The Inductance of a Wire Hoop

Many textbooks and webpages quote a formula for the inductance of a thin wire hoop, but few actually give the derivation. ***David Gibson*** *had been confused by formulas that appeared to differ by a factor of two but has eventually decided where the subtle difference lies. This leads to an interesting possibility for the design of a wideband loop antenna with a lower Q-factor than is normally achievable.*

Occasionally, there are simple problems that one fails to get to grips with, and for me, this is one of them, which I have revisited several times, in puzzlement.

In my PhD thesis, and for years afterwards, I mis-quoted the formula for the self-inductance of a wire hoop as including an $8r/w$ term[1], without anybody noticing that it should be $8d/w$. In CREG Journal 77 [Gibson, 2012] I corrected that error, but did not spot a further mistake, concerning the placement of the brackets, which has eluded me all these years. The correct formula is

$$L \approx \mu_0 r\left(\ln\frac{8d}{w} - 2\right), \text{ with } w << d \qquad (1)$$

The factor of two difference between $8d/w$ and $8r/w$ has puzzled me for a long while. The correct form, widely quoted and derived in, for example, [Ramo *et al*, 1984; §4.7, p190] and [Paul, 2009; §4.1] uses $8d/w$. But in [Clemmow, 1973; §4.4.1, p144] – my textbook of choice for electromagnetic theory – the term *seemed* to be given as $8r/w$. After careful checking, I have now eventually decided that it is *not* a misprint – i.e. *both* formulas are correct!

The subtle point I had missed is that Clemmow's derivation is based on first obtaining the mutual inductance between two infinitesimally thin hoops of radius r, spaced by Δr, resulting in a term of $8r/\Delta r$. The next stage (which I missed, having interpreted $8r/\Delta r$ as $8r/w$) considers the case where one filament is located at the *centre* of a thin wire of width $2\Delta r$, which represents a uniformly-distributed current, whilst the other is along the inner edge of the wire, representing the boundary of the surface over which we integrate the flux density to calculate the inductance. This results in the term being $8r/a \equiv 8d/w$.

There is a bit of a conceptual difficulty here: why *should* the mutual inductance between two filaments a distance Δr apart be the same as the self-inductance of a single hoop of width $2\Delta r$? The answer is that the same magnetic fields arise.

To some extent, the missing factor of two is not significant. Noting that $\ln 2 \approx 0.7$, we can write the log term with only a small difference in the formulation as

$$\ln\frac{8d}{w} - 2 \approx \ln\frac{8r}{w} - 1.3$$

We can also manipulate the term to give

$$\ln\frac{8d}{w} - 2 = \ln\frac{d}{w} + \ln 8 - 2 \approx \ln\frac{d}{w} + 0.08$$

which is a helpful simplification, meaning that we can re-write the inductance as

$$L \approx \mu_0 r \ln\frac{d}{w}, \text{ with } w << d \qquad (2)$$

Given that the traditional formula (1) is itself only an approximation, I am not sure why the above simplification is not more widely used.

Mutual Inductance

The observation about mutual inductance leads to an interesting situation, which I described in CREGJ 77. Suppose we connect two concentric hoops in parallel (see Figure). What is the overall inductance? In this situation, we cannot treat the two hoops as filamentary – each has its own self-inductance – and there is a mutual inductance between them, given by using $r/\Delta r$ in (1) or (2) instead of d/w. In simple circuit terms, we could say ***what is the inductance of two inductors, L_1 and L_2, connected in parallel, which have a mutual inductance of M?***

The exact solution to that problem is left as an exercise[2] for the reader. But the result is simplified by defining $L = \sqrt{(L_1 L_2)}$ and noting that if the two hoops are nearly the same size (i.e. $\Delta r/r << 1$) then their self-inductances approximate to L, giving

$$L_{total} \approx \tfrac{1}{2}(L + M) \qquad (3)$$

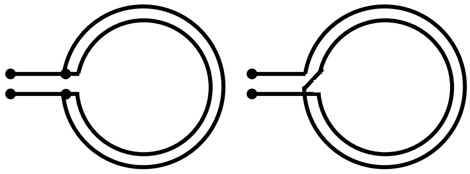
Thus, with no coupling (i.e. $M = 0$), the inductance is halved, and with full coupling (i.e. $M = L$) it is unchanged from L.

This provides an interesting method of reducing the inductance of a hoop. We know from the formula for self-inductance that one method is simply to increase the width of the wire. But another method now appears to be to connect two concentric hoops in parallel, a distance Δr apart. We can substitute (3) in (2) using $r/\Delta r$ to obtain M, to give

$$L_{total} \approx \mu_0 r \ln\left(r/\sqrt{a\Delta r}\right) \qquad (4)$$

where, notably, the effective width of the hoop is the geometric mean of a and Δr.

We can analyse a similar abstract circuit with L_1 and L_2 in series and show that L_{total} is now four times the value in (3), which is what we would expect by adding the usual N^2 term to (4). The Q-factor will be unaltered because the hoop resistance will also be four times greater. The salient difference is that series-connected hoops may suffer from a greater degree of self-capacitance effects.



Two concentric hoops, in parallel and in series – same reduced Q-factor but different inductance.

It seems reasonable to assume that similar arguments of flux coupling and mutual inductance will apply to co-axial hoops. We can further infer that for any multi-turn loop, we should not use (1) or (2) without modifying w to take account of the 'extent' of the full winding[3]. The salient point is that the turns need to be spaced and, perhaps, connected in parallel.

Concluding Remarks

The traditional formula for the inductance of a wire hoop contains an $8d/w$ term (1), which can be simplified to (2). A similar formula applies to the mutual inductance of two concentric hoops, with the term being $8r/\Delta r$ in (1). For two concentric hoops in parallel, the inductance is (3) and (4). Thus, there is a reduction in the Q-factor; it is lower than if a single hoop were formed of the same mass of wire.

References

Clemmow, PC (1973). *An Introduction to Electromagnetic Theory*. Cambridge: Cambridge University Press. ISBN 0-521-09815-7

Gibson, D (2012). Loop Antennas v. Ferrite Rods: A Case Study, CREGJ **77**, pp10-14. March 2012

Paul, Clayton R (2009). *Inductance: Loop and Partial*. John Wiley. ISBN 978-0-470-46188-4

1 The hoop's radius is r, and d is its diameter. The wire's radius is a, and w its diameter.

2 Unless I missed a trick, it is surprisingly tedious. The answer is $(L_1L_2-M^2)/(L_1+L_2-2M)$

3 There are plenty of approximate formulas for the inductance of squat solenoids, and the present note gives another perspective.