Supplementary Material: Modelling Ecosystem Adaptation and Dangerous Rates of Global Warming

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S1 Solving the model analytically at constant temperature

The model can be solved analytically at equilibrium at a constant environmental temperature T_0, when the population distribution in the phase space defined by T_{opt} is approximately Gaussian. At equilibrium, the governing equation becomes

$$0 = \nu\left(G_{max}\left[1 - \frac{1}{2}\left(\frac{T_0 - T_{opt}}{T_w}\right)^2\right]\left(1 - N_{tot}\right) - 1\right) + \Lambda\frac{\partial^2 \nu}{\partial T_{opt}^2}. \tag{S1}$$

For simplicity, the axes can be translated by setting $x = T_0 - T_{opt}$, and grouping constants by defining $C = G_{max}(1 - N_{tot})$. The governing equation becomes

$$\frac{\partial^2 \nu}{\partial x^2} = \frac{\nu}{\Lambda}\left(1 - C\left[1 - \frac{x^2}{2T_w^2}\right]\right). \tag{S2}$$

Assuming the solution is a Gaussian, then it can be written in the form $\nu_g = \frac{A}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$, where A and σ are constants. The second order derivative of ν_g is

$$\frac{d^2\nu_g}{dx^2} = \left(-\frac{1}{\sigma^2} + \frac{x^2}{\sigma^4}\right)\nu_g. \tag{S3}$$

The constant σ can be found by correlating the coefficients of the independent variables in the function multiplying ν and ν_g on the right-hand side of the two governing equations, S2 and S3.

Matching the x^2 coefficients gives

$$\sigma^2 = \sqrt{\frac{2\Lambda T_\omega^2}{C}}. \tag{S4}$$

Matching the constant coefficients gives

$$\sigma^2 = \frac{\Lambda}{C - 1}. \tag{S5}$$

σ^2 can be found by setting equations S4 and S5 equal to each other. Rearranging gives Λ in terms of C:

$$\Lambda = \frac{2T_\omega^2(C-1)^2}{C}. \tag{S6}$$

Finding C in terms of Λ gives

$$C = 1 + \frac{\Lambda \pm \sqrt{8\Lambda T_\omega^2 + \Lambda^2}}{4T_\omega^2}. \tag{S7}$$

Substituting this into equation S5 gives

$$\sigma^2 = \frac{4T_\omega^2}{1 + \sqrt{\frac{8T_\omega^2}{\Lambda} + 1}}. \tag{S8}$$

We take the positive root so that the variance is positive. A can be chosen so that the integral of the solution covers the correct total area, $A = N_{tot}$. Using $N_{tot} = 1 - \frac{C}{G_{max}}$ and equation S5 to eliminate C,

$$A = \frac{\sigma^2(G_{max} - 1) - \Lambda}{\sigma^2 G_{max}}. \tag{S9}$$

Therefore, the approximate analytical solution is given by

$$\nu(T_{opt}) = \left(\frac{\sigma^2(G_{max} - 1) - \Lambda}{\sigma^3 G_{max}\sqrt{2\pi}} \right) \exp\left[-\frac{(T_0 - T_{opt})^2}{2\sigma^2} \right]. \tag{S10}$$

S2 Solving the model analytically for a linear temperature change

An analytical approximation to the solution can be found for a linear temperature change of the form $T = T_0 + \epsilon t$. The population distribution in the phase space defined by T_{opt} remains close to Gaussian for a low rate of temperature change, i.e. when ϵ is small. Therefore, an analytical solution can be found by approximating the distribution as a Gaussian travelling wave. This solution holds for the dynamic equilibrium found after the initial stages of change seen in figure 2.

For a linear temperature change, the governing equation is

$$\frac{\partial \nu}{\partial t} = \nu\left(G_{max}\left[1 - \frac{1}{2}\left(\frac{T_0 + \epsilon t - T_{opt}}{T_w} \right)^2 \right]\left(1 - N_{tot} \right) - 1 \right) + \Lambda \frac{\partial^2 \nu}{\partial T_{opt}^2}. \tag{S11}$$

For simplicity, constants can be grouped by defining $C = G_{max}(1 - N_{tot})$, and the independent variable can be grouped by defining $y(t, T_{opt} = T_0 + \epsilon t - T_{opt})$, leading to a simplified governing equation,

$$\frac{\partial^2 \nu}{\partial T_{opt}^2} - \frac{1}{\Lambda}\frac{\partial \nu}{\partial t} = \frac{\nu}{\Lambda}\left(1 - C\left[1 - \frac{y^2}{2T_w^2} \right] \right). \tag{S12}$$

Assuming the solution is a Gaussian travelling wave, then it can be written in the form $\nu_g = \frac{A}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-b)^2}{2\sigma^2}}$, where A, b and σ are constants. This solution can be substituted into the simplified governing equation to find the constants. The derivatives of ν_g are $\frac{\partial \nu_g}{\partial t} = -\epsilon \frac{y-b}{\sigma^2}\nu_g$ and $\frac{\partial^2 \nu_g}{\partial T_{opt}^2} = (-\frac{1}{\sigma^2} + \frac{(y-b)^2}{\sigma^4})\nu_g$. Substituting this into the left-hand side of equation S12 gives

$$\frac{\partial^2 \nu_g}{\partial T_{opt}^2} - \frac{1}{\Lambda}\frac{\partial \nu_g}{\partial t} = \nu_g\left(\frac{1}{\sigma^2}\left[\frac{\epsilon}{\Lambda}(y-b) - 1\right] + \frac{1}{\sigma^4}(y-b)^2\right). \tag{S13}$$

The constants b and σ can then be found by correlating the coefficients of the independent variables in the function multiplying ν and ν_g on the right-hand side of the two governing equations, S12 and S13.

Matching the y^2 coefficients gives

$$\sigma^2 = \sqrt{\frac{2\Lambda T_\omega^2}{C}}. \tag{S14}$$

Matching the y coefficients gives

$$b = \frac{\epsilon\sigma^2}{2\Lambda}. \tag{S15}$$

Matching the constant coefficients gives

$$\frac{C-1}{\Lambda} - \frac{1}{\sigma^2}\left(\frac{\epsilon}{\Lambda}b + 1\right) + \frac{b^2}{\sigma^4} = 0. \tag{S16}$$

Use equations S14 and S15 to substitute expressions for b and σ in equation S16:

$$\frac{C-1}{\Lambda} - \sqrt{\frac{C}{2\Lambda T_\omega^2}} + \frac{\epsilon^2}{\Lambda^2}\left(\frac{1}{4} - \frac{1}{2}\right) = 0. \tag{S17}$$

This is a polynomial in $C^{\frac{1}{2}}$. Finding $C^{\frac{1}{2}}$ in terms of Λ gives

$$C^{\frac{1}{2}} = \sqrt{\frac{\Lambda}{8T_\omega^2}} \pm \sqrt{\frac{\Lambda}{8T_\omega^2} + \frac{\epsilon^2}{4\Lambda} + 1}. \tag{S18}$$

Substituting this into equation S14 gives a variance

$$\sigma^2 = \frac{4T_\omega^2}{1 \pm \sqrt{1 + \frac{2T_\omega^2\epsilon^2}{\Lambda^2} + \frac{8T_\omega^2}{\Lambda}}}. \tag{S19}$$

We take the positive root so that the variance is consistent with the form found for a constant environmental temperature. A can be chosen so that the integral of the solution covers the correct total area, $A = N_{tot}$. Using $N_{tot} = 1 - \frac{C}{G_{max}}$ and equations S16 and S15 to eliminate C and b,

$$A = 1 - \frac{1}{G_{max}}\left(\frac{\epsilon^2}{4\Lambda} + \frac{\Lambda}{\sigma^2} + 1\right). \tag{S20}$$

Therefore, the approximate analytical solution is given by

$$\nu(T_{opt}) = \frac{1}{G_{max}\sqrt{2\pi\sigma^2}}\left(G_{max} - \frac{\epsilon^2}{4\Lambda} - \frac{\Lambda}{\sigma^2} - 1\right)\exp\left[\frac{-\left(T_0 + \epsilon\tau - T_{opt} - \frac{\epsilon\sigma^2}{2\Lambda}\right)^2}{2\sigma^2}\right]. \tag{S21}$$

S3 Solving the model numerically

To solve the model numerically, the governing equation, equation 2, must first be discretised. The model can be discretised along the trait axis, by dividing the axis up and sampling at n points. The second order derivative can then be written using a centred in space scheme. The governing equation for the i^{th} point is therefore

$$\frac{\partial \nu_i}{\partial t} = \nu_i\left(g_i(T_{opt,i})\Big(1 - \underbrace{h\sum_{i=0}^{n}\nu_i}_{N_{tot}}\Big) - \gamma\right) + \frac{\lambda}{h^2}\Big(\big(\nu_{i+1} - \nu_i\big) - \big(\nu_i - \nu_{i-1}\big)\Big), \tag{S22}$$

where h is the spacing between T_{opt} of adjacent sampling points. A full solution can then be found using the Runge-Kutta 4^{th} order algorithm.