(S_{-i}) \to [0,1]$ *for* $i \in \{1, \ldots, N\}$*. Then* $\hat{\nu} = \langle \hat{\nu}_1, \ldots, \hat{\nu}_N \rangle$ *is an equilibrium in beliefs under ambiguity (EUA) if for all* $i \in \{1, \ldots, N\}$

$$\emptyset \neq supp(\hat{\nu}_i) \subseteq \times_{j \neq i} \arg\max_{s_j \in S_j} \int u_j(s_j, s_{-j})\ d\hat{\nu}_j,$$

with $\int u_j(s_j, s_{-j})\ d\hat{\nu}_j$ *being the Choquet integral of the strategy* s_j *given the capacity* $\hat{\nu}_j$*.*

If $supp(\hat{\nu}_i)$ contains just one element $\hat{s}_i$ for all $i \in \{1, \ldots, N\}$ then $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_N)$ is called singleton equilibrium in beliefs under ambiguity. Eichberger and Kelsey (2014) provide the following equilibrium existence result for games with positive externalities and increasing differences.

Theorem 5.1. *Let* $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ *be a game with positive externalities and increasing differences. Then for any exogenous ambiguity-attitudes* $\alpha_1, \ldots, \alpha_N$*, maximal degrees of ambiguity* $\lambda_1, \ldots, \lambda_N$ *and minimal degrees am ambiguity* $\gamma_1, \ldots, \gamma_N$*, the game* Γ *has a singleton equilibrium in beliefs under ambiguity in JP-capacities* $\nu = \langle \nu_1, \ldots, \nu_N \rangle$*, where* $\nu_i = \alpha_i\mu_i + (1-\alpha_i)\bar{\mu}_i$ *such that the convex capacity* μ_i *has maximal degree of ambiguity at most* λ_i *and minimal degree of ambiguity at least* γ_i *for* $i \in \{1, \ldots, n\}$*.*

5.3.3 Comparative Statics

Eichberger and Kelsey (2014) have two comparative statics results. Firstly, after strenghtening the payout assumptions to positive aggregate externalities, they show

[4] See Choquet (1954) and Schmeidler (1989).

that for any exogenous upper and lower bounds for maximal and minimal degrees of ambiguity the highest and lowest equilibrium of the game is increasing in optimism: more optimism leads to higher strategies in equilibrium. We show in the next section that this result breaks down when preferences are of the α-MEU type.

In their second comparative statics result, after strenthening the payout assumptions to having a unique maximizer, they show that for sufficient ambiguity as well as extreme ambiguity attitude, the EUA will be unique.

They thus show that perceived ambiguity and ambiguity attitude have distinct effects on the set of equilibria: whereas more optimism leads to higher equilibria, more perceived ambiguity eventually results in the uniqueness of the equilibrium.

Comparative Statics in Ambiguity Attitude

For their first comparative statics result the authors need the additional assumption that the game has positive aggregate externalities.[5]

Definition 5.5 (Eichberger and Kelsey (2014))**.** *A game $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N\rangle$ has positive aggregate externalities if $u_i(s_i, s_{-i}) = u_i(s_i, f_i(s_i))$, where u_i is increasing in f_i and $f_i : S_{-i} \to \mathbb{R}$ is increasing in all arguments for all $i \in \{1, \dots, N\}$.*

Positive aggregate externalities implies that S_{-i} does not just have a lattice structure but has a complete ordering.

Theorem 5.2 (Eichberger and Kelsey (2014))**.** *Let $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N\rangle$ be a game with positive aggregate externalities and increasing differences. Assume that beliefs are represented by JP-capacities and let $\alpha = (\alpha_1, \dots, \alpha_N)$ denote the vector of ambiguity-attitudes. Let $\underline{s}(\alpha)$ ($\overline{s}(\alpha)$) denote the lowest (highest) equilibrium strategy profile when the minimal (maximal) degree of ambiguity is γ (λ). Then $\underline{s}(\alpha)$ and $\overline{s}(\alpha)$ are decreasing functions of α_i for all $i \in \{1, \dots, N\}$.*

Comparative Statics in Perceived Ambiguity

Again, the authors need an additional assumption.

Assumption 1. For $i \in \{1, \dots, N\}$, let $u_i(s_i, \bar{s}_{-i})$ and $u_i(s_i, \underline{s}_{-i})$ have a unique maximizer, i.e. $|\arg\max_{s_i \in S_i} u_i(s_i, \bar{s}_{-i})| = 1$ and $|\arg\max_{s_i \in S_i} u_i(s_i, \underline{s}_{-i})| = 1$.

[5] We highlight in the next section that indeed this assumption is needed for their result, i.e. it breaks down if we only assume positive externalities without "aggregate".

This assumpion enables them to prove the following Proposition.

Proposition 5.1 (Eichberger and Kelsey (2014)). *Let* $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ *be a game with positive externalities and increasing differences that satisfies Assumption 1. There exist* $\overline{\alpha}$ ($\underline{\alpha}$) *and* $\overline{\gamma}$ *such that if the minimal degree of ambiguity is* $\gamma(\mu_i) \geq \overline{\gamma}$ *and* $\alpha_i \geq \overline{\alpha}$ ($\alpha_i \leq \underline{\alpha}$) *for* $i \in \{1, \ldots, N\}$ *then there is a unique singleton EUA with an equilibrium strategy profile that is smaller (larger) than the smallest (largest) equilibrium profile without ambiguity.*

5.4 Our Multiple Prior Approach

In this section we analyse whether the results from Eichberger and Kelsey (2014) can be replicated for α-MEU preferences. We show that equilibrium existence and the second comparative statics result still hold in our more general framework. We show that the first comparative statics result breaks down: there are games with positive externalities and increasing differences in which the highest equilibrium is not increasing in optimism when players have α-MEU preferences.[6] We present a neccessary and sufficient condition on the prior sets which guarantee that the comparative statics result holds. Furthermore we show that if the payoffs of the game are positive affinely related the comparative statics result always holds.

5.4.1 Equilibrium Existence

We use the support and equilibrium notion introduced in the previous article. It is the natural extension of the approach in Eichberger and Kelsey (2014) to the multiple prior model and thus α-MEU.

We restate our equilibrium notion of the previous chapter. In equilibrium the support of the prior set is non-empty and only contains the opponents' best responses given their beliefs. Note that the following definition and the subsequent theorem do not assume a constant ambiguity attitude but the more general ambiguity attitude functions $a_i : S_{-i} \to [0, 1]$.

[6] The assumption of positive aggregate externalities does not change this result.

Definition 5.6 (Equilibrium under Ambiguity). *Let $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N\rangle$ be a normal-form game. The tuple $(\mathcal{C}_i, a_i)_{i=1}^N$ is an Equilibrium under Ambiguity if for all $1 \leq i \leq N$*

$$\emptyset \neq supp(\mathcal{C}_i) \subseteq \times_{j\neq i} R_j(\mathcal{C}_j, a_j),$$

where $R_j(\mathcal{C}_j, a_j) = \arg\max_{s_j \in S_j} [V(s_j|\mathcal{C}_j, a_j)]$ is the best response correspondence.

If $supp(\mathcal{C}_i)$ contains just a single element $\hat{s}_i \in S_i$ for all $i \in \{1, \ldots, N\}$ we refer to the equilibrium as a singleton equilibrium and $\hat{s} = (\hat{s}_i, \ldots, \hat{s}_N)$ as its strategy profile.

In the previous article we prove equilibrium existence for normal-form games for any exogenous perceived ambiguity and ambiguity attitude. In this article we consider a special case of normal-form games, thus equilibrium existence follows directly from that equilibrium existence result. The following theorem is more specific as it guarantees the existence of a *singleton* equilibrium for any exogenous perceived ambiguity and ambiguity attitude. This is achieved thanks to the lattice structure of the strategy space. Recall that for a prior set $\mathcal{C}_i \subseteq \Delta(S_{-i})$, $[\mathcal{C}_i]$ is the set of translations of $\mathcal{C}_i$ within $\Delta(S_{-i})$: it contains the prior sets that represent the same perceived ambiguity as $\mathcal{C}_i$.

Theorem 5.3 (Equilibrium Existence). *Let $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N\rangle$ be a normal-form game with positive externalities and increasing differences. Then for any exogenously given $([\mathcal{C}_1], \ldots, [\mathcal{C}_N])$ such that $supp(\mathcal{C}_i) \neq \emptyset$ for all $i \in \{1, \ldots, N\}$ and ambiguity-attitudes $a_1, \ldots, a_N$ there exist $\mathcal{C}_1^*, \ldots, \mathcal{C}_N^*$ with $\mathcal{C}_i^* \in [\mathcal{C}_i]$ for all $i \in \{1, \ldots, N\}$ such that $(\mathcal{C}_i^*, a_i)_{i=1}^N$ is a singleton Equilibrium under Ambiguity.*

By restricting preferences to constant ambiguity attitude functions we achieve existence of a singleton EUA for α-MEU preferences.

5.4.2 Comparative Statics in Ambiguity Attitude

This section contains the arguably most interesting and also probably at first counter-intuitive results. We show that more optimism can lead to lower equilibria in games with positive externalities and increasing differences. This insight highlights the difference in modelling capabilities of the approach in Eichberger and Kelsey (2014) and our own.

		Player 2		
		s_{21}	s_{22}	s_{23}
Player 1	s_{11}	$(8, 8)$	$(0, 7)$	$(-4, 0)$
	s_{12}	$(7, 0)$	$(1, 1)$	$(-1, 0)$
	s_{13}	$(0, -4)$	$(0, -3)$	$(0, 0)$

Table 5.1: Highest Equilibrium not increasing in Optimism.

Example 5.1. Consider the game in Table 5.1. It is a symmetric game with positive externalities and increasing differences.[7]

Let $Q = (\frac{17}{24}, 0, \frac{7}{24})$ and $Q' = (\frac{9}{16}, \frac{7}{16}, 0)$ be two probability distributions in $\Delta(S_{-i})$ and define $\mathcal{C} = Conv(Q, Q')$, the convex hull of Q and Q'. Then $supp(\mathcal{C}) = \{s_{-i1}\}$. Straightworward calculations result in the following for $i \in \{1, 2\}$:

$$V(s_{i1}|\ \mathcal{C}, 1) = \min_{P \in \mathcal{C}} \int u(s_i, s_{-i}) dP = \frac{9}{2}$$
$$V(s_{i1}|\ \mathcal{C}, 0) = \max_{P \in \mathcal{C}} \int u(s_i, s_{-i}) dP = \frac{9}{2}$$
$$V(s_{i2}|\ \mathcal{C}, 1) = \min_{P \in \mathcal{C}} \int u(s_i, s_{-i}) dP = \frac{35}{8}$$
$$V(s_{i2}|\ \mathcal{C}, 0) = \min_{P \in \mathcal{C}} \int u(s_i, s_{-i}) dP = \frac{14}{3}.$$

Thus $V(s_{i1}|\ \mathcal{C}, 1) > V(s_{i2}|\ \mathcal{C}, 1)$ and $V(s_{i1}|\ \mathcal{C}, 0) < V(s_{i2}|\ \mathcal{C}, 0)$ for $i \in \{1, 2\}$. Therefore $(\mathcal{C}, 1)_{i=1}^2$ is a singleton equilibrium under ambiguity, but $(\mathcal{C}, 0)_{i=1}^2$ is not. Thus under pessimism ($\alpha = 1$) and belief $[\mathcal{C}]$ for both players the highest strategy combination constitutes an EUA, but under optimism ($\alpha = 0$) it does not. Thus, given these beliefs, the highest equilibrium is not increasing in optimism in this game.

The example shows that the highest equilibrium is not always increasing in optimism when players have α-MEU preferences. Under what conditions can this phenomenon occur? Eichberger and Kelsey (2014) show that it cannot occur when the prior set is the core of a convex capacity as this case corresponds to preferences given a JP-capacity. Are there other shapes or structures of prior sets that guarantee the comparative statics result?

In the following we present a necessary and sufficient condition for the prior sets

[7] It automatically has aggregate externalities as it is a 2-player game.

that answers this question. We also present a sufficient condition on the payouts for the highest/lowest equilibrium to increase in optimism.

Results on Beliefs: For which Beliefs is the Highest/Lowest Equilibrium guaranteed to increase in Optimism?

The following definition introduces a class of prior sets that we call Positive Externality Lattice Stochastic Dominance (PELSD) prior sets. It turns out that they play a crucial role in figuring out when the comparative statics result in ambiguity attitude holds.

Definition 5.7. *Let S be a finite state space with a lattice structure and let $\mathcal{C} \subseteq \Delta(S)$ be a prior set. We say that $\mathcal{C}$ is of the Positive Externality Lattice-Stochastic Dominance (PELSD) type if for all acts f with positive externalities it holds that there exists a $P \in M_f(\mathcal{C})$ and a $Q \in m_f(\mathcal{C})$ such that P lattice stochastically dominates Q.*

Note that the property PELSD is location independent, i.e. a prior set $\mathcal{C}$ has PELSD type if and only if $\mathcal{C}'$ is of the PELSD type for all $\mathcal{C}' \in [\mathcal{C}]$. As an example of a prior set that is not of the PELSD type, consider the prior set $\mathcal{C} = Conv(Q, Q')$ from the above example. Consider the act $f = 1_{s_{21}}0$. Then $M_f(\mathcal{C}) = Q$ and $m_f(\mathcal{C}) = Q'$. But Q does not lattice stochastically dominate Q' since $Q(\{s_{21}, s_{22}\}) < Q'(\{s_{21}, s_{22}\})$. Thus $\mathcal{C}$ is not of the PELSD type.

The following theorem shows that the highest/lowest EUA is guaranteed to increase in optimism if and only if all players' preferences can be represented by prior sets that are of the PELSD type.

Theorem 5.4. *Let $S = (S_1 \times \cdots \times S_N)$ be a set with a lattice-structure. Consider games Γ with pure-strategy space S and assume that players have α-MEU preferences. The following are equivalent:*

1. *All player have prior sets of the PELSD type.*
2. *The highest/lowest equilibrium is increasing in optimism for all $\Gamma \in \Gamma_{peid}(N, S_i; 1 \leq i \leq N)$.*

The following two lemmas characterize two classes of prior sets that are of the PELSD type under positive aggregate externalities. The first class are cores of convex capacities. In combination with Theorem 5.4 this fact proves the first comparative statics result of Eichberger and Kelsey (2014). In the Appendix we also provide an alternative (and much less complex) proof of their comparative statics result. The second class are ball-shaped prior sets.

Lemma 5.1. *Under positive aggregate externalities, cores of convex capacities are of the PELSD type.*

Lemma 5.2. *Under positive aggregate externalities, prior sets of the type $B_\epsilon(P)$, i.e. balls, are of the PELSD type.*

If positive aggregate externalities is not assumed, both of the above lemmas fail when the state space is sufficiently large. This is because whether a prior set $\mathcal{C}$ is of the PELSD type depends on the structure of the state space. A prior set $\mathcal{C}$ may be of the PELSD type if S has a complete order, but not when S is just a lattice. The reason being that there are more acts with positive externalities when S is not ordered. The assumption of positive aggregate externalities turns out to make the set of acts with positive externalities sufficiently small for cores of convex capacities and balls to always be of the PELSD type.

Result on Payouts: For which Games is the Highest/Lowest Equilibrium guaranteed to increase in Optimism?

The following theorem provides a sufficient condition on the payouts of the game for the highest/lowest equilibrium to be increasing in optimism, regardless of the prior set.

Theorem 5.5. *Let $\Gamma = \langle N; S_i, u_i : 1 \leq i \leq N \rangle$ be a game with positive externalities and increasing differences. Assume that the strategies are affinely related for all players and that players have α-MEU preferences. Then the highest/lowest equilibrium is increasing in optimism.*

The proof follows from the fact that the evaluation functional is linear for positive affinely related strategies as α-MEU functionals are constant-linear. A reverse of this theorem is not possible, i.e. there exist games in which the strategies are not affinely related but where the highest/lowest equilibrium is increasing in optimism for all exogenous perceived ambiguities.

5.4.3 Equilibrium Uniqueness for Sufficient Ambiguity

The payouts of the game allow multiple Nash Equilibria. Eichberger and Kelsey (2014) show that, with the additional assumption of a unique maximizer, the game has a unique EUA in JP-capacities when the perceived ambiguity is sufficiently large and ambiguity attitude sufficiently extreme. In the following we replicate this result in our framework. For this we need to define what sufficiently large ambiguity means. It is clear that the prior set $\mathcal{C}_i = \Delta(S_{-i})$ represents the maximum possible perceived ambiguity for P_i. Thus a set which is *close* to $\Delta(S_{-i})$ also represents a large perceived ambiguity. Recall that $\Delta(S_{-i}) = Conv(P_{s_{-i}}|s_{-i} \in S_{-i})$. We define the closeness of some $\mathcal{C}_i$ to $\Delta(S_{-i})$ via the maximum distance of $\mathcal{C}_i$ to the degenerate lotteries $P_{s_{-i}}$. The term $\min_{s_{-i}\in S_{-i}} \max_{P\in\mathcal{C}_i} P(s_{-i})$ measures this. If this term is close to 1, then $\mathcal{C}_i$ is close to $\Delta(S_{-i})$. For exogenous perceived ambiguity $[\mathcal{C}_i]$ we say that $[\mathcal{C}_i]$ is close to $\Delta(S_{-i})$ if $\min_{\mathcal{C}_i'\in[\mathcal{C}_i]} \min_{s_{-i}\in S_{-i}} \max_{P\in\mathcal{C}_i'} P(s_{-i})$ is close to 1.

Theorem 5.6. *Let $\Gamma = \langle N; S_i, u_i : 1 \le i \le N\rangle$ be a game with positive externalities and increasing differences for which Assumption 1 holds. There exist $\overline{\alpha}$ (respectively $\underline{\alpha}$) with $0 < \underline{\alpha} \le \overline{\alpha} < 1$ and $\epsilon > 0$ such that if for $([\mathcal{C}_1], \dots, [\mathcal{C}_N])$ it holds that $\min_{\mathcal{C}_i'\in[\mathcal{C}_i]} \min_{s_{-i}\in S_{-i}} \max_{P\in\mathcal{C}_i'} P(s_{-i}) > 1-\epsilon$ for all $i \in \{1, \dots, N\}$, there is a unique singleton EUA with an equilibium strategy profile which is smaller (larger) than the smallest (largest) equilibrium strategy without ambiguity.*

The theorem states that for sufficiently extreme ambiguity attitude and for $[\mathcal{C}_i]$ sufficiently close to $\Delta(S_{-i})$ for all $i \in \{1, \dots, N\}$, the EUA is unique. Furthermore it is bigger/smaller than the biggest/smallest equilibrium without ambiguity.

5.5 Conclusion

In this article, we extend the studies of Eichberger and Kelsey (2014) to α-MEU preferences. We adapt their notion of support and equilibrium in beliefs under ambiguity to our framework. As Eichberger and Kelsey (2014), we show existence of a singleton equilibrium, even for non-constant ambiguity-attitudes, in games with positive externalities and increasing differences.

We also replicate their comparative statics result on perceived ambiguity. Most importantly, we show that their comparative statics result on ambiguity attitude breaks down for α-MEU preferences. Thus our approach allows richer dynamics and modelling of more behaviour patterns. This does not mean that our approach is superior, rather it highlights some interesting limitations and capabilities of these two models.

5.6 Appendix

Proof of Theorem 5.3. All that needs to be shown is the existence of a singleton equlibrium, as equilibrium existence is implied by Theorem 4.3 of the previous chapter. Assume that $(\mathcal{C}_i^*, a_i)_{i=1}^N$ with $\mathcal{C}_i^* \in [\mathcal{C}_i]$ for all $i \in \{1, \dots, N\}$ is an EUA. Recall that the payouts induce a lattice structure over S and an ordering over S_i for $i \in \{1, \dots, N\}$. Let $\hat{s}_i$ be the highest best response of P_i given $(\mathcal{C}_i^*, a_i)$ and define $\hat{s} = (\hat{s}_1, \dots, \hat{s}_N)$. Consider the prior sets $(\mathcal{C}_{[\mathcal{C}_i]}^{\hat{s}_{-i}})_{i=1}^N$ induced by $\hat{s}$, where $\mathcal{C}_{[\mathcal{C}_i]}^{\hat{s}_{-i}} \in [\mathcal{C}_i]$ is constructed as in the proof of Theorem 4.3 of the previous chapter.

Case 1: $(\mathcal{C}_{[\mathcal{C}_i]}^{\hat{s}_{-i}}, a_i)_{i=1}^N$ is an EUA. Then we are finished since $supp(\mathcal{C}^{\hat{s}_{-i}}) = \{\hat{s}_{-i}\}$ for all $i \in \{1, \dots, N\}$ by construction of $\mathcal{C}^{\hat{s}_{-i}}$, which implies that the EUA is singleton.

Case 2: $(\mathcal{C}^{\hat{s}_{-i}}, a_i)_{i=1}^N$ is not an EUA. Then there is an incentive for some P_i to deviate to some strategy $\tilde{s}_i \in S_i$. Going from $\mathcal{C}_i^*$ to $\mathcal{C}_{[\mathcal{C}_i]}^{\hat{s}_{-i}}$ means that player i perceives the opponents to play higher strategies. Thus due to increasing differences this incentive to deviate must be to a higher strategy, so $\tilde{s}_i > \hat{s}_i$.

For the strategy combination $\tilde{s} = (\hat{s}_1, \dots, \hat{s}_{i-1}, \tilde{s}_i, \hat{s}_{i+1}, \dots, \hat{s}_N)$ consider the prior sets $(\mathcal{C}_{[\mathcal{C}_i]}^{\tilde{s}_{-i}})_{i=1}^N$ induced by $\tilde{s}$.

Case 2.1: $(\mathcal{C}_{[\mathcal{C}_1]}^{\hat{s}_{-1}}, a_1, \dots, \mathcal{C}_{[\mathcal{C}_{i-1}]}^{\hat{s}_{-(i-1)}}, a_{i-1}, \mathcal{C}_{[\mathcal{C}_i]}^{\tilde{s}_{-i}}, a_i, \mathcal{C}_{[\mathcal{C}_{i+1}]}^{\hat{s}_{-(i+1)}}, a_{i+1}, \dots, \mathcal{C}_{[\mathcal{C}_N]}^{\hat{s}_{-N}}, a_N)$ is an EUA, then we are finished as it is singleton as in Case 1.

Case 2.2: $(\mathcal{C}_{[\mathcal{C}_1]}^{\hat{s}_{-1}}, a_1, \dots, \mathcal{C}_{[\mathcal{C}_{i-1}]}^{\hat{s}_{-(i-1)}}, a_{i-1}, \mathcal{C}_{[\mathcal{C}_i]}^{\tilde{s}_{-i}}, a_i, \mathcal{C}_{[\mathcal{C}_{i+1}]}^{\hat{s}_{-(i+1)}}, a_{i+1}, \dots, \mathcal{C}_{[\mathcal{C}_N]}^{\hat{s}_{-N}}, a_N)$ is not an EUA. Then the step in Case 2 can be repeated.

Since Γ is a finite game and the incentive to deviate is always *upwards* in S this process has to finish after a finite amount of steps. This proves the existence of a singleton EUA.

□

Proof of Theorem 5.6. For $i \in \{1, \dots, N\}$ consider the case $\mathcal{C}_i = \Delta(S_{-i})$ and $\alpha_i = 0$. The functional $V(s_i|\mathcal{C}_i, \alpha_i)$, due to positive externalities, then puts all the weight on the scenario where the opponents all play their highest strategy. The unique maximizer assumption implies that $|\arg\max_{s_i \in S_i} V(s_i|\Delta(S_{-i}), 0)| = 1$.

Consider a sequence of prior sets $(\mathcal{C}_{ij})_{j\in\mathbb{N}}$, $\mathcal{C}_{ij} \subseteq \Delta(S_{-i})$ for all $j \in \mathbb{N}$, which converges to $\Delta(S_{-i})$ in the sense that $\min_{s_{-i} \in S_{-i}} \max_{P \in \mathcal{C}_{ij}} P(s_{-i}) \xrightarrow[j\to\infty]{} 1$.

Furthermore assume that $\alpha_{ij} \xrightarrow[j\to\infty]{} 0$. Thus $(\mathcal{C}_{ij}, \alpha_{ij}) \xrightarrow[j\to\infty]{} (\Delta(S_{-i}), 0)$. V is continuous in both entries,[8] thus

$$V(s_i|\mathcal{C}_{ij},\alpha_{ij}) \underset{j\to\infty}{\longrightarrow} V(s_i|\Delta(S_{-i}),0)$$

for all $s_i \in S_i$. It follows that there exists a $j^* \in \mathbb{N}$ such that for all $j \geq j^*$ and for all $\mathcal{C}'_{ij} \in [\mathcal{C}_{ij}]$

$$|\arg\max_{s_i\in S_i} V(s_i|\mathcal{C}'_{ij},\alpha_{ij})| = |\arg\max_{s_i\in S_i} V(s_i|\Delta(S_{-i}),0)| = 1.$$

Define $\epsilon_i = 1 - \min_{\mathcal{C}'_i\in[\mathcal{C}_{ij^*}]} \min_{s_{-i}\in S_{-i}} \max_{P\in\mathcal{C}'_i} P(s_{-i})$ and $\epsilon = \min_{i\in\{1,\dots,N\}} \epsilon_i$. Define $\underline{\alpha}_i = \alpha_{ij^*}$ and $\underline{\alpha} = \min_{i\in\{1,\dots,N\}} \underline{\alpha}_i$.

Then for all beliefs $([\mathcal{C}_i],\alpha_i)_{i=1}^N$ with $\min_{\mathcal{C}'_i\in[\mathcal{C}_i]} \min_{s_{-i}\in S_{-i}} \max_{P\in\mathcal{C}'_i} P(s_{-i}) > 1-\epsilon$ and $\alpha_i \geq \underline{\alpha}$ there exists just a single equilibrium in which the prior set is $\mathcal{C}_i^{\hat{s}_i} \in [\mathcal{C}_i]$ with $\hat{s}_i = \arg\max_{s_i\in S_i} V(s_i|\Delta(S_{-i}),0)$.

We have thus shown the existence of ϵ and $\underline{\alpha}$ as required. We use the same idea with $\alpha_{ij} \underset{j\to\infty}{\longrightarrow} 1$ to show the existence of $\overline{\alpha}$.

$\square$

Proofs of Section 4.2

In order to prove Theorem 5.4 some definitions are necessary.

$\Gamma_{f,g}(S)$ denotes a 2-act single-person decision problem with positive externalities and increasing differences, the ordering of acts $f > g$ and a state space $S = (s_1,\dots,s_n)$. Thus $f(s_1) \geq \dots \geq f(s_n)$, $g(s_1) \geq \dots \geq g(s_n)$ and $f(s_1)-g(s_1) \geq \dots \geq f(s_n)-g(s_n)$. We denote by $\Gamma_{peid}(S)$ the set of such 2-act single-person decision problems with positive externalities and increasing differences with state space S.

Let $span(\Delta(S))$ be the vector-space spanned by $\Delta(S)$, i.e. $span(\Delta(S)) = \{P \in \mathbb{R}^{|S|} | \sum_{s\in S} P(s) = 1\}$. For $P \in span(\Delta(S))$, which may have negative values, and act f, $\int f\ dP$ denotes the "expected value" of f given P:

$$\int f\ dP = \sum_{s\in S} f(s)P(s).$$

[8] Continuous in the sense that small changes in $(\mathcal{C},\alpha)$ lead to small changes in $V(\cdot|\mathcal{C},\alpha)$.

For an act f and probability distribution $Q \in \Delta(S)$, I^f_Q denotes the vector-space spanned by the indifference set of f containing Q, i.e.

$$I^f_Q := \{P \in span(\Delta(S)) | \int f \; dP = \int f \; dQ\}.$$

With slight abuse of terminology we refer to I^f_Q as the indifference set of f through Q. For a prior set $\mathcal{C} \subseteq \Delta(S)$, with a slight abuse of notation, we denote by $I^f_{M_f(\mathcal{C})}$ the highest indifference set of the act f that intersects $\mathcal{C}$ and $I^f_{m_f(\mathcal{C})}$ the lowest indifference set that intersects $\mathcal{C}$.

For a prior set $\mathcal{C} \subseteq \Delta(S)$ and two acts f and g we define

$$\overline{P}_{f,g}(\mathcal{C}) := \{P \in span(\Delta(S)) | u_P(f) = u_{M_f(\mathcal{C})}(f) \text{ and } u_P(g) = u_{M_g(\mathcal{C})}(g)\} = \bigcap_{h \in \{f,g\}} I^h_{M_h(\mathcal{C})}$$

$$\underline{P}_{f,g}(\mathcal{C}) := \{P \in span(\Delta(S)) | u_P(f) = u_{m_f(\mathcal{C})}(f) \text{ and } u_P(g) = u_{m_g(\mathcal{C})}(g)\} = \bigcap_{h \in \{f,g\}} I^h_{m_h(\mathcal{C})}.$$

The sets $\overline{P}_{f,g}(\mathcal{C})$ and $\underline{P}_{f,g}(\mathcal{C})$ are the intersections of the maximum and minimum indifference sets of the two acts f and g, given $\mathcal{C}$. Note that indifference sets are $n-2$ dimensional and the sets $\overline{P}_{f,g}(\mathcal{C})$ and $\underline{P}_{f,g}(\mathcal{C})$ are $n-3$ dimensional, unless they are positive affinely related in which case the indifference curves coincide such that these sets are also $n-2$ dimensional. For a decision problem $\Gamma_{f,g}(S) \in \Gamma_{peid}(S)$ and prior set $\mathcal{C}$ the sets $\overline{P}_{f,g}(\mathcal{C})$ and $\underline{P}_{f,g}(\mathcal{C})$ are well-defined and non-empty.

For $\Gamma_{f,g}(S)$ we define

$$I^0_{f,g} := \{P \in span(\Delta(S)) | \int f \; dP = \int g \; dP\}$$
$$I^+_{f,g} := \{P \in span(\Delta(S)) | \int f \; dP > \int g \; dP\}$$
$$I^-_{f,g} := \{P \in span(\Delta(S)) | \int f \; dP < \int g \; dP\}.$$

5.6.1 Helplemmas

The following lemmas are helplemmas to prove Theorem 5.4.

Lemma 5.3. *Consider a prior set* $\mathcal{C} \subseteq \Delta(S)$ *and a decision problem* $\Gamma_{f,g}(S) \in \Gamma_{peid}(S)$. *For any* $P \in M_f(\mathcal{C}), Q \in M_g(\mathcal{C}), R \in \overline{P}_{f,g}(\mathcal{C})$ *it holds that*

$$\int f - g \; dP \geq \int f - g \; dR \geq \int f - g \; dQ.$$

If in addition $M_f(\mathcal{C}) \cap M_g\mathcal{C} = \emptyset$, *the inequalities are strict. Similarly for any* $p \in m_f(\mathcal{C}), q \in m_g(\mathcal{C}), r \in \underline{P}_{f,g}(\mathcal{C})$

$$\int f - g \; dp \leq \int f - g \; dr \leq \int f - g \; dq.$$

If in addition $m_f(\mathcal{C}) \cap m_g\mathcal{C} = \emptyset$, *the inequalities are strict.*

Proof. It holds that

$$\int f \; dP = \int f \; dR \geq \int f \; dQ.$$

The inequality follows from the fact that $Q \in \mathcal{C}$ and the definition of $M_f(\mathcal{C})$. Similarly

$$\int g \; dP \leq \int g \; dR = \int g \; dQ.$$

It follows from these two inequalitiesthat

$$\int f - g \; dP \geq \int f - g \; dR \geq \int f - g \; dQ.$$

If $M_f(\mathcal{C}) \cap m_f(\mathcal{C}) = \emptyset$, all inequalities are strict. The second part can be shown in an analogue way. □

Lemma 5.4. *Let* $f, g : S \to \mathbb{R}$ *be two acts. Every intersection of indifference sets of the two acts* f *and* g *is a subset of some indifference set of the act* $f - g$.

Proof. For some $P \in \mathcal{C}$ consider the two indifference curves $I_P(f)$ and $I_P(g)$ and assume that $Q \in I_P(f) \cup I_P(g)$. Then

$$\begin{aligned}\int f - g\ dP &= \int f\ dP - \int g\ dP \\ &= \int f\ dQ - \int g\ dQ \\ &= \int f - g\ dQ.\end{aligned}$$

This shows that P and Q are element of the same indifference set of the act $f - g$. □

A consequence of Lemma 4.2 is that $\overline{P}_{f,g}(\mathcal{C})$ and $\underline{P}_{f,g}(\mathcal{C})$ each lie on some indifference set of the act $f - g$. Furthermore both $\overline{P}_{f,g}(\mathcal{C})$ and $\underline{P}_{f,g}(\mathcal{C})$ are always subsets of one of the sets $I_0^{f,g}$, $I_+^{f,g}$ or $I_-^{f,g}$.

Lemma 5.5. *Let $f, g : S \to \mathbb{R}$ be two acts with positive externalities. Then f and g exhibit increasing differences if and only if the act $f - g$ is an act with positive externalities.*

Proof. This follows directly from the definition of increasing differences. □

We say that for $\Gamma_{f,g}(\mathcal{S}) \in \Gamma_{peid}(S)$ and prior sets $\mathcal{C}$ the optimal act is increasing in optimism if $\arg\max_{f,g} \alpha \int f\ dm_f(\mathcal{C}) + (1-\alpha) \int f\ dM_f(\mathcal{C})$ is increasing in optimism. Recall that α reflects the degree of pessimism and $1 - \alpha$ the degree of optimism.

Lemma 5.6. *Consider $\Gamma_{f,g}(S) \in \Gamma_{peid}(S)$ and a prior set $\mathcal{C} \subseteq \Delta(S)$. Then the following are equivalent:*

1. *The optimal act is not increasing in optimism.*

2. $$\int f\ dM_f(\mathcal{C}) < \int g\ dM_g(\mathcal{C}) \quad \text{and} \int f\ dm_f(\mathcal{C}) > \int g\ dm_g(\mathcal{C}).$$

3. $$\overline{P}_{f,g}(\mathcal{C}) \subseteq I^-_{f,g}, \qquad \underline{P}_{f,g}(\mathcal{C}) \subseteq I^+_{f,g}.$$

Proof. 1. $\Rightarrow$ 2. From 1. it follows that there exists $\alpha_1 < \alpha_2$ such that

$$\alpha_1 \int f\ dm_f(\mathcal{C}) + (1-\alpha_1) \int f\ dM_f(\mathcal{C}) < \alpha_1 \int g\ dm_g(\mathcal{C}) + (1-\alpha_1) \int g\ dM_g(\mathcal{C})$$
$$\alpha_2 \int f\ dm_f(\mathcal{C}) + (1-\alpha_2) \int f\ dM_f(\mathcal{C}) > \alpha_2 \int g\ dm_g(\mathcal{C}) + (1-\alpha_2) \int g\ dM_g(\mathcal{C}).$$

Since $\alpha \int h\ dm_h(\mathcal{C}) + (1-\alpha) \int h\ dM_f(\mathcal{C})$ is strictly decreasing in α for any act h, 2. follows.

2. $\Rightarrow$ 3. By the definition of $\overline{P}_{f,g}(\mathcal{C}), \underline{P}_{f,g}(\mathcal{C})$ we have that 2. is equivalent to $\int f - g\ d\overline{P}_{f,g}(\mathcal{C}) < 0$ and $\int f - g\ d\underline{P}_{f,g}(\mathcal{C}) > 0$, thus 3. holds.

3. $\Rightarrow$ 1. Given the belief $\mathcal{C}$, the pessimist ($\alpha = 1$) prefers f to g and the optimist ($\alpha = 0$) prefers g to f, thus 1. holds.

□

Lemma 5.7. *Let S be a state space with a lattice structure and assume that $\mathcal{C} \subseteq \Delta(S)$. The following are equivalent:*

1. *$\mathcal{C}$ is of the PELSD type.*
2. *Given $\mathcal{C}$, the optimal act is increasing in optimism for all decision problems $\Gamma_{f,g}(S) \in \Gamma_{peid}(S)$.*

Proof. 1. $\Rightarrow$ 2. Assume 2. fails. Then due to Lemma 5.6 there exists $\Gamma_{f,g}(S) \in \Gamma_{peid}(S)$ such that $\int f\ dM_f(\mathcal{C}) < \int g\ dM_g(\mathcal{C})$ and $\int f\ dm_f(\mathcal{C}) > \int g\ dm_g(\mathcal{C})$. Lemma 5.4 implies that $\int f - g\ d\overline{P}_{f,g}(\mathcal{C}) < \int f - g\ d\underline{P}_{f,g}(\mathcal{C})$. Lemma 5.3 implies that $\int f - g\ dP < \int f - g\ dQ$ for all $P \in M_g(\mathcal{C}), Q \in m_g(\mathcal{C})$. The act $f - g$ is an act with positive externalities due to Lemma 5.5. We have thus found an act g for which there exists no $P \in M_g(\mathcal{C}), Q \in m_G(\mathcal{C})$ such that P lattice-stochastically dominates Q. Thus 1. fails.

2. $\Rightarrow$ 1. Assume 1. fails. Then there exists an act f with positive externalities such that for all $P \in M_f(\mathcal{C}), Q \in m_f(\mathcal{C})$ P does not lattice-stochastically dominate Q. This implies the existence of an $s' \in S$ such that for $E = \{s \in S | s \geq s'\}$ we have that

$$\max_{P \in M_f(\mathcal{C})} P(E) < \min_{P \in m_f(\mathcal{C})} P(E).$$

There now exist real numbers $a > b$ such that for the act $h = a_E b$ it holds that

$$\max_{P \in M_f(\mathcal{C})} \int h \; dP < 0 < \min_{P \in m_f(\mathcal{C})} \int h \; dP.$$

Note that h is an act with positive externalities. There exists an act f' which is strictly increasing in S such that

$$\max_{P \in M_{f'}(\mathcal{C})} \int h \; dP < 0 < \min_{P \in m_{f'}(\mathcal{C})} \int h \; dP. \tag{5.2}$$

Such an act f' exists since a small change in an act changes the evaluation of the worst/best scenario by a small amount, i.e. there are no discontinuities of $\int f \; dM_f(\mathcal{C})$ for changes in f.

Since f' is strictly increasing in S, there exists a $c > 0$ such that $cf' - h$ is an act with positive externalities. Define $g = cf' - h$. Note that $M_{cf'}(\mathcal{C}) = M_{f'}(\mathcal{C})$ and $m_{cf'}(\mathcal{C}) = m_{f'}(\mathcal{C})$, so (5.2) holds for the act cf' as well.

It holds that $h = cf' - g$ and $\Gamma_{cf',g}(S)$ is a decision problem with positive externalities and increasing differences due to Lemma 5.5. Lemma 5.3 implies that $\int h \; d\overline{P}_{cf',g}(\mathcal{C}) < 0 < \int h \; d\underline{P}_{cf',g}(\mathcal{C})$. This implies $\int cf' \; dM_{cf'}(\mathcal{C}) < \int g \; dM_g(\mathcal{C})$ and $\int cf' \; dm_{cf'}(\mathcal{C}) > \int g \; dm_g(\mathcal{C})$. Thus the optimal act is not increasing in optimism for $\Gamma_{cf',g}(\mathcal{C})$.

□

Proof Theorem 5.4. 2. $\implies$ 1.: Assume that 1. fails. Assume that the perceived ambiguities are represented by $[\mathcal{C}_1], \dots, [\mathcal{C}_N]$. Without loss ofgenerality assume that player 1 has exogenous perceived ambiguity $[\mathcal{C}_1]$ which is not of the PELSD type.

Consider the set $\mathcal{C}_1^{\bar{s}_{-1}}$ with $\bar{s}_{-1}$ being the highest strategy combination in S_{-i}. Lemma 5.7 implies the existence of some $\Gamma_{f,g}(S_{-1}) \in \Gamma_{peid}(S_{-1})$ such that the optimal act is decreasing in optimism, given $\mathcal{C}_1^{\bar{s}_{-1}}$. This implies

$$\int f \; dM_f(\mathcal{C}_1^{\bar{s}_{-1}}) < \int g \; dM_g(\mathcal{C}_1^{\bar{s}_{-1}}) \quad \text{and} \int f \; dm_f(\mathcal{C}_1^{\bar{s}_{-1}}) > \int g \; dm_g(\mathcal{C}_1^{\bar{s}_{-1}}).$$

Let $S_1 = (s_{11}, s_{12}, \dots, s_{1|S_1|})$ be the ordered strategy set of player 1. Construct the game Γ as follows: Define $s_{11}(s_{-1}) = f(s_{-1})$, $s_{12}(s_{-1}) = g(s_{-1})$ for all $s_{-1} \in S_{-1}$ and $s_{1j} = g(s_{1j}) - \epsilon$ with $\epsilon > 0$ for all $j \in \{3, \dots, |S_1|\}$ and for all $s_{-1} \in S_{-1}$. Thus s_{12} strictly dominates the strategies $s_{13}, \dots, s_{1N}$ such that these strategies do not play a role in determining the equilibrium.

For simplicity define the payouts of all other players to be a constant c. This game Γ is a game with positive externalities and increasing differences.

Given belief $\mathcal{C}_1^{\bar{s}_{-1}}$ and complete pessimism $\alpha_1 = 1$ for player 1, as well as arbitrary ambiguity attitudes for all other players, the highest EUA is

$$(\mathcal{C}_1^{\bar{s}_{-1}}, 1, \mathcal{C}_2^{\bar{s}_{-2}}, \alpha_2, \ldots, \mathcal{C}_N^{\bar{s}_{-N}}, \alpha_N),$$

i.e. the supports of the prior sets contain only the highest strategy combination of the other players.

Now, ceteris paribus, consider the case $\alpha_1 = 0$. Player 1 now has an incentive to deviate from s_{11} to s_{12} due to the way Γ is constructed. Thus the above EUA is not an EUA in this case. The highest equilibrium is thus not increasing in optimism for Γ.

2. $\Longrightarrow$ 1.: When all players have PELSD preferences then due to Lemma 5.7 there can never be an incentive to play a strategy when optimism increases. Thus the highest/lowest equilibrium must be increasing in optimism.

□

The following proof of Lemma 5.1 provides a much easier proof of the comparative statics result on ambiguity attitude of Eichberger and Kelsey (2014).

Proof of Lemma 5.1. It is sufficient to show that for $\mathcal{C} = Core(\nu)$ for some convex capacity $\nu : \mathcal{P}(S) \to [0, 1]$ the optimal act is decreasing in optimism for any $\Gamma_{f,g}(S) \in \Gamma_{peid}(S)$.

Recall that positive aggregate externalities implies that S is ordered. This in combination with increasing differences implies that the act $f - g$ has positive externalities. This implies that the acts f, g and $f - g$ are pairwise comonotonic.

It thus follows from the comonotonic independence axiom that

$$V(f|\mathcal{C}, \alpha) - V(g|\mathcal{C}, \alpha) = V(f - g|\mathcal{C}, \alpha)$$

for all $\alpha \in [0, 1]$. Since $V(f - g|\mathcal{C}, \alpha)$ is increasing in optimism $(1 - \alpha)$, there can never be an incentive to switch from the higher act f to the lower act g when optimism increases. This rules out the possibility that the optimal act is increasing in optimism. Thus cores of convex capacities are always of the PELSD type.

□

Proof of Theorem 5.5. Acts that are affinely related are always comonotonic. Furthermore due to increasing differences, for P_i and two strategies $s_{ij}, s_{ik} \in S_i$ with $s_{ij} > s_{ik}$, the act $s_{ij} - s_{ik}$ is comonotonic to s_{ij} and s_{ik}.

It thus holds that

$$V(s_{ij}|\mathcal{C}_i, \alpha_i) - V(s_{ik}|\mathcal{C}_i, \alpha_i) = V(s_{ij} - s_{ik}|\mathcal{C}_i, \alpha_i)$$

for all $(\mathcal{C}_i, \alpha_i)$. The term $V(s_{ij} - s_{ik}|\mathcal{C}_i, \alpha_i)$ is decreasing in α_i. Thus as α_i decreases there can never be an incentive to deviate to a lower strategy. Therefore, if $(\mathcal{C}_1, \alpha_1, \ldots, \mathcal{C}_N, \alpha_N)$ is the highest equilibrium of the game, a decrease in some α_i can never result in a lower highest equilibrium. Thus the highest equilibrium is increasing in optimism $1 - \alpha$. The same reasoning is used for the lowest equilibrium. □

6 Conclusion

This thesis aims to contribute to the literature on decision-making under ambiguity and the study of strategic interaction between players that perceived strategic ambiguity. Crucial questions of this field of research are:

What is perceived ambiguity? What is ambiguity attitude?

In our articles we suggest new approaches to these questions and apply them to games. The thesis can be separated into two parts. The first two articles are on axiomatic decision theory. We suggest a new conceptual framework for ambiguity aversion in the first article. We view this contribution as the strongest in this thesis. The results sparked strong interest amongst the audience at the 2018 Risk, Uncertainty and Decision (RUD) conference. Especially the characterizations of balanced and exact capacities were acknowledged by renowned decision theorists including Ithzak Gilboa, Massimo Marinacci and Peter Klibanoff (David Schmeidler acknowledged the result via email). Nonetheless there is a lot of work to be done. Whether the results are model-free and how to extend the approach to the Savage framework are open questions that we aim to tackle in the near future.

The second article is the newest and least exploited work. The most interesting contribution is the insight that a weaker set of axioms is sufficient for the SEU model in the framework of Anscombe and Aumann which implies that a separation of beliefs and tastes is a consequence rather than an assumption that needs to be made. Our aim for future research is to extend this result to the framework of Savage as well as figure out what consequences Weak Monotonicity has on other important preference classes.

The second part of the thesis is on perceived ambiguity and games. Our approach to perceived ambiguity in the multiple prior model is intuitive and unifies the existing measures of perceived ambiguity that we are aware of. The application of this concept to games provides a very general model capable of modelling a huge amount of phenomena. Critics may put forward that it is too general. We respond to this critique by pointing out that one can always add more assumptions on preferences, such as an axiom on monotonicity or some version of independence, to rule out undesirable consequences. The equilibrium existence result however holds for our general framework. The proof of that theorem is in our opinion the most interesting mathematical result of the thesis (and the idea of the proof rather beautiful).

The fourth article provides an application of our model and is a generalization of Eichberger and Kelsey (2014). Interestingly we find that more optimism does not necessarily lead to higher equilibria when preferences are α-MEU. The reason is that perceived ambiguity, reflected by the prior set, can intervene in this dynamic: for certain prior sets, a low strategy can *benefit more* from an increase in optimism than a higher strategy. We characterize the prior sets for which this phenomenon can occur, building on a new concept of first-order stochastic dominance for lattice structures.

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