Renormalization in Piecewise Isometries

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I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

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Abstract

Interval exchange transformations (IET) are bijective piecewise translations of an interval divided into a finite partition of subintervals. *Piecewise isometries* (PWIs) are generalizations of IETs to higher dimension where a region is split into a number of convex sets and these are rearranged using isometries. Although PWIs are higher dimensional generalizations of IETs, their generic dynamical properties seem to be quite different. In this thesis we consider embeddings of IETs into PWIs in order to understand their similarities and differences.

We investigate *translated cone exchange transformations*, a new family of piecewise isometries and renormalize its first return map to a subset of its partition. As a consequence we show that the existence of an embedding of an interval exchange transformation into a map of this family implies the existence of infinitely many bounded invariant sets. We also prove the existence of infinitely many periodic islands, accumulating on the real line, as well as non-ergodicity of our family of maps close to the origin.

We derive some necessary conditions for existence of embeddings using combinatorial, topological and measure theoretic properties of IETs. In particular, we prove that continuous embeddings of minimal 2-IETs into orientation preserving PWIs are necessarily trivial and that any 3-PWI has at most one non-trivially continuously embedded minimal 3-IET with the same underlying permutation. Furthermore, we introduce a family of 4-PWIs with apparent abundance of invariant nonsmooth fractal curves supporting IETs, that limit to a trivial embedding of an IET.

Finally, we prove that almost every interval exchange transformation, with an associated translation surface of genus $g \geq 2$, can be non-trivially and isometrically embedded into a family of piecewise isometries. In particular, this proves the existence of invariant curves for piecewise isometries, reminiscent of KAM curves for area preserving maps, which are not unions of circle arcs or line segments.

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Semper nobis viam invenire.

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Chapter 1

Introduction

In 1888 the French mathematician Henri Poincaré submitted a famous memoir [47], to the *Acta Mathematica*, for a prized competition in honour of King Oscar II of Sweden and Norway. In his paper he made a remarkable contribution towards the understanding of Hamiltonian systems with practical implications extending to celestial mechanics, particularly, the stability of the solar system (see [28]). His original work, however, contained a fundamental error which he would only later correct. This gave rise to the first published example of chaotic behaviour in a deterministic system, giving birth to the field of *Dynamical Systems*.

1.1 Dynamical systems, chaos and hyperbolicity

Although time evolving systems have been studied for hundreds of years, since mathematics was first used to model the dynamics of the surrounding natural phenomena, it was in the 19th century that Poincaré pioneered the qualitative theory of ordinary differential equations realizing for the first time that even simple deterministic systems could give rise to very complex behaviour. It is the main endeavour of the theory of dynamical systems, to formalize and explain this complexity.

A dynamical system is a formalization of a law describing the time evolution of a point in an underlying space. Time can be considered continuous or discrete, which leads to the description of different families of dynamical systems. In discrete time a *dynamical system*, on a space X, is a map $f : X \to X$. Hence, in this case, the dynamical properties of a system can be studied by understanding the repeated iteration of the map f.

Defining, measuring and understanding the mechanisms causing the emergence of complex behaviour is central to the field of dynamical systems. The term *chaos* is commonly used to describe such behaviour, but it actually encompasses a number of different definitions. One of the most useful is called *Devaney chaos*.

Devaney's definition of chaos [27]. Let (X, d) be a metric space. A map $f : X \to X$ is said to be Devaney-chaotic on X if it satisfies the following conditions:
(1) Sensitive dependence on initial conditions: There exists a $\delta > 0$ such that, for any $x \in X$ and $\epsilon > 0$, there exist some $y \in X$ satisfying $d(x, y) < \epsilon$ and $m \in \mathbb{N}$ such that $d(f^m(x), f^m(y)) > \delta$.
(2) Topological transitivity: For any pair of open sets $U, V \subset X$ there exists an $m \in \mathbb{N}$ such that $f^m(U) \cap V \neq \emptyset$.
(3) The set of periodic orbits is dense in X.

Other definitions include *Lyapunov chaos* (the existence of a positive *Lyapunov exponent*) and *topological chaos* (positive *topological entropy*). In fact, Lyapunov exponents measure the exponential rate of divergence of orbits of nearby points, while topological entropy measures the asymptotic exponential growth rate of distinguishable orbits as iterations of a map tend to infinity.

The latter definitions are fitting tools to characterize *hyperbolic dynamics*. Informally, hyperbolicity in a smooth dynamical system is characterized by the existence of expanding and contracting directions for its derivative. The presence of these directions leads, as time evolves, to an exponential deviation between orbits of nearby points. The resulting stretching and folding of the phase space gives rise to complex long term behaviour in such systems.

1.2 Renormalization in dynamical systems

Hyperbolic dynamics may be, perhaps, the best studied sub-field of dynamical systems, however, is it possible to observe complex behaviour in non-hyperbolic systems, with both Lyapunov exponents and topological entropy equal to zero?

The answer is in fact, yes! In fact conditions (1) and (2) in the definition of Devaney chaos do not imply the existence of any sort of hyperbolicity and indeed there are such systems which still exhibit topological transitivity and sensitivity to initial conditions. A natural question that arises is then what is the mechanism causing the emerging complexity in such systems? To answer this question we need to focus on the most powerful tool we have to study such dynamical systems: *renormalization*.

Informally, renormalization is the study of the self-similarity of a system at different spatial scales.

Quoting Artur Avila on his survey [13] titled "Dynamics of Renormalization Operators":

> *"It is a remarkable characteristic of some classes of low-dimensional dynamical systems that their long time behaviour at a short spatial scale is described by an induced dynamical system in the same class. The renormalization operator that relates the original and the induced transformations can then be iterated, and a basic theme is that certain features (such as hyperbolicity, or the existence of an attractor) of the resulting 'dynamics in parameter space' impact the behaviour of the underlying systems."*

Consider a family of dynamical systems $\mathcal{F} = \{f_\mu : X \to X\}$ parametrized by $\mu \in \mathcal{P}$, where $\mathcal{P}$ is called the parameter space of $\mathcal{F}$. A *renormalization scheme* for $\mathcal{F}$ is a decreasing chain of subsets of X,

$$X = Y_0(\mu) \supset Y_1(\mu) \supset Y_2(\mu) \supset ...,$$

together with a renormalization operator $\mathcal{R} : \mathcal{P} \to \mathcal{P}$ such that the first return map of a point in $Y_{n+1}(\mu)$ under iteration by $f_{\mathcal{R}^n(\mu)} : Y_n(\mu) \to Y_n(\mu)$ is given by $f_{\mathcal{R}^{n+1}(\mu)} : Y_{n+1}(\mu) \to Y_{n+1}(\mu)$.

In general, renormalizable dynamical systems are not Lyapunov chaotic. The reason for this is that after renormalization, each iteration corresponds to several iterations of the original map. In this way if a Lyapunov exponent were positive, it would increase after each successive induction and eventually diverge. Therefore this cannot happen in a renormalizable system. In contrast with the dynamics of the underlying renormalizable map, the renormalization dynamics itself tends to display hyperbolicity, which allows for the use of strong techniques from ergodic theory to aid in its study.

Renormalization can be a powerful tool in the study of nonlinear maps (see [13]), such as diffeomorphisms of the circle [51], one-frequency Schrödinger cocycles [14] and analytic unimodal maps [22]. It can also be a useful concept in the absence of continuity of the map, indeed, an example of this is given by *interval exchange transformations.*

An *interval exchange transformation (IET)* is a bijective piecewise order preserving isometry f of an interval $I \subset \mathbb{R}$. Specifically I is partitioned into subintervals $\{I_\alpha\}_{\alpha \in \mathcal{A}}$, indexed over a finite alphabet $\mathcal{A}$ of $d \geq 2$ symbols, so that the restriction of f to each subinterval is a translation. An IET f is determined by a vector

$\lambda \in \mathbb{R}_+^{\mathcal{A}}$, with coordinates λ_α determining the lengths of the subintervals I_α, and a permutation π which describes the ordering of the subintervals before and after applying f.

IETs were defined by Keane [37] and studied for instance in [4, 18, 29, 54, 55]. Masur and Veech [42, 54] established unique ergodicity of typical IETs while Avila and Forni [15] showed that a typical IET is either weakly mixing or an irrational rotation. It is known that IETs (and suspension flows over IETs with roof function of bounded variation) are not strongly mixing [23, 36].

A *translation surface* (as defined in [15]), is a surface with a finite number of conical singularities endowed with an atlas such that coordinate changes are given by translations in $\mathbb{R}^2$. Given an IET it is possible to associate, via a suspension construction, a translation surface, with genus only depending on the combinatorial properties of the underlying IET (see [54]). Indeed these maps are deeply related to geodesic flows on flat surfaces, Teichmüller flows in moduli spaces of Abelian differentials and polygonal billiards [42].

Another important example of the power of renormalization in the absence of non-linearity is that of *Piecewise Isometries* (PWIs), higher dimensional generalizations of IETs. The subject of this thesis is the study of the dynamics of PWIs with emphasis on renormalization and their relation with IETs.

1.3 Background on interval exchange transformations

We recall some notions of the theory of interval exchange transformations following [17], [53] and [56].

1.3.1 Definition

As in [17, 56], let $\mathcal{A}$ be an alphabet on $d \geq 2$ symbols, and let $I \subset \mathbb{R}$ be an interval having 0 as left endpoint. In what follows we use the notation $\mathbb{R}^{\mathcal{A}} \simeq \mathbb{R}^d$ and $\mathbb{R}_+^{\mathcal{A}} \simeq \mathbb{R}_+^d$. We choose a partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ of I into subintervals which we assume to be closed on the left and open on the right. An *interval exchange transformation* (IET) is a bijection of I defined by two data

(1) A vector $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}_+^{\mathcal{A}}$ with coordinates corresponding to the lengths of the subintervals, that is, for all $\alpha \in \mathcal{A}$, $\lambda_\alpha = |I_\alpha|$. We write $I = I(\lambda) = [0, |\lambda|)$, where $|\lambda| = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha$.

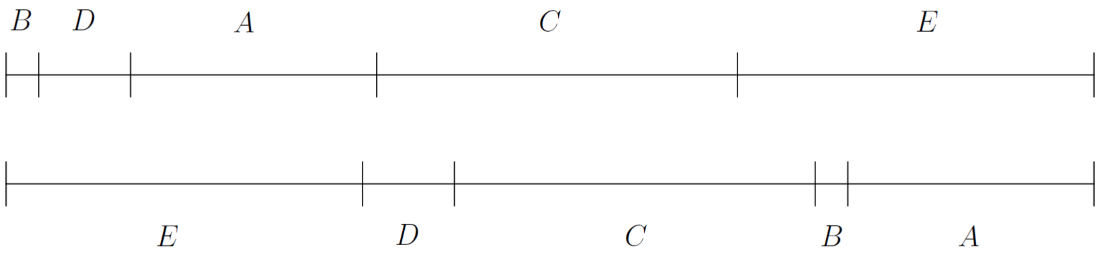


Figure 1.1: An illustrative depiction of the action of a 5-IET, with $\pi_0(\{A,B,C,D,E\}) = \{3,1,4,2,5\}$ and $\pi_1(\{A,B,C,D,E\}) = \{5,4,3,2,1\}$ on the interval.

(2) A pair $\pi = \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix}$ of bijections $\pi_\varepsilon : \mathcal{A} \to \{1, ..., d\}$, $\varepsilon = 0, 1$, describing the ordering of the subintervals I_α before and after the application of the map. This is represented as

$$\pi = \begin{pmatrix} \alpha_1^0 & \alpha_2^0 & ... & \alpha_d^0 \\ \alpha_1^1 & \alpha_2^1 & ... & \alpha_d^1 \end{pmatrix}.$$

We call π a *permutation* and identify it, at times, with its *monodromy invariant* $\tilde{\pi} = \pi_1 \circ \pi_0^{-1} : \{1, ...d\} \to \{1, ...d\}$. We denote by $\mathfrak{S}(\mathcal{A})$ the set of irreducible permutations, that is $\pi \in \mathfrak{S}(\mathcal{A})$ if and only if $\tilde{\pi}(\{1, ..., k\}) \neq \{1, ..., k\}$ for $1 \leq k < d$.

Define a linear map $\Omega_\pi : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ by

$$(\Omega_\pi(\lambda'))_{\alpha \in \mathcal{A}} = \sum_{\pi_1(\beta) < \pi_1(\alpha)} \lambda'_\beta - \sum_{\pi_0(\beta) < \pi_0(\alpha)} \lambda'_\beta. \tag{1.3.1}$$

Given a permutation $\pi \in \mathfrak{S}(\mathcal{A})$ and $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ the interval exchange transformation associated with this data is the map $f_{\lambda,\pi}$ that rearranges I_α according to π, that is

$$f_{\lambda,\pi}(x) = x + \upsilon_\alpha, \tag{1.3.2}$$

for any $x \in I_\alpha$, where $\upsilon_\alpha = (\Omega_\pi(\lambda))_\alpha$. We write $f = f_{\lambda,\pi}$ and also denote an IET by the pair $(I, f_{\lambda,\pi})$.

1.3.2 Rauzy induction

We will assume throughout the rest of this thesis that (λ, π) satisfies the *infinite distinct orbit condition (IDOC)*, first introduced by Keane in [37]. The pair (λ, π) satisfies the IDOC if the orbits of the endpoints of the subintervals $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ are as disjoint as possible

$$f_{\lambda,\pi}^n\left(\sum_{\pi_0(\varsigma) < \pi_0(\alpha)} \lambda_\varsigma\right) \neq \sum_{\pi_0(\varsigma) < \pi_0(\beta)} \lambda_\varsigma,$$

for all $n \geq 1$ and $\alpha, \beta \in \mathcal{A}$ with $\pi_0(\beta) \neq 1$. In particular the IDOC implies minimality of $f_{\lambda,\pi}$, that is, every orbit is dense in the interval.

We define *Rauzy induction* (also known as *Rauzy-Veech induction*) as in [56]. Let $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A})$. For $\varepsilon = 0, 1$, denote by β_ε the last symbol in the expression of π_ε, that is

$$\beta_\varepsilon = \pi_\varepsilon^{-1}(d) = \alpha_d^\varepsilon. \tag{1.3.3}$$

Assume the intervals I_{β_0} and I_{β_1} have different lengths. We say that (λ, π) is of *type* 0 if $\lambda_{\beta_0} > \lambda_{\beta_1}$ and is of *type* 1 if $\lambda_{\beta_0} < \lambda_{\beta_1}$. The largest interval is called *winner* and the smallest *loser* of (λ, π). Let $I^{(1)}$ be the interval obtained by removing the loser from $I(\lambda)$:

$$I^{(1)} = [0, |\lambda| - \min(|\lambda_{\beta_0}|, |\lambda_{\beta_1}|)\}) \,. \tag{1.3.4}$$

The first return map of $f_{\lambda,\pi}$ to the subinterval $I^{(1)}$ is again an IET, $f_{\lambda^{(1)},\pi^{(1)}}$, where the parameters $(\lambda^{(1)}, \pi^{(1)})$ are defined as follows. If (λ, π) is of type 0 then

$$\pi^{(1)} = \begin{pmatrix} \pi_0^{(1)} \\ \pi_1^{(1)} \end{pmatrix} = \begin{pmatrix} \alpha_1^0 & \dots & \alpha_{k-1}^0 & \alpha_k^0 & \alpha_{k+1}^0 & \dots & & \dots & \beta_0 \\ \alpha_1^1 & \dots & \alpha_{k-1}^1 & \beta_0 & \beta_1 & & \alpha_{k+1}^1 & \dots & \alpha_{d-1}^1 \end{pmatrix}. \tag{1.3.5}$$

where $k \in \{1, ..., d-1\}$ is defined by $\alpha_k^1 = \beta_0$, and $\lambda^{(1)} = (\lambda_\alpha^{(1)})_{\alpha \in \mathcal{A}}$, where

$$\lambda_\alpha^{(1)} = \lambda_\alpha \text{ for } \alpha \neq \beta_0, \text{ and } \lambda_{\beta_0}^{(1)} = \lambda_{\beta_0} - \lambda_{\beta_1}.$$

If (λ, π) is of type 1 then

$$\pi^{(1)} = \begin{pmatrix} \pi_0^{(1)} \\ \pi_1^{(1)} \end{pmatrix} = \begin{pmatrix} \alpha_1^0 & \dots & \alpha_{k-1}^0 & \beta_1 & \beta_0 & & \alpha_{k+1}^0 & \dots & \alpha_{d-1}^0 \\ \alpha_1^1 & \dots & \alpha_{k-1}^1 & \alpha_k^1 & \alpha_{k+1}^1 & \dots & & \dots & \beta_1 \end{pmatrix}. \tag{1.3.6}$$

where $k \in \{1, ..., d-1\}$ is defined by $\alpha_k^0 = \beta_1$, and $\lambda^{(1)} = (\lambda_\alpha^{(1)})_{\alpha \in \mathcal{A}}$, where

$$\lambda_\alpha^{(1)} = \lambda_\alpha \text{ for } \alpha \neq \beta_1, \text{ and } \lambda_{\beta_1}^{(1)} = \lambda_{\beta_1} - \lambda_{\beta_0}. \tag{1.3.7}$$

The map $\mathcal{R} : \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A}) \to \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A})$ defined by $\mathcal{R}(\lambda, \pi) = (\lambda^{(1)}, \pi^{(1)})$ is called the *Rauzy induction map.*

The IDOC condition assures that the iterates $\mathcal{R}^n$ are defined for all $n \geq 0$. We denote

$$\mathcal{R}^n(\lambda, \pi) = (\lambda^{(n)}, \pi^{(n)}), \tag{1.3.8}$$

with $\pi^{(n)} = (\; \pi_0^{(n)} \quad \pi_1^{(n)} \;)^T$. Furthermore we denote by $\beta_{\varepsilon,n}$ the last symbol in the expression of $\pi_\varepsilon^{(n)}$, by $\varepsilon(n)$ the type of $f_{\lambda^{(n)},\pi^{(n)}}$, by $I^{(n)}$ its domain and by $\{I_\alpha^{(n)}\}_{\alpha \in \mathcal{A}}$ its partition in subintervals, for $n \geq 0$. We also denote the translation vector of $f_{\lambda^{(n)},\pi^{(n)}}$ by $\upsilon^{(n)} = \Omega_{\pi^{(n)}}(\lambda^{(n)})$.

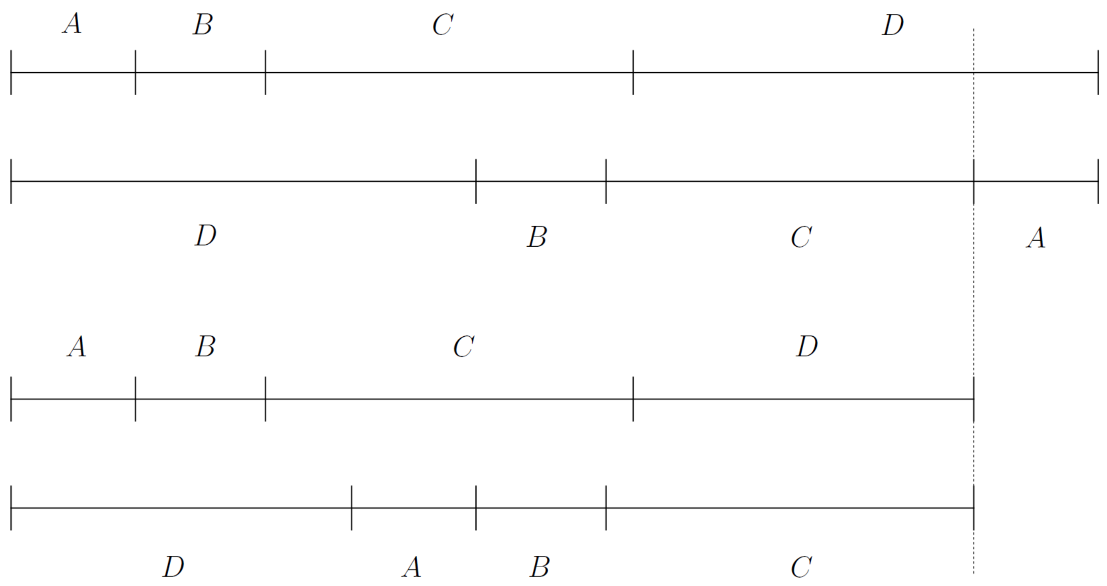


Figure 1.2: A schematic illustration of Rauzy-Veech induction on a 4-IET described by a pair (λ, π) of type 0 (depicted on top), as $\lambda_D > \lambda_A$. In the bottom we can see the IET, obtained by Rauzy-Veech induction, which is described by parameters $\lambda^{(1)} = (\lambda_A, \lambda_B, \lambda_C, \lambda_D - \lambda_A)$ and $\pi^{(1)}$ given by (1.3.5).

1.3.3 Rauzy classes

The *Rauzy class* (see [56]) of a permutation $\pi \in \mathfrak{S}(\mathcal{A})$, is the set $\mathfrak{R}(\pi)$ of all $\pi^{(1)} \in \mathfrak{S}(\mathcal{A})$ such that there exist $\lambda, \lambda^{(1)} \in \mathbb{R}_+^{\mathcal{A}}$ and $n \in \mathbb{N}$ such that $\mathcal{R}^n(\lambda, \pi) = (\lambda^{(1)}, \pi^{(1)})$. A Rauzy class $\mathfrak{R}$ can be visualized in terms of a directed labelled graph, the *Rauzy graph* (see [53]). Its vertices are in bijection with $\mathfrak{R}$ and it is formed by edges that connect permutations which are obtained one from another by (1.3.5) and (1.3.6) labeled respectively by 0 or 1 according to the type of the induction. A *path* $\varrho = (\varrho_1, ..., \varrho_n)$ is a sequence of compatible edges of the Rauzy graph, that is, such that the starting vertex of ϱ_{i+1} is the ending vertex of ϱ_i, $i = 1, ..., n-1$. We say a path is *closed* if the starting vertex of ϱ_1 is the ending vertex of ϱ_n. The set of all paths in this graph is denoted by $\Pi(\mathfrak{R})$.

1.3.4 Rauzy cocycle

We define the *Rauzy cocycle* as in [17]. Let $(\mathfrak{X}, \mu)$ be a probability space. A linear cocycle is a pair (T, A), where $T : \mathfrak{X} \to \mathfrak{X}$ and $A : \mathfrak{X} \to GL(p, \mathbb{R})$, viewed as a linear skew-product $(x, v) \mapsto (T(x), A(x) \cdot v)$ on $\mathfrak{X} \times \mathbb{R}^p$, $p \in \mathbb{N}$. Notice that $(T, A)^n = (T^n, A^{(n)})$, where

$$A^{(n)}(x) = A(T^{n-1}(x)) \cdot ... \cdot A(x), \quad n \geq 0.$$

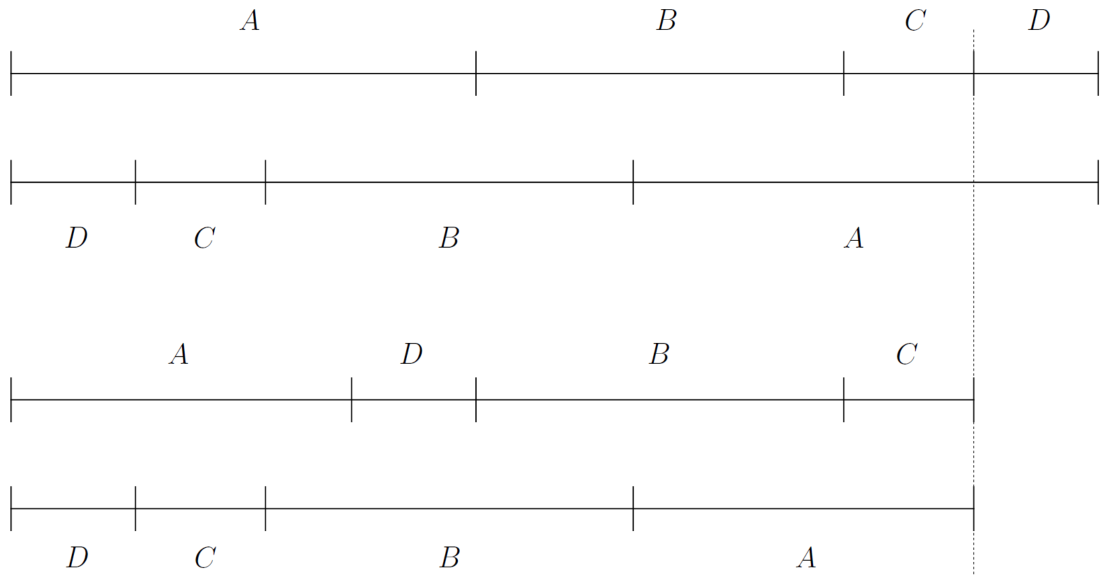


Figure 1.3: A schematic illustration of Rauzy-Veech induction on a 4-IET described by a pair (λ, π) of type 1 (depicted on top), as $\lambda_D > \lambda_A$. In the bottom we can see the IET, obtained by Rauzy-Veech induction, which is described by parameters $\lambda^{(1)} = (\lambda_A - \lambda_D, \lambda_B, \lambda_C, \lambda_D)$ and $\pi^{(1)}$ given by (1.3.6).

In what follows, we use the notation $SL(\mathcal{A}, \mathbb{Z}) \simeq SL(d, \mathbb{Z})$. Let $\|\cdot\|$ denote a matrix norm on $SL(\mathcal{A}, \mathbb{Z})$. Let $\log^+ y = \max\{\log(y), 0\}$ for any $y > 0$. If μ is an ergodic probability measure for T and

$$\int_{\mathfrak{X}} \log^+ \|A(x)\| d\mu(x) < +\infty,$$

we say (T, A) is a measurable cocycle.

Denote by $E_{\alpha\beta}$ the elementary matrix $(\delta_{i\alpha}\delta_{j\beta})_{i\geq 1, j\leq d}$ and let $\mathfrak{R} \subseteq \mathfrak{S}(\mathcal{A})$ be a Rauzy class. To any given path $\varrho \in \Pi(\mathfrak{R})$ we associate a matrix $B_P(\varrho) \in SL(\mathcal{A}, \mathbb{Z})$, defined inductively as follows

i) If ϱ is an edge labeled by 0, set $B_P(\varrho) = \mathbf{1}_d + E_{\beta_1\beta_0}$, with $\mathbf{1}_d$ denoting the $d \times d$ identity matrix;

ii) If ϱ is an edge labeled by 1, set $B_P(\varrho) = \mathbf{1}_d + E_{\beta_0\beta_1}$;

iii) If $\varrho = (\varrho_1, ..., \varrho_n)$, set $B_P(\varrho) = B_P(\varrho_n)...B_P(\varrho_1)$.

We denote by $\varrho(\lambda, \pi) \in \Pi(\mathfrak{R}(\pi))$, the edge in the Rauzy graph starting at π labeled by the type of (λ, π).

Define the function $B_R : \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R} \to SL(\mathcal{A}, \mathbb{Z})$ such that $B_R(\lambda, \pi) = B_P(\varrho(\lambda, \pi))$. The *Rauzy cocycle* is the linear cocycle over the Rauzy induction $(\mathcal{R}, B_R)$ on $\mathbb{R}_+^{\mathcal{A}} \times$

$\mathfrak{R} \times \mathbb{R}^{\mathcal{A}}$. Note that $(\mathcal{R}, B_R)^n = (\mathcal{R}^n, B_R^{(n)})$, where

$$B_R^{(0)}(\lambda,\pi) = \mathbf{1}_d, \quad B_R^{(n)}(\lambda,\pi) = B_R(\lambda^{(n-1)},\pi^{(n-1)}) \cdot \ldots \cdot B_R(\lambda^{(1)},\pi^{(1)}) \cdot B_R(\lambda,\pi), \tag{1.3.9}$$

for $n \geq 1$, with $(\lambda^{(n)},\pi^{(n)})$ as in (1.3.8). Note that, we have

$$\lambda = \left(B_R^{(n)}(\lambda,\pi)\right)^* \cdot \lambda^{(n)},$$

for all $n \geq 0$, where * denotes the transpose operator.

We now stress an important property of the Rauzy cocycle (see [56]). For any $n \geq 0$ and $x \in I^{(n)}$, let $r^n_{\lambda,\pi}(x)$ denote the first return time of x by $f_{\lambda,\pi}$ to $I^{(n)}$. Note that $r^n_{\lambda,\pi}(x)$ is constant on each $I^{(n)}_\alpha$, for any $\alpha \in \mathcal{A}$. We write $r^n_{\lambda,\pi}(I^{(n)}_\alpha)$ to mean $r^n_{\lambda,\pi}(x)$, for any $x \in I^{(n)}_\alpha$.

Each entry $\left(B_R^{(n)}(\lambda,\pi)\right)_{\alpha,\beta}$ of the matrix $B_R^{(n)}(\lambda,\pi)$ counts the number of visits of $I^{(n)}_\alpha$ to the interval I_β up to the $r^n_{\lambda,\pi}(I^{(n)}_\alpha)$-th iterate of $f_{\lambda,\pi}$. That is for every $\alpha,\beta \in \mathcal{A}$ and every $n \geq 1$,

$$\left(B_R^{(n)}(\lambda,\pi)\right)_{\alpha,\beta} = \text{card}\left\{0 \leq j < r^n_{\lambda,\pi}(I^{(n)}_\alpha) : f_{\lambda,\pi}(I^{(n)}_\alpha) \subset I_\beta\right\}.$$

1.4 Renormalization in piecewise isometries

One of the central problems in dynamical systems is to investigate renormalization of certain classes of maps. In this thesis we study piecewise isometries with emphasis on renormalization of a family of these maps and also use techniques from the theory of renormalization of IETs to solve a problem in the dynamics of piecewise isometries.

Piecewise isometries (PWIs) are higher dimensional generalizations of one dimensional interval exchange transformations. Both IETs and PWIs arise in a number of applications. For example, PWIs in two dimension have been found in models used for signal processing and digital filters [7, 25, 26, 38], for Hamiltonian systems [49, 50], for printing processes [2] or for other types of geometric dynamics [48]. PWIs exhibit complex and diverse dynamical behaviour that is far less understood than, and more sophisticated than that of IETs. There are many results that suggest generic choices of parameters for IETs give ergodicity while many examples suggest that this is rarely the case for PWIs in dimension two or more.

Piecewise isometries have been defined on higher dimensional spaces and Riemannian manifolds (see [8, 33]). In this thesis we consider orientation preserving planar piecewise isometries with respect to the standard euclidean metric, which we now define as follows. Let X be a subset of $\mathbb{C}$ and $\mathcal{P} = \{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a finite partition of X into convex sets (or *atoms*), that is $\bigcup_{\alpha\in\mathcal{A}} X_\alpha = X$ and $X_\alpha \cap X_\beta = \emptyset$ for $\alpha \neq \beta$. Given a *rotation vector* $\theta \in \mathbb{T}^{\mathcal{A}}$ (with $\mathbb{T}^{\mathcal{A}}$ denoting the torus $\mathbb{R}^{\mathcal{A}}/2\pi\mathbb{Z}^{\mathcal{A}}$) and a *translation vector* $\eta \in \mathbb{C}^{\mathcal{A}}$, we say (X, T) is a *piecewise isometry* if T is such that

$$T(z) := T_\alpha(z) = e^{i\theta_\alpha} z + \eta_\alpha, \text{ if } z \in X_\alpha, \tag{1.4.1}$$

so that T is a piecewise isometric rotation or translation (see [30]).

PWIs occur naturally in the dynamics of Hamiltonian systems with periodic kicks [41, 50] as well as outer billiards [48]. Many examples of PWIs have been studied in recent years; for example, in [19], the authors studied a class of piecewise rotations on the square and numerically computed box-counting dimensions, correlation dimensions and complexity of the symbolic language produced by the system. Adler, Kitchens and Tresser [1] investigated a specific class of nonergodic piecewise affine maps of the torus and gave a decomposition into three invariant sets whose dynamics is very different. They showed that the map on one of these invariant set is minimal, uniquely ergodic and an odometer; they also demonstrated the existence of a full Lebesgue measure set of periodic points. In [40] the authors studied the renormalizability of a one-parameter family of piecewise isometries of a rhombus with a fixed rotational component. It was proved by Buzzi [20] that piecewise isometries have zero topological entropy.

In general, for a given PWI it is helpful to define a partition of X into a *regular* and an *exceptional set* [10]. If we consider the zero measure set given by the union $\mathcal{E}$ of all preimages of the set of discontinuities D, then we say its closure $\overline{\mathcal{E}}$ (which may be of positive measure) is called the *exceptional set* for the map. The complement of the exceptional set is called the *regular set* for the map and consists of disjoint polygons or disks that are periodically coded by their itinerary through the atoms of the PWI. As an example, in [1] the authors show, for a particular transformation where the rotations are rational, that the regular set has full Lebesgue measure and as a consequence, the exceptional set has zero Lebesgue measure. However as highlighted in [8] there is numerical evidence that the exceptional set may have positive Lebesgue measure for typical PWIs. In [34], the author shows that this is the case for certain rectangle-exchange transformations.

Even when the exceptional set has positive Lebesgue measure, as noted in [10] there is numerical evidence that Lebesgue measure on the exceptional set may not be ergodic - there can be invariant curves that prevent trajectories from spreading across the whole of the exceptional set. In the same paper, a planar PWI whose generating map is a permutation of four cones was investigated, and coexistence of an infinite number of periodic components and of an uncountable number of transitive components was proved. On these transitive components it was noted that the dynamics is conjugate to a transitive interval exchange. In [5, 10], similar maps were examined and the existence of a large number of these invariant curves, apparently nowhere smooth, are investigated.

This suggests that renormalization in a general family of PWIs should be connected to that of IETs. Although renormalization of IETs has been well studied over the past years, renormalization of PWIs is still far from understood.

In [1] Adler, Kitchens and Tresser find renormalization operators for three rational rotation parameters for a non ergodic piecewise affine map of the Torus. Lowenstein and Vivaldi [40] gave a computer assisted proof of the renormalization of a family of piecewise isometries of a rhombus with one translation parameter and a fixed rational rotation parameter. They show that recursive constructions of first-return maps on an appropriate sub-domain produce a scaleddown replica of this domain, but described by a parameter given by a renormalization operator, related to a map of generalised Lüroth type, a piecewise-affine version of Gauss map. Their renormalization process is organized by a graph, particularly there are ten distinct renormalization scenarios corresponding to as many closed circuits in this graph. These results however rely on fixing the rotation component such that its coefficients belong to a quadratic number field in order to perform computer assisted calculations. Recently, Hooper [35] investigated a two dimensional parameter space of polygon exchange maps, a family of PWIs with no rotation, invariant under a renormalization operation, related to corner percolation and Truchet tillings, where each map admits a return map affinely conjugate to a map in the same family. In [3] the authors show how to construct minimal rectangle exchange maps, associated to Pisot numbers, using a cut-and-project method and prove that these maps are renormalizable. The maps described in these papers are PWIs with no rotational component, exhibiting very particular behaviour among typical PWIs, making it difficult to generalize their techniques.

In this thesis we study the dynamics of piecewise isometries using renormalization techniques. In particular we introduce a new notion of renormalization to study a class of PWIs called Translation Cone Exchange Transformations. We also introduce the notion of embedding IETs into PWIs and use IET renormalization techniques to establish the existence of invariant curves for PWIs which are not the union of line segments or circle arcs.

1.4.1 Translated cone exchange transformations

In Chapter 2 of this thesis, we introduce and renormalize a particular family of PWIs - *Translated Cone Exchange Transformations (TCEs)*.

Set $\omega = (\omega_1, ..., \omega_d) \in \mathbb{W}$, where $\mathbb{W}$ is the open polytope defined by

$$\mathbb{W} = \left\{ \omega \in \mathbb{R}_+^d : 0 < \sum_{j=1}^{d} \omega_j < \pi \right\}. \tag{1.4.2}$$

Note that we have

$$\vartheta = \frac{\pi}{2} - \frac{|\omega|}{2}, \tag{1.4.3}$$

where $|\omega|$ is the ℓ_1 norm of ω.

To introduce the family of TCEs, consider a partition of the upper half plane $\mathbb{H}$ into $d+2$ cones

$$\mathcal{P} = \{P_0, P_1, \ldots, P_d, P_{d+1}\},$$

where $P_j = \{z \in \mathbb{C} : \arg(z) \in W_j\}$, and W_j for $j = 0, \ldots, d+1$ are defined as

$$W_j = \begin{cases} [0, \vartheta), & \text{for } j = 0, \\ [\vartheta, \vartheta + \omega_1], & \text{for } j = 1, \\ (\vartheta + \sum_{k=1}^{j-1} \omega_k, \vartheta + \sum_{k=1}^{j} \omega_k], & \text{for } j \in \{2, ..., d\}, \\ (\pi - \vartheta, \pi], & \text{for } j = d+1. \end{cases}$$

We set $\nu = \tan(\vartheta)$. Note that ν depends on $|\omega|$, and when necessary to stress this dependence we write $\nu = \nu(|\omega|)$.

Denote the ray in $\mathbb{H}$ passing through the origin and with slope $a \in \mathbb{R}$ by

$$L_a = \{z \in \mathbb{H} : \mathrm{Im}(z) = a\mathrm{Re}(z)\}. \tag{1.4.4}$$

We denote by $\partial\mathcal{P}$ the union of the boundaries of the elements of the partition $\mathcal{P}$ and by L_ν and $L_{-\nu}$, respectively, the rays $\overline{P_0} \cap \overline{P_1}$ and $\overline{P_d} \cap \overline{P_{d+1}}$.

Let $G : \mathbb{H} \to \mathbb{H}$ be the following family of translation maps

$$G(z) = \begin{cases} z - 1, & z \in P_0, \\ z - \eta', & z \in P_j, \ j \in \{1, ..., d\}, \\ z + \eta, & z \in P_{d+1}, \end{cases}$$

depending on the parameters ϑ, η and η' with $\vartheta > 0$, $\eta \in \mathbb{R}^+ \backslash \mathbb{Q}$ and $0 < \eta' < \eta$.

Consider a permutation $\pi \in \mathfrak{S}(\{1, ..., d\})$ with a monodromy invariant $\tilde{\pi}$, and let $\theta_j(\omega, \tilde{\pi})$ be the angle associated to the monodromy invariant $\tilde{\pi}$ for the cone P_j for $j = 1, \ldots, d$. We have

$$\theta_j(\omega, \tilde{\pi}) = \sum_{\tilde{\pi}(k) < \tilde{\pi}(j)} \omega_k - \sum_{k<j} \omega_k. \tag{1.4.5}$$

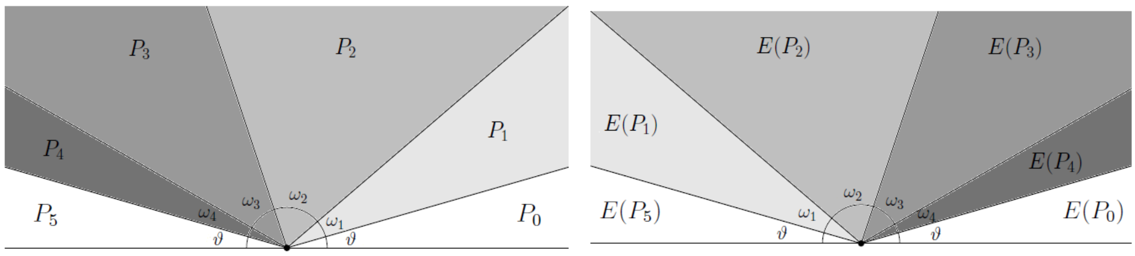


Figure 1.4: On the left a partition $\mathcal{P}$ with $d = 5$. On the right the action of map E on this partition with $\tilde{\pi}(1) = 4$, $\tilde{\pi}(2) = 3$, $\tilde{\pi} = 2$ and $\tilde{\pi} = 1$.

Let $E : \mathbb{H} \to \mathbb{H}$ be the following family of maps

$$E(z) = \begin{cases} z, & z \in P_0 \cup P_{d+1}, \\ ze^{i\theta_j(\omega,\tilde{\pi})}, & z \in P_j, \ j \in \{1, ..., d\}, \end{cases}$$

depending on $\theta_j(\omega, \tilde{\pi})$. This map also depends on ω and ϑ as the partition elements P_j depend on these parameters. Note that we have

$$\vartheta + \arg\left(E(z)\right) = f_{\omega,\pi}(\arg(z) - \vartheta),$$

for $z \in P_j$, $j = 1, ..., d$, where $\arg : \mathbb{C} \to [0, 2\pi)$ is the argument function. Hence E exchanges these cones according to the monodromy invariant $\tilde{\pi}$.

From the translation and exchange families of maps we get our family of TCEs, $F : \mathbb{H} \to \mathbb{H}$, given by

$$F(z) = G \circ E(z).$$

The dynamics of F restricted to P_0 is a translation to the left by 1 while the dynamics of F restricted to P_{d+1} is a translation to the right by η, via the action of the translation map G. The rest of the cones are all permuted, according to a monodromy invariant $\tilde{\pi}$, by the exchange map E and horizontally translated by η' by the translation map G.

Note that TCEs are cone isometry transformations for which the map induced by projection onto the circle at infinity $\hat{F}$ (see [11]) is invertible. F is defined on $\mathbb{H} \subseteq \mathbb{C}$, partitioned into $d+2$ cones by $\mathcal{P}$, hence it is a cone exchange transformation. $\hat{F}$ is an interval exchange transformation with interval partition given by $\{W_0, ..., W_{d+1}\}$ and combinatorial data given by the bijection $\tilde{\pi}'$, where $\tilde{\pi}'(0) = 0$, $\tilde{\pi}'(d+1) = d+1$, and $\tilde{\pi}'(j) = \tilde{\pi}'(j)$, for $j = 1, ..., d$.

Let us introduce some notation. We define the *central cone* P_c of F as

$$P_c = P_1 \cup ... \cup P_d,$$

the *first hitting time of* $z \in \mathbb{H}$ *to* P_c, as the map $k : \mathbb{H} \to \mathbb{N}$ given by

$$k(z) = \inf\{n \geq 1 : F^n(z) \in P_c\}, \tag{1.4.6}$$

and the *first return map of* $z \in P_c$ *to* P_c, as the map $F_c : P_c \to P_c$ such that

$$F_c(z) = F^{k(z)}(z). \tag{1.4.7}$$

The typical notion of renormalization may not capture all possible self similar behaviour in PWIs. TCEs apparently exhibit invariant regions on which the dynamics is self similar after rescaling. Thus, we say a TCE is renormalizable if F_c, the first return map to P_c described above, is conjugated to itself by a scaling map.

1.4.2 Embedding interval exchange transformations into piecewise isometries

It is a general belief that the phase space of typical Hamiltonian systems is divided into regions of regular and chaotic motion [21]. Area preserving maps which can be obtained as Poincaré sections of Hamiltonian systems, exhibit this property as well, with KAM curves splitting the domain into regions of chaotic and periodic dynamics (see for instance [44]). A general and rigorous treatment of this has been however missing.

PWIs, which are area preserving maps that have been studied as linear models for the standard map (see [6]), can exhibit a similar phenomenon. Unlike IETs which are typically ergodic, there is numerical evidence, as noted in [10], that Lebesgue measure on the exceptional set is typically not ergodic in some families of PWIs - there can be non-smooth invariant curves that prevent trajectories from spreading across the whole of the exceptional set. These curves were first observed in [5] for an isolated parameter and later found in [10] to be apparently abundant for a large family of PWIs. For cases where the exceptional set is a union of annuli a small perturbation in the rotational parameters causes it to decompose into invariant curves and periodic orbits, a phenomenon that is reminiscent of KAM curves. An understanding of these invariant curves would thus shed light on the ergodic properties of PWIs and would be an important first step towards the study of the dynamical behaviour shared by generic PWIs and systems which are modelled by these. A proof of their existence however remained elusive for more than a decade.

The first progress was made in [9], where a planar PWI, with a rational rotation vector, whose generating map is a permutation of four cones was investigated, and the existence of an uncountable number of invariant polygonal curves on which the dynamics is conjugate to a transitive interval exchange was proved. The methods used however are based on calculations in a rational cyclotomic field and do not generalize for typical choices of parameters.

We relate the existence of invariant curves to the general problem of embedding IET dynamics within PWIs, of which we give rigorous definitions.

An injective map $\gamma : I \to X$ is a *piecewise continuous embedding* of (I, f) into (X, T) if $\gamma|_{I_\alpha}$ is a homeomorphism for each $\alpha \in \mathcal{A}$ such that $\gamma(I_\alpha) \subset X_\alpha$ and

$$\gamma \circ f(x) = T \circ \gamma(x), \tag{1.4.8}$$

for all $x \in I$. In this case note that $\gamma(I) \subset X$ is an invariant set for (X, T).

If γ is a piecewise continuous embedding that is continuous on I, we say it is a *continuous embedding* (or *embedding* when this does not cause any ambiguity). Otherwise we say it is a *discontinuous embedding*.

We say γ is a *differentiable embedding* if it is a piecewise continuous embedding and $\gamma|_{I_\alpha}$ is continuously differentiable. We characterize certain differentiable embeddings as, in some sense, trivial. Given $I' \subseteq I$ we say a map $\gamma : I' \to \mathbb{C}$ is an *arc map* if there exists $\xi \in \mathbb{C}$, $r, a > 0$ and $\varphi \in [0, 2\pi)$ such that for all $x \in I'$,

$$\gamma(x) = re^{i(ax+\varphi)} + \xi.$$

We say an embedding $\gamma : I \to \mathbb{C}$ of an IET into a PWI is an *arc embedding* if there exists a finite partition of I into subintervals such that the restriction of γ to each subinterval is an arc map. We say an embedding γ of an IET into a PWI is a *linear embedding* if γ is a piecewise linear map. Moreover an embedding is *non-trivial* if it is not an arc embedding or a linear embedding. Figure 1.5 shows an illustration of a non-trivial embedding.

From the definitions it is clear that the image $\gamma(I)$ of an embedding is an invariant curve for the underlying PWI and that if the embedding is non-trivial this curve is not the union of line segments or circle arcs. For any IET it is straightforward to construct a PWI in which it is trivially embedded. The same is not true for non-trivial embeddings, for which results have been much scarcer.

1.5 Main Theorems

The main results of this thesis are the following. We consider first the family of TCEs and then more general PWIs.

Translated cone exchange transformations. Recall, from (1.4.7), the definition of F_c, the first return map under F to P_c. In Theorem A we renormalize TCEs, in the sense defined in subsection 1.4.1, for all rotation parameters and for

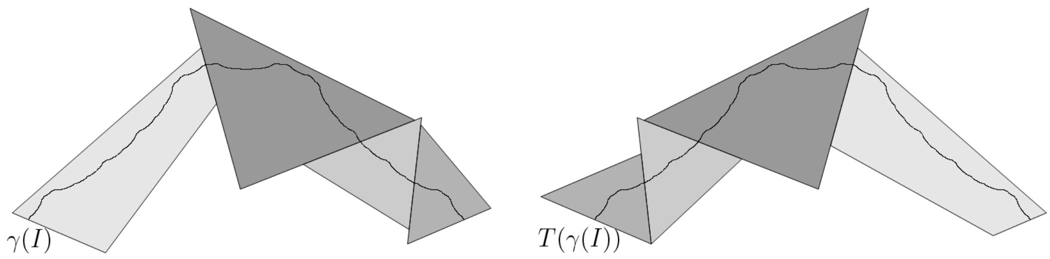


Figure 1.5: An illustration of the action of a PWI T with rotation vector $\theta \approx (4.85, 0.92, 1.31, 1.28)$ on its partition and on an invariant curve $\gamma(I)$. The map γ, estimated using technical tools from this chapter, is a non-trivial embedding of a self-inducing IET associated to the monodromy invariant $\tilde{\pi}(j) = 4 - (j-1)$, $j = 1, ..., 4$ and a translation vector of algebraic irrationals $\lambda \approx (0.43, 0.34, 0.12, 0.11)$.

infinitely many translational parameters. We show that for a set of parameters, the first return map under a TCE to P_c, is self-similar by a scaling factor Φ^2 where

$$\Phi = (\sqrt{5} - 1)/2.$$

Theorem A *For all $\omega \in \mathbb{W}$, $\eta = 1/(k + \Phi)$ and $\eta' = 1 - k\eta$ with $k \in \mathbb{N}$, there is an open set U containing the origin such that F is renormalizable for all $z \in U$, that is*

$$F_c(\Phi^2 z) = \Phi^2 F_c(z). \tag{1.5.1}$$

The proof of this theorem is centred around a one dimensional approach to the study of these PWIs. In particular we define sequences coding information related to the first return map of a given line contained in the cone P_c. We are able to relate the renormalizability of a map of this family with the periodicity of these sequences and indeed, for the parameters in the statement of the theorem, these are proved to be periodic. As a consequence of this we show that for these parameters F_c is a PWI with respect to a partition $\mathcal{P}_{F_c}$ of countably many atoms.

We say a collection of atoms $\mathcal{B} \subseteq \mathcal{P}$ is a *barrier* for a PWI $(T, \mathcal{P})$ if $X \backslash \mathcal{B}$ is the union of two disjoint connected components A_1, A_2 such that

$$A_1 \cap T(A_2) = T(A_1) \cap A_2 = \emptyset,$$

and for any $P \in \mathcal{P}$ such that $P \subseteq A_j$ and $T(P) \cap \overline{\mathcal{B}} \cap \overline{A_j} = \emptyset$ then $T(P) \cap \mathcal{B} = \emptyset$, for $j = 1, 2$.

The first condition says that the image by T of a connected component A_j cannot intersect the other component A_{3-j} while the second condition guarantees that if the image by T of an atom in A_j does not intersect the boundary shared between $\overline{\mathcal{B}}$ and $\overline{A_j}$ then it does not intersect $\mathcal{B}$.

Recall, from (1.4.4), the definition of L_a, $a \in \mathbb{R}$.

For $\omega \in \mathbb{W}$, $\eta = 1/(k+\Phi)$ and $\eta' = 1 - k\eta$, $k \in \mathbb{N}$. We denote by $\mathfrak{A}(\eta, \eta')$ the subset of $\mathbb{W}$ such that for all $\omega \in \mathfrak{A}(\eta, \eta')$ there are $d' \geq 2$, $\lambda \in \mathbb{R}_+^{d'}$, $\pi \in \mathfrak{S}(\{1, ..., d'\})$ and a continuous embedding γ of $f_{\lambda,\pi} : I \to I$ into $F_c : P_c \to P_c$ such that

i) the collection $\mathcal{B} = \{P \in \mathcal{P}_{F_c} : P \cap \gamma(I) \neq \emptyset\}$, is a barrier for F_c,

ii) $\gamma(0) \in L_{-\nu}$ and $\lim_{a \to |\lambda|} \gamma(a) \in L_\nu$,

iii) $\gamma(I) \subset \Phi^2 U$, where U is the open set from Theorem A.

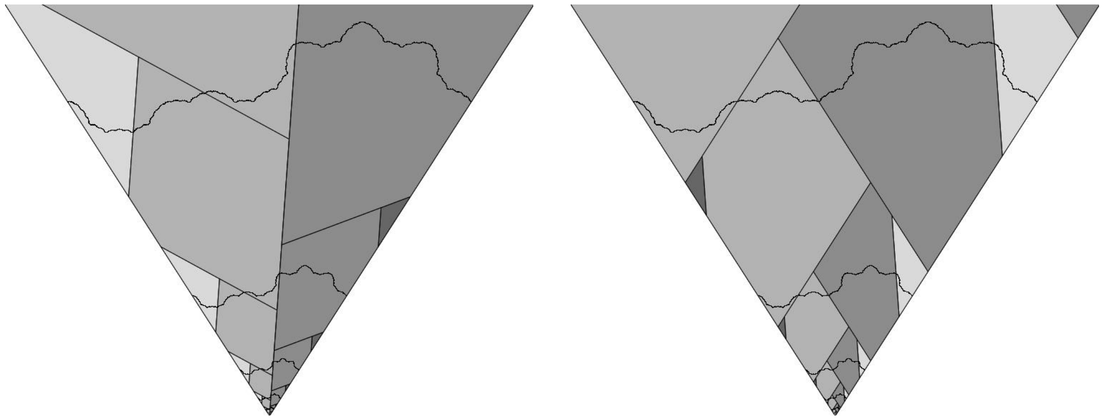
Informally this is the set of parameters $\omega \in \mathbb{W}$ such that the associated map F_c admits an embedding of an IET which image is strictly contained in atoms of the partition of F_c which form a barrier. Condition ii) guarantees that the endpoints of $\gamma(I)$ are contained in the boundary of P_c, respectively in $L_{-\nu}$ and L_ν. Condition iii) is a technical requirement guarantees the applicability of Theorem A to a useful domain in the next theorem.

Although numerical experiments (see Section 3.4.3), support that the set $\mathfrak{A}(\eta, \eta')$ should be non-empty for the parameters in consideration, it is not known whether this is true. Indeed, this is related to one of the greatest open questions in the field: whether typical families of PWIs support embeddings of IETs.

In the next theorem we show, as a consequence of renormalization of TCEs, that the existence of one continuous embedding of an IET into a first return map F_c of a TCE, satisfying the property that the image of the embedding is contained in a barrier, implies the existence of infinitely many embeddings of the same IET into F_c, as well as infinitely many bounded and forward invariant regions. This shows in particular that if one non trivial embedding exists then the results from Chapter 3 for 2,3-PWIs do not generalize for PWIs with partitions with a higher number of atoms. We prove that for $\omega \in \mathfrak{A}(\eta, \eta')$ there are infinitely many sets, bounded away from 0 and infinity, which are forward invariant by F_c and that there exist infinitely many continuous embeddings of IETs into F_c.

Theorem B *Let $\eta = 1/(k+\Phi)$, $\eta' = 1 - k\eta$ with $k \in \mathbb{N}$ and assume that $\mathfrak{A}(\eta, \eta')$ is non-empty. For all $\omega \in \mathfrak{A}(\eta, \eta')$,*

i) There exist sets $V_1, V_2, ...,$ which are forward invariant for F_c and $y^* > 0$ such that for all $z \in P_c$, satisfying $0 < \mathrm{Im}(z) < y^*$, there is an $n \in \mathbb{N}$ for which $z \in V_n$.



(a) (b)

Figure 1.6: A schematic representation of the action of F_c on the cone P_c close to the origin, for parameters $\tilde{\pi}(1,2) = (2,1)$, $\omega = (0.5, \pi - 2.5)$, $\eta = \Phi$ and $\eta' = 1 - \Phi$. F_c is a PWI with respect to a partition of infinitely many atoms, which correspond to the polygons depicted in the figure (A). In (B) the image of this partition by F_c can be seen. Each curve in both figures corresponds to the orbit of a given point. By Theorem A if the orbit remains close to the origin then there are infinitely many replica of this orbit accumulating on the origin. It is still an open question whether the closure of this orbit is the image of an embedding of an IET into F_c.

ii) For all $n \in \mathbb{N}$ there exist constants $0 < \underline{b}_n < \overline{b}_n$ such that for all $z \in V_n$ and $k \in \mathbb{N}$,

$$\underline{b}_n < |F^k(z)| < \overline{b}_n. \tag{1.5.2}$$

iii) There exist infinitely many continuous embeddings of IETs into F_c.

The proof of Theorem B relies on the Jordan curve Theorem, and on the properties of the barrier containing the image of the embedding, to prove the existence of one invariant set V_1, and relies on the renormalizability of F, established in Theorem A, to show that this implies the existence of infinitely many such sets.

This result, and the study of TCEs in general, gives a strong motivation to study the existence of embeddings of IETs into PWIs which we develop in the following two chapters.

Embeddings of IETs into PWIs. In Chapter 3 we establish necessary conditions which PWIs must satisfy in order to support an embedding of an IET. As a consequence we prove the following theorem.

Theorem C *A minimal 2-IET has no non-trivial continuous embedding into a 2-PWI.*

This is a surprising result, considering that numerical evidence supports the existence of non-trivial embeddings of d-IETs into d-PWIs for $d \geq 4$. We will later see in Chapter 4 that this is indeed established. The proof of Theorem C reveals that for $d = 2$ there are not enough parametric degrees of freedom to allow for this to occur in this case. As another consequence of the techniques developed in Chapter 3 we also prove the following theorem.

Theorem D *A* 3*-PWI has at most one non-trivially continuously embedded minimal* 3*-IET with the same underlying permutation.*

Once again this result contrasts with the numerical evidence for the case $d = 4$ which suggests that there can be an abundance of embeddings of IETs for the same PWI.

The proofs of these Theorems C and D rely on the use combinatorial properties of IETs to prove that in order for a PWI to realize a continuous embedding of an IET with the same permutation, its parameters must satisfy a necessary condition: the *parametric connecting equation.* Making use of a generalization of Rauzy-Veech induction to PWIs this allows for arguments which give strong parametric restrictions for PWIs supporting non-trivial embeddings of IETs.

In Chapter 4 we prove that a full measure set of IETs admit non-trivial embeddings into a class of PWIs thus also establishing the existence of invariant curves for PWIs which are not unions of circle arcs or line segments.

Theorem E *For almost every IET $(I, f_{\lambda,\pi})$ satisfying $g(\mathfrak{R}) \geq 2$, there exists a set $\mathcal{W} \subseteq \mathbb{T}^{\mathcal{A}}$, of dimension $g(\mathfrak{R})$, such that for all $\theta \in \mathcal{W}$ there is a family $\mathcal{F}_\theta$, of PWIs with rotation vector θ, and a map $\gamma_\theta : I \to \mathbb{C}$, which is a non-trivial and isometric embedding of $(I, f_{\lambda,\pi})$ into any $(X, T) \in \mathcal{F}_\theta$. Furthermore $\gamma_\theta(I)$ is an invariant curve for (X, T) which is not the union of circle arcs or line segments.*

To prove this result we inductively define, associated to a given IET, a sequence of piecewise linear parametrized curves, which we call the *breaking sequence*, dependent on a rotation vector $\theta \in \mathbb{T}^{\mathcal{A}}$. In particular for its construction we define the *breaking operator*, which acts on piecewise linear maps from I to $\mathbb{C}$ by rotating particular segments of their image by a given angle. The construction also involves the *Rauzy cocycle*, an important tool in the theory of IET renormalization. We then show that each element of the breaking sequence is a *quasi-embedding* (a rigorous notion defined in Section 4.2) of the underlying IET into a certain sequence of piecewise isometric maps related to Rauzy induction. Provided the breaking sequence converges to a topological embedding of the interval, this is enough to show that its limit is an embedding of the underlying IET into a family of PWIs. Hence the following step is to use tools from the theory of IET renormalization and measurable cocycles such as *Zorich cocycle* [58] and *Oseledets Theorem* [45] to prove this is the case for almost every (λ, π) and for θ contained in a submanifold of $\mathbb{T}^{\mathcal{A}}$. After some further parameter exclusion to guarantee that the embedding is non-trivial we finally conclude the proof of Theorem E.

1.6 Organization

This thesis is organized as follows, in Chapter 2 we introduce and investigate the dynamics of *translated cone exchange transformations*, introduced in Section 1.4.1. We renormalize its first return map to a subset of its partition. As a consequence we prove Theorems A and B. We also prove the existence of infinitely many periodic islands, accumulating on the real line, as well as non-ergodicity of our family of maps close to the origin.

In Chapter 3 we derive some necessary conditions for existence of continuous and discontinuous embeddings of IETs into PWIs, using combinatorial, topological and measure theoretic properties of IETs. We use some of these techniques to prove Theorems C and D. We also introduce a family of 4-PWIs with apparent abundance of invariant non-smooth curves supporting IETs, that limit to a trivial embedding of an IET.

In Chapter 4 we introduce the breaking operator and breaking sequence of curves. We prove that these curves are *quasi-embedded* into a family of PWIs and use tools from the theory of IET renormalization to establish several resuts leading to the proof of Theorem E.

Finally in Chapter 5 we present some concluding remarks and discuss possible directions for future work.

The material in Chapter 3 has been published in *Ergodic Theory and Dynamical Systems* [12].

Chapter 2

Dynamics of Translated Cone Exchange Transformations

In this chapter, we investigate *translated cone exchange transformations* and renormalize its first return map to a subset of its partition. As a consequence we show that the existence of an embedding of an interval exchange transformation into a map of this family implies the existence of infinitely many bounded invariant sets. We also prove the existence of infinitely many periodic islands, accumulating on the real line, as well as non-ergodicity of our family of maps close to the origin.

This chapter is organized as follows. In Section 2.1 we investigate a family of maps related to IETs. In Section 2.2 we study the sequence of bifurcation points and the bifurcation sequence for the family of maps introduced in the previous section making use of the theory of continued fractions. In Section 2.3 we introduce two sequences that we designate by dynamical sequences that will be an important tool to prove our main theorems. We derive inductive formulas to compute these sequences. In Section 2.4 we study the dynamics of the first return map to the central cone P_c. Finally, in Sections 2.5 and 2.6 we prove theorems A and B.

2.1 Bifurcation points

Recall, from Section 1.4.1, the definition of Translated Cone Exchange Transformations (TCEs). In this section we study a specific family of maps g_ℓ, closely related to IETs, on the interval $I = [0, 1 + \eta]$ with $\eta \in \mathbb{R}^+ \backslash \mathbb{Q}$. Orbits of points of the map g_ℓ are connected to those of a TCE under certain conditions. Indeed the real part of iterates, under a TCE, of a point outside the cone P_c can be described by iterates of a point on the interval I under the map g_ℓ.

We will introduce the left and right bifurcation points and bifurcation sets for this family of maps.

Consider the interval $I = [0, 1+\eta]$ and the following family of maps

$$g_\ell(x) = \begin{cases} x+\eta, & x \in I_1(\ell) \\ x, & x \in I_c(\ell) \\ x-1, & x \in I_2(\ell). \end{cases} \tag{2.1.1}$$

with $I_1(\ell) = [0,1]$, $I_c(\ell) = (1, 1+\ell)$ and $I_2(\ell) = [1+\ell, 1+\eta]$ and $0 \le \ell < \eta$. To simplify notation we will only include the argument when it is necessary, otherwise we just refer to these intervals as I_j, for $j = 1, 2, c$. Note that when $\ell = 0$ we have $I_c = \emptyset$ and we set $I_2(\ell) = (1, 1+\eta]$ therefore

$$g_0(x) = \begin{cases} x+\eta, & x \in [0,1] \\ x-1, & x \in (1, 1+\eta]. \end{cases}$$

Recall $k(z)$ as in (1.4.6). Given $\vartheta \in (0, \pi/2)$, consider the *trapping region*

$$\mathcal{R}_{\eta,\vartheta} = \{z \in \mathbb{H}\backslash P_c : \mathrm{Re}(z) + \mathrm{Im}(z)\cot(\vartheta) \in [-1, \eta] \text{ and } 2\mathrm{Im}(z)\cot(\vartheta) \le \eta\}. \tag{2.1.2}$$

It is simple to see, by definition of F, that for any $z \in P_c$ such that $2\mathrm{Im}(F(z))\cot(\vartheta) \le \eta$, either $F(z) \in P_c$, or $F(z) \in \mathbb{H}\backslash P_c$, in which case we get

$$-1 - \mathrm{Im}(z)\cot(\vartheta) \le \mathrm{Re}(F(z)) \le \eta - \mathrm{Im}(z)\cot(\vartheta),$$

and thus $F(z) \in \mathcal{R}_{\eta,\vartheta}$.

In this way we may think $\mathcal{R}_{\eta,\vartheta}$ as a region of $\mathbb{C}$ where orbits of points $z \in P_c$, with sufficiently small imaginary part, are confined until they return to P_c.

The next lemma relates iterates of our family of maps F with iterates of g_ℓ for some values of z.

Lemma 2.1.1 *For any $\eta > 0$, $\vartheta \in (0, \pi/2)$ and $z \in \mathcal{R}_{\eta,\vartheta}$ we have*

$$F^n(z) = s^{-1} \circ g^n_{2\mathrm{Im}(z)\cot(\vartheta)} \circ s(\mathrm{Re}(z)) + i\mathrm{Im}(z), \tag{2.1.3}$$

for all $n \le k(z)$, where $s(x) = x + 1 + \ell/2$.

Proof. As $z \in \mathcal{R}_{\eta,\vartheta}$ we have $z \in P_0 \cap \mathcal{R}_{\eta,\vartheta}$ or $z \in P_{d+1} \cap \mathcal{R}_{\eta,\vartheta}$. By the definitions of P_0 and P_{d+1}, in both cases we have $\mathrm{Re}(F(z)) = s^{-1} \circ g_{2\mathrm{Im}(z)\cot(\vartheta)} \circ s(\mathrm{Re}(z))$. It is direct to see that for $n \le k(z)$ we have $F^n(z) \in \mathcal{R}_{\eta,\vartheta}$ and thus repeating the previous argument n times we get (2.1.3). □

We define the *first hitting time* of x to $\overline{I_c(\ell)}$ as the map $n_\ell : I \to \mathbb{N}$ given by

$$n_\ell(x) = \inf\{n \ge 1 : g^n_\ell(x) \in \overline{I_c(\ell)}\}, \tag{2.1.4}$$

and the *first hitting map* of x to $\overline{I_c(\ell)}$, as the map

$$r_\ell(x) = g_\ell^{n_\ell(x)}(x). \tag{2.1.5}$$

For our next lemma we need also to consider the map

$$r'_\ell(x) = \begin{cases} r_\ell(x), & x \notin \overline{I_c(\ell)}, \\ x, & x \in \overline{I_c(\ell)}. \end{cases} \tag{2.1.6}$$

Recall the first return to the central cone map F_c from (1.4.7).

Lemma 2.1.2 *Let $\eta \in \mathbb{R}^+ \backslash \mathbb{Q}$ and $0 < \vartheta < \frac{\pi}{2}$. If $z \in P_c$ with $2\mathrm{Im}(F(z))\cot(\vartheta) \leq \eta$, then*

$$F_c(z) = s^{-1} \circ r'_{2\mathrm{Im}(F(z))\cot(\vartheta)} \circ s(\mathrm{Re}(F(z))) + i\mathrm{Im}(F(z)). \tag{2.1.7}$$

Proof. It is clear that if $F(z) \in P_c$, then we have (2.1.7), so we may assume $F(z) \in \mathbb{H}\backslash P_c$ and therefore $F(z) \in \mathcal{R}_{\eta,\vartheta}$. From Lemma 2.1.1 it follows that (2.1.3) holds for all $n \leq k(F(z))$. It is simple to see that

$$k(F(z)) = n_{2\mathrm{Im}(F(z))\cot(\vartheta)}(\mathrm{Re}(F(z))),$$

and thus by definition of r'_ℓ we get (2.1.7) as intended. □

Let $\eta \in \mathbb{R}^+ \backslash \mathbb{Q}$ and $I = [0, 1+\eta]$. Let $N \in \mathbb{N}$. Define

$$d^-(N) = \begin{cases} 1, & \text{if } g_0^n(1) > 1 \text{ for all } 1 \leq n \leq N, \\ 1 - \max_{1\leq n\leq N}\left\{g_0^n(1) \leq 1\right\}, & \text{otherwise,} \end{cases}$$

and

$$d^+(N) = \begin{cases} \eta, & \text{if } g_0^n(1) < 1 \text{ for all } 1 \leq n \leq N, \\ \min_{1\leq n\leq N}\left\{g_0^n(1) \geq 1\right\} - 1, & \text{otherwise.} \end{cases}$$

We want now to investigate orbits by g_0 of points which are in a small neighbourhood of 1. We prove the next lemmas.

Lemma 2.1.3 *Assume that $\eta \in \mathbb{R}^+ \backslash \mathbb{Q}$.*

i) If $N \geq 0$ and $0 \leq \ell < d^+(N)$, then for all $0 \leq n \leq N$ we have

$$g_0^n(1-\ell) = g_0^n(1) - \ell. \tag{2.1.8}$$

ii) If $N \geq 2$ and $0 \leq \ell \leq d^-(N)$, then for all $2 \leq n \leq N$ we have

$$g_0^n(1+\ell) = g_0^n(1) + \ell. \tag{2.1.9}$$

Proof. To simplify notation we denote $d^+ = d^+(N)$ and $d^- = d^-(N)$. Let us prove i) by induction on n. It is clear that (2.1.8) holds for $n \in \{0, 1\}$. We now assume (2.1.8) holds for $1 \leq n < N$ and we show it holds for $n + 1$.

It follows from the definitions of d^- and d^+ that $g_0^n(1) \notin (1 - d^-, 1 + d^+)$, for $1 \leq n \leq N$, thus $g_0^n(1) \leq 1 - d^-$ or $g_0^n(1) \geq 1 + d^+$.

If $g_0^n(1) \leq 1 - d^-$, then as $\ell \geq 0$ and since we are assuming (2.1.8) holds for n we have $g_0^n(1 - \ell) \leq 1 - d^-$. Therefore $g_0^n(1 - \ell) \in [0, 1]$ and since $g_0^n(1) \in [0, 1]$ we get

$$g_0^{n+1}(1 - \ell) = g_0^{n+1}(1) - \ell.$$

If $g_0^n(1) \geq 1 + d^+$, then as $\ell < d^+$ and since we are assuming (2.1.8) holds for n we have $g_0^n(1 - \ell) > 1$. Therefore $g_0^n(1) \in (1, 1 + \eta]$ and thus

$$g_0^{n+1}(1 - \ell) = g_0^n(1) - \ell - 1.$$

Since $g_0^n(1) \in (1, 1 + \eta]$, we get that (2.1.8), holds for $n + 1$ and we finish the proof of i).

The proof of ii) is similar to the proof of i) so we omit it. □

Given $\ell > 0$ and $x \in I \backslash [1, 1 + \ell]$, we define

$$\mathfrak{d}^-(x, n_\ell(x)) = 1 - \max_{0 \leq n \leq n_\ell(x)} \{g_0^n(x) \leq 1\}. \tag{2.1.10}$$

Lemma 2.1.4 *Assume $0 < \ell' < \ell$, $x \in I \backslash [1, 1 + \ell]$ and $x' \in (x - (\ell - \ell'), x + \mathfrak{d}^-(x, n_\ell(x)))$. Then for all $n \leq n_\ell(x)$ we have*

$$g_\ell^n(x) - g_{\ell'}^n(x') = x - x'. \tag{2.1.11}$$

Moreover, $n_{\ell'}(x') \geq n_\ell(x)$.

Proof. To simplify notation we denote $\mathfrak{d}^- = \mathfrak{d}^-(x, n_\ell(x))$. We proceed by induction on n. It is clear that (2.1.11) holds for $n = 0$. Now assume (2.1.11) holds for $n < n_\ell(x)$ and we prove it for $n + 1$ instead.

As $n < n_\ell(x)$ we have $g_\ell^n(x) \notin [1, 1 + \ell]$. Since we are assuming (2.1.11) holds for n, we get

$$g_{\ell'}^n(x') \in (g_\ell^n(x) - (\ell - \ell'), g_\ell^n(x) + \mathfrak{d}^-).$$

If $g_\ell^n(x) < 1$, then $g_\ell^n(x) \leq 1 - \mathfrak{d}^-$ and thus $g_{\ell'}^n(x') \in (1 - \mathfrak{d}^- - (\ell - \ell'), 1)$.

Otherwise, if $g_\ell^n(x) > 1 + \ell$ then $g_{\ell'}^n(x') \in (1 + \ell', 1 + \ell + \mathfrak{d}^-)$, thus we have $g_{\ell'}^n(x') \in I_j$ if and only if $g_{\ell'}^n(x) \in I_j$, for $j = 0, 1$ and $g_{\ell'}^n(x') \notin [1, 1 + \ell']$. Therefore by (2.1.1) we get $g_\ell^{n+1}(x) - g_{\ell'}^{n+1}(x') = x - x'$. This proves (2.1.11) for $n \leq n_\ell(x)$.

Since $g^n_{\ell'}(x') \notin [1, 1+\ell']$ for $n < n_\ell(x)$ we have $n_{\ell'}(x') \geq n_\ell(x)$ and we finish our proof. □

In the beginning of this section we defined the first hitting map of x to $\overline{I_c(\ell)}$, as $r_\ell(x) = g_\ell^{n_\ell(x)}(x)$, where $n_\ell(x)$ is the first hitting time of x to $\overline{I_c(\ell)}$. We want now to investigate when is $1+\ell$ mapped to 1 under $r_\ell(x)$ and when is 1 mapped to $1+\ell$. Note that these are the endpoints of the central interval $\overline{I_c(\ell)}$. We define the following points and sets.

We say ℓ is a *right bifurcation point* if $r_\ell(1+\ell) = 1$, ℓ is a *left bifurcation point* if $r_\ell(1) = 1+\ell$ and ℓ is a *bifurcation point* if it is either a left or right bifurcation point.

The *left/right bifurcation sets* are defined respectively as

$$\Lambda_L = \{0 < \ell \leq \eta : \text{for all } l < \ell,\ n_\ell(1) < n_l(1)\},$$

and

$$\Lambda_R = \{0 < \ell \leq \eta : \text{for all } l < \ell,\ n_\ell(1+\ell) < n_l(1+l)\}.$$

The main result of this section is the next theorem, relating bifurcation points with the bifurcation sets.

Theorem 2.1.5 *ℓ is a left (resp. right) bifurcation point if and only if $\ell \in \Lambda_L$ (resp. $\ell \in \Lambda_R$). Furthermore, $\ell \mapsto n_\ell(1)$ and $\ell \mapsto n_\ell(1+\ell)$ are decreasing functions of ℓ and the sets Λ_R, Λ_L are discrete with 0 as the only possible point of accumulation.*

Proof.

First assume that $r_\ell(1) = 1+\ell$ and $l < \ell$. By the definitions of n_ℓ and r_ℓ we have, for $1 \leq n < n_\ell(1)$, that either $g^n_\ell(1) < 1$ or $g^n_\ell(1) > 1+\ell$. As $l < \ell$, by (2.1.1) we have for $1 \leq n < n_l(1)$, $g^n_l(1) < 1$ or $g^n_l(1) > 1+l$. Thus $n_\ell(1) \leq n_l(1)$ and $g_l^{n_l(1)}(1) = g_\ell^{n_\ell(1)}(1)$. Since $g_\ell^{n_\ell(1)}(1) = 1+\ell$ and $1+\ell > 1+l$ this shows $g_l^{n_l(1)}(1) > 1+l$ and thus $n_\ell(1) < n_l(1)$. This proves that if ℓ is a left bifurcation point, then $\ell \in \Lambda_L$.

Now assume that $r_\ell(1) \neq 1+\ell$. As η is irrational we must have $r_\ell(1) \in (1, 1+\ell)$, therefore there is $0 < \ell' < \ell$ such that $g_\ell^{n_\ell(1)}(1) = 1+\ell'$.

We show, by induction on n, that for all $l \in [\ell', \ell]$ and $0 \leq n \leq n_\ell(1)$

$$g^n_l(1) = g^n_\ell(1). \tag{2.1.12}$$

It is clear that (2.1.12) holds for $n = 0$. We assume it holds for $n < n_\ell(1)$ and we prove it for $n+1$. As $n < n_\ell(1)$ we have $g^n_\ell(1) \notin (1, 1+\ell)$, and since $g^n_l(1) = g^n_\ell(1)$, this implies that $g_l^n(1) \notin (1, 1+l)$, thus by (2.1.1) we have that (2.1.12) must hold for $n+1$.

Since (2.1.12) holds for $n = n_\ell(1)$ we have $g_l^{n_\ell(1)}(1) = 1+\ell'$ and thus $n_\ell(1) = n_l(1)$ for all $l \in [\ell', \ell]$.

Thus, there is $l < \ell$ such that $n_\ell(1) \geq n_l(1)$. This proves that ℓ is a left bifurcation point if and only if $\ell \in \Lambda_L$. Note that it also shows that $\ell \mapsto n_\ell(1)$ is a decreasing function of ℓ.

By Lemma 2.1.4, for all $l < \ell$ and $0 \leq n \leq n_\ell(1+\ell)$ we have

$$g_l^n(1+l) = g_\ell^n(1+\ell) - (\ell - l). \tag{2.1.13}$$

From which follows that $g_l^{n_\ell(1+\ell)}(1+l) = g_\ell^{n_\ell(1+\ell)}(1+\ell) - (\ell - l)$. If $r_\ell(1+\ell) = 1$, as $r_\ell(1+\ell) = g_\ell^{n_\ell(1+\ell)}(1+\ell)$, this implies

$$g_l^{n_\ell(1+\ell)}(1+l) = 1 - (\ell - l) \notin \overline{I_c(l)},$$

thus, $n_\ell(1+\ell) < n_l(1+l)$. Then for all $l < \ell$, we have $n_\ell(1+\ell) < n_l(1+l)$. This proves that if ℓ is a right bifurcation point then $\ell \in \Lambda_R$,

If $r_\ell(1+\ell) \neq 1$, as η is irrational we must have $r_\ell(1+\ell) \in (1, 1+\ell)$, therefore there is an $0 < \ell' < \ell$ such that $g_\ell^{n_\ell(1+\ell)}(1+\ell') = 1$.

Now take $l \in [\ell', \ell)$. By (2.1.13) we get

$$g_l^{n_\ell(1+\ell)}(1+l) = 1 + l - \ell' \in [1, 1+l),$$

hence $g_l^{n_\ell(1+\ell)}(1+l) \in \overline{I_c(l)}$ and we have $n_\ell(1+\ell) = n_l(1+l)$. Thus, there is a $l < \ell$ such that $n_\ell(1+\ell) \geq n_l(1+l)$. This proves that if $\ell \in \Lambda_R$ then ℓ is a right bifurcation point. This proves that ℓ is a right bifurcation point if and only if $\ell \in \Lambda_R$. Note that it also shows that $\ell \mapsto n_\ell(1+\ell)$ is a decreasing function of ℓ.

Since $\ell \mapsto n_\ell(1)$ and $\ell \mapsto n_\ell(1+\ell)$ are decreasing functions of ℓ and are also integer valued functions this implies that the sets Λ_L and Λ_R are discrete and each has at most one point of accumulation, which has to be 0. □

2.2 Bifurcation sequence

In this section we study the sequence of bifurcation points for the family g_l (in (2.1.1)). We will first recall some elements of the theory of continued fractions, and compute the sequence of errors of the semiconvergents of $\eta = 1/(k + \Phi)$, where $\Phi = (\sqrt{5} - 1)/2$ and $k \in \mathbb{N}$. We will then relate the bifurcation sequence with the theory of continued fractions by showing that this sequence is equal to the sequence of errors of the semiconvergents of η.

Throughout this section we assume that $\eta \in (0,1)$ is an irrational real number with continued fraction expansion $\eta = [0, \eta_1, \eta_2, ...]$. Consider the sequence of its *convergents* given by

$$\left(\frac{p_n}{q_n}\right)_{n\geq 0}, \text{ where } \frac{p_0}{q_0} = \frac{0}{1} \text{ and } \frac{p_n}{q_n} = [0, \eta_1, ..., \eta_n].$$

For all $n \geq 0$ it is well known that

$$\begin{aligned} p_{n+2} &= p_n + \eta_{n+2} p_{n+1}, \\ q_{n+2} &= q_n + \eta_{n+2} q_{n+1}. \end{aligned} \tag{2.2.1}$$

Define the sequence of *upper semiconvergents* of η as

$$\left(\frac{p'_n}{q'_n}\right)_n = ([0,1], ..., [0, \eta_1], [0, \eta_1, \eta_2, 1], ..., [0, \eta_1, \eta_2, \eta_3], ...).$$

which is the sequence of best rational approximations of η by above , this is, any other fraction $\frac{a}{b} \neq \frac{p'_n}{q'_n}$, with $1 \leq b \leq q'_n$, satisfies $a - b\eta > p'_n - q'_n\eta$ (see for instance [39]).

The sequence of errors of approximation of the upper semiconvergents smaller than η is given by

$$\Gamma'_n = \left(p'_{n+\eta_1-1} - q'_{n+\eta_1-1}\eta\right)_n.$$

Analogously, we define the sequence of *lower semiconvergents* of η as

$$\left(\frac{p''_n}{q''_n}\right)_n = (0, [0, \eta_1, 1], ..., [0, \eta_1, \eta_2], [0, \eta_1, \eta_2, \eta_3, 1], ..., [0, \eta_1, \eta_2, \eta_3, \eta_4], ...).$$

which is the sequence of best rational approximations of η by below, this is, any other fraction $\frac{a}{b} \neq \frac{p''_n}{q''_n}$, with $1 \leq b \leq q''_n$, satisfies $b\eta - a > q''_n\eta - p''_n$.

The sequence of errors of approximation of the lower semiconvergents is given by

$$\Gamma''_n = (q''_n\eta - p''_n)_n.$$

Note that Γ' and Γ'' are monotonic sequences of positive real numbers that converge to 0. Finally, we define recursively the intercalation of Γ'_n and Γ''_n as Γ_n given by

$$\Gamma_0 = \max(\Gamma'_0, \Gamma''_0), \quad \Gamma_n = \max\left((\Gamma' \cup \Gamma'') \setminus \bigcup_{k=0}^{n-1} \Gamma_k\right), \quad n \geq 1.$$

In the next lemma, we compute explicitly the sequences Γ_n, Γ'_n and Γ''_n.

Lemma 2.2.1 *Let $\Phi = (\sqrt{5}-1)/2$, $k \in \mathbb{N}$ and $\eta = 1/(k+\Phi)$. For all $n \geq 0$ we have*

$$\Gamma'_n = \eta\Phi^{2n+1}, \tag{2.2.2}$$

$$\Gamma''_n = \eta\Phi^{2n}, \tag{2.2.3}$$

and

$$\Gamma_n = \begin{cases} \Gamma'_{(n-1)/2}, & \text{if } n \text{ is odd}, \\ \Gamma''_{n/2}, & \text{if } n \text{ is even}. \end{cases} \tag{2.2.4}$$

Proof. Let $(F_n)_{n\geq 0}$, be the Fibonacci sequence, given by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2},$$

for $n \geq 2$.

We begin by proving, by induction on n, that for all $n \geq 0$,

$$\eta\Phi^{2n+1} = F_{2n+1} - (F_{2n+1}k + F_{2n})\eta, \tag{2.2.5}$$

and that for all $n \geq 1$,

$$\eta\Phi^{2n} = (F_{2n}k + F_{2n-1})\eta - F_{2n}. \tag{2.2.6}$$

Clearly, (2.2.5) holds for $n = 0$ and (2.2.6) holds for $n = 1$. Assuming that (2.2.5) holds for n, (2.2.6) holds for $n+1$ and using $1 - \Phi = \Phi^2$, we get $F_{2n+3} - (F_{2n+3}k + F_{2n+2})\eta = \eta\Phi^{2n+3}$. The proof of (2.2.6) is similar to the proof of (2.2.5) and so we omit it.

Using the fact that $(F_n)_{n\geq 0}$ is the Fibonacci sequence and some elementary properties of continued fractions it can be easily proved by induction on n that

$$p_n = F_n, \quad q_n = F_n k + F_{n-1}. \tag{2.2.7}$$

Finally we show that (2.2.2) holds for all $n \geq 0$. It is clear that $\Gamma'_0 = 1 - k\eta = \eta\Phi$ and $\Gamma''_n = \eta\Phi^{2n}$ for $n = 0, 1$. Hence (2.2.2) holds for $n = 0$, and (2.2.3) holds for $n = 0, 1$.

Now assume (2.2.2) holds for all $0 \leq n \leq N$ and (2.2.3) for all $0 \leq n \leq N+1$. We now prove that (2.2.2) holds for $0 \leq n \leq N+1$ and (2.2.3) for all $0 \leq n \leq N+2$. We have $\Gamma''_0 > \Gamma'_0 > \Gamma''_1 > \ldots > \Gamma''_N > \Gamma'_N > \Gamma''_{N+1}$.

Thus, we have (2.2.4) for $n \leq 2(N+1)$, also

$$p''_n = p_{2n}, \quad q''_n = q_{2n}, \tag{2.2.8}$$

for $1 \leq n \leq N+1$, and

$$p'_{n+\eta_1-1} = p_{2n+1}, \quad q'_{n+\eta_1-1} = q_{2n+1}, \tag{2.2.9}$$

for $0 \leq n \leq N$.

By (2.2.5) and (2.2.7), we get $p_{2N+3} - \eta q_{2N+3} = \eta\Phi^{2N+3} > 0$. Thus, from (2.2.8) and (2.2.9) we have $p'_{N+\eta_1} = p_{2N+3}, q'_{N+\eta_1} = q_{2N+3}$ and we get $\Gamma'_{N+1} = \eta\Phi^{2N+3}$. This proves (2.2.2) for $0 \le n \le N+1$.

Now, by (2.2.6) and (2.2.7), $\eta q_{2N+4} - p_{2N+4} = \eta\Phi^{2N+4}$. Thus, from (2.2.8) and (2.2.9) we have $p''_{N+2} = p_{2N+4}, \quad q''_{N+2} = q_{2N+4}$ and we get $\Gamma''_{N+2} = \eta\Phi^{2N+4}$. This proves now (2.2.3) for $0 \le n \le N+2$.

This completes our proof. □

Let

$$k'_0 = \eta_1 + 1, \quad k'_n = \min\{k \ge 1 : g_0^{k'_{n-1}}(1) < g_0^k(1) < 1\}, \tag{2.2.10}$$

for all $n \ge 1$, $s'_n = 1 - g_0^{k'_n}(1)$, and consider the the sequence S' given by

$$S' = (s'_n)_{n\ge 0}.$$

We have $k'_n = \min\{k \ge 1 : 1 - s'_{n-1} < g_0^k(1 - s'_{n-1}) < 1\} + k'_{n-1}$. Also let

$$k''_1 = \eta_1 + 2, \quad k''_n = \min\{k \ge 1 : 1 < g_0^k(1) < g_0^{k''_{n-1}}(1)\}, \tag{2.2.11}$$

for all $n \ge 2$, $s''_0 = \eta$ and $s''_n = g_0^{k''_n}(1) - 1$, for $n \ge 1$. We define another sequence S'' as

$$S'' = (s''_n)_{n\ge 0},$$

Note that $k''_n = \min\{k \ge 1 : 1 < g_0^k(s''_{n-1}+1) < s''_{n-1}+1\} + k''_{n-1}$. We are interested in studying the bifurcation sets Λ_L and Λ_R of g_0. By Theorem 2.1.5 they are discrete with 0 as the only possible point of accumulation, hence they can be regarded as decreasing sequences, which we now define. Let the *right bifurcation sequence* $\Lambda' = (\Lambda'_n)_n$ be given by

$$\Lambda'_0 = \max(\Lambda_R), \quad \Lambda'_n = \max\left(\Lambda_R \setminus \bigcup_{k=0}^{n-1} \Lambda'_k\right), \quad n \ge 1,$$

the *left bifurcation sequence* $\Lambda'' = (\Lambda''_n)_n$ by

$$\Lambda''_0 = \max(\Lambda_L), \quad \Lambda''_n = \max\left(\Lambda_L \setminus \bigcup_{k=0}^{n-1} \Lambda''_k\right), \quad n \ge 1,$$

and finally we define recursively the sequence of all bifurcation points of g_0, Λ_n (it follows from Theorem 2.1.5 that it is equal to the intercalation of Λ' and Λ'')

$$\Lambda_0 = \max(\Lambda'_0, \Lambda''_0), \quad \Lambda_n = \max\left((\Lambda_R \cup \Lambda_L) \setminus \bigcup_{k=0}^{n-1} \Lambda_k\right), \quad n \ge 1.$$

In the next lemma we relate the sequences s'_n and s''_n with Λ'_n and Λ''_n for all $n \ge 0$.

Lemma 2.2.2 *The sequences S', S'' are equal to Λ', Λ'', respectively.*

Proof. We first prove by induction on n, that $s''_n = \Lambda''_n$, for all $n \in \mathbb{N}$. It is clear that $s''_0 = \eta = \Lambda''_0$. Assuming $s''_n = \Lambda''_n$, we have $k''_n = \min\{k \geq 1 : 1 < g_0^k(1) < 1 + \Lambda''_n\}$, and $n_\ell = k''_{n+1}$, for all $\ell \in [s''_{n+1}, \Lambda''_n)$. This shows that $s''_{n+1} \geq \Lambda''_{n+1}$. As $g_0^{k''_{n+1}}(1) = 1 + s''_{n+1}$, we get that $s''_{n+1} = \Lambda''_{n+1}$. This proves that S'' is equal to Λ''.

We now prove, by induction on n, that $s'_n = \Lambda'_n$, for all $n \in \mathbb{N}$. It is clear that $s'_0 = 1 - \eta_1\eta = \Lambda'_0$, where η_1 is the first coefficient in the continued fraction expansion of η. Assume $s'_n = \Lambda'_n$. Let ℓ be a constant such that $s'_{n+1} \leq \ell < s'_n$. Since $s'_n = d^-(k'_{n+1}-1)$, $\ell < d^-(k'_{n+1}-1)$, where $k'_{n+1}-1 > k'_0 - 1 \geq 2$ and with $k = k'_{n+1}$, by Lemma 2.1.3, we have $g_0^{k'_{n+1}}(1+\ell) = g_0(g_0^{k'_{n+1}-1}(1)+\ell)$ for $s'_{n+1} \leq \ell < s'_n$ and since $g_0^{k'_{n+1}}(1) = 1 - s'_{n+1}$ and $\eta < 1$, we have $g_0^{k'_{n+1}-1}(1) = 1 - \eta - s'_{n+1}$. Combining these, we get

$$g_0^{k'_{n+1}}(1+\ell) = 1 - s'_{n+1} + \ell. \tag{2.2.12}$$

By Lemma 2.1.3 we have $g_0^k(1+\ell) = g_0^k(1) + \ell$, for all $1 < k < k'_{n+1}$. Note that $g_0^k(1) \notin (g_0^{k'_n}(1), 1)$. Combining these we get $g_0^k(1+\ell) \notin (1 - s'_n + \ell, 1 + \ell)$, and since η is irrational and $\ell < s'_n$, this gives

$$g_0^k(1+\ell) \notin [1, 1+\ell],$$

for all $k < k'_{n+1}$. Thus from (2.2.12), we get that s'_{n+1} is the largest value ℓ can take such that $g_0^{k'_{n+1}}(1+\ell) = 1$. Since we have $s'_n = \Lambda'_n$, this proves that $s'_{n+1} = \Lambda'_{n+1}$, and so, that S' is equal to Λ'.

□

In the next theorem we relate the sequences of errors of upper/lower semiconvergents of η with the right/left bifurcation sequences of η.

Theorem 2.2.3 *Assume $\eta \in (0,1)$ is an irrational number. The sequences Λ' and Λ'' are equal to Γ' and Γ'', respectively. Moreover, the associated bifurcation sequence Λ is equal to the sequence Γ of errors of the semiconvergents of η.*

Proof. Let $v : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by

$$v(m,n) = \begin{cases} n, & m \leq 1, \\ \eta_2 + \eta_4 + ... + \eta_m + n, & m > 0 \text{ and } m \text{ is even}, \\ \eta_3 + \eta_5 + ... + \eta_m + n, & m > 1 \text{ and } m \text{ is odd}. \end{cases}$$

for $m, n \in \mathbb{N}$.

Since η is irrational, this shows that we have

$$\bigcup_{m \geq 0 \text{ even}} \{v(m,n)\}_{0 \leq n \leq \eta_{m+2}} = \bigcup_{m \geq 1 \text{ odd}} \{v(m,n)\}_{0 \leq n \leq \eta_{m+2}} = \mathbb{N}.$$

From this we get that $\Gamma''_k = \Lambda''_k$ and $\Gamma'_k = \Lambda'_k$ for all $k \in \mathbb{N}$ if and only if for all even $m \geq 0$

$$\Gamma''_{v(m,n)} = \Lambda''_{v(m,n)},$$

for $0 \leq n \leq \eta_{m+2}$, and

$$\Gamma'_{v(m+1,n)} = \Lambda'_{v(m+1,n)},$$

for $0 \leq n \leq \eta_{m+3}$.

It is well known (see for instance [39]) that $[0, \eta_1, ..., \eta_m] = (np_m + p_{m-1})/(nq_m + q_{m-1})$, for all $m, n \in \mathbb{N}$. From this it follows that for all even $m \geq 0$ we have

$$\Gamma''_{v(m,n)} = (q_m\eta - p_m) - n(p_{m+1} - q_{m+1}\eta),$$

for $0 \leq n \leq \eta_{m+2}$, and

$$\Gamma'_{v(m+1,n)} = (p_{m+1} - q_{m+1}\eta) - n(q_{m+2}\eta - p_{m+2}),$$

for $0 \leq n \leq \eta_{m+3}$. Combining the four expressions above it follows that $\Gamma''_k = \Lambda''_k$ and $\Gamma'_k = \Lambda'_k$ for all $k \in \mathbb{N}$ if and only if for all even $m \geq 0$ we have

$$\Lambda''_{v(m,n)} = (q_m\eta - p_m) - n(p_{m+1} - q_{m+1}\eta), \tag{2.2.13}$$

for $0 \leq n \leq \eta_{m+2}$, and

$$\Lambda'_{v(m+1,n)} = (p_{m+1} - q_{m+1}\eta) - n(q_{m+2}\eta - p_{m+2}), \tag{2.2.14}$$

for $0 \leq n \leq \eta_{m+3}$.

We now prove, by induction on m, that (2.2.13) and (2.2.14) hold for all even $m \geq 0$. Before, we prove by induction on n, that

$$\Lambda''_{v(0,n)} = (q_0\eta - p_0) - n(p_1 - q_1\eta), \tag{2.2.15}$$

for $0 \leq n \leq \eta_2$.

We have $v(0,0) = 0$, thus $\Lambda''_{v(0,0)} = \Lambda''_0$. Since $s''_0 = \eta$ and $(p_0, q_0) = (0, 1)$, by Lemma 2.2.2 we have $\Lambda''_0 = q_0\eta - p_0$. Thus, (2.2.15) holds for $n = 0$. Now fix $n < \eta_{m+2}$. We assume (2.2.15) holds for n and prove it for $n + 1$ instead.

We have that (2.2.15) is equivalent to $g_0^{1+n(1+\eta_1)}(1) - 1 = \eta - n(1 - \eta_1\eta)$, therefore since we are assuming (2.2.15) holds for n we get

$$g_0^{1+(n+1)(1+\eta_1)}(1) - 1 = g_0(g_0^{\eta_1}(1 + \Lambda''_{v(0,n)})). \tag{2.2.16}$$

Recall the definition of d^-. We have $d^-(N) = 1 - \max_{1\leq k\leq N}\left\{g_0^k(1) \leq 1\right\}$, for any $N \geq 2$. Note that $d^-(\eta_1) = 1 - (\eta_1 - 1)\eta$, therefore $\Lambda''_{v(0,n)} \leq d^-(\eta_1)$.

Assume now that $\eta_1 \geq 2$. Applying Lemma 2.1.3 with $\ell = \Lambda''_{v(0,n)}$ and $N = \eta_1$ we get $g_0^{\eta_1}(1 + \Lambda''_{v(0,n)}) = g_0^{\eta_1}(1) + \Lambda''_{v(0,n)}$. Since $1 - g_0^{\eta_1+1}(1) = s'_0$, $s'_0 = 1 - \eta_1\eta$ and $\eta < 1$ we have $g_0^{\eta_1}(1) = (\eta_1 - 1)\eta$. Combining this we get

$$g_0^{\eta_1}(1 + \Lambda''_{v(0,n)}) = 1 - \eta + (\eta_1 - (n+1)(1 - \eta_1\eta)),$$

which combined with (2.2.16) gives

$$g_0^{1+(n+1)(1+\eta_1)}(1) = 1 + (\eta_1 - (n+1)(1 - \eta_1\eta)), \tag{2.2.17}$$

which is smaller than $1 + \Lambda''_{v(0,n)}$.

If $\eta_1 = 1$ it is clear from (2.2.16) that we get (2.2.17) as well. Since Γ'' is the sequence of best rational approximations of η by below and we have $\Lambda''_{v(0,n)} = \Gamma''_{v(0,n)}$ and $g_0^{1+(n+1)(1+\eta_1)}(1) - 1 = \Gamma''_{v(0,n)+1}$, we must have

$$1 + (n+1)(1 + \eta_1) = \min\{k \geq 1 : 1 < g_0^k(1) < g_0^{1+n(1+\eta_1)}(1)\},$$

and thus by Lemma 2.2.2 and (2.2.17) we have $\Lambda''_{v(0,n+1)} = (q_0\eta - p_0) - (n+1)(p_1 - q_1\eta)$. This completes the proof that (2.2.15) holds for $0 \leq n \leq \eta_{m+2}$.

In a similar way, it can be proved that

$$\Lambda'_{v(1,n)} = (p_1 - q_1\eta) - n(q_2\eta - p_2),$$

for all $0 \leq n \leq \eta_3$, so we omit the proof.

We now assume that for $0 \leq n \leq \eta_{m+2}$, we have

$$\Lambda''_{v(m,n)} = (q_m\eta - p_m) - n(p_{m+1} - q_{m+1}\eta), \tag{2.2.18}$$

and that for $0 \leq n \leq \eta_{m+3}$, we have

$$\Lambda'_{v(m+1,n)} = (p_{m+1} - q_{m+1}\eta) - n(q_{m+2}\eta - p_{m+2}). \tag{2.2.19}$$

and prove that we have

$$\Lambda''_{v(m+2,n)} = (q_{m+2}\eta - p_{m+2}) - n(p_{m+3} - q_{m+3}\eta), \tag{2.2.20}$$

for all $0 \leq n \leq \eta_{m+4}$, and

$$\Lambda'_{v(m+3,n)} = (p_{m+3} - q_{m+3}\eta) - n(q_{m+4}\eta - p_{m+4}). \tag{2.2.21}$$

for all $0 \leq n \leq \eta_{m+5}$.

First we prove (2.2.20), by induction on n, for all $n \leq \eta_{m+4}$.

Since $v(m+2,0) = v(m,\eta_{m+2})$, by (2.2.18) and (2.2.1), we get $\Lambda''_{v(m+2,0)} = q_{m+2}\eta - p_{m+2}$. Thus, (2.2.20) holds for $n=0$.

Fix $n < \eta_{m+4}$. We assume that (2.2.20) holds for n and prove it for $n+1$ instead. Recall the definition of Λ''. With $K''(n) = p_{m+2} + q_{m+2} + n(p_{m+3} + q_{m+3})$, we have that (2.2.20) is equivalent to $g_0^{K''(n)}(1) - 1 = q_{m+2}\eta - p_{m+2} - n(p_{m+3} - q_{m+3}\eta)$, and combining these we get

$$g_0^{K''(n+1)}(1) = g_0(g_0^{(p_{m+3}+q_{m+3}-1)}(1 + \Lambda''_{v(m+2,n)})). \tag{2.2.22}$$

From (2.2.19) we get

$$\Lambda'_{v(m+1,n)} = \Gamma'_{v(m+1,n)},$$

for $0 \le n \le \eta_{m+3}$. It follows from this identity and fact that the upper semi-convergents of η are its best rational approximations by above, that $v(m+1,\eta_{m+3}-1)$ is the largest integer such that $k'_{v(m+1,\eta_{m+3}-1)} < q_{m+3}+p_{m+3}$, where k' is as in (2.2.10). Recall the definition of d^-. We have $d^-(N) = 1 - \max_{1\le k\le N}\left\{g_0^k(1) \le 1\right\}$, for any $N \ge 2$, thus $d^-(q_{m+3} + p_{m+3} - 1) = s'_{\eta_{m+3}-1}$, and by Lemma 2.2.2, (2.2.19) and (2.2.1) we get

$$d^-(q_{m+3} + p_{m+3} - 1) = p_{m+3} - q_{m+3}\eta + q_{m+2}\eta - p_{m+2}.$$

Therefore $\Lambda''_{v(m+2,n)} < d^-(q_{m+3} + p_{m+3} - 1)$. Applying Lemma 2.1.3 with $\ell = \Lambda''_{v(m+2,n)}$ and $N = p_{m+3} + q_{m+3} - 1$ yields

$$g_0^{p_{m+3}+q_{m+3}-1}(1 + \Lambda''_{v(m+2,n)}) = g_0^{p_{m+3}+q_{m+3}-1}(1) + \Lambda''_{v(m+2,n)}. \tag{2.2.23}$$

By Lemma 2.2.2, (2.2.19) and (2.2.1) we have $1 - g_0^{p_{m+3}+q_{m+3}}(1) = p_{m+3} - q_{m+3}\eta$, and since $\eta < 1$, we get

$$g_0^{p_{m+3}+q_{m+3}}(1) = 1 - \eta - (p_{m+3} - q_{m+3}\eta).$$

Combining this identity with (2.2.22) and (2.2.23) we have

$$g_0^{K''(n+1)}(1) = g_0(1 - \eta - (p_{m+3} - q_{m+3}\eta) + \Lambda''_{v(m+2,n)}),$$

and since (2.2.20) holds for n we get

$$g_0^{K''(n+1)}(1) = 1 + (q_{m+2}\eta - p_{m+2}) - (n+1)(p_{m+3} - q_{m+3}\eta), \tag{2.2.24}$$

which is smaller than $1 + \Lambda''_{v(m+2,n)}$.

Since Γ'' is the sequence of best rational approximations of η by below and we have $\Lambda''_{v(m+2,n)} = \Gamma''_{v(m+2,n)}$ and $g_0^{K(n+1)}(1) - 1 = \Gamma''_{v(m+2,n)+1}$, we must have

$$K''(n+1) = \min\{k \ge 1 : 1 < g_0^k(1) < g_0^{K''(n)}(1)\},$$

and thus by Lemma 2.2.2 and (2.2.24) we have

$$\Lambda''_{v(m+2,n+1)} = (q_{m+2}\eta - p_{m+2}) - (n+1)(p_{m+3} - q_{m+3}\eta).$$

This completes the proof that (2.2.20) holds for $0 \leq n \leq \eta_{m+4}$.

We now prove (2.2.21) by induction on n, for all $n \leq \eta_{m+5}$.

Since $v(m+3,0) = v(m+1, \eta_{m+3})$, by (2.2.1) and (2.2.19), we get $\Lambda'_{v(m+3,0)} = p_{m+3} - q_{m+3}\eta$ and so (2.2.21) holds for $n = 0$.

Fix $n < \eta_{m+5}$. We assume that (2.2.21) holds for n and prove it for $n+1$ instead.

Recall the definition of Λ'. With $K'(n) = p_{m+3} + q_{m+3} + n(p_{m+4} + q_{m+4})$, we have that (2.2.21) is equivalent to $1 - g_0^{K'(n)}(1) = p_{m+3} - q_{m+3}\eta - n(q_{m+4}\eta - p_{m+4})$, and we get

$$g_0^{K'(n+1)}(1) = g_0(g_0^{(p_{m+4}+q_{m+4}-1)}(1 - \Lambda'_{v(m+3,n)})). \tag{2.2.25}$$

From (2.2.20) we get $\Lambda''_{v(m+2,n)} = \Gamma''_{v(m+2,n)}$, for $0 \leq n \leq \eta_{m+4}$. It follows from this identity and from the fact that the lower semi-convergents of η are its best rational approximations by below, that $v(m+2, \eta_{m+4} - 1)$ is the largest integer such that $k''_{v(m+2,\eta_{m+4}-1)} < q_{m+4} + p_{m+4}$, where k'' is as in (2.2.11).

Recall the definition of d^+. We have $d^+(N) = \min_{1\leq k\leq N} \left\{g_0^k(1) \geq 1\right\} - 1$, for any $N \geq \eta_1 + 1$. Thus, $d^+(q_{m+4} + p_{m+4} - 1) = s''_{\eta_{m+4}-1}$. By Lemma 2.2.2, (2.2.20) and (2.2.1) we get $d^+(q_{m+4} + p_{m+4} - 1) = q_{m+4}\eta - p_{m+4} + p_{m+3} - q_{m+3}\eta$. Therefore $\Lambda'_{v(m+3,n)} < d^+(q_{m+4} + p_{m+4} - 1)$. Applying Lemma 2.1.3 with $\ell = 1 - \Lambda'_{v(m+3,n)}$ and $N = p_{m+4} + q_{m+4} - 1$ we get

$$g_0^{p_{m+4}+q_{m+4}-1}(1 - \Lambda'_{v(m+3,n)}) = g_0^{p_{m+4}+q_{m+4}-1}(1) - \Lambda'_{v(m+3,n)}.$$

By Lemma 2.2.2, (2.2.20) and (2.2.1) we have $g_0^{p_{m+4}+q_{m+4}}(1) - 1 = q_{m+4}\eta - p_{m+4}$, and since $\eta < 1$ we get

$$g_0^{p_{m+4}+q_{m+4}}(1) = 1 - \eta + (q_{m+4}\eta - p_{m+4}).$$

Combining the two above identities with (2.2.25) and since (2.2.21) holds for n we get

$$g_0^{K'(n+1)}(1) = 1 - \left[(p_{m+3} - q_{m+3}\eta) - (n+1)(q_{m+4}\eta - p_{m+4})\right], \tag{2.2.26}$$

which is larger than $1 - \Lambda'_{v(m+3,n)}$.

Since Γ' is the sequence of best rational approximations of η by above and we have $\Lambda'_{v(m+3,n)} = \Gamma'_{v(m+3,n)}$ and $1 - g_0^{K'(n+1)}(1) = \Gamma'_{v(m+3,n)+1}$, we must have

$$K'(n+1) = \min\{k \geq 1 : g_0^{K'(n)}(1) < g_0^k(1) < 1\},$$

and thus by Lemma 2.2.2 and (2.2.26) we have

$$\Lambda'_{v(m+3,n+1)} = (p_{m+3} - q_{m+3}\eta) - (n+1)(q_{m+4}\eta - p_{m+4}).$$

This completes the proof that (2.2.21) holds for $0 \leq n \leq \eta_{m+5}$.

This proves (2.2.13) and (2.2.14) and therefore Λ' is equal to the sequence Γ' and Λ'' is equal to the sequence Γ''. By definition of Λ and Γ, this implies that $\Gamma = \Lambda$ as well. This finishes our proof.

□

Recall, from (2.2.10) and (2.2.11), the definitions of k' and k''. Also recall our definitions of first hitting time $n_\ell(x)$ of x to $\overline{I_c(\ell)}$, in (2.1.4), and the first hitting map r_ℓ, in (2.1.5). We want now to relate these to Γ' and Γ''. This is done in the next theorem.

Theorem 2.2.4 *Let $0 < \ell \leq \eta$ and let $n_1, n_2 \in \mathbb{N}$ be such that $\Gamma'_{n_1+1} \leq \ell < \Gamma'_{n_1}$ and $\Gamma''_{n_2+1} \leq \ell < \Gamma''_{n_2}$. Then $r_\ell(1+\ell) = 1 + \ell - \Gamma'_{n_1+1}$ and $r_\ell(1) = 1 + \Gamma''_{n_2+1}$. Furthermore $n_\ell(1+\ell) = k'_{n_1+1}$ and $n_\ell(1) = k''_{n_2+1}$.*

Proof. We prove only that $r_\ell(1+\ell) = 1 + \ell - \Gamma'_{n_1+1}$. As the proof for the other case is similar, we omit it.

By Theorem 2.2.3, we have $\Gamma' = \Lambda'$, therefore $\Gamma'_{n_1+1} \leq \ell < \Gamma'_{n_1}$ implies that $\Lambda'_{n_1+1} \leq \ell < \Lambda'_{n_1}$. Also, combining Theorem 2.2.3 with Lemma 2.2.2, we have $S' = \Gamma'$ and we get

$$g_0^{k'_{n_1+1}}(1) = 1 - \Gamma'_{n_1+1}. \tag{2.2.27}$$

As $k_{n_1+1} > 2$ and $\Gamma'_{n_1+1} \leq d^-(k'_{n_1+1})$, applying Lemma 2.1.3 we get

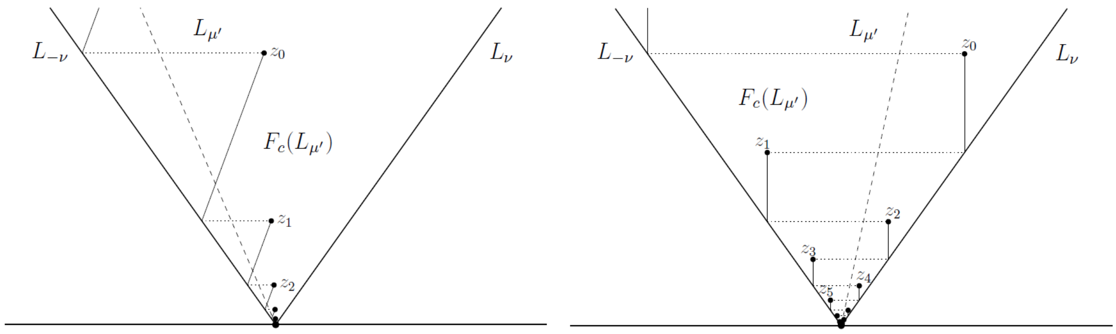
$$g_0^{k'_{n_1+1}}(1 + \Gamma'_{n_1+1}) = g_0^{k'_{n_1+1}}(1) + \Gamma'_{n_1+1}.$$

From these two identities we get $g_0^{k'_{n_1+1}}(1+\Gamma'_{n_1+1}) = 1$, thus $n_{\Gamma'_{n_1+1}}(1+\Gamma'_{n_1+1}) = k'_{n_1+1}$. Therefore $r_\ell(1+\ell) = g_0^{k'_{n_1+1}}(1+\ell)$.

As $\ell < d^-(k_{n_1+1}-1)$ by Lemma 2.1.3, $g_0^{k'_{n_1+1}-1}(1+\ell) = g_0^{k'_{n_1+1}-1}(1) + \ell$. Applying g_0 on both sides and combining with (2.2.27), we get $g_0^{k'_{n_1+1}}(1+\ell) = 1 + \ell - \Gamma'_{n_1+1}$. This finishes our proof. □

2.3 Dynamical sequences

In this section, we introduce the *dynamical sequences* $(y_n)_n$ and $(u_n)_n$. We define these sequences abstractly, independently from any dynamical interpretation. These will be an important tool in order to prove our main theorems and will be later



(a) (b)

Figure 2.1: An illustration of the action of map F_c. The dashed line is $L_{\mu'}$ and the union of disjoint line segments is $F_c(L_{\mu'})$. Also marked are the points $z_n = (u_n - 1/2)\ell(y_n) + iy_n$. (A) shows points z_n such that the corresponding sequence $u_n(\mu)$ satisfies $u_n(\mu) = u_{n+1}(\mu)$ for all $n \geq 1$. (B) shows points z_n such that the corresponding sequence $u_n(\mu)$ satisfies $u_n(\mu) = u_{n+2}(\mu)$ for all $n \geq 0$.

related to the dynamics of our family of maps. Indeed (y_n) will be the sequence of imaginary parts of the discontinuities of a map ρ (defined in the following section) containing information related to the first return under our transformation F to the central cone P_c , while (u_n) is the sequence of ratios of the horizontal jumps produced by discontinuities of ρ relative to the cone width $\ell(y_n)$.

In this section we show inductive formulas to compute these sequences and prove that for some choice of parameters $(u_n)_n$ is periodic with period at most 2.

Let

$$\ell(y) = \frac{2y}{\nu}, \tag{2.3.1}$$

and denote

$$C_\mu^+ = \frac{2\mu}{\mu + \nu}, \quad C_\mu^- = \frac{2\mu}{\mu - \nu}. \tag{2.3.2}$$

We now inductively define our sequences $(y_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ and $(\kappa_n)_{n\in\mathbb{N}}$ depending on the parameters $\nu > 0$, $|\mu| > \nu$, $\eta \in (0,1)\backslash\mathbb{Q}$ and $0 < \eta' < \eta$. We will denote these sequences by $(y_n(\mu))_{n\in\mathbb{N}}$, $(u_n(\mu))_{n\in\mathbb{N}}$ and $(\kappa_n(\mu))_{n\in\mathbb{N}}$ when it is important to stress the dependence on the parameter μ.

Set

$$y_0 = \eta' \frac{\mu\nu}{\mu + \nu}. \tag{2.3.3}$$

Note that as $\eta' > 0$ we have $y_0 > 0$. Since η is irrational, (Γ''_n) (see Section 2.2) is an infinite sequence and converges to 0. Furthermore by the definitions of ℓ and y_0 we have $\ell(y_0) > 0$. Thus there exists a smallest natural number κ_0 such that

$\Gamma''_{\kappa_0} < \ell(y_0)$. Set $\Upsilon_0 = \Gamma''_{\kappa_0}$ and

$$u_0 = \frac{\Upsilon_0}{\ell(y_0)}.$$

For $n \geq 0$ assume we defined y_n, u_n, κ_n and Υ_n and that at least one of the conditions $y_n > 0$ or $u_n = 1/C_\mu^+$ holds. Set

$$y_{n+1} = \begin{cases} (1 - C_\mu^+ u_n) y_n, & \text{if } u_n < 1/C_\mu^+, \\ (1 - C_\mu^-(1-u_n)) y_n, & \text{if } u_n > 1/C_\mu^+, \\ 0, & \text{if } u_n = 1/C_\mu^+. \end{cases} \tag{2.3.4}$$

Since η is irrational, (Γ'_n) and (Γ''_n) (see Section 2.2) are infinite sequences and converge to 0, furthermore if $u_n \neq 1/C_\mu^+$, by (2.3.2) and (2.3.4) we have $\ell(y_{n+1}) > 0$, thus there are integers k' and k'' such that $\Gamma'_k < \ell(y_{n+1})$ and $\Gamma''_k < \ell(y_{n+1})$ respectively. We set

$$\kappa_{n+1} = \begin{cases} \min\{k \in \mathbb{N} : \Gamma''_k < \ell(y_{n+1})\}, & \text{if } u_n < 1/C_\mu^+, \\ \min\{k \in \mathbb{N} : \Gamma'_k < \ell(y_{n+1})\}, & \text{if } u_n > 1/C_\mu^+, \\ \kappa_n, & \text{if } u_n = 1/C_\mu^+. \end{cases}$$

If $u_n \leq 1/C_\mu^+$ set $\Upsilon_{n+1} = \Gamma''_{\kappa_{n+1}}$, else if $u_n > 1/C_\mu^+$ set

$$\Upsilon_{n+1} = \begin{cases} 1, & \text{if } \ell(y_{n+1}) > 1, \\ 1 - \left(1 + \left[\frac{1-\ell(y_{n+1})}{\eta}\right]\right)\eta, & \text{if } \Gamma'_0 < \ell(y_{n+1}) \leq 1, \\ \Gamma'_{\kappa_{n+1}}, & \text{if } \ell(y_{n+1}) < \Gamma'_0, \end{cases} \tag{2.3.5}$$

where $[\cdot]$, denotes the integer part of a real number. Finally set

$$u_{n+1} = \begin{cases} \dfrac{\Upsilon_{n+1}}{\ell(y_{n+1})}, & \text{if } u_n < 1/C_\mu^+, \\ 1 - \dfrac{\Upsilon_{n+1}}{\ell(y_{n+1})}, & \text{if } u_n > 1/C_\mu^+, \\ 0, & \text{if } u_n = 1/C_\mu^+. \end{cases} \tag{2.3.6}$$

The following lemma characterizes the sequence $(y_n)_{n\in\mathbf{N}}$.

Lemma 2.3.1 *Given $\nu > 0$, $|\mu| > \nu$, $\eta \in (0,1)\backslash\mathbb{Q}$ and $0 < \eta' < \eta$, the sequence $(y_n)_{n\in\mathbf{N}}$ with $\mathbf{N} = \{n \in \mathbb{N} : y_n > 0\}$ is strictly decreasing and it is either finite or converges to* 0.

Proof. If $u_n < 1/C_\mu^+$ or $u_n > 1/C_\mu^+$ then $(1 - C_\mu^+ u_n) \in (0,1)$ or $(1 - C_\mu^-(1-u_n)) \in (0,1)$, respectively, and by (2.3.4), $y_{n+1} < y_n$. If $u_n = 1/C_\mu^+$ then, by definition,

$y_{n+1} = 0$ and thus $n+1 \notin \mathbf{N}$. This shows that $(y_n)_{n\in\mathbf{N}}$ is strictly decreasing and either $\mathbf{N}$ is finite or for all $n \in \mathbb{N}$ we have $y_{n+1} < y_n$.

We now show that if $y_n > 0$ we have $y_n \to 0$. Assume by contradiction that (y_n) does not converge to 0. Since it is strictly decreasing there must exist $y' > 0$ such that $y_n \to y'$. Since for all $n \in \mathbb{N}$, $u_n \neq 1/C_\mu^+$, we must have that either $u_n < 1/C_\mu^+$ or $u_n > 1/C_\mu^+$ for infinitely many values of $n \in \mathbb{N}$.

Assume the first case holds. Then there is a subsequence $(u_{n(l)})_{l\in\mathbb{N}}$ such that $u_{n(l)} < 1/C_\mu^+$ for all $l \in \mathbb{N}$. Since $y_n \to y'$ we have in particular that

$$\lim_{l\to+\infty} y_{n(l)+1} = \lim_{l\to+\infty} y_{n(l)} = y'$$

and by (2.3.4) $y_{n(l)+1} = (1 - C_\mu^+ u_{n(l)})y_{n(l)}$, thus, we must have $1 - C_\mu^+ u_{n(l)} \to 1$ and therefore $u_{n(l)} \to 0$. Hence by the definition of u_n and since $\ell(y_{n(l)}) \to \ell(y')$, we have $\Gamma''_{\kappa_{n(l)}+1} \to 0$.

Since η is irrational, $\{\Gamma''_n\}$ is an infinite sequence and converges to 0. Thus there exists an unique natural number k' such that $\Gamma''_{k'+1} < \ell(y') \leq \Gamma''_{k'}$, and by the definition of κ_n we must have $\kappa_{n(l)} \to k'$. Thus we get $\Gamma''_{k'+1} = 0$, which implies that η is rational which is a contradiction.

The proof is analogous if $u_n > 1/C_\mu^+$ for infinitely many values of $n \in \mathbb{N}$, hence we omit it. □

Given $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we introduce the following maps:

$$\chi_n(x) = \begin{cases} x & \text{, if } u_n < 1/C_\mu^+, \\ 1-x & \text{, if } u_n > 1/C_\mu^+, \\ 1 & \text{, if } u_n = 1/C_\mu^+, \end{cases} \quad \text{and} \quad \psi_n(x) = \begin{cases} C_\mu^+ & \text{, if } u_n < 1/C_\mu^+, \\ C_\mu^- & \text{, if } u_n > 1/C_\mu^+, \\ 1 & \text{, if } u_n = 1/C_\mu^+. \end{cases}$$

The next lemma, which follows direcly from combining the above expressions with (2.3.4), (2.3.5) and (2.3.6), gives recursive expressions for y_n and u_n.

Lemma 2.3.2 *Given $\nu > 0$ and $\mu \in \mathbb{R}$ satisfying $|\mu| > \nu$, for all $n \in \mathbf{N}\backslash\{0\}$ we have*

$$y_n(\mu) = (1 - \psi_{n-1}\chi_{n-1}(u_{n-1}(\mu)))y_{n-1}(\mu).$$

Moreover if $u_{n-1}(\mu) \neq 1/C_\mu^+$, we have

$$\chi_{n-1}(u_n(\mu)) = \frac{\Upsilon_n(\mu)}{\Upsilon_{n-1}(\mu)} \frac{\chi_{n-2}(u_{n-1}(\mu))}{1 - \psi_{n-1}\chi_{n-1}(u_{n-1}(\mu))}.$$

Next theorem provides, under some conditions on η and η', a closed form expression for the sequence $\{u_n\}$ and shows that it is periodic with period at most 2. We will denote

$$\bar{\mu} = \frac{\nu}{\Phi^3}.$$

Theorem 2.3.3 *Assume $\nu > 0$, $\eta = 1/(k + \Phi)$ and $\eta' = 1 - k\eta$ with $k \in \mathbb{N}$. Let $\mu \in \mathbb{R}$ be such that $|\mu| \geq \nu$. If $|\mu| > \bar{\mu}$, then $u_n(\mu) = u_{n+2}(\mu)$ for all $n \geq 0$, in particular*

$$u_n(\mu) = \frac{1}{C_\mu^+ \Phi} \text{ and } \ell(y_n(\mu)) = C_\mu^+ \eta \Phi^{n+1}, \text{ if } n \text{ is even,} \tag{2.3.7}$$

and

$$u_n(\mu) = 1 - \frac{1}{C_\mu^- \Phi} \text{ and } \ell(y_n(\mu)) = C_\mu^- \eta \Phi^{n+1}, \text{ if } n \text{ is odd.} \tag{2.3.8}$$

If $|\mu| \leq \bar{\mu}$, then $u_n(\mu) = u_{n+1}(\mu)$ for all $n \geq 1$. In particular, if $-\bar{\mu} < \mu < -\nu$, then for all $n \geq 1$,

$$u_n(\mu) = 1 - \frac{\Phi}{C_\mu^-} \text{ and } \ell(y_n(\mu)) = C_\mu^- \eta \Phi^{2n}. \tag{2.3.9}$$

If $\nu < \mu < \bar{\mu}$, then for all $n \geq 0$,

$$u_n(\mu) = \frac{\Phi}{C_\mu^+} \text{ and } \ell(y_n(\mu)) = C_\mu^+ \eta \Phi^{2n+1}. \tag{2.3.10}$$

Proof.

Let us first investigate C_μ^+ and C_μ^- as in (2.3.2). It is clear that

$$\frac{1}{\Phi} < 2 < C_\mu^- = \frac{2\mu}{\mu - \nu} < \frac{2}{1 - \Phi^3} = \frac{1}{\Phi^2},$$

where we used the fact that $\partial C_\mu^- / \partial \mu = -\nu/(\mu - \nu) < 0$, as long as $\nu > 0$ and also $\Phi^2 = 1 - \Phi$. Now, if $\mu < -\bar{\mu} < -\nu < 0$ we have $1 < C_\mu^- < 2 < 1/\Phi^2$. Since $\mu < -\bar{\mu}$ we get

$$\mu < -\bar{\mu} = -\frac{\nu}{\Phi^3} = -\frac{\nu}{2\Phi - 1},$$

which is equivalent to $2\mu\Phi < \mu - \nu$, and since $\mu < 0$ we get $C_\mu^- > \frac{2\mu}{(\mu - \nu)} > \frac{1}{\Phi}$.

Since C_μ^+ and C_μ^- are Hölder conjugate we have $\frac{1}{1-\Phi^2} < C_\mu^+ < \frac{1}{1-\Phi}$. Combining this we get that if $|\mu| > \bar{\mu}$, then

$$\frac{1}{\Phi} < C_\mu^+ < \frac{1}{\Phi^2}, \quad \frac{1}{\Phi} < C_\mu^- < \frac{1}{\Phi^2}. \tag{2.3.11}$$

We now prove by induction on n that if $|\mu| > \bar{\mu}$, we have (2.3.7) and (2.3.8) for all $n \geq 0$.

From (2.3.3) we have $\ell(y_0(\mu)) = 2\mu(1 - k\eta)/(\mu + \nu)$. Since $\eta\Phi = 1 - k\eta$ we get $\ell(y_0(\mu)) = C_\mu^+ \eta \Phi$. For $|\mu| > \bar{\mu}$ we get from (2.3.11) that

$$\eta < \ell(y_0(\mu)) < \eta/\Phi \leq 1.$$

Hence we have that $\kappa_0 = 0$, and by Lemma 2.2.1 and (2.3.5) we get $\Upsilon_0(\mu) = \Gamma_0$ and by definition of u_0 we have

$$u_0 = \frac{\Gamma_0}{\ell(y_0(\mu))} = \frac{\Gamma_0 \nu}{2y_0} = \frac{\Gamma_0}{2\mu\eta'}(\mu + \nu) = \frac{\eta}{C_\mu^+ \eta'} = \frac{1}{C_\mu^+ \Phi}.$$

Thus we get (2.3.7) for $n = 0$.

Since $\Phi < 1$ and (2.3.7) holds for $n = 0$, we have $u_0(\mu) > 1/C_\mu^+$, hence by Lemma 2.3.2 we have

$$\ell(y_1(\mu)) = (1 - C_\mu^-(1 - u_0(\mu)))\ell(y_0(\mu)). \tag{2.3.12}$$

Simple computations show that

$$1 - C_\mu^-(1 - u_0(\mu)) = 1 - C_\mu^-(1 - \frac{\eta}{C_\mu^+\eta'}) = \frac{C_\mu^-}{C_\mu^+}\Phi, \tag{2.3.13}$$

where we used Hölder conjugacy of C_μ^+ and C_μ^- several times to simplify the expression. By (2.3.13) and (2.3.12) we have $\ell(y_1(\mu)) = \frac{C_\mu^-}{C_\mu^+}\Phi\ell(y_0(\mu))$ and since (2.3.7) holds for $n = 0$, we get

$$\ell(y_1(\mu)) = C_\mu^-\eta\Phi^2. \tag{2.3.14}$$

Now, by Lemma 2.3.2 we have

$$u_1(\mu) = 1 - \frac{\Upsilon_1(\mu)}{\eta}\frac{u_0(\mu)}{1 - C_\mu^-(1 - u_0(\mu))}, \tag{2.3.15}$$

and by (2.3.14) and (2.3.11), since $|\mu| > \bar{\mu}$, we get $\eta\Phi < \ell(y_1(\mu)) < \eta$. This together with Lemma 2.2.1 and (2.3.5) shows that $\Upsilon_1(\mu) = \eta\Phi$, and from (2.3.13) and (2.3.15) we get $u_1(\mu) = 1 - 1/(C_\mu^-\Phi)$. Together with (2.3.14) this shows (2.3.8) holds for $n = 1$.

Let $n \geq 0$ be an even number. We now assume that (2.3.7) holds for n, (2.3.8) holds for $n + 1$ and prove that (2.3.7) holds for $n + 2$ and (2.3.8) holds for $n + 3$.

Note that since we assume (2.3.7) holds for n and (2.3.8) for $n + 1$, by (2.3.11) we have

$$\eta\Phi^n < \ell(y_n(\mu)) < \eta\Phi^{n-1}, \quad \eta\Phi^{n+1} < \ell(y_{n+1}(\mu)) < \eta\Phi^n. \tag{2.3.16}$$

Since $1/\Phi > 1$ we have $u_1(\mu) = 1 - \frac{1}{d\Phi} < 1 - \frac{1}{C_\mu^-} = \frac{1}{C_\mu^+}$, thus $u_{n+1}(\mu) < 1/C_\mu^+$, since we also have $u_n(\mu) > 1/C_\mu^+$ we get from Lemma 2.3.2 that

$$\ell(y_{n+2}(\mu)) = (1 - C_\mu^+u_1(\mu))(1 - C_\mu^-(1 - u_0(\mu)))\ell(y_n(\mu)). \tag{2.3.17}$$

After a simple computation we have

$$1 - C_\mu^+u_1(\mu) = \frac{\Phi}{d-1} = \frac{C_\mu^+}{C_\mu^-}\Phi. \tag{2.3.18}$$

Combining (2.3.13), (2.3.17) and (2.3.18) we get

$$\ell(y_{n+2}(\mu)) = \Phi^2\ell(y_n(\mu)). \tag{2.3.19}$$

Since we assume (2.3.7) for n, we get from (2.3.19) that

$$\eta\Phi^{n+2} < \ell(y_{n+2}(\mu)) < \eta\Phi^{n+1}. \tag{2.3.20}$$

We now prove that (2.3.7) holds for $n+2$. Since (2.3.7) holds for n, from (2.3.19) we get $\ell(y_{n+2}(\mu)) = C^+_\mu\eta\Phi^{n+3}$. To see that $u_{n+2}(\mu) = u_0(\mu)$ note that by Lemma 2.2.1, by the definition of Υ_{n+1} and by (2.3.16) and (2.3.20), we have $\Upsilon_{n+1}(\mu) = \Gamma_{n+1}$ and $\Upsilon_{n+2}(\mu) = \Gamma_{n+2}$. Together with $u_n(\mu) > 1/C^+_\mu$ and $u_{n+1}(\mu) < 1/C^+_\mu$, Lemma 2.3.2 implies

$$u_{n+2}(\mu) = \frac{\Gamma_{n+2}}{\Gamma_{n+1}}\frac{1-u_{n+1}(\mu)}{1-C^+_\mu u_{n+1}(\mu)} = \Phi\frac{\frac{1}{C^-_\mu\Phi}}{1-C^+_\mu+\frac{C^+_\mu}{C^-_\mu\Phi}} = \frac{1}{C^+_\mu\Phi}.$$

Finally we prove that (2.3.8) holds for $n+3$. Since (2.3.8) holds for $n+1$ and (2.3.7) holds for $n+2$, from (2.3.19) we get $\ell(y_{n+3}(\mu)) = C^-_\mu\eta\Phi^{n+4}$. By (2.3.11) this gives

$$\eta\Phi^{n+3} < \ell(y_{n+3}(\mu)) < \eta\Phi^{n+2}.$$

To see that $u_{n+3}(\mu) = u_1(\mu)$ note that the above inequalities, by Lemma 2.2.1 and by the definition of Υ_{n+1}, we have $\Upsilon_{n+3}(\mu) = \Gamma_{n+3}$. Together $u_{n+1}(\mu) < 1/C^+_\mu$ and $u_{n+2}(\mu) > 1/C^+_\mu$, by Lemma 2.3.2 this gives

$$u_{n+3}(\mu) = 1-\frac{\Gamma_{n+3}}{\Gamma_{n+2}}\frac{u_{n+2}(\mu)}{1-C^-_\mu(1-u_{n+2}(\mu))} = 1-\Phi\frac{\frac{1}{C^+_\mu\Phi}}{1-C^-_\mu(1-\frac{1}{C^+_\mu\Phi})} = 1-\frac{1}{C^-_\mu\Phi}.$$

Therefore if $|\mu| > \bar{\mu}$, (2.3.7) and (2.3.8) holds for all $n \geq 0$ and thus $u_n(\mu) = u_{n+2}(\mu)$ for all $n \geq 0$.

We now prove that if $-\bar{\mu} < \mu < -\nu$, we have

$$\Phi < C^-_\mu < \frac{1}{\Phi}. \tag{2.3.21}$$

Since $\eta < 1$ and $C^-_\mu > 1$ the left inequality follows. Since $\mu > -\bar{\mu}$ and $C^-_{-\bar{\mu}} = 1/\Phi$ we get $C^-_\mu < 1/\Phi$.

We now prove by induction on n that if $-\bar{\mu} < \mu < -\nu$, we have (2.3.9) for all $n \geq 1$. From (2.3.3) and $\eta\Phi = 1-k\eta$ we get $\ell(y_0(\mu)) = C^+_\mu\eta\Phi$. Since $\mu < -\nu$, we have $\Phi < \ell(y_0(\mu)) < +\infty$, hence $\kappa_0 = 0$ and by Lemma 2.2.1 we get $\Upsilon_0(\mu) = \Gamma_0$. Thus by Lemma 2.3.2 we have

$$u_0(\mu) = \frac{1}{C^+_\mu\Phi},$$

from which we get $u_0(\mu) > 1/C^+_\mu$, hence (2.3.12-2.3.15) hold. By (2.3.14) and (2.3.21), since $\mu > -\bar{\mu}$, we have $\eta\Phi^3 < \ell(y_1(\mu)) < \eta\Phi$. This together with Lemma

2.2.1 and by the definition of Υ_{n+1} shows that $\Upsilon_1(\mu) = \Gamma_3$, and from (2.3.13) and (2.3.15) we get $u_1(\mu) = 1 - \Phi/C_\mu^-$. Hence (2.3.9) holds for $n = 1$. We now assume that (2.3.9) holds for n and prove that (2.3.9) holds for $n+1$. Since $\Phi < 1$ we have $1 - \Phi/C_\mu^- > 1 - 1/C_\mu^- = 1/C_\mu^+$, thus by Lemma 2.3.2,

$$\ell(y_{n+1}(\mu)) = (1 - C_\mu^-(1 - u_1(\mu)))\ell(y_n(\mu)),$$

and as (2.3.9) holds for n, combining this with (2.3.21) we get

$$\eta\Phi^{2n+1} < \ell(y_n(\mu)) < \eta\Phi^{2n-1} \text{ and } \eta\Phi^{2n+3} < \ell(y_{n+1}(\mu)) < \eta\Phi^{2n+1}.$$

Therefore by Lemma 2.2.1 and by the definition of Υ_{n+1}, $\Upsilon_n(\mu) = \Gamma_{2n+1}$ and $\Upsilon_{n+1}(\mu) = \Gamma_{2n+3}$. By Lemma 2.3.2 we get

$$1 - u_{n+1}(\mu) = \frac{\eta\Phi^{2n+1}}{\eta\Phi^{2n-1}} \frac{1 - u_n(\mu)}{1 - C_\mu^-(1 - u_n(\mu))} = \Phi^2 \frac{1 - u_n(\mu)}{\Phi^2} = 1 - u_n(\mu).$$

Therefore if $-\bar{\mu} < \mu < -\nu$, then (2.3.9) holds for $n \geq 1$ and thus $u_n(\mu) = u_{n+1}(\mu)$ for all $n \geq 1$.

We now prove by induction on n that if $\nu < \mu < \bar{\mu}$, (2.3.10) holds for all $n \geq 0$.

We have

$$C_\nu^+ = 1 \text{ and } C_{\bar{\mu}}^+ = \frac{2\bar{\mu}}{\bar{\mu} + \nu} = \frac{1}{\Phi},$$

Since $\mu \mapsto C_\mu^+$ is a continuous map for $\nu < \mu < \bar{\mu}$ this gives

$$1 < C_\mu^+ < \frac{1}{\Phi}. \tag{2.3.22}$$

From (2.3.3) and $\eta\Phi = 1 - k\eta$ we get $\ell(y_0(\mu)) = C_\mu^+ \eta\Phi$. Since $\mu < \bar{\mu}$, from (2.3.22) we get $\eta\Phi < \ell(y_0(\mu)) < \eta$.

Hence we also have $\eta\Phi^2 < \ell(y_0(\mu)) < \eta$. Therefore by Lemma 2.2.1 $\kappa_0 = 0$ and we get $\Upsilon_0(\mu) = \Gamma_2$. Thus by Lemma 2.3.2 we have

$$u_0 = \frac{\Gamma_0}{\ell(y_0(\mu))} = \frac{\eta\Phi^2}{C_\mu^+ \eta\Phi} = \frac{\Phi}{C_\mu^+},$$

and thus (2.3.10) holds for $n = 0$. We now assume that (2.3.10) holds for n and prove that (2.3.9) holds for $n+1$. Since $u_n(\mu) = u_0(\mu) < 1/C_\mu^+$ we have,

$$\ell(y_{n+1}(\mu)) = (1 - C_\mu^+ u_0(\mu))\ell(y_n(\mu)) = \Phi^2 \ell(y_n(\mu)).$$

Since we assume (2.3.10) holds for n, then we have

$$\ell(y_n(\mu)) = C_\mu^+ \eta\Phi^{2n+1}.$$

From these two identities, combined with (2.3.22), we get

$$\eta\Phi^{2n+2} < \ell(y_n(\mu)) < \eta\Phi^{2n} \text{ and } \eta\Phi^{2n+4} < \ell(y_{n+1}(\mu)) < \eta\Phi^{2n+2}.$$

Therefore by Lemma 2.2.1 and by the definition of Υ_{n+1} we have $\Upsilon_n(\mu) = \Gamma_{2(n+1)}$ and $\Upsilon_{n+1}(\mu) = \Gamma_{2(n+2)}$. By Lemma 2.3.2 we get

$$u_{n+1}(\mu) = \frac{\Gamma_{n+1}}{\Gamma_n}\frac{u_n(\mu)}{1 - C^+_\mu u_n(\mu)} = \frac{\eta\Phi^{2n+2}}{\eta\Phi^{2n}}\frac{\frac{\Phi}{C^+_\mu}}{1-\Phi} = \frac{\Phi}{C^+_\mu}$$

Therefore if $\nu < \mu < \bar{\mu}$ then (2.3.10) holds for $n \geq 1$ and thus if $|\mu| \leq \bar{\mu}$ then $u_n = u_{n+1}$ for all $n \geq 1$. This completes the proof. □

2.4 Dynamics of the first return map to the central cone

In this section we introduce a map, denoted by ρ, containing information related to the first return under our transformation F to the central cone P_c and we show how it can be computed using tools from sections 2.2 and 2.3. This gives a dynamical meaning to the sequences introduced in Section 2.3, (y_n) is the sequence of imaginary parts of the discontinuities of the map ρ, while (u_n) is the sequence of ratios of the horizontal jumps produced by discontinuities of ρ relative to the cone width $\ell(y_n)$.

From Section 1.4.1 recall (1.4.3) and the definition of $\nu = \tan(\vartheta)$. Note that ν depends on $|\omega|$, and when necessary to stress this dependence we write $\nu = \nu(|\omega|)$. Let $\mu' \in \mathbb{R}$, be such that $|\mu'| > \nu$. Recall that we denote the upper-half plane of $\mathbb{C}$ by $\mathbb{H}$. With this notation we can write

$$P_c = \{z \in \mathbb{H} : -\nu\mathrm{Re}(z) < \mathrm{Im}(z) \wedge \mathrm{Im}(z) > \nu\mathrm{Re}(z)\}.$$

Note that by (2.3.1), $\ell(y)$ is the length of the line segment $P_c \cap \{z \in \mathbb{H} : \mathrm{Im}(z) = y\}$. From Section 1.4.1 recall (1.4.4). Particularly we denote by L_ν and $L_{-\nu}$, respectively, the lines $\overline{P_0} \cap \overline{P_1}$ and $\overline{P_d} \cap \overline{P_{d+1}}$ and also a ray $L_{\mu'}$ of slope μ' lying on P_c,

$$\begin{aligned} L_\nu &= \{z \in \mathbb{H} : \mathrm{Im}(z) = \nu\mathrm{Re}(z)\}, \\ L_{-\nu} &= \{z \in \mathbb{H} : \mathrm{Im}(z) = -\nu\mathrm{Re}(z)\}, \\ L_{\mu'} &= \{z \in \mathbb{H} : \mathrm{Im}(z) = \mu'\mathrm{Re}(z)\}. \end{aligned}$$

Let L''_S be the image of $L_{\mu'}$ by F, denote its slope by μ and consider also the ray L_μ of slope μ lying on P_c, this is:

$$\begin{aligned} L_\mu &= \{z \in \mathbb{H} : \mathrm{Im}(z) = \mu\mathrm{Re}(z)\}, \\ L''_S &= \{z \in \mathbb{H} : \mathrm{Im}(z) = \mu\mathrm{Re}(z) + \big(1 + \frac{\mu}{\nu}\big)y_0\}, \end{aligned}$$

where y_0 is as in (2.3.3).

Let R_θ be a rotation by an angle θ centred at the origin. If μ' is such that $L_{\mu'}$ is contained in P_j, $j = 1, ..., d$ then we have $L_\mu = R_{\theta_j}(L_{\mu'})$, with $\theta_j = \theta_j(\omega, \tilde{\pi})$, where $\tilde{\pi}$ is the monodromy invariant associated to the TCE. Thus μ and μ' are related by the expression

$$\mu' = \frac{\mu - \tan(\theta_j)}{1 + \mu \tan(\theta_j)}.$$

or, equivalently,

$$\mu = \frac{\mu' + \tan(\theta_j)}{1 - \mu' \tan(\theta_j)}.$$

The image by F of a point $z' \in L_{\mu'}$ is a point $z \in L''_S$ where

$$\mathrm{Im}(z) = \gamma(\mu, \mu')\mathrm{Im}(z'), \quad \gamma(\mu, \mu') = \sqrt{\left(1 + \frac{1}{\mu'^2}\right) / \left(1 + \frac{1}{\mu^2}\right)}\,.$$

Note that by the definition of μ, since $|\mu'| > \nu$, we also have $|\mu| > \nu$. Now for $y > 0$, let $\xi_S(y)$ denote the point $z \in L_{\mu'}$, such that $\mathrm{Im}(F(z)) = y$. This point is unique and is given by $z = (\mu'^{-1} + i)\gamma(\mu, \mu')^{-1}y$.

Recall the first return to the central cone map F_c from (1.4.7). Define the first return of $\xi_S(y)$ to P_c as the map $\rho : \mathbb{R}^+ \to P_c$ given by

$$\rho(y) = F_c(\xi_S(y)).$$

By the definition of F_c we have

$$F_c(z) = \rho(\mathrm{Im}(F(z))),$$

for $z \in P_c$. Thus, the study of the map ρ and F_c are very closely related.

Let

$$\mathcal{D} = \{y > 0 : \rho \text{ is discontinuous at } y\}.$$

Theorem 2.4.2 relates the sequence $\{y_n\}_{n\in\mathbf{N}}$ with $\mathcal{D}$, and characterizes the map ρ.

Recall our definitions of first hitting time $n_\ell(x)$ of x to $\overline{I_c(\ell)}$, in (2.1.4), and the map r'_ℓ, in (2.1.6). Before stating and proving this theorem we need the following lemma.

Lemma 2.4.1 *i) Assume there is an $n_1 \in \mathbb{N}$ and constants $\delta \le \Gamma'_{n_1+1}$, ℓ and ℓ' such that $\Gamma'_{n_1+1} \le \ell < \ell' < \ell + \delta \le \Gamma'_{n_1}$, then $n_\ell(1 + \ell') = k'_{n_1+1}$ and*

$$r'_\ell(1 + \ell') = 1 + \ell' - \Gamma'_{n_1+1}. \tag{2.4.1}$$

ii) Assume there is an $n_d \in \mathbb{N}$ and constants ℓ and ℓ' such that $0 < \ell' \le \Gamma''_{n_d+1} \le \ell < \Gamma''_{n_d}$, then $n_\ell(1 - \ell') = k''_{n_d+1}$ and

$$r'_\ell(1 - \ell') = 1 - \ell' + \Gamma''_{n_d+1}. \tag{2.4.2}$$

Proof. We begin by proving i). First note that as $1+\ell' \notin (1,1+\ell)$ we have

$$r'_\ell(1+\ell') = r_\ell(1+\ell'). \tag{2.4.3}$$

Also it is clear that for $1 \le n < n_{\ell'}(1+\ell')$ we have $g_0^n(1+\ell') \notin [1,1+\ell']$, and since $\ell \le \ell'$ this shows that $g_0^n(1+\ell') \notin [1,1+\ell]$ as well. Thus $n_\ell(1+\ell') \ge n_{\ell'}(1+\ell')$.

Since $\Gamma'_{n_1+1} \le \ell' < \Gamma'_{n_1}$, by Theorem 2.2.4 we have

$$g_0^{n_{\ell'}(1+\ell')}(1+\ell') = 1+\ell' - \Gamma'_{n_1+1},$$

and as $\ell \le \ell' \le \ell+\delta$ this implies that $g_0^{n_{\ell'}(1+\ell')}(1+\ell') \in [1,1+\ell]$, thus $n_\ell(1+\ell') = n_{\ell'}(1+\ell')$ and from (2.4.3) we get (2.4.1). Since by Theorem 2.2.4 we have $n_{\ell'}(1+\ell') = k'_{n_1+1}$ this shows that $n_\ell(1+\ell') = k'_{n_1+1}$ as well.

We now prove ii). Note that as $1-\ell' < 1$ we have

$$r'_\ell(1-\ell') = r_\ell(1-\ell'). \tag{2.4.4}$$

By the definition of d^+, since $\ell' < \ell$ we have $\ell' < d^+(n_\ell(1-\ell'))$, hence, by Lemma 2.1.3 we get

$$g_0^{n_\ell(1-\ell')}(1-\ell') = r_\ell(1) - \ell'.$$

As $\Gamma''_{n_d+1} \le \ell < \Gamma''_{n_d}$ we can apply Theorem 2.2.4 from whence we obtain

$$g_0^{n_\ell(1-\ell')}(1-\ell') = 1-\ell' + \Gamma''_{n_d+1}. \tag{2.4.5}$$

As $0 < \ell \le \Gamma''_{n_d+1}$ we get $g_0^{n_\ell(1-\ell')}(1-\ell') \in [1,1+\ell']$, hence $n_{\ell'}(1-\ell') = n_\ell(1-\ell')$ which implies that $r_\ell(1-\ell') = g_0^{n_\ell(1-\ell')}(1-\ell')$. Thus, combining (2.4.4) and (2.4.5) we get (2.4.2). Since by Theorem 2.2.4 we have $n_\ell(1) = k''_{n_d+1}$ and combined with (2.4.5) this proves that $n_\ell(1-\ell') = k''_{n_d+1}$ as well. □

Theorem 2.4.2 *Assume $\eta \in (0,1)\backslash\mathbb{Q}$ and $|\mu'| > \nu > 0$. Then ρ is a piecewise affine map of slope μ^{-1}. The set $\mathcal{D}$ is equal to the union of all points in the sequence $(y_n)_{n\in\mathbf{N}}$. Furthermore, for all $n \in \mathbf{N}$; if $\rho(y_n) \in L_\nu$, for $y_{n+1} \le y < y_n$ we have*

$$\rho(y) = F^{k(\xi_S(y_n))}(\xi_S(y)) - \Upsilon_n, \tag{2.4.6}$$

if $\rho(y_n) \in L_{-\nu}$, for $y_{n+1} \le y < y_n$ we have

$$\rho(y) = F^{k(\xi_S(y_n))}(\xi_S(y)) + \Upsilon_n. \tag{2.4.7}$$

Also $\rho(y_n) \in L_\nu$ (resp. $\rho(y_n) \in L_{-\nu}$) if and only if $u_{n-1} > 1/C^+_\mu$ (resp. $u_{n-1} < 1/C^+_\mu$).

Proof. We begin by proving, by induction on n, that for all $n \in \mathbf{N}$

$$\operatorname{card}\big\{\mathcal{D} \cap \{y \in \mathbb{R}^+ : y > y_n\}\big\} = n, \tag{2.4.8}$$

$\rho(y_n) \in L_\nu \cup L_{-\nu}$ and that for all $y < y_n$, we have

$$k(\xi_S(y)) > k(\xi_S(y_n)). \tag{2.4.9}$$

For all $n \in \mathbf{N}$, we prove that the map $\rho_n : [0, y_n) \to \mathbb{H}$ such that

$$\rho_n(y) = F^{k(\xi_S(y_n))}(\xi_S(y)), \tag{2.4.10}$$

is an affine map of slope μ^{-1}. Furthermore, if $\rho(y_n) \in L_\nu$ (resp. $\rho(y_n) \in L_{-\nu}$) then for all $y < y_n$ we have

$$F^{k(\xi_S(y'))}(\xi_S(y)) = \rho_n(y) - \Upsilon_n \tag{2.4.11}$$

$$\Big(\text{resp. } F^{k(\xi_S(y'))}(\xi_S(y)) = \rho_n(y) + \Upsilon_n\Big), \tag{2.4.12}$$

where $y' = y_{n+1}$ if $n+1 \in \mathbf{N}$ and $y' = y_n/2$ otherwise. For $y_{n+1} \le y < y_n$ we have (2.4.6) (resp. (2.4.7)).

We first show that for $n = 0$ we have (2.4.8), (2.4.9), $\rho(y_0) \in L_{-\nu}$ and that ρ_0 is an affine map of slope μ^{-1}.

Note that for all $y \ge 0$ we have

$$F(\xi_S(y)) = (\mu^{-1} + i)y - \left(\frac{1}{\mu} + \frac{1}{\nu}\right) y_0, \tag{2.4.13}$$

which is an affine map of slope μ^{-1}. By (2.3.3) we have that $\rho(y_0) \in L_{-\nu}$ and thus $y_0 \in \mathcal{D}$.

As $L''_S \cap \{z \in \mathbb{H} : \operatorname{Im}(z) > y_0\} \subseteq P_c$, for $y > y_0$ we have (2.4.8) and

$$\rho(y) = F(\xi_S(y)).$$

Thus by (2.4.13) we have that ρ_0 is an affine map of slope μ^{-1}. Note that for $y < y_0$ we have $F(\xi_S(y)) \in P_{d+1}$ and thus we have (2.4.9) as well.

It is clear that if $u_0 > 1/C_\mu^+$ (resp. $u_0 < 1/C_\mu^+$) then $\rho(y_1) \in L_\nu$ (resp. $\rho(y_1) \in L_{-\nu}$). Now assume, for $n \in \mathbf{N}$ that $\rho(y_n) \in L_\nu$ (resp. $\rho(y_n) \in L_{-\nu}$), $u_{n-1} > 1/C_\mu^+$ (resp. $u_{n-1} < 1/C_\mu^+$), that (2.4.8) and (2.4.9) are true and ρ_n is an affine map of slope μ^{-1}. We show that (2.4.8) and (2.4.9) hold for $n+1$. If $\rho(y_n) \in L_\nu$ (resp. $\rho(y_n) \in L_{-\nu}$) then for all $y < y_n$ we have (2.4.11) (resp. (2.4.12)) and for

$y_{n+1} \leq y < y_n$ we have (2.4.6) (resp. (2.4.7)). In particular if $y_{n+1} > 0$ then ρ_{n+1} is an affine map of slope μ^{-1} and $\rho(y_{n+1}) \in L_\nu \cup L_{-\nu}$.

Assume that $\rho(y_n) \in L_\nu$. We begin by proving that there is $\tilde{y} < y_n$ such that for $\tilde{y} \leq y < y_n$ we have $\rho(y) = F^{k(\xi_S(\tilde{y}))}(\xi_S(y))$ and (2.4.6).

Since ρ_n is an affine map of slope μ^{-1} and $\rho(y_n) \in L_\nu$, for $y < y_n$ we have

$$\rho_n(y) = y_n\left(\frac{1}{\nu} - \frac{1}{\mu}\right) + \frac{1}{\mu}y + iy. \tag{2.4.14}$$

We now consider that $\ell(y_n) \leq \Gamma'_0$. As in the other case the proof is similar we will omit it for brevity. By the definition of κ_n we have

$$\Gamma'_{\kappa_n} < \ell(y_n) \leq \Gamma'_{\kappa_n - 1}. \tag{2.4.15}$$

As $\Gamma'_0 \leq \eta$ we get $\ell(y_n) \leq \eta$, hence by the definition of ℓ we get (2.1.7) for $z = \xi_S(y)$ and combining Lemma 2.1.2 with $\rho(y) = F_c(\xi_S(y))$, by (2.4.9) and (2.4.10) we get

$$\mathrm{Re}(\rho(y)) = s^{-1} \circ r'_{\ell(y)}\left(1 + \frac{\ell(y)}{2} + \mathrm{Re}(\rho_n(y))\right). \tag{2.4.16}$$

Recall the sequence $(\Upsilon_n)_{n\in\mathbf{N}}$ as in (2.3.5). Take $0 < \delta' < \Upsilon_n$ and

$$\tilde{y} = \max\left(y_n - \left(\frac{1}{\nu} - \frac{1}{\mu}\right)^{-1}\delta', \frac{\nu\Upsilon_n}{2}\right).$$

Note that we have $\tilde{y} < y_n$, since by (2.3.1) and (2.4.15), we have $\frac{\nu\Gamma'_{\kappa_n}}{2} < y_n$ and as $|\mu| > \nu$ we also have $(1/\nu - 1/\mu)^{-1} > 0$.

We now show that for $\tilde{y} \leq y < y_n$ we have

$$\Gamma'_{\kappa_n} \leq \ell(y) < \frac{\ell(y)}{2} + \mathrm{Re}(\rho_n(y)) < \ell(y) + \delta \leq \Gamma'_{\kappa_n - 1}, \tag{2.4.17}$$

with

$$\delta = \max(\Gamma'_{\kappa_n - 1} - \ell(y), \Gamma'_{\kappa_n}). \tag{2.4.18}$$

First note that as $y \geq \tilde{y} \geq \nu\Gamma'_{\kappa_n}$ we have $\Gamma'_{\kappa_n} \leq \ell(y)$. As $\rho_n(y) \in P_0$ we have $\mathrm{Re}(\rho_n(y)) > \ell(y)/2$ and thus $\ell(y) < \ell(y)/2 + \mathrm{Re}(\rho_n(y))$.

By (2.4.14) and the definition of ℓ we have

$$\frac{\ell(y)}{2} + \mathrm{Re}(\rho_n(y)) = \ell(y) + \left(\frac{1}{\nu} - \frac{1}{\mu}\right)(y_n - y). \tag{2.4.19}$$

As $|\mu| > \nu$ we have $(1/\nu + 1/\mu) > 0$, thus, as $y < y_n$ we get that $\ell(y)/2 + \mathrm{Re}(\rho_n(y)) < 2y_n/\nu$, which combined with (2.4.15) and (2.3.1) shows that

$$\frac{\ell(y)}{2} + \mathrm{Re}(\rho_n(y)) < \ell(y) + (\Gamma'_{\kappa_n} - \ell(y)).$$

Since $\delta' < \Gamma'_{\kappa_n}$ we have $y \geq \tilde{y} > y_n - (1/\nu - 1/\mu)^{-1}\Gamma'_{\kappa_n+1}$ and from (2.4.18) and (2.4.19) we get

$$\frac{\ell(y)}{2} + \mathrm{Re}(\rho_n(y)) < \ell(y) + \delta.$$

Finally note that if $\ell(y) > \Gamma'_{\kappa_n-1} - \Gamma'_{\kappa_n}$ then $\ell(y) + \delta = \Gamma'_{\kappa_n-1}$ and if $\ell(y) \leq \Gamma'_{\kappa_n-1} - \Gamma'_{\kappa_n}$ then

$$\ell(y) + \delta = \ell(y) + \Gamma'_{\kappa_n-1} \leq \Gamma'_{\kappa_n-1}.$$

This shows that (2.4.17) holds true.

Therefore the conditions for applying Lemma 2.4.1 i) are satisfied. With $\ell = \ell(y)$ and $\ell' = \ell(y)/2 + \mathrm{Re}(\rho_n(y))$ we get

$$r'_{\ell(y)}\left(1 + \frac{\ell(y)}{2} + \mathrm{Re}(\rho_n(y))\right) = 1 + \frac{\ell(y)}{2} + \mathrm{Re}(\rho_n(y)) - \Gamma'_{\kappa_n},$$

and $n_{\ell(y)}(1 + \ell(y)/2 + \mathrm{Re}(\rho_n(y))) = k'_{\kappa_n}$, where $n_{\ell(y)}$ and $r'_{\ell(y)}$ are as in (2.1.4) and (2.1.6) respectively.

Combining this with (2.4.16) and noting that $\mathrm{Im}(\rho(y)) = \mathrm{Im}(\rho_n(y)) = y$ we get (2.4.6) for $y \in [\tilde{y}, y_n)$. Since $k(\xi_S(y)) = n_{\ell(y)}(1 + \ell(y)/2 + \mathrm{Re}(\rho_n(y))) + 1$ we get that for $y \in [\tilde{y}, y_n)$, $k(\xi_S(y)) = k'_{\kappa_n} + 1$, and thus $k(\xi_S(\tilde{y})) = k(\xi_S(y))$ and $\rho(y) = F^{k(\xi_S(\tilde{y}))}(\xi_S(y))$.

Denote

$$\mathfrak{d}^- = \mathfrak{d}^-\left(1 + \frac{\ell(\tilde{y})}{2} + \mathrm{Re}(F(\xi_S(\tilde{y}))), n_{\ell(\tilde{y})}\left(1 + \frac{\ell(\tilde{y})}{2} + \mathrm{Re}(F(\xi_S(\tilde{y})))\right)\right),$$

and let

$$\Delta(y, \tilde{y}) = \frac{\ell(y)}{2} + \mathrm{Re}(F(\xi_S(y))) - \frac{\ell(\tilde{y})}{2} - \mathrm{Re}(F(\xi_S(\tilde{y}))).$$

we will show that

$$F^{k(\xi_S(\tilde{y}))}(\xi_S(y)) = \rho_n(y) - \Upsilon_n, \tag{2.4.20}$$

for all $y < y_n$.

Let us first prove (2.4.20) for all $y < y_n$. Since it holds for $y \in [\tilde{y}, y_n)$, we are left to prove it for $y < \tilde{y}$.

Note first that by (2.4.13), we have

$$\Delta(y, \tilde{y}) = \left(\frac{1}{\nu} + \frac{1}{\mu}\right)^{-1}(y - \tilde{y}) < 0,$$

and since $\mathfrak{d}^- \geq 0$ we have $\Delta(y, \tilde{y}) < \mathfrak{d}^-$. Combining this with (2.3.1), we get for $y < \tilde{y}$,

$$-(\ell(\tilde{y}) - \ell(y)) < \Delta(y, \tilde{y}) < \mathfrak{d}^-.$$

From these inequalities and Lemma 2.1.4 we get that for $n \leq n_{\ell(\tilde{y})}(1 + \ell(\tilde{y})/2 + \mathrm{Re}(F(\xi_S(\tilde{y}))))$

$$g^n_{\ell(y)}\left(1 + \frac{\ell(y)}{2} + \mathrm{Re}(F(\xi_S(y)))\right) = g^n_{\ell(\tilde{y})}\left(1 + \frac{\ell(\tilde{y})}{2} + \mathrm{Re}(F(\xi_S(\tilde{y})))\right) + \Delta(y, \tilde{y}). \tag{2.4.21}$$

Recalling the definition of $\mathcal{R}_{\eta,\vartheta}$ in (2.1.2) and also that

$$n_{\ell(\tilde{y})}(1 + \ell(\tilde{y})/2 + \mathrm{Re}(F(\xi_S(\tilde{y})))) = k(\xi_S(\tilde{y})) - 1,$$

by Lemma 2.1.2 we have $F(\xi_S(\tilde{y})) \in \mathcal{R}_{\eta,\vartheta}$ and

$$s^{-1} \circ g^{k(\xi_S(\tilde{y}))-1}_{\ell(\tilde{y})}\left(1 + \frac{\ell(\tilde{y})}{2} + \mathrm{Re}(F(\xi_S(\tilde{y})))\right) = \mathrm{Re}(\rho(\tilde{y})).$$

By Lemma 2.1.1, combining the previous identity with (2.4.21) gives

$$F^{k(\xi_S(\tilde{y}))}(\xi_S(y)) = \mathrm{Re}(\rho(\tilde{y})) - \frac{1}{\mu}(\tilde{y} - y) + iy,$$

and since (2.4.6) holds true for $y = \tilde{y}$, by (2.4.14) we also have

$$\mathrm{Re}(\rho(\tilde{y})) = y_n\left(\frac{1}{\nu} - \frac{1}{\mu}\right) + \frac{1}{\mu}\tilde{y} - \Gamma'_{\kappa_n}.$$

Combining the two expressions above and (2.4.14) we get $F^{k(\xi_S(\tilde{y}))}(\xi_S(y)) = \rho_n(y) - \Gamma'_{\kappa_n}$, which together with (2.3.5) gives (2.4.20) as intended.

We now prove that for all $y_{n+1} \leq y < y_n$,

$$k(\xi_S(y)) = k(\xi_S(\tilde{y})). \tag{2.4.22}$$

By Lemma 2.1.4, $n_{\ell(\tilde{y})}(1+\ell(\tilde{y})/2+\mathrm{Re}(F(\xi_S(\tilde{y})))) \leq n_{\ell(y)}(1+\ell(y)/2+\mathrm{Re}(F(\xi_S(y))))$, for $y \leq \tilde{y}$, thus $k(\xi_S(y)) \geq k(\xi_S(\tilde{y}))$.

For all $y \in [\tilde{y}, y_n)$, since $k(\xi_S(y)) = k(\xi_S(\tilde{y}))$, to prove (2.4.22) for $y_{n+1} \leq y < y_n$ it is enough instead to show that

$$F^{k(\xi_S(\tilde{y}))}(\xi_S(y)) \in P_c. \tag{2.4.23}$$

Begin by noting that by (2.4.20) we have

$$F^{k(\xi_S(\tilde{y}))}(\xi_S(y)) = y_n\left(\frac{1}{\nu} - \frac{1}{\mu}\right) + \frac{1}{\mu}y - \Gamma'_{\kappa_n} + iy. \tag{2.4.24}$$

Combining (2.3.2) with the definitions of y_{n+1}, κ_{n+1}, Υ_{n+1} and u_{n+1}, we get

$$y_{n+1} = \begin{cases} 0\,, & y_n = \left(\frac{1}{\nu} - \frac{1}{\mu}\right)^{-1}\Gamma'_{\kappa_n}, \\ y_n - \left(\frac{1}{\nu} - \frac{1}{\mu}\right)^{-1}\Gamma'_{\kappa_n}\,, & y_n > \left(\frac{1}{\nu} - \frac{1}{\mu}\right)^{-1}\Gamma'_{\kappa_n}, \\ \left(\frac{1}{\nu} + \frac{1}{\mu}\right)^{-1}\Gamma'_{\kappa_n} - \frac{\mu-\nu}{\mu+\nu}y_n\,, & y_n < \left(\frac{1}{\nu} - \frac{1}{\mu}\right)^{-1}\Gamma'_{\kappa_n}. \end{cases} \tag{2.4.25}$$

It is clear from (2.4.25) and using $|\mu| > \nu$ that $y_{n+1} > 0$ if $y_n \neq (1/\nu - 1/\mu)^{-1}\Gamma'_{\kappa_n}$ and $y_{n+1} = 0$ otherwise.

We consider the three separate cases in (2.4.25).

If $y_n = (1/\nu - 1/\mu)^{-1}\Gamma'_{\kappa_n}$, by (2.4.24) we have $F^{k(\xi_S(\tilde{y}))}(\xi_S(y)) = y/\mu + iy$, which, since $|\mu| > \nu$ proves (2.4.23).

If $y_n > (1/\nu - 1/\mu)^{-1}\Gamma'_{\kappa_n}$ it follows from (2.4.24) and $|\mu| > \nu$ that $-y/\nu < \mathrm{Re}(F^{k(\xi_S(\tilde{y}))}(\xi_S(y)))$, also it follows from (2.4.24) that

$$\mathrm{Re}(F^{k(\xi_S(\tilde{y}))}(\xi_S(y))) = (y_n - y)\left(\frac{1}{\nu} - \frac{1}{\mu}\right) - \Gamma'_{\kappa_n} + \frac{1}{\nu}y,$$

and since $y \geq y_{n+1}$, we get from (2.4.25) that $\mathrm{Re}(F^{k(\xi_S(\tilde{y}))}(\xi_S(y))) \leq y/\nu$, proving (2.4.23) in this case.

Finally, if $y_n < (1/\nu - 1/\mu)^{-1}\Gamma'_{\kappa_n}$, it follows from (2.4.24) and $|\mu| > \nu$ that $\mathrm{Re}(F^{k(\xi_S(\tilde{y}))}(\xi_S(y))) < y/\nu$, and from (2.4.24) that

$$\mathrm{Re}(F^{k(\xi_S(\tilde{y}))}(\xi_S(y))) = y_n\left(\frac{1}{\nu} - \frac{1}{\mu}\right) + y\left(\frac{1}{\nu} + \frac{1}{\mu}\right) - \Gamma'_{\kappa_n} - \frac{1}{\nu}y.$$

Since $y \geq y_{n+1}$, we get from the above expression and (2.4.25) that

$$\mathrm{Re}(F^{k(\xi_S(\tilde{y}))}(\xi_S(y))) \geq -y/\nu,$$

and thus (2.4.23).

This shows that for all $y_{n+1} \leq y < y_n$ we have (2.4.22).

From (2.4.22) it follows that (2.4.8) holds for $n+1$. It also follows that

$$F^{k(\xi_S(\tilde{y}))}(\xi_S(y)) = F^{k(\xi_S(y'))}(\xi_S(y)),$$

hence by (2.4.20) we have that (2.4.11) holds for all $y < y_n$. Also from (2.4.22) it follows that for all $y_{n+1} \leq y < y_n$, $\rho(y) = F^{k(\xi_S(y'))}(\xi_S(y))$ and thus from (2.4.11) we get (2.4.6) as well.

Finally note that if $y_{n+1} > 0$, then $y' = y_{n+1}$ and hence by (2.4.6) ρ_{n+1} is an affine map of slope μ^{-1}. As $\rho(y_n) \in L_\nu$ we have $u_{n-1} > 1/C_\mu^+$, hence by (2.4.25) and the definitions of y_n and u_n it is straightforward to check that $\rho(y_{n+1}) \in L_\nu$ (resp. $\rho(y_{n+1}) \in L_{-\nu}$) if and only if $y_n > \left(\nu^{-1} - \mu^{-1}\right)^{-1}\Gamma'_{\kappa_n}$ (resp. $y_n < \left(\nu^{-1} - \mu^{-1}\right)^{-1}\Gamma'_{\kappa_n}$) if and only if $u_n > 1/C_\mu^+$ (resp. $u_n < 1/C_\mu^+$).

The proof for the case $\rho(y_n) \in L_{-\nu}$ is similar to the previous one and so we omit it.

By (2.4.6), (2.4.7) and (2.4.13) we get that $\rho(y)$ is a an affine map of slope μ^{-1} for all $y_{n+1} \leq y < y_n$, $n \in \mathbf{N}$, hence by Lemma 2.3.1 it is a picewise affine map in $[0, y_0]$. Also by Lemma 2.3.1 and (2.4.8) it follows that the set of discontinuities $\mathcal{D}$ is equal to the union of all $\{y_n\}_{n\in\mathbf{N}}$. □

2.5 Proof of Theorems A and B

In this section we prove our main results, theorems A and B.

Set $x_n(\mu) = \mathrm{Re}(\rho(y_n^-(\mu)))$. By Theorem 2.4.2 and by the definition of Υ_n, for all $n \in \mathbf{N}$, we have

$$\rho(y_n^-(\mu)) = \begin{cases} \dfrac{y_n(\mu)}{\nu} - \Upsilon_n(\mu) + iy_n(\mu), & \rho(y_n(\mu)) \in L_\nu, \\ \Upsilon_n(\mu) - \dfrac{y_n(\mu)}{\nu} + iy_n(\mu), & \rho(y_n(\mu)) \in L_{-\nu}, \end{cases}$$

which by the definitions of ℓ and u_n gives

$$u_n(\mu) = \frac{x_n(\mu)}{\ell(y_n(\mu))} + \frac{1}{2}, \text{ for all } n \in \mathbf{N}. \tag{2.5.1}$$

2.5.1 Proof of Theorem A

Let $(y_n(\mu))$ be the sequence associated to $L''_S(\mu)$. Recall that by (1.4.3) we have $\vartheta = (\pi - |\omega|)/2$.

We begin by proving that there is a positive real number $\bar{y_1}$ such that, for all μ satisfying $|\mu| > \tan(\vartheta) = \nu$, we have $y_1(\mu) \geq \bar{y_1}$. Let φ, $\varphi' \in [\vartheta, \pi - \vartheta]$ be such that

$$\mu = \tan(\varphi) \quad \text{and} \quad \mu' = \tan(\varphi'). \tag{2.5.2}$$

Let $L_{\mu'} \subseteq P_j$, we define

$$\gamma_j(\varphi) = |\cos(\theta_j) - \sin(\theta_j)\cot(\varphi)|^{-1}, \tag{2.5.3}$$

and

$$\gamma'_j(\varphi') = |\cos(\theta_j) - \sin(\theta_j)\cot(\varphi')|,$$

where $\theta_j = \theta_j(\omega, \tilde{\pi})$ and $\tilde{\pi}$ is the monodromy invariant associated to the TCE. By the definition of μ' we can see that

$$\gamma(\mu, \mu') = \gamma_j(\varphi) = \gamma'_j(\varphi'). \tag{2.5.4}$$

Recall from (2.3.3) that

$$y_0(\mu) = \eta' \frac{\mu\nu}{\mu + \nu}.$$

Hence using (2.5.3), we have

$$y_0(\tan(\varphi))\gamma_j(\varphi)^{-1} = \eta'\nu\,|\cos(\theta_j)| \left| \frac{1 - \tan(\theta_j)\cot(\varphi)}{1 + \nu\cot(\varphi)} \right|. \tag{2.5.5}$$

Let

$$\bar{y_0} = \min_{j \in \{1,\ldots,d\}} \left\{ \inf_{\varphi \in W_j} \{y_0(\tan(\varphi))\gamma_j(\varphi)\} \right\}. \tag{2.5.6}$$

Fix $j \in J = \{1 \leq j \leq d : \theta_j = \pi/2\}$. By (2.5.5), if $\varphi \neq \pi/2$, we have

$$y_0(\tan(\varphi))\gamma_j(\varphi) = \eta'\nu \left| \frac{\cot(\varphi)}{1 + \nu\cot(\varphi)} \right| > 0.$$

We now show that $\pi/2 \notin W_j$. Assume that $\varphi = \pi/2 \in W_j$. Note that from the definition of $L_{\mu'}$ and (2.5.2) we get $\varphi' = \varphi - \theta_j$. Therefore, since $\theta_j = \pi/2$, we have $\varphi' = 0$, which is impossible since $\nu = \tan(\vartheta) > 0$ and $\varphi' \in [\vartheta, \pi - \vartheta]$. Thus, we get

$$\bar{y_0} = \min_{j \notin J} \left\{ \inf_{\varphi \in W_j} \{ y_0(\tan(\varphi))\gamma_j(\varphi) \} \right\}.$$

Now fix $j \in \{1, ..., d\}$. Since $\varphi' \in [\vartheta, \pi - \vartheta]$ we have $\varphi' > \arctan(\nu)$, and thus, since $\varphi' = \varphi - \theta_j$, we have $\varphi - \theta_j > \arctan(\nu)$. Thus, φ is bounded away from θ_j and this bound depends only on ν. Therefore $\tan(\theta_j) \neq \tan(\varphi)$ and thus there is $\tilde{c}(\nu, j) > 0$ such that

$$|1 - \tan(\theta_j)\cot(\varphi)| > \tilde{c}(\nu, j).$$

Since $\varphi \in [\arctan(\nu), \pi - \arctan(\nu)]$ we have $|\nu\cot(\varphi) \leq 1|$, thus we also have $|1 + \nu\cot(\varphi)| \leq 2$. From this and the above inequality we get

$$\eta'\nu\,|\cos(\theta_j)| \left| \frac{1 - \tan(\theta_j)\cot(\varphi)}{1 + \nu\cot(\varphi)} \right| \geq \frac{\eta'\nu}{2}\tilde{c}(\nu, j)\,|\cos(\theta_j)| > 0.$$

Combining this with (2.5.5) and (2.5.6) we get

$$\bar{y_0} \geq \min_{j \in \{1,\dots,n\}} \left\{ \frac{1}{2}\tilde{c}(\nu, j)\,|\cos(\theta_j)| \right\} > 0.$$

Thus, for all $\nu > 0$, we have $\bar{y_0} > 0$.

Note that from (2.5.2) and the definitions of C_μ^+ and C_μ^-, we can write $\mathfrak{D}(\varphi) = C_\mu^-/C_\mu^+$ as a function of φ as

$$\mathfrak{D}(\varphi) = \frac{1 + \nu\cot(\varphi)}{1 - \nu\cot(\varphi)}.$$

Define the interval $W^\varphi = [\arctan(\nu), \pi - \arctan(\bar{\mu})]$. Note that $\mathfrak{D}(\varphi)$ is a positive, continuous and decreasing function of $\varphi \in W^\varphi$. Since $\varphi \leq \pi - \arctan(\bar{\mu})$, we have

$$\mathfrak{D}(\varphi) \geq \frac{1 + \nu(-\Phi^3/\nu)}{1 - \nu(-\Phi^3/\nu)} = \frac{1 - \Phi^3}{1 + \Phi^3} = \Phi,$$

since $\Phi^2 = 1 - \Phi$. Thus we obtain

$$\inf_{\varphi \in W^\varphi} \mathfrak{D}(\varphi) \geq \Phi. \tag{2.5.7}$$

It follows from Theorem 2.3.3 that $y_1 = y_0 C_\mu^- \Phi / C_\mu^+$ if $\mu \geq -\bar{\mu}$ and $y_1 = y_0\Phi^2$ if $\mu < -\bar{\mu}$. This implies that for all $\varphi \in W^\varphi$, we have that

$$y_1(\tan(\varphi)) = \begin{cases} \Phi^2 y_0(\tan(\varphi)), & \mu < \bar{\mu}, \\ \mathfrak{D}(\varphi) y_0(\tan(\varphi)) & \mu \geq \bar{\mu}. \end{cases}$$

By (2.5.6) this gives

$$\min_{j\in\{1,\dots,d\}} \left\{ \inf_{\varphi\in W_j} \{y_1(\tan(\varphi))\gamma_j(\varphi)\} \right\} \geq \min\left(\Phi^2, \inf_{\varphi\in W^\varphi} \mathfrak{D}(\varphi)\right) \bar{y_0}$$

Define $\bar{y_1} = \Phi^2 \bar{y_0}$. Note that since $\bar{y_0} > 0$, we have $\bar{y_1} > 0$ as well. From the above inequality and (2.5.7) we get

$$y_1(\mu) \geq \min_{j\in\{1,\dots,d\}} \left\{ \inf_{\varphi\in W_j} \{y_1(\tan(\varphi))\gamma_j(\varphi)\} \right\} \geq \bar{y_1}. \tag{2.5.8}$$

Define $U = \{z \in P_c : \mathrm{Im}(z) < \bar{y_1}\}$. We now prove (1.5.1) for $z \in U$. Let μ' be such that $z \in L_{\mu'}$, then $\Phi^2 z \in L_{\mu'}$, hence by the definition of $\gamma(\mu, \mu')$ and as $F_c(z) = \rho(\mathrm{Im}(F(z)))$ we have

$$\frac{1}{\Phi^2} F_c(\Phi^2 z) = \frac{1}{\Phi^2}\rho(\gamma(\mu,\mu') y \Phi^2), \tag{2.5.9}$$

Set $y' = \gamma(\mu, \mu')y$. From (2.5.4) and (2.5.8) we have

$$y_1(\mu) = \gamma(\mu,\mu')\gamma_j(\mu)^{-1} y_1(\mu) \geq \gamma(\mu,\mu')\bar{y_1}, \tag{2.5.10}$$

for j such that $(x, y) \in P_j$. Since $\mathrm{Im}(\rho(y')) = y'$, by (2.5.9) and (2.5.10), to prove (1.5.1) it is enough to prove that

$$\mathrm{Re}(\rho(y'\Phi^2)) = \Phi^2 \mathrm{Re}(\rho(y')), \tag{2.5.11}$$

for $y' < y_1(\mu)$. We prove (2.5.11) for $y' < y_1(\mu)$. Recall that $y_1 = y_1(\mu)$. By (2.5.10), there must be an $n \geq 1$, such that

$$y_{n+1}(\mu) \leq y' < y_n(\mu). \tag{2.5.12}$$

Recall from Theorem 2.4.2 that $\rho(y')$ is a piecewise affine map of constant slope μ^{-1} and it is continuous if y' satisfies (2.5.12). From this we have

$$\rho(y') = \rho(y_{n+1}) - \frac{y_{n+1} - y'}{\mu},$$

and combining this with (2.5.1) and by the definition of ℓ, we have

$$\mathrm{Re}(\rho(y')) = (2u_n(\mu) - 1)\frac{y_{n+1}(\mu)}{\nu} - \frac{y_{n+1}(\mu) - y'}{\mu}. \tag{2.5.13}$$

Now multiplying (2.5.12) by Φ^2 we get

$$y_{n+1}(\mu)\Phi^2 \leq y'\Phi^2 < y_n(\mu)\Phi^2,$$

thus by Theorem 2.3.3 we have

$$\begin{cases} y_{n+2}(\mu) \leq y'\Phi^2 < y_{n+1}(\mu) & , \text{ if } |\mu| < \bar{\mu} \\ y_{n+3}(\mu) \leq y'\Phi^2 < y_{n+2}(\mu) & , \text{ if } |\mu| \geq \bar{\mu}. \end{cases}$$

By a similar argument to the used to prove (2.5.1), from the above inequalities we get

$$\mathrm{Re}(\rho(y'\Phi^2)) = \begin{cases} (2u_{n+1}(\mu) - 1)\dfrac{y_{n+2}(\mu)}{\nu} - \dfrac{y_{n+2}(\mu) - y'\Phi^2}{\mu} & , \text{ if } |\mu| < \bar{\mu} \\ (2u_{n+2}(\mu) - 1)\dfrac{y_{n+3}(\mu)}{\nu} - \dfrac{y_{n+3}(\mu) - y'\Phi^2}{\mu} & , \text{ if } |\mu| \geq \bar{\mu}, \end{cases}$$

applying Theorem 2.3.3 to this expression gives

$$\mathrm{Re}(\rho(y'\Phi^2)) = (2u_n(\mu) - 1)\frac{y_{n+1}(\mu)\Phi^2}{\nu} - \frac{y_{n+1}(\mu)\Phi^2 - y'\Phi^2}{\mu}.$$

Comparing this identity with (2.5.13) we get (2.5.11). This completes our proof.

□

Recall our definition of first return map F_c of $z \in P_c$ to the central cone P_c. Before proving Theorem B we need the following result showing that in the conditions of Theorem A, F_c is a PWI with respect to a partition of countably many atoms.

Theorem 2.5.1 *For all $\omega \in \mathbb{W}$, $\eta = 1/(k + \Phi)$ and $\eta' = 1 - k\eta$ with $k \in \mathbb{N}$, F_c is a piecewise isometry with respect to a partition of countably many atoms.*

Proof. We begin by noting that F_c is a PWI since it is the first return map under F to P_c which is a union of elements of the partition of F. We now prove that the partition of F_c has countably many atoms. Assume by contradiction that there is $N \in \mathbb{N}$, a partition $\{Q_j\}_{j\in\{0,\dots,N-1\}}$ of P_c, and $\theta_j(\omega, \tilde{\pi})$, η_j for $j \in \{0, ..., N-1\}$ such that

$$F_c(z) = e^{i\theta_j(\omega,\tilde{\pi})}z + \eta_j, \quad z \in Q_j.$$

By Theorem A there is an open set U of P_c, containing the origin, where F_c is renormalizable. Consider the set $U' = U\backslash\Phi^2 U$ and take $j' \in \{0, ..., N-1\}$ such that $U' \cap P_{j'} \neq \emptyset$. Since η and η' are irrational numbers, we have that $F_c(z) = e^{i\theta_{j'}(\omega,\tilde{\pi})}z + \eta_{j'}$ for $z \in U' \cap P_{j'}$, Define the sequence $(\tilde{U}_k)_{k\geq 0}$, where

$$\tilde{U}_0 = U' \cap P_{j'} \quad \text{and} \quad \tilde{U}_k = \Phi^{2(k-1)}\tilde{U}_0 \backslash \Phi^{2k}\tilde{U}_0, \text{ for } k \geq 1.$$

For every $k \geq 0$ and all $z \in \tilde{U}_k$ we have that $\Phi^{-2k}z \in \tilde{U}_0$. Since $\tilde{U}_k \subseteq U$, we can renormalize F_c, k times to get

$$F_c(z) = \Phi^{2k}F_c(\Phi^{-2k}z) = e^{i\theta_{j'}(\omega,\tilde{\pi})}z + \Phi^{2k}\eta_{j'}.$$

Since $\eta_{j'} \neq 0$, $\Phi^{2k}\eta_{j'}$ takes countably many different values, hence for each k there must be a j_k such that for $z \in \tilde{U}_k$ we have $z \in P_{j_k}$ and $j_k \neq j_{k'}$ for $k \neq k'$. But $j_k \in \{0, ..., N-1\}$ hence there must exist $k' \neq k''$ such that $j_{k'} = j_{k''}$, which is a contradiction. This finishes our proof. □

2.5.2 Proof of Theorem B

We begin by proving that P_c can be separated into two connected regions C_b and C_u, which are forward invariant for F_c, such that C_b is bounded and C_u is unbounded.

By the proof of Theorem A there exists a $\overline{y_1} > 0$ and an open set

$$U = \{z \in P_c : \text{Im}(z) < \overline{y_1}\}, \tag{2.5.14}$$

such that we have (1.5.1) for all $z \in U$.

Since $\eta = 1/(k+\Phi)$ and $\eta' = 1 - k\eta$ with $k \in \mathbb{N}$, by Theorem 2.5.1, F_c is a PWI with respect to a partition of countably many atoms which we denote $\mathcal{P}_{F_c}$. Furthermore, since $\omega \in \mathfrak{A}(\eta, \eta')$, there exist $d' \geq 2$, $\lambda \in \mathbb{R}_+^{d'}$, $\pi \in S_{d'}$ and a continuous embedding γ, of $f_{\lambda,\pi} : I \to I$ into $F_c : P_c \to P_c$, such that $\gamma(I) \subset \Phi^2 U$, $\gamma(0) \in L_{-\nu}$, $\gamma(|\lambda|) \in L_\nu$ and

$$\mathcal{B} = \{P \in \mathcal{P}_{F_c} : P \cap \gamma(I) \neq \emptyset\},$$

is a barrier for F_c. Let

$$\mathfrak{L}_\nu = \{z \in L_\nu : \text{Im}(z) \leq \text{Im}(\gamma(|\lambda|))\}, \quad \mathfrak{L}_{-\nu} = \{z \in L_{-\nu} : \text{Im}(z) \leq \text{Im}(\gamma(0))\}.$$

Since $\gamma(|\lambda|) \in L_\nu$ and $\gamma(0) \in L_{-\nu}$ we have that $\gamma(|\lambda|) \in \mathfrak{L}_\nu$ and $\gamma(0) \in \mathfrak{L}_{-\nu}$ respectively. As γ is a homeomorphism of I, $\mathfrak{L}_\nu \cap \gamma(I) = \gamma(|\lambda|)$ and $\mathfrak{L}_{-\nu} \cap \gamma(I) = \gamma(0)$, we have that $J = \mathfrak{L}_\nu \cup \mathfrak{L}_{-\nu} \cup \gamma(I)$ is homeomorphic to a circle, hence by the Jordan curve Theorem $\mathbb{C}\backslash J$ consists of two connected components, a bounded C_b' and an unbounded C_u'.

Take $C_b = \overline{C_b'} \cap P_c$ and $C_u = C_u' \cap P_c$. We now show that for any $P \in \mathcal{B}$ we have $F_c(P \cap C_u) \subseteq C_u$ and $F_c(P \cap C_b) \subseteq C_b$.

Let $P \in \mathcal{B}$. Note that the restriction $F_c|_P$ of F_c to P is an orientation preserving isometry. Furthermore since γ is a continuous embedding it is order preserving, hence $F_c|_{P\cap\gamma(I)}$ is order preserving as well. Thus it is possible to construct an orientation preserving homeomorphism $\tilde{\gamma} : \mathbb{C} \to \mathbb{C}$ such that $\tilde{\gamma}|_P = F_c|_P$. $\tilde{\gamma}$ must map C_b into C_b and C_u into C_u. In particular if $z \in P \cap C_u$ (resp. $z \in P \cap C_b$) then $F_c(z) = \tilde{\gamma}(z) \in C_u$ (resp. C_b).

We now show that $F_c(C_u) \subseteq C_u$. Note that since $\mathcal{B}$ is a barrier, $P_c \backslash \mathcal{B}$ is the union of two disjoint connected components A_u, A_b. Since $\gamma(I) \subset \bigcup_{A\in\mathcal{B}} A$, these regions must be contained in C_u or C_b. Without loss of generality assume $A_u \subseteq C_u$ and $A_b \subseteq C_b$.

Assume by contradiction that there is a $z \in C_u$ such that $F_c(z) \notin C_u$. Since for any $P \in \mathcal{B}$ we have $F_c(P \cap C_u) \subseteq C_u$, we must have $z \in B_u$. Since $\mathcal{B}$ is a barrier we have that $F_c(z) \notin A_b$, thus we must have $F_c(z) \in C_b \backslash A_b$. Let $P \subseteq A_u$ be the atom of the partition $\mathcal{P}_{F_c}$ such that $z \in P$. Since $F_c(z) \in C_b \backslash A_b$ we have $F_c(P) \cap \mathcal{B} \neq \emptyset$ and since $\mathcal{B}$ is a barrier this implies that $F_c(P) \cap (\overline{\mathcal{B}} \cap \overline{A_u}) \neq \emptyset$.

As $\overline{B_u} \subseteq \overline{C_u}$ we have that either $F_c(P) \cap \gamma(I) \neq \emptyset$ or $F_c(P) \cap C_u \neq \emptyset$. In the later case, as $F_c(z) \in C_b$, $F_c(P)$ is connected and C_u and C_b are disjoint we have that $F_c(P) \cap \overline{C_b} \cap \overline{C_u} \neq \emptyset$ and hence $F_c(P) \cap \gamma(I) \neq \emptyset$ as well. As γ is bijective this is only possible if $A \in \mathcal{B}$ which contradicts $P \subseteq A_u$.

Similarly we can see that $F_c(C_b) \subseteq C_b$. We will omit this part for brevity of the argument.

We now construct sets $V_1, V_2, ...$, which are forward invariant by F_c. We first define a set $V_1 \subseteq U$ and show that $F_c(V_1) \subseteq V_1$.

Let $\gamma' = \Phi^{-2}\gamma$, we show that $\gamma' : I \to \Phi^{-2}\gamma(I)$ is a continuous embedding of $f_{\lambda,\pi}$ into F_c. Since $\gamma(I) \subseteq \Phi^2 U$, by Theorem A we have (1.5.1) for all $z \in \Phi^{-2}\gamma(I)$. Hence for all $x \in I$ we have

$$F_c \circ \gamma'(x) = \Phi^{-2} F_c \circ \gamma(x).$$

Combining this with (1.4.8), which holds as γ is an embedding, we get

$$F_c \circ \gamma'(x) = \gamma' \circ f(x),$$

for all $x \in I$.

As before $\gamma'(I)$ separates P_c into two disjoint connected components, one bounded C''_b and other unbounded C''_u. Take $V_1 = C''_b \cap C_u$. Since $\gamma'(I) \subset U$ we have $C''_b \subseteq U$ and thus $V_1 \subseteq U$. To see that V_1 is forward invariant by F_c, note that if $z \in C''_b$, then $\Phi^2 z \in C_b$ and hence $F_c(\Phi^2 z) \in C_b$. Since $C''_b \subseteq U$, by Theorem A we have $F_c(z) \in \Phi^{-2}C_b \subseteq C_b''$. Thus $F_c(C_b'') \subseteq C_b''$ and as $F_c(C_u) \subseteq C_u$ we get that $F_c(V_1) \subseteq V_1$ as intended.

Take $V_n = \Phi^{2(n-1)}V_1$, for $n \geq 2$. To see that V_n is forward invariant by F_c, take $z \in V_n$, then $\Phi^{2(n-1)}z \in V_1 \subseteq U$. Hence by Theorem A we have

$$F_c(z) = \Phi^{2(n-1)}F_c(\Phi^{-2(n-1)}z),$$

and thus $F_c(z) \in V_n$.

We now prove that

$$\bigcup_{n=1}^{+\infty} V_n = C_b''\backslash\{0\}. \tag{2.5.15}$$

First we show, by induction on n, that for all $n \geq 1$ we have

$$V_1 \cup ... \cup V_n = C_b'' \cap \Phi^{2(n-1)}C_u. \tag{2.5.16}$$

It is simple to see that (2.5.16) holds for $n = 1$. We assume (2.5.16) holds for n and show it holds for $n + 1$. By (2.5.16) we get

$$V_1 \cup ... \cup V_{n+1} = (C_b'' \cap \Phi^{2(n-1)}C_u) \cup (\Phi^{2n}C_b'' \cap \Phi^{2n}C_u).$$

As $\Phi^{2n}C_b'' = \Phi^{2(n-1)}C_b$ we have that

$$C_b'' = \Phi^{2n}C_b'' \cup (C_b'' \cap \Phi^{2(n-1)}C_u),$$

and as $\Phi^{2(n-1)}C_u \subseteq \Phi^{2n}C_u$ we have

$$\Phi^{2n}C_u = \Phi^{2n}C_u \cup (C_b'' \cap \Phi^{2(n-1)}C_u).$$

Combining the three expressions above we get that (2.5.16) is true for $n + 1$, as intended.

Since $\gamma(I) \subseteq \Phi^2 U$, we have that $P_c\backslash\Phi^2 U \subseteq C_u$, hence, by (2.5.14), if $\mathrm{Im}(z) > \overline{y_1}\Phi^2$ then $z \in C_u$. Similarly it can be seen that if $\mathrm{Im}(z) > \overline{y_1}\Phi^{2n}$, then $z \in C_u\Phi^{2(n-1)}$. Therefore, as $\Phi < 1$, for all $z \in P_c\backslash\{0\}$, there is an $n \in \mathbb{N}$ such that $z \in \Phi^{2(n-1)}C_u$. Combining this with (2.5.16) we get (2.5.15).

We now show that there exists an $m \in \mathbb{N}$ such that $\Phi^{2m}U \subseteq C_b''$. Let

$$y' = \inf_{x \in I} \{\mathrm{Im}(\gamma(x))\}.$$

Note that as γ' is an embedding we must have $y' > 0$. Hence there must be an $m \in \mathbb{N}$ such that $y' > \overline{y_1}\Phi^{2m}$. Thus $\gamma(I) \subset P_c \backslash \Phi^{2m} U$. As $P_c \backslash \Phi^{2m} U$ is unbounded we must have $C_u \subseteq P_c \backslash \Phi^{2m} U$ and hence $\Phi^{2m} U \subseteq C_b''$.

To conclude the proof of i), take $y^* = \overline{y_1}\Phi^{2m}$. For any $z \in P_c$, such that $0 < \mathrm{Im}(z) < y^*$, by (2.5.14), as $\Phi^{2m} U \subseteq C_b''$ we have $z \in C_b' \backslash \{0\}$. Hence by (2.5.15) there must be a $n \in \mathbb{N}$ such that $z \in V_n$.

We now prove ii). We show that for all $n \geq 1$ we have

$$V_n \subseteq \Phi^{2(n-1)} U \backslash \Phi^{2(m+n)} U. \tag{2.5.17}$$

Note that we have

$$\Phi^{2m} U \subseteq C_b'' \subseteq U,$$

therefore as $C_b = \Phi^2 C_b''$ we get

$$\Phi^{2(m+1)} U \subseteq C_b \subseteq \Phi^2 U,$$

hence $C_u \subseteq P_c \backslash \Phi^{2(m+1)} U$ and thus

$$C_b'' \cap C_u \subseteq (P_c \backslash \Phi^{2(m+1)} U) \cap U.$$

Therefore $V_1 \subseteq U \backslash \Phi^{2(m+1)} U$. As $V_n = \Phi^{2n} C_b'' \cap \Phi^{2n} C_u$ we get (2.5.17) as intended.

We now show that for any $n \in \mathbb{N}$ there exist constants $0 < \underline{b}_n < \overline{b}_n$ such that for all $z \in V_n$ and $k \in \mathbb{N}$ we have (1.5.2). Let

$$\underline{b}_n = \overline{y_1}\Phi^{2(n+m)} \sin(\vartheta), \tag{2.5.18}$$

$$\overline{b}_n = \left(\left| 1 + \overline{y_1}\Phi^{2(n-1)} \cot(\vartheta) \csc(\vartheta) \right|^2 + \overline{y_1}^2 \Phi^{4(n-1)} \csc^2(\vartheta) \right)^{\frac{1}{2}}. \tag{2.5.19}$$

As $\vartheta < \pi/2$ it is straightforward to check that $0 < \underline{b}_n < \overline{b}_n$.

We first show that $|F^k(z)| \geq \underline{b}_n$ for all $k \in \mathbb{N}$. Recall the definition of $\gamma(\mu, \mu')$. For $1 \leq k \leq k(z)$ we have

$$\mathrm{Im}(F^k(z)) = \gamma \mathrm{Im}(z). \tag{2.5.20}$$

Let $j \in \{1, ..., d\}$ be such that $z \in P_j$, by (2.5.3) and (2.5.4) we have

$$\gamma = \frac{\sin(\arg(z))}{\sin(\arg(z) - \theta_j)},$$

as $\{\arg(z), \arg(z) - \theta_j\} \subset [\vartheta, \pi - \vartheta]$, this shows

$$\sin(\vartheta) \leq \gamma \leq \csc(\vartheta). \tag{2.5.21}$$

Combining (2.5.20) and (2.5.21) we get $\min_{k\leq k(z)} \mathrm{Im}(F^k(z)) \geq \sin(\vartheta)\mathrm{Im}(z)$. As $z \in V_n$, by (2.5.14) and (2.5.17) we have

$$\overline{y_1}\Phi^{2(n+m)} < \mathrm{Im}(z) < \overline{y_1}\Phi^{2(n-1)}. \tag{2.5.22}$$

Combining the inequalities above we get

$$|F^k(z)| \geq \min_{k\leq k(z)} \mathrm{Im}(F^k(z)) \geq \overline{y_1}\Phi^{2(n+m)}\sin(\vartheta),$$

hence, by (2.5.18) we get that $|F^k(z)| \geq \underline{b}_n$ for all $k \leq k(z)$. Since $F^k(z) = F_c(z) \in V_n$ this holds for all $k \in \mathbb{N}$.

We now prove that $|F^k(z)| \leq \overline{b}_n$ for all $k \in \mathbb{N}$. Recall the definition of trapping region in (2.1.2). If $\mathrm{Im}(F(z)) \leq \eta/(2\cot(\vartheta))$, then $F(z) \in \mathcal{R}_{\eta,\vartheta}$ and by Lemma 2.1.1, we get that for $k \leq k(z)$

$$|\mathrm{Re}(F^k(z))| \leq |1 + \mathrm{Im}(F^k(z))\cot(\vartheta)|. \tag{2.5.23}$$

If $\mathrm{Im}(F(z)) > \eta/(2\cot(\vartheta))$, we get

$$|\eta - \mathrm{Im}(F(z))\cot(\vartheta)| < |1 + \mathrm{Im}(F(z))\cot(\vartheta)|,$$

and combining this with the definition of F we get that (2.5.23) holds in this case as well.

By (2.5.21), (2.5.20), (2.5.22) and noting that $\csc(\vartheta) > 1$, for $0 \leq k \leq k(z)$ we have

$$|\mathrm{Im}(F^k(z))| \leq \csc(\vartheta)\overline{y_1}\Phi^{2(n-1)}.$$

Combining this with (2.5.23) we get

$$|\mathrm{Re}(F^k(z))| \leq |1 + \overline{y_1}\Phi^{2(n-1)}\cot(\vartheta)\csc(\vartheta)|.$$

From the two inequalities above we obtain

$$|F^k(z)| \leq \left(\left|1 + \overline{y_1}\Phi^{2(n-1)}\cot(\vartheta)\csc(\vartheta)\right|^2 + \overline{y_1}^2\Phi^{4(n-1)}\csc^2(\vartheta)\right)^{\frac{1}{2}}.$$

hence, by (2.5.19) we get that $|F^k(z)| \leq \overline{b}_n$ for all $k \leq k(z)$. Since $F^k(z) = F_c(z) \in V_n$ this holds for all $k \in \mathbb{N}$.

Finally we prove iii). Let $\gamma_n(x) = \Phi^{2n}\gamma(x)$, for all $x \in I$. We show that for all $n \in \mathbb{N}$, γ_n is an embedding of $f_{\lambda,\pi}$ into F_c.

As γ is an embedding it is clear that $\gamma_n : I \to \Phi^{2n}\gamma(I)$ is a homeomorphism. Since $\gamma(I) \subset U$ we have that $\gamma_n(I) \subset \Phi^{2n}U$, hence by Theorem A we get

$$F_c \circ \gamma_n(x) = \Phi^{2n}F_c \circ \gamma(x),$$

for all $x \in I$. Since $\gamma_n = \Phi^{2n}\gamma$ by (1.4.8) we also have

$$\Phi^{2n} F_c \circ \gamma(x) = \gamma_n \circ f_{\lambda,\pi}(x),$$

for all $x \in I$. Combining the identities above we get

$$F_c \circ \gamma_n(x) = \gamma_n \circ f_{\lambda,\pi}(x),$$

for all $x \in I$, and hence γ_n is an embedding of $f_{\lambda,\pi}$ into F_c. $\square$

2.6 Infinitely many periodic islands and non-ergodicity

In this section we prove the existence of infinitely many periodic islands, accumulating on the real line, as well as non-ergodicity of Translated Cone Exchange Transformations (TCEs) close to the origin.

An *horizontal periodic orbit* is a periodic orbit $\mathcal{O}$, such that there is an $h \in \mathbb{R}$ for every $z_k \in \mathcal{O}$ such that $\mathrm{Im}(z_k) = h$ for all $k \in \mathbb{N}$. We say h is the *height* of the orbit. An *horizontal periodic island* is a periodic island that contains an horizontal periodic orbit.

Recall the open polytope $\mathbb{W}$ defined in (1.4.2). Let $\mathcal{R}(\tilde{\pi})$ denote the set of all $\omega \in \mathbb{W}$ such that for some $j \in \{1, ..., d\}$ we have

$$\left| \sum_{\tilde{\pi}(k) > \tilde{\pi}(j)} \omega_k - \sum_{k<j} \omega_k \right| < \omega_j. \tag{2.6.1}$$

Given a permutation $\pi \in \mathfrak{S}(\{1, ..., d\})$ with monodromy invariant $\tilde{\pi}$, let $J_{\mathcal{R}}(\tilde{\pi})$ be the set of all $j \in \{1, ..., d\}$ such that (2.6.1) holds for some $\omega \in \mathbb{W}$.

Define the sets $\zeta_-(d)$ (resp. $\zeta_+(d)$) of all monodromy invariants $\tilde{\pi} : \{1, ..., d\} \to \{1, ..., d\}$ such that $\tilde{\pi}(\{1, ..., k\}) \neq \{1, ..., k\}$ for $1 \leq k < d$, there is a $j' \in J_{\mathcal{R}}(\tilde{\pi})$ and a $j'' \in \{1, ..., d\}$ such that $j' < j''$ and $\tilde{\pi}(j'') < \tilde{\pi}(j')$ (resp. $j' > j''$ and $\tilde{\pi}(j'') > \tilde{\pi}(j')$). Denote by $\zeta(d)$ their union $\zeta_-(d) \cup \zeta_+(d)$.

In this section we prove the following theorem, which states that there is a non-empty open set of rotation parameters for which TCEs have infinitely many horizontal periodic islands accumulating on the real line.

Theorem 2.6.1 *Let $\tilde{\pi} \in \zeta(d)$, $\eta = 1/(k + \Phi)$ and $\eta' = 1 - k\eta$, for some $k \in \mathbb{N}$. There is a non-empty open set $\mathcal{W} \subseteq \mathbb{W} \cap \mathcal{R}(\tilde{\pi})$ such that for all $\omega \in \mathcal{W}$, F has infinitely many horizontal periodic islands accumulating on the real line.*

As a result we get that for the same parameter set, TCEs are not ergodic in a neighbourhood of the origin.

Theorem 2.6.2 *Let $\tilde{\pi} \in \zeta(d)$, $\omega \in \mathbb{W} \cap \mathcal{R}(\tilde{\pi})$, $\eta = 1/(k+\Phi)$ and $\eta' = 1 - k\eta$, for some $k \in \mathbb{N}$. If U is an invariant set for F_c that contains a neighbourhood of the origin then the restriction of F_c to U does not have a dense orbit. In particular F is not ergodic with respect to Lebesgue measure.*

We begin by proving Theorem 2.6.3, which states that periodic points of a TCE are contained in periodically coded islands formed by unions of invariant circles.

We introduce *reflective interval exchange transformations*, relate them to TCEs and prove Theorem 2.6.6 which shows that for a family of TCEs for every $n \in \mathbf{N}$ such that u_n belongs to a certain interval $I_{P(\mu_j)}$ there is a horizontal periodic orbit for the TCE. The final part of the section contains the proof of Theorems 2.6.1 and 2.6.2.

We define the itinerary of a point $z \in \mathbb{H}$, under F, to be $i(z) = i_0 i_1 ...$, with

$$i_k = \begin{cases} 0, \text{ if } F^k(z) \in P_0, \\ j, \text{ if } F^k(z) \in P_j, \ j = 1, ..., d, \\ d+1, \text{ if } F^k(z) \in P_{d+1}, \end{cases}$$

for $k \in \mathbb{N}$. Given $\delta > 0$, denote by $S^\delta(z)$, the circle of radius δ centred at z. Let $m'_j(k)$ be the number of js in the k-th first symbols of the itinerary of p, for $j = 1, ..., d$. In the next theorem we prove that for η irrational, every periodic orbit that does not fall on the boundary of the partition must have a family of invariant manifolds. These are unions of circles centred on the periodic point parametrized by their radii.

Theorem 2.6.3 *Let $p \in \mathbb{H} \backslash \bigcup_{j'=0}^{k} F^{-j'}(\partial\mathcal{P})$ be a periodic point of F of period k. Assume $\eta \in \mathbb{R}^+ \backslash \mathbb{Q}$. There exists $\epsilon > 0$ such that for all $0 < \delta < \epsilon$ the union $\bigcup_{r=0}^{k-1} S^\delta(F^r(p))$ is an invariant set for F. The orbit of any $z \in \bigcup_{r=0}^{k-1} S^\delta(F^r(p))$ is dense on this set if and only if $m'_1(k)\theta_1(\omega,\tilde{\pi}) + ... + m'_d(k)\theta_d(\omega,\tilde{\pi}) \in \pi \cdot \mathbb{R} \backslash \mathbb{Q}$.*

Proof. We begin by showing that the itinerary of p contains at least one symbol in $\{1, ..., d\}$. Assume by contradiction that $i(p)$ is a periodic sequence of 0s and $d+1$s. It is clear that

$$F^k(p) = F^{m'_0(k)+m'_{d+1}(k)}(p) = z + m'_{d+1}(k)\eta - m'_0(k).$$

Since p is a periodic point of F of period k we have $z = F^k(z) = m'_{d+1}(k)\eta - m'_0(k) + z$. Therefore we get that $\eta = m'_0(k)/m'_{d+1}(k) \in \mathbb{Q}$, contradicting the assumption that η is irrational.

Hence we can assume $i_0(p) \in \{1, ..., d\}$ without loss of generality, since we can choose to start the periodic orbit at the first iterate that falls in P_j for some $j = 1, ..., d$. Since $p \in \mathbb{H} \backslash \bigcup_{j'=0}^{k} F^{-j'}(\partial \mathcal{P})$, then p belongs to some open cell U_k in the k-th refinement of the partition. Since all points in this cell will share the first k addresses in the itinerary, we have $i_0(p)...i_k(p) = i_0(z)...i_k(z)$ for $z \in U_k$. Therefore $F^k : U_k \to \mathbb{C}$ is such that

$$F^k(z) = e^{i\theta'(\omega,\vartheta)}z + t'(\omega, \vartheta, \eta, \eta'),$$

for some functions $\theta' : [0, \pi)^2 \to [0, \pi)$ and $t' : [0, \pi)^2 \times \mathbb{R}_+^2 \to \mathbb{R}$. Since $F^k(p) = p$ we have

$$p = \frac{t'(\omega, \vartheta, \eta, \eta')}{1 - e^{i\theta'(\omega,\vartheta)}}.$$

From this it is easy to check that we can rewrite

$$F^k(z) = e^{i\theta'}(z - p) + pe^{i\theta'} + t = e^{i\theta'}(z - p) + p,$$

and we get

$$|F^k(z) - p| = |e^{i\theta'}(z - p) + p - p| = |z - p|. \tag{2.6.2}$$

This implies that F^k is invariant in the largest circle with center p contained in U_k.

Take $\epsilon > 0$ such that $B_\epsilon(p) \subseteq U_k$. We now see that for $l = 1, ..., k - 1$ we have $F^l(B_\epsilon(p)) = B_\epsilon(F^l(p))$.

From (2.6.2) we have $|F^k(z) - p| = |z - p| < \epsilon$ which implies that $F^k(z) \in B_\epsilon(p)$. Therefore $F^k(B_\epsilon(p)) \subseteq B_\epsilon(p)$. This implies that for all $r \in \mathbb{N}$, we have $F^{rk}(z) \in B_\epsilon(p)$, hence we also have for $l = 1, ..., k - 1$ that $i(F^l(z)) = i(F^{rk+l}(z))$. Therefore every $z \in B_\epsilon(p)$ has the same itinerary of p. It follows that $B_\epsilon(F^l(p))$ is also an invariant set for F^l, since we can repeat the above argument for $l = 1, ..., k - 1$ and conclude $F^l(B_\epsilon(p)) = B_\epsilon(F^l(p))$.

For any $0 < \delta < \epsilon$ we know that $z \in S^\delta(p)$ if and only if $z = p + \delta e^{i\nu'}$ for some $\nu' \in [0, 2\pi)$. Since $F^k(z) = \delta e^{i(\theta'+\nu')} + p$, we have $F^k(S^\delta(p)) \subseteq S^\delta(p)$. Therefore $F^k(S^\delta(p)) = S^\delta(p)$, since the reverse inclusion is clear. We can repeat this argument for $l = 1, ..., k - 1$ and conclude that $F^l(S^\delta(p)) = S^\delta(F^l(p))$ is an invariant set for F^l. Therefore $\bigcup_{r=0}^{k-1} S^\delta(F^r(p))$ is an invariant set for F.

Finally we prove that the orbit of any $z \in \bigcup_{r=0}^{k-1} S^\delta(F^r(p))$ is dense on this set if and only if $m'_1(k)\theta_1(\omega, \tilde{\pi}) + ... + m'_d(k)\theta_d(\omega, \tilde{\pi}) \in \pi \cdot \mathbb{R} \backslash \mathbb{Q}$. Note that

$$\theta'(\omega, \vartheta) = m'_1(k)\theta_1(\omega, \tilde{\pi}) + ... + m'_d(k)\theta_d(\omega, \tilde{\pi}).$$

We also have that F^k acts as a rotation by an angle θ' in $S^\delta(p)$, so the orbit of F^k is dense if and only if $m'_1(k)\theta_1(\omega, \tilde{\pi}) + ... + m'_d(k)\theta_d(\omega, \tilde{\pi}) \in \pi \cdot \mathbb{R} \backslash \mathbb{Q}$. The statement for F follows by $F^l(S^\delta(p)) = S^\delta(F^l(p))$. $\square$

Recall the definition of interval exchange transformation (IET) in the Introduction. We will adopt this definition throughout the remainder of this chapter. Given $\omega \in \mathbb{R}^d_+$, $\tilde{\pi} : \{1, ..., d\} \to \{1, ..., d\}$, we say an IET $f_{\omega,\tilde{\pi}}$ is *reflective* if there is a point $x \in I$ such that $f_{\omega,\pi}(x) = |\omega| - x$. Where $|\omega|$ denotes the ℓ_1 norm of ω.

Recall, from the Introduction, that $\mathcal{R}(\tilde{\pi})$ denotes the parameter region of all $\omega \in \mathbb{R}^d_+$ such that for some $j \in \{1, ..., d\}$ we have (2.6.1). The following lemma gives an alternative characterization of this set.

Lemma 2.6.4 *Let $\omega \in \mathbb{R}^d_+$ and $\pi \in \mathfrak{S}(\{1, ..., d\})$. Then $f_{\omega,\pi}$ is reflective if and only if $\omega \in \mathcal{R}(\tilde{\pi})$.*

Proof. Consider the map $\tilde{f} : I \to I$ such that $\tilde{f}(x) = |\omega| - f_{\omega,\pi}(x)$, for $x \in I$. By definition of this property, $f_{\omega,\pi}$ is reflective if and only if $\tilde{f}$ has a fixed point. Note that for all $j \in \{1, ..., d\}$ the restriction of $\tilde{f}$ to I_j is an orientation reversing continuous bijection, hence $\tilde{f}$ has a fixed point if and only if there is a $j \in \{1, ..., d\}$ such that $\tilde{f}(I_j) \cap I_j \neq \emptyset$. It is simple to see that this condition is satisfied if and only if (2.6.1) holds. Thus $f_{\omega,\pi}$ is reflective if and only if $\omega \in \mathcal{R}(\tilde{\pi})$ as desired. □

Recall, from the Introduction, that given $\pi \in \mathfrak{S}(\{1, ..., d\})$, $J_{\mathcal{R}}(\tilde{\pi})$ is the set of all $j \in \{1, ..., d\}$ such that (2.6.1) holds, for some $\omega \in \mathbb{R}^d_+$.

Given $\omega \in \mathbb{W} \cap \mathcal{R}(\tilde{\pi})$ and $j \in J_{\mathcal{R}}(\tilde{\pi})$ set

$$\mu_j(\omega, \tilde{\pi}) = \tan\left(\frac{\pi + \theta_j(\omega, \tilde{\pi})}{2}\right). \tag{2.6.3}$$

We omit, for simplicity, the arguments of $\mu_j(\omega, \tilde{\pi})$ when this does not cause ambiguity.

Lemma 2.6.5 *Let $\pi \in \mathfrak{S}(\{1, ..., d\})$, $\omega \in \mathbb{W} \cap \mathcal{R}(\tilde{\pi})$, $j \in J_{\mathcal{R}}(\tilde{\pi})$ and $\mu_j(\omega, \tilde{\pi})$ as in (2.6.3). We have $L_{-\mu_j} \subseteq P_j$ and for all $z \in L_{-\mu_j}$ we have $\mathrm{Im}(F(z)) = \mathrm{Im}(z)$.*

Proof. We begin by showing that there is a $j \in \{1, ..., d\}$ and a $\varphi \in W_j$ such that

$$f_{\omega,\pi}(\varphi - \vartheta) = \pi - \vartheta - \varphi, \tag{2.6.4}$$

with ϑ as in (1.4.3). Since $\omega \in \mathcal{R}(\tilde{\pi})$ we have that $f_{\omega,\pi}$ is a reflective IET, hence there is a $j \in \{1, ..., d\}$ and a $\varphi' \in I_j$ such that $f_{\omega,\pi}(\varphi') = |\omega| - \varphi'$. Since $|\omega| = \pi - 2\vartheta$, by taking $\varphi = \varphi' + \vartheta$ we get (2.6.4). We show that for $z \in L_{\tan(\varphi)}$ we have $\mathrm{Im}(F(z)) = \mathrm{Im}(z)$. By the definition of the map E and by (1.4.5), for $z \in P_c$ we have

$$E(z) = |z| \exp\left[i\left(\vartheta + f_{\omega,\pi}(\arg(z) - \vartheta)\right)\right]. \tag{2.6.5}$$

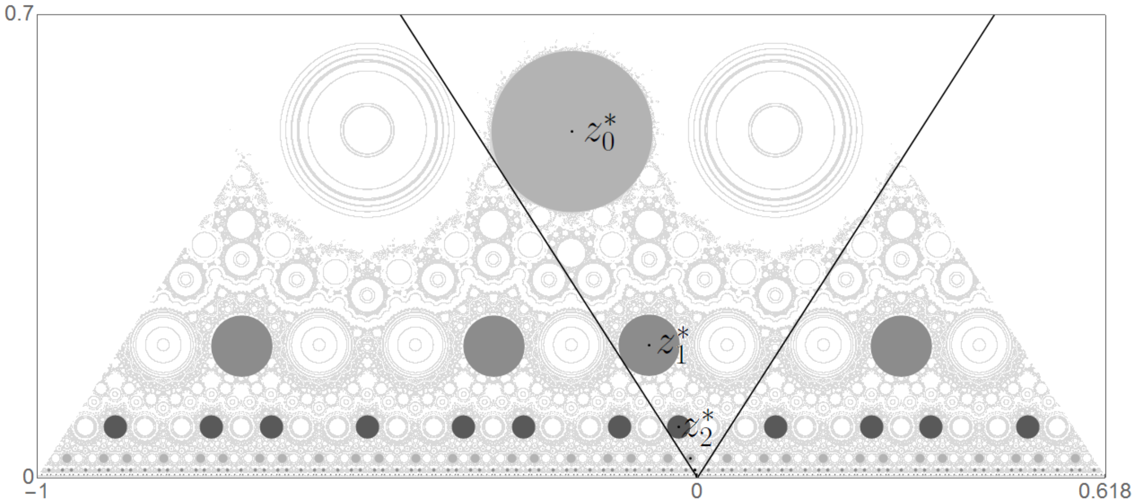


Figure 2.2: Periodic structures of the TCE with parameters $d = 2$, $\omega = (0.7, \pi - 2.7)$, $\tilde{\pi}(1, 2) = (2, 1)$, $\eta = \Phi$ and $\eta' = \Phi^2$. The lines represented are L_ν and $L_{-\nu}$ and the differently coloured disks are periodic islands, formed by invariant circles, containing periodic points $z_0^*, z_1^*, \ldots$. In light grey the first 10^4 iterates of the orbits of 320 points can be seen.

In particular for $z \in L_{\tan(\varphi)}$, by the definition of F, (2.6.4) and (2.6.5) we have

$$F(z) = |z|e^{i(\pi-\varphi)} - \eta'.$$

From (2.6.5) it follows that $\text{Im}(z) = |z|\sin(\varphi) = \text{Im}(F(z))$. We now prove that $\tan(\varphi) = -\mu_j$. By comparing the two identities above we get

$$\varphi = \frac{\pi - \theta_j(\omega, \tilde{\pi})}{2}.$$

Therefore, by (2.6.5) the slope of L''_S is equal to $\tan(\pi - \varphi)$ which coincides with μ_j. Thus $\tan(\varphi) = -\mu_j$, which completes the proof. □

Given $\nu > 0$ and μ such $|\mu| > \nu$, let

$$P(\mu) = \{z \in P_c : -\frac{\text{Im}(z)}{|\mu|} < \text{Re}(z) < \frac{\text{Im}(z)}{|\mu|}\}.$$

Define the interval $I_{P(\mu)}$ as

$$I_{P(\mu)} = \begin{cases} \left(1/C_\mu^-, 1/C_\mu^+\right), & \mu > \nu, \\ \left(1/C_\mu^+, 1/C_\mu^-\right), & \mu < -\nu. \end{cases}$$

The following theorem shows that a simple condition for the existence of a horizontal periodic island, as defined in the Introduction, for a TCE. A visual depiction of this can be seen in Figure 2.2.

Theorem 2.6.6 *Let $\pi \in \mathfrak{S}(\{1,...,d\})$, $\omega \in \mathbb{W} \cap \mathcal{R}(\tilde{\pi})$, $j \in J_{\mathcal{R}}(\tilde{\pi})$ and $\mu_j(\omega, \tilde{\pi})$ as in* (2.6.3)*. For every $n \in \mathbf{N}$ such that $p_n(\mu_j) \in I_{P(\mu_j)}$, F has a horizontal periodic orbit at height $\hat{y_n}$, for a certain $y_{n+1}(\mu_j) < \hat{y_n} < y_n(\mu_j)$. If $L_{\mu_j'} \cap \partial\mathcal{P} = \emptyset$, then F has an horizontal periodic island.*

Proof. Since $\pi \in \mathfrak{S}(\{1,...,d\}$ and $\omega \in \mathbb{W} \cap \mathcal{R}(\tilde{\pi})$, by Lemma 2.6.5 we have for all $z \in L_{-\mu_j}$ that $\text{Im}(F(z)) = \text{Im}(z)$. Recall (2.5.1). We begin by proving that if for some $n \in \mathbf{N}$ we have $u_n(\mu_j) \in I_{P(\mu_j)}$, then $x_n(\mu_j) + iy_n(\mu_j) \in P(\mu_j)$. By the definition of ℓ and from (2.5.1) we have

$$u_n(\mu_j) = \frac{1}{\ell(y_n(\mu_j))}\left(\frac{y_n(\mu_j)}{\nu} + x_n(\mu_j)\right).$$

From this, we have $x_n(\mu_j) + iy_n(\mu_j) \in P(\mu_j)$, if and only if we have

$$\frac{1}{2}\left(1 - \frac{\nu}{|\mu|}\right) < u_n(\mu_j) < \frac{1}{2}\left(1 + \frac{\nu}{|\mu|}\right).$$

By (2.3.2) it is direct to see that these inequalities are satisfied if and only if $u_n(\mu_j) \in I_{P(\mu_j)}$.

We now prove that if $u_n(\mu_j) \in I_{P(\mu_j)}$, there is an $\hat{y_n}$ satisfying

$$y_{n+1}(\mu_j) < \hat{y_n} < y_n(\mu_j), \tag{2.6.6}$$

such that $\xi_S(\hat{y_n})$ is a horizontal periodic orbit of F at height $\hat{y_n}$.

We split the proof in two cases $\mu_j > \nu$ and $\mu_j < -\nu$, but omit the $\mu_j < -\nu$ case as it is analogous to the other case.

Assume $\mu_j > \nu$. As for $y > 0$, $\xi_S(y) \in L_{-\mu_j}$ we have $\text{Re}(\xi_S(y)) = -y/\mu_j$, moreover as $u_n(\mu_j) \in I_{P(\mu_j)}$ we have $x_n(\mu_j) + iy_n(\mu_j) \in P(\mu_j)$ and hence $x_n > -y_n/\mu_j$. Since $x_n(\mu_j) = \text{Re}(\rho(y_n^-))$ this shows that

$$\text{Re}(\xi_S(y_n^-)) < \text{Re}(\rho(y_n^-)).$$

As $\mu_j > \nu$ and $u_n(\mu_j) \in I_{P(\mu_j)}$ we have $u_n(\mu_j) < 1/C^+_{\mu_j}$, hence by Theorem 2.4.2 we get that $\rho(y_{n+1}) \in L_{-\nu}$. As $\xi_S(y_{n+1}) \in \text{int}(P_c)$ we get

$$\text{Re}(\rho(y_{n+1})) < \text{Re}(\xi_S(y_{n+1})).$$

By Theorem 2.4.2, $\rho(y)$ is an affine map for $y_{n+1} \le y < y_n$ and the map $y \mapsto \xi_S(y)$ is also affine, in particular both maps are continuous for $y_{n+1} \le y < y_n$. Therefore by the two inequalities above, there must be a $\hat{y_n}$ satisfying (2.6.6) such that

$$\text{Re}(\rho(\hat{y_n})) = \text{Re}(\xi_S(\hat{y_n})).$$

As $\xi_S(\hat{y_n}) \in L_{-\mu_j}$, by Lemma 2.6.5 we have that

$$\mathrm{Im}(\xi_S(\hat{y_n})) = \mathrm{Im}(F(\xi_S(\hat{y_n}))) = \hat{y_n}.$$

By Theorem 2.4.2, $\mathrm{Im}(\rho(\hat{y_n})) = \hat{y_n}$, hence by the two identities above we get that $\rho(\hat{y_n}) = \xi_S(\hat{y_n})$. Thus by the definition of ρ, $\xi_S(\hat{y_n})$ is a periodic orbit for F. Moreover by Lemma 2.1.1 we have that the imaginary part of $\xi_S(\hat{y_n})$ remains constant, and equal to $\hat{y_n}$, throughout its orbit, hence it is an horizontal periodic orbit for F.

Finally we show that if $L_{-\mu_j} \cap \partial\mathcal{P} = \emptyset$, then F has a periodic island that contains this periodic orbit. Since $\xi_S(\hat{y_n}) \in L_{-\mu_j}$ and $L_{-\mu_j} \cap \partial\mathcal{P} = \emptyset$ we can apply Theorem 2.6.3 which shows that this orbit shadows a periodic island which is formed by the union of infinitely many invariant circles. □

We now prove Theorems 2.6.1 and 2.6.2.

2.6.1 Proof of Theorem 2.6.1

We divide the proof in two cases $\tilde{\pi} \in \zeta_-(d)$ (resp. $\zeta_+(d)$) and prove that there is a non-empty open set $\mathcal{W}_- \subseteq \mathbb{W} \cap \mathcal{R}(\tilde{\pi})$ (resp. $\mathcal{W}_+$) such that for all $\omega \in \mathcal{W}_-$ (resp. $\mathcal{W}_+$), F has infinitely many horizontal periodic islands accumulating on the origin. Having proved this, taking $\mathcal{W} = \mathcal{W}_- \cup \mathcal{W}_+$ gives the desired result.

We begin by considering the case $\tilde{\pi} \in \zeta_-(d)$. Given $j \in J_{\mathcal{R}}(\tilde{\pi})$, consider the set

$$J_{\zeta_-}(j, \tilde{\pi}) = \{j'' \in \{1, ..., d\} : j < j'' \text{ and } \tilde{\pi}(j'') < \tilde{\pi}(j')\}.$$

Since $\tilde{\pi} \in \zeta_-(d)$, we can take $j' \in J_{\mathcal{R}}(\tilde{\pi})$ such that $J_{\zeta_-}(j', \tilde{\pi}) \neq \emptyset$ and take $j'' \in J_{\zeta_-}(j', \tilde{\pi})$.

Let $\mu_{j'}(\omega, \tilde{\pi})$ be as in (2.6.3). Consider the set $\mathcal{V}_-$ of all $\omega \in \mathbb{W} \cap \mathcal{R}(\tilde{\pi})$, such that:

$$|\omega| \notin \left\{\frac{2\pi}{n}\right\}_{n\geq 1}, \quad \frac{\mu_{j'}(\omega, \tilde{\pi})}{\nu(|\omega|)} < -1 \quad \text{and} \quad \frac{\mu_{j'}(\omega, \tilde{\pi}) + \nu(|\omega|)}{\mu_{j'}(\omega, \tilde{\pi}) - \nu(|\omega|)} < \Phi. \tag{2.6.7}$$

We now show that if $|\omega| \notin \{2\pi/n\}_{n\geq 1}$, there is a $\delta > 0$ such that for $\theta_{j'}(\omega/|\omega|, \tilde{\pi}) \in (1 - \delta, 1)$, we have (2.6.7).

Since the map $r \mapsto (r+1)/(r-1)$ is continuous for all $r \in \mathbb{R}\backslash\{-1\}$ and zero for $r = -1$, there is an $\epsilon > 0$, such that for all $\omega \in \mathcal{V}_-$ such that if:

$$\frac{\mu_{j'}(\omega, \tilde{\pi})}{\nu(|\omega|)} \in (-1 - \epsilon, -1), \tag{2.6.8}$$

then we have (2.6.7). By (2.6.3) we have

$$\frac{\mu_{j'}(\omega, \tilde{\pi})}{\nu(|\omega|)} = \tan\left(\frac{\pi + \theta_{j'}(\omega, \tilde{\pi})}{2}\right) / \tan\left(\frac{\pi - |\omega|}{2}\right).$$

Using linearity of $\omega \mapsto \theta_{j'}(\omega, \tilde{\pi})$ and simple trigonometric identities, from the above identity, we get

$$\frac{\mu_{j'}(\omega,\tilde{\pi})}{\nu(|\omega|)} = -\cot\left(|\omega|\frac{\theta_{j'}(\omega/|\omega|,\tilde{\pi})}{2}\right)\tan\left(\frac{|\omega|}{2}\right).$$

Since $\omega \mapsto \theta_{j'}(\omega/|\omega|, \tilde{\pi})$ is independent of $|\omega|$ and we have $|\omega| \notin \{2\pi/n\}_{n\geq 1}$, the map $\theta \mapsto -\cot(|\omega|\theta/2)\tan(|\omega|/2)$ is continuous and therefore there is a $\delta > 0$ such that for $\theta_{j'}(\omega/|\omega|, \tilde{\pi}) \in (1-\delta, 1)$, we have (2.6.8) and thus (2.6.7).

We now show that there is a nonempty open set $\mathcal{W}'_- \subseteq \mathcal{V}_-$. To do this we construct an open set $\mathcal{W}'_-$ such that for $\omega \in \mathcal{W}'_-$ we have $\theta_{j'}(\omega/|\omega|, \tilde{\pi}) \in (1-\delta, 1)$. By (1.4.5) and (2.6.1), it suffices to show there is an $\tilde{\omega} \in \mathcal{V}_-$ such that we have

$$\sum_{\tilde{\pi}(k)<\tilde{\pi}(j')} \tilde{\omega}_k - \sum_{k<j'} \tilde{\omega}_k > |\tilde{\omega}|(1-\delta), \tag{2.6.9}$$

$$\left| \sum_{\tilde{\pi}(k)>\tilde{\pi}(j')} \tilde{\omega}_k - \sum_{k<j'} \tilde{\omega}_k \right| < \tilde{\omega}_{j'}. \tag{2.6.10}$$

Since the above inequalities are strict, we have that there is a neighbourhood $\mathcal{W}'_- \subseteq \mathcal{V}_-$ of $\tilde{\omega}$, such that both inequalities are true for all $\omega \in \mathcal{W}'_-$.

We now prove there is $\tilde{\omega} \in \mathcal{V}_-$ satisfying (2.6.9) and (2.6.10). Assume first that $d = 2$ and take $\tilde{\omega}$ such that $\tilde{\omega}_{j'} = |\tilde{\omega}|\delta/2$ and $\tilde{\omega}_{j''} = |\tilde{\omega}|(1-\delta/2)$. Since $j'' \in J_{\zeta_-}(j', \tilde{\pi})$, we have $j < j''$ and $\tilde{\pi}(j'') < \tilde{\pi}(j')$, we have $j' = 1$ and $j'' = 2$, hence

$$\sum_{\tilde{\pi}(k)<\tilde{\pi}(j')} \tilde{\omega}_k - \sum_{k<j'} \tilde{\omega}_k = |\tilde{\omega}|(1-\delta/2),$$

thus (2.6.9) holds. We also have

$$\sum_{\tilde{\pi}(k)>\tilde{\pi}(j')} \tilde{\omega}_k - \sum_{k<j'} \tilde{\omega}_k = 0,$$

hence, since $\tilde{\omega}_{j'} > 0$, we get (2.6.10) as well.

Now assume $d > 2$ and set $\tilde{\omega} = (\tilde{\omega}_j)_{j=1,\dots,d}$, where

$$\tilde{\omega}_j = \begin{cases} |\tilde{\omega}|\delta/6, & j = j', \\ |\tilde{\omega}|(1-\delta/4), & j = j'', \\ \dfrac{|\tilde{\omega}|\delta}{12(d-2)}, & j \neq j', j''. \end{cases} \tag{2.6.11}$$

We show that (2.6.9) is true for $\tilde{\omega}$. Since $j'' \in J_{\zeta_-}(j', \tilde{\pi})$ we have

$$\sum_{k<j'} \tilde{\omega}_k + \sum_{\tilde{\pi}(k)\geq\tilde{\pi}(j')} \tilde{\omega}_k \leq 2|\tilde{\omega}| - 2\tilde{\omega}_{j''}.$$

By (2.6.11) we have $2|\tilde{\omega}| - 2\tilde{\omega}_{j''} = |\tilde{\omega}|\delta/2$, hence by the inequality above we have

$$\sum_{k<j'} \tilde{\omega}_k + \sum_{\tilde{\pi}(k)\geq\tilde{\pi}(j')} \tilde{\omega}_k < |\tilde{\omega}|\delta,$$

which is equivalent to (2.6.9).

We now show that (2.6.10) is true for $\tilde{\omega}$. Since for $k \in \{j', j''\}$ we have $\tilde{\pi}(k) \leq \tilde{\pi}(j')$ and $k > j'$ we have

$$\left| \sum_{\tilde{\pi}(k)>\tilde{\pi}(j')} \tilde{\omega}_k - \sum_{k<j'} \tilde{\omega}_k \right| < \sum_{k\neq j',j''} \tilde{\omega}_k.$$

By (2.6.11) we have $\tilde{\omega}_{j'} = |\tilde{\omega}|\delta/6$ and $\sum_{k\neq j',j''} \tilde{\omega}_k = \delta/12$ hence by the inequality above we have that (2.6.10) is true for $\tilde{\omega}$.

We now prove that for $\omega \in \mathcal{W}'_-$, F has infinitely many horizontal periodic orbits accumulating on the origin. By Theorem 2.6.6 it suffices to show that for infinitely many $n \in \mathbb{N}$ we have $u_n(\mu_{j'}(\omega, \tilde{\pi})) \in I_{P(\mu_{j'}(\omega,\tilde{\pi}))}$. Note that we have

$$\frac{C^-_{\mu_{j'}(\omega,\tilde{\pi})}}{C^+_{\mu_{j'}(\omega,\tilde{\pi})}} = \frac{\mu_{j'}(\omega, \tilde{\pi}) + \nu(|\omega|)}{\mu_{j'}(\omega, \tilde{\pi}) - \nu(|\omega|)},$$

hence since $\omega \in \mathcal{W}'_- \subseteq \mathcal{V}_-$ we have

$$\frac{C^-_{\mu_{j'}(\omega,\tilde{\pi})}}{C^+_{\mu_{j'}(\omega,\tilde{\pi})}} < \Phi < 1. \tag{2.6.12}$$

Assume first that $-\bar{\mu} < \mu_{j'} < -\nu$, with $\bar{\mu} = \frac{\nu}{\Phi^3}$. Using Hölder conjugacy of $C^+_{\mu_{j'}}$ and $C^-_{\mu_{j'}}$ it can be seen that (2.6.12) is equivalent to

$$\frac{1}{C^+_{\mu_{j'}}} < 1 - \frac{\Phi}{C^-_{\mu_{j'}}} < \frac{1}{C^-_{\mu_{j'}}}.$$

By Theorem 2.3.3 (2.3.9), for all $n \geq 1$ we have that $u_n(\mu_{j'}) = 1 - \Phi/C^-_{\mu_{j'}}$, hence by the inequality above we get $u_n(\mu_{j'}) \in I_{P(\mu_{j'})}$ for infinitely many $n \in \mathbb{N}$. Now assume $\mu_{j'} \leq -\bar{\mu}$. It can be seen that (2.6.12) is equivalent to:

$$\frac{1}{C^+_{\mu_{j'}}} < \frac{1}{C^+_{\mu_{j'}}\Phi} < \frac{1}{C^-_{\mu_{j'}}}.$$

By Theorem 2.3.3 (2.3.7), for all even $n \in \mathbb{N}$ we have that $u_n(\mu_{j'}) = (C^+_{\mu_{j'}}\Phi)^{-1}$, hence by the inequality above we get $u_n(\mu_{j'}) \in I_{P(\mu_{j'})}$ for infinitely many $n \in \mathbb{N}$.

We now show that there is a non-empty open set $\mathcal{W}_- \subseteq \mathbb{W} \cap \mathcal{R}(\tilde{\pi})$ such that for all $\omega \in \mathcal{W}_-$, F has infinitely many horizontal periodic islands accumulating on the origin. By Theorem 2.6.6 it suffices to show that there is a non-empty open set $\mathcal{W}_- \subseteq \mathcal{W}'_-$ such that for all $\omega \in \mathcal{W}_-$ we have $L_{\mu'_{j'}(\omega,\tilde{\pi})} \cap \partial\mathcal{P} = \emptyset$.

Consider the sets

$$\mathcal{H}_k = \left\{ \omega \in \mathbb{W} : |\omega| - \theta_{j'}(\omega, \tilde{\pi}) - 2\sum_{j \le k} \omega_j = 0 \right\},$$

for $k = 0, 1, ..., d$. Note that we have $L_{\mu'_{j'}(\omega,\tilde{\pi})} \cap \partial\mathcal{P} \neq \emptyset$ if for some $k \in \{0, 1, ..., d\}$ we have

$$-\mu_{j'}(\omega, \tilde{\pi}) = \tan\left(\frac{\pi - |\omega|}{2} + \sum_{j \le k} \omega_j \right).$$

By (2.6.3) and the two identities above it follows that we have $L_{\mu'_{j'}(\omega,\tilde{\pi})} \cap \partial\mathcal{P} \neq \emptyset$ if and only if $\omega \in \mathcal{H}_k$ for some $k \in \{0, 1, ..., d\}$.

Set $\mathcal{W}_- = \mathcal{W}'_- \setminus \bigcup_{k=0}^{d} \mathcal{H}_k$. Since $\mathcal{H}_k$ are codimension 1 closed subsets of $\mathbb{W}$, we have that $\mathcal{W}_-$ is a non-empty open set and since for $\omega \in \mathcal{W}_-$ we have $L_{\mu'_{j'}(\omega,\tilde{\pi})} \cap \partial\mathcal{P} = \emptyset$, F has infinitely many horizontal periodic islands accumulating on the origin.

We now consider the case $\tilde{\pi} \in \zeta_+(d)$. This case is mostly analogous to the previous one, so for brevity we will only outline the proof.

Given $j \in J_{\mathcal{R}}(\tilde{\pi})$, consider the set

$$J_{\zeta_+}(j, \tilde{\pi}) = \{ j'' \in \{1, ..., d\} : j > j'' \text{ and } \tilde{\pi}(j'') > \tilde{\pi}(j') \}.$$

Take $j' \in J_{\mathcal{R}}(\tilde{\pi})$ such that $J_{\zeta_+}(j', \tilde{\pi}) \neq \emptyset$ and take $j'' \in J_{\zeta_+}(j', \tilde{\pi})$.

Consider the set $\mathcal{V}_+$, of all $\omega \in \mathbb{W} \cap \mathcal{R}(\tilde{\pi})$, such that:

$$|\omega| \notin \left\{ \frac{2\pi}{n} \right\}_{n \ge 1}, \quad \frac{\mu_{j'}(\omega, \tilde{\pi})}{\nu(|\omega|)} > 1 \quad \text{and} \quad \frac{\mu_{j'}(\omega, \tilde{\pi}) - \nu(|\omega|)}{\mu_{j'}(\omega, \tilde{\pi}) + \nu(|\omega|)} < \Phi.$$

By a similar argument to the previous case, if $|\omega| \notin \{2\pi/n\}_{n \ge 1}$, there is a $\delta > 0$ such that for $\theta_{j'}(\omega/|\omega|, \tilde{\pi}) \in (-1, -1 + \delta)$, the expression above is satisfied.

To find a nonempty open set $\mathcal{W}'_+ \subseteq \mathcal{V}_-$ by (1.4.5) and (2.6.1), it suffices to show there is an $\tilde{\omega} \in \mathcal{V}_+$ such that we have (2.6.10) and:

$$\sum_{\tilde{\pi}(k) < \tilde{\pi}(j')} \tilde{\omega}_k - \sum_{k < j'} \tilde{\omega}_k < |\tilde{\omega}|(-1 + \delta).$$

Indeed it can be seen that both this inequality and (2.6.10) hold for the same choice of $\tilde{\omega}$ of the previous case.

We prove that for $\omega \in \mathcal{W}'_+$, F has infinitely many horizontal periodic orbits accumulating on the origin. By Theorem 2.6.6 it suffices to show that for infinitely many $n \in \mathbb{N}$ we have $u_n(\mu_{j'}(\omega, \tilde{\pi})) \in I_{P(\mu_{j'}(\omega,\tilde{\pi}))}$. Note that we have

$$\frac{C^+_{\mu_{j'}(\omega,\tilde{\pi})}}{C^-_{\mu_{j'}(\omega,\tilde{\pi})}} = \frac{\mu_{j'}(\omega, \tilde{\pi}) - \nu(|\omega|)}{\mu_{j'}(\omega, \tilde{\pi}) + \nu(|\omega|)},$$

hence since $\omega \in \mathcal{W}'_+ \subseteq \mathcal{V}_+$ we have

$$\frac{C^+_{\mu_{j'}(\omega,\tilde{\pi})}}{C^-_{\mu_{j'}(\omega,\tilde{\pi})}} < \Phi < 1. \tag{2.6.13}$$

Assume first that $\nu < \mu_{j'} < \bar{\mu}$. It can be seen that (2.6.13) is equivalent to

$$\frac{1}{C^-_{\mu_{j'}}} < 1 - \frac{1}{C^-_{\mu_{j'}}\Phi} < \frac{1}{C^+_{\mu_{j'}}}.$$

By Theorem 2.3.3 (2.3.8), for all odd n we have that $u_n(\mu_{j'}) = 1 - (C^-_{\mu_{j'}}\Phi)^{-1}$, hence by the inequality above we get $u_n(\mu_{j'}) \in I_{P(\mu_{j'})}$ for infinitely many $n \in \mathbb{N}$. Now assume $\mu_{j'} \geq \bar{\mu}$. It can be seen that (2.6.13) is equivalent to:

$$\frac{1}{C^-_{\mu_{j'}}} < \frac{\Phi}{C^+_{\mu_{j'}}} < \frac{1}{C^+_{\mu_{j'}}}.$$

By Theorem 2.3.3 (2.3.10), for all $n \in \mathbb{N}$ we have that $u_n(\mu_{j'}) = (C^+_{\mu_{j'}}\Phi)^{-1}$, hence by the inequality above we get $u_n(\mu_{j'}) \in I_{P(\mu_{j'})}$ for all $n \in \mathbb{N}$.

Setting $\mathcal{W}_+ = \mathcal{W}'_+ \setminus \bigcup_{k=0}^{d} \mathcal{H}_k$, we get that for $\omega \in \mathcal{W}_+$ we have $L_{\mu'_{j'}(\omega,\tilde{\pi})} \cap \partial\mathcal{P} = \emptyset$, hence by Theorem 2.6.6 F has infinitely many horizontal periodic islands at heights which converge to 0, hence accumulating on the real line. □

2.6.2 Proof of Theorem 2.6.2

Let U be an invariant set for F_c that contains a neighbourhood of the origin. By Theorem 2.6.1 it contains infinitely many periodic islands. Suppose there is a point $z \in U$ with a dense orbit in U. Then $\{F_c^n(z)\}_n$ can get arbitrarily close to a periodic point z', this implies that for some $m \in \mathbb{N}$, $F_c^m(z)$ is contained in a periodic island. Hence its orbit is contained in a circle thus contradicting the hypothesis that the orbit of z is dense in U. □

Chapter 3

Embeddings of Interval Exchange Transformations into Piecewise Isometries

Recall the definitions of Interval Exchange Transformation (IET) and Piecewise Isometry (PWI) in the Introduction.

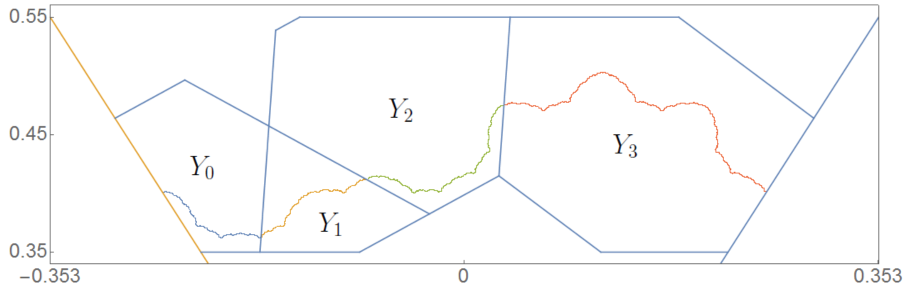
In this chapter, we discuss the general problem of embedding IET dynamics within PWIs with a particular focus on the regularity of this embedding for two dimensional PWIs. In particular, we consider conditions for this embedding to be trivial or non-trivial. Our main results are as follows.

- In Theorem 3.2.4 we use combinatorial properties of IETs to prove that in order for a PWI realize a continuous embedding of an IET with the same permutation, its parameters must satisfy a necessary condition: the *parametric connecting equation* (3.2.10).

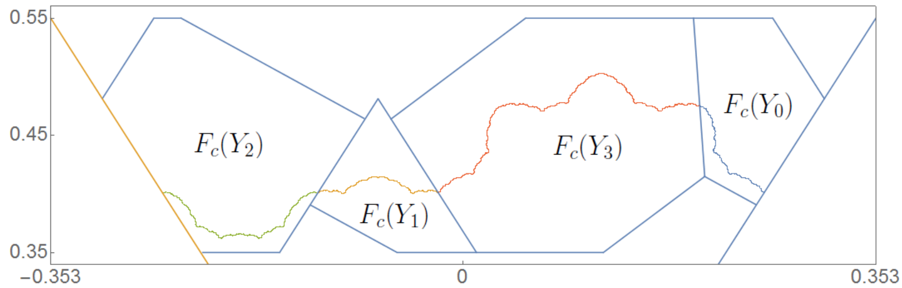
- As a consequence of this, Theorem C, states that all continuous embeddings of minimal 2-IETs are trivial and Theorem D asserts that a 3-PWI has at most one non-trivially continuously embedded minimal 3-IET with the same underlying permutation.

- Given an IET embedded into a PWI we use the derived *tangent exchange map* (3.3.1) to prove Theorem 3.3.1, which gives a necessary condition on the parameters of a PWI such that there is a continuous embedding of an IET into that PWI.

We introduce a specific example F (3.4.3) of a translated cone exchange transformation that has a trivially embedded IET on the boundary. Recalling F_c, a first return map under F to a subset of the phase space P_c we observe invariant regions



(a)



(b)

Figure 3.1: An illustration of the action of a piecewise isometry F_c (see (3.4.5)), on the image of a non-trivial embedding $Y = \bigcup_{\alpha\in\mathcal{A}} Y_\alpha$, with $\mathcal{A} = \{0, 1, 2, 3\}$ of a minimal 4-IET. (A) An invariant set Y where each Y_α, is contained in a polygon. Points in each polygon are mapped isometrically by F_c to a subset of the region $\{z \in \mathbb{C} : 0.35 < \mathrm{Im}(z) < 0.55\}$. (B) Image of Y and the polygons in (A) under F_c.

bounded by invariant curves (Figure 3.7) and perform numerical experiments to verify the conditions of Theorems 3.2.4 and 3.3.1. We introduce a PWI T' (see (3.4.1)) on 3 atoms that apparently exhibits a single invariant curve that is a non-trivial embedding of a 3-IET into T'. Using this we make specific conjectures about the nature of non-trivial embeddings of IETs in PWIs.

This chapter is organized as follows. In Section 3.1 we consider possible embeddings of a transitive IET into a PWI, and make some definitions regarding their regularity. We identify trivial cases of embedding as where the image of the embedding is either a union of lines or of arcs of the same radius. Furthermore, we extend the Rauzy-Veech induction for IETs to PWIs that admit continuous embeddings of IETs. In Section 3.2 we introduce some combinatorial conditions on the embedding of an IET into a PWI and state a necessary condition for existence of continuous embeddings. Using these techical tools, we prove that only trivial embeddings of 2-IETs are possible and that a 3-PWI has at most one non-trivially continuous embedded 3-IET with the same underlying permutation. In Section 3.3 we turn to ergodic properties of the embeddings and in Theorem 3.3.1 give a necessary condition for embedding in terms of average returns. In Section 3.4 we introduce concrete examples of PWIs and show numerical results. We introduce a PWI on 3 atoms, illustrate some examples of orbits for this piecewise isometry and numerically estimate the parameters of a 3-IET which is embedded into this PWI. We also introduce a particular planar translated cone exchange transformation illustrated in Figure 3.4. This transformation has a trivially embedded 2-IET on a line that we call the *baseline* and arbitrarily close to this baseline there are non-trivial rotations. The dynamics of points close to this baseline is remarkably rich. In particular, numerical simulations suggest that the baseline is an accumulation for non-smooth invariant curves that are non-trivial embeddings of 4-IETs in the 4-PWI. We illustrate some examples of orbits for this piecewise isometry and show numerical evidence for abundance of periodic orbits for certain regions of the parameters. We show that the parameters of this map satisfy the restrictions from Theorem 3.2.4. We numerically verify that the condition from Theorem 3.3.1 is satisfied.

The material in this chapter has been published in *Ergodic Theory and Dynamical Systems* [12].

3.1 Symbolic, topological and differentiable embeddings

In this section we introduce some definitions of various regularity properties that characterize an embedding of an IET into a PWI. The weakest of these is a symbolic embedding. Furthermore, we extend Rauzy-Veech induction for IETs to PWIs that admit continuous embeddings of IETs.

Consider a d-IET $(I, f_{\lambda,\pi})$ which we sometimes denote by (I, f) when parameters are clear from context. For a point $x \in I$ we define the *itinerary* or *symbolic encoding* of x by the IET as

$$\iota(x) = \alpha_0\alpha_1 \ldots \in \mathcal{A}^{\mathbb{N}}, \tag{3.1.1}$$

where $\alpha_k \in \mathcal{A}$ is such that $f^k(x) \in I_\alpha$ if and only if $\alpha_k = \alpha$.

Similarly, suppose that (X, T) is a d-PWI with atoms $\{X_\alpha\}_{\alpha\in\mathcal{A}}$. We define the itinerary of a point $z \in X$ by the PWI as

$$\iota'(z) = \alpha'_0\alpha'_1 ... \subset \mathcal{A}^{\mathbb{N}} \tag{3.1.2}$$

where $\alpha'_k \in \mathcal{A}$ is such that $T^k(z) \in X_\alpha$ if and only if $\alpha'_k = \alpha$.

We now introduce some definitions that will be used throughout this chapter.

An injective map $\gamma : I \to X$ is a *symbolic embedding* of (I, f) into (X, T) if $\gamma(I) \subset X$ is an invariant set for (X, T) and there is a labeling of the atoms such that

$$\iota' \circ \gamma(x) = \iota(x) \quad \text{for all } x \in I.$$

Recall, from the Introduction Section 1.4.2, the definitions of piecewise continuous, continuous, differentiable, arc, linear, trivial and non-trivial embeddings.

Note that if (I, f) has a piecewise continuous embedding γ into (X, T) then it is also a symbolic embedding, but the converse does not necessarily hold (to see this, note that $\gamma(I)$ need not be closed if it is a disconnected union of disjoint orbits).

It follows immediately from the definitions of linear and arc embeddings that if a piecewise continuous embedding of (I, f) by γ into (X, T) is a linear embedding then there are $z_\alpha, \upsilon_\alpha \in \mathbb{C}$ such that

$$\gamma|_{I_\alpha}(x) = z_\alpha + \upsilon_\alpha x, \tag{3.1.3}$$

for all $x \in I$, while if it is an arc embedding, then there are $\xi_\alpha \in \mathbb{C}$, $r_\alpha > 0$ and $a_\alpha, \varphi_\alpha \in \mathbb{R}$ such that

$$\gamma|_{I_\alpha}(x) = \xi_\alpha + r_\alpha \exp[i(a_\alpha x + \varphi_\alpha)], \tag{3.1.4}$$

for all $x \in I$.

The following lemma shows that there exist some basic relations for the parameters defining trivial embeddings which are automatically satisfied.

Lemma 3.1.1 *For any d-IET (I, f) there exists a trivial continuous embedding $\gamma : I \to X$ of (I, f) into a d-PWI (X, T), which can be either a linear embedding or an arc embedding. Suppose in addition that (I, f) is minimal. (a) If γ is a linear embedding then $|\upsilon_\alpha|$ is independent of α. (b) If γ is an arc embedding then r_α and a_α are independent of α.*

Proof. Assume without loss of generality that $I \subset [0, \pi)$. Note that there exists a linear embedding with rectangular atoms such that $T(x+iy) = f(x)+iy$, and there exists an arc embedding such that $T(re^{i\theta}) = re^{if(\theta)}$.

We now prove (a) and (b). Fix $x \in I_\beta$ for some $\beta \in \mathcal{A}$. Since (I, f) is minimal, for all $\varsigma \in \mathcal{A}\backslash\{\beta\}$ there is a $N_\varsigma > 0$ such that $f^{N_\varsigma}(x) = x+v \in I_\varsigma$, with $v = \sum_{k=0}^{N_\varsigma-1} \upsilon_{\alpha_k(x)}$.

We begin by proving (a). Assume that γ is a linear embedding of (I, f) into (X, T) as in (3.1.3). We show that $|\upsilon_\beta| = |\upsilon_\varsigma|$. By (1.4.1), (1.4.8) and (3.1.3) we have

$$e^{i\theta_\beta}(z_\beta + \upsilon_\beta x) + \eta_\beta = z_\varsigma + \upsilon_\varsigma(x + v). \tag{3.1.5}$$

Differentiating (3.1.5) with respect to x gives $e^{i\theta_\varsigma} v_\beta = v_\varsigma$, thus $|v_\beta| = |v_\varsigma|$.

We now prove (b). Assume that γ is an arc embedding of (I, f) into (X, T) as in (3.1.4). We show that $a_\beta = a_\varsigma$ and $r_\beta = r_\varsigma$. Combining (1.4.8), (3.1.4) and differentiating with respect to x we get

$$ir_\beta a_\beta \exp[i(\theta_\beta + a_\beta x + b_\beta)] = ir_\varsigma a_\varsigma \exp[i(a_\varsigma x + a_\varsigma \upsilon + b_\varsigma)],$$

and taking modulus gives

$$r_\beta |a_\beta| = r_\varsigma |a_\varsigma|, \tag{3.1.6}$$

while the argument gives

$$\theta_\beta + a_\beta x + b_\beta = a_\varsigma x + a_\varsigma \upsilon + b_\varsigma \mod 2\pi. \tag{3.1.7}$$

Note that (3.1.7) holds for any $x \in f^{-N_\varsigma}(I_\varsigma) \cap I_\beta$. Since this set contains an interval, (3.1.7) must hold for infinitely many values of x, hence we get $a_\beta = a_\varsigma$. Together with (3.1.6) this shows that $r_\beta = r_\varsigma$, completing the proof. □

Recall from the definition of IET that a permutation π is a pair of bijections $\pi_\varepsilon : \mathcal{A} \to \{1, ..., d\}$, $\varepsilon = 0, 1$.

Given an IET $(I, f_{\lambda,\pi})$, consider the points

$$x_0 = 0, \quad x_j = \sum_{k=1}^{j} \lambda_k, \quad 1 \leq j \leq d. \tag{3.1.8}$$

Note that $I = [x_0, x_d)$ and that $I_{\pi_0^{-1}(j)} = [x_{j-1}, x_j)$ for $1 \leq j \leq d$.

The next theorem allow us to characterize the existence of continuous or discontinuous embeddings in terms of the preimages of interior discontinuities of f.

Theorem 3.1.2 *Assume that (I, f) is a d-IET with intervals $I_{\pi_0^{-1}(j)} = [x_{j-1}, x_j)$ for $j = 1, \ldots d$. There exists a d-PWI (X, T), such that (I, f) has a discontinuous embedding into (X, T) if and only if*

$$f^{-1}(\{x_1, ..., x_{d-1}\}) \cap \{x_0, ..., x_d\} \neq \emptyset.$$

Proof. Let $I = I_{\pi_0^{-1}(1)} \cup ... \cup I_{\pi_0^{-1}(d)}$, with $I_j = [x_{j-1}, x_j), j \in \{1, ..., d\}$.

We begin by proving that if there is $j' \in \{1, ..., d-1\}$ such that $f^{-1}(x_{j'}) \in \{x_0, ..., x_d\}$, then there exists a d-PWI (X, T), such that (I, f) has a discontinuous embedding into (X, T).

By Lemma 3.1.1 there is a continuous embedding of (I, f) by γ' into a d-PWI (X', T') with $Y' = \gamma(I) \subset X'$ invariant set for (X', T'). Note that since this embedding is trivial we can take X' to be a compact set. Therefore it has a finite diameter, which we denote as $|X'|$.

Set $Y'_{\pi_0^{-1}(j)} = Y' \cap X'_{\pi_0^{-1}(j)}$ for $j = 1, ..., d$ and let

$$X_{\pi_0^{-1}(j)} = \begin{cases} X'_{\pi_0^{-1}(j)}, & \text{if } j \leq j', \\ X'_{\pi_0^{-1}(j)} + 2|X'|, & \text{if } j > j', \end{cases}$$

with $X = X_{\pi_0^{-1}(j)} \cup ... \cup X_{\pi_0^{-1}(j)}$. Define the maps

$$T_{\pi_0^{-1}(j)}(z) = \begin{cases} T'_{\pi_0^{-1}(j)}(z), & \text{if } j \leq j', \\ T'_{\pi_0^{-1}(j)}(z - 2|X'|) + 2|X'|, & \text{if } j > j'. \end{cases}$$

If $T(z) = T_{\pi_0^{-1}(j)}(z)$, for $z \in X_{\pi_0^{-1}(j)}$, with $j = 1, ..., d$, then (X, T) defines a d-PWI.

Define the function $\gamma : I \to X$ as

$$\gamma(x) = \begin{cases} \gamma'(x), & x < x_{j'}, \\ \gamma'(x) + 2|X'|, & x \geq x_{j'}. \end{cases}$$

Set $Y = \gamma(I)$. The map $\gamma : I \to Y$ is bijective and it is simple to check that (I, f) has a piecewise continuous embedding by γ into (X, T). Moreover, note that the restriction of γ to $I_{\pi_0^{-1}(j)}$ is continuous for $j = 1, ..., d$, but γ has a discontinuity at $x = x_{j'}$. Thus, the embedding is discontinuous.

Now assume there is no $x_j \in \{x_1, ..., x_{d-1}\}$ such that $f^{-1}(x_j) \in \{x_0, ..., x_d\}$ and there exists a d-PWI (X, T), such that (I, f) has a discontinuous embedding by γ into (X, T).

Since the restriction of γ to $I_{\pi_0^{-1}(j)}$ is continuous for all $j = 1, ..., d$, the set of discontinuities of γ must be contained in $\{x_1, ..., x_{d-1}\}$. Assume $j' \in \{1, ..., d-1\}$ is such that γ is discontinuous at $x_{j'}$. Let

$$\underline{z_{j'}} = \lim_{x \to x_{j'}^-} \gamma(x), \quad \overline{z_{j'}} = \lim_{x \to x_{j'}^+} \gamma(x)$$

and $l \in \{1, ..., d\}$ be such that $x_{j'} \in f(I_{\pi_0^{-1}(l)})$. Set $Y = \gamma(I)$ and $Y_{\pi_0^{-1}(j)} = X_{\pi_0^{-1}(j)} \cap Y$ for $j = 1, ..., d$. Then $\{\underline{z_{j'}}, \overline{z_{j'}}\} \subset T(Y_{\pi_0^{-1}(l)})$. Since $f^{-1}(x_{j'}) \notin \{x_0, ..., x_d\}$, we have

$$T^{-1}(\{\underline{z_{j'}}, \overline{z_{j'}}\}) \cap \{\gamma(x_0), ..., \gamma(x_d)\} = \emptyset.$$

Thus there must be an $l' \in \{1, ..., d\}$ such that $\{\underline{z_{j'}}, \overline{z_{j'}}\} \subset Y_{\pi_0^{-1}(l')}$. Therefore the restriction of γ' to $I_{\pi_0^{-1}(l')}$ must be discontinuous, contradicting γ being a piecewise continuous embedding of (I, f) into (X, T). This completes the proof. $\square$

Recall Rauzy-Veech induction from the Introduction, Section 1.3.2. Particularly recall (1.3.3), (1.3.4) and that we say that (λ, π) is of *type* 0 if $\lambda_{\beta_0} > \lambda_{\beta_1}$ and is of *type* 1 if $\lambda_{\beta_0} < \lambda_{\beta_1}$. The largest interval is called *winner* and the smallest *loser* of (λ, π).

We now extend Rauzy-Veech induction to PWIs which admit embeddings of IETs as follows. Assume $(I, f_{\lambda,\pi})$ has an embedding by γ into (X, T). Define the map $\mathcal{S}(T)$ as the first return map under T to X^*, where

$$X^* = \begin{cases} \bigcup_{\alpha\neq\beta_0} X_\alpha \cup (X_{\beta_0} \cap T(X_{\beta_1})), & \text{if } (\lambda,\pi) \text{ has type } 0, \\ \bigcup_{\alpha\neq\beta_0} X_\alpha, & \text{if } (\lambda,\pi) \text{ has type } 1. \end{cases}$$

Note that $(X^*, \mathcal{S}(T))$ is again a d'-PWI since it is a first return map or a PWI to a convex subset of X. However it is now possible that $d' \neq d$. Denote by $\mathcal{A}'$ an alphabet with d' symbols and denote by $\{X^*_{\alpha'}\}_{\alpha'\in\mathcal{A}'}$ the partition of X^*. It is simple to see that there is a collection of d symbols $\mathcal{A} \subseteq \mathcal{A}'$, possibly after relabeling, such that $X^*_{\alpha'} \cap \gamma(I^{(1)}) \neq \emptyset$ if and only if $\alpha' \in \mathcal{A}$. Define $X' = \bigcup_{\alpha\in\mathcal{A}} X^*_\alpha$. Now, $(X', \mathcal{S}(T))$ is a d-PWI. We show, in the following theorem, that a continuous embedding of (I, f) into (X, T) also embeds $(I', \mathcal{R}(f))$ into $(X', \mathcal{S}(T))$.

Theorem 3.1.3 *Assume that a d-IET $(I, f_{\lambda,\pi})$, such that $I_{\beta_0} \neq f(\beta_1)$, has a continuous embedding by γ into a d-PWI (X, T). Then $(I', \mathcal{R}(f))$ has a continuous embedding by γ into $(X', \mathcal{S}(T))$.*

Proof. We prove that for all $x \in I'$ we have

$$\gamma \circ f_{\lambda^{(1)},\pi^{(1)}}(x) = \mathcal{S}(T) \circ \gamma(x). \tag{3.1.9}$$

Assume first that (λ, π) has type 0. Let $I^{(1)}_{\pi_0^{-1}(j)} = I_{\pi_0^{-1}(j)}$ for $j \neq d$ and $I^{(1)}_{\beta_0} = I_{\beta_0} \backslash f(I_{\beta_1})$. It is well known (see [56]) that

$$f_{\lambda^{(1)},\pi^{(1)}}(x) = \begin{cases} f^2(x), & x \in I^{(1)}_{\beta_1}, \\ f(x), & x \in I^{(1)}_{\pi_0^{-1}(j)},\ \pi_0^{-1}(j) \neq \beta_1. \end{cases} \tag{3.1.10}$$

We now show that we have

$$\mathcal{S}(T)(z) = \begin{cases} T^2(z), & z \in \gamma(I^{(1)}_{\beta_1}), \\ T(z), & z \in \gamma(I^{(1)}_{\pi_0^{-1}(j)}),\ \pi_0^{-1}(j) \neq \beta_1. \end{cases} \tag{3.1.11}$$

Note that $f(I^{(1)}_{\pi_0^{-1}(j)}) \subset I^{(1)}$, for $\pi_0^{-1}(j) \neq \beta_1$. Thus, by (1.4.8) we have $T(\gamma(I^{(1)}_{\pi_0^{-1}(j)})) \subset \gamma(I^{(1)})$, and we get (3.1.11) for $z \in \gamma(I^{(1)}_{\pi_0^{-1}(j)})$ and $\pi_0^{-1}(j) \neq \beta_1$.

Since $f(I^{(1)}_{\beta_1}) = f(I_{\beta_1}) \not\subset I^{(1)}$ and $f^2(I^{(1)}_{\beta_1}) \subset f(I_{\beta_0}) \subset I^{(1)}$, by (1.4.8) we have $T(\gamma(I^{(1)}_{\beta_1})) = T(\gamma(I_{\beta_1})) \not\subset \gamma(I^{(1)})$ and $T^2(\gamma(I^{(1)}_{\beta_1})) \subset T(\gamma(I_{\beta_0})) \subset \gamma(I^{(1)})$, and thus we have (3.1.11).

Noting that $x \in I_{\pi_0^{-1}(j)}$ if and only if $\gamma(x) \in \gamma(I^{(1)}_{\pi_0^{-1}(j)})$, for $j = 1, ..., d$, and combining (1.4.8), (3.1.10) and (3.1.11) we get (3.1.9).

Assume now that (λ, π) has type 1. Let $I^{(1)}_{\pi_0^{-1}(j)} = I_{\pi_0^{-1}(j)}$ for $1 \leq j < \pi_0(\beta_1)$, $I^{(1)}_{\beta_1} = I_{\beta_1} \backslash f^{-1}(I_{\beta_0})$, $I^{(1)}_{\pi_0^{-1}(\pi_0(\beta_1)+1)} = f^{-1}(I_{\beta_0})$ and $I^{(1)}_{\pi_0^{-1}(j)} = I_{\pi_0^{-1}(j-1)}$ for $\pi_0(\beta_1) + 1 < j \leq d$. It is clear that

$$f_{\lambda^{(1)},\pi^{(1)}}(x) = \begin{cases} f^2(x), & x \in I^{(1)}_{\pi_0^{-1}(\pi_0(\beta_1)+1)}, \\ f(x), & x \in I^{(1)}_{\pi_0^{-1}(j)},\ \pi_0(j) \neq \pi_0(\beta_1) + 1. \end{cases} \tag{3.1.12}$$

By a similar argument it can be proved that

$$\mathcal{S}(T)(z) = \begin{cases} T^2(z), & z \in \gamma(I^{(1)}_{\pi_0^{-1}(\pi_0(\beta_1)+1)}), \\ T(z), & z \in \gamma(I^{(1)}_{\pi_0^{-1}(j)}),\ \pi_0(j) \neq \pi_0(\beta_1) + 1. \end{cases} \tag{3.1.13}$$

Since $x \in I_{\pi_0^{-1}(j)}$ if and only if $\gamma(x) \in \gamma(I^{(1)}_{\pi_0^{-1}(j)})$, for $j = 1, ..., d$, combining (1.4.8), (3.1.12) and (3.1.13) we get (3.1.9). □

3.2 Connecting equations and embeddings of $2, 3$-interval exchange transformations

In this section we introduce, a graph for a given permutation. We use its combinatorial and topological properties to obtain a necessary condition for the parameters of a PWI to be a continuous embedding of an IET into a PWI described by the same permutation.

We then prove that only trivial embeddings of 2-IETs are possible and that a 3-PWI has at most one non-trivially continuous embedded 3-IET with the same underlying permutation.

Given $(\lambda, \pi) \in \mathbb{R}^{\mathcal{A}}_+ \times \mathfrak{S}(\mathcal{A})$, let $f_{\lambda,\pi}(x) : I \to I$ be minimal IET with $I = I_{\pi_0^{-1}(1)} \cup ... \cup I_{\pi_0^{-1}(d)}$. As before we write at times $f = f_{\lambda,\pi}$. Recall (1.3.2). Define functions $f_j(x) = x + \upsilon_{\pi_0^{-1}(j)}$, for $x \in \bar{I}$, and $j = 1, ..., d$, then $f(x) = f_j(x)$, for $x \in I_{\pi_0^{-1}(j)}$.

We extend π_ε to O by setting $\pi_\varepsilon(O) = 0$, for $\varepsilon = 0, 1$ and define f_0 as the identity map in I. Recall from the Introduction that we denote by $\tilde{\pi}$ the monodromy invariant $\pi_1 \circ \pi_0^{-1}$ and hence with this extension we have $\tilde{\pi}(0) = 0$.

For $j \in \mathbb{Z}$ we write $[j] = j \mod d + 1$. For x_j with $0 \leq j \leq d$ as in (3.1.8) we have the following

$$f_{\tilde{\pi}^{-1}([j])}(x_{[\tilde{\pi}^{-1}(j)-1]}) = f_{\tilde{\pi}^{-1}([j-1])}(x_{\tilde{\pi}^{-1}([j-1])}), \tag{3.2.1}$$

where $j = 0, ..., d$. Note that as the domain of each map f_j, $j = 0, ..., d$ is the closed interval $\bar{I}$, the maps are defined at the endpoints x_j.

We now define a directed graph $\mathcal{G}_\pi$ in $d+1$ vertices $v_0, ..., v_d$ such that there is an edge

$$v_{\tilde{\pi}^{-1}([i-1])} \to v_{\tilde{\pi}^{-1}([j-1])} \tag{3.2.2}$$

if $\tilde{\pi}^{-1}([j-1]) = [\tilde{\pi}^{-1}(i) - 1]$, with $i, j \in \{0, ..., d\}$.

The graph $\mathcal{G}_\pi$, as we will shortly see, identifies the end-points of adjacent intervals after rearrangement by a d-IET with base permutation π. The next proposition characterizes the topology of $\mathcal{G}_\pi$.

Proposition 3.2.1 *Given a permutation $\pi \in \mathfrak{S}(\mathcal{A})$, the directed graph $\mathcal{G}_\pi$ is a disjoint union of directed cyclic subgraphs.*

Proof. Since $\mathcal{G}_\pi$ is a finite graph, it has a finite number of connected components, hence it suffices to prove that every connected component of $\mathcal{G}_\pi$ is a cyclic graph.

Consider a vertex v_q, with $q \in \{0, ..., d\}$. There is a unique $i_0 = [\tilde{\pi}(q) + 1] \in \{0, ..., d\}$, such that $\tilde{\pi}^{-1}([i_0 - 1]) = q$. Define the map $\psi : \{0, ..., d\} \to \{0, ..., d\}$ as $\psi(n) = \tilde{\pi}([\tilde{\pi}^{-1}([n-1]) + 1])$. Note that ψ is a bijection, hence $i_1 = \psi(i_0)$ is the unique $i_1 \in \{0, ..., d\}$ satisfying

$$\tilde{\pi}^{-1}([i_0 - 1]) = [\tilde{\pi}^{-1}(i_1) - 1].$$

Thus, there is an edge $v_q \to v_{\tilde{\pi}^{-1}([i_1-1])}$.

We now form a sequence $(i_k)_{k\in\mathbb{N}}$ where $i_0 = [\tilde{\pi}(q_1) + 1]$ and $i_k = \psi(i_{k-1})$, for $k \geq 1$. Since ψ is a bijection between finite sets $(i_k)_{k\in\mathbb{N}}$ must be a periodic sequence. If ψ has period $d+1$, then $\mathcal{G}_\pi$ is a cyclic graph. Otherwise, ψ has period $p \leq d$. This implies that the vertices v_n, for $n \in (i_k)_{0,...,p-1}$ and the edges connecting them form a connected and directed cyclic subgraph. Since the point $q \in \{0, ..., d\}$ was chosen without loss of generality, this shows that connected subgraphs of $\mathcal{G}_\pi$ are cycles. This completes the proof. □

Proposition 3.2.2 *Let (I, f) be a d-IET with respect to an irreducible permutation π. The directed graph $\mathcal{G}_\pi$ has an edge $v_p \to v_q$ if and only if*

$$x_p = f_p^{-1} \circ f_{\tilde{\pi}^{-1}([\tilde{\pi}(p)+1])}(x_q). \tag{3.2.3}$$

Proof. Let $p = \tilde{\pi}^{-1}([i-1])$ and $q = \tilde{\pi}^{-1}([j-1])$, for some $i, j \in \{0, ..., d\}$. From (3.2.1) we have $f_{\tilde{\pi}^{-1}([i])}(x_{[\tilde{\pi}^{-1}(i)-1]}) = f_{\tilde{\pi}^{-1}([i-1])}(x_{\tilde{\pi}^{-1}([i-1])})$, which is equivalent to

$$f_{\tilde{\pi}^{-1}([i])}(x_{\tilde{\pi}^{-1}([j-1])}) = f_{\tilde{\pi}^{-1}([i-1])}(x_{\tilde{\pi}^{-1}([i-1])}), \tag{3.2.4}$$

if and only if $\tilde{\pi}^{-1}([j-1]) = [\tilde{\pi}^{-1}(i)-1]$, that is, if $v_p \to v_q$. From (3.2.4) we get (3.2.3), which completes the proof. □

Now assume (I,f) has a continuous embedding by γ into a d-PWI (X,T) with $Y=\gamma(I)$ and $Y_{\pi_0^{-1}(j)} = X_{\pi_0^{-1}(j)} \cap Y$, such that $T(z) = T_{\pi_0^{-1}(j)}(z)$, for $z \in Y_{\pi_0^{-1}(j)}, j = 1,..,d$. with

$$T_{\pi_0^{-1}(j)}(z) = e^{i\theta_{\pi_0^{-1}(j)}} z + \eta_{\pi_0^{-1}(j)}, \quad z \in \mathbb{C}, \quad j = 1,..,d. \tag{3.2.5}$$

Define T_O as the identity map in $\mathbb{C}$. Let $z_j = \gamma(x_j)$, for $j = 0,...,d$. Equations (3.2.1) are preserved under topological conjugacy and can be written for T as

$$T_{\pi_1^{-1}([j])}(z_{[\tilde{\pi}^{-1}(j)-1]}) = T_{\pi_1^{-1}([j-1])}(z_{\tilde{\pi}^{-1}([j-1])}), \quad j = 0,...,d. \tag{3.2.6}$$

We call (3.2.6) the *connecting equations*. The next corollary follows from Proposition 3.2.2 and from the topological conjugacy of (Y,T) and (I,f).

Corollary 3.2.3 *Assume a d-IET (I,f) has a continuous embedding by γ into a d-PWI (X,T). The directed graph $\mathcal{G}_\pi$ has an edge $v_p \to v_q$ if and only if*

$$z_p = T^{-1}_{\pi_0^{-1}(p)} \circ T_{\pi_1^{-1}([\tilde{\pi}(p)+1])}(z_q).$$

Let $\mathfrak{c}_0 \in \{0,...,d\}$. We define a *connecting sequence* $(\mathfrak{c}_k)_{k\in\mathbb{N}}$ for $\mathfrak{c}_0$, with $\mathfrak{c}_k = q_{k-1}$, where q_{k-1} is such that $v_{\mathfrak{c}_{k-1}} \to v_{q_{k-1}}$. By Proposition 3.2.1, the connected component of $\mathcal{G}_\pi$ containing $v_{\mathfrak{c}_0}$ must be a directed cyclic graph. Thus, $(\mathfrak{c}_k)_{k\in\mathbb{N}}$ is a well defined periodic sequence with period $s(\mathfrak{c}_0) \le d+1$.

With $\sigma : \{0,...,d\} \to \{0,...,d\}$ such that,

$$\sigma(p) = [\tilde{\pi}^{-1}(\tilde{\pi}(p)+1)) - 1],$$

it is simple to see by (3.2.2) that $\mathfrak{c}_k = \sigma(\mathfrak{c}_{k-1})$ and hence the number of distinct orbits of σ is equal to the number of connected components of $\mathcal{G}_\pi$. The map σ was first introduced by Veech in [54].

Recall that a *translation surface* (as defined in [15]), with genus g, is a surface with a finite number κ of conical singularities endowed with an atlas such that coordinate changes are given by translations in $\mathbb{R}^2$.

Recall the definition of Rauzy class in Section 1.3.3. Given an IET it is possible to associate, via a suspension construction, a translation surface, with g and κ depending only on the Rauzy class of the permutation of the underlying IET (see for instance [54]). It is known (see [56]) that the number of distinct orbits of σ is constant on each Rauzy class and determines g and κ of the associated translation surface. In particular, for the *hyperelliptic Rauzy class*, that is the Rauzy class containing the permutation with monodromy invariant $\tilde{\pi}(j) = d+1-j$ for all $j = 1, ..., d$, σ has a single orbit if d is even and has two distinct orbits if d is odd.

We define the *connecting map* for $\mathfrak{c}_0$ as

$$\mathcal{C}_{\mathfrak{c}_0}(z) = T^{-1}_{\pi_0^{-1}(\mathfrak{c}_0)} \circ T_{\pi_1^{-1}([\tilde{\pi}(\mathfrak{c}_0)+1])} \circ ... \circ T^{-1}_{\pi_0^{-1}(\mathfrak{c}_{s(\mathfrak{c}_0)-1})} \circ T_{\pi_1^{-1}([\tilde{\pi}(\mathfrak{c}_{s(\mathfrak{c}_0)-1})+1])}(z), \quad z \in \mathbb{C}.$$

It follows from Corollary 3.2.3 that $z_{\mathfrak{c}_0}$ is a fixed point of $\mathcal{C}_{\mathfrak{c}_0}$, thus, $\mathcal{C}_{\mathfrak{c}_0}(z_{\mathfrak{c}_0}) = z_{\mathfrak{c}_0}$. We have

$$\left(e^{i\Theta_\pi(\mathfrak{c}_0)} - 1\right) z_{\mathfrak{c}_0} + \mathcal{C}_{\mathfrak{c}_0}(0) = 0, \tag{3.2.7}$$

and

$$\Theta_\pi(\mathfrak{c}_0) = \sum_{k=0}^{s(\mathfrak{c}_0)-1} \theta_{\pi_1^{-1}([\tilde{\pi}(\mathfrak{c}_k)+1])} - \theta_{\pi_0^{-1}(\mathfrak{c}_k)}.$$

Now (3.2.7) either imposes a restriction on γ, if $\Theta_\pi(\mathfrak{c}_0) \neq 0$, by forcing

$$\gamma(x_{\mathfrak{c}_0}) = \left(1 - e^{i\Theta_\pi(\mathfrak{c}_0)}\right)^{-1} \mathcal{C}_{\mathfrak{c}_0}(0), \tag{3.2.8}$$

or if $\Theta_\pi(\mathfrak{c}_0) = 0$ it imposes a restriction on the parameters $\eta_{\pi_0^{-1}(j)}$, $\theta_{\pi_0^{-1}(j)}$, $j = 1, ..., d$, by

$$\mathcal{C}_{\mathfrak{c}_0}(0) = 0. \tag{3.2.9}$$

Note that $\mathcal{C}_{\mathfrak{c}_0}(0)$ can be seen as a sum where each term is η_j times a coefficient depending only on $\theta_{\pi_0^{-1}(j)}, ..., \theta_{\pi_0^{-1}(j)}$.

Denote the coefficient of $\eta_{\pi_0^{-1}(j)}$ in $\mathcal{C}_{\mathfrak{c}_0}(0)$ by $r_{\pi_0^{-1}(j)}(\theta_{\pi_0^{-1}(1)}, ..., \theta_{\pi_0^{-1}(d)})$ for $j = 1, ..., d$. Note that by linearity in $\eta_{\pi_0^{-1}(j)}$, (3.2.9) can be written as

$$\sum_{j=1}^{d} \eta_{\pi_0^{-1}(j)} r_{\pi_0^{-1}(j)}(\theta_{\pi_0^{-1}(1)}, ..., \theta_{\pi_0^{-1}(d)}) = 0. \tag{3.2.10}$$

We call (3.2.10) the *parametric connecting equation* for $\mathfrak{c}_0$.

In the following theorem we show that if $\mathcal{G}_\pi$ is connected then the parameters of the PWI satisfy the parametric connecting equation.

Theorem 3.2.4 *Assume a d-IET (I, f) has a continuous embedding by γ into a d-PWI (X, T). If $\mathcal{G}_\pi$ is a connected graph, then the parameters $\eta_{\pi_0^{-1}(j)}$, $\theta_{\pi_0^{-1}(j)}$, $j = 1, ..., d$ satisfy the parametric connecting equation* (3.2.10).

Proof. Since $\mathcal{G}_\pi$ is a connected graph, by Proposition 3.2.1 it must be a directed cyclic graph. The connecting sequence for $\mathfrak{c}_0 = 0$ is well defined and has period

$d+1$. Since the map $n \mapsto \tilde{\pi}^{-1}([\tilde{\pi}(n)+1])$ is a bijection between finite sets we must have

$$\Theta_\pi(0) = \sum_{k=0}^{d} \theta_{\pi_1^{-1}([\tilde{\pi}(\mathfrak{c}_k)+1])} - \sum_{k=0}^{d} \theta_{\pi_0^{-1}(\mathfrak{c}_k)} = 0.$$

Thus, there are functions $r_{\pi_0^{-1}(j)}(\theta_{\pi_0^{-1}(1)}, ..., \theta_{\pi_0^{-1}(d)})$ for $j = 1, ..., d$, not identically 0, satisfying (3.2.10). □

The following example shows two permutations, one for which the graph $\mathcal{G}_\pi$ is disconnected and a permutation that yields a connected $\mathcal{G}_\pi$ and a parametric connecting equation that can in principle allow the existence of non-trivial embeddings.

Example 3.2.5 *Consider a permutation π with monodromy invariant $\tilde{\pi}$ such that $\tilde{\pi}(1,2,3) = (2,3,1)$. It is simple to see, either by checking directly or by noting that π is in the hyperelliptic Rauzy class for $d = 3$, that $\mathcal{G}_\pi$ is not a connected graph. The connecting sequence for 1 is constant and equal to 1, thus, from (3.2.7) we get*

$$(e^{i(\theta_{\pi_0^{-1}(2)} - \theta_{\pi_0^{-1}(1)})} - 1)\gamma(x_1) + (\eta_{\pi_0^{-1}(2)} - \eta_{\pi_0^{-1}(1)})e^{-i\theta_{\pi_0^{-1}(1)}} = 0. \qquad (3.2.11)$$

Consider the permutation with monodromy invariant $\tilde{\pi'}$ such that $\tilde{\pi'}(1,2,3,4) = (4,2,1,3)$. It is clear that in this case $\mathcal{G}_{\pi'}$ is a connected graph. Indeed π' is in the hyperelliptic Rauzy class for $d = 4$. The connecting sequence for 0 is $p = (0,2,3,1,4,...)$ and we have the connecting map

$$\begin{aligned}\mathcal{C}_0(z) =& T^{-1}_{\pi_0^{-1}(0)} \circ T_{\pi_0^{-1}(3)} \circ T^{-1}_{\pi_0^{-1}(2)} \circ T_{\pi_0^{-1}(4)} \circ T^{-1}_{\pi_0^{-1}(3)} \circ \\ & \circ T_{\pi_0^{-1}(2)} \circ T^{-1}_{\pi_0^{-1}(1)} \circ T_{\pi_0^{-1}(0)} \circ T^{-1}_{\pi_0^{-1}(4)} \circ T_{\pi_0^{-1}(1)}(z).\end{aligned}$$

From this we get the following parametric connecting equation

$$\begin{aligned}&\eta_{\pi_0^{-1}(1)}(e^{-i\theta_{\pi_0^{-1}(1)}} - e^{i(\theta_{\pi_0^{-1}(4)} - \theta_{\pi_0^{-1}(1)})}) + \eta_{\pi_0^{-1}(2)}(e^{i(\theta_{\pi_0^{-1}(4)} - \theta_{\pi_0^{-1}(2)})} - e^{i(\theta_{\pi_0^{-1}(3)} - \theta_{\pi_0^{-1}(2)})}) + \\ &\eta_{\pi_0^{-1}(3)}(1 - e^{i(\theta_{\pi_0^{-1}(4)} - \theta_{\pi_0^{-1}(2)})}) + \eta_{\pi_0^{-1}(4)}(e^{i(\theta_{\pi_0^{-1}(3)} - \theta_{\pi_0^{-1}(2)})} - e^{-i\theta_{\pi_0^{-1}(1)}}) = 0.\end{aligned} \qquad (3.2.12)$$

In Section 3.4 we will discuss an example of a PWI satisfying (3.2.12). In particular we present some numerical results which suggest that there exist non-trivial embeddings of d-IETs into d-PWIs, for $d = 3$ and $d = 4$.

In the remainder of this section we prove Theorem C, which states that there are no non-trivial continuous embeddings of minimal 2-interval exchange transformations into orientation preserving planar PWIs and Theorem D which asserts that a 3-PWI has at most one non-trivially continuously embedded minimal 3-IET with the same underlying permutation.

3.2.1 Proof of Theorem C

Let $(I, f_{\lambda,\pi})$ be a minimal 2-IET different from the identity with $\lambda = \{\lambda_{\pi_0^{-1}(1)}, \lambda_{\pi_0^{-1}(2)}\} \in \mathbb{R}_+^2$. Assume there is a continuous embedding of (I, f) by γ into a 2-PWI (X, T) with partition $\{X_{\pi_0^{-1}(1)}, X_{\pi_0^{-1}(2)}\}$.

Set $Y = \gamma(I)$ and $Y_{\pi_0^{-1}(j)} = Y \cap X_{\pi_0^{-1}(j)}$ for $j = 1, 2$. There are $\theta_{\pi_0^{-1}(j)} \in [0, 2\pi)$ and $\eta_{\pi_0^{-1}(j)} \in \mathbb{C}$, such that

$$T_{\pi_0^{-1}(j)}(z) = e^{i\theta_{\pi_0^{-1}(j)}} z + \eta_{\pi_0^{-1}(j)}, \ z \in \mathbb{C}, \quad j = 1, 2,$$

and the restriction of T to Y is given by

$$T(z) = T_{\pi_0^{-1}(j)}(z), \ z \in Y_{\pi_0^{-1}(j)}, \quad j = 1, 2.$$

Since f is not the identity, π is a permutation with monodromy invariant $\tilde{\pi}(1, 2) = (2, 1)$ and $\mathcal{G}_\pi$ is a connected graph, the connecting sequence for $\mathfrak{c}_0 = 0$ is $p = (0, 1, 2, ...)$. This gives the connecting map

$$\mathcal{C}_0(z) = T^{-1}_{\pi_0^{-1}(0)} \circ T_{\pi_0^{-1}(2)} \circ T^{-1}_{\pi_0^{-1}(1)} \circ T_{\pi_0^{-1}(0)} \circ T^{-1}_{\pi_0^{-1}(2)} \circ T_{\pi_0^{-1}(1)}(z).$$

By Theorem 3.2.4, the parameters $\eta_{\pi_0^{-1}(1)}$, $\eta_{\pi_0^{-1}(2)}$, $\theta_{\pi_0^{-1}(1)}$, and $\theta_{\pi_0^{-1}(2)}$ must satisfy the parametric connecting equation, which can be written as

$$\eta_{\pi_0^{-1}(1)}(e^{-\theta_{\pi_0^{-1}(1)}} - e^{\theta_{\pi_0^{-1}(2)} - \theta_{\pi_0^{-1}(1)}}) + \eta_{\pi_0^{-1}(2)}(1 - e^{-\theta_{\pi_0^{-1}(1)}}) = 0. \tag{3.2.13}$$

Multiplying by $e^{i\theta_{\pi_0^{-1}(1)}}$, (3.2.13) becomes

$$\eta_{\pi_0^{-1}(2)}(1 - e^{i\theta_{\pi_0^{-1}(1)}}) = \eta_{\pi_0^{-1}(1)}(1 - e^{i\theta_{\pi_0^{-1}(2)}}). \tag{3.2.14}$$

Since $T_{\pi_0^{-1}(j)}$ is not the identity map (3.2.14) is true if either both sides equal 0 or not.

In the case that both sides are equal to zero, we have the following cases:

i) If $\theta_{\pi_0^{-1}(1)} = \theta_{\pi_0^{-1}(2)} = 0 \mod 2\pi$, then $T_{\pi_0^{-1}(j)}(z) = z + \eta_{\pi_0^{-1}(j)}, \ z \in Y_{\pi_0^{-1}(j)}$. Since we are assuming that f is minimal and Y is compact it follows that T has dense orbits. This implies that there is $s \in \mathbb{R}$ such that $\eta_{\pi_0^{-1}(1)} = s\eta_{\pi_0^{-1}(2)}$. For such a transformation, invariant sets must be unions of lines. This implies that γ is a trivial linear embedding.

ii) If $\eta_{\pi_0^{-1}(1)} = \eta_{\pi_0^{-1}(2)} = 0$, then $T_{\pi_0^{-1}(j)}(z) = e^{i\theta_{\pi_0^{-1}(j)}} z, \ z \in Y_{\pi_0^{-1}(j)}$. Since we are assuming that f is minimal, the orbits of T must be dense and in such a transformation, invariant sets must be unions of circle arcs. This implies that γ is a trivial circle arc embedding.

iii) Finally, if $\eta_{\pi_0^{-1}(j)} = 0$ and $\theta_{\pi_0^{-1}(j)} = 0 \mod 2\pi$, for $j = 1$ or 2 then $T_{\pi_0^{-1}(j)}$ is equal to the identity and hence T can not be conjugated to a minimal IET.

In the case that both sides of equation (3.2.14) are different than 0, there must exist $\eta \in \mathbb{C}$ such that $\eta_{\pi_0^{-1}(j)} = \eta(1 - e^{i\theta_{\pi_0^{-1}(j)}})$, $j = 1, 2$. This implies

$$T_j(z) = (z - \eta)e^{i\theta_{\pi_0^{-1}(j)}} + \eta$$

which is conjugate by $L(z) = z + \eta$, to the map

$$\tilde{T}(z) = e^{i\theta_{\pi_0^{-1}(j)}} z, \ z \in Y_{\pi_0^{-1}(j)} - \eta, \ j = 1, 2.$$

and thus γ is an arc embedding. This completes the proof. □

3.2.2 Proof of Theorem D

Given $(\lambda, \pi) \in \mathbb{R}^3_+ \times \mathfrak{S}(\mathcal{A})$, assume there is a minimal 3-IET $(I, f_{\lambda,\pi})$ which is continuously embedded by γ into a 3-PWI (X, T), with partition $\{X_{\pi_0^{-1}(1)}, X_{\pi_0^{-1}(2)}, X_{\pi_0^{-1}(3)}\}$ and

$$T(z) = e^{i\theta_{\pi_0^{-1}(j)}} z + \eta_{\pi_0^{-1}(j)}, \quad z \in X_{\pi_0^{-1}(j)}.$$

Let $Y = \gamma(I)$. We show that $(I, f_{\lambda,\pi})$ and γ are either unique or that the embedding is trivial.

Assume first that π is a permutation with monodromy invariant $\tilde{\pi}(1, 2, 3) = (2, 3, 1)$. Recall that this is the permutation π in Example 3.2.5. By (3.2.11) we have $|\Theta_\pi(j)| = |\Theta_\pi| = |\theta_{\pi_0^{-1}(2)} - \theta_{\pi_0^{-1}(1)}|$ for $j = 0, ..., 3$.

If $\Theta_\pi = 0$ then $\theta_{\pi_0^{-1}(1)} = \theta_{\pi_0^{-1}(2)}$, and by (3.2.11) we get $\eta_{\pi_0^{-1}(1)} = \eta_{\pi_0^{-1}(2)}$.

Consider the 2-IET $(I, f_{\lambda',\pi'})$, where $\lambda' = (\lambda_{\pi_0^{-1}(1)} + \lambda_{\pi_0^{-1}(2)}, \lambda_{\pi_0^{-1}(3)})$ and π' is the permutation (12). Consider the 2-PWI (X, T'), with base partition $\{X'_{\pi_0^{-1}(1)}, X'_{\pi_0^{-1}(2)}\}$, where $X'_{\pi_0^{-1}(1)} = X_{\pi_0^{-1}(1)} \cup X_{\pi_0^{-1}(2)}$ and $X'_{\pi_0^{-1}(2)} = X_{\pi_0^{-1}(3)}$ and

$$T'(z) = e^{i\theta'_{\pi_0^{-1}(j)}} z + \eta'_{\pi_0^{-1}(j)}, \quad z \in X'_{\pi_0^{-1}(j)},$$

with $\theta'_{\pi_0^{-1}(1)} = \theta_{\pi_0^{-1}(1)}$, $\theta'_{\pi_0^{-1}(2)} = \theta_{\pi_0^{-1}(3)}$, $\eta'_{\pi_0^{-1}(1)} = \eta_{\pi_0^{-1}(1)}$ and $\eta'_{\pi_0^{-1}(2)} = \eta_{\pi_0^{-1}(3)}$. It is simple to see now that $f_{\lambda',\pi'} = f_{\lambda,\pi}$ and $T' = T$, thus by Theorem C the embedding of $(I, f_{\lambda,\pi})$ must be trivial.

If $\Theta_\pi \neq 0$, (3.2.8) gives

$$\gamma(x_j) = \left(1 - e^{i\Theta_\pi(j)}\right)^{-1} \mathcal{C}_j(0), \quad j = 0, ..., 3. \tag{3.2.15}$$

Since $\mathcal{C}_j(0)$ does not depend of λ, by (3.2.15) we have that for any $\lambda' \in \mathbb{R}^3_+$, such that $(I, f_{\lambda',\pi})$ is minimal, any continuous embedding γ' into (X, T) must satisfy

$\gamma'(x'_j) = \gamma(x_j)$. Since the restriction of T to Y must be invertible and every $z \in Y$ must have a dense orbit in Y this shows that $\lambda' = \lambda$ and $\gamma' = \gamma$.

We omit the proof for $\tilde{\pi}(1,2,3) = (3,1,2)$ as it can be done in a similar way to the previous case.

Finally, assume that π has a monodromy invariant given by $\tilde{\pi}(1,2,3) = (3,2,1)$. Then $\mathcal{G}_\pi$ is not a connected graph. The connecting sequence for 1 is equal to $(1,3,...)$, and from (3.2.7) we get

$$\begin{aligned}&(\exp\left[-i(\theta_{\pi_0^{-1}(3)} + \theta_{\pi_0^{-1}(1)} - \theta_{\pi_0^{-1}(2)})\right] - 1)\gamma(x_1) + \\ &e^{-i\theta_{\pi_0^{-1}(1)}}\left[e^{-i\theta_{\pi_0^{-1}(3)}}(\eta_{\pi_0^{-1}(2)} - \eta_{\pi_0^{-1}(3)}) - \eta_{\pi_0^{-1}(1)}\right] = 0.\end{aligned} \tag{3.2.16}$$

We have $|\Theta_\pi(j)| = |\Theta_\pi| = |\theta_{\pi_0^{-1}(3)} + \theta_{\pi_0^{-1}(1)} - \theta_{\pi_0^{-1}(2)}|$ for $j = 0, ..., 3$.

If $\Theta_\pi = 0$ then by (3.2.16) we get

$$\theta_{\pi_0^{-1}(2)} = \theta_{\pi_0^{-1}(1)} + \theta_{\pi_0^{-1}(3)}, \qquad \eta_{\pi_0^{-1}(2)} = \eta_{\pi_0^{-1}(1)}e^{i\theta_{\pi_0^{-1}(3)}} + \eta_{\pi_0^{-1}(3)}. \tag{3.2.17}$$

Note that $I_{\pi_0^{-1}(3)} = f_{\lambda,\pi}(I_{\pi_1^{-1}(3)})$ if and only if $\lambda_{\pi_0^{-1}(1)} = \lambda_{\pi_0^{-1}(3)}$. In this case we have that the restriction of $f_{\lambda,\pi}$ to $I_{\pi_0^{-1}(2)}$ is equal to the identity map. Since $f_{\lambda,\pi}$ is minimal we must have $I_{\pi_0^{-1}(3)} \neq f_{\lambda,\pi}(I_{\pi_1^{-1}(3)})$, thus by Theorem 3.1.3 there is a continuous embedding of $(I^{(1)}, f_{\lambda^{(1)},\pi^{(1)}})$ by γ into $(X', \mathcal{S}(T))$.

We now prove that this embedding is trivial.

Assume that (λ, π) has type 1. Let $I_{\pi_0^{-1}(j)}$ be as in the proof of Theorem 3.1.3. By (3.1.13) we have

$$\mathcal{S}(T)(z) = \begin{cases} e^{i\theta_{\pi_0^{-1}(1)}}z + \eta_{\pi_0^{-1}(1)}, & z \in \gamma(I^{(1)}_{(\pi_0^{(1)})^{-1}(1)}), \\ e^{i(\theta_{\pi_0^{-1}(1)} + \theta_{\pi_0^{-1}(3)})}z + (\eta_{\pi_0^{-1}(1)}e^{i\theta_{\pi_0^{-1}(3)}} + \eta_{\pi_0^{-1}(3)}), & z \in \gamma(I^{(1)}_{(\pi_0^{(1)})^{-1}(2)}), \\ e^{i\theta_{\pi_0^{-1}(2)}}z + \eta_{\pi_0^{-1}(2)}, & z \in \gamma(I^{(1)}_{(\pi_0^{(1)})^{-1}(3)}). \end{cases} \tag{3.2.18}$$

Consider the 2-IET $(I, f_{\lambda'',\pi''})$, with $\lambda'' = (\lambda_{\pi_0^{-1}(1)} - \lambda_{\pi_0^{-1}(3)}, \lambda_{\pi_0^{-1}(2)} + \lambda_{\pi_0^{-1}(3)})$, $\tilde{\pi}''(1,2) = (2,1)$, and the map $T'' : \gamma(I^{(1)}) \to \gamma(I^{(1)})$ such that

$$T''(z) = e^{i\theta_{(\pi''_0)^{-1}(j)}}z + \eta_{(\pi''_0)^{-1}(j)}, \quad z \in Y''_{(\pi''_0)^{-1}(j)},$$

where $Y''_{(\pi''_0)^{-1}(1)} = \gamma(I^{(1)}_{(\pi_0^{(1)})^{-1}(1)})$ and $Y''_{(\pi''_0)^{-1}(2)} = \gamma(I^{(1)}_{(\pi_0^{(1)})^{-1}(2)} \cup I^{(1)}_{(\pi_0^{(1)})^{-1}(3)})$. It is simple to see now that $f_{\lambda'',\pi''} = f_{\lambda^{(1)},\pi^{(1)}}$ and by (3.2.17) and (3.2.18) we have $T''(z) = \mathcal{S}(T(z))$, for all $z \in \gamma(I^{(1)})$. Therefore by Theorem C the embedding of $(I^{(1)}, f_{\lambda^{(1)},\pi^{(1)}})$ by γ into $(X', \mathcal{S}(T))$ must be trivial. By (1.4.8) we have that for $x \in I_{\pi_0^{-1}(3)}$ we have $\gamma(x) = e^{i\theta_{\pi_0^{-1}(1)}}\gamma(x - \lambda_{\pi_0^{-1}(2)} - \lambda_{\pi_0^{-1}(3)}) + \eta_{\pi_0^{-1}(1)}$ thus the embedding of $(I, f_{\lambda,\pi})$ by γ into (X, T) must be trivial as well.

We omit the proof for the case when (λ, π) has type 0 as it can be done in a similar case to the previous case.

Finally, if $\Theta_\pi \neq 0$, by (3.2.8), $\gamma(x_j)$ is determined by (3.2.15). Since $\mathcal{C}_j(0)$ does not depend of λ, we have that for any $\lambda' \in \mathbb{R}^3_+$, such that $(I, f_{\lambda',\pi})$ is minimal, any continuous embedding γ' into (X, T) must satisfy $\gamma'(x'_j) = \gamma(x_j)$. Since the restriction of T to Y must be invertible and every $z \in Y$ must have a dense orbit in Y this shows that $\lambda' = \lambda$ and $\gamma' = \gamma$. □

3.3 Ergodic condition for the existence of piecewise continuous embeddings

In this section we give a necessary condition for the existence of piecewise continuous embeddings of uniquely ergodic IETs into planar PWIs.

Recall from Section 1.3.1 the definitions of IET let I_α and υ_α. Given a d-IET (I, f), suppose we have a piecewise continuous embedding γ of this map into a d-PWI (X, T) and suppose that $T(z) = T_\alpha(z)$, for $z \in X_\alpha$ with $T_\alpha(z) = e^{i\theta_\alpha} z + \eta_\alpha$.

Recall the definition of itinerary in (3.1.1). Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. For $x \in I$ and $y \in S^1$ we define the *tangent exchange map* $\Psi : I \times S^1 \to I \times S^1$ as the skew-product given by

$$\Psi(x, y) = (f(x), y + \theta_{\alpha_0(x)}). \tag{3.3.1}$$

The dynamics of this map contains information on the angle of tangents of an embedding when iterated by the underlying PWI. It will be the main technical tool to prove Theorem 3.3.1.

For $n \in \mathbb{N}$ we have

$$\Psi^n(x, y) = (f^n(x), y + C^{(n)}(x)),$$

where $C^{(\cdot)} : \mathbb{Z} \times I \to S^1$ is the *rotational cocycle* for this embedding, given by

$$C^{(0)}(x) = 0, \quad C^{(n)}(x) = \theta_{\alpha_0(x)} + ... + \theta_{\alpha_0(f^{n-1}(x))} \mod 2\pi,$$

for $x \in I, n \geq 0$, and

$$C^{(n)}(x) = -C^{(-n)}(x) \mod 2\pi,$$

for $n < 0$, where $\alpha_0(x)$ is the piecewise constant map such that $\alpha_0(x) = \alpha$ when $x \in I_\alpha$. Informally, the rotational cocycle keeps track of the angle of a line passing through a point $\gamma(x)$ when iterated by T.

For $x \in I_\alpha$ we define the *first return time of x by f to I_α* as

$$n_\alpha(x) = \inf\{k \geq 1 : f^k(x) \in I_\alpha\}.$$

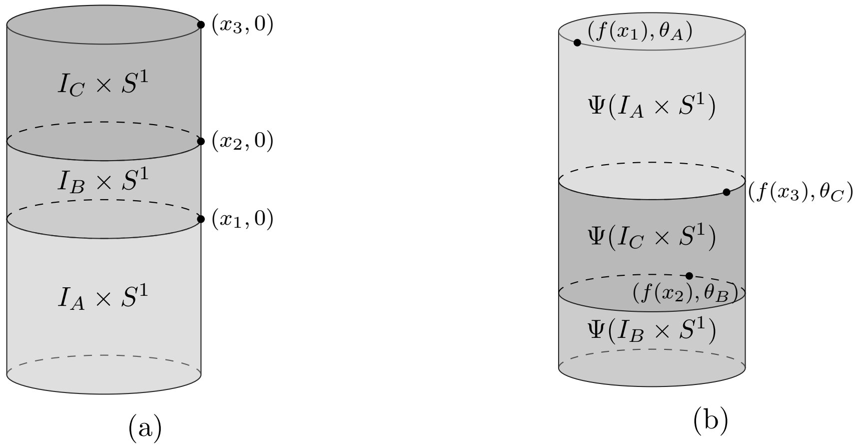


Figure 3.2: A schematic representation of the action of a tangent exchange map Ψ, as in (3.3.1), on a cylinder with $\tilde{\pi}(1,2,3) = (3,1,2)$. (A) The partitioned space $I \times S^1$ in three subcylinders $I_\alpha \times S^1$. The x_j are equal to $\sum_{\pi_0(\alpha) \leq j} \lambda_\alpha$ respectively for $j = 1,2,3$ and the points $(x_j, 0)$ are represented. (B) The action of the map Ψ on $I \times S^1$ and on the points $(x_j, 0)$ which map to $(f_{\lambda,\pi}(x_j), \theta_{\pi_0^{-1}(j)})$ respectively for $j = 1,2,3$.

If f is minimal, then $n_\alpha(x)$ is finite. The *first return map of x by f to I_α*, $f'_\alpha : I_\alpha \to I_\alpha$ is then a well defined d-IET and is given by

$$f'_\alpha(x) = f^{n_\alpha(x)}(x), \quad x \in I_\alpha. \tag{3.3.2}$$

For $\alpha \in \mathcal{A}$, we define the cocycle $N_\alpha^{(\cdot)} : \mathbb{Z} \times I_\alpha \to \mathbb{Z}$ as

$$N_\alpha^{(0)}(x) = 0, \quad N_\alpha^{(k)}(x) = n_\alpha(x) + n_\alpha\left(f'_\alpha(x)\right) + \dots + n_\alpha\left(f'^{k-1}_\alpha(x)\right),$$

for $x \in I_\alpha$ and $k > 0$. For $n < 0$ we set $N_\alpha^{(k)}(x) = -N_\alpha^{(-k)}(x)$.

Define the sequence $(p(n))_{n \geq 1}$ by

$$p(1) = \min\{k \geq 1 : f^k(0) \in I_{\pi_0^{-1}(1)}\},$$

and

$$p(n) = \min\{k > p(n-1) : f^k(0) < f^{p(n-1)}(0)\}, \quad n > 1.$$

Note that if f is minimal then $f^{p(n)}(0) \to 0$, as $n \to +\infty$. Let

$$m_\alpha(n) = \text{card}\{f^k(0) \in I_\alpha : k \leq n\},$$

with $n \in \mathbb{N}, \alpha \in \mathcal{A}$, and

$$k_\alpha = \min\left\{k \geq 0 : f^k(0) \in I_\alpha\right\}.$$

Denote $x'_j = f^{k_{\pi_0^{-1}(j)}}(0)$, $y'_j = C^{(k_{\pi_0^{-1}(j)})}(0)$. For $n \in \mathbb{N}$ and $j = 1, ..., d$, define the sequences

$$c_\alpha(n) = y'_{\pi_0(\alpha)} + C^{\left(N_\alpha^{(n)}(x'_{\pi_0(\alpha)})+1\right)}(x'_{\pi_0(\alpha)}),$$

and

$$e_\alpha(n) = \sum_{k=0}^{m_\alpha(n)-1} \exp[-ic_\alpha(k)]. \tag{3.3.3}$$

The sequence $e_\alpha(p(n))$ can be seen as a the displacement by rotation of a point $\gamma(x'_{\pi_0(\alpha)})$, up to the n-th return to X_α. The limit of $e_\alpha(p(n))$, when $n \to +\infty$, need not exist in general.

Consider, for $\alpha \in \mathcal{A}$, the limiting average of the sequence $e_\alpha(p(n))$,

$$\overline{\xi}_\alpha = \lim_{n\to+\infty} \frac{1}{m_\alpha(p(n))} e_\alpha(p(n)). \tag{3.3.4}$$

Note that this limit need not exist in general. By Weyl's criterion if $c_\alpha(n)$ is uniformly distributed $\mod 2\pi$ then $\overline{\xi}_\alpha = 0$. However this need not hold in general: a numerical study, in Sections 3.4.1 and 3.4.3, presents a non-trivial example where the $\overline{\xi}_\alpha$'s are non-zero. The following theorem shows that for a piecewise continuous embedding of a uniquely ergodic $(I, f_{\lambda,\pi})$, as long as the limit (3.3.4) is finite, the condition (3.3.6) tells us that the average of displacements by rotation and by translations, weighted by the lengths λ_α's, must cancel out so that orbits remain bounded.

Theorem 3.3.1 *Assume that $(I, f_{\lambda,\pi})$ is a uniquely ergodic d-IET that has a piecewise continuous embedding by γ into a d-PWI (X, T) with $X \subseteq \mathbb{C}$, where*

$$T(z) = e^{i\theta_\alpha} z + \eta_\alpha, \tag{3.3.5}$$

for $z \in X_\alpha$ and $\alpha \in \mathcal{A}$. If there are $\overline{\xi}_\alpha \in \mathbb{C}$ such that (3.3.4) holds, then

$$\sum_{\alpha\in\mathcal{A}} \left(\eta_\alpha - \gamma(0)(1 - e^{i\theta_\alpha})\right) \overline{\xi}_\alpha \lambda_\alpha = 0, \tag{3.3.6}$$

where we recall that λ_α denotes the length of the subinterval I_α, for $\alpha \in \mathcal{A}$.

Proof. We begin by proving that there is an orientation preserving PWI $(\tilde{X}, \tilde{T})$, conjugated by a translation to (X, T), such that $(I, f_{\lambda,\pi})$ has a piecewise continuous embedding by $\tilde{\gamma}$ into $(\tilde{X}, \tilde{T})$ with $\tilde{\gamma}(0) = 0$.

Let $\tilde{X} = \{z \in \mathbb{C} : z + \gamma(0) \in X\}$, and $q : X \to \tilde{X}$ be such that $q(z) = z - \gamma(0)$. Let

$$\tilde{T}(z) = q \circ T \circ q^{-1}(z),$$

for $z \in \tilde{X}$. The homeomorphism $\tilde{\gamma} = q \circ \gamma$ conjugates (I, f) to $(\tilde{\gamma}(I), \tilde{T})$, with $\tilde{\gamma}(I) \subseteq \tilde{X}$ invariant for $(\tilde{X}, \tilde{T})$. Moreover, $\tilde{\gamma}(0) = q(\gamma(0)) = 0$. Note that we have

$$\tilde{T}(z) = e^{i\tilde{\theta}_\alpha} z + \tilde{\eta}_\alpha,$$

for $z \in \tilde{X}_\alpha$, where $\tilde{X}_\alpha = \{z \in \mathbb{C} : z+\gamma(0) \in X_\alpha\}$, $\tilde{\theta}_\alpha = \theta_\alpha$ and $\tilde{\eta}_\alpha = \eta_\alpha - \gamma(0)(1-e^{i\theta_\alpha})$.

We now prove that

$$\lim_{n \to +\infty} \sum_{\alpha \in \mathcal{A}} \tilde{\eta}_\alpha e_\alpha(p(n)) = 0. \tag{3.3.7}$$

Since $(I, f_{\lambda,\pi})$ has a piecewise continuous embedding by $\tilde{\gamma}$ into $(\tilde{X}, \tilde{T})$, we have

$$\tilde{\gamma}(x + \upsilon_\alpha) = e^{i\theta_\alpha} \tilde{\gamma}(x) + \tilde{\eta}_\alpha, \tag{3.3.8}$$

for $x \in I_\alpha, \alpha \in \mathcal{A}$. Let $\tilde{Y} = \tilde{\gamma}(I)$, $\tilde{Y}_\alpha = \tilde{Y} \cap \tilde{X}_\alpha$ and $\tilde{\gamma}_\alpha : I_\alpha \to \tilde{Y}_\alpha$ be the restriction of $\tilde{\gamma}$ to I_α. From (3.3.8) we get

$$\tilde{\gamma}_\alpha(x) = e^{-i\theta_\alpha}(\tilde{\gamma}_\beta(x + \upsilon_\alpha) - \tilde{\eta}_\alpha),$$

where $x \in f_{\lambda,\pi}^{-1}(I_\beta)$, and $\alpha, \beta \in \mathcal{A}$.

Recall the itinerary of x as in (3.1.1). It can be proved by induction that for $x \in I$, $n \in \mathbb{N}$ we have

$$\tilde{\gamma}_{\alpha_0}(x) = \exp\left[-i\sum_{k=0}^{n-1} \theta_{\alpha_k}\right] \tilde{\gamma}_{\alpha_n}(f_{\lambda,\pi}^n(x)) - \sum_{k=0}^{n-1} \tilde{\eta}_{\alpha_k} \exp\left[-i\sum_{l=0}^{k} \theta_{\alpha_l}\right], \tag{3.3.9}$$

Since $\tilde{\gamma}(0) = 0$, taking $x = 0$ in (3.3.9) we get

$$\exp\left[-i\sum_{k=0}^{n-1} \theta_{\alpha_k}\right] \tilde{\gamma}_{\alpha_n}(f_{\lambda,\pi}^n(0)) - \sum_{k=0}^{n-1} \tilde{\eta}_{\alpha_k} \exp\left[-i\sum_{l=0}^{k} \theta_{\alpha_l}\right] = 0, \tag{3.3.10}$$

for $n \in \mathbb{N}$.

Note that $\tilde{\gamma}_\alpha : I_\alpha \to \tilde{Y}_\alpha$ is a homeomorphism for $\alpha \in \mathcal{A}$. By continuity of $\tilde{\gamma}_{\pi_0^{-1}(1)}$ and (3.3.10)

$$\left|\tilde{\gamma}_{\pi_0^{-1}(1)}(f_{\lambda,\pi}^{p(n)}(0)) - \tilde{\gamma}_{\pi_0^{-1}(1)}(0)\right| = \left|\sum_{k=0}^{p(n)-1} \eta_{\alpha_k} \exp\left[-i\sum_{l=0}^{k} \theta_{\alpha_l}\right]\right| \xrightarrow{n \to +\infty} 0. \tag{3.3.11}$$

By (3.3.3), (3.3.11) is equivalent to (3.3.7).

We now show that

$$\sum_{\alpha \in \mathcal{A}} \tilde{\eta}_\alpha \overline{\xi}_\alpha \lambda_\alpha = 0. \tag{3.3.12}$$

Since $(I, f_{\lambda,\pi})$ is uniquely ergodic with respect to Lebesgue measure,

$$\lim_{n\to+\infty} \frac{m_\alpha(p(n))}{p(n)} = \frac{\lambda_\alpha}{|I|}, \tag{3.3.13}$$

for $\alpha \in \mathcal{A}$.

Note that (3.3.4) is equivalent to

$$e_\alpha(p(n)) = m_\alpha(p(n))\overline{\xi}_\alpha + o(p(n)), \quad \alpha \in \mathcal{A}. \tag{3.3.14}$$

Combining (3.3.13) and (3.3.14) we have

$$e_\alpha(p(n)) = p(n)\frac{m_\alpha(p(n))}{p(n)} \frac{1}{m_\alpha(p(n))} e_\alpha(p(n)) = (p(n) + o(p(n)))\frac{\lambda_\alpha}{|I|}\overline{\xi}_\alpha,$$

for $\alpha \in \mathcal{A}$, and we get

$$\sum_{\alpha\in\mathcal{A}} \tilde{\eta}_\alpha e_\alpha(p(n)) = \sum_{\alpha\in\mathcal{A}} (p(n) + o(p(n)))\tilde{\eta}_\alpha \lambda_\alpha \overline{\xi}_\alpha. \tag{3.3.15}$$

Since $(I, f_{\lambda,\pi})$ has a piecewise continuous embedding into (X, T), (3.3.7) holds. Thus (3.3.15) implies that

$$\lim_{n\to+\infty} \sum_{\alpha\in\mathcal{A}} (p(n) + o(p(n)))\tilde{\eta}_\alpha \lambda_\alpha \overline{\xi}_\alpha = 0,$$

which can only hold if (3.3.12) is true, as desired. Finally note that (3.3.12) is equivalent to (3.3.6), and the proof is complete. □

Condition (3.3.4) is not simple to verify in general since $c_\alpha(n)$ is determined by two cocycles. However under some assumption on θ_α we can identify $c_\alpha(n)$ with an orbit of a point by interval exchange map and compute the $\overline{\xi}_\alpha$ as spatial averages using the ergodic theorem.

Corollary 3.3.2 *Assume that (I, f) is a uniquely ergodic d-IET with a piecewise continuous embedding by γ into a d-PWI (X, T) as in (3.3.5). Let χ_{I_α} denote the characteristic function of I_α. If*

$$\theta_\alpha = \frac{2\pi}{|I|}\upsilon_\alpha, \tag{3.3.16}$$

for $\alpha \in \mathcal{A}$, then

$$\int_I \left(\sum_{\alpha\in\mathcal{A}} \left(\eta_\alpha - \gamma(0)(1 - e^{i\theta_\alpha})\right) \chi_{I_\alpha}(f^{-1}(x)) \right) e^{-2\pi i x} dx = 0. \tag{3.3.17}$$

Proof. Let $f'_\alpha : I_\alpha \to I_\alpha$ be as in (3.3.2). With $f_{\lambda,\pi} = f$, by (3.3.16) we have

$$c_\alpha(n) = \frac{2\pi}{|I|} f_{\lambda,\pi} \circ f'^{n}_\alpha(x'_{\pi_0(\alpha)}),$$

Since $(I, f_{\lambda,\pi})$ is uniquely ergodic, it follows that (I_α, f'_α) is also uniquely ergodic. Thus, the ergodic theorem implies that

$$\lim_{n\to+\infty} \frac{1}{m_\alpha(p(n))} \sum_{k=0}^{m_\alpha(p(n))-1} \exp\left[-\frac{2\pi i}{|I|} f_{\lambda,\pi} \circ f'^{k}_\alpha(x'_{\pi_0(\alpha)})\right] = \frac{1}{\lambda_\alpha} \int_{f_{\lambda,\pi}(I_\alpha)} \exp\left[-2\pi i x\right] dx, \tag{3.3.18}$$

for $\alpha \in \mathcal{A}$.

Let $\overline{\xi}_\alpha = \frac{1}{\lambda_\alpha} \int_{f_{\lambda,\pi}(I_\alpha)} \exp\left[-2\pi i x\right] dx$, for $\alpha \in \mathcal{A}$. Combined with (3.3.3) and (3.3.18) we get

$$\lim_{n\to+\infty} \frac{1}{m_\alpha(p(n))} e_\alpha(p(n)) = \overline{\xi}_\alpha,$$

for $\alpha \in \mathcal{A}$, and thus by Theorem 3.3.1 we must satisfy (3.3.6) which is equivalent to (3.3.17). This completes the proof. □

3.4 Evidence of non-trivial embeddings of interval exchange transformations into piecewise isometries

In this section we present some numerical evidence of non-trivial continuous embeddings of IETs in PWIs. In order to do this we first define a PWI on 3 atoms that apparently exhibits a single invariant curve that is the image of a non-trivial embedding of a 3-IET. We also show some numerical evidence that a family of Translated Cone Exchange Transformations apparently supports many non-trivial embeddings of 4-IETs.

3.4.1 A piecewise isometry with an embedded three interval exchange

We now present an example of a 3-PWI for which numerical evidence suggests the existence of a non-trivial embedded 3-IET.

Let $\mathcal{A} = \{1, 2, 3\}$, $\omega' = 1.3$, $\vartheta' = 0.75$, $z'_0 = 0$, $z'_1 = 0, 0.215998 + i0.168125$, $z'_2 = 0.491520 + i0.051612$, $z'_3 = 0.586452$ and the convex sets

$$Q'_1 = \{z \in \mathbb{C} : \mathrm{Im}(e^{i\omega'}(z - z'_1)) < 0\},$$
$$Q'_2 = \{z \in \mathbb{C} : \mathrm{Im}(e^{-i\vartheta'}(z - z'_2)) > 0 \text{ and } \mathrm{Im}(e^{i\omega'}(z - z'_1)) \geq 0\},$$
$$Q'_3 = \{z \in \mathbb{C} : \mathrm{Im}(e^{-i\vartheta'}(z - z'_2)) \leq 0 \text{ and } \mathrm{Im}(e^{i\omega'}(z - z'_1)) \geq 0\}.$$

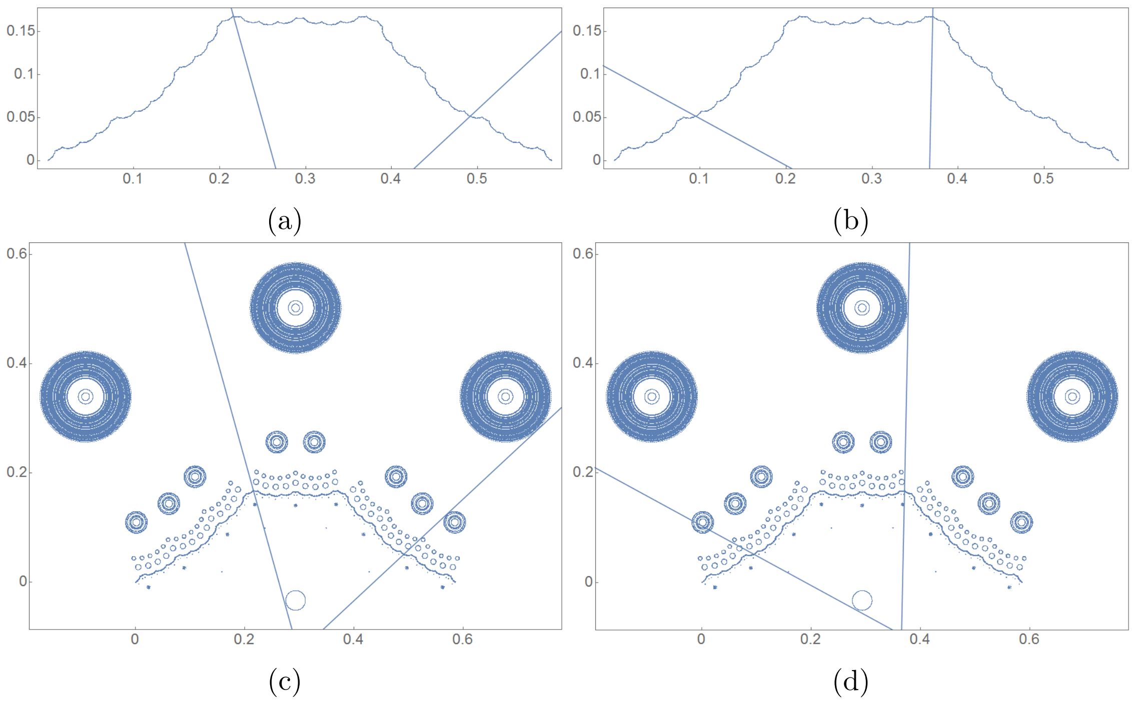


Figure 3.3: An illustration of the action of the piecewise isometry T'. (A) An invariant set Y' and the partition $\{Q'_\alpha\}_{\alpha=1,2,3}$. (B) Image of Y' by T'. (C) Orbits of 40 points, including z_0, (ignoring a transient) under T' and the partition $\{Q'_\alpha\}_{\alpha=1,2,3}$. (D) Image of the orbits and the partition in (C) by T'.

Consider the PWI $T' : \mathbb{C} \to \mathbb{C}$ such that

$$T'(z) = e^{i\theta'_\alpha} z + \eta'_\alpha, \quad z \in Q'_\alpha, \tag{3.4.1}$$

for $\alpha = 1, 2, 3$, where

$$\theta'_\alpha = \begin{cases} 4.460361, & \alpha = 1, \\ 0.800153, & \alpha = 2, \\ 0.995933, & \alpha = 3, \end{cases}$$
$$\eta'_\alpha = \begin{cases} z'_3 - e^{i\theta'_1} z'_1, & \alpha = 1, \\ e^{i\theta'_3}(z'_3 - z'_2) - e^{i\theta'_2} z'_1, & \alpha = 2, \\ e^{i\theta'_3} z'_2, & \alpha = 3, \end{cases} \tag{3.4.2}$$

and set $Y' = \overline{\{T'^n(z'_0)\}_{n\in\mathbb{N}}}$. These parameters are constructed according to certain renormalization properties of the IET. Figure 3.3 shows the action of the map T', in particular in Figure 3.3 (A) we can see Y' and in Figure 3.3 (B) its image by T'. Consider the family $\mathcal{F}_3$ of 3-IETs $f_{\lambda,\pi'} : I \to I$ given by subdividing the interval into four intervals of lengths $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+$ with base permutation π', with $\tilde{\pi}'(1, 2, 3) = (3, 2, 1)$.

We can partition Y' by setting $Y'_\alpha = Y' \cap Q'_\alpha$, for $\alpha = 1,2,3$. The length $l'_\alpha = Leb(Y'_\alpha)$ of each Y'_α can be numerically estimated to be

$$l'_1 = 0.3910666426, l'_2 = 0.4553369973, l'_3 = 0.1535963601.$$

Fix $\lambda = (l'_1, l'_2, l'_3)$ and consider the IET $(I, f_{\lambda,\pi'}) \in \mathcal{F}_3$. Numerical evidence suggests that there is a continuous embedding of $(I, f_{\lambda,\pi'})$ into $(\mathbb{C}, T')$, by a map $\gamma' : I \to Y'$ with $Y' \subseteq \mathbb{C}$, such that $\gamma'(0) = z'_0$. Note that $\mathcal{G}_{\pi'}$ is not a connected graph so we are not in the conditions of Theorem 3.2.4. However by (3.4.2) it is simple to check that (3.2.6) and (3.2.8) are satisfied. Indeed numerical verification shows that $\iota'_k(T'(\gamma'(0))) = \iota_k(f_{\lambda,\pi'}(0))$ for all $k \leq 6 \times 10^4$, supporting that γ' is a symbolic embedding.

We can also verify numerically that the condition in Theorem 3.3.1 holds for this case. We estimate $\overline{\xi}'_\alpha \simeq \frac{e_\alpha(p(8))}{m_\alpha(p(8))}$ where $\overline{\xi}'_1 \simeq -0.453 + 0.651i$, $\overline{\xi}'_2 \simeq 0.326 + 0.669i$ and $\overline{\xi}'_3 \simeq 0.417 + 0.679i$. For these estimates we get

$$\left| \sum_{\alpha \in \mathcal{A}} \eta'_\alpha \overline{\xi}'_\alpha \lambda_\alpha - \gamma'(0) \sum_{\alpha \in \mathcal{A}} (1 - e^{i\theta'_\alpha}) \overline{\xi}'_\alpha \lambda_\alpha \right| \simeq 1.19 \times 10^{-5}.$$

3.4.2 A planar piecewise isometry with four cones

Consider the following family of PWIs that include a linear embedding of a 2-IET and, apparently an infinite number of non-trivial embeddings of 4-IETs.

For any $\vartheta \in (0, \frac{\pi}{2})$ and $\omega_1 \in (0, \pi - 2\vartheta)$ and $\eta \in \mathbb{R}^+$ we consider a partition of $\mathbb{C}$ into four atoms

$$\begin{aligned} P_0 &= \{z \in \mathbb{C} : \arg(z) \in [-\vartheta, \vartheta)\} \cup \{0\}, \\ P_1 &= \{z \in \mathbb{C} : \arg(z) \in [\vartheta, \omega_1 + \vartheta)\}, \\ P_2 &= \{z \in \mathbb{C} : \arg(z) \in [\omega_1 + \vartheta, \pi - \vartheta)\}, \\ P_3 &= \{z \in \mathbb{C} : \arg(z) \in [\pi - \vartheta, 2\pi - \vartheta)\}, \end{aligned}$$

and define a map $T : \mathbb{C} \to \mathbb{C}$ by $T(z) = T_\alpha(z)$, for $z \in P_\alpha$, where

$$F_\alpha(z) = \begin{cases} z - 1, & z \in \alpha = 0, \\ ze^{i\varpi_1} - (1 - \eta), & z \in \alpha = 1, \\ ze^{i\varpi_2} - (1 - \eta), & z \in \alpha = 2, \\ z + \eta, & z \in \alpha = 3, \end{cases} \tag{3.4.3}$$

and $\varpi_1 = \pi - 2\vartheta - \omega_1$, $\varpi_2 = -\omega_1$. An example is illustrated in Figure 3.4. We chose $\mathcal{A} = \{0, 1, 2, 3\}$ to label the atoms P_α for this map tho emphasize that this is a Translated cone exchange transformation for parameters $\tau(1, 2) = (2, 1)$, $\omega = (\omega_1, \pi - \omega_1 - 2\vartheta)$, $\eta = \Phi$ and $\eta' = 1 - \Phi$

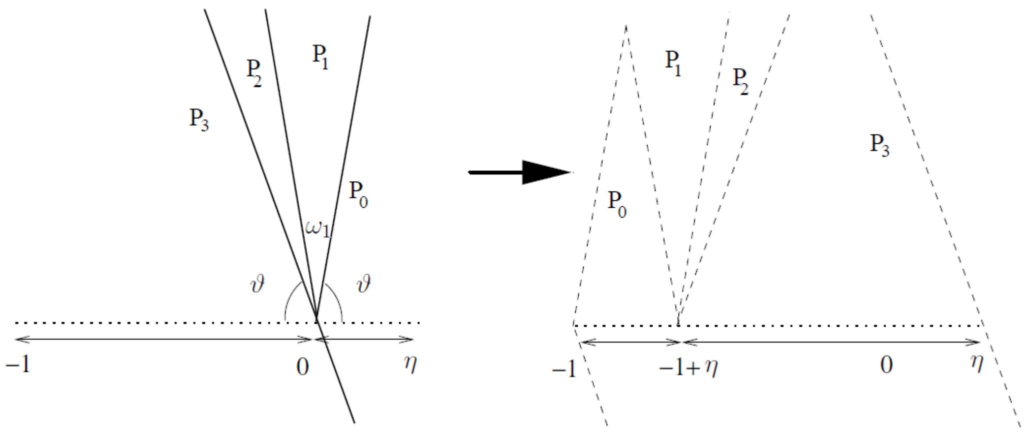


Figure 3.4: Schematic representation of a family of 4-PWIs $F : \mathbb{C} \to \mathbb{C}$ with atoms given by the four cones P_α, and three parameters: ω_1, ϑ and η. The atoms P_0 and P_3 are translated by F while P_1 and P_2 are rotated about their vertices then translated. The map on the baseline $[-1, \eta)$ is a 2-IET.

We define the maximal invariant set for this map as $X \subset \mathbb{C}$. Note that F restricted to the real line reduces to a 2-IET on $[-1, \eta)$ that equivalent to interchange of intervals of length 1 and η. We refer to this as the *baseline* transformation.

This map is such that all vertices of atoms that touch the baseline are mapped to the baseline. This means that although F is not invertible, it is locally bijective near the base line. The middle cones P_1 and P_2 are swapped by two rotations and after this, P_1 and P_2 are translated by $-(1 - \eta)$.

Recall the first return map to $P_c = P_1 \cup P_2$, $F_c : P_1 \cup P_2 \to P_1 \cup P_2$,

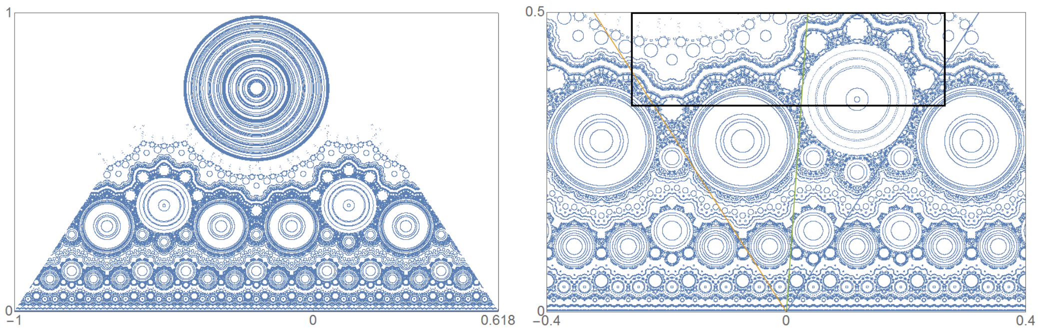
$$F_c(z) = F^{k(z)}(z). \tag{3.4.4}$$

where $k(z) = \inf\{k \geq 1 : F^k(z) \in P_1 \cup P_2\}$. If η is irrational then every point enters $P_1 \cup P_2$ after a finite number of iterates, and hence in this case F_c can be used to characterise all orbits of the map.

For typical choices of parameters ω_1, ϑ and η it seems that the dynamics of F defined by (3.4.3) (and hence of R) is very rich. Figure 3.5 (A) shows typical trajectories (after a transient), for two hundred randomly selected points and $(\omega_1, \vartheta, \eta) = (0.5, 1, \frac{\sqrt{5}-1}{2})$. Details of some invariant sets are then shown in Figure 3.5 (B). These numerical simulations illustrate that (as expected [9, 10]) the map F has an abundance of periodic islands for typical values of the parameters.

Figure 3.6 (A) shows the orbits of 5 points (ignoring a transient) under F_c, for $(\omega_1, \vartheta, \eta) = (0.5, 1, \frac{\sqrt{5}-1}{2})$. Details of this are shown in Figure 3.6 (B) and (C) in the areas $[-0.04, -0.01] \times [0.16, 0.21]$ and $[-0.0016, -0.01] \times [0.16, 0.165]$ respectively.

These figures show the diverse types of behaviour that can be found in the



(a) (b)

Figure 3.5: (A) Orbits of 200 points (ignoring a transient) by F, for $(\omega_1, \vartheta, \eta) = (0.5, 1, \frac{\sqrt{5}-1}{2})$. (B) Details of (A) in the area $[-0.4, 0.4] \times [0, 0.5]$. The cone indicates the location of $P_1 \cup P_2$. In this and later figures, orbits of length 10^5 are generated after removing a transient of 100 iterates. The maximal invariant set appears to have a highly complex boundary, but it does appear to include a polygon containing the baseline. The boxed region contains what seem to be many invariant non smooth curves.

invariant sets of F_c (and hence F). They show what seem to be non-trivial embedded IETs as well as invariant sets of higher dimension. There are also periodic islands to which the return map is a rotation.

Numerical results show that for some parameters we can observe non smooth invariant curves for the dynamics of the map F_c as defined in equation (3.4.4). These curves appear to have a dynamics similar to that of an interval exchange transformation. These curves can bound invariant regions that exhibit quite complex dynamics. We now construct one such region: set $\omega_1 = 0.5$, $\vartheta = 1$, $\eta = \frac{\sqrt{5}-1}{2}$ and $\eta' = 1 - \eta$. Consider the points

$$z_0 = r_0 e^{i(\pi-\vartheta)}, \quad z_1 = r_1 e^{i(\pi-\vartheta)},$$

with $r_0 = 0.470$ and $r_1 = 0.503$ and denote the orbit closures of these points as Ξ' and Ξ''. These are contained in the boxed region in Figure 3.5 (B) and are also represented in Figure 3.7 where it can be seen that both Ξ' and Ξ'' appear to be non-trivial continuous embeddings of IETs. Now consider the sets

$$Q'_L = \{z \in \mathbb{C} : \arg(z) = \pi - \vartheta \text{ and } r_0 \leq |z| \leq r_1\},$$

$$Q'_R = \{z \in \mathbb{C} : \arg(z) = \vartheta \text{ and } r_0 \leq |z| \leq r_1\}.$$

If Ξ' and Ξ'' are invariant curves that are embeddings of IETs, then the set $\partial\Xi = Q'_L \cup Q'_R \cup \Xi' \cup \Xi''$ is a Jordan curve. Denote by Ξ the closure of the region bounded by $\partial\Xi$. Numerical investigations suggest that Ξ is an invariant region for F_c. Let

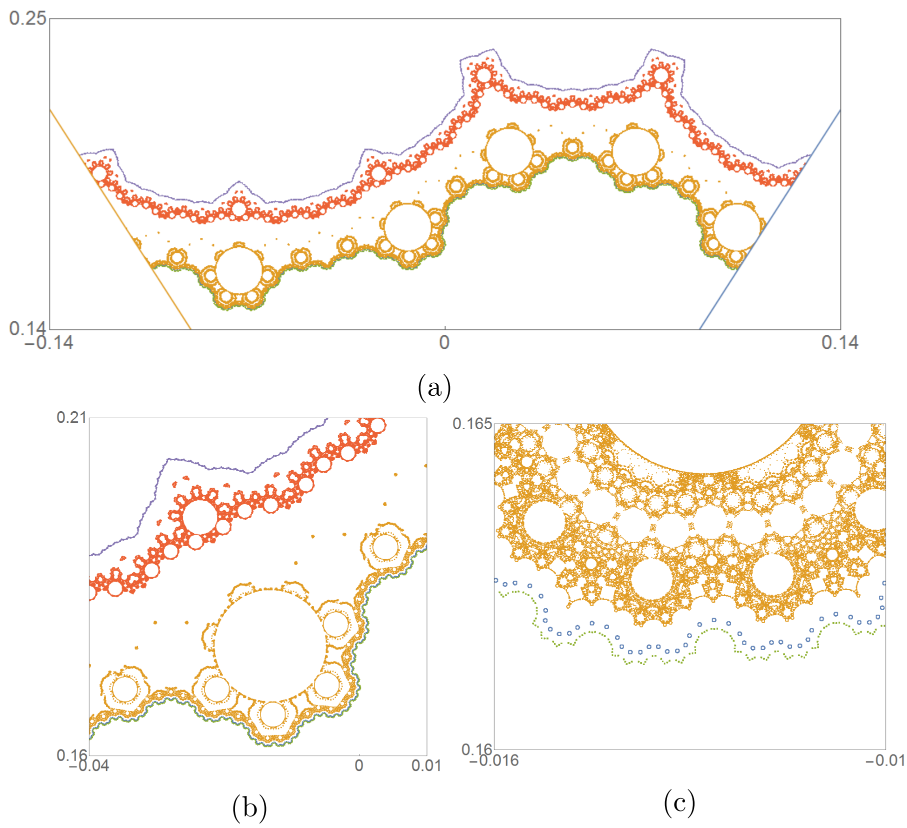


Figure 3.6: (A) Orbits of 5 points (ignoring a transient) under F_c, for $(\omega_1, \vartheta, \eta) = (0.5, 1, \frac{\sqrt{5}-1}{2})$ in the area $[-0.14, 0.14] \times [0, 0.25]$. (B) Details of (A) in the area $[-0.04, -0.01] \times [0.16, 0.21]$. (C) Details of (A) in the area $[-0.0016, -0.01] \times [0.16, 0.165]$. Observe a complex pattern of periodic islands, the presence of non-trivially embedded IETs as well as orbits with more complex structure.

$\Xi_k = Q_k \cap \Xi$, where

$$Q_0 = \{z \in \mathbb{C} : \mathrm{Im}(e^{-i(\omega_1+\vartheta)}(z + (2\eta - 1)e^{i\omega_1})) > 0\},$$
$$Q_1 = \{z \in \mathbb{C} : \mathrm{Im}(e^{-i(\omega_1+\vartheta)}(z + (2\eta - 1)e^{i\omega_1})) \leq 0 \text{ and } \mathrm{Im}(e^{i(\vartheta-\omega_1)}(z - (1-\eta)e^{i\omega_1})) < 0\},$$
$$Q_2 = \{z \in \mathbb{C} : \mathrm{Im}(e^{i(\vartheta-\omega_1)}(z - (1-\eta)e^{i\omega_1})) \geq 0 \text{ and } \mathrm{Im}(e^{-i(\omega_1+\vartheta)}z) > 0\},$$
$$Q_3 = \{z \in \mathbb{C} : \mathrm{Im}(ze^{-i(\omega_1+\vartheta)}) \leq 0\}.$$

Using the property of the golden mean $1 - \eta = \eta^2$ it can be seen that $F_c(z) = (F_c)_\alpha(z)$, for $z \in \Xi_\alpha$ where

$$(F_c)_\alpha(z) = \begin{cases} ze^{i\varpi_2} + \eta^3, & \alpha = 0, \\ ze^{i\varpi_2} - \eta^4, & \alpha = 1, \\ ze^{i\varpi_2} - \eta^2, & \alpha = 2, \\ ze^{i\varpi_1} + \eta^3, & \alpha = 3. \end{cases} \tag{3.4.5}$$

The subsets Ξ_α, $\alpha = 0, ..., 3$ and the action of F_c in this set are depicted in Figure 3.7. Note that that F_c acts isometrically on each Ξ_α, but since these sets are not

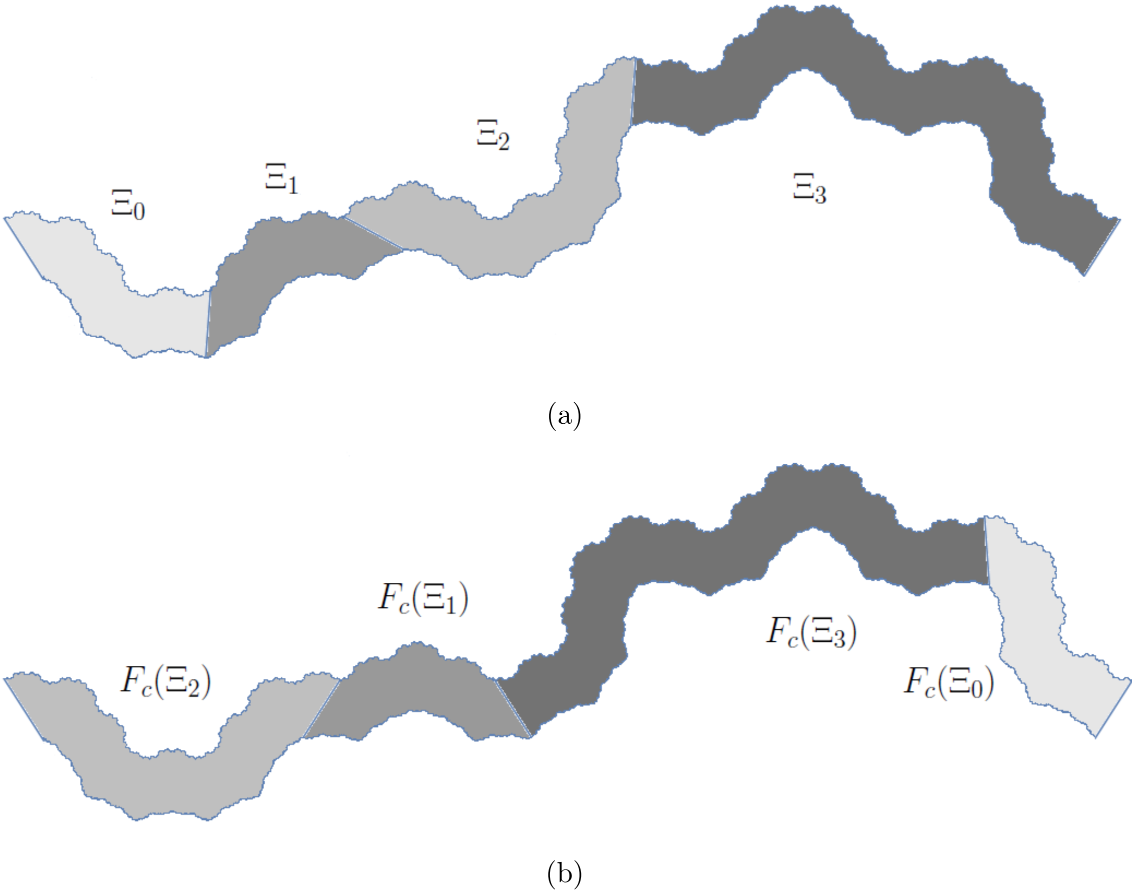


Figure 3.7: (A) The presumably invariant region $\Xi = \Xi_0 \cup \Xi_1 \cup \Xi_2 \cup \Xi_3$. (B) Image of Ξ by F_c.

convex (Ξ, F_c) is not a 4-PWI, but it is simple to construct a 4-PWI $(\mathbb{C}, S)$ such that Ξ is invariant under S and the restriction of S to Ξ is equal to F_c, by partitioning $\mathbb{C} = \bigcup_{\alpha \in \mathcal{A}} Q_\alpha$ and setting $S(z) = (F_c)_\alpha(z)$, for $z \in Q_\alpha$. One can verify that S satisfies the parametric connecting equation (3.2.12), therefore satisfying a necessary condition for the existence of an IET that can be continuously embedded by γ in $(\mathbb{C}, S)$, with $Y = \gamma(I) \subseteq \Xi$ also invariant under F_c.

3.4.3 A piecewise isometry with an embedded four interval exchange

Finally, we show that the map F_c in (3.4.5) is an example of a 4-PWI for which numerical evidence suggests the existence of a non-trivial embedded 4-IET.

Consider the family $\mathcal{F}_4$ of four-interval exchange maps $f_{\lambda,\pi} : I \to I$ given by subdividing the interval into four intervals of lengths $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4_+$ with base permutation π with monodromy invariant satisfying $\tilde{\pi}(1, 2, 3, 4) = (4, 2, 1, 3)$.

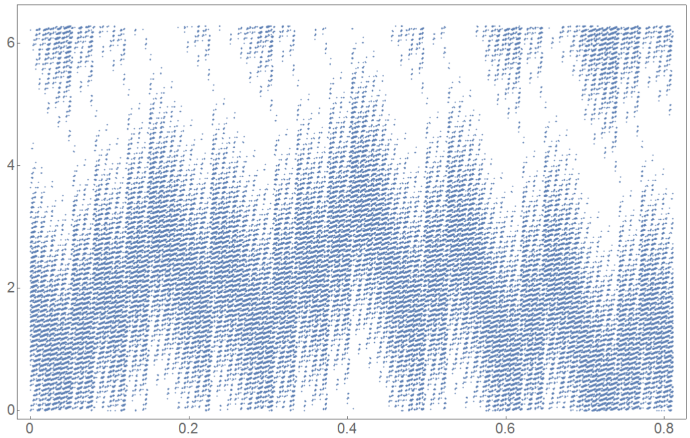


Figure 3.8: First 10^5 points of the orbit of $(0,0)$, by the tangent exchange map Ψ given by $\lambda = (l_0, l_1, l_2, l_3)$, $\tilde{\pi}(1,2,3,4) = (4,2,1,3)$, $\theta_\alpha = \varpi_2$, for $\alpha = 0,1,2$ and $\theta_3 = \varpi_1$. Observe the apparent lack of ergodicity as expected for a non-trivial embedding.

Note that on the real axis $\mathrm{Im}(z) = 0$ is a trivial embedding of the (degenerate) four-interval exchange where $\lambda = (\eta, 0, 0, 1)$. Let

$$Y = \overline{\{F_c^n(0.416i)\}_{n\in\mathbb{N}}}.$$

This defines an invariant set which is portrayed in Figure 3.1 that appears to be an embedding of an IET. We can partition Y by setting $Y_\alpha = Y \cap \Xi_\alpha$, for $\alpha = 0, ...3$. The length or Lebesgue one dimensional measure $l_\alpha = Leb(Y_\alpha)$ of each Y_α can be numerically estimated to be

$$l_0 = 0.1217970148, l_1 = 0.1329352086, l_2 = 0.2008884081, l_3 = 0.3550989199$$

Fix $\lambda = (l_0, l_1, l_2, l_3)$ and consider the IET $(I, f_{\lambda,\pi}) \in \mathcal{F}_4$. Numerical evidence suggests that there is a continuous embedding of $(I, f_{\lambda,\pi})$ into $(\mathbb{C}, S)$, by a map $\gamma : I \to Y$ with $Y \subseteq \Xi$, such that $\gamma(0) = r_0 e^{i\theta_0}$, with $r_0 = 0.47665$, and $\theta_0 = 0.68165\pi$. Indeed numerical verification shows that $\iota'_k R(h(0)) = \iota_k(f_{\lambda,\pi}(0))$ for all $k \le 10^5$, supporting that γ is a symbolic embedding.

We can also verify numerically that the condition in Theorem 3.3.1 holds for this case. Estimating $\overline{\xi}_\alpha \simeq \frac{e_\alpha(p(8))}{m_\alpha(p(8))}$ where $\overline{\xi}_0 \simeq 0.718 + 0.125i$, $\overline{\xi}_1 \simeq 0.538 - 0.512i$, $\overline{\xi}_2 \simeq 0.460 - 0.438i$ and $\overline{\xi}_3 \simeq 0.300 - 0.562i$. For these estimates we get

$$\left| \sum_{\alpha\in\mathcal{A}} \eta_\alpha \overline{\xi}_\alpha \lambda_\alpha - \gamma(0) \sum_{\alpha\in\mathcal{A}} (1 - e^{i\theta_\alpha}) \overline{\xi}_\alpha \lambda_\alpha \right| \simeq 6.30 \times 10^{-6},$$

where $\theta_\alpha = \varpi_2$, for $\alpha = 0,1,2$ and $\theta_3 = \varpi_1$.

Figure 3.8 shows 10^5 points of the orbit of $(0, 0)$, by tangent exchange map Ψ associated to S, which is consistent with the orbit being dense but not having nonuniform distribution on the cylinder $I \times S^1$.

In this chapter, we discussed the general problem of embedding IET dynamics within PWIs with a particular focus conditions for this embedding to be trivial or non-trivial, leaving still open the question of whether a non-trivial embedding of an IET into a PWI can exist at all. In the next chapter we answer this question by establishing that almost every IET, with an associated translation surface of genus $g \geq 2$, can be non-trivially embedded in a family of PWIs.

Chapter 4

Existence of non-trivial embeddings of Interval Exchange Transformations into Piecewise Isometries

In Section 1.4.2 we introduced continuous embeddings of an IET into a PWI (a curve γ satisfying (1.4.8), continuous on I). Then, on Section 2.5 we used them in the proof of one of our main results, Theorem B. On Chapter 3 we derived necessary conditions for the existence of embeddings. We proved that continuous embeddings of minimal 2-IETs into orientation preserving PWIs are necessarily trivial and that any 3-PWI has at most one non-trivially continuously embedded minimal 3-IET with the same underlying permutation. We gave numerical evidence of the existence of non-trivial embeddings of 3 and 4-IETs into PWIs, however we did not provide a rigorous proof.

In this chapter we prove that a full measure set of IETs admit non-trivial embeddings into a class of PWIs thus also establishing the existence of invariant curves for PWIs which are not unions of circle arcs or line segments.

This chapter is organized as follows. We start by introducing a sequence of piecewise linear curves determined by parameters $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A})$ and $\theta \in \mathbb{T}^{\mathcal{A}}$. These curves are related to the Rauzy induction (recall Section 1.3.2) of the IET parametrized by the same parameters (λ, π). We prove several technical lemmas which lead to the proof that each curve in the breaking sequence is *quasi-embedded* in a certain PWI. Finally we use tools from the theory of IET renormalization to prove key results which lead to the proof of Theorem E.

4.1 Breaking sequence

In this section we define the *breaking sequence*, a sequence of curves associated to IET parameters $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A})$ and a rotational parameter $\theta \in \mathbb{T}^{\mathcal{A}}$ via the *breaking operator*, an operator acting on the space of piecewise linear curves. We then relate the dynamics of a breaking sequence and that of the underlying IET.

Given $\ell > 0$ we denote by $\mathcal{PL}(\ell)$ the class of continuous piecewise linear maps $\gamma : [0, \ell) \to \mathbb{C}$ such that all x in the domain of differentiability of γ, satisfy $|\gamma(x)'| = 1$. Note that for any $\gamma \in \mathcal{PL}(\ell)$, its image $\gamma([0, \ell))$ has an arc length equal to ℓ.

We say that a sequence of mutually disjoint intervals $(J_n)_n$ is an *ordered sequence of disjoint intervals* if whenever $m < m'$, we have $x < x'$ for all $x \in J_m$ and $x' \in J_{m'}$.

Moreover, given a collection of mutually disjoint intervals $\mathcal{J}$, we say an ordered sequence of intervals $\{J_n\}$ is an *ordering* of $\mathcal{J}$ if for all $J \in \mathcal{J}$ there is a unique m such that $J_m = J$. Note that if $\mathcal{J}$ is a finite collection then it has a unique ordering.

We now define the *breaking operator*, which acts on $\mathcal{PL}(\ell)$. Given a sequence of subintervals of $[0, \ell)$, it takes a curve and rotates, by a fixed angle, the pieces corresponding to these subintervals. This is represented in Figure 4.1.

Consider a map $\gamma \in \mathcal{PL}(\ell)$, a real number $\varphi \in [-\pi, \pi)$, and an ordered sequence of disjoint intervals $J = (J_k)_{0 \le k \le r-1}$ of equal length $\Delta \in (0, \ell/r)$. We write $J_k = [y_k, y_k + \Delta) \subset \mathbb{R}$, where $y_k + \Delta \le y_{k+1}$ and $k \in \{0, ..., r-1\}$.

Let $B([0, \ell), \mathbb{C})$ denote the space of bounded maps from $[0, \ell)$ to $\mathbb{C}$. We define the *breaking operator* $\mathfrak{Br}(\varphi, J) : \mathcal{PL}(\ell) \to B([0, \ell), \mathbb{C})$ as

$$\mathfrak{Br}(\varphi, J) \cdot \gamma(x) = \begin{cases} \gamma(x), & x \in [0, y_0), \\ \gamma(x) \cdot e^{i\varphi} + \overline{\epsilon}_k(\varphi, J), & x \in [y_k, y_k + \Delta), \\ \gamma(x) + \underline{\epsilon}_k(\varphi, J), & x \in [y_k + \Delta, y_{k+1}), \end{cases} \tag{4.1.1}$$

for $k \in \{0, ..., r-1\}$, where $y_r = \ell$,

$$\overline{\epsilon}_0 = \gamma(x_0)(1 - e^{i\varphi}), \quad \overline{\epsilon_k} = \gamma(y_k)(1 - e^{i\varphi}) + \underline{\epsilon}_{k-1}, \tag{4.1.2}$$

and also

$$\underline{\epsilon}_0 = (\gamma(y_0) - \gamma(y_0 + \Delta))(1 - e^{i\varphi}), \quad \underline{\epsilon}_k = \overline{\epsilon}_k - \gamma(y_k + \Delta)(1 - e^{i\varphi}). \tag{4.1.3}$$

The above expressions for $\underline{\epsilon}_k$ and $\overline{\epsilon}_k$ are constructed in a way so that the action of the breaking operator preserves the continuity of a curve. Indeed in our next lemma we show that for all $\ell > 0$ and $\varphi \in [-\pi, \pi)$, $\mathfrak{Br}(\varphi, J)$ maps $\mathcal{PL}(\ell)$ into a subset of $\mathcal{PL}(\ell)$.

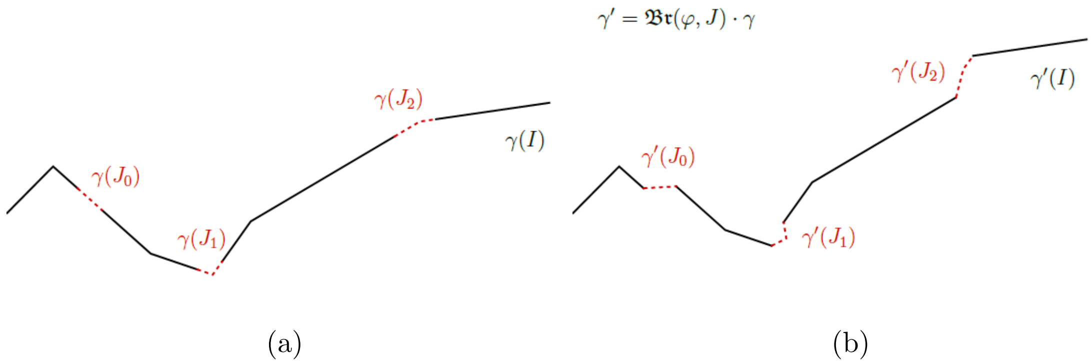


Figure 4.1: An illustrative representation of the action of the operator $\mathfrak{Br}$ on a piecewise linear curve $\gamma : I \to \mathbb{C}$. In (a) we can observe $\gamma(I)$. The dashed segments correspond to the images by γ of segments from an ordered sequence of disjoint intervals $J = (J_n)_{n=0,1,2}$; (b) shows the image $\mathfrak{Br}(\varphi, J)\cdot\gamma(I)$, with $\varphi = \pi/4$. Note how the breaking operator acts by only rotating the segments $\gamma(J_n)$ by φ while keeping the transformed curve continuous.

Lemma 4.1.1 *If $\ell > 0$, $\gamma \in \mathcal{PL}(\ell)$, $\varphi \in [-\pi, \pi)$ and J is an ordered sequence of disjoint subintervals of $[0, \ell)$ with length $\Delta > 0$, then $\mathfrak{Br}(\varphi, J)(\mathcal{PL}(\ell)) \subseteq \mathcal{PL}(\ell)$.*

Proof.

Let $\gamma \in \mathcal{PL}(\ell)$. It is clear that $\mathfrak{Br}(\varphi, J)\cdot\gamma$ is piecewise linear and continuous. In particular, it is semi-differentiable, that is, it admits both left and right derivatives for every point. Denote by ∂_- and ∂_+ its left and right derivative, respectively.

Given $x \in (0, \ell)$ we have

$$|\partial_- (\mathfrak{Br}(\varphi, J)\cdot\gamma)(x)| = |\partial_+ (\mathfrak{Br}(\varphi, J)\cdot\gamma)(x)| = |\partial_+\gamma(x)|.$$

Since $\gamma \in \mathcal{PL}(\ell)$, $|\partial_+\gamma'(x)| = 1$ and hence, if $\mathfrak{Br}(\varphi, J)\cdot\gamma$ is differentiable at x we must have $|(\mathfrak{Br}(\varphi, J)\cdot\gamma)'(x)| = 1$. This finishes our proof.

□

We will later need the estimate in the next lemma.

Lemma 4.1.2 *Let $\ell > 0$, $\gamma \in \mathcal{PL}(\ell)$, $\varphi \in [-\pi, \pi)$, $\Delta < \ell$ be a positive constant and J be an ordered sequence of disjoint intervals of length Δ. For all $k \in \mathbb{N}$ we have*

$$\max(|\overline{\epsilon}_k|, |\underline{\epsilon}_k|) \le 2\ell \sin\left|\frac{\varphi}{2}\right|.$$

Proof. Let r be the number of subintervals in J and $J = ([y_k, y_k + \Delta))_{0\le k<r}$.

By inserting (4.1.3) in (4.1.2) it is clear that, for any $1 \le k < r$, we have

$$\overline{\epsilon}_k = (\gamma(y_k) - \gamma(y_{k-1} + \Delta))(1 - e^{i\varphi}) + \overline{\epsilon}_{k-1},$$

and applying the triangle inequality we get, for any $1 \leq k < r$, that

$$|\overline{\epsilon}_k| \leq |1 - e^{i\varphi}| \left[|\gamma(y_k)| + \sum_{l=0}^{k-1} |\gamma(y_l) - \gamma(y_l + \Delta)| \right].$$

As $|1 - e^{i\varphi}| = |\sin(\varphi/2)|$, $y_k \leq \ell - (r-k)\Delta$ and $|\gamma(y_l) - \gamma(y_l + \Delta)| \leq \gamma([y_k, y_k + \Delta)) \leq \Delta$ we get as $r\Delta \leq \ell$

$$|\overline{\epsilon}_k| \leq 2\ell \sin\left|\frac{\varphi}{2}\right|.$$

It is also clear from (4.1.2) and (4.1.3) applying the triangle inequality that for any $1 \leq k < r$ we have

$$|\underline{\epsilon}_k| \leq |1 - e^{i\varphi}| \sum_{l=0}^{k} |\gamma(y_l) - \gamma(y_l + \Delta)|,$$

and in a similar way as before we can prove that $|\underline{\epsilon}_k| \leq k\Delta \sin|\varphi/2| \leq \ell \sin\left|\frac{\varphi}{2}\right|$.

□

Recall we say an ordered sequence of intervals $\{J_n\}$ is an *ordering* of a collection of mutually disjoint intervals $\mathcal{J}$, if for all $J \in \mathcal{J}$ there is a unique m such that $J_m = J$.

Also recall the definition of IET and notation introduced in Section 1.3.1. The n-th iterate of the Rauzy induction map $\mathcal{R} : \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A}) \to \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A})$ is defined for any $n \geq 0$ and is denoted by

$$\mathcal{R}^n(\lambda, \pi) = (\lambda^{(n)}, \pi^{(n)}),$$

with $\pi^{(n)} = (\ \pi_0^{(n)}\ \ \pi_1^{(n)}\ )^T$. Furthermore we denote by $\beta_{\varepsilon,n}$ the last symbol in the expression of $\pi_\varepsilon^{(n)}$, by $\varepsilon(n)$ the type of $f_{\lambda^{(n)},\pi^{(n)}}$, by $I^{(n)}$ its domain and by $\{I_\alpha^{(n)}\}_{\alpha\in\mathcal{A}}$ its partition in subintervals, for $n \geq 0$. Also recall that $r^n_{\lambda,\pi}(I_\alpha^{(n)})$ denotes the first return time of any $x \in I_\alpha^{(n)}$ by $f_{\lambda,\pi}$ to $I^{(n)}$.

Given $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A})$, consider the collection of sets

$$\mathcal{J}(n) = \left\{ f^k_{\lambda,\pi}\left(I^{(n-1)} \backslash I^{(n)}\right)\right\}_{0 \leq k < r(n-1)}, \tag{4.1.4}$$

where $r(n-1) = r^{n-1}_{\lambda,\pi}\left(I^{(n-1)}_{\beta_{0,n-1}}\right)$ and

$$\beta_{\varepsilon,m} = \left(\pi_\varepsilon^{(m)}\right)^{-1}(d).$$

It is clear that for all $n \geq 1$, $r(n-1)$ is equal to the smallest $r \geq 1$ such that $f^k_{\lambda,\pi}(I^{(n-1)} \backslash I^{(n)}) \subset I^{(n)}$. We denote the ordering of $\mathcal{J}(n)$ by $J^{(n)} = (J_k^{(n)})_{0 \leq k < r(n-1)}$, for all $n \geq 1$.

Let $\mathbb{Z}^{\mathcal{A}} \simeq \mathbb{Z}^d$ and $\mathbb{T}^{\mathcal{A}} \simeq \mathbb{T}^d$ be the d-dimensional torus $\mathbb{R}^{\mathcal{A}}/2\pi\mathbb{Z}^{\mathcal{A}}$. Furthermore, let $p : \mathbb{R}^{\mathcal{A}} \to \mathbb{T}^{\mathcal{A}}$ be the natural projection,

$$p(v) = \left((v)_\alpha \mod 2\pi\right)_{\alpha\in\mathcal{A}}, \quad \text{for all } v \in \mathbb{R}^{\mathcal{A}}.$$

We sometimes use the notation $p(v) = v \mod 2\pi$.

Recall the definition of Rauzy cocycle in Section 1.3.4. We introduce the *projection of the Rauzy cocycle on* $\mathbb{T}^{\mathcal{A}}$ as the application $B_{\mathbb{T}^{\mathcal{A}}} : \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R} \times \mathbb{T}^{\mathcal{A}} \to \mathbb{T}^{\mathcal{A}}$ such that $B_{\mathbb{T}^{\mathcal{A}}}(\lambda, \pi) \cdot \theta = p(B_R(\lambda, \pi) \cdot v)$, for any $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R}$, $n \geq 0$ and $\theta \in \mathbb{T}^{\mathcal{A}}$, with $v \in p^{-1}(\theta)$. Note that, as B_R is an integral cocycle, for any $v, v' \in p^{-1}(\theta)$ we have $p(B_R(\lambda, \pi) \cdot v) = p(B_R(\lambda, \pi) \cdot v')$ and thus the map $B_{\mathbb{T}^{\mathcal{A}}}$ is well defined. We also use the notation

$$B^{(n)}_{\mathbb{T}^{\mathcal{A}}}(\lambda, \pi) \cdot \theta = B^{(n)}_R(\lambda, \pi) \cdot v \mod 2\pi, \tag{4.1.5}$$

for any $n \geq 0$ and $\theta \in \mathbb{T}^{\mathcal{A}}$, with $v \in p^{-1}(\theta)$.

Given $\theta \in \mathbb{T}^{\mathcal{A}}$ let

$$\theta^{(0)} = \theta, \quad \theta^{(n)} = B^{(n)}_{\mathbb{T}^{\mathcal{A}}}(\lambda, \pi) \cdot \theta. \tag{4.1.6}$$

We define the *breaking sequence* as a sequence of piecewise linear curves $(\gamma^{(n)}_\theta(x))_n \in \mathcal{PL}(\ell)$, such that

$$\begin{aligned} &\gamma^{(0)}_\theta(x) = x, \\ &\gamma^{(n)}_\theta(x) = \mathfrak{Br}\left(\theta^{(n-1)}_{\beta_{1,n-1}}, J^{(n)}\right) \cdot \gamma^{(n-1)}_\theta(x), \end{aligned} \tag{4.1.7}$$

for all $x \in [0, |\lambda|)$ and $n \geq 1$.

Each map in the breaking sequence is a curve parametrized by its arclength and is obtained by successively applying the breaking operator with angles $\theta^{(n-1)}_{\beta_{1,n-1}}$ and segments $J^{(n)}$. Note that the number of these segments will increase while their lengths will decrease as $n \to +\infty$. In this way this sequence of curves is related both to the IET $f_{\lambda,\pi}$ and to a PWI with rotation vector θ. A representation of a breaking sequence of curves can be observed in Figure 4.2.

Denote by $\Theta_{\lambda,\pi}$ the set of all $\theta \in \mathbb{T}^{\mathcal{A}}$ such that for all $n \geq 0$, $\gamma^{(n)}_\theta : I \to \mathbb{C}$ is an injective map. Throughout the rest of this section we will consider $\gamma^{(n)} = \gamma^{(n)}_\theta$ with $\theta \in \Theta_{\lambda,\pi}$.

The monodromy invariant of the permutation $\pi^{(m)}$ is the bijection

$$\tilde{\pi}^{(m)} : \{1, \cdots, d\} \to \{1, \cdots, d\},$$

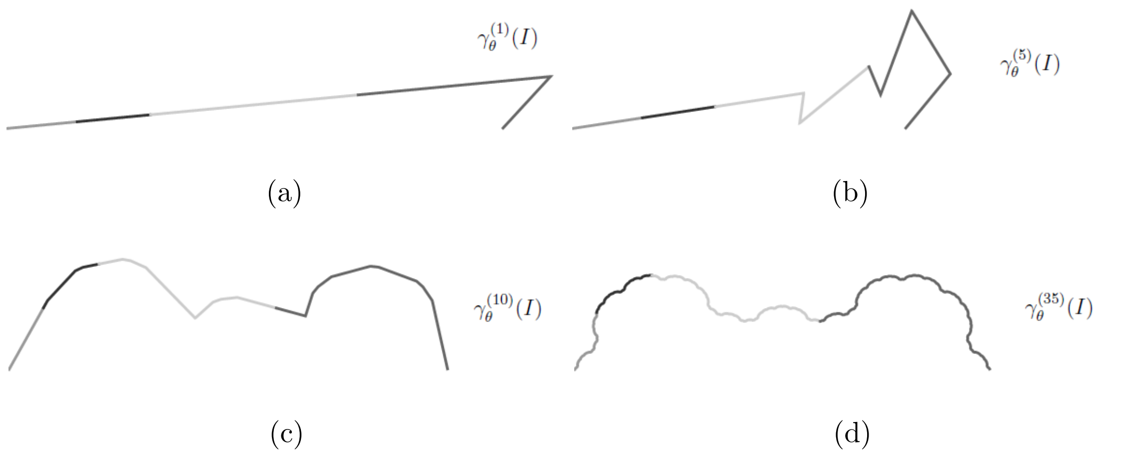


Figure 4.2: A representation of 4 curves in the breaking sequence $(\gamma_\theta^{(n)})_{n\geq 0}$ associated to parameters $\lambda \approx (0.43, 0.34, 0.12, 0.11)$, a permutation $\pi = (\pi_0, \pi_1)$, with $\pi_0(A, B, C, D) = (1, 2, 3, 4)$ and $\pi_1(A, B, C, D) = (4, 3, 2, 1)$ and $\theta \approx (3.872, 5.110, 0.531, 0.553)$. Each figure is the image of the interval I, by a map $\gamma^{(n)}$, composed with a rotation which assures that both endpoints lie on the Real axis. Each of the differently shaded segments correspond respectively, from left to right, to $\gamma^{(n)}(I_\alpha)$, $\alpha = A, B, C, D$. In (a), (b), (c) and (d) we can see $\gamma^{(n)}(I)$ respectively for $n = 1, 5, 10, 35$.

such that $\tilde{\pi}^{(m)} = \pi_1^{(m)} \circ (\pi_0^{(m)})^{-1}$. We denote its inverse by $\hat{\pi}^{(m)} = \pi_0^{(m)} \circ (\pi_1^{(m)})^{-1}$. We write

$$x_{\varepsilon,j}^{(m)} = \textstyle\sum_{\pi_\varepsilon^{(m)}(\alpha)\leq j} \lambda_\alpha^{(m)}, \tag{4.1.8}$$

for $\varepsilon = 0, 1$, where $x_{0,j}^{(m)}$ denotes the j-th endpoint of the partition associated to to $f_{\lambda^{(m)},\pi^{(m)}}$, this is $\{I_\alpha^{(m)}\}_{\alpha\in\mathcal{A}}$, and $x_{1,j}^{(m)}$ denotes the j-th endpoint of the image of this partition by $f_{\lambda^{(m)},\pi^{(m)}}$. Furthermore we denote their image by $\gamma^{(n)}$ as $\gamma_{\varepsilon,j}^{n,m} = \gamma^{(n)}\left(x_{\varepsilon,j}^{(m)}\right)$.

We may now define points $\xi_j^{n,m} \in \mathbb{C}$ recursively as follows

$$\begin{aligned} \xi_d^{n,m} &= \gamma_{0,d}^{n,m}, \\ \xi_j^{n,m} &= \exp\left\{ i\theta^{(m)}_{(\pi_1^{(m)})^{-1}(j+1)} \right\} \left(\gamma^{n,m}_{0,\hat{\pi}^{(m)}(j+1)-1} - \gamma^{n,m}_{0,\hat{\pi}^{(m)}(j+1)} \right) + \xi_{j+1}^{n,m}. \end{aligned} \tag{4.1.9}$$

For all $\alpha \in \mathcal{A}$, $n \in \mathbb{N}$, $0 \leq m \leq n$ and $z \in \mathbb{C}$, we define a map,

$$\hat{T}_\alpha^{(n,m)}(z) = e^{i\theta_\alpha^{(m)}} \left(z - \gamma^{n,m}_{0,\pi_0^{(m)}(\alpha)} \right) + \xi^{n,m}_{\pi_1^{(m)}(\alpha)}. \tag{4.1.10}$$

The isometries $\hat{T}_\alpha^{(n,m)}$ act on the segments $\gamma^{(n)}(I_\alpha^{(m)})$ by rearranging their order according to the permutation $\pi^{(m)}$, via rotations by angles $\theta_\alpha^{(m)}$. The right endpoint $\gamma^{n,m}_{0,\hat{\pi}^{(m)}(d)}$ of the segment $\gamma^{(n)}(I^{(m)}_{\beta_{1,m}})$ is mapped to the right endpoint $\xi_d^{n,m}$ of $\gamma^{(n)}(I^{(m)})$.

For $j < d$, the right endpoint $\gamma^{n,m}_{0,\hat{\pi}^{(m)}(j)}$ of $\gamma^{(n)}(I^{(m)}_{\hat{\pi}^{(m)}(j)})$ is mapped to the left endpoint $\xi^{n,m}_j$ of the image by $\hat{T}^{(n,m)}_{\hat{\pi}^{(m)}(j+1)}$ of $\gamma^{(n)}(I^{(m)}_{\hat{\pi}^{(m)}(j+1)})$. In this way, the union over $\alpha \in \mathcal{A}$, of all $\hat{T}^{(n,m)}_\alpha(\gamma^{(n)}(I^{(m)}_\alpha))$ is a continuous curve which *a priori* may not coincide with $\gamma^{(n)}(I^{(m)})$.

We also define inductively a map $T^{(n,m)}_\alpha$ as follows:

$$T^{(n,n)}_\alpha(z) = \hat{T}^{(n,n)}_\alpha(z). \tag{4.1.11}$$

For $z \in \mathbb{C}$, $0 < m \leq n$, if $\varepsilon(m-1) = 0$ then

$$T^{(n,m-1)}_\alpha(z) = \begin{cases} T^{(n,m)}_\alpha(z), & \alpha \neq \beta_{1,m-1}, \\ \left(T^{(n,m)}_{\beta_{0,m-1}}\right)^{-1} \circ T^{(n,m)}_\alpha(z), & \alpha = \beta_{1,m-1}, \end{cases} \tag{4.1.12}$$

if $\varepsilon(m-1) = 1$ then

$$T^{(n,m-1)}_\alpha(z) = \begin{cases} T^{(n,m)}_\alpha(z), & \alpha \neq \beta_{0,m-1}, \\ T^{(n,m)}_\alpha \circ \left(T^{(n,m)}_{\beta_{1,m-1}}\right)^{-1}(z), & \alpha = \beta_{0,m-1}. \end{cases} \tag{4.1.13}$$

Finally, we define a map $T^{(n,m)} : \gamma^{(n)}\left(I^{(m)}\right) \to \mathbb{C}$ by

$$T^{(n,m)}(z) = T^{(n,m)}_\alpha(z), \quad z \in \gamma^{(n)}\left(I^{(m)}_\alpha\right).$$

To understand the inductive procedure used to define $T^{(n,m)}$, consider first the map $f_{\lambda^{(m)},\pi^{(m)},\alpha} : \mathbb{R} \to \mathbb{R}$ such that $f_{\lambda^{(m)},\pi^{(m)},\alpha}(x) = x + v^{(m)}_\alpha$. If $\theta = 0$, by the definition of breaking sequence, $\gamma^{(n)}_\theta(x) = x$, for all $x \in I$ and $n \geq 0$. Consequently, we have $\gamma^{n,m}_{\varepsilon,j} = x^{(m)}_{\varepsilon,j}$, $\xi^{n,m}_j = x^{(m)}_{1,j}$ and thus, for all $z \in \mathbb{C}$, we have

$$\hat{T}^{(n,m)}_\alpha(z) = f_{\lambda^{(m)},\pi^{(m)},\alpha}(\mathrm{Re}(z)) + i\mathrm{Im}(z).$$

For $0 < m \leq n$ and $\varepsilon(m-1) = 0$, (4.1.12) gives

$$f_{\lambda^{(m-1)},\pi^{(m-1)},\alpha}(\mathrm{Re}(z)) = \begin{cases} f_{\lambda^{(m)},\pi^{(m)},\alpha}(\mathrm{Re}(z)), & \alpha \neq \beta_{1,m-1}, \\ f^{-1}_{\lambda^{(m)},\pi^{(m)},\beta_{0,m-1}} \circ f_{\lambda^{(m)},\pi^{(m)},\alpha}(\mathrm{Re}(z)), & \alpha = \beta_{1,m-1}, \end{cases}$$

and as $f_{\lambda^{(m)},\pi^{(m)}}(x) = f_{\lambda^{(m)},\pi^{(m)},\alpha}(x)$, when $x \in I^{(m)}_\alpha$, these identities can be easily verified to be equivalent to Rauzy induction in this case. An analogous set of identities can also be obtained for the case $\varepsilon(m-1) = 1$. Also note that for this example we have $\hat{T}^{(n,m)}_\alpha = T^{(n,m)}_\alpha$. This is no coincidence and indeed later we will prove that this identity holds in general. In this way (4.1.12) and (4.1.13) are a generalization of Rauzy induction and hence $(T^{(n,m)})_{n\geq 0}$ is a sequence of maps defined on $\gamma^{(n)}(I^{(m)})$ which preserves this inductive structure.

For the remainder of this section we prove several lemmas which serve as technical tools for our next section where we explore the relation between $\hat{T}_\alpha^{(n,m)}$, $T_\alpha^{(n,m)}$, $\gamma^{(n)}$ and $f_{\lambda^{(m)},\pi^{(m)}}$. The following lemma gives useful expressions for compositions of $\hat{T}_\alpha^{(n,m)}$ which are related to the inductive procedure used to define $T_\alpha^{(n,m)}$.

Lemma 4.1.3 *For all $n \geq 1$, $0 < m \leq n$ and $z \in \mathbb{C}$ if $\varepsilon(m-1) = 0$ then*

$$\left(\hat{T}^{(n,m)}_{\beta_{0,m-1}}\right)^{-1} \circ \hat{T}^{(n,m)}_{\beta_{1,m-1}}(z) = e^{i\theta^{(m-1)}_{\beta_{1,m-1}}} \left(z - \gamma^{n,m-1}_{0,\hat{\pi}^{(m-1)}(d)-1}\right) + \gamma^{n,m-1}_{1,d-1}.$$

and if $\varepsilon(m-1) = 1$ then

$$\hat{T}^{(n,m)}_{\beta_{0,m-1}} \circ \left(\hat{T}^{(n,m)}_{\beta_{1,m-1}}\right)^{-1}(z) = e^{i\theta^{(m-1)}_{\beta_{0,m-1}}} \left(z - \gamma^{n,m-1}_{0,d-1}\right) + \xi^{n,m}_{\tilde{\pi}^{(m-1)}(d)-1}.$$

Proof. Assume first $\varepsilon(m-1) = 0$. It is clear that $\pi_0^{(m-1)} = \pi_0^{(m)}$, $\pi_1^{(m)}(\beta_{1,m-1}) = \pi_1^{(m)}(\beta_{0,m}) + 1$ and we get

$$\xi^{n,m}_{\pi_1^{(m)}(\beta_{1,m-1})} - \xi^{n,m}_{\pi_1^{(m)}(\beta_{0,m-1})} = e^{i\theta^{(m)}_{\beta_{1,m-1}}} \left(\gamma^{n,m}_{0,\hat{\pi}^{(m-1)}(d)} - \gamma^{n,m}_{0,\hat{\pi}^{(m-1)}(d)-1}\right). \tag{4.1.14}$$

directly from the definition of $\xi_j^{n,m}$ with $j = \pi_1^{(m)}(\beta_{0,m})$.

From (4.1.6) we can write

$$\theta_\alpha^{(m)} = \begin{cases} \theta_\alpha^{(m-1)}, & \alpha \neq \beta_{1,m-1}, \\ \theta^{(m-1)}_{\beta_{1,m-1}} + \theta^{(m-1)}_{\beta_{0,m-1}}, & \alpha = \beta_{1,m-1}, \end{cases} \tag{4.1.15}$$

Now, since for any $j < d$ we have $\gamma_{0,j}^{n,m-1} = \gamma_{0,j}^{n,m}$, from the above relations using (4.1.10) we prove our lemma in this case.

Now assume $\varepsilon(m-1) = 1$. It is cleat that $\pi_1^{(m-1)} = \pi_1^{(m)}$ and $\pi_0^{(m)}(\beta_{1,m-1}) = \pi_0^{(m)}(\beta_{0,m-1}) - 1$. With $j = \tilde{\pi}^{(m-1)}(d) - 1$, it is straightforward from the definition of $\xi_j^{n,m}$ that

$$\xi^{n,m}_{\tilde{\pi}^{(m-1)}(d)-1} = e^{i\theta^{(m)}_{\beta_{0,m-1}}} \left(\gamma^{n,m}_{0,\pi_0^{(m)}(\beta_{1,m-1})} - \gamma^{n,m}_{0,\pi_0^{(m)}(\beta_{0,m-1})}\right) + \xi^{n,m}_{\tilde{\pi}^{(m-1)}(d)}. \tag{4.1.16}$$

By (1.3.7) and (4.1.8) we have

$$\gamma_{0,j}^{n,m-1} = \begin{cases} \gamma_{0,j}^{n,m}, & 0 \leq j < \hat{\pi}^{(m)}(d), \\ \gamma_{0,j+1}^{n,m}, & \hat{\pi}^{(m)}(d) \leq j < d, \end{cases} \tag{4.1.17}$$

which in particular, by (4.1.9) gives $\gamma_{0,d-1}^{n,m-1} = \xi_d^{n,m}$. Also, by (4.1.6) we have

$$\theta_\alpha^{(m)} = \begin{cases} \theta_\alpha^{(m-1)}, & \alpha \neq \beta_{0,m-1}, \\ \theta^{(m-1)}_{\beta_{0,m-1}} + \theta^{(m-1)}_{\beta_{1,m-1}}, & \alpha = \beta_{0,m-1}, \end{cases} \tag{4.1.18}$$

The second statement in the lemma follows from combining this with (4.1.16) using the definition of $\hat{T}_\alpha^{(n,m)}$.

□

Before proving our next lemma, note that we can write (1.3.5) as

$$\hat{\pi}^{(m-1)}(j) = \begin{cases} \hat{\pi}^{(m)}(\tilde{\pi}^{(m)}(d)+1), & j=d, \\ \hat{\pi}^{(m)}(j+1), & \tilde{\pi}^{(m-1)}(d) < j < d, \\ \hat{\pi}^{(m)}(j), & j \leq \tilde{\pi}^{(m)}(d). \end{cases} \tag{4.1.19}$$

The proofs of our next two lemmas consist of simple computations using our formulas and definitions. We highlight the main steps but do not present exhaustive proofs.

Lemma 4.1.4 *Let $n \geq 1$ and $0 < m \leq n$. If $\varepsilon(m-1) = 0$ and $\xi_{d-1}^{n,m-1} = \gamma_{1,d-1}^{n,m-1}$, then*

$$\hat{T}_\alpha^{(n,m-1)}(z) = \hat{T}_\alpha^{(n,m)}(z). \tag{4.1.20}$$

for all $z \in \mathbb{C}$ and $\alpha \in \mathcal{A}\backslash\{\beta_{1,m-1}\}$.

Proof.

For $\tilde{\pi}^{(m)}(d) < j < d$, from the definition of $\hat{\pi}^{(m-1)}$ and since $\gamma_{0,j}^{n,m-1} = \gamma_{0,j}^{n,m}$ we can write

$$\gamma_{0,\hat{\pi}^{(m-1)}(j)}^{n,m-1} - \gamma_{0,\hat{\pi}^{(m-1)}(j)}^{n,m-1} = \gamma_{0,\hat{\pi}^{(m)}(j+1)-1}^{n,m} - \gamma_{0,\hat{\pi}^{(m)}(j+1)}^{n,m}.$$

Since $\pi_0^{(m-1)} = \pi_0^{(m)}$, $\left(\pi_1^{(m-1)}\right)^{-1}(j) = \left(\pi_1^{(m)}\right)^{-1}(j+1)$, and as $j < d$, using (4.1.15) we get

$$\theta^{(m-1)}_{\left(\pi_1^{(m-1)}\right)^{-1}(j)} = \theta^{(m)}_{\left(\pi_1^{(m)}\right)^{-1}(j+1)}.$$

As $\xi_{d-1}^{n,m-1} = \gamma_{1,d-1}^{n,m-1}$ and $\gamma_{1,d-1}^{n,m-1} = \gamma_{0,d}^{n,m}$, the two expressions above give for $\tilde{\pi}^{(m)}(d) \leq j < d$

$$\xi_j^{n,m-1} = \xi_{j+1}^{n,m}. \tag{4.1.21}$$

Now assume $\alpha \in \mathcal{A}$ is such that $\pi_1^{(m)}(\alpha) > \tilde{\pi}^{(m)}(d)+1$. By (4.1.21) we get $\xi^{n,m}_{\pi_1^{(m)}(\alpha)} = \xi^{n,m-1}_{\pi_1^{(m)}(\alpha)-1}$, and since by (1.3.5), we have $\pi_1^{(m-1)}(\alpha) = \pi_1^{(m)}(\alpha) - 1$, this gives

$$\xi^{n,m}_{\pi_1^{(m)}(\alpha)} = \xi^{n,m-1}_{\pi_1^{(m-1)}(\alpha)}.$$

Since $\gamma_{0,j}^{n,m-1} = \gamma_{0,j}^{n,m}$ the proof of the lemma in this case follows from the definition of $\xi_j^{n,m}$ and $\hat{T}_\alpha^{n,m}$.

Note that $\pi_1^{(m-1)}(\beta_{0,m-1}) = \pi_1^{(m)}(\beta_{1,m-1})$ and thus it follows from (4.1.21) that $\xi^{n,m-1}_{\pi_1^{(m-1)}(\beta_{0,m-1})} = \xi^{n,m}_{\pi_1^{(m)}(\beta_{1,m-1})}$. By (4.1.14) we get

$$\xi^{n,m-1}_{\pi_1^{(m-1)}(\beta_{0,m-1})} - \xi^{n,m}_{\pi_1^{(m)}(\beta_{0,m-1})} = e^{i\theta^{(m)}_{\beta_{1,m-1}}} \left(\gamma^{n,m}_{0,\hat{\pi}^{(m-1)}(d)} - \gamma^{n,m}_{0,\hat{\pi}^{(m-1)}(d)-1}\right). \tag{4.1.22}$$

Since $\xi^{n,m-1}_{d-1} = \gamma^{n,m-1}_{1,d-1}$, we have

$$\gamma^{n,m-1}_{0,d} - \gamma^{n,m-1}_{1,d-1} = e^{i\theta^{(m-1)}_{\beta_{1,m-1}}} \left(\gamma^{n,m-1}_{0,\hat{\pi}^{(m-1)}(d)} - \gamma^{n,m-1}_{0,\hat{\pi}^{(m-1)}(d)-1}\right), \tag{4.1.23}$$

which combined with (4.1.15) and (4.1.22), using the fact that $\gamma^{n,m-1}_{1,d-1} = \gamma^{n,m}_{0,d}$, $\gamma^{n,m-1}_{0,j} = \gamma^{n,m}_{0,j}$, when $j < d$ and the definition of $\xi^{n,m}_j$ proves the lemma for $\alpha = \beta_{0,m-1}$.

From (4.1.15), (4.1.19) and (4.1.23), as $\pi_1^{(m)}(\beta_{1,m-1}) = \pi_1^{(m)}(\beta_{0,m})+1$ and $\gamma^{n,m}_{0,d} = \gamma^{n,m-1}_{1,d-1}$, a trivial computation gives

$$\xi^{n,m}_{\tilde{\pi}^{(m)}(d)+1} = e^{i\theta^{(m)}_{\beta_{0,m}}} \left(\gamma^{n,m-1}_{0,d} - \gamma^{n,m}_{0,d}\right) + \xi^{n,m}_{\tilde{\pi}^{(m)}(d)}.$$

By (4.1.19), (4.1.21) and from the definition of $\xi^{n,m-1}_{\tilde{\pi}^{(m)}(d)-1}$ we get

$$\xi^{n,m-1}_{\tilde{\pi}^{(m)}(d)-1} = e^{i\theta^{(m-1)}_{\beta_{0,m}}} \left(\gamma^{n,m-1}_{0,d-1} - \gamma^{n,m-1}_{0,d}\right) + \xi^{n,m}_{\tilde{\pi}^{(m)}(d)+1}.$$

Combining this with (4.1.15), (4.1.19) and noting that $\gamma^{n,m}_{0,d-1} = \gamma^{n,m-1}_{0,d-1}$, the relation

$$\xi^{n,m-1}_{\tilde{\pi}^{(m)}(d)-1} = \xi^{n,m}_{\tilde{\pi}^{(m)}(d)-1}.$$

simply follows from the definition of $\xi^{n,m}_j$ for $j = \tilde{\pi}^{(m)}(d) - 1$.

We now prove by induction on j that

$$\xi^{n,m-1}_j = \xi^{n,m}_j. \tag{4.1.24}$$

for $1 \le j < \tilde{\pi}^{(m)}(d)$.

Since $\pi_0^{(m-1)} = \pi_0^{(m)}$, we get by (4.1.19) that $(\pi_1^{(m-1)})^{-1}(j) = (\pi_1^{(m)})^{-1}(j)$, and as $j < d$, by (4.1.15) we have

$$\theta^{(m-1)}_{\left(\pi_1^{(m-1)}\right)^{-1}(j)} = \theta^{(m)}_{\left(\pi_1^{(m)}\right)^{-1}(j)}, \tag{4.1.25}$$

Combined with (4.1.19) this gives

$$\xi^{n,m-1}_{j-1} = \exp\left\{i\theta^{(m)}_{(\pi_1^{(m)})^{-1}(j)}\right\} \left(\gamma^{n,m}_{0,\hat{\pi}^{(m)}(j)-1} - \gamma^{n,m}_{0,\hat{\pi}^{(m)}(j)}\right) + \xi^{n,m}_j,$$

which, as $\xi_j^{n,m-1} = \xi_j^{n,m}$, by (4.1.9) shows that $\xi_{j-1}^{n,m-1} = \xi_{j-1}^{n,m}$, proving (4.1.24).

Now assume $\alpha \in \mathcal{A}$ is such that $\pi_1^{(m)}(\alpha) < \tilde{\pi}^{(m)}(d)$. From (4.1.24), since $\pi_1^{(m-1)}(\alpha) = \pi_1^{(m)}(\alpha)$ we get $\xi^{n,m-1}_{\pi_1^{(m-1)}(\alpha)} = \xi^{n,m}_{\pi_1^{(m)}(\alpha)}$. This, combined with (4.1.25) and the definition of $\hat{T}_\alpha^{(n,m)}$, proves our statement for $\pi_1^{(m)}(\alpha) < \tilde{\pi}^{(m)}(d)$.

Since $\pi_1^{(m)}(\beta_{0,m-1}) = \tilde{\pi}^{(m)}(d)$ and since we proved (4.1.20) for $\alpha = \beta_{0,m-1}$ and $\pi_1^{(m)}(\alpha) > \tilde{\pi}^{(m)}(d) + 1$, we get (4.1.20), for all $\pi_1^{(m)}(\alpha) \neq \tilde{\pi}^{(m)}(d) + 1$. As $\pi_1^{(m)}(\beta_{1,m-1}) = \pi_1^{(m)}(\beta_{0,m}) + 1$ and $\pi_1^{(m)}(\beta_{1,m}) = \tilde{\pi}^{(m)}(d) + 1$ we have (4.1.20) for all $\alpha \in \mathcal{A}\backslash\{\beta_{1,m-1}\}$. □

Note that by (1.3.6) we can write

$$\hat{\pi}^{(m-1)}(j) = \begin{cases} \hat{\pi}^{(m)}(j) - 1, & \hat{\pi}^{(m)}(j) > \hat{\pi}^{(m)}(d) + 1, \\ d, & \hat{\pi}^{(m)}(j) = \hat{\pi}^{(m)}(d) + 1, \\ \hat{\pi}^{(m)}(j), & \hat{\pi}^{(m)}(j) < \hat{\pi}^{(m)}(d) + 1. \end{cases} \tag{4.1.26}$$

The following lemma provides a result similar to that of Lemma 4.1.4, but for the case $\varepsilon(m-1) = 1$. The main difference, compared to the previous case, comes from the fact that $\xi_{d-1}^{n,m-1}$ does not, beforehand, coincide with $\gamma_{1,d-1}^{n,m-1}$, although we will later establish this equality.

Lemma 4.1.5 *Let $n \geq 1$ and $0 < m \leq n$. If $\varepsilon(m-1) = 1$ and $\xi_{d-1}^{n,m-1} = \xi_{d-1}^{n,m}$, then for all $z \in \mathbb{C}$ and $\alpha \in \mathcal{A}\backslash\{\beta_{0,m-1}, \beta_{1,m-1}\}$ we have*

$$\hat{T}_\alpha^{(n,m-1)}(z) = \hat{T}_\alpha^{(n,m)}(z). \tag{4.1.27}$$

and

$$\hat{T}^{(n,m-1)}_{\beta_{0,m-1}}(z) = \hat{T}^{(n,m)}_{\beta_{0,m-1}} \circ \left(\hat{T}^{(n,m)}_{\beta_{1,m-1}}\right)^{-1}(z). \tag{4.1.28}$$

Proof.

By (4.1.17) and (4.1.26), for all j such that $\hat{\pi}^{(m)}(j) \notin \{\hat{\pi}^{(m)}(d), \hat{\pi}^{(m)}(d)+1\}$, we get

$$\gamma^{n,m-1}_{0,\hat{\pi}^{(m-1)}(j)} = \gamma^{n,m}_{0,\hat{\pi}^{(m)}(j)}, \tag{4.1.29}$$

similarly, $\gamma^{n,m-1}_{0,\hat{\pi}^{(m-1)}(j)-1} = \gamma^{n,m}_{0,\hat{\pi}^{(m)}(j)-1}$. In particular, for any $j \notin \{\tilde{\pi}^{(m)}(\hat{\pi}^{(m)}(d)+1), d\}$ we have

$$\gamma^{n,m-1}_{0,\hat{\pi}^{(m-1)}(j)-1} - \gamma^{n,m-1}_{0,\hat{\pi}^{(m-1)}(j)} = \gamma^{n,m}_{0,\hat{\pi}^{(m)}(j)-1} - \gamma^{n,m}_{0,\hat{\pi}^{(m)}(j)}. \tag{4.1.30}$$

As $\pi_1^{(m-1)} = \pi_1^{(m)}$ and by (4.1.18), for all $j < d$ we have

$$\theta^{(m-1)}_{(\pi_1^{(m-1)})^{-1}(j)} = \theta^{(m)}_{(\pi_1^{(m)})^{-1}(j)}. \tag{4.1.31}$$

We now prove, by induction on j, that

$$\xi_j^{n,m-1} = \xi_j^{n,m}. \tag{4.1.32}$$

for $\tilde{\pi}^{(m-1)}(d) \leq j < d$.

We have $\xi_{d-1}^{n,m-1} = \xi_{d-1}^{n,m}$. Take $\tilde{\pi}^{(m-1)}(d) < j < d$. As $\tilde{\pi}^{(m)}(\hat{\pi}^{(m)}(d)+1) = \tilde{\pi}^{(m-1)}(d)$, we have that $j \notin \{\tilde{\pi}^{(m)}(\hat{\pi}^{(m)}(d)+1), d\}$, hence by (4.1.30) and (4.1.31) we get $\xi_{j-1}^{n,m-1} = \xi_{j-1}^{n,m}$. This shows that for any $\tilde{\pi}^{(m-1)}(d) \leq j < d$, (4.1.32) holds.

By (4.1.17) and (4.1.26) we have $\gamma_{0,\hat{\pi}^{(m-1)}(d)}^{n,m-1} = \gamma_{0,\hat{\pi}^{(m)}(d)+1}^{n,m}$ and $\gamma_{0,\hat{\pi}^{(m-1)}(d)-1}^{n,m-1} = \gamma_{0,\hat{\pi}^{(m)}(d)-1}^{n,m}$, thus by (4.1.9) we get

$$\xi_{d-1}^{n,m-1} = e^{i\theta_{\beta_{1,m-1}}^{(m-1)}} \left(\gamma_{0,\hat{\pi}^{(m)}(d)-1}^{n,m} - \gamma_{0,\hat{\pi}^{(m)}(d)+1}^{n,m}\right) + \gamma_{0,d}^{n,m-1},$$

since $\xi_{d-1}^{n,m-1} = \xi_{d-1}^{n,m}$ and $\xi_d^{n,m} = \gamma_{0,d-1}^{n,m-1}$, by combining this with the definition of $\xi_{d-1}^{n,m}$ and (4.1.31) we get

$$\gamma_{0,d-1}^{n,m-1} - \gamma_{0,d}^{n,m-1} = e^{i\theta_{\beta_{1,m-1}}^{(m-1)}} \left(\gamma_{0,\hat{\pi}^{(m)}(d)}^{n,m} - \gamma_{0,\hat{\pi}^{(m)}(d)+1}^{n,m}\right). \tag{4.1.33}$$

By (4.1.9), with $j = \tilde{\pi}^{(m-1)}(d) - 1$, we have

$$\xi_{\tilde{\pi}^{(m-1)}(d)-1}^{n,m-1} = e^{i\theta_{\beta_{0,m-1}}^{(m-1)}} \left(\gamma_{0,d-1}^{n,m-1} - \gamma_{0,d}^{n,m-1}\right) + \xi_{\tilde{\pi}^{(m-1)}(d)}^{n,m-1},$$

which by and (4.1.18), (4.1.32) and (4.1.33) gives

$$\xi_{\tilde{\pi}^{(m-1)}(d)-1}^{n,m-1} = e^{i\theta_{\beta_{0,m-1}}^{(m)}} \left(\gamma_{0,\hat{\pi}^{(m)}(d)}^{n,m} - \gamma_{0,\hat{\pi}^{(m)}(d)+1}^{n,m}\right) + \xi_{\tilde{\pi}^{(m-1)}(d)}^{n,m},$$

and as, by (4.1.26), $\tilde{\pi}^{(m-1)}(d) = \tilde{\pi}^{(m)}(\hat{\pi}^{(m)}(d)+1)$, combined with (4.1.9) this shows that $\xi_{\tilde{\pi}^{(m-1)}(d)-1}^{n,m-1} = \xi_{\tilde{\pi}^{(m-1)}(d)-1}^{n,m}$.

Now assume, by induction in j, that for some $j < \tilde{\pi}^{(m-1)}(d)$ we have $\xi_j^{n,m-1} = \xi_j^{n,m}$. It is straightforward to see, by definition of $\xi_j^{n,m}$, (4.1.30) and (4.1.31) that $\xi_{j-1}^{n,m-1} = \xi_{j-1}^{n,m}$. Since we had proved before that (4.1.32) holds for $\tilde{\pi}^{(m-1)}(d) \leq j < d$, this shows that (4.1.32) is true for all $j < d$.

Now, consider $\alpha \in \mathcal{A}\backslash\{\beta_{0,m-1}, \beta_{1,m-1}\}$. By taking $j = \pi_1^{(m)}(\alpha)$ we get $j \notin \{\tilde{\pi}^{(m)}(\hat{\pi}^{(m)}(d)+1), d\}$ and by (4.1.29) we obtain $\gamma_{0,\pi_0^{(m-1)}(\alpha)}^{n,m-1} = \gamma_{0,\pi_0^{(m)}(\alpha)}^{n,m}$, and thus by (4.1.31), (4.1.32) and (4.1.10) we get (4.1.27).

By (4.1.10), for all $z \in \mathbb{C}$, we get

$$\hat{T}_{\beta_{0,m-1}}^{(n,m)} \circ \left(\hat{T}_{\beta_{1,m-1}}^{(n,m)}\right)^{-1}(z) = \xi_{\pi_1^{(m)}(\beta_{0,m-1})}^{n,m} +$$
$$e^{i\theta_{\beta_{0,m-1}}^{(m)}} \left[e^{-i\theta_{\beta_{1,m-1}}^{(m)}} \left(z - \xi_{\pi_1^{(m)}(\beta_{1,m-1})}^{n,m}\right) + \gamma_{0,\pi_0^{(m)}(\beta_{1,m-1})}^{n,m} - \gamma_{0,\pi_0^{(m)}(\beta_{0,m-1})}^{n,m}\right],$$

which by Lemma 4.1.3 gives

$$\hat{T}^{(n,m)}_{\beta_{0,m-1}} \circ \left(\hat{T}^{(n,m)}_{\beta_{1,m-1}}\right)^{-1}(z) = e^{i\theta^{(m-1)}_{\beta_{0,m-1}}}\left(z - \gamma^{n,m-1}_{0,d-1}\right) + \xi^{n,m}_{0,\tilde{\pi}^{(m-1)}(d)-1},$$

combined with (4.1.32) and (4.1.9) for $j = \tilde{\pi}^{(m-1)}(d) - 1$, this gives

$$\hat{T}^{(n,m)}_{\beta_{0,m-1}} \circ \left(\hat{T}^{(n,m)}_{\beta_{1,m-1}}\right)^{-1}(z) = e^{i\theta^{(m-1)}_{\beta_{0,m-1}}}\left(z - \gamma^{n,m-1}_{0,d}\right) + \xi^{n,m}_{0,\tilde{\pi}^{(m-1)}(d)}.$$

By (4.1.10) this shows that (4.1.28) holds. □

Consider now $J = (J_k)_{0\le k<r}$, with $r \in \mathbb{N}$, an ordered sequence of disjoint subintervals of I. Let I' be a subinterval of I, we denote

$$J \cap I' = \{J_k \cap I' : J_k \cap I' \neq \emptyset\}_{0\le k<r}.$$

Recall we denote by $J^{(n+1)}$ the ordering of $\left\{f^k_{\lambda,\pi}\left(I^{(n)}\backslash I^{(n+1)}\right)\right\}_{0\le k<r(n)}$, where $r(n) = r^n_{\lambda,\pi}\left(I^{(n)}_{\beta_{0,n}}\right)$.

Given $n \in \mathbb{N}$ we define a sequence $(k(m))_{0\le m\le n+1}$ of indices of $J^{(n+1)}$ as follows. Set $k(n+1) = 0$. For $0 \le m < n+1$ let $k(m)$ be equal to the number of disjoint subintervals in $J^{(n+1)} \cap I^{(m)}$. It is clear we have

$$J^{(n+1)} \cap (I^{(m-1)}\backslash I^{(m)}) = (J_k)_{k(m)\le k<k(m-1)},$$

for $0 < m \le n+1$.

Denote by $\beta(m) = \beta_{1-\varepsilon(m),m}$ that is, the loser of $(\lambda^{(m)}, \pi^{(m)})$. In the following two lemmas we give a description of $J^{(n+1)}$ that will later be used. We believe that these are elementary results however we could not find them in the literature and thus we present a proof.

Lemma 4.1.6 *For all $n \ge 0$, $0 < m \le n+1$ and $0 \le k < r(m-1)$, if $J_k \cap (I^{(m-1)}\backslash I^{(m)}) \neq \emptyset$ then $J_k \subseteq I^{(m-1)}\backslash I^{(m)}$.*

Proof. Assume, by contradiction, that there is a $J_k = J'_k \sqcup J''_k \in J^{(n+1)}$ such that $J'_k \cap (I^{(m-1)}\backslash I^{(m)}) = \emptyset$ and $J''_k \subseteq I^{(m-1)}\backslash I^{(m)}$. Take $l \ge 0$ such that $f^{-l}_{\lambda,\pi}(J'_k) \subseteq I^{(n)}\backslash I^{(n+1)}$. It is simple to check, given two points $x' \in f^{-l}_{\lambda,\pi}(J'_k)$ and $x'' \in I^{(n)}_{\beta_{0,n}}\backslash f^{-l}_{\lambda,\pi}(J'_k)$, that $r^n_{\lambda,\pi}(x') \neq r^n_{\lambda,\pi}(x'')$, which, as $I^{(n)}\backslash I^{(n+1)} \subseteq I^{(n)}_{\beta_{0,n}}$ contradicts the fact that $r^n_{\lambda,\pi}$ is constant on $I^{(n)}_{\beta_{0,n}}$. □

Lemma 4.1.7 *For all $n \ge 0$ we have*

$$J^{(n+1)} \cap (I^{(n)}\backslash I^{(n+1)}) = (I^{(n)}\backslash I^{(n+1)}),$$

furthermore for all $0 < m \leq n$ *we have*

$$J^{(n+1)} \cap (I^{(m-1)} \backslash I^{(m)}) = f_{\lambda^{(m-1)},\pi^{(m-1)}} \left(J^{(n+1)} \cap I^{(m)}_{\beta(m-1)} \right). \tag{4.1.34}$$

In particular there exists a $k'(m) > 0$ *such that for all* $k(m) \leq k < k(m-1)$ *we have*

$$J_k = f_{\lambda^{(m-1)},\pi^{(m-1)}} \left(J_{k-k'(m)} \right). \tag{4.1.35}$$

Proof. Note that we have $J_0 = I^{(n)} \backslash I^{(n+1)}$ from whence the first statement follows.

Assume that $J^{(n+1)} \cap I^{(m-1)} \backslash I^{(m)} \neq \emptyset$, as otherwise the result holds trivially, and take $J_k \in J^{(n+1)}$ such that $J_k \cap (I^{(m-1)} \backslash I^{(m)}) \neq \emptyset$. By Lemma 4.1.6 we have $J_k \subseteq I^{(m-1)} \backslash I^{(m)}$ and thus it follows from the definition of $J^{(n+1)}$ that there is an $l \geq 1$ such that $f^l_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n)} \backslash I^{(n+1)}) = J_k$. Furthermore the pre-image by $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ of J_k is contained in $I^{(m)}_{\beta(m-1)}$ and it is a term $J_{k'}$, with $k' < k$, in the sequence $J^{(n+1)}$. The difference $k'(m) = k - k'$ is independent of the choice of J_k, from which (4.1.35) follows. Observing that $J^{(n+1)} \cap I^{(m)}_{\beta(m-1)} = (J_k)_{k \in K}$, with $K = \{k(m) - k'(m), ..., k(m-1) - k'(m)\}$, and combining this with (4.1.35) we obtain (4.1.34), thus completing the proof. □

4.2 Existence of a quasi-embedding

In this section we introduce the notion of *quasi-embedding* and use it to relate the dynamics of $f_{\lambda^{(m)},\pi^{(m)}}$ with that of $T^{(n,m)}$ for any $n \geq 0$ and $0 \leq m \leq n$. Recall that $\gamma^{(n)} = \gamma^{(n)}_\theta$ with $\theta \in \Theta_{\lambda,\pi}$, where $\gamma^{(n)}_\theta$ is as in (4.1.7).

We say that $f_{\lambda^{(m)},\pi^{(m)}}$ is *quasi-embedded* into $T^{(n,m)}$, or that $\gamma^{(n)}$ is a *quasi-embedding* of $f_{\lambda^{(m)},\pi^{(m)}}$ into $T^{(n,m)}$, for $x \in I' \subseteq I$ if

$$T^{(n,m)}(\gamma^{(n)}(x)) = \gamma^{(n)}(f_{\lambda^{(m)},\pi^{(m)}}(x)). \tag{4.2.1}$$

Intuitively this means that $T^{(n,m)}$ and $f_{\lambda^{(m)},\pi^{(m)}}$ are *nearly* topologically conjugate, the conjugacy failing only for points in $I \backslash I'$.

The following theorem establishes that $T^{(n,m)}_\alpha = \hat{T}^{(n,m)}_\alpha$ and that $\gamma^{(n)}$ is a quasi-embedding of $f_{\lambda^{(m)},\pi^{(m)}}$ into $T^{(n,m)}$ except for points in a subinterval which decreases with n.

Theorem 4.2.1 *For all* $n \geq 0$ *and* $0 \leq m \leq n$, $\gamma^{(n)}$ *is a quasi-embedding of* $f_{\lambda^{(m)},\pi^{(m)}}$ *into* $T^{(n,m)}$ *for* $x \in I^{(m)} \backslash f^{-1}_{\lambda^{(m)},\pi^{(m)}} \left(I^{(n)} \right)$. *Furthermore for all* $\alpha \in \mathcal{A}$ *and* $z \in \mathbb{C}$ *we have*

$$T^{(n,m)}_\alpha(z) = \hat{T}^{(n,m)}_\alpha(z). \tag{4.2.2}$$

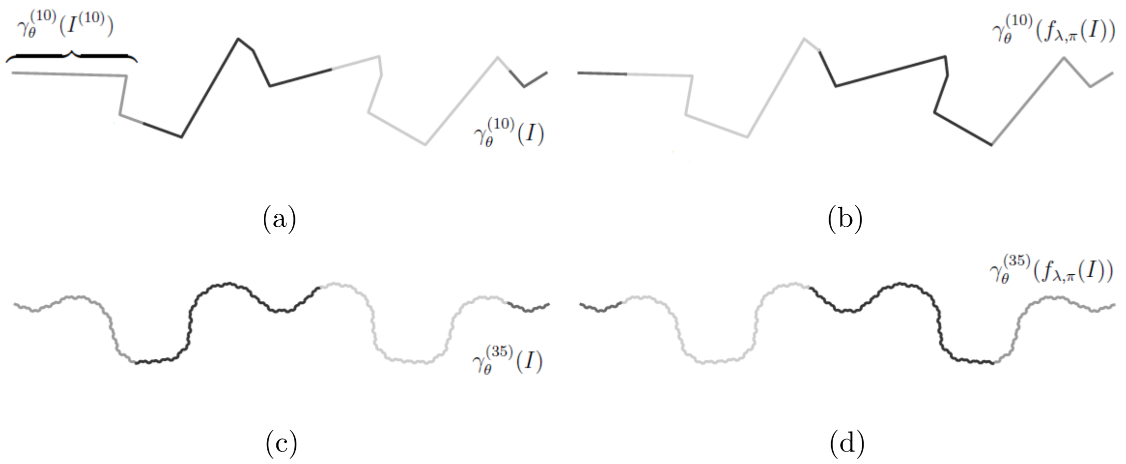


Figure 4.3: A representation of curves in the breaking sequence $(\gamma_\theta^{(n)})_{n\geq 0}$ associated to parameters $\lambda \approx (0.222, 0.343, 0.377, 0.058)$, a permutation $\pi = (\pi_0, \pi_1)$, with $\pi_0(A, B, C, D) = (1, 2, 3, 4)$ and $\pi_1(A, B, C, D) = (4, 3, 2, 1)$ and $\theta \approx (0.905, 5.501, 0.169, 0.067)$.
Figures (a) and (c) show respectively the curves $\gamma_\theta^{(10)}(I)$ and $\gamma_\theta^{(35)}(I)$. Each of the differently shaded segments correspond respectively, from left to right, to $\gamma^{(n)}(I_\alpha)$, $\alpha = A, B, C, D$, $n = 10, 35$. Also note that the segment, under the braces in figure (a), corresponds to $\gamma_\theta^{(10)}(I^{(10)})$.
Figures (b) and (d) show respectively the curves $\gamma_\theta^{(10)}(f_{\lambda,\pi}(I))$ and $\gamma_\theta^{(35)}(f_{\lambda,\pi}(I))$. Each of the differently shaded segments correspond respectively, from left to right, to $\gamma^{(n)}(f_{\lambda,\pi}(I_\alpha))$, $\alpha = D, C, B, A$, $n = 10, 35$.
By comparing figures (a) and (b) note that for any $I' \subseteq I$ $\gamma^{(10)}(f_{\lambda,\pi}(I'))$ can be obtained from applying the piecewise isometry $T^{(10,0)}$ to $\gamma^{(10)}(I')$ as long as $I' \cap f_{\lambda,\pi}^{-1}(I^{(10)}) = \emptyset$, in agreement with Theorem 4.2.1. A similar fact is true for figures (c) and (d), however since $f_{\lambda,\pi}^{-1}(I^{(35)})$ is small in this case, it is no longer apparent that the conjugacy fails for points in $f_{\lambda,\pi}^{-1}(I^{(35)})$.

A visual representation of this result can be found in Figure 4.3. Throughout the rest of this section we prove several lemmas that will later be used in the proof of Theorem 4.2.1 in Section 4.2.1.

Our next lemma is a particular case of Theorem 4.2.1 where $n \geq 1$ and $m = n-1$. We study separately the cases $\varepsilon(m-1) = 0$ and the cases $\varepsilon(m-1) = 1$ as it can be seen from (4.1.12) and (4.1.13) that the expressions for $T_\alpha^{(n,m)}$ are different in these two cases.

Lemma 4.2.2 *Let $n \geq 1$ and $\alpha \in \mathcal{A}$. Then $\gamma^{(n)}$ is a quasi-embedding of $f_{\lambda^{(n-1)},\pi^{(n-1)}}$ into $T^{(n,n-1)}$ for $x \in I^{(n-1)} \setminus f^{-1}_{\lambda^{(n-1)},\pi^{(n-1)}}\left(I^{(n)}\right)$. Furthermore for all $z \in \mathbb{C}$ we have*

$$T_\alpha^{(n,n-1)}(z) = \hat{T}_\alpha^{(n,n-1)}(z). \tag{4.2.3}$$

Proof.

We distinguish the cases $\varepsilon(n-1)=0$ and $\varepsilon(n-1)=1$.

Given $n \geq 1$ assume $\varepsilon(n-1)=0$. Lemma 4.1.3 for $m=n$ combined with (4.1.12) gives

$$T^{(n,n-1)}_{\beta_{1,n-1}}(z) = e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\left(z-\gamma^{n,n-1}_{0,\hat{\pi}^{(n-1)}(d)-1}\right)+\gamma^{n,n-1}_{1,d-1}. \tag{4.2.4}$$

for all $z \in \mathbb{C}$.

By Lemma 4.1.7, $J^{(n)} = (I^{(n-1)}\backslash I^{(n)})$. Let $x \in I^{(n-1)}_{\beta_{1,n-1}}\backslash f^{-1}_{\lambda^{(n-1)},\pi^{(n-1)}}(I^{(n)})$. Since we have $f_{\lambda^{(n-1)},\pi^{(n-1)}}(x) \in I^{(n-1)}\backslash I^{(n)}$, it follows from our definitions of breaking operator and breaking sequence that

$$\gamma^{(n)}(f_{\lambda^{(n-1)},\pi^{(n-1)}}(x)) = f_{\lambda^{(n-1)},\pi^{(n-1)}}(x)e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}+|I^{(n)}|(1-e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}), \tag{4.2.5}$$

since $x^{(n-1)}_{1,d-1}=|I^{(n)}|$, $\gamma^{n,n-1}_{0,\hat{\pi}^{(n-1)}(d)-1}=x^{(n-1)}_{0,\hat{\pi}^{(n-1)}(d)-1}$ as $\gamma^{(n)}(x)=x$ and $|I^{(n)}|=\gamma^{n,n-1}_{1,d-1}$, this gives

$$\gamma^{(n)}(f_{\lambda^{(n-1)},\pi^{(n-1)}}(x))-e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\gamma^{(n)}(x) = \gamma^{n,n-1}_{1,d-1}-e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\gamma^{n,n-1}_{0,\hat{\pi}^{(n-1)}(d)-1}. \tag{4.2.6}$$

As $I^{(n-1)}_{\alpha}\backslash f^{-1}_{\lambda^{(n-1)},\pi^{(n-1)}}(I^{(n)})=\emptyset$ for $\alpha \neq \beta_{1,n-1}$, (4.2.4) together with (4.2.6) gives

$$T^{(n,n-1)}(\gamma^{(n)}(x)) = \gamma^{(n)}(f_{\lambda^{(n-1)},\pi^{(n-1)}}(x)),$$

which proves that the map $\gamma^{(n)}$ is a quasi-embedding of $f_{\lambda^{(n-1)},\pi^{(n-1)}}$ into $T^{(n,n-1)}$ for $x \in I^{(n-1)}_{\alpha}\backslash f^{-1}_{\lambda^{(n-1)},\pi^{(n-1)}}\left(I^{(n)}\right)$.

By continuity of $f_{\lambda^{(n-1)},\pi^{(n-1)}}$ in $I^{(n-1)}_{\beta_{1,n-1}} = [x^{(n-1)}_{0,\hat{\pi}^{(n-1)}(d)-1}, x^{(n-1)}_{0,\hat{\pi}^{(n-1)}(d)})$ and from (4.2.6) we get

$$\gamma^{n,n-1}_{1,d-1}-e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\gamma^{n,n-1}_{0,\hat{\pi}^{(n-1)}(d)-1}=\gamma^{n,n-1}_{1,d}-e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\gamma^{n,n-1}_{0,\hat{\pi}^{(n-1)}(d)},$$

which combined with (4.2.4), (4.1.9) and (4.1.10) gives (4.2.3) for $\alpha=\beta_{1,n-1}$.

Since $\xi^{n,n-1}_{d-1} = \gamma^{n,n-1}_{1,d-1}$, by Lemma 4.1.4 we prove the second statement in our lemma for all $\alpha \in \mathcal{A}$.

Now assume $\varepsilon(n-1)=1$. It follows directly from our definitions of $\hat{T}^{(n,m)}_{\alpha}(z)$, $T^{(n,n)}_{\alpha}(z)$ and $\xi^{n,m}_{j}$ using (4.1.13) that

$$T^{(n,n-1)}_{\beta_{1,n-1}}(z) = e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\left(z-\gamma^{n,n}_{0,\hat{\pi}^{(n)}(d)}\right)+\gamma^{n,n}_{1,d}, \tag{4.2.7}$$

for all $z \in \mathbb{C}$. Again, in this case we also have $J^{(n)} = \{I^{(n-1)}\backslash I^{(n)}\}$ and we can use (4.2.5) as before which since $\theta^{(n-1)}_{\beta_{1,n-1}} = \theta^{(n)}_{\beta_{1,n}}$, $\gamma^{n,n}_{1,d} = x^{(n)}_{1,d}$ and $\gamma^{n,n}_{0,\hat{\pi}^{(n)}(d)} = x^{(n)}_{0,\hat{\pi}^{(n)}(d)}$, gives

$$\gamma^{(n)}(f_{\lambda^{(n-1)},\pi^{(n-1)}}(x))-e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\gamma^{(n)}(x) = \gamma^{n,n}_{1,d}-e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\gamma^{n,n}_{0,\hat{\pi}^{(n)}(d)}, \tag{4.2.8}$$

for all $x \in I^{(n-1)}_{\beta_{1,n-1}} \backslash f^{-1}_{\lambda^{(n-1)},\pi^{(n-1)}}(I^{(n)})$. As $I^{(n-1)}_{\alpha} \backslash f^{-1}_{\lambda^{(n-1)},\pi^{(n-1)}}(I^{(n)}) = \emptyset$ for $\alpha \neq \beta_{1,n-1}$, combining (4.2.7) and (4.2.8) we prove the first statement in the lemma.

By continuity of $f_{\lambda^{(n-1)},\pi^{(n-1)}}$ in $I^{(n-1)}_{\beta_{1,n-1}} = [x^{(n-1)}_{0,\hat{\pi}^{(n-1)}(d)-1}, x^{n-1}_{0,\hat{\pi}^{(n-1)}(d)})$ and from (4.2.8) we can relate the image by $\gamma^{(n)}$ of the d-th endpoint of the partitions associated to $f_{\lambda^{(n-1)},\pi^{(n-1)}}$ and $f_{\lambda^{(n)},\pi^{(n)}}$ as follows

$$\gamma^{n,n-1}_{1,d} - e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\gamma^{n,n-1}_{0,\hat{\pi}^{(n-1)}(d)} = \gamma^{n,n}_{1,d} - e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\gamma^{n,n}_{0,\hat{\pi}^{(n)}(d)}.$$

As $\gamma^{n,n-1}_{1,d} = \xi^{n,n-1}_{d}$, this together with (4.2.7) and (4.1.10), proves (4.2.3) for $\alpha = \beta_{1,n-1}$. Using the definition of $\xi^{n,m}_{j}$ this can be rewritten as

$$\xi^{n,n-1}_{d-1} = e^{i\theta^{(n-1)}_{\beta_{1,n-1}}}\left(\gamma^{n,n-1}_{0,\hat{\pi}^{(n-1)}(d)-1} - \gamma^{n,n}_{0,\hat{\pi}^{(n)}(d)}\right) + \xi^{n,n}_{d},$$

and since $\gamma^{n,n-1}_{0,\hat{\pi}^{(n-1)}(d)-1} = \gamma^{n,n}_{0,\hat{\pi}^{(n)}(d)-1}$, by (4.1.9) and (4.1.18) we get that $\xi^{n,n-1}_{d-1} = \xi^{n,n}_{d-1}$. Hence by Lemma 4.1.5, (4.1.10), (4.1.11) and (4.1.13) we prove the second statement in the lemma for all $\alpha \in \mathcal{A}$.

□

Recall we denote by $J^{(n+1)}$ the ordering of $\left\{f^{k}_{\lambda,\pi}\left(I^{(n)}\backslash I^{(n+1)}\right)\right\}_{0\leq k<r(n)}$, where $r(n) = r^{n}_{\lambda,\pi}\left(I^{(n)}_{\beta_{0,n}}\right)$. Given $0 < m \leq n+1$, by Lemma 4.1.7 there exist $0 < k(m) < k(m-1)$ such that

$$J^{(n+1)} \cap (I^{(m-1)}\backslash I^{(m)}) = (J_k)_{k(m)\leq k<k(m-1)},$$

and there exists $k'(m) > 0$ such that

$$J_k = f_{\lambda^{(m-1)},\pi^{(m-1)}}(J_{k-k'(m)}).$$

In particular we have the following relations

$$[x^{(m)}_{0,d}, y_{k(m)}) = f_{\lambda^{(m-1)},\pi^{(m-1)}}\left([f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x^{(m)}_{0,d}), y_{k(m)-k'(m)})\right), \tag{4.2.9}$$

$$[y_{k(m+1)-1} + \Delta, x^{(m-1)}_{0,d}) = f_{\lambda^{(m-1)},\pi^{(m-1)}}\left([y_{k(m+1)-1-k'(m)}, x^{(m-1)}_{0,\hat{\pi}^{(m-1)}(d)})\right), \tag{4.2.10}$$

recalling we denote $J_k = [y_k, y_k + \Delta)$, for all $k(m) \leq k < k(m+1)$ we have

$$J_k = f_{\lambda^{(m-1)},\pi^{(m-1)}}\left([y_{k(m)-k'(m)}, y_{k(m)-k'(m)} + \Delta)\right), \tag{4.2.11}$$

and denoting $J'_k = [y_k + \Delta, y_{k+1})$, for all $k(m) \leq k < k(m+1) - 1$ we get

$$J'_k = f_{\lambda^{(m-1)},\pi^{(m-1)}}\left([y_{k(m)-k'(m)} + \Delta, y_{k(m)+1-k'(m)})\right). \tag{4.2.12}$$

With the assumptions that $T^{(n,m-1)}_{\alpha} = \hat{T}^{(n,m-1)}_{\alpha}$ and that $\gamma^{(n)}$ and $\gamma^{(n+1)}$ are quasi-embeddings of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ respectively into $T^{(n,m-1)}$ for all $x \in I^{(m-1)}\backslash$

$f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}\left(I^{(n)}\right)$ and into $T^{(n+1,m-1)}$ for some point in $I^{(m)}_{\beta_{1,m-1}}$, in the next two lemmas we extend the quasi-embedding $\gamma^{(n+1)}$ for points in a larger subinterval of $I^{(m)}_{\beta_{1,m-1}}$. With the hypothesis that (4.2.1) holds for some $y \in J'_{k-k'(m)-1}$, the first lemma extends this quasi-embedding for points $x \in J'_{k-k'(m)-1}$ such that $x > y$, while the second provides a similar extension for points in $J_{k-k'(m)}$.

Lemma 4.2.3 *Given $n \geq 0$ and $0 < m \leq n+1$ assume that $\gamma^{(n)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n,m-1)}$ for $x \in I^{(m-1)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}\left(I^{(n)}\right)$, that for all $\alpha \in \mathcal{A}$ and $z \in \mathbb{C}$,*

$$T^{(n,m-1)}_{\alpha}(z) = \hat{T}^{(n,m-1)}_{\alpha}(z). \tag{4.2.13}$$

and that with $\hat{x} = f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x^{(m)}_{0,d})$ we have

$$T^{(n+1,m-1)}_{\beta_{1,m-1}}(z) = e^{i\theta^{(m-1)}_{\beta_{1,m-1}}}(z - \gamma^{(n+1)}(\hat{x})) + \gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(\hat{x})). \tag{4.2.14}$$

Furthermore for $k(m) \leq k \leq k(m+1)$ assume that $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n+1,m-1)}$ for $y \in I^{(m-1)}_{\beta_{1,m-1}} \cap J'_{k-k'(m)-1}$.

If $y_k < x^{(m-1)}_{0,d}$, then $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n+1,m-1)}$ for all $x \in [y, y_{k-k'(m)}]$. If $y_k \geq x^{(m-1)}_{0,d}$ then $\gamma^{(n+1)}$ is a quasi-embedding for $x \in [y, x^{(m-1)}_{0,d})$.

Proof. As $x \in I^{(m-1)}_{\beta_{1,m-1}}$, by (4.2.9), (4.2.10) and (4.2.12) we have $f_{\lambda^{(m-1)},\pi^{(m-1)}}(x) \in [f_{\lambda^{(m-1)},\pi^{(m-1)}}(y), y_k]$, thus by (4.1.1), (4.1.7) and continuity of $\gamma^{(n+1)}$ we get

$$\gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) = \gamma^{(n)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) + \underline{\epsilon}_{k-1}.$$

Since $\gamma^{(n)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n,m-1)}$ for $x \in [y, y_{k-k'(m)}]$ we have

$$\gamma^{(n)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) = T^{(n,m-1)}(\gamma^{(n)}(x)).$$

Combining these two formulas and using (4.2.13) we obtain

$$\gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) = e^{i\theta^{(m-1)}_{\beta_{1,m-1}}}(\gamma^{(n)}(x) - \gamma^{(n)}(y)) + \gamma^{(n)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(y)) + \underline{\epsilon}_{k-1}.$$

Finally, using the definitions of breaking operator and breaking sequence one gets

$$\gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) = e^{i\theta^{(m-1)}_{\beta_{1,m-1}}}(\gamma^{(n+1)}(x) - \gamma^{(n+1)}(y)) + \gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(y)). \tag{4.2.15}$$

Since $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n+1,m-1)}$ for $y \in I^{(m-1)}_{\beta_{1,m-1}} \cap J'_{k-k'(m)-1}$ we have

$$\gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(y)) = T^{(n+1,m-1)}(\gamma^{(n+1)}(y)),$$

which combined with (4.2.14) gives that for any $z \in \mathbb{C}$,

$$T^{(n+1,m-1)}_{\beta_{1,m-1}}(z) = e^{i\theta^{(m-1)}_{\beta_{1,m-1}}}(z - \gamma^{(n+1)}(y)) + \gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(y)).$$

Combined with (4.2.15), we get

$$\gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) = T^{(n+1,m-1)}(\gamma^{(n+1)}(x)). \tag{4.2.16}$$

for all $x \in [y, y_{k-k'(m)}]$ and therefore $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n+1,m-1)}$ in this interval. Moreover, it can be proved in a similar way that if $y_k \geq x^{(m-1)}_{0,d}$, then (4.2.16) holds for all $x \in [y, x^{(m-1)}_{0,d})$. □

Lemma 4.2.4 *Given $n \geq 0$ and $0 < m \leq n+1$ assume that $\gamma^{(n)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n,m-1)}$ for $x \in I^{(m-1)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}\left(I^{(n)}\right)$, that for all $\alpha \in \mathcal{A}$, $z \in \mathbb{C}$ we have (4.2.13), and that with $\hat{x} = f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x^{(m)}_{0,d})$ we have (4.2.14).*

Furthermore for $k(m) \leq k \leq k(m+1)-1$ assume that $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n+1,m-1)}$ for $y \in J_{k-k'(m)}$.

If $y_k + \Delta \neq x^{(m-1)}_{0,d}$ then $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n+1,m-1)}$ for $x \in [y, y_{k-k'(m)} + \Delta]$. If $y_k + \Delta = x^{(m-1)}_{0,d}$, then $\gamma^{(n+1)}$ is a quasi-embedding for $x \in [y, x^{(m-1)}_{0,d})$.

Proof.

Assume first that $y_k + \Delta \neq x^{(m-1)}_{0,d}$ and take $x \in [y, y_{k-k'(m)} + \Delta]$. As $(I^{(m-1)} \backslash I^{(m)}) \cap J_k \neq \emptyset$, by Lemma 4.1.6 we must have $y_k + \Delta < x^{(m-1)}_{0,d}$ and thus $x \in I^{(m-1)}_{\beta_{1,m-1}}$. By (4.2.11) we have $f_{\lambda^{(m-1)},\pi^{(m-1)}}(x) \in [f_{\lambda^{(m-1)},\pi^{(m-1)}}(y), y_k + \Delta]$, hence by (4.1.1), (4.1.7) and continuity of $\gamma^{(n+1)}$ we get

$$\gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) = \gamma^{(n)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) e^{i\theta^{(n)}_{\beta_{1,n}}} + \bar{\epsilon}_k.$$

As $[y, y_{k-k'(m)} + \Delta] \subseteq I^{(m-1)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}\left(I^{(n)}\right)$, we have that $\gamma^{(n)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n,m-1)}$ for $x \in [y, y_{k-k'(m)} + \Delta]$ from whence we have

$$\gamma^{(n)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) = T^{(n,m-1)}(\gamma^{(n)}(x)).$$

Combining these two formulas and using (4.2.13) we obtain

$$\gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) =$$
$$\left[e^{i\theta^{(m-1)}_{\beta_{1,m-1}}}(\gamma^{(n)}(x) - \gamma^{(n)}(y)) + \gamma^{(n)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(y))\right] e^{i\theta^{(n)}_{\beta_{1,n}}} + \bar{\epsilon}_k.$$

As before, using the definitions of breaking operator and breaking sequence one gets (4.2.15). We omit the conclusion of the proof as it is completely analogous to that of Lemma 4.2.3. □

4.2.1 Proof of Theorem 4.2.1

We now prove Theorem 4.2.1. The proof is structured as follows. The theorem holds trivially in the case $n \geq 0$ and from Lemma 4.2.2 in the case $n \geq 1$ and $m = n-1$. Next we assume, by induction on m, that given a fixed $n \geq 1$, the theorem is true for $T^{(n,m)}$, with $0 \leq m \leq n$ and also for $T^{(n+1,m)}$, with $0 < m \leq n+1$ and we prove it for $T^{(n+1,m-1)}$.

We prove that $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ is quasi-embedded, by $\gamma^{(n+1)}$, into $T^{(n+1,m-1)}$ in $I^{(m-1)}_{\beta_{1,m-1}} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$ by induction in k, considering separate subintervals of $J^{(n+1)}$. In particular, we apply Lemmas 4.2.3 and 4.2.4 in an alternate way to extend the quasi-embedding throughout the interval. It follows that our theorem is true for $x \in I^{(m-1)}_{\beta_{1,m-1}} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$.

To prove it is true for $I^{(m-1)}_{\alpha} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$, with $\alpha \neq \beta_{1,m-1}$, we separate the two cases $\varepsilon(m-1) = 0$ and $\varepsilon(m-1) = 1$.

Proof.

Both statements in our theorem are trivial to prove for $n \geq 0$ and $m = n$, as $I^{(m)}_{\alpha} \backslash f^{-1}_{\lambda^{(m)},\pi^{(m)}}\left(I^{(n)}\right) = \emptyset$. For $m = n-1$, both statements follow directly from Lemma 4.2.2.

Given $n \geq 0$, we now assume the following.

(H1). For all $0 \leq m' \leq n$ and $\alpha \in \mathcal{A}$ that $\gamma^{(n)}$ is a quasi-embedding of $f_{\lambda^{(m')},\pi^{(m')}}$ into $T^{(n,m')}$ for $x \in I^{(m')} \backslash f^{-1}_{\lambda^{(m')},\pi^{(m')}}\left(I^{(n)}\right)$, and that for all $z \in \mathbb{C}$,

$$T^{(n,m')}_{\alpha}(z) = \hat{T}^{(n,m')}_{\alpha}(z).$$

(H2). Given $0 < m \leq n+1$, we also assume that for all $\alpha \in \mathcal{A}$ that $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m)},\pi^{(m)}}$ into $T^{(n+1,m)}$ for $x \in I^{(m)} \backslash f^{-1}_{\lambda^{(m)},\pi^{(m)}}\left(I^{(n+1)}\right)$, and that for $z \in \mathbb{C}$,

$$T^{(n+1,m)}_{\alpha}(z) = \hat{T}^{(n+1,m)}_{\alpha}(z). \tag{4.2.17}$$

We need to relate the breaking sequence at the $(m-1)$-step of the Rauzy induction with our map $T^{(n+1,m-1)}_{\alpha}$.

Case 1. Fix $\alpha = \beta_{1,m-1}$. The Rauzy induction is either of type 1 or type 0 and we have $\beta_{\epsilon,m} = (\pi^{(m)}_{\epsilon})^{-1}(d)$. We prove now that $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n+1,m-1)}$ for all $x \in I^{(m-1)}_{\alpha} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$ that is

$$\gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)) = T^{(n+1,m-1)}(\gamma^{(n+1)}(x)). \tag{4.2.18}$$

Step 1. We begin by showing that we have (4.2.14), with $\hat{x} = f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x^{(m)}_{0,d})$. Assume first that $\varepsilon(m-1) = 0$. From (4.1.12) and (4.2.17), we have

$$T^{(n+1,m-1)}_{\alpha}(z) = \left(\hat{T}^{(n+1,m)}_{\beta_{0,m-1}}\right)^{-1} \circ \hat{T}^{(n+1,m)}_{\alpha}(z),$$

for all $z \in \mathbb{C}$. By definition of $T^{(n+1,m-1)}_{\alpha}$ and Lemma 4.1.3 we get (4.2.14).

Assume now that $\varepsilon(m-1) = 1$. In this case, we have $f_{\lambda^{(m-1)},\pi^{(m-1)}}(x) = f_{\lambda^{(m)},\pi^{(m)}}(x)$ for $x \in I^{(m)}_{\alpha}$. In particular, if $x \in I^{(m)}_{\alpha} \backslash f^{-1}_{\lambda^{(m)},\pi^{(m)}}(I^{(n+1)})$, then $x \in I^{(m)}_{\alpha} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$ as well.

By (H2) and (4.1.18) we get

$$\xi^{n+1,m}_{\pi^{(m)}_1(\alpha)} - e^{i\theta^{(m)}_{\alpha}} \gamma^{n+1,m}_{0,\pi^{(m)}_0(\alpha)} = \gamma^{(n+1)}\left(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)\right) - e^{i\theta^{(m-1)}_{\alpha}} \gamma^{(n+1)}(x), \quad (4.2.19)$$

for all $x \in I^{(m)}_{\alpha} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$. By (4.1.13) and (4.2.17) we have $T^{(n+1,m-1)}_{\alpha} = \hat{T}^{(n+1,m)}_{\alpha}$, hence by (4.2.19), (4.1.10) we get (4.2.18) for $x \in I^{(m)}_{\alpha} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$.

Since $\gamma^{(n+1)}$ is a continuous map and $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ is continuous at $\hat{x}$, we get (4.2.18) for $x = \hat{x}$. Since $T^{(n+1,m-1)}_{\alpha} = \hat{T}^{(n+1,m)}_{\alpha}$, (4.2.14) holds as well.

Step 2. Recall we denote by $J^{(n+1)}$ the ordering of $\left\{f^k_{\lambda,\pi}\left(I^{(n)} \backslash I^{(n+1)}\right)\right\}_{0 \le k < r(n)}$ and that we have the relations (4.2.9)-(4.2.12).

By Lemma 4.1.6, $J_{k(m)-1} \subseteq I^{(m)}$ and $J_{k(m)} \subseteq I^{(m-1)} \backslash I^{(m)}$. Thus, either $y_{k(m)-1} + \Delta \le x^{(m)}_{0,d} < y_{k(m)}$ or $x^{(m)}_{0,d} = y_{k(m)}$.

Assuming first that $y_{k(m)-1} + \Delta \le x^{(m)}_{0,d} < y_{k(m)}$, from (4.2.14) we get that $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{n+1,m-1}$ for $y = f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x^{(m)}_{0,d})$, that is

$$\gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(y)) = T^{(n+1,m-1)}(\gamma^{(n+1)}(y)). \quad (4.2.20)$$

Since we are assuming (H1) we can apply Lemma 4.2.3, and thus we have (4.2.18) either for all $x \in I^{(m-1)}_{\alpha}$ if $y_{k(m)} = x^{(m-1)}_{0,d}$, or for all $x \in [f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x^{(m)}_{0,d}), y_{k(m)-k'(m)}]$ if $y_{k(m)} < x^{(m-1)}_{0,d}$. In particular we have (4.2.20) with $y = y_{k(m)-k'(m)}$.

Now assume that $x^{(m)}_{0,d} = y_{k(m)}$. By (4.2.14) we also have (4.2.20) with $y = y_{k(m)-k'(m)}$. Therefore by Lemma 4.2.3 we have (4.2.18) either for all $x \in I^{(m-1)}_{\alpha}$ if $y_{k(m)} + \Delta = x^{(m-1)}_{0,d}$, or for all $x \in [f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x^{(m)}_{0,d}), y_{k(m)-k'(m)} + \Delta]$ if $y_{k(m)} + \Delta < x^{(m-1)}_{0,d}$.

Step 3. Now assume, by induction on k, for $k(m)+1 \le k \le k(m+1)$, and with $y_{k-1} + \Delta < x^{(m-1)}_{0,d}$, that $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{n+1,m-1}$ for all $x \in [f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x^{(m)}_{0,d}), y_{k-k'(m)-1} + \Delta]$. In particular we have (4.2.20) with $y = y_{k-k'(m)-1} + \Delta$. Thus by Lemma 4.2.3 we have (4.2.18) either for all $x \in I^{(m-1)}_{\alpha}$ if $y_k \ge$

$x_{0,d}^{(m-1)}$, or for all $x \in [f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x_{0,d}^{(m)}), y_{k-k'(m)}]$ if $y_k < x_{0,d}^{(m-1)}$. In particular we get that $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{n+1,m-1}$ for $y = y_{k-k'(m)}$. Since we are assuming (H1) we can apply Lemma 4.2.4 and thus we have (4.2.18) either for all $x \in I_\alpha^{(m-1)}$ if $y_k + \Delta = x_{0,d}^{(m-1)}$, or for all $x \in [f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x_{0,d}^{(m)}), y_{k-k'(m)} + \Delta]$ if $y_k \neq x_{0,d}^{(m-1)}$. Since $f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}([x_{0,d}^{(m)}, x_{0,d}^{(m-1)})) = I_\alpha^{(m-1)} \cap f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I_{\beta_{0,m-1}}^{(m-1)})$, this shows that we have (4.2.18) for all $x \in I_\alpha^{(m-1)} \cap f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I_{\beta_{0,m-1}}^{(m-1)})$. In particular if $\varepsilon(m-1) = 0$, this shows that $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{n+1,m-1}$ for all $x \in I_\alpha^{(m-1)}$. If $\varepsilon(m-1) = 1$, since $f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I_{\beta_{0,m-1}}^{(m-1)}) = I_\alpha^{(m-1)} \backslash I_\alpha^{(m)}$ and we already proved that (4.2.18) holds for all $x \in I_\alpha^{(m)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$, this shows that it is true for all $x \in I_\alpha^{(m-1)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$.

Step 4. Combining (4.2.18) and (4.2.14), for any $x \in I_\alpha^{(m-1)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$ and $z \in \mathbb{C}$ replacing $x = x_{0,\hat{\pi}^{(m-1)}(d)}^{(m-1)} - \delta$ and taking $\delta \to 0^+$, we get

$$T_\alpha^{(n+1,m-1)}(z) = e^{i\theta_\alpha^{(m-1)}}\left(z - \gamma_{0,\hat{\pi}^{(m-1)}(d)}^{n+1,m-1}\right) + \gamma_{0,d}^{n+1,m-1}, \qquad (4.2.21)$$

and this can be written as

$$T_{\beta_{1,m-1}}^{(n+1,m-1)}(z) = \hat{T}_{\beta_{1,m-1}}^{(n+1,m-1)}(z).$$

In the next cases we establish a relation between $T_\alpha^{(n+1,m-1)}$, when $\alpha \neq \beta_{1,m-1}$ and the breaking sequence at the step $n+1$.

Note first that since we are assuming that $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m)},\pi^{(m)}}$ into $T^{(n+1,m)}$ for $x \in I_\alpha^{(m)} \backslash f^{-1}_{\lambda^{(m)},\pi^{(m)}}\left(I^{(n+1)}\right)$ it follows that for these values of x we have

$$T^{(n+1,m)}(\gamma^{(n+1)}(x)) = \gamma^{(n+1)}(f_{\lambda^{(m)},\pi^{(m)}}(x)). \qquad (4.2.22)$$

Case 2. Set $\alpha \neq \beta_{1,m-1}$ and $\varepsilon(m-1) = 0$. Since $f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x_{0,d}^{(m)}) = x_{0,\hat{\pi}^{(m-1)}(d)-1}^{(m-1)}$, by (4.2.14) we get

$$T_{\beta_{1,m-1}}^{(n+1,m-1)}(z) = e^{i\theta_{\beta_{1,m-1}}^{(m-1)}}\left(z - \gamma_{0,\hat{\pi}^{(m-1)}(d)-1}^{n+1,m-1}\right) + \gamma_{0,d-1}^{n+1,m-1},$$

which by (4.2.21) and (4.1.9) shows that $\xi_{d-1}^{n+1,m-1} = \gamma_{1,d-1}^{n+1,m-1}$. Hence by Lemma 4.1.4 we get that $T_\alpha^{(n+1,m-1)} = \hat{T}_\alpha^{(n+1,m-1)}$.

By (4.1.12) and (4.2.17) we get that $T_\alpha^{(n+1,m-1)} = T_\alpha^{(n+1,m)}$ and by (4.2.22) we get

$$T_\alpha^{(n+1,m-1)}(\gamma^{(n+1)}(x)) = \gamma^{(n+1)}(f_{\lambda^{(m)},\pi^{(m)}}(x)),$$

for $x \in I_\alpha^{(m)} \backslash f^{-1}_{\lambda^{(m)},\pi^{(m)}}(I^{(n+1)})$. Since $f_{\lambda^{(m-1)},\pi^{(m-1)}}(x) = f_{\lambda^{(m)},\pi^{(m)}}(x)$, for $x \in I_\alpha^{(m)}$, we get

$$T_\alpha^{(n+1,m-1)}(\gamma^{(n+1)}(x)) = \gamma^{(n+1)}(f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)), \qquad (4.2.23)$$

for all $x \in I_\alpha^{(m)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$. In particular, for $\alpha \neq \beta_{0,m-1}$ we get (4.2.23) for $x \in I_\alpha^{(m-1)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$.

Now take $\alpha = \beta_{0,m-1}$ and $x \in (I_\alpha^{(m-1)} \backslash I_\alpha^{(m)}) \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$.

Since $f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x) \in I^{(m-1)}_{\beta_{1,m-1}}$, we get by (4.2.18),

$$\gamma^{n+1}(x) = T^{(n+1,m-1)}_{\beta_{1,m-1}}(\gamma^{(n+1)}(f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x)))$$

and since $x \in I_\alpha^{(m-1)}$, by (4.1.12) this gives

$$T_\alpha^{(n+1,m-1)}(\gamma^{(n+1)}(x)) = T^{(n+1,m)}_{\beta_{1,m-1}}(\gamma^{(n+1)}(f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x))).$$

As $I^{(m)}_{\beta_{1,m-1}} = I^{(m-1)}_{\beta_{1,m-1}}$ and $f^2_{\lambda^{(m-1)},\pi^{(m-1)}}(x') = f_{\lambda^{(m)},\pi^{(m)}}(x')$, for $x' \in I_{\beta_{1,m-1}}$, we get that $f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x) \in I^{(m-1)}_{\beta_{1,m-1}} \backslash f^{-1}_{\lambda^{(m)},\pi^{(m)}}(I^{(n+1)})$ and $f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x) = f^{-1}_{\lambda^{(m)},\pi^{(m)}} \circ f_{\lambda^{(m-1)},\pi^{(m-1)}}(x)$, thus by (4.2.22) we get (4.2.23) for $x \in I_\alpha^{(m-1)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$.

Case 3. Now assume $\varepsilon(m-1) = 1$ and $\alpha \neq \beta_{1,m-1}$. By (4.1.13) and (4.2.17) we have $T^{(n+1,m-1)}_{\beta_{1,m-1}} = \hat{T}^{(n+1,m)}_{\beta_{1,m-1}}$, and combining this with (4.1.18) and (4.1.10) we get

$$T^{(n+1,m-1)}_{\beta_{1,m-1}}(z) = e^{i\theta^{(m-1)}_{\beta_{1,m-1}}}(z - \gamma^{n+1,m}_{0,\hat{\pi}^{(m)}(d)}) + \xi_d^{n+1,m},$$

for any $z \in \mathbb{C}$. As $\gamma^{n+1,m-1}_{1,d} = \xi^{n+1,m-1}_d$, from (4.2.21) we get

$$\xi_d^{n+1,m-1} - e^{i\theta^{(m-1)}_{\beta_{1,m-1}}}\gamma^{n+1,m-1}_{0,\hat{\pi}^{(m-1)}(d)} = \xi_d^{n+1,m} - e^{i\theta^{(m-1)}_{\beta_{1,m-1}}}\gamma^{n+1,m}_{0,\hat{\pi}^{(m-1)}(d)},$$

which by (4.1.9) with $j = d-1$, gives

$$\xi^{n+1,m-1}_{d-1} = e^{i\theta^{(m-1)}_{\beta_{1,m-1}}}\left(\gamma^{n+1,m-1}_{0,\hat{\pi}^{(m-1)}(d)-1} - \gamma^{n+1,m}_{0,\hat{\pi}^{(m-1)}(d)}\right) + \xi_d^{n+1,m}.$$

Recalling (4.1.29) we have $\gamma^{n+1,m-1}_{0,\hat{\pi}^{(m-1)}(d)-1} = \gamma^{n+1,m}_{0,\hat{\pi}^{(m)}(d)-1}$ and again by (4.1.9) we get that $\xi^{n+1,m-1}_{d-1} = \xi^{n+1,m}_{d-1}$. Thus, by Lemma 4.1.5, (4.1.10),(4.1.11) and (4.1.13) we obtain $T_\alpha^{(n+1,m-1)} = \hat{T}_\alpha^{(n+1,m-1)}$.

By a reasoning analogous to the case $\varepsilon(m-1) = 0$, we have that (4.2.23) is true for all $x \in I_\alpha^{(m)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$. In particular, for $\alpha \neq \beta_{0,m-1}$ we get (4.2.23) for all $x \in I_\alpha^{(m-1)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$.

Now take $\alpha = \beta_{0,m-1}$ and $x \in I_\alpha^{(m-1)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$.

Since $f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x) \in I^{(m-1)}_{\beta_{1,m-1}}$, we get by (4.2.18) and (4.1.13),

$$\gamma^{n+1}(x) = T^{(n+1,m-1)}_{\beta_{1,m-1}}(\gamma^{(n+1)}(f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x)))$$

and since $x \in I^{(m-1)}_{\alpha}$, by (4.1.13) this gives

$$T^{(n+1,m-1)}_{\alpha}(\gamma^{(n+1)}(x)) = T^{(n+1,m)}_{\alpha}(\gamma^{(n+1)}(f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x))).$$

As $f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(m-1)}_{\alpha}) = I^{(m-1)}_{\alpha}$ and $f^{2}_{\lambda^{(m-1)},\pi^{(m-1)}}(x') = f_{\lambda^{(m)},\pi^{(m)}}(x')$, for $x' \in I_{\alpha}$, we get that $f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x) \in I^{(m-1)}_{\alpha} \backslash f^{-1}_{\lambda^{(m)},\pi^{(m)}}(I^{(n+1)})$ and that

$$f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(x) = f^{-1}_{\lambda^{(m)},\pi^{(m)}} \circ f_{\lambda^{(m-1)},\pi^{(m-1)}}(x),$$

thus by (4.2.22) we get (4.2.23) for $x \in I^{(m-1)}_{\alpha} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}(I^{(n+1)})$.

Conclusion. We proved that for all $z \in \mathbb{C}$,

$$T^{(n+1,m-1)}_{\alpha}(z) = \hat{T}^{(n+1,m-1)}_{\alpha}(z),$$

and from (4.2.23) we get for all $\alpha \in \mathcal{A}$ that $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m-1)},\pi^{(m-1)}}$ into $T^{(n+1,m-1)}$ for $x \in I^{(m-1)} \backslash f^{-1}_{\lambda^{(m-1)},\pi^{(m-1)}}\left(I^{(n+1)}\right)$. Thus for all $0 \leq m \leq n+1$ and $\alpha \in \mathcal{A}$ we have that (4.2.17) and (4.2.22) hold and therefore $\gamma^{(n+1)}$ is a quasi-embedding of $f_{\lambda^{(m)},\pi^{(m)}}$ into $T^{(n+1,m)}$ for $x \in I^{(m)} \backslash f^{-1}_{\lambda^{(m)},\pi^{(m)}}\left(I^{(n+1)}\right)$.

This shows that for all $n \geq 0$, $0 \leq m \leq n$ and $\alpha \in \mathcal{A}$ that $\gamma^{(n)}$ is a quasi-embedding of $f_{\lambda^{(m)},\pi^{(m)}}$ into $T^{(n,m)}$ for $x \in I^{(m)} \backslash f^{-1}_{\lambda^{(m)},\pi^{(m)}}\left(I^{(n)}\right)$ and for all $z \in \mathbb{C}$ we have (4.2.2). This finishes our proof.

□

4.3 Existence of embeddings of interval exchange transformations into piecewise isometries

In this section we prove the existence of non-trivial embeddings of IETs into PWIs. Recall the definition of the breaking sequence of curves $\gamma^{(n)}_{\theta}$ with $\theta \in \Theta_{\lambda,\pi}$, in (4.1.7). We introduce the family $\mathcal{F}_{\theta}$ of PWIs which are θ-adapted to an IET (λ, π) and show that when $\gamma^{(n)}_{\theta}$ converges to a topological embedding γ_{θ}, then the latter is an isometric embedding of $(I, f_{\lambda,\pi})$ into any θ-adapted PWI. We recall some classical notions of the theory of IETs, in particular the Zorich cocycle and the characterization of its Oseledets flags and associated Lyapunov spectrum, as well as the translation surface of genus $g(\mathfrak{R})$ associated to an IET.

We introduce a submanifold $W^{\delta}_{[\lambda],\pi}$ of the torus $\mathbb{T}^{\mathcal{A}}$ related to the Oseledets flags of the Zorich cocycle for the underlying IET and determine a bound for the sequence $\theta^{(n)}$ when $\theta \in W^{\delta}_{[\lambda],\pi}$. This result together with Theorem 4.3.1 are the key ingredients in the proof that for a full measure set of IETs, if $\theta \in W^{\delta}_{[\lambda],\pi}$, then $\gamma_\theta^{(n)}$ converges to a Lipschitz map γ_θ, which is an isometric embedding of $(I, f_{\lambda,\pi})$ into any θ-adapted PWI. The resulting embedding may, however, be trivial. Thus we define a submanifold $\mathcal{W}^{\delta}_{[\lambda],\pi} \subset W^{\delta}_{[\lambda],\pi}$ which we show, has full measure when $g(\mathfrak{R}) \geq 2$, for which the embedding γ_θ is guaranteed to be non-trivial.

Given $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A})$, recall we denote by $\Theta_{\lambda,\pi}$ the set of all $\theta \in \mathbb{T}^{\mathcal{A}}$ such that for all $n \geq 0$, $\gamma_\theta^{(n)} : I \to \mathbb{C}$ is an injective map. Let $\Theta'_{\lambda,\pi}$ denote the set of all $\theta \in \Theta_{\lambda,\pi}$ for which there exists a topological embedding $\gamma_\theta : I \to \mathbb{C}$ such that for all $x \in I$,

$$\gamma_\theta(x) = \lim_{n \to +\infty} \gamma_\theta^{(n)}(x).$$

Furthermore, given $\theta \in \Theta'_{\lambda,\pi}$, we say that a PWI $T : X \to X$ together with a partition $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is *θ-adapted to* (λ, π) if for all $\alpha \in \mathcal{A}$,

i) $X_\alpha \supseteq \gamma_\theta(I_\alpha)$;

ii) with $x_j = x^{(0)}_{0,j}$, and

$$T_\alpha(z) = e^{i\theta_\alpha} \left(z - \gamma_\theta(x_{\pi_0(\alpha)-1})\right) + \gamma_\theta \left(f_{\lambda,\pi}(x_{\pi_0(\alpha)-1})\right), \tag{4.3.1}$$

for all $z \in \mathbb{C}$, we have $T(z) = T_\alpha(z)$, for all $z \in X_\alpha$.

We denote the family of PWIs which are *θ-adapted to* (λ, π) by $\mathcal{F}_\theta$.

Recall that we say there is a *embedding* of an IET $(I, f_{\lambda,\pi})$ into a PWI (X, T) if there exists a topological embedding $\gamma : I \to \mathbb{C}$ such that for all $x \in I$,

$$\gamma \circ f_{\lambda,\pi}(x) = T \circ \gamma(x).$$

Given $x \in I$, consider a family $\Pi(x)$ of points $0 = t_0 < t_1 < ... < t_N = x$. Given $\theta \in \Theta'_{\lambda,\pi}$ define a map $\mathcal{L}_\theta : I \to \mathbb{R}_+$ by

$$\mathcal{L}_\theta(x) = \sup_{(t_0,\dots,t_N) \in \Pi(x)} \sum_{j=0}^{N-1} |\gamma_\theta(t_{j+1}) - \gamma_\theta(t_j)| \, .$$

We say a map γ_θ is an *isometric embedding* of an IET $(I, f_{\lambda,\pi})$ into a PWI (X, T) if it is an embedding and $\mathcal{L}_\theta(x) = x$ for all $x \in I$.

The following theorem states that when $\gamma_\theta^{(n)}$ converges to a topological embedding γ_θ it is also an isometric embedding of $(I, f_{\lambda,\pi})$ into any PWI which is θ-adapted to (λ, π). The proof follows from estimates related to the facts that the restriction of any PWI in $\mathcal{F}_\theta$ to $\gamma_\theta(I)$ can be approximated by the map $T^{(n,0)}$ with increasing precision as $n \to +\infty$, and that Theorem 4.2.1 guarantees that γ^n is a quasi-embedding of $f_{\lambda,\pi}$ into $T^{(n,0)}$ for points in $I \backslash f_{\lambda,\pi}^{-1}(I^{(n)})$ which implies that the conjugacy between these two maps only fails to hold for points in an interval which is decreasing with n.

Theorem 4.3.1 *Let $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A})$, $\theta \in \Theta'_{\lambda,\pi}$ and (X,T) be a PWI θ-adapted to (λ, π). Then γ_θ is an isometric embedding of $(I, f_{\lambda,\pi})$ into (X,T).*

Proof. For any map $g : I \to \mathbb{C}$ denote $\|g\|_\infty = \sup_{x \in I} |g(x)|$.

As $f_{\lambda,\pi}$ is a bijective map we have

$$\|\gamma_\theta \circ f_{\lambda,\pi} - \gamma_\theta^{(n)} \circ f_{\lambda,\pi}\|_\infty = \|\gamma_\theta - \gamma_\theta^{(n)}\|_\infty,$$

which as $\theta \in \Theta'_{\lambda,\pi}$, shows that

$$\lim_{n \to +\infty} \|\gamma_\theta \circ f_{\lambda,\pi} - \gamma_\theta^{(n)} \circ f_{\lambda,\pi}\|_\infty = 0. \tag{4.3.2}$$

From (4.1.10) and Theorem 4.2.1, for any $\alpha \in \mathcal{A}$ and $x \in I$ we have

$$T_\alpha^{(n,0)}(\gamma_\theta^{(n)}(x)) = e^{i\theta_\alpha} \left(\gamma_\theta^{(n)}(x) - \gamma_{0,\pi_0(\alpha)-1}^{n,0} \right) + \gamma_{1,\pi_1(\alpha)-1}^{n,0},$$

and by (4.3.1) applying the triangle inequality we get

$$\begin{aligned}&\|T_\alpha^{(n,0)} \circ \gamma_\theta^{(n)} - T_\alpha \circ \gamma_\theta\|_\infty \leq \\ &\|\gamma_\theta^{(n)} - \gamma_\theta\|_\infty + |\gamma_\theta(x_{\pi_0(\alpha)-1}) - \gamma_{0,\pi_0(\alpha)-1}^{n,0}| + |\gamma_{1,\pi_1(\alpha)-1}^{n,0} - \gamma_\theta\left(f_{\lambda,\pi}(x_{\pi_0(\alpha)-1})\right)|,\end{aligned}$$

which, as $\theta \in \Theta'_{\lambda,\pi}$, shows that

$$\lim_{n \to +\infty} \|T^{(n,0)} \circ \gamma_\theta^{(n)} - T \circ \gamma_\theta\|_\infty = 0. \tag{4.3.3}$$

By Theorem 4.2.1, $\gamma^{(n)}$ is a quasi-embedding of $f_{\lambda,\pi}$ into $T^{(n,0)}$ for all $x \in I \backslash f_{\lambda,\pi}^{-1}(I^{(n)})$ and thus we have

$$\gamma^{(n)}\left(f_{\lambda,\pi}(x)\right) = T^{(n,0)}\left(\gamma^{(n)}(x)\right),$$

in particular this gives

$$\|\gamma^{(n)} \circ f_{\lambda,\pi} - T^{(n,0)} \circ \gamma^{(n)}\|_\infty \leq \sup_{x \in f_{\lambda,\pi}^{-1}(I^{(n)})} \left|\gamma^{(n)}\left(f_{\lambda,\pi}(x)\right) - T^{(n,0)}\left(\gamma^{(n)}(x)\right)\right|.$$

For a sufficiently large $N > 0$ we have $f_{\lambda,\pi}^{-1}(I^{(n)}) \subseteq I_{\pi_1^{-1}(1)}$, whenever $n > N$.

As $T^{(n,0)}(\gamma_\theta^{(n)}(x^{(n)}_{\hat{\pi}^{(n)}(1)})) = x_1^{(n)} \in \overline{I^{(n)}}$ and since $T^{(n,0)}_{\pi_1^{-1}(1)}$ is an isometry, we get that

$$\sup_{x\in f^{-1}_{\lambda,\pi}(I^{(n)})} \left|T^{(n,0)}\left(\gamma^{(n)}(x)\right)\right| \leq 2|I^{(n)}|.$$

Since $\sup_{x\in f^{-1}_{\lambda,\pi}(I^{(n)})} \left|\gamma^{(n)}\left(f_{\lambda,\pi}(x)\right)\right| \leq |I^{(n)}|$ and $|I^{(n)}| \to 0$ as $n \to +\infty$, this shows that

$$\lim_{n\to+\infty} \|\gamma^{(n)} \circ f_{\lambda,\pi} - T^{(n,0)} \circ \gamma^{(n)}\|_\infty = 0. \tag{4.3.4}$$

By the triangle inequality we have

$$\|\gamma_\theta \circ f_{\lambda,\pi} - T \circ \gamma_\theta\|_\infty \leq$$
$$\|\gamma_\theta \circ f_{\lambda,\pi} - \gamma_\theta^{(n)} \circ f_{\lambda,\pi}\|_\infty + \|\gamma^{(n)} \circ f_{\lambda,\pi} - T^{(n,0)} \circ \gamma^{(n)}\|_\infty + \|T^{(n,0)} \circ \gamma_\theta^{(n)} - T \circ \gamma_\theta\|_\infty.$$

Taking the limit as $n \to +\infty$ and by (4.3.2), (4.3.3) and (4.3.4) we get

$$\gamma_\theta \circ f_{\lambda,\pi}(x) = T \circ \gamma_\theta(x),$$

for all $x \in I$, which proves that γ_θ is an embedding of $(I, f_{\lambda,\pi})$ into (X, T).

Finally, given $x \in I$, consider $0 = t_0 < t_1 < ... < t_N = x$. For all $n \geq 0$, $\gamma_\theta^{(n)} \in \mathcal{PL}(|I|)$ from which follows that $\left|\gamma_\theta^{(n)}(t_{j+1}) - \gamma_\theta^{(n)}(t_j)\right| = |t_{j+1} - t_j|$, for any $j = 0, ..., N-1$. Hence, as $\theta \in \Theta'_{\lambda,\pi}$, we get

$$x = \sum_{j=0}^{N-1} \left|\gamma_\theta^{(n)}(t_{j+1}) - \gamma_\theta^{(n)}(t_j)\right| \to \sum_{j=0}^{N-1} \left|\gamma_\theta(t_{j+1}) - \gamma_\theta(t_j)\right|, \text{ as } n \to +\infty,$$

which shows that $\mathcal{L}_\theta(x) = x$ finishing our proof. □

Following [15, 17], let $\mathbb{P}_+^{\mathcal{A}} = \mathbb{P}(\mathbb{R}_+^{\mathcal{A}}) \simeq \mathbb{P}_+^{\mathcal{A}}$ denote the projectivization of $\mathbb{R}_+^{\mathcal{A}}$. Let $\mathfrak{R} \subseteq \mathfrak{S}(\mathcal{A})$ be a Rauzy class. Since $\mathcal{R}$ commutes with dilations on $\mathbb{R}_+^{\mathcal{A}}$ it projectivizes to a map $\mathcal{R}_R : \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R} \to \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ called the *Rauzy renormalization map* which is defined in the complement of countably many hyperplanes. Moreover we have that if $[\lambda] = [\lambda']$, then $B_R(\lambda', \pi) = B_R(\lambda, \pi)$ for any $\pi \in \mathfrak{R}$, hence the application $([\lambda], \pi) \mapsto B_R([\lambda], \pi)$ is well defined. We refer to this cocycle as the Rauzy cocycle as well.

An induction scheme $\mathcal{S} : \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R} \to \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R}$ is an *acceleration of Rauzy induction* if there exists an integral application $m : \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R} \to \mathbb{Z}_+$, such that for every $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R}$ we have $m(a\lambda, \pi) = m(\lambda, \pi)$ for all $a > 0$ and

$$\mathcal{S}(\lambda, \pi) = \mathcal{R}^{m(\lambda,\pi)}(\lambda, \pi).$$

It is immediate to see that $\mathcal{S}$ also commutes with dilations on $\mathbb{R}_+^{\mathcal{A}}$ and hence it projectivizes to a map $\mathcal{S}_R : \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R} \to \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ which we call an *acceleration of*

Rauzy renormalization. Moreover we have that if $A : \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R} \to SL(\mathcal{A}, \mathbb{Z})$ defines a cocycle over $\mathcal{S}$, then its projectivization $([\lambda], \pi) \to A([\lambda], \pi)$ is well defined.

A *flag*, on an N-dimensional vector space F, is a decreasing family of vector subspaces $\{F^j\}_{j=1,...,k+1}$, with $k \leq N$,

$$F = F^1 \supsetneq F^2 \supsetneq ... \supsetneq F^k \supsetneq \{0\} = F^{k+1}.$$

The flag is said to be *complete* if $k = N$ and $\dim F^j = N + 1 - j$, for all $j = 1, ..., N$.

The following well known result follows from Oseledets Theorem [45].

Theorem 4.3.2 *Let $\mathfrak{R} \subseteq \mathfrak{S}(\mathcal{A})$ be a Rauzy class, $\mathcal{S}_R : \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R} \to \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ be an acceleration of Rauzy renormalization which is measurable with respect to an ergodic measure $m_{\mathfrak{R}}$ and let $A : \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R} \to SL(\mathcal{A}, \mathbb{Z})$ be a $m_{\mathfrak{R}}$-measurable cocycle over $\mathcal{S}_R$.*

There exist $\kappa(\mathfrak{R}) \in \mathbb{N}$, real numbers $\nu_1(\mathfrak{R}) > ... > \nu_{\kappa(\mathfrak{R})}(\mathfrak{R})$ and for $m_{\mathfrak{R}}$-almost every $([\lambda], \pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ there exists a flag $\mathbb{R}^{\mathcal{A}} = V_{[\lambda],\pi}^1 \supsetneq ... \supsetneq V_{[\lambda],\pi}^{\kappa(\mathfrak{R})} \supsetneq \{0\} = V_{[\lambda],\pi}^{\kappa(\mathfrak{R})+1}$ such that $A([\lambda], \pi) \cdot V_{[\lambda],\pi}^j = V_{\mathcal{S}_R([\lambda],\pi)}^j$ and

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A^{(n)}([\lambda], \pi) \cdot v\| = \nu_j(\mathfrak{R}),$$

for all $v \in V_{[\lambda],\pi}^j \backslash V_{[\lambda],\pi}^{j+1}$, $j = 1, ..., \kappa(\mathfrak{R})$.

The spaces $V_{[\lambda],\pi}^j$ are called *Oseledets subspaces* and the numbers $\nu_j(\mathfrak{R})$ are called the *Lyapunov exponents* of the cocycle. The integer $\dim V_{[\lambda],\pi}^j - \dim V_{[\lambda],\pi}^{j+1}$ is called the *multiplicity* of the Lyapunov exponent $\nu_j(\mathfrak{R})$ and it is constant in a full measure set. The *Lyapunov spectrum* of the cocycle is the set of its Lyapunov exponents counted with multiplicity.

In [54], Veech proved that Rauzy renormalization admits an absolutely continuous ergodic measure. This measure, however is not finite and thus the Rauzy cocycle is not measurable with respect to it.

In [58] Zorich defined an acceleration of Rauzy induction as follows. Given $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A})$, let $n(\lambda, \pi)$ denote the smallest $n \in \mathbb{N}$ such that $\varepsilon(n) \neq \varepsilon(0)$ and set

$$\mathcal{Z}(\lambda, \pi) = \mathcal{R}^{n(\lambda,\pi)}(\lambda, \pi).$$

The map $\mathcal{Z}$ is called *Zorich induction* and it projectivizes to a map $\mathcal{Z}_R : \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R} \to \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ called *Zorich renormalization.*

Theorem 4.3.3 ([58]) *Let $\mathfrak{R} \subset \mathfrak{S}(\mathcal{A})$ be a Rauzy class. Then $\mathcal{Z}_R : \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R} \to \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ admits a unique ergodic absolutely continuous probability measure $\mu_{\mathfrak{R}}$. Its density is positive and analytic.*

Define the matrix function $B_Z : \mathbb{R}^{\mathcal{A}} \times \mathfrak{R} \to SL(\mathcal{A}, \mathbb{Z})$ by

$$B_Z(\lambda, \pi) = B_R(\lambda^{(n(\lambda,\pi)-1)}, \pi^{(n(\lambda,\pi)-1)}) \cdot ... \cdot B_R(\lambda^{(1)}, \pi^{(1)}) \cdot B_R(\lambda, \pi).$$

The *Zorich cocycle* is the linear cocycle over the Zorich induction $(\mathcal{Z}, B_Z)$ on $\mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R} \times \mathbb{R}^{\mathcal{A}}$. Its projectivization $(\mathcal{Z}_R, B_Z)$ is well defined and also called Zorich cocycle.

Let $\|\cdot\|$ denote a matrix norm on $SL(\mathcal{A}, \mathbb{Z})$ and let $\|A\|_0 = \max\{\|A\|, \|A\|^{-1}\}$ for any $A \in SL(\mathcal{A}, \mathbb{Z})$. Recall we denote $\log^+ y = \max\{\log(y), 0\}$ for any $y > 0$.

Theorem 4.3.4 ([58]) *Let $\mathfrak{R} \subset \mathfrak{S}(\mathcal{A})$ be a Rauzy class. Then*

$$\int_{\mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}} \log^+ \|B_Z\|_0 d\mu_{\mathfrak{R}} < +\infty.$$

In particular B_Z is a measurable cocycle with respect to $\mu_{\mathfrak{R}}$.

Recall the linear map Ω_π in (1.3.1). Let H_π be the image subspace of Ω_π, that is, $H_\pi = \Omega_\pi(\mathbb{R}^{\mathcal{A}})$. From [15, 55] it follows that

$$B_R(\lambda, \pi) \cdot H_\pi = H_{\pi^{(1)}}, \tag{4.3.5}$$

from which follows that $\dim H_\pi$ only depends on the Rauzy class $\mathfrak{R} \subset \mathfrak{S}(\mathcal{A})$ of π.

A *translation surface* (as defined in [15]), is a surface with a finite number of conical singularities endowed with an atlas such that coordinate changes are given by translations in $\mathbb{R}^2$. Given $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R}$ it is possible (see for instance [54]) to associate, via a suspension construction, a translation surface, with genus $g(\mathfrak{R}) \geq 1$ and κ singularities depending only on $\mathfrak{R}$. Moreover $\dim H_\pi = 2g(\mathfrak{R})$.

By (4.3.5), it is immediate to see that H_π is an invariant subspace for both Rauzy and Zorich cocycles. Hence we can consider restrictions $B_R([\lambda], \pi)|_{H_\pi}$ and $B_Z([\lambda], \pi)|_{H_\pi}$ as integral cocycles over $\mathcal{R}_R$ and $\mathcal{Z}_R$ respectively, which we call *restricted* Rauzy and Zorich cocycles. To simplify the notation we, at times, write $B_R([\lambda], \pi)$ and $B_Z([\lambda], \pi)$ instead of $B_R([\lambda], \pi)|_{H_\pi}$ and $B_Z([\lambda], \pi)|_{H_\pi}$.

As a consequence of theorems 4.3.2 and 4.3.4, for any Rauzy class $\mathfrak{R} \subset \mathfrak{S}(\mathcal{A})$ there exist $k(\mathfrak{R}) \in \mathbb{N}$ such that for $\mu_{\mathfrak{R}}$-almost every $([\lambda], \pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ there exists a flag of Oseledets subspaces $H_\pi = F^1_{[\lambda],\pi} \supsetneq ... \supsetneq F^{k(\mathfrak{R})}_{[\lambda],\pi} \supsetneq \{0\} = F^{k(\mathfrak{R})+1}_{[\lambda],\pi}$ with an associated Lyapunov spectrum

$$\vartheta_1(\mathfrak{R}) > ... > \vartheta_{k(\mathfrak{R})}(\mathfrak{R}).$$

In [58] it is shown that $k(\mathfrak{R}) \leq 2g(\mathfrak{R})$ and that $\vartheta_j(\mathfrak{R}) = -\vartheta_{k(\mathfrak{R})+1-j}(\mathfrak{R})$, for all $j = 1, ..., k(\mathfrak{R})$. In [16] the authors proved that the Lyapunov spectrum of the restricted Zorich cocycle is *simple* on every Rauzy class, that is, all Lyapunov exponents have multiplicity 1. Consequently, the spectral properties of the restricted Zorich cocycle can be summarized as follows.

Theorem 4.3.5 *Let $\mathfrak{R} \subset \mathfrak{S}(\mathcal{A})$ be a Rauzy class. There exist Lyapunov exponents,*

$$\vartheta_1(\mathfrak{R}) > ... > \vartheta_{g(\mathfrak{R})}(\mathfrak{R}) > 0 > \vartheta_{g(\mathfrak{R})+1}(\mathfrak{R}) = -\vartheta_{g(\mathfrak{R})}(\mathfrak{R}) > ... > \vartheta_{2g(\mathfrak{R})}(\mathfrak{R}) = -\vartheta_1(\mathfrak{R}),$$

and, for $\mu_{\mathfrak{R}}$-almost every $([\lambda], \pi) \in \mathbb{P}^{\mathcal{A}}_{+} \times \mathfrak{R}$, there exists a complete flag

$$H_\pi = F^1_{[\lambda],\pi} \supsetneq ... \supsetneq F^{2g(\mathfrak{R})}_{[\lambda],\pi} \supsetneq \{0\} = F^{2g(\mathfrak{R})+1}_{[\lambda],\pi},$$

such that $B_Z([\lambda], \pi)|_{H_\pi} \cdot F^j_{[\lambda],\pi} = F^j_{\mathcal{Z}_R([\lambda],\pi)}$. For all $v \in F^j_{[\lambda],\pi} \backslash F^{j+1}_{[\lambda],\pi}$, $j = 1, ..., 2g(\mathfrak{R})$,

$$\lim_{n\to+\infty} \frac{1}{n} \log \|B_Z([\lambda], \pi)|_{H_\pi} \cdot v\| = \vartheta_j(\mathfrak{R}).$$

We say $([\lambda], \pi) \in \mathbb{P}^{\mathcal{A}}_{+} \times \mathfrak{R}$ is *generic* if $([\lambda], \pi)$ is in the full measure set of $\mathbb{P}^{\mathcal{A}}_{+} \times \mathfrak{R}$ from Theorem 4.3.5.

Let $\|\cdot\|_1 : SL(\mathcal{A}, \mathbb{Z}) \to \mathbb{R}_+$ be the norm,

$$\|A\|_1 = \sum_{\alpha\in\mathcal{A}} \sum_{\beta\in\mathcal{A}} |A_{\alpha\beta}|.$$

Denote by Leb the Lebesgue measure in $\mathbb{P}^{\mathcal{A}}_{+}$ and by $c_{\mathfrak{R}}$ the counting measure in a Rauzy class $\mathfrak{R}$. The following theorem is a restatement of a result by Marmi, Moussa and Yoccoz [43] and gives a bound for the growth of the Zorich cocycle for a full measure set of $([\lambda], \pi)$. The proof can be found in Section 4.7 in [43].

Theorem 4.3.6 ([43]) *For Leb $\times$ $c_{\mathfrak{R}}$-almost every $([\lambda], \pi) \in \mathbb{P}^{\mathcal{A}}_{+} \times \mathfrak{R}$ and $\varepsilon' > 0$, there exists $C_{\varepsilon'} > 0$ such that for any $m \geq 0$,*

$$\|B_Z(\mathcal{Z}^m_R([\lambda], \pi))\|_1 < C_{\varepsilon'} \|B^{(m)}_Z([\lambda], \pi)\|_1^{\varepsilon'}$$

Given $([\lambda], \pi) \in \mathbb{P}^{\mathcal{A}}_{+} \times \mathfrak{R}$ and $m \geq 0$, denote the sum of the m first Zorich acceleration times by

$$s^m([\lambda], \pi) = \sum_{k<m} n(\mathcal{Z}^k_R([\lambda], \pi)).$$

So far the choice of vector norm $\|\cdot\|$ has not been relevant as Theorem 4.3.2 does not depend on any particular choice. However in what follows we consider $\|\cdot\|$ to be the euclidean norm.

In the following lemma we combine estimates from theorems 4.3.5 and 4.3.6 to obtain an important bound for the growth of the Rauzy cocycle, restricted to $F^{g(\mathfrak{R})+1}_{[\lambda],\pi} \backslash \{0\}$, for a full measure set of parameters.

Lemma 4.3.7 *For Leb* $\times\, c_{\mathfrak{R}}$*-almost every* $([\lambda],\pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$*, there exists* $K \geq 1$ *such that for all* $v \in F_{[\lambda],\pi}^{g(\mathfrak{R})+1} \backslash \{0\}$ *we have*

$$\sum_{n=0}^{+\infty} \|B_R^{(n)}([\lambda],\pi)\cdot v\| < K\|v\|. \tag{4.3.6}$$

Proof.

By Theorem 4.3.5, for $\mu_{\mathfrak{R}}$-almost every $([\lambda],\pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ and any $0 < \eta < 1$ there exists $K_\eta > 0$ such that for every $m \geq 0$,

$$\|B_Z^{(m)}\left([\lambda],\pi\right)\|_1 < K_\eta e^{\eta^{-1}\vartheta_1(\mathfrak{R})m}$$

As, by Theorem 4.3.4, $\mu_{\mathfrak{R}}$ has positive density, this also holds for Leb $\times\, c_{\mathfrak{R}}$-a.e. $([\lambda],\pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$. Combined with Theorem 4.3.6, for $\varepsilon' = \frac{1}{4}\eta^2\vartheta_{g(\mathfrak{R})}(\mathfrak{R})/\vartheta_1(\mathfrak{R})$, this gives

$$\|B_Z\left(\mathcal{Z}_R^m([\lambda],\pi)\right)\|_1 < K_\eta C_{\varepsilon'} e^{\frac{1}{4}\eta\vartheta_{g(\mathfrak{R})}(\mathfrak{R})m}, \tag{4.3.7}$$

for Leb $\times\, c_{\mathfrak{R}}$-a.e. $([\lambda],\pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$.

By Theorem 4.3.5 we also get that for Leb $\times\, c_{\mathfrak{R}}$-a.e. $([\lambda],\pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ there exists $K'_\eta > 0$, such that, for any $v \in F_{[\lambda],\pi}^{g(\mathfrak{R})+1} \backslash \{0\}$ we have

$$\|B_Z^{(m)}\left([\lambda],\pi\right)\cdot v\| < K'_\eta e^{-\eta\vartheta_{g(\mathfrak{R})}(\mathfrak{R})m}\|v\|. \tag{4.3.8}$$

Let $\mathcal{E}_\eta$ denote the set of $([\lambda],\pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ for which there exists $K''_\eta > 0$ such that

$$\|B_Z\left(\mathcal{Z}_R^m([\lambda],\pi)\right)\|_1^2 \cdot \|B_Z^{(m)}\left([\lambda],\pi\right)\cdot v\| < K''_\eta e^{-\frac{1}{2}\eta\vartheta_{g(\mathfrak{R})}(\mathfrak{R})m}\|v\|, \tag{4.3.9}$$

for all $v \in F_{[\lambda],\pi}^{g(\mathfrak{R})+1} \backslash \{0\}$ and $m \geq 0$. By combining (4.3.7) and (4.3.8) we get that $\mathcal{E}_\eta$ is a set of full Leb $\times\, c_{\mathfrak{R}}$ measure.

Now, fix $0 < \eta < 1$ and $([\lambda],\pi) \in \mathcal{E}_\eta$. For $n \geq 0$, let

$$M(n) = \max\left\{m \geq 0 : s^m([\lambda],\pi) \leq n\right\}.$$

Also, given positive integers $k_1 < k_2$ we denote

$$B_R^{(k_1,k_2)}([\lambda],\pi) = B_R([\lambda^{(k_2)}],\pi^{(k_2)}) \cdot B_R([\lambda^{(k_2-1)}],\pi^{(k_2-1)}) \cdot \ldots \cdot B_R([\lambda^{(k_1)}],\pi^{(k_1)}).$$

We have

$$\|B_R^{(n)}([\lambda],\pi)\cdot v\| \leq \max_{s^{M(n)}([\lambda],\pi)\leq k<n} \left\| B_R^{\left(s^{M(n)}([\lambda],\pi),k\right)}([\lambda],\pi)\cdot B_Z^{(M(n))}([\lambda],\pi)\cdot v\right\|. \tag{4.3.10}$$

It is clear that we have

$$\max_{s^{M(n)}([\lambda],\pi)\le k<n}\left\|B_R^{\left(s^{M(n)}([\lambda],\pi),k\right)}([\lambda],\pi)\right\|_1 \le \left\|B_Z\left(\mathcal{Z}_R^{M(n)}([\lambda],\pi)\right)\right\|_1,$$

hence, from (4.3.10), for all $n \ge 0$ we get

$$\|B_R^{(n)}([\lambda],\pi)\cdot v\| \le \|B_Z(\mathcal{Z}_R^{M(n)}([\lambda],\pi))\|_1 \cdot \|B_Z^{(M(n))}([\lambda],\pi)\cdot v\|,$$

which combined with the fact that for all $m \ge 0$ we have

$$n(\mathcal{Z}_R^m([\lambda],\pi)) \le \|B_Z\left(\mathcal{Z}_R^m([\lambda],\pi)\right)\|_1,$$

gives

$$\sum_{n=0}^{+\infty}\|B_R^{(n)}([\lambda],\pi)\cdot v\| \le \sum_{m=0}^{+\infty}\|B_Z\left(\mathcal{Z}_R^m([\lambda],\pi)\right)\|_1^2\cdot\|B_Z^{(m)}([\lambda],\pi)\cdot v\|.$$

This, combined with (4.3.9), which holds since $([\lambda],\pi)\in\mathcal{E}_\eta$, shows that by taking $K=\max\{K''_\eta(1-e^{-1/2\eta\vartheta_{g(\mathfrak{R})}(\mathfrak{R})})^{-1},1\}$ we get (4.3.6) as intended. □

Recalling (4.1.5) note that for any $\lambda,\lambda'\in\mathbb{R}_+^{\mathcal{A}}$ such that $[\lambda]=[\lambda']$ we have $B_{\mathbb{T}^{\mathcal{A}}}(\lambda,\pi)=B_{\mathbb{T}^{\mathcal{A}}}(\lambda',\pi)$ and thus $B_{\mathbb{T}^{\mathcal{A}}}(\lambda,\pi)$ admits a projectivization which we denote $B_{\mathbb{T}^{\mathcal{A}}}([\lambda],\pi)$ and also call projection of the Rauzy cocycle on $\mathbb{T}^{\mathcal{A}}$.

Recall that $p:\mathbb{R}^{\mathcal{A}}\to\mathbb{T}^{\mathcal{A}}$ is the natural projection,

$$p(v)=\left((v)_\alpha \mod 2\pi\right)_{\alpha\in\mathcal{A}},\quad \text{for all } v\in\mathbb{R}^{\mathcal{A}}.$$

The *flat torus* is the torus $\mathbb{T}^{\mathcal{A}}$ viewed as a Riemannian manifold equipped with the *flat Riemannian metric*, this is, the pushforward under p of the euclidean metric in $\mathbb{R}^{\mathcal{A}}$. The flat Riemannian metric induces a distance on the torus $d_{\mathbb{T}^{\mathcal{A}}}:\mathbb{T}^{\mathcal{A}}\times\mathbb{T}^{\mathcal{A}}\to\mathbb{R}_+$ such that

$$d_{\mathbb{T}^{\mathcal{A}}}(\theta,\theta')=\inf\left\{\|v-v'\| : v\in p^{-1}(\theta),\ v'\in p^{-1}(\theta')\right\},$$

for any $\theta,\theta'\in\mathbb{T}^{\mathcal{A}}$.

Given $\delta>0$ and a generic $([\lambda],\pi)\in\mathbb{P}_+^{\mathcal{A}}\times\mathfrak{R}$, let

$$E_{[\lambda],\pi}^\delta=\left\{v\in F_{[\lambda],\pi}^{g(\mathfrak{R})+1}\backslash\{0\} : \|v\|<\delta\right\},$$

and let $W_{[\lambda],\pi}^\delta=p\left(E_{[\lambda],\pi}^\delta\right)$.

Recall (4.1.6), which given $\theta\in\mathbb{T}^{\mathcal{A}}$ defines a sequence $(\theta^{(n)})_{n\ge0}$ on $\mathbb{T}^{\mathcal{A}}$ which is used to construct the breaking sequence $(\gamma_\theta^{(n)})_{n\ge0}$. The following lemma states that for a full measure set of $([\lambda],\pi)$, and for sufficiently small $\delta>0$, and all $\theta\in W_{[\lambda],\pi}^\delta$ the sum of all $d_{\mathbb{T}^{\mathcal{A}}}(\theta^{(n)},0)$ is bounded.

Lemma 4.3.8 *For Leb* $\times\, c_{\mathfrak{R}}$*-almost every* $([\lambda],\pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$*, there exists* $K \geq 1$ *and* $\delta > 0$ *such that for all* $\theta \in W^{\delta}_{[\lambda],\pi}$ *we have*

$$\sum_{n=0}^{+\infty} d_{\mathbb{T}^{\mathcal{A}}}(\theta^{(n)}, 0) < K d_{\mathbb{T}^{\mathcal{A}}}(\theta, 0). \tag{4.3.11}$$

Proof. By Lemma 4.3.7, for Leb $\times\, c_{\mathfrak{R}}$-a.e. $([\lambda],\pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$, there exists $K > 1$ such that for all $v \in E^{\delta}_{[\lambda],\pi}$, with $\delta = \pi \cdot K^{-1}$, for all $n \geq 0$ we have

$$\|B_R^{(n)}([\lambda],\pi) \cdot v\| < \pi,$$

Moreover, is is clear that if $\|v\| < \pi$ we have $d_{\mathbb{T}^{\mathcal{A}}}(p(v), 0) = \|v\|$, thus, for all $n \geq 0$ we have

$$d_{\mathbb{T}^{\mathcal{A}}}\left(p\left(B_R^{(n)}([\lambda],\pi)\cdot v\right), 0\right) = \left\|B_R^{(n)}([\lambda],\pi)\cdot v\right\|, \tag{4.3.12}$$

Also note that as $\delta \leq \pi$, the restriction $p|_{E^{\delta}_{[\lambda],\pi}} : E^{\delta}_{[\lambda],\pi} \to W^{\delta}_{[\lambda],\pi}$ is a bijection and thus $p^{-1}(\theta) \cap E^{\delta}_{[\lambda],\pi}$ contains a single point which we denote by $p_\delta^{-1}(\theta)$. Take $\theta \in W^{\delta}_{[\lambda],\pi}$. It is clear by (4.1.5) that we have

$$B^{(n)}_{\mathbb{T}^{\mathcal{A}}}([\lambda],\pi)\cdot\theta = p\left(B_R^{(n)}([\lambda],\pi)\cdot p_\delta^{-1}(\theta)\right),$$

which combined with (4.3.12) yields $d_{\mathbb{T}^{\mathcal{A}}}(B^{(n)}_{\mathbb{T}^{\mathcal{A}}}([\lambda],\pi)\cdot\theta, 0) = \|B_R^{(n)}([\lambda],\pi)\cdot p_\delta^{-1}(\theta)\|$, for all $n \geq 0$. By (4.1.6) and Lemma 4.3.7 this gives (4.3.11) finishing our proof. □

We say a map $\gamma : I \to \mathbb{C}$ is Lipschitz if $\{(\mathrm{Re}(\gamma(x)), \mathrm{Im}(\gamma(x))) : x \in I\}$ is the graph of a Lipschitz map. The following theorem shows that for a generic $([\lambda],\pi)$ and sufficiently small $\delta > 0$, when $\theta \in W^{\delta}_{[\lambda],\pi}$ the sequence $\gamma_\theta^{(n)}$ converges to a a Lipschitz map γ_θ which is an isometric embedding of $(I, f_{\lambda,\pi})$ into any PWI that is θ-adapted to (λ,π).

Theorem 4.3.9 *For Leb* $\times\, c_{\mathfrak{R}}$*-almost every* $([\lambda],\pi) \in \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$*, there exists a* $\delta > 0$ *such that for all* $\theta \in W^{\delta}_{[\lambda],\pi}$ *there exists a Lipschitz map* $\gamma_\theta : I \to \mathbb{C}$*, which is an isometric embedding of* $(I, f_{\lambda,\pi})$ *into any PWI that is* θ*-adapted to* (λ,π)*.*

Proof. Consider the space $\mathcal{C}(I,\mathbb{C})$ of continuous maps from the interval I, to $\mathbb{C}$. Note that this is a Banach space for the supremum norm $\|.\|_\infty$. We also have that $\gamma_\theta^{(n)} \in \mathcal{C}(I,\mathbb{C})$ for all $n \geq 0$, since $\gamma_\theta^{(0)}$ is continuous and, by Lemma 4.1.1, $\mathfrak{Br}\left(\theta^{(n-1)}_{\beta_{1,n-1}}, J^{(n)}\right)\cdot\mathcal{C}(I,\mathbb{C}) \subseteq \mathcal{C}(I,\mathbb{C})$.

Take any $\varphi \in (0,\pi/2)$. By Lemma 4.3.8, there exists a set $\mathcal{E} \subseteq \mathbb{P}_+^{\mathcal{A}} \times \mathfrak{R}$ of full Leb $\times\, c_{\mathfrak{R}}$ measure such that for every $([\lambda],\pi) \in \mathcal{E}$, there exists $K \geq 1$ and $0 < \delta < \varphi K^{-1}$ such that for all $\theta \in W^{\delta}_{[\lambda],\pi}$ we have (4.3.11).

Take $([\lambda],\pi)\in\mathcal{E}$ and $\theta\in W^{\delta}_{[\lambda],\pi}$. For all $x\in I$ we have

$$\left|\gamma_\theta^{(n+1)}(x)-\gamma_\theta^{(n)}(x)\right| = \left|\mathfrak{Br}\left(\theta^{(n)}_{\beta_{1,n}}, J^{(n+1)}\right)\cdot\gamma_\theta^{(n)}(x)-\gamma_\theta^{(n)}(x)\right|.$$

Denoting, as in (4.1.4), by $r(n)$ the number of intervals of $J^{(n+1)}$, by (4.1.1) this gives

$$\left|\gamma_\theta^{(n+1)}(x)-\gamma_\theta^{(n)}(x)\right| \le \max_{k<r(n)}\{|\underline{\epsilon}_k|,|\overline{\epsilon}_k|\}+\sup_{x\in I}|\gamma_\theta^{(n)}(x)(1-e^{i\theta^{(n)}_{\beta_{1,n}}})|.$$

Since

$$\sup_{x\in I}\left|\gamma_\theta^{(n)}(x)(1-e^{i\theta^{(n)}_{\beta_{1,n}}})\right| \le 2|\lambda|\sin\left(\theta^{(n)}_{\beta_{1,n}}/2\right),$$

by Lemma 4.1.2 we get

$$\left|\gamma_\theta^{(n+1)}(x)-\gamma_\theta^{(n)}(x)\right| \le 4|\lambda|\sin\left(\theta^{(n)}_{\beta_{1,n}}/2\right).$$

Therefore, as $\theta^{(n)}_{\beta_{1,n}}\le d_{\mathbb{T}^{\mathcal{A}}}(\theta^{(n)},0)$ there exists $C>0$ such that for all $n\ge 0$,

$$\left|\gamma_\theta^{(n+1)}(x)-\gamma_\theta^{(n)}(x)\right| \le C|\lambda| d_{\mathbb{T}^{\mathcal{A}}}(\theta^{(n)},0).$$

Now take m, $n\in\mathbb{N}$ such that $m>n$. Note that we have

$$\|\gamma_\theta^{(m)}-\gamma_\theta^{(n)}\|_\infty \le \sum_{k=0}^{m-n-1}\|\gamma_\theta^{(m-k)}-\gamma_\theta^{(m-k-1)}\|_\infty,$$

and therefore

$$\|\gamma_\theta^{(m)}-\gamma_\theta^{(n)}\|_\infty \le C|\lambda|\sum_{k=n}^{m-1} d_{\mathbb{T}^{\mathcal{A}}}(\theta^{(k)},0), \tag{4.3.13}$$

From (4.3.11) by taking a sufficiently large $N>0$ and considering $N<n<m$ the righthand side of (4.3.13) can be made arbitrarily small. Thus $\{\gamma_\theta^{(n)}\}_{n\ge 0}$ is a Cauchy sequence in $\mathcal{C}(I,\mathbb{C})$ and therefore it must converge to a unique limit $\gamma_\theta\in\mathcal{C}(I,\mathbb{C})$.

As for all $n\ge 0$, $\gamma_\theta^{(n)}\in\mathcal{C}(I,\mathbb{C})$, by (4.1.7) it is simple to see that for any $x,y\in I$, $x\ne y$, we have

$$\frac{\left|\mathrm{Im}(\gamma_\theta^{(n)}(x))-\mathrm{Im}(\gamma_\theta^{(n)}(y))\right|}{\left|\mathrm{Re}(\gamma_\theta^{(n)}(x))-\mathrm{Re}(\gamma_\theta^{(n)}(y))\right|} \le \tan\left(\sum_{n=1}^{+\infty}\theta^{(n-1)}_{\beta_{1,n-1}}\right).$$

For any map $\gamma: I\to\mathbb{C}$, its Lipschitz constant $L(\gamma)$ is given by

$$L(\gamma)=\sup_{x,y\in I,\ x\ne y}\frac{|\mathrm{Im}(\gamma(x))-\mathrm{Im}(\gamma(y))|}{|\mathrm{Re}(\gamma(x))-\mathrm{Re}(\gamma(y))|}.$$

Hence, in particular we get,

$$\arctan(L(\gamma_\theta^{(n)})) \leq \sum_{n=0}^{+\infty} d_{\mathbb{T}^{\mathcal{A}}}(\theta^{(n)}, 0),$$

which, as $\delta < \varphi K^{-1}$, by (4.3.11) gives $\arctan(L(\gamma_\theta^{(n)})) \leq \varphi$. Clearly $L(\gamma_\theta) \leq \sup_{n\geq 0} L(\gamma_\theta^{(n)})$, and as $\varphi < \pi/2$ this shows that $L(\gamma_\theta) < +\infty$ and thus γ_θ is a Lipschitz map. In particular it is continuous and injective and thus a topological embedding.

This proves that $W^\delta_{[\lambda],\pi} \subseteq \Theta'_{\lambda,\pi}$ and therefore by Theorem 4.3.1, for any $\theta \in W^\delta_{[\lambda],\pi}$, γ_θ is an isometric embedding of $(I, f_{\lambda,\pi})$ into any PWI that is θ-adapted to (λ, π). □

Recall from Chapter 3, that we can extend Rauzy-Veech induction to PWIs which admit embeddings of IETs as follows. Assume $(I, f_{\lambda,\pi})$ has an embedding by γ_θ into (X, T). Define the map $\mathcal{S}(T)$ as the first return map under T to X^*, where

$$X^* = \begin{cases} \bigcup_{\alpha\neq\beta_0} X_\alpha \cup (X_{\beta_0} \cap T(X_{\beta_1})), & \text{if } (\lambda,\pi) \text{ has type } 0,\\ \bigcup_{\alpha\neq\beta_0} X_\alpha, & \text{if } (\lambda,\pi) \text{ has type } 1. \end{cases}$$

It is clear that $(X^*, \mathcal{S}(T))$ is again a d'-PWI, with possibly $d' \neq d$. Denote by $\mathcal{A}'$ an alphabet with d' symbols and denote by $\{X^*_{\alpha'}\}_{\alpha'\in\mathcal{A}'}$ the partition of X^*. It is simple to see that there is a collection of d symbols $\mathcal{A} \subseteq \mathcal{A}'$, possibly after relabeling, such that $X^*_{\alpha'} \cap \gamma_\theta(I^{(1)}) \neq \emptyset$ if and only if $\alpha' \in \mathcal{A}$. Define $X' = \bigcup_{\alpha\in\mathcal{A}} X^*_\alpha$.

Now, $(X', \mathcal{S}(T))$ is $\theta^{(1)}$-adapted to $(\lambda^{(1)}, \pi^{(1)})$ and, by Theorem 3.1.3 in Chapter 3, the restriction of γ_θ to $I^{(1)}$ is an embedding of $(I^{(1)}, f_{\lambda^{(1)},\pi^{(1)}})$ into $(X', \mathcal{S}(T))$.

It is thus possible to iterate this procedure by setting $(X^{(0)}, \mathcal{S}^{(0)}(T)) = (X, T)$, and $\left(X^{(n)}, \mathcal{S}^{(n)}(T)\right) = \left((X^{(n-1)})', \mathcal{S}(\mathcal{S}^{(n-1)}(T))\right)$ for $n \geq 1$. The following lemma easily follows from Theorem 4.3.1.

Lemma 4.3.10 *Let $(\lambda, \pi) \in \mathbb{R}^{\mathcal{A}}_+ \times \mathfrak{R}$, $\theta \in \Theta'_{\lambda,\pi}$ and (X, T) be a PWI θ-adapted to (λ, π). Then for all $n \geq 0$, $(X^{(n)}, \mathcal{S}^{(n)}(T))$ is $\theta^{(n)}$-adapted to $(\lambda^{(n)}, \pi^{(n)})$ and the restriction of γ_θ to $I^{(n)}$ is an embedding of $(I^{(n)}, f_{\lambda^{(n)},\pi^{(n)}})$ into $(X^{(n)}, \mathcal{S}^{(n)}(T))$.*

Given a generic $([\lambda], \pi) \in \mathbb{P}^{\mathcal{A}}_+ \times \mathfrak{R}$ and $\delta > 0$ note that $W^\delta_{[\lambda],\pi}$ defines a $g(\mathfrak{R})$-dimensional submanifold embedded in the torus $\mathbb{T}^{\mathcal{A}}$. Pulling back the flat metric by the embedding map it is possible to construct a $g(\mathfrak{R})$-volume form and thus define a positive measure $m_{g(\mathfrak{R})}$ on $W^\delta_{[\lambda],\pi}$.

Denote the projection on $\mathbb{T}^{\mathcal{A}}$ of the Oseledets subspace $F^{2g(\mathfrak{R})}_{[\lambda],\pi}$ by $W^{SS}_{[\lambda],\pi} = p\left(F^{2g(\mathfrak{R})}_{[\lambda],\pi}\right)$. Note that $W^{SS}_{[\lambda],\pi}$ is a 1-dimensional submanifold embedded in $\mathbb{T}^{\mathcal{A}}$.

For any $n \geq 0$ and $\theta \in \mathbb{T}^{\mathcal{A}}$ let $B^{(-n)}_{\mathbb{T}^{\mathcal{A}}}([\lambda], \pi) \cdot \theta = \left\{ \theta' \in \mathbb{T}^{\mathcal{A}} : B^{(n)}_{\mathbb{T}^{\mathcal{A}}}([\lambda], \pi) \cdot \theta' = \theta \right\}$. Consider

$$\mathcal{W}^{\delta}_{[\lambda],\pi} = W^{\delta}_{[\lambda],\pi} \backslash \left(W^{SS}_{[\lambda],\pi} \cup \bigcup_{n=0}^{+\infty} B^{(-n)}_{\mathbb{T}^{\mathcal{A}}}([\lambda], \pi) \cdot 0 \right).$$

Recall the definitions of arc, linear and non-trivial embeddings in the Introduction. The following theorem establishes that for any Rauzy class $\mathfrak{R}$ such that $g(\mathfrak{R}) \geq 2$ and for a full measure set of $([\lambda], \pi) \in \mathbb{P}^{\mathcal{A}}_{+} \times \mathfrak{R}$, when $\theta \in \mathcal{W}^{\delta}_{[\lambda],\pi}$ for sufficiently small $\delta > 0$, γ_θ is a non-trivial isometric embedding of $(I, f_{\lambda,\pi})$ into any PWI (X, T) that is θ-adapted to (λ, π). Since $(I, f_{\lambda,\pi})$ is topologically conjugated to the restriction of (X, T) to the image of the embedding $\gamma_\theta(I)$ we have that the latter map is one-to-one and therefore $\gamma_\theta(I)$ is an invariant set for (X, T). Moreover $\gamma_\theta(I)$ is a curve which is not a union of line segments or circle arcs. Thus, Theorem E follows directly from our next result.

Theorem 4.3.11 *For any Rauzy class $\mathfrak{R}$ satisfying $g(\mathfrak{R}) \geq 2$ and Leb $\times\, c_{\mathfrak{R}}$-almost every $([\lambda], \pi) \in \mathbb{P}^{\mathcal{A}}_{+} \times \mathfrak{R}$, there exists $\delta > 0$ such that $\mathcal{W}^{\delta}_{[\lambda],\pi}$ is a set of full $m_{g(\mathfrak{R})}$-measure in $W^{\delta}_{[\lambda],\pi}$ and for all $\theta \in \mathcal{W}^{\delta}_{[\lambda],\pi}$ there exists a Lipschitz map $\gamma_\theta : I \to \mathbb{C}$, which is a non-trivial isometric embedding of $(I, f_{\lambda,\pi})$ into any PWI that is θ-adapted to (λ, π).*

Proof.

As for any $\delta > 0$ we have $\mathcal{W}^{\delta}_{[\lambda],\pi} \subseteq W^{\delta}_{[\lambda],\pi}$, by Theorem 4.3.9 for Leb $\times\, c_{\mathfrak{R}}$-almost every $([\lambda], \pi) \in \mathbb{P}^{\mathcal{A}}_{+} \times \mathfrak{R}$, there exists $\delta > 0$ such that for all $\theta \in \mathcal{W}^{\delta}_{[\lambda],\pi}$ there exists a Lipschitz map $\gamma_\theta : I \to \mathbb{C}$, which is an isometric embedding of $(I, f_{\lambda,\pi})$ into any PWI that is θ-adapted to (λ, π).

Note that $\bigcup_{n=0}^{+\infty} B^{(-n)}_{\mathbb{T}^{\mathcal{A}}}([\lambda], \pi) \cdot 0$ is a countable set, $\dim(W^{SS}_{[\lambda],\pi}) = 1$ and $\dim(W^{\delta}_{[\lambda],\pi}) = g(\pi)$. Thus, when $g(\mathfrak{R}) \geq 2$ we have that $\mathcal{W}^{\delta}_{[\lambda],\pi}$ is a set of full $m_{g(\mathfrak{R})}$-measure in $W^{\delta}_{[\lambda],\pi}$.

For $\theta \in \mathcal{W}^{\delta}_{[\lambda],\pi}$, assume by contradiction that γ_θ is an arc embedding of $(I, f_{\lambda,\pi})$ into a PWI (X, T) that is θ-adapted to (λ, π). There exists $x' > 0$ such that the restriction of γ_θ to $[0, x')$ is an arc map. Moreover, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $I^{(n)} \subseteq [0, x')$. As γ_θ is an isometric embedding and $\gamma_\theta(0) = 0$, there is an $r > 0$ and a $\varphi \in [0, 2\pi)$ such that for all $x \in I^{(n)}$ we have

$$\gamma_\theta(x) = r(e^{i(r^{-1}x + \varphi)} - e^{i\varphi}). \tag{4.3.14}$$

By Lemma 4.3.10, for any $n \geq N$, $(X^{(n)}, \mathcal{S}^{(n)}(T))$ is a PWI $\theta^{(n)}$-adapted to $(\lambda^{(n)}, \pi^{(n)})$ and the restriction of γ_θ to $I^{(n)}$ is an isometric embedding of $(I^{(n)}, f_{\lambda^{(n)},\pi^{(n)}})$

into $(X^{(n)}, \mathcal{S}^{(n)}(T))$. Hence we have

$$\gamma_\theta(f_{\lambda^{(n)},\pi^{(n)}}(x)) = e^{i\theta_\alpha^{(n)}}\left(\gamma_\theta(x) - \gamma_\theta\left(x^{(n)}_{\pi_0^{(n)}(\alpha)-1}\right)\right) + \gamma_\theta\left(f_{\lambda^{(n)},\pi^{(n)}}\left(x^{(n)}_{\pi_0^{(n)}(\alpha)-1}\right)\right), \tag{4.3.15}$$

for all $\alpha \in \mathcal{A}$, any $x \in I_\alpha^{(n)}$ and any $n \geq N$.

Recall that we denote $\upsilon^{(n)} = \Omega_{\pi^{(n)}}(\lambda^{(n)})$. Let $M > 0$ be such that for all $m \geq M$ we have $s^m([\lambda], \pi) > N$. From (4.3.14), (4.3.15) and (1.3.2) we have

$$\theta^{(s^m([\lambda]\pi))} = p\left(r^{-1}\upsilon^{(s^m([\lambda],\pi))}\right). \tag{4.3.16}$$

By the proof of Theorem 4.3.9 we have $\delta < \pi$ and thus, the restriction $p|_{E^\delta_{[\lambda],\pi}}$: $E^\delta_{[\lambda],\pi} \to W^\delta_{[\lambda],\pi}$ is a bijection and thus $p^{-1}(\theta) \cap E^\delta_{[\lambda],\pi}$ contains a single point which we denote by $p_\delta^{-1}(\theta)$. As $\theta \in \mathcal{W}^\delta_{[\lambda],\pi}$, by (4.3.16) we get

$$\upsilon^{(s^m([\lambda],\pi))} = B_Z^{(m)}([\lambda], \pi) \cdot p_\delta^{-1}(\theta). \tag{4.3.17}$$

By the results in [57] Section 5.3, it is known that $F^{2g(\mathfrak{R})}_{[\lambda],\pi}$ is equal to the linear span of $\{\upsilon^{(0)}\}$ in $\mathbb{R}^\mathcal{A}$ and thus by (4.3.17) and Theorem 4.3.5 we get that $p_\delta^{-1}(\theta) \in F^{2g(\mathfrak{R})}_{[\lambda],\pi}$ and consequently $\theta \in W^{SS}_{[\lambda],\pi}$ which contradicts our assumption $\theta \in \mathcal{W}^\delta_{[\lambda],\pi}$. Therefore γ_θ is not an arc embedding.

Now, for $\theta \in \mathcal{W}^\delta_{[\lambda],\pi}$, assume by contradiction that γ_θ is a linear embedding of $(I, f_{\lambda,\pi})$ into a PWI (X, T) that is θ-adapted to (λ, π). As γ_θ is an isometric embedding and $\gamma_\theta(0) = 0$ for a sufficiently large $N \in \mathbb{N}$ there is $\varphi \in [0, 2\pi)$ such that

$$\gamma_\theta(x) = e^{i\varphi}x, \tag{4.3.18}$$

for all $x \in I^{(N)}$.

By Lemma 4.3.10, $(X^{(N)}, \mathcal{S}^{(N)}(T))$ is a PWI $\theta^{(N)}$-adapted to $(\lambda^{(N)}, \pi^{(N)})$ and the restriction of γ_θ to $I^{(N)}$ is an isometric embedding of $(I^{(N)}, f_{\lambda^{(N)},\pi^{(N)}})$ into $(X^{(N)}, \mathcal{S}^{(N)}(T))$. Hence we have (4.3.15) which combined with (4.3.18) shows that $\theta^{(N)} = 0$. Therefore $\theta \in \bigcup_{n=0}^{+\infty} B^{(-n)}_{\mathbb{T}^\mathcal{A}}(\lambda, \pi) \cdot 0$, which contradicts $\theta \in \mathcal{W}^\delta_{[\lambda],\pi}$. Thus γ_θ is not a linear embedding.

This proves that γ_θ is a non-trivial isometric embedding of $(I, f_{\lambda,\pi})$ into (X, T). □

Chapter 5

Concluding remarks

In Chapter 2 we introduced Translated Cone Exchange Transformations and found in Theorem A that they are renormalizable for all rotational parameters and for countably many translational parameters. This was the first time a renormalization scheme was found to work for generic rotations, however as a tradeoff it has a limited scope with respect to the remaining parameters. A natural way forward is to generalize the techniques developed to a wider class of algebraic parameters.

We have highlighted that embeddings of IETs into PWIs present a number of subtle and mathematically rich problems associated with the regularity or otherwise of these embeddings.

In Theorem B we showed, as a consequence of Theorem A, that the existence of an embedding of a IET into a Translated Cone Exchange Transformation, which is contained in a *barrier*, results in the existence of infinitely many embeddings as well as the existence of invariant bounded regions. However we do not provide a proof that such an embedding may be contained in a barrier. Numerical evidence suggests that this is reasonable to expect and, in fact, it is our expectation that the techniques from Chapter 4 can be adapted to show that this is indeed the case.

In Chapter 3, Theorem C shows that there are no non-trivial continuous embeddings of a minimal d-IET into a d-PWI, for $d = 2$, while Theorem 3.3.1 gives a condition for the existence of a piecewise continuous embedding. For $d = 4$ there are PWIs that seem to have an abundance of non-trivial embeddings of d-IETs. It seems to be much harder to find a 3-PWI that exhibits non-trivial embeddings of 3-IETs and to do so requires much parametric fine tuning, a fact that is justified by Theorem D which shows that any 3-PWI has at most one non-trivially continuously embedded minimal 3-IET with the same underlying permutation. We suspect that typical non-trivial embeddings have a tangent exchange map that is minimal but not ergodic.

The region Ξ discussed in Section 3.4 seems to contain periodic islands, embedded IETs and other invariant sets that are neither. It is a challenge to describe these other invariant sets in a coherent way. Regarding the IETs embedded in Ξ we conjecture that all minimal nearby IETs in $\mathcal{F}_4$ are continuously (or at least symbolically) embedded.

In Chapter 4 we proved that a full measure set of IETs admit non-trivial embeddings into a class of PWIs, by using techniques from the theory of IET renormalization and measurable cocycles. In particular we prove the existence of invariant curves for PWIs which are not unions of circle arcs or line segments, solving a long-standing conjecture in the field. This novel technique allows the use of tools, from the theory of IETs, to study dynamics of PWIs from this class. Note that for 2-IETs we necessarily have $g(\mathfrak{R}) = 1$ and indeed Theorem C shows that the condition $g(\mathfrak{R}) \geq 2$, in the statement of Theorem E, is in fact sharp. Also note that Theorem E does not establish the existence of embeddings of 3-IETs into 3-PWIs, as in this case we necessarily have $g(\mathfrak{R}) = 1$ as well. Although this does not follow directly from our results, the techniques developed in Chapter 4, coupled with the fact that the Zorich cocycle has a non-trivial central Oseledets space in this case, present a natural path to possibly establish this in the future.

The results from this thesis open up a number of interesting lines of enquiry:

- Are there non-trivial embeddings of 3-IETs into 3-PWIs? The necessary condition $g(\mathfrak{R}) \geq 2$ in the statement of Theorem E implies that this result only applies to IETs with $d \geq 4$. Is it possible to generalize these techniques to prove the existence of a wider class of embeddings?

- How can the symmetries exhibited by these invariant curves be explained by the renormalization dynamics of the underlying IET?

- For a given IET (I, f), what is the structure of the PWIs that carry continuous embeddings of (I, f), and how can the regularity of the continuous embeddings be characterised within this class?

- If an IET has a non trivial embedding into a PWI, must its rotation parameters be irrationals? How does this relate to the behaviour of the rotational cocycle?

- For a given PWI, what are the arithmetic properties and structure of the IETs

(I, f) that are embedded within this PWI? Moreover, what is the structure of parametrizations of d-PWIs that embed the same given IET?

- How can these techniques be used to understand rational orbits in the neighbourhood of an embedding?

Particularly, developing on the last point, a natural line of investigation that opens up is to use these techniques to determine, in a large family of PWIs with non-zero rotational parameters, the existence of PWIs which exceptional sets have positive Lebesgue measure.

One of the central problems in dynamical systems theory is to investigate their measure theoretic properies. Although IETs have been well studied over the past years, the measure theoretic properties of PWIs are still far from understood.

In [32] Goetz studied a piecewise rotation with two atoms, with an exceptional set resembling a Sierpinsky gasket and shows that it has zero Lebesgue measure. Adler, Kitchens and Tresser [1] showed for a particular transformation where the rotations are rational that the regular set has full Lebesgue measure and, as a consequence, the exceptional set has zero Lebesgue measure. In [24] Cheung, Goetz and Quas studied a simple family of piecewise isometries of the plane parameterized by an angle parameter. They investigate the periodic islands around a particular family of periodic orbits and demonstrate that, for all angle parameters that are irrational multiples of π, the islands have asymptotic density in the plane of $3 \log 2 - \pi^2/8$.

Poggiaspalla [46] studied a class of renormalizable PWIs associated to primitive substitutions and computed the Hausdorff dimension of an invariant set, contained in the exceptional set, as a ratio $-\log(\Lambda)/\log(\lambda)$, where Λ is the largest eigenvalue of the substitution's incidence matrix and λ is the renormalization scaling factor. Recently, Hooper [35] investigated a family of polygon exchange maps, with no rotational parameters, invariant under a renormalization operation, related to Truchet tillings. He shows that for almost all parameters, the polygon exchange map has the property that almost every point is periodic. However, there is a dense set of irrational parameters for which this fails. By choosing parameters carefully, the measure of non-periodic points can be made arbitrarily close to full measure.

The above described papers made progress in understanding the exceptional set in particular families of PWIs. However these results are mostly dependent on particular choices of rotational parameters with convenient arithmetic properties. The results in [46] concern a possibly more general family of PWIs however with strong restrictions regarding its symbolic dynamics.

However, now that the existence of embeddings of IETs into PWIs is established this suggests a new approach to this problem. Particularly this provides tools to study the Lebesgue measure of the regular set of a PWI in a neighbourhood of a non-trivial embedding of an IET and also to investigate stability under perturbations of an embedding of an IET into a PWI. Together, these investigations may give global information regarding the abundance of embeddings of IETs in a given PWI from a generic family, in this manner giving bounds for the Lebesgue measure of the exceptional set of typical PWIs from this family.

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